Some characterizations of function spaces around mixed Morrey spaces

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Chapter 1

Introduction

1.1 Mixed Morrey spaces

Function spaces collect functions having the same properties such as continuity, differentiability, and integrability, and so on. By characterizing the properties of function spaces, many fields of mathematics, for example, Fourier analysis and PDE, etc., have been developed. One of the most fundamental function spaces is the Lebesgue space $L^p(\mathbb{R}^n)$. For $0 < p \leq \infty$, we define the L^p norm $\|\cdot\|_{L^p}$ by

$$\|f\|_{L^p} = \|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \quad (0 < p < \infty),$$

where f is a measurable function on $\mathbb{R}^n$. If $p = \infty$, we interpret this expression as

$$\|f\|_{L^\infty} = \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

The Lebesgue space $L^p(\mathbb{R}^n)$ is the set of all measurable functions f for which $\|f\|_p < \infty$. Based on this space, the theory of analysis has advanced remarkably. However, integrability is not enough to describe sufficiently many properties of functions we require. Thus, function spaces having more fine properties were needed, and so during the 20th century many authors introduced a lot of function spaces such as Sobolev spaces, Orlicz spaces, Morrey spaces, Lorentz spaces, Hardy spaces, mixed Lebesgue spaces, Besov spaces, Triebel–Lizorkin spaces, and so on.

One of the important function spaces in this thesis is the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$. This is defined as follows: Let $0 < q \leq p < \infty$. Define the *Morrey norm* $\|\cdot\|_{\mathcal{M}^p_q}$ by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup\left\{ |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \;:\; Q \text{ is a cube in } \mathbb{R}^n \right\}$$

for a measurable function f. The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all measurable functions f for which $\|f\|_{\mathcal{M}^p_q}$ is finite.

Morrey spaces were introduced by C.B.Morrey Jr. in 1938 to investigate the local behavior of solutions to second order elliptic partial differential equation [97]. Morrey spaces cover Lebesgue spaces so that Morrey spaces describe nice properties more than Lebesgue spaces. For instance, Morrey spaces can handle the function $|x|^{-\frac{n}{p}}$ which fails to belong to Lebesgue spaces. We recall the fundamental properties of this space and the boundedness results for basic operators in harmonic analysis in Subsection 2.1.1.

Meanwhile, in [11], Benedek and Panzone introduced mixed Lebesgue spaces and investigated some properties in 1961. Let $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty]^n$. Then define the *mixed Lebesgue norm* $\|\cdot\|_{\vec{p}}$ or $\|\cdot\|_{(p_1,\ldots,p_n)}$ by

$$\begin{aligned}\|f\|_{L^{\vec{p}}} &= \|f\|_{\vec{p}} = \|f\|_{(p_1,\ldots,p_n)} \\ &\equiv \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_n)|^{p_1} \mathrm{d}x_1\right)^{\frac{p_2}{p_1}} \mathrm{d}x_2\right)^{\frac{p_3}{p_2}} \cdots \mathrm{d}x_n\right)^{\frac{1}{p_n}},\end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a measurable function. If $p_j = \infty$, then we have to make appropriate modifications. We define the *mixed Lebesgue space* $L^{\vec{p}}(\mathbb{R}^n) = L^{(p_1,\ldots,p_n)}(\mathbb{R}^n)$ to be the set of all measurable functions f on $\mathbb{R}^n$ with $\|f\|_{\vec{p}} < \infty$.

Since functions belonging to mixed Lebesgue spaces have the different integrability for each direction, we expect that they can characterize functions more subtly. In Subsection 2.1.2, we summarize their properties and the classical results for this space.

The author defined the mixed Morrey space $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ combining the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ and the mixed Lebesgue space $L^{\vec{q}}(\mathbb{R}^n)$ in [106].

Let $\vec{q} = (q_1, \ldots, q_n) \in (0, \infty]^n$ and $p \in (0, \infty)$ satisfy

$$\sum_{j=1}^n \frac{1}{q_j} \geq \frac{n}{p}.$$

Then define the *mixed Morrey norm* $\|\cdot\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}$ by

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \equiv \sup\left\{|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \|f\chi_Q\|_{\vec{q}} : \; Q \text{ is a cube in } \mathbb{R}^n\right\}$$

for all measurable functions f on $\mathbb{R}^n$. We define the *mixed Morrey space* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to be the set of all measurable function f on $\mathbb{R}^n$ satisfying $\|f\|_{\mathcal{M}^p_{\vec{q}}} < \infty$.

In [106], the author investigated the basic properties (for example, embedding, completeness) and the mapping properties of the Hardy–Littlewood maximal operator, fractional integral operators, and singular integral operators. We recall these properties in Subsection 2.1.3.

In this thesis, based on the studies in [106], we summarize the further studies of mixed Morrey spaces and the related spaces in [55, 106, 107, 108]. The next section is devoted to a summary of three topics and main theorems treated in each chapter.

1.2 Three topics and main theorems

From Chapter 3 to Chapter 5, we treat three topics related to mixed Morrey spaces. In this section, we give an overview of these topics and main results for each topic. First, we introduce the idea of commutators generated by functions and operators and consider the boundedness results for commutators on mixed Morrey spaces in Subsection 1.2.1. Subsection 1.2.2 is devoted to giving the concept of pointwise multiplier spaces, and we characterize them in terms of Morrey spaces. Finally, in Subsection 1.2.3, we consider the characterization by atomic decomposition for mixed Morrey spaces.

1.2.1 Main theorem on the boundedness of commutators generated by BMO functions and fractional integral operators on mixed Morrey spaces

The idea of commutators generated by functions and operators was introduced by Coifman, Rochberg, and Weiss [20] and many authors investigated the boundedness and compactness results for these operators. In particular, we treat commutators generated by functions and fractional integral operators. Let $0 < \alpha < n$. Define the fractional integral operator I_α of order α by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as long as the right-hand side makes sense. The commutator $[a, I_\alpha]$ is given by

$$[a, I_\alpha](f)(x) = a(x) I_\alpha f(x) - I_\alpha(af)(x) = \int_{\mathbb{R}^n} \frac{a(x)-a(y)}{|x-y|^{n-\alpha}} f(y) \mathrm{d}y$$

for a measurable function a and $x \in \mathbb{R}^n$ as long as the integral makes sense. Moreover, to describe our main theorem, we recall the BMO class. Define the BMO norm by

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f(y) - f_Q| \mathrm{d}y$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where

$$f_Q = \frac{1}{|Q|} \int_Q f(y) \mathrm{d}y$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$. Then, $\mathrm{BMO}(\mathbb{R}^n)$ is the set of all functions f modulo constants satisfying $\|f\|_{\mathrm{BMO}} < \infty$.

We recall the classical results on the boundedness of commutators on Morrey spaces. In 1991, Di Fazio and Ragusa [24] gave the necessary and sufficient condition on a function b for the boundedness of commutator $[b, I_\alpha]$ on Morrey spaces. Although there was a little additional assumption in their result, Shirai removed it in [140]. Our aim in Chapter 3 is to generalize these results to mixed Morrey spaces. Here we state our main theorem in Chapter 3.

Theorem 1.2.1. *Let* $0 < \alpha < n, 1 < p < \frac{n}{\alpha}, \frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$, *and* $\frac{n}{r} \le \sum_{j=1}^{n} \frac{1}{s_j}$. *Also, assume that*

$$\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q_j}{p} = \frac{s_j}{r} \quad (j = 1, \ldots, n).$$

Then, the following conditions are equivalent:

(a) $b \in \mathrm{BMO}(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ *is bounded from* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ *to* $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$.

(c) $[b, I_\alpha]$ *is bounded from* $\widetilde{\mathcal{M}}^p_{\vec{q}}(\mathbb{R}^n)$ *to* $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$.

(d) $[b, I_\alpha]$ *is bounded from* $\widetilde{\mathcal{M}}^p_{\vec{q}}(\mathbb{R}^n)$ *to* $\mathcal{M}^r_1(\mathbb{R}^n)$.

Here, $\widetilde{\mathcal{M}}^p_{\vec{q}}(\mathbb{R}^n)$ *is the* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$*-closure of* $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

We compare our results with the classical ones. Usually, when we handle commutators, the sharp maximal operator is a useful tool as was done in [24, 129, 130]. The sharp maximal operator, which is defined in [34], is a good operator to control the singularity of the integral operators. To control the sharp maximal operator, we use the so-called good λ-inequality described in [141]. However, the layer cake formula, which is also described in [141], is not available in the mixed-norm setting. So we make use of the dyadic local sharp maximal operator defined in [84] together with a key formula [84, Theorem 2.2]. By using these ingredients, we established the estimate for the sharp maximal operator on mixed Morrey spaces in Section 3.2. In Section 3.3, we give the proof of Theorem 1.2.1. We note that our method does not employ the predual spaces of Morrey spaces which were used in the previous works ([24, 140]).

1.2.2 Main theorems on the characterization of Morrey spaces associated with Banach lattice in terms of pointwise multiplier spaces

Given Banach spaces $E_1(\mathbb{R}^n)$ and $E_2(\mathbb{R}^n)$ of measurable functions defined on $\mathbb{R}^n$, we define $\mathrm{PWM}(E_1(\mathbb{R}^n), E_2(\mathbb{R}^n))$ as follows: A measurable function g is a *pointwise multiplier* from $E_1(\mathbb{R}^n)$ to $E_2(\mathbb{R}^n)$ if the pointwise product $f \cdot g$ belongs to $E_2(\mathbb{R}^n)$ for each $f \in E_1(\mathbb{R}^n)$ and there exists a constant $M > 0$ such that

$$\|f \cdot g\|_{E_2(\mathbb{R}^n)} \le M \|f\|_{E_1(\mathbb{R}^n)}. \tag{1.1}$$

One defines a norm on $\mathrm{PWM}(E_1(\mathbb{R}^n), E_2(\mathbb{R}^n))$ by

$$\|g\|_{\mathrm{PWM}(E_1,E_2)} \equiv \inf\{M > 0 : (1.1) \text{ holds for all } f \in E_1(\mathbb{R}^n)\}$$

for $g \in \mathrm{PWM}(E_1(\mathbb{R}^n), E_2(\mathbb{R}^n))$.

A typical example is

$$\mathrm{PWM}(L^p(\mathbb{R}^n), L^1(\mathbb{R}^n)) = L^{p'}(\mathbb{R}^n),$$

where $p' = \dfrac{p}{p-1}$ is a conjugate exponent of p. This example is easily obtained from the Hölder inequality.

In [82], to investigate the solution of the Navier–Stokes equation, Lemarié-Rieusset used that pointwise multiplier spaces from Besov spaces to Lebesgue spaces coincide Morrey spaces.

Theorem 1.2.2 (cf. [82])**.** *Let $1 \le p < \infty$ and $0 < s \le \dfrac{n}{p}$. Then*

$$\mathrm{PWM}(\dot{B}^s_{p1}(\mathbb{R}^n), L^p(\mathbb{R}^n)) \approx \mathcal{M}^{\frac{n}{s}}_{p}(\mathbb{R}^n)$$

with equivalence of norms.

Lemarié-Rieusset obtained Theorem 1.2.2 for $n = 3$ and $p = 2$ [82, Lemma 6]. A passage to the general case is a minor modification, so that we give a proof in Subsection 4.2.2.

Our main result generalizes this one by replacing Lebesgue spaces with abstract Banach lattices. Recall that a *Banach* (*function*) *lattice* on $\mathbb{R}^n$ is a Banach space $(E, \|\cdot\|_E)$ contained in the linear space of all measurable functions, such that, for all $f, g \in E$, the implication "$|f| \le |g| \Rightarrow \|f\|_E \le \|g\|_E$" holds. Then, we have also to generalize Morrey spaces and Besov spaces. To simplify the discussion, in this subsection and Chapter 4, we let $E(\mathbb{R}^n)$ be a Banach lattice be such that $\|f(\cdot - x)\|_E = \|f\|_E$ for all $f \in E(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Based on [58, Definition 2.6], we define $\mathcal{M}^p_E(\mathbb{R}^n)$ to be the set of all measurable functions f for which

$$\|f\|_{\mathcal{M}^p_E} \equiv \sup_Q |Q|^{\frac{1}{p}} \left(\frac{1}{\|\chi_Q\|_E} \|f\chi_Q\|_E \right)$$

is finite, where Q moves over all cubes whose edges are parallel to the coordinate axes. Note that if we take $E(\mathbb{R}^n) = L^q(\mathbb{R}^n)$, $\mathcal{M}^p_E(\mathbb{R}^n)$ coincides with the classical Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$.

Next, we define 2-microlocal Besov spaces. First, we recall the class $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$:

Definition 1.2.3 (Weight class $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$)**.** Let $\alpha_1, \alpha_2, \alpha_3 \in [0, \infty)$. The *class* $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ of weights is defined as the set of all the sequences of the measurable functions $w = \{w_j\}_{j=-\infty}^{\infty}$ satisfying the following conditions:

1. There exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,

$$0 < w_j(x) \le C w_j(y)(1 + 2^j|x - y|)^{\alpha_3}.$$

2. For all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,

$$2^{-\alpha_1} w_j(x) \le w_{j+1}(x) \le 2^{\alpha_2} w_j(x).$$

Such a sequence $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ is called an admissible weight sequence.

The (homogeneous) generalized 2-microlocal Besov spaces are usually defined by the use of the Fourier transform as follows. For $f \in L^1(\mathbb{R}^n)$, define its Fourier transform and its inverse Fourier transform by

$$\mathcal{F}f(\xi) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-\mathrm{i}x\cdot\xi} \mathrm{d}x, \quad \mathcal{F}^{-1}f(x) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi) \mathrm{e}^{\mathrm{i}x\cdot\xi} \mathrm{d}\xi.$$

By a well-known method, we can extend $\mathcal{F}, \mathcal{F}^{-1}$ naturally to the Schwartz distribution space $\mathcal{S}'(\mathbb{R}^n)$.

Definition 1.2.4 (Generalized 2-microlocal Besov spaces)**.** Let $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. Let $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ satisfy

$$\chi_{B(4)\setminus B(2)} \le \varphi \le \chi_{B(8)\setminus B(\frac{3}{2})}.$$

and define $\varphi_j(x) = \varphi(2^{-j}x)$. Let $0 < p, q \le \infty$. Then for $f \in \mathcal{S}'(\mathbb{R}^n)$ define

$$\|f\|_{\dot{B}^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n,w)} \equiv \left(\sum_{j=-\infty}^{\infty} 2^{js} \left\| w_j \mathcal{F}^{-1}[\varphi_j \mathcal{F} f] \right\|_p^q \right)^{\frac{1}{q}}.$$

The *generalized* 2*-microlocal Besov space* $\dot{B}^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, w)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{\dot{B}^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n,w)}$ is finite.

Note that if we take $w_j(x) = 1$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, we obtain classical Besov spaces.

To state our result, also recall the definition of dyadic cubes. For $j \in \mathbb{Z}$ and $m \equiv (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, we define $Q_{jm} \equiv \prod_{j=1}^{n} \left[\frac{m_j}{2^j}, \frac{m_j+1}{2^j} \right)$. Denote by $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ the set of such cubes. The elements in $\mathcal{D}$ are called *dyadic cubes*.

Our main result in Chapter 4 is the following assertion.

Theorem 1.2.5. *Let $E(\mathbb{R}^n)$ and $F(\mathbb{R}^n)$ be Banach lattices such that*

$$\|\chi_{Q_{jm}}\|_F \lesssim \|\chi_{Q_{jm}}\|_E |Q_{j0}|^{-\frac{1}{p}} \quad (j \in \mathbb{Z}, m \in \mathbb{Z}^n).$$

Set

$$w_j \equiv \|\chi_{Q_{j0}}\|_E |Q_{j0}|^{-1-\frac{1}{p}} \quad (j \in \mathbb{Z}).$$

Then $\dot{B}^{0,\mathrm{mloc}}_{11}(\mathbb{R}^n, w)$ is continuously embedded into $F(\mathbb{R}^n)$ and

$$\mathrm{PWM}(\dot{B}^{0,\mathrm{mloc}}_{11}(\mathbb{R}^n, w), E(\mathbb{R}^n)) \approx \mathcal{M}^p_E(\mathbb{R}^n)$$

with equivalence of norms.

In Section 4.2, we give the proofs of our results. Lemarié-Rieusset used the wavelet decomposition to show the corresponding result since he considered the assertion based on $L^2(\mathbb{R}^n)$. Meanwhile for the proof of Theorem 1.2.5, we employ the atomic decomposition for classical Besov spaces and 2-microlocal Besov spaces. Section 4.3 is devoted to apply our result for various function spaces, which are Orlicz spaces (Subsection 4.3.1), Lorentz spaces (Subsection 4.3.2), mixed Lebesgue spaces (Subsection 4.3.3), and mixed Morrey spaces (Subsection 4.3.4). In addition, the definition of each function space is given in Section 2.2.

1.2.3 Main theorems on the atomic decomposition for mixed Morrey spaces

One of the characterization methods of function spaces is to decompose functions or distributions into linear sums of elementary ones. In Chapter 5, we discuss the decomposition results for mixed Morrey spaces. In particular, we concentrate on the decomposition by the atom, which is a function with a compact support, a suitable norm estimate, and the moment condition. The decomposition results by atoms for classical Morrey spaces were proved by Iida, Sawano, and Tanaka in [66].

Denote by $\mathcal{Q}(\mathbb{R}^n)$ the set of all cubes in $\mathbb{R}^n$. Our first result is the following construction result about the functions in mixed Morrey spaces.

Theorem 1.2.6. *Suppose that the parameters $p, \vec{q}, s, \vec{t}$ satisfy*

$$1 < p < s < \infty, \quad 1 < \max\{q_1, \ldots, q_k\} < t_k < \infty \quad (k = 1, \ldots, n),$$

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n}{s} \le \sum_{j=1}^{n} \frac{1}{t_j}.$$

Assume that $\{a_j\}_{j=1}^\infty \subset \mathcal{M}^s_{\vec{t}}(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$, and $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ fulfill

$$\|a_j\|_{\mathcal{M}^s_{\vec{t}}} \le |Q_j|^{\frac{1}{s}}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\vec{q}}} < \infty.$$

Then $f = \sum_{j=1}^{\infty} \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \le C_{p,\vec{q},s,\vec{t}} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

The next assertion concerns the decomposition of functions in $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. For $d \ge 0$, denote by $\mathcal{P}_d(\mathbb{R}^n)$ the set of all polynomial functions with degree less than or equal to d, so that $\mathcal{P}(\mathbb{R}^n) \equiv \bigcup_{d=0}^{\infty} \mathcal{P}_d(\mathbb{R}^n)$. It is clear that $\mathcal{P}_{-1}(\mathbb{R}^n) = \{0\}$. The set $\mathcal{P}_K(\mathbb{R}^n)^{\perp}$ denotes the set of measurable function f for which

$$(1+|\cdot|^2)^{\frac{K}{2}} f \in L^1(\mathbb{R}^n) \quad \text{and} \quad \int_{\mathbb{R}^n} x^{\alpha} f(x) \mathrm{d}x = 0$$

for any $\alpha \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha| \le K$. Such a function f is said to satisfy the *moment condition of order* K. In this case, one also writes $f \perp \mathcal{P}_K(\mathbb{R}^n)$.

Theorem 1.2.7 (cf. [60])**.** *Suppose that the real parameters* $p, \vec{q}, K$ *satisfy*

$$1 < p < \infty, \quad \vec{q} \in (0,\infty)^n, \quad \frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad K \in \mathbb{N}_0 \cap \left(\frac{n}{q_0} - n - 1, \infty\right),$$

where $q_0 = \min(q_1, \ldots, q_n)$. *Let* $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. *Then there exists a triplet* $\{a_j\}_{j=1}^{\infty} \subset L^{\infty}(\mathbb{R}^n) \cap \mathcal{P}_K^{\perp}(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$, *and* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ *such that* $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *and that, for any* $v > 0$

$$|a_j| \le \chi_{Q_j}, \quad \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{\vec{q}}} \le C_v \|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

Here the constant $C_v > 0$ *is independent of* f.

Note that applying Theorems 1.2.6 and 1.2.7 for $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, we obtain norm estimate

$$C^{-1} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\vec{q}}} \le \|f\|_{\mathcal{M}^p_{\vec{q}}} \le C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\vec{q}}}$$

for some $C > 1$. Here λ_j and Q_j are same ones which appear in Theorem 1.2.7.

Although Theorem 1.2.7 can be given as a corollary of the abstract results in [60], we can give a direct proof without using Herz spaces which were used in the abstract setting of [60].

Moreover, we can prove the general decomposition theorems using Hardy-mixed Morrey spaces (Theorems 5.1.7 and 5.1.8 to follow). We can show that Hardy-mixed Morrey spaces coincide with mixed Morrey spaces for $\vec{q} > 1$ (Proposition 5.3.1). Therefore, Theorems 5.1.7 and 5.1.8 include Theorems 1.2.6 and 1.2.7, respectively.

As an application, we show the Olsen inequality for the fractional integral operator I_{α} acting on mixed Morrey spaces.

Theorem 1.2.8. *Suppose that the parameters* $\alpha, p, \vec{q}, p^*, \vec{q}^*, s, \vec{t}$ *satisfy*

$$1 < p, p^*, s < \infty, \quad 1 < \vec{q}, \vec{q}^*, \vec{t} < \infty,$$

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n}{p^*} \le \sum_{j=1}^{n} \frac{1}{q_j^*}, \quad \frac{n}{s} \le \sum_{j=1}^{n} \frac{1}{t_j},$$

$$\max\{t_1, \dots, t_j\} < q_j^*, \quad \frac{1}{p} > \frac{\alpha}{n}, \quad \frac{1}{p^*} \le \frac{\alpha}{n},$$

for each $j = 1, 2, \dots, n$, *and that*

$$\frac{1}{s} = \frac{1}{p^*} + \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t_j}{s} = \frac{q_j}{p} \quad (j = 1, 2, \dots, n).$$

Then for all $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ *and* $g \in \mathcal{M}^{p^*}_{\vec{q}^*}(\mathbb{R}^n)$

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^s_{\vec{t}}} \le C \|g\|_{\mathcal{M}^{p^*}_{\vec{q}^*}} \cdot \|f\|_{\mathcal{M}^p_{\vec{q}}},$$

where the constant C *is independent of* f *and* g.

As the special case of $q_i = q$ and $t_i = t$ for all $i = 1, \dots, n$, this result recaptures the one for classical Morrey spaces [134, Proposition 1.8].

In Section 5.2, we establish the boundedness result for the maximal operator to prove Theorem 1.2.6. We observe the characterization of Hardy-Morrey spaces in Section 5.3. Section 5.4 is devoted to the proof of the main theorems. In Subsections 5.4.1 and 5.4.2, we consider the reconstruction theorems for mixed Morrey spaces and Hardy-mixed Morrey spaces, respectively. Meanwhile in Subsection 5.4.3, we prove the decomposition theorems for mixed Morrey spaces and Hardy-mixed Morrey spaces. At last, in Section 5.5, we establish Olsen's inequality for mixed Morrey spaces by applying the atomic decompositions.

1.3 Notation

Throughout this thesis, we use the following notation.

1. The letters $\vec{p}, \vec{q}, \vec{r}, \dots$ denote the n-tuples of the numbers in $[0, \infty]$ $(n \ge 1)$, that is, $\vec{p} = (p_1, \dots, p_n), \vec{q} = (q_1, \dots, q_n), \vec{r} = (r_1, \dots, r_n)$.

2. The inequality, for example, $0 < \vec{p} < \infty$ means that $0 < p_j < \infty$ for each $j = 1, \dots, n$.

3. For $\vec{p} = (p_1, \dots, p_n)$ and $r \in \mathbb{R} \setminus \{0\}$, let

$$\frac{1}{\vec{p}} = \left(\frac{1}{p_1}, \dots, \frac{1}{p_n}\right), \quad \frac{\vec{p}}{r} = \left(\frac{p_1}{r}, \dots, \frac{p_n}{r}\right), \quad \vec{p}' = (p_1', \dots, p_n'),$$

   where $p_j' = \dfrac{p_j}{p_j - 1}$ is the conjugate exponent of p_j $(j = 1, \dots, n)$.

4. Let $Q = Q(x, r)$ be a cube having center x and radius r, whose sides are parallel to the coordinate axes. In particular, if $x = 0$, then we write $Q(r)$. The symbol $\mathcal{Q}$ denotes the set of all cubes.

5. Denote by $B(x, r)$ the ball centered at x and radius $r > 0$. We shall write $B(r) = B(0, r)$ as before.

6. The symbols $|Q|$ denotes the volume of the cube Q and $\ell(Q)$ denotes the side length of the cube Q.

7. For given a cube Q and $k > 0$, $k\,Q$ means a cube concentric to Q with sidelength $k\,\ell(Q)$.

8. For $j \in \mathbb{Z}$ and $m \equiv (m_1, \ldots, m_n) \in \mathbb{Z}^n$, we define $Q_{jm} \equiv \prod_{k=1}^{n} \left[\frac{m_k}{2^j}, \frac{m_k+1}{2^j}\right)$.

   Denote by $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ the set of such cubes. The elements in $\mathcal{D}$ are called *dyadic cubes*.

9. By $A \lesssim B$, we denote that $A \leq CB$ for some constant $C > 0$, and $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

10. When we need to emphasize or keep in mind that the constant C depends on the parameter α, β, etc, we write $C = C_{\alpha,\beta}$.

11. We write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

12. We use "$\cdot$" for functions; $f = f(\cdot)$. In particular, when we only use "$\cdot$" for j-th coordinate, we write $f = f(\cdot_j)$.

13. Let E be a measurable set in $\mathbb{R}^n$. Then, χ_E denotes the characteristic function for E.

14. Let E be a measurable set in $\mathbb{R}^n$ and f be a measurable function. Then,

$$\|f\|_{L^p(E)} \equiv \|f\chi_E\|_p.$$

15. Let w be a nonnegative measurable function. Then, $\|\cdot\|_{L^p(w)}$ denote the weighted Lebesgue norm, that is, for a measurable function f,

$$\|f\|_{L^p(w)} \equiv \|fw\|_p.$$

16. The norm $\|\cdot\|_*$ denote the operator norm.

17. We define $L^0(\mathbb{R}^n)$ as the set of all measurable functions on $\mathbb{R}^n$.

18. $\mathcal{S}(\mathbb{R}^n)$ denote the set of all rapidly decreasing functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ its dual (that is, the set of all tempered distributions on $\mathbb{R}^n$).

19. Let $L^{\vec{q}}_{\rm c}(\mathbb{R}^n)$ denote the set of all $L^{\vec{q}}(\mathbb{R}^n)$ functions with compact support.

Chapter 2

Preliminaries

In this chapter, we provide the definition and some properties of function spaces which are used in this thesis. In Section 2.1, we recall mixed Morrey spaces and their related spaces. First, we review Morrey spaces and mixed Lebesgue spaces in Subsections 2.1.1 and 2.1.2, respectively. After that we summarize the definition and some results of mixed Morrey spaces investigated in [106]. At the end of this section, we consider the predual spaces of mixed Morrey spaces. We use the predual spaces in Chapter 5. Section 2.2 is devoted to introducing some function spaces which we apply to in Section 4. At first, we recall Besov spaces in Subsection 2.2.1. We also consider generalized 2-microlocal Besov spaces which generalize the weighted Besov spaces in Subsection 2.2.2. In the last two subsections, we recall Lorentz spaces and Orlicz spaces, which are other generalizations of Lebesgue spaces.

2.1 Mixed Morrey spaces and related spaces

2.1.1 Classical Morrey spaces and fundamental results

In this subsection, we recall the definition and some properties of the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$. For their proofs, we refer to [117].

Definition 2.1.1. Let $0 < q \le p < \infty$. Define the *Morrey norm* $\|\cdot\|_{\mathcal{M}^p_q}$ by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup\left\{ |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \;:\; Q \text{ is a cube in } \mathbb{R}^n \right\}$$

for a measurable function f. The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all measurable functions f for which $\|f\|_{\mathcal{M}^p_q}$ is finite.

Remark 2.1.2. We can also define the following norm:

$$\|f\|^{\text{ball}}_{\mathcal{M}^p_q} \equiv \sup\left\{ |B|^{\frac{1}{p}-\frac{1}{q}} \left(\int_B |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \;:\; B \text{ is a ball in } \mathbb{R}^n \right\}.$$

Then, using the fact

$$B(x,r) \subset Q(x,r) \subset B(x,\sqrt{n}r),$$

we see that the norms $\|f\|_{\mathcal{M}^p_q}$ and $\|f\|^{\mathrm{ball}}_{\mathcal{M}^p_q}$ are equivalent. Thus, we will use the suitable one and denote it by the same notation $\|\cdot\|_{\mathcal{M}^p_q}$.

First of all, we point out the fundamental properties of Morrey spaces.

Theorem 2.1.3. *Let $1 \le q \le p < \infty$. Then, the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is a Banach space.*

The next proposition suggests that Morrey spaces are generalizations of Lebesgue spaces.

Proposition 2.1.4. *Let $0 < p < \infty$. Then, $\mathcal{M}^p_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.*

The relation of two different Morrey spaces is as follows.

Proposition 2.1.5. *Let $0 < q_1 \le q_2 \le p < \infty$. Then, we have*

$$L^p(\mathbb{R}^n) = \mathcal{M}^p_p(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_2}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_1}(\mathbb{R}^n).$$

Next, we consider the examples of functions belonging to Morrey spaces.

Example 2.1.6 ([126, Exercise 6.17])**.** Let Q be a cube in $\mathbb{R}^n$. Then, for $0 < q < p < \infty$

$$\|\chi_Q\|_{\mathcal{M}^p_q} = |Q|^{\frac{1}{p}}.$$

Example 2.1.7 ([79, Lemma 4.1])**.** Let $0 < q < p < \infty$. Then, $|x|^{-\frac{n}{p}} \in \mathcal{M}^p_q(\mathbb{R}^n)$.

Note that $|x|^{-\frac{n}{p}}$ does not belong to $L^p(\mathbb{R}^n)$ for $0 < p < \infty$. Therefore, the Lebesgue space $L^p(\mathbb{R}^n)$ is proper subset of the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$.

Finally, we recall the studies of Morrey spaces. Morrey spaces were introduced by C.B.Morrey Jr. in 1938 to investigate the local behavior of solutions to second order elliptic partial differential equation [97]. Later, many authors investigated Morrey spaces. In 1960s, Campanato introduced and studied Campanato spaces which coincide with the many function spaces, the BMO space, Lipschitz spaces, Hölder spaces, and Morrey spaces. Peetre [114] gave a survey of Morrey spaces and Campanato spaces in 1969. In this survey, he also investigated the boundedness of the singular integral operators. Singular integral operators on Morrey spaces have several definitions via preduals and weight theory. See [19, 120, 121]. As the last point in the first development, Adams pointed out that the fractional integral operator I_α is bounded on Morrey spaces in [1].

Theorem 2.1.8. *Let $0 < \alpha < n, 1 < q \le p < \infty$, and $1 < s \le r < \infty$. Assume that*

$$\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{p}{q} = \frac{r}{s}.$$

Then, for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$,

$$\|I_\alpha f\|_{\mathcal{M}^r_s} \lesssim \|f\|_{\mathcal{M}^p_q}.$$

The next turning point is that Chiarenza and Frasca proved the boundedness of the Hardy–Littlewood maximal operator on Morrey spaces in 1987 [19]. The Hardy–Littlewood maximal operator is one of the most important operators in harmonic analysis. This operator is defined for all measurable functions f and $x \in \mathbb{R}^n$ as

$$Mf(x) = \sup_Q \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| \mathrm{d}y,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$. The basic idea of this operator was introduced by Hardy and Littlewood [54] in the language of cricket. Frankly speaking, this operator is taking the largest average of a function over all cubes containing x. The technique of averaging naturally arises in many situations, so it is very significant. The boundedness result of the Hardy–Littlewood maximal operator on Morrey spaces is as follows.

Theorem 2.1.9. *Let $1 < q \le p < \infty$. Then*

$$\|Mf\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q}$$

for all $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.

Using the maximal operator, Di Fazio and Ragusa [24] and Shirai [140] proved commutators generated by BMO functions and the fractional integral operator I_α are bounded on Morrey spaces (these statements will appear in Theorems 3.1.2 and 3.1.3).

Concerning the duality, Long proved that the block space $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is a predual space of the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ in [87]. Zorko characterized the predual space by means of the atomic decomposition in [152]. In [73], Kalita constructed another predual space of Morrey spaces which is the same space as Zorko's spaces with norm equivalence. Furthermore, by using the theory of capacities, the third predual space was defined by Adams and Xiao [2]. See also a recent survey by Rosenthal and Triebel [122].

Furthermore, Morrrey spaces were generalized by many authors in various directions. First of all, we take up generalized Morrey spaces. A definition which are still often used goes back to Zorko's paper [152]. Later, Mizuhara [93], Nakai [98], and Guliyev [43] defined and investigated generalized Morrey spaces, respectively, and many authors studied them in [22, 30, 44, 45, 79, 53]. For an application to partial differential equations, we refer to [3, 23, 78, 86]. See also a survey [127]. Next, we turn to the weight theory of Morry spaces. We have two different definitions of weighted Morrey spaces which are Samko type [123] and Komori–Shirai type [80]. We regard the weight as the one for functions in the former, as the one for measures in the latter. Each of them is investigated in [103, 104] and [65, 139, 151]. Furthermore, both of them appear in the study of partial differential equations [32, 51, 52, 138]. In addition, Morrey spaces for non-doubling measures were defined by Sawano and Tanaka and investigated the boundedness property of some operators [124, 130, 132], and variable exponent Morrey spaces were studied in [4, 48, 49, 94, 95].

2.1.2 Mixed Lebesgue spaces and fundamental results

In this subsection, we recall the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ which is introduced by Benedek and Panzone in [11]. Since the proofs are overall elementary, we omit the details; see [11].

Definition 2.1.10. Let $\vec{p} = (p_1, \dots, p_n) \in (0, \infty]^n$. Then define the *mixed Lebesgue norm* $\|\cdot\|_{\vec{p}}$ or $\|\cdot\|_{(p_1,\dots,p_n)}$ by

$$\begin{aligned}\|f\|_{L^{\vec{p}}} &= \|f\|_{\vec{p}} = \|f\|_{(p_1,\dots,p_n)}\\ &\equiv \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \dots, x_n)|^{p_1} \mathrm{d}x_1 \right)^{\frac{p_2}{p_1}} \mathrm{d}x_2 \right)^{\frac{p_3}{p_2}} \cdots \mathrm{d}x_n \right)^{\frac{1}{p_n}},\end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a measurable function. If $p_j = \infty$, then we have to make appropriate modifications. We define the *mixed Lebesgue space* $L^{\vec{p}}(\mathbb{R}^n) = L^{(p_1,\dots,p_n)}(\mathbb{R}^n)$ to be the set of all measurable functions f on $\mathbb{R}^n$ with $\|f\|_{\vec{p}} < \infty$.

Note that if each $p_i = p$, then $L^{\vec{p}}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, so mixed Lebesgue spaces generalize classical spaces. This space has properties similar to classical Lebesgue space.

Proposition 2.1.11. *For $1 \le \vec{p} \le \infty$, $L^{\vec{p}}(\mathbb{R}^n)$ is a Banach space.*

Proposition 2.1.12 (Hölder's inequality)**.** *Let $1 < \vec{p}, \vec{q} < \infty$ and define $\vec{r}$ so that $\frac{1}{\vec{p}} + \frac{1}{\vec{q}} = \frac{1}{\vec{r}}$. If $f \in L^{\vec{p}}(\mathbb{R}^n), g \in L^{\vec{q}}(\mathbb{R}^n)$, then $fg \in L^{\vec{r}}(\mathbb{R}^n)$, and*

$$\|fg\|_{\vec{r}} \le \|f\|_{\vec{p}} \|g\|_{\vec{q}}.$$

Proposition 2.1.13. *Let $0 < \vec{p} \le \infty$. The mixed Lebesgue norm has the dilation relation: for all $f \in L^{\vec{p}}(\mathbb{R}^n)$ and $t > 0$,*

$$\|f(t\cdot)\|_{\vec{p}} = t^{-\sum_{j=1}^{n} \frac{1}{p_j}} \|f\|_{\vec{p}}.$$

The mapping

$$(x_2, \dots, x_n) \in \mathbb{R}^{n-1} \mapsto \|f\|_{(p_1)}(x_2, \dots, x_n) \equiv \left(\int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{p_1} \mathrm{d}x_1 \right)^{\frac{1}{p_1}}$$

is a measurable function and defined on $\mathbb{R}^{n-1}$. Moreover, we define

$$\|f\|_{\vec{q}} = \|f\|_{(p_1,\dots,p_j)} \equiv \left\| \left[\|f\|_{(p_1,\dots,p_{j-1})} \right] \right\|_{(p_j)},$$

where $\|f\|_{(p_1,\dots,p_{j-1})}$ denotes $|f|$, if $j = 1$ and $\vec{q} = (p_1, \dots, p_j), j \le n$. Note that $\|f\|_{\vec{q}}$ is a measurable function of $(x_{j+1}, \dots, x_n)$ for $j < n$.

Next, we consider the examples of $L^{\vec{p}}(\mathbb{R}^n)$.

Example 2.1.14. Let Q be a cube. Then, for $0 < \vec{p} \le \infty$,

$$\|\chi_Q\|_{\vec{p}} = |Q|^{\frac{1}{n}(\frac{1}{p_1}+\cdots+\frac{1}{p_n})}. \tag{2.1}$$

This identity is important to consider some inequalities of mixed Morrey spaces.

Example 2.1.15. Let $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $\{a_m\}_{m\in\mathbb{Z}^n} \subset \mathbb{C}$. Define

$$f(x) = \sum_{m\in\mathbb{Z}^n} a_m \chi_{m+[0,1]^n}(x).$$

Then,

$$\|f\|_{\vec{p}} = \left(\sum_{m_n\in\mathbb{Z}} \cdots \left(\sum_{m_1\in\mathbb{Z}} \left|a_{(m_1,\ldots,m_n)}\right|^{p_1}\right)^{\frac{p_2}{p_1}} \cdots\right)^{\frac{1}{p_n}}. \tag{2.2}$$

We can consider the right-hand side of (2.2) as a mixed sequence norm, which computes each ℓ^{p_i}-norm with respect to m_i. We denote it by $\|\{a_m\}_{m\in\mathbb{Z}^n}\|_{\ell^{(p_1,\ldots,p_n)}}$:

$$\begin{aligned} \|\{a_m\}_{m\in\mathbb{Z}^n}\|_{\ell^{(p_1,\ldots,p_n)}} &= \|a_{(m_1,\ldots,m_n)}\|_{\ell^{(p_1,\ldots,p_n)}} \\ &\equiv \left(\sum_{m_n\in\mathbb{Z}} \cdots \left(\sum_{m_2\in\mathbb{Z}} \left(\sum_{m_1\in\mathbb{Z}} \left|a_{(m_1,\ldots,m_n)}\right|^{p_1}\right)^{\frac{p_2}{p_1}}\right)^{\frac{p_3}{p_2}} \cdots\right)^{\frac{1}{p_n}}. \end{aligned}$$

Furthermore, this norm is also defined inductively:

$$\|a_{(m_1,\ldots,m_n)}\|_{\ell^{(p_1,\ldots,p_j)}} \equiv \left\| \left[\|a_{(m_1,\ldots,m_n)}\|_{\ell^{(p_1,\ldots,p_{j-1})}} \right] \right\|_{\ell^{(p_j)}},$$

where $\|a_{(m_1,\ldots,m_n)}\|_{\ell^{(p_1,\ldots,p_{j-1})}} = |a_{(m_1,\ldots,m_n)}|$ if $j = 1$ and

$$\|a_{(m_1,\ldots,m_n)}\|_{\ell^{(p_j)}} \equiv \left(\sum_{m_j\in\mathbb{Z}} |a_{(m_1,\ldots,m_n)}|^{p_j}\right)^{\frac{1}{p_j}}$$

for $j = 1, \ldots, n$.

We survey the studies of mixed Lebesgue spaces. Benedek and Panzone investigated fundamental properties (completeness, duality, reflexivity, etc.), a counterpart to the Riesz–Thorin interpolation theorem, the boundedness of the fractional integral operator, and so on. After this paper, there are a lot of studies for mixed Lebesgue spaces. In 1975, Bagby showed the boundedness of the Hardy–Littlewood maximal operator for the functions taking values in the mixed Lebesgue spaces [10]. In particular, Stöckert considered that the strong maximal operator is bounded on mixed Lebesgue spaces [144]. The author, in [106], gave another simple proof using the above Bagby's result. See also [35, 57] for this boundedness. Singular integral operators were studied in [35, 142] by Fernandez, and by Stevanov and Torres, respectively. Additionally, there exist many remarkable works which concern Hörmander–Mikhlin theorem [7], multivariate rearrangements [15, 37], the inclusion problem [42], the theory of variable exponents [59], and the interpolation theory [63, 91]. Recently, Huang and Yang summarized a recent series of investigation on function spaces with mixed norms [62].

From the viewpoint of applications, this mixed norm serves to describe decay at infinity for each direction. For example, we consider a bounded measurable function f on $\mathbb{R}^3$ satisfying

$$|f(x)| \le \frac{C}{|x_1|^{1/10}|x'|^{10}}, \quad |x_1| > 1, |x'| > 1, C > 0$$

for $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^2$, where $x' = (x_2, x_3)$. Then, $f \in L^p(\mathbb{R}^3)$ if $p > 10$, that is, the fast decaying directions are completely ignored in the classical Lebesgue norm. So, in this framework of Lebesgue spaces, this function f is identified with the same as a function g satisfying

$$|g(x)| \le \frac{C}{|x|^{1/10}}, \quad |x| > 1, C > 0.$$

Meanwhile, using the mixed Lebesgue norm, we can see $f \in L^{\vec{p}}(\mathbb{R}^n)$ for $p_1 > 10$ and $p_2, p_3 > \frac{1}{10}$. Employing this idea, many authors established and analyzed solutions for partial differential equations, abstract elliptic and parabolic equations, the Navier–Stokes equation, and etc. [28, 112, 113].

2.1.3 Mixed Morrey spaces and some results

In this subsection, based on [106], we collect some properties and boundedness results of fundamental operators in harmonic analysis. All the proofs are in [106].

Definition 2.1.16. Let $\vec{q} = (q_1, \ldots, q_n) \in (0, \infty]^n$ and $p \in (0, \infty]$ satisfy

$$\sum_{j=1}^{n} \frac{1}{q_j} \ge \frac{n}{p}.$$

Then define the *mixed Morrey norm* $\|\cdot\|_{\mathcal{M}^p_{\vec{q}}}$ by

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \equiv \sup\left\{ |Q|^{\frac{1}{p} - \frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \|f\chi_Q\|_{\vec{q}} : \ Q \text{ is a cube in } \mathbb{R}^n \right\}$$

for all measurable functions f on $\mathbb{R}^n$. We define the *mixed Morrey space* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to be the set of all measurable functions f on $\mathbb{R}^n$ satisfying $\|f\|_{\mathcal{M}^p_{\vec{q}}} < \infty$.

First, we remark the relations of Morrey spaces and mixed Lebesgue spaces. If each $q_i = q \in (0, \infty)$, then $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n)$ with coincidence of norm. In particular, choose $\vec{q} \in (0,\infty]^n$ satisfying $\dfrac{1}{p} = \dfrac{1}{n}\sum_{j=1}^{n}\dfrac{1}{q_j}$ so that $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) = L^{\vec{q}}(\mathbb{R}^n)$ with coincidence of norm.

Note that Scapellato and Ragusa [137] introduced the same named space "Mixed Morrey space", which is different from our space, essentially.

We give the properties of the *mixed Morrey spaces.* Just as with mixed Lebesgue spaces, we see that mixed Morrey norm has the following dilation relation:

$$\|f(t\cdot)\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = t^{-\frac{n}{p}}\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \quad (f \in L^0(\mathbb{R}^n), t > 0), \tag{2.3}$$

for $\vec{q} \in (0,\infty]^n$ and $p \in (0,\infty]$ with $\dfrac{n}{p} \le \displaystyle\sum_{j=1}^{n}\dfrac{1}{q_j}$. The embedding properties are as follows:

Proposition 2.1.17. ([106, Proposition 3.2]) *Let $0 < \vec{q} \le \vec{r} \le \infty$, $0 < p < \infty$, and assume $\dfrac{1}{r_1} + \cdots + \dfrac{1}{r_n} \ge \dfrac{n}{p}$. Then,*

$$\mathcal{M}^p_{\vec{r}}(\mathbb{R}^n) \subset \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n).$$

Let us give some examples.

Example 2.1.18. By Example 2.1.7, $f(x) = |x|^{-\frac{n}{p}} \in \mathcal{M}^p_q(\mathbb{R}^n)$ if $q < p$. Let $\vec{q} = (q_1, \ldots, q_n)$. Using Proposition 2.1.17, we have

$$\mathcal{M}^p_{\widetilde{q}}(\mathbb{R}^n) = \mathcal{M}^p_{\underbrace{(\widetilde{q},\ldots,\widetilde{q})}_{n \text{ times}}}(\mathbb{R}^n) \subset \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n),$$

where $\widetilde{q} = \max(q_1, \ldots, q_n)$. Thus, if $\max(q_1, \ldots, q_n) = \widetilde{q} < p$,

$$f(x) = |x|^{-\frac{n}{p}} \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n).$$

However, the condition

$$\max(q_1, \ldots, q_n) = \widetilde{q} < p \tag{2.4}$$

is a sufficient condition but is not a necessary condition for $f(x) = |x|^{-\frac{n}{p}} \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. In fact, consider the case $\vec{s} = (s_1, \underbrace{\infty, \ldots, \infty}_{(n-1) \text{ times}})$ and $s_1 < \dfrac{p}{n}$. Then, by Proposition 2.1.13,

$$\begin{aligned}
\|f\|_{\mathcal{M}^p_{\vec{s}}(\mathbb{R}^n)} &= \sup_{Q=Q(x,r)} |Q(x,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{s_j}\right)}\|f\chi_{Q(x,r)}\|_{\vec{s}} \\
&= \sup_{r>0} |Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{s_j}\right)}\|f\chi_{Q(0,r)}\|_{\vec{s}} \\
&= \sup_{r>0} |Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{s_j}\right)}\|f\chi_{Q(0,1)}\|_{\vec{s}} \times r^{\sum_{j=1}^n \frac{1}{s_j}} \times r^{-\frac{n}{p}} \\
&\le \left(\int_{-1}^{1} |x_1|^{-\frac{n}{p}s_1}\mathrm{d}x_1\right)^{\frac{1}{s_1}}.
\end{aligned}$$

Since $s_1 < \frac{p}{n}$, $\|f\|_{\mathcal{M}^p_{\vec{s}}(\mathbb{R}^n)} < \infty$ and $f \in \mathcal{M}^p_{\vec{s}}(\mathbb{R}^n)$. But $\vec{s}$ does not satisfy (2.4).

Example 2.1.19. Let $0 < \vec{q} \le \infty$ and assume that $q_j < p_j$ if $p_j < \infty$ and that $q_j \le \infty$ if $p_j = \infty$ $(j = 1, \ldots, n)$. Let

$$\sum_{j=1}^{n} \frac{1}{p_j} = \frac{n}{p}. \tag{2.5}$$

Then, we have

$$f(x) = \prod_{j=1}^{n} |x_j|^{-\frac{1}{p_j}} \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n).$$

Furthermore, condition (2.5) is a necessary and sufficient condition for $f(x) = \prod_{j=1}^{n} |x_j|^{-\frac{1}{p_j}}$ to be a member in $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. In fact, let $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ and $f \neq 0$. Applying (2.3), we have

$$\|f(t\cdot)\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = t^{-\frac{n}{p}} \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \quad (t > 0). \tag{2.6}$$

On the other hand, since $f(tx) = t^{-\sum_{j=1}^{n} \frac{1}{p_j}} f(x)$,

$$\|f(t\cdot)\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = t^{-\sum_{j=1}^{n} \frac{1}{p_j}} \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}. \tag{2.7}$$

By (2.6) and (2.7), for all $t > 0$,

$$t^{-\sum_{j=1}^{n} \frac{1}{p_j}} = t^{-\frac{n}{p}}.$$

Thus, we obtain (2.5).

Example 2.1.20. Let Q be a cube and $\vec{q} \in (0, \infty]^n$. Then,

$$\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = |Q|^{\frac{1}{p}}.$$

To check this, put $\sum\limits_{j=1}^{n} \frac{1}{q_j} = \bar{q}$. First, using (2.1), we get

$$\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = \sup_{R \in \mathcal{Q}} |R|^{\frac{1}{p} - \frac{\bar{q}}{n}} \|\chi_Q \chi_R\|_{\vec{q}} \ge |Q|^{\frac{1}{p} - \frac{\bar{q}}{n}} \|\chi_Q\|_{\vec{q}} = |Q|^{\frac{1}{p} - \frac{\bar{q}}{n}} |Q|^{\frac{\bar{q}}{n}} = |Q|^{\frac{1}{p}}.$$

Meanwhile, thanks to Proposition 2.1.17,

$$\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \le \|\chi_Q\|_{\mathcal{M}^p_{\max(q_1, \ldots, q_n)}(\mathbb{R}^n)} = |Q|^{\frac{1}{p}}.$$

Combining the above two inequalities, we obtain

$$\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = |Q|^{\frac{1}{p}}.$$

We turn to the boundedness results for some operators on mixed Morrey spaces. First, we consider the Hardy–Littlewood maximal operator. The boundedness of the Hardy–Littlewood maximal operator in classical Morrey spaces is showed by Chiarenza and Frasca in 1987 [19] (see Theorem 2.1.9 above).

Theorem 2.1.21 ([106, Theorem 4.5])**.** *Let $1 < \vec{q} < \infty$ and $1 < p < \infty$ satisfy $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$. Then*

$$\|Mf\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}}$$

for all $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

Next, we give the boundedness result of the fractional integral operator I_α. Its boundedness in classical Morrey spaces is proved by Adams [1] (see, Theorem 2.1.8).

Theorem 2.1.22 ([106, Theorem 1.11])**.** *Let $0 < \alpha < n, 1 < \vec{q}, \vec{s} < \infty$ and $1 < p, r < \infty$. Assume that $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$ and $\frac{n}{r} \le \sum_{j=1}^{n} \frac{1}{s_j}$. Also, assume that*

$$\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{\vec{q}}{p} = \frac{\vec{s}}{r}.$$

Then, for $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$,

$$\|I_\alpha f\|_{\mathcal{M}^r_{\vec{s}}} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

Finally, we recall the boundedness results for singular integral operators. A singular integral operator T with a kernel $k(x, y)$ is defined as an L^2-bounded operator which satisfies the following conditions:

(1) There exists a constant $C > 0$ such that $|k(x, y)| \le \dfrac{C}{|x - y|^n}$.

(2) There exist $\varepsilon > 0$ and $C > 0$ such that

$$|k(x, y) - k(z, y)| + |k(y, x) - k(y, z)| \le C \frac{|x - z|^\varepsilon}{|x - y|^{n+\varepsilon}},$$

if $|x - y| \ge 2|x - z|$ with $x \ne y$.

(3) If $f \in L^\infty_c(\mathbb{R}^n)$, the set of all compactly supported L^∞-functions, then

$$Tf(x) = \int_{\mathbb{R}^n} k(x, y) f(y) \mathrm{d}y \quad (x \notin \mathrm{supp}(f)).$$

Keeping in mind that T extends to a bounded linear operator on $\mathcal{M}^p_q(\mathbb{R}^n)$ [19], we obtain the following theorem.

Theorem 2.1.23 ([106, Theorem 1.12]). *Let $1 < \vec{q} < \infty$ and $1 < p < \infty$ satisfy*

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}.$$

Then, if we restrict T to $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, which is initially defined on $\mathcal{M}^p_{\min(q_1,\dots,q_n)}(\mathbb{R}^n)$,

$$\|Tf\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}$$

for $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

At last in this subsection, we describe the Fefferman–Stein vector-valued inequality on mixed Morrey spaces. This inequality was first considered by Fefferman and Stein in [33].

Proposition 2.1.24 ([106, Theorem 1.8]). *Let $1 < \vec{q}, p < \infty$, $\frac{n}{p} \le \sum\limits_{j=1}^{n} \frac{1}{q_j}$, and $1 < r \le \infty$. Then*

$$\left\| \left(\sum_{j=1}^{\infty} (Mf_j)^r \right)^{1/r} \right\|_{\mathcal{M}^p_{\vec{q}}} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{1/r} \right\|_{\mathcal{M}^p_{\vec{q}}}$$

for all sequences of measurable functions $\{f_j\}_{j=1}^{\infty}$.

See [130, Theorem 2.2] and [146, Lemma 2.5] for the case of classical Morrey spaces.

2.1.4 Predual spaces of mixed Morrey spaces

In this subsection, we introduce the predual spaces of mixed Morrey spaces following the idea of Long [87]. We need these space in Section 5.

Definition 2.1.25. Let $1 \le p < \infty$ and $\frac{n}{p} \le \sum\limits_{j=1}^{n} \frac{1}{q_j}$. A measurable function A is said to be a $(p, \vec{q})$-block if there exists a cube Q that supports A such that

$$\|A\|_{\vec{q}} \le |Q|^{\frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right) - \frac{1}{p}}.$$

Note that the idea of blocks was introduced by Taibleson and Weiss to investigate the a.e. convergence of the Fourier series in [145]. Based on this functions, we define the following function spaces.

Definition 2.1.26. Let $1 \le p < \infty$ and $\frac{n}{p} \le \sum\limits_{j=1}^{n} \frac{1}{q_j}$. Define the function space $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$ as the set of all $f \in L^p(\mathbb{R}^n)$ for which f is realized as the sum $f = \sum\limits_{j=0}^{\infty} \lambda_j A_j$ with some

$\lambda = \{\lambda_j\}_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0)$ and a sequence $\{A_j\}_{j\in\mathbb{N}_0}$ of $(p', \vec{q}\,')$-blocks. The norm $\|f\|_{\mathcal{H}^{p'}_{\vec{q}\,'}}$ for $f \in \mathcal{H}^{p'}_{\vec{q}\,'}(\mathbb{R}^n)$ is defined as

$$\|f\|_{\mathcal{H}^{p'}_{\vec{q}\,'}} \equiv \inf_{\lambda} \|\lambda\|_{\ell^1},$$

where $\lambda = \{\lambda_j\}_{j\in\mathbb{N}_0}$ runs over all admissible expressions

$$f = \sum_{j=0}^{\infty} \lambda_j A_j,\ \{\lambda_j\}_{j\in\mathbb{N}_0} \in \ell^1,\ A_j \text{ is a } (p', \vec{q}\,')\text{-block for all } j \in \mathbb{N}_0. \tag{2.8}$$

Note that if $q_1 = \cdots = q_n$, then the notion of $(p, \vec{q})$-block and the one of $\mathcal{H}^{p'}_{\vec{q}\,'}(\mathbb{R}^n)$ coincide the classical ones.

Remark 2.1.27. As is easily verified by Hölder's inequality, any $(p, \vec{q})$-block has L^p norm less than 1;

$$\|A\|_p \le 1$$

for all blocks A. Due to this fact, the series $f = \sum_{j=0}^{\infty} \lambda_j A_j$ in Definition 2.1.26 converges in the topology of $L^p(\mathbb{R}^n)$. In fact, let $n < m$. Then,

$$\left\| \sum_{j=0}^{m} \lambda_j A_j - \sum_{j=0}^{n} \lambda_j A_j \right\|_p = \left\| \sum_{j=n+1}^{m} \lambda_j A_j \right\|_p \le \sum_{j=n+1}^{m} |\lambda_j| \|A_j\|_p \le \sum_{j=n+1}^{m} |\lambda_j| \longrightarrow 0$$

as $n, m \to 0$. Thus, this series converges in the topology of $L^p(\mathbb{R}^n)$.

We shall see some properties of the space $\mathcal{H}^{p'}_{\vec{q}\,'}(\mathbb{R}^n)$.

Lemma 2.1.28. *Let $1 \le p < \infty$ and $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$. If A is a $(p', \vec{q}\,')$-block, then*

$$\|A\|_{\mathcal{H}^{p'}_{\vec{q}\,'}} \le 1.$$

Proof. In (2.8), simply choose

$$A_0 = A, A_1 = A_2 = \cdots = 0, \lambda_0 = 1, \lambda_1 = \lambda_2 = \cdots = 0.$$

∎

Lemma 2.1.29. *Let $1 \le p < \infty$ and $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$. Let A be an $L^{\vec{q}\,'}(\mathbb{R}^n)$ function supported on a cube Q. Then*

$$\|A\|_{\mathcal{H}^{p'}_{\vec{q}\,'}} \le \|A\|_{\vec{q}\,'} |Q|^{\frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right) - \frac{1}{p}}. \tag{2.9}$$

Proof. Set $B \equiv \frac{|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)}}{\|A\|_{\vec{q}'}}A$, and assume that A is not zero for almost everywhere. Then B is supported on a cube Q and by virtue of the facts that $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{\vec{q}}+\frac{1}{\vec{q}'}=(1,1,\ldots,1)$,

$$\|B\|_{\vec{q}'} = |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)} = |Q|^{\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q'_j}\right)-\frac{1}{p'}}.$$

Hence, B is a $(p',\vec{q}')$-block. By Lemma 2.1.28, $\|B\|_{\mathcal{H}^{p'}_{\vec{q}'}} \le 1$. Thus, we obtain (2.9). ∎

Recall that $L^{\vec{q}}_{\rm c}(\mathbb{R}^n)$ denotes the set of all $L^{\vec{q}}(\mathbb{R}^n)$ function with compact support. By Lemma 2.1.29, the elements of $L^{\vec{q}}_{\rm c}(\mathbb{R}^n)$ can be regarded as a $(p',\vec{q}')$-block modulo multiplicative constants. From this fact, we also consider the density for the space $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$.

Lemma 2.1.30. *The space $L^{\vec{q}}_{\rm c}(\mathbb{R}^n)$ is dense in $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$. In particular, the space $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$ is separable.*

Proof. We shall verify that for all $g \in \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$, the sequence $\{h_j\}_{j=1}^{\infty} \subset L^{\vec{q}}_{\rm c}(\mathbb{R}^n)$ exists such that $\lim_{j\to\infty} h_j = g$ in $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$. Since $g \in \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$, there exist $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1$ and $\{b_j\}_{j=1}^{\infty}$ which is a sequence of $(p',\vec{q}')$-blocks such that

$$g = \sum_{k=1}^{\infty} \lambda_k b_k.$$

For each $j \in \mathbb{N}$, let

$$h_j \equiv \sum_{k=1}^{j} \lambda_k b_k.$$

Then $h_j \in L^{\vec{q}}_{\rm c}(\mathbb{R}^n)$ and since

$$|g-h_j| \le \sum_{k=j+1}^{\infty} |\lambda_k b_k|,$$

we have

$$\|g-h_j\|_{\mathcal{H}^{p'}_{\vec{q}'}} \le \sum_{k=j+1}^{\infty} |\lambda_k| \to 0 \quad (j\to\infty).$$

∎

The following theorem is an extension of the result by Long [87] to mixed Morrey spaces.

Theorem 2.1.31. *Suppose that $1 \le p < \infty$ and $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$.*

(i) *Any $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ defines a continuous functional L_f by*

$$L_f : \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n) \ni g \longmapsto \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x \in \mathbb{C}$$

on $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$.

(ii) *Conversely, every continuous functional L on $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$ can be realized with $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.*

(iii) *The correspondence*

$$\tau : \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) \ni f \longmapsto L_f \in \left(\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)\right)^*$$

is an isomorphism. Furthermore,

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| : g \in \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n), \|g\|_{\mathcal{H}^{p'}_{\vec{q}'}} = 1\right\} \tag{2.10}$$

and

$$\|g\|_{\mathcal{H}^{p'}_{\vec{q}'}} = \max\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| : f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n), \|f\|_{\mathcal{M}^p_{\vec{q}}} = 1\right\}. \tag{2.11}$$

Proof. (i) Since $g \in \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$, for any $\varepsilon > 0$, there exist a non-negative sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$ and a sequence $\{g_j\}_{j\in\mathbb{N}}$ of $(p', \vec{q}')$-blocks such that

$$g = \sum_{j=1}^{\infty} \lambda_j g_j,$$

and

$$\sum_{j=1}^{\infty} \lambda_j \le \|g\|_{\mathcal{H}^{p'}_{\vec{q}'}} + \varepsilon,$$

where each g_j is supported on Q_j. Then, by Hölder's inequality,

$$\begin{aligned}|L_f(g)| = \left|\int_{\mathbb{R}^n} f(x)\sum_{j=1}^{\infty} \lambda_j g_j(x)\mathrm{d}x\right| &\le \sum_{j=1}^{\infty} \lambda_j \int_{\mathbb{R}^n} |f(x)g_j(x)|\mathrm{d}x \\ &\le \sum_{j=1}^{\infty} \lambda_j \|f\chi_{Q_j}\|_{\vec{q}}\|g_j\|_{\vec{q}'}.\end{aligned}$$

Since each g_j is a $(p', \vec{q}')$-block which is supported on Q_j, it follows that

$$|L_f(g)| \le \sum_{j=1}^{\infty} \lambda_j |Q_j|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{\infty}\frac{1}{q_j}\right)} \|f\chi_{Q_j}\|_{\vec{q}}$$
$$\le \|f\|_{\mathcal{M}^p_{\vec{q}}} \sum_{j=1}^{\infty} \lambda_j \le \|f\|_{\mathcal{M}^p_{\vec{q}}} \left(\|g\|_{\mathcal{H}^{p'}_{\vec{q}'}} + \varepsilon \right).$$

Since ε is arbitrary, we have

$$|L_f(g)| \le \|f\|_{\mathcal{M}^p_{\vec{q}}} \|g\|_{\mathcal{H}^{p'}_{\vec{q}'}}. \tag{2.12}$$

Thus, we conclude

$$\|L_f\|_* \le \|f\|_{\mathcal{M}^p_{\vec{q}}}, \tag{2.13}$$

where $\|\cdot\|_*$ denotes the operator norm.

(ii) We take a cube Q_0 and let $Q_j \equiv 2^j Q_0$ for $j \in \mathbb{N}$. For the sake of the simplicity, we write

$$L^{\vec{q}'}(Q_j) \equiv \left\{ f \in L^{\vec{q}'}(\mathbb{R}^n) : f \text{ is supported on } Q_j \right\}.$$

According to the proof of Lemma 2.1.29, since we can regard the element of $L^{\vec{q}'}(Q_j)$ as a $(p', \vec{q}')$-block modulo multiplicative constant, the functional $g \longmapsto L(g)$ is well defined and bounded on $L^{\vec{q}'}(Q_j)$. Thus, by the $L^{\vec{q}} - L^{\vec{q}'}$ duality [11], there exists $f_j \in L^{\vec{q}}(Q_j)$ such that

$$L(g) = \int_{Q_j} f_j(x) g(x) \mathrm{d}x$$

for all $g \in L^{\vec{q}'}(Q_j)$. By the uniqueness of this theorem, we can find $L^{\vec{q}}_{\rm loc}(\mathbb{R}^n)$-function f such that

$$f|_{Q_j} = f_j \quad \text{a.e.}$$

for any j. We shall prove $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. For a fixed cube Q and the above f, we set

$$g \equiv (\overline{\mathrm{sgn} f}) |f|^{q_1-1} \chi_Q \|f\chi_Q\|_{(q_1)}^{q_2-q_1} \|f\chi_Q\|_{(q_1,q_2)}^{q_3-q_2} \cdots \|f\chi_Q\|_{(q_1,\dots,q_{n-1})}^{q_n-q_{n-1}}.$$

A simple calculation shows

$$\int_Q f(x) g(x) \mathrm{d}x = \|f\chi_Q\|_{\vec{q}}^{q_n}, \quad \|g\|_{\vec{q}'} = \|f\chi_Q\|_{\vec{q}}^{q_n-1}.$$

Then we can write

$$|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)} \|f\chi_Q\|_{\vec{q}} = |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)} \left(\int_Q f(x) g(x) \mathrm{d}x \right)^{\frac{1}{q_n}}$$
$$= |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{q_j}\right)} (L(g))^{\frac{1}{q_n}}. \tag{2.14}$$

Meanwhile, thanks to Lemma 2.1.29,

$$|L(g)| \le \|L\|_*|Q|^{\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)-\frac{1}{p}}\|g\|_{\vec{q}'} = \|L\|_*|Q|^{\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)-\frac{1}{p}}\|f\chi_Q\|_{\vec{q}}^{q_n-1}. \quad (2.15)$$

Using (2.14) and (2.15), we obtain

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \le \|L\|_*, \quad (2.16)$$

so that $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. Hence, we conclude that L is realized as $L = L_f$ for $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ at least on $g \in L^{\vec{q}}_c(\mathbb{R}^n)$. Since $L^{\vec{q}'}_c(\mathbb{R}^n)$ is dense in $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$ by Lemma 2.1.30, we can obtain the desired result.

(iii) Thanks to (2.13), (2.16) and (ii), it follows that τ is an isomorphism. We shall check (2.10). By virtue of (2.12), we have

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \ge \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| : g \in \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n), \|g\|_{\mathcal{H}^{p'}_{\vec{q}'}} = 1\right\}.$$

Fix a cube Q. We can assume that $f \not\equiv 0$ on Q. Then, let

$$g \equiv (\overline{\mathrm{sgn} f})|f|^{q_1-1}\chi_Q\|f\chi_Q\|_{(q_1)}^{q_2-q_1}\|f\chi_Q\|_{(q_1,q_2)}^{q_3-q_2}\cdots\|f\chi_Q\|_{(q_1,\dots,q_{n-1})}^{q_n-q_{n-1}},$$

and

$$h \equiv \frac{|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)}}{\|g\|_{\vec{q}'}} g.$$

By Lemma 2.1.29, we see that $\|h\|_{\mathcal{H}^{p'}_{\vec{q}'}} \le 1$. Therefore,

$$\begin{aligned}\int f(x)h(x)\mathrm{d}x = \frac{|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)}}{\|g\|_{\vec{q}'}}\int_Q f(x)g(x)\mathrm{d}x &= \frac{|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)}}{\|f\chi_Q\|_{\vec{q}}^{q_n-1}}\|f\chi_Q\|_{\vec{q}}^{q_n} \\ &= |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)}\|f\chi_Q\|_{\vec{q}}.\end{aligned}$$

Taking the supremum over the all functions $h \in \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$ satisfying $\|h\|_{\mathcal{H}^{p'}_{\vec{q}'}} \le 1$, we obtain

$$|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)}\|f\chi_Q\|_{\vec{q}} \le \sup\left\{\left|\int_{\mathbb{R}^n} f(x)h(x)\mathrm{d}x\right| : h \in \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n), \|h\|_{\mathcal{H}^{p'}_{\vec{q}'}} = 1\right\}.$$

Thus, we conclude

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \le \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| : g \in \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n), \|g\|_{\mathcal{H}^{p'}_{\vec{q}'}} = 1\right\},$$

so that we have (2.10). Meanwhile, by (2.12), we have

$$\|g\|_{\mathcal{H}^{p'}_{\vec{q}'}} \ge \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| : f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n), \|f\|_{\mathcal{M}^p_{\vec{q}}} = 1\right\}.$$

Using the Hahn–Banach theorem, we learn that there exists a functional $\tilde{L} \in \left(\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)\right)^*$ such that $\|\tilde{L}\|_* = 1$ and $\|g\|_{\mathcal{H}^{p'}_{\vec{q}'}} = \tilde{L}(g)$. Since $\tilde{L} \in \left(\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)\right)^*$, by (ii), there is a function $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ such that $\tilde{L}(g) = L_f(g)$, and that

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} = \|L_f\|_* = \|\tilde{L}\|_* = 1.$$

Thus, we obtain (2.11).

∎

Finally, we give an example of functions in the predual spaces $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$.

Example 2.1.32. Let $1 < p < \infty$ and $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$. Then, we have

$$\|\chi_Q\|_{\mathcal{H}^{p'}_{\vec{q}'}} = |Q|^{\frac{1}{p'}}.$$

Applying Lemma 2.1.29 to the function χ_Q, we have

$$\|\chi_Q\|_{\mathcal{H}^{p'}_{\vec{q}'}} \le \|\chi_Q\|_{\vec{q}'} |Q|^{\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right) - \frac{1}{p}} = |Q|^{1-\frac{1}{p}} = |Q|^{\frac{1}{p'}}.$$

Meanwhile, let $f(x) \equiv \dfrac{\chi_Q(x)}{\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}}}$. Then, $\|f\|_{\mathcal{M}^p_{\vec{q}}} = 1$. Thanks to Theorem 2.1.31 (iii), we obtain

$$\|\chi_Q\|_{\mathcal{H}^{p'}_{\vec{q}'}} \ge \int_{\mathbb{R}^n} f(x)\chi_Q(x)\mathrm{d}x = \frac{1}{\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}}} \int_{\mathbb{R}^n} \chi_Q(x)\mathrm{d}x = |Q|^{1-\frac{1}{p}} = |Q|^{\frac{1}{p'}}.$$

Thus, we conclude $\|\chi_Q\|_{\mathcal{H}^{p'}_{\vec{q}'}} = |Q|^{\frac{1}{p'}}$.

2.2 Other function spaces

2.2.1 Besov spaces

Besov spaces have a lot of studies. First, Besov introduced this space using differences in 1959 [13]. Peetre characterized this space by using Fourier transform in 1967 [115]. After that Besov spaces have been investigated by many authors until now. As for the relation with Morrey sapces, Kozono and Yamazaki introduced Besov-Morrey spaces which were introduced to apply to Navier–Stokes equations [81]. Furthermore, there are many textbooks including these spaces [116, 126, 147, 148].

In this subsection, we recall the definition and some fundamental properties of Besov spaces. All of the proofs are referred to books [116, 126, 147, 148] or a survey paper [128].

In this thesis, we employ the well-known definition by the Fourier transform. For $\tau \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, define $\tau(D)f \equiv \mathcal{F}^{-1}[\tau \cdot \mathcal{F}f]$. Next, we define the Littlewood–Paley decomposition. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfy

$$\chi_{B(4)\setminus B(2)} \le \varphi \le \chi_{B(8)\setminus B(\frac{3}{2})}. \tag{2.17}$$

Then define $\varphi_j \equiv \varphi(2^{-j}\cdot)$. The j-th Littlewood–Paley decomposition is the operator $\varphi_j(D)$. Now we are ready to define the (homogeneous) Besov space $\dot{B}^s_{pr}(\mathbb{R}^n)$ for $1 \le p < \infty$, $1 \le r \le \infty$ and $s \in \mathbb{R}$ using the polynomial space $\mathcal{P}(\mathbb{R}^n)$.

Definition 2.2.1. Let $s \in \mathbb{R}$, $1 \le p, r \le \infty$. We define

$$\|f\|_{\dot{B}^s_{pr}} \equiv \left(\sum_{j=-\infty}^{\infty} (2^{js} \|\varphi_j(D)f\|_{L^p})^r \right)^{\frac{1}{r}}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$. The (homogeneous) *Besov space* $\dot{B}^s_{pq}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ for which the norm $\|f\|_{\dot{B}^s_{pq}}$ is finite.

Remark that in the above definition, we choose φ so that the norm of $\dot{B}^s_{pq}(\mathbb{R}^n)$ depends on φ. However, we can verify that $\dot{B}^s_{pq}(\mathbb{R}^n)$ is independent of φ as the set.

Theorem 2.2.2 ([126, Theorem 2.1])**.** *In Definition 2.2.1, we obtain the equivalent norms for the admissible choice of* φ*.*

We recall the elementary properties for Besov spaces.

Theorem 2.2.3 ([126, Theorem 2.4])**.** *Let* $1 \le p, q \le \infty$ *and* $s \in \mathbb{R}$*. Then,* $\dot{B}^s_{pq}(\mathbb{R}^n)$ *is complete, that is,* $\dot{B}^s_{pq}(\mathbb{R}^n)$ *is Banach space.*

The embedding properties are as follows.

Proposition 2.2.4.

(1) ([126, Proposition 2.2]) *For* $s \in \mathbb{R}, 1 \le p \le \infty$ *and* $1 \le r_1 \le r_2 \le \infty$*, we have*

$$\dot{B}^s_{pr_1}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{pr_2}(\mathbb{R}^n).$$

(2) ([126, Theorem 2.5]) *For* $s \in \mathbb{R}, 1 \le p_1 \le p_2 \le \infty$ *and* $1 \le r_1 \le r_2 \le \infty$*,*

$$\dot{B}^s_{p_1 r_1}(\mathbb{R}^n) \hookrightarrow \dot{B}^{s-(\frac{n}{p_1}-\frac{n}{p_2})}_{p_2 r_2}(\mathbb{R}^n).$$

Moreover, we consider the relation between Besov spaces and Lebesgue spaces.

Proposition 2.2.5.

(1) ([126, Theorem 4.4]) *For* $1 \le p \le q \le \infty$, $\dot{B}^{\frac{n}{p}-\frac{n}{q}}_{p1}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$.

(2) ([126, Proposition 2.1]) *For* $1 \le p \le \infty$, $\dot{B}^0_{p1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{p\infty}(\mathbb{R}^n)$.

(3) ([126, Exercise 3.8]) *For* $1 < p < \infty$, $\dot{B}^0_{p2}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$. *Moreover, by duality, for* $1 < p \le 2$, $L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{p2}(\mathbb{R}^n)$.

(4) ([126, Exercise 3.8]) *For* $1 \le p \le 2$, $\dot{B}^0_{pp}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$. *Moreover, by duality, for* $2 \le p \le \infty$, $L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{pp}(\mathbb{R}^n)$.

In Section 4, we use the Besov space $\dot{B}^s_{p1}(\mathbb{R}^n)$ characterized by atoms. So we only describe the characterization of the Besov space $\dot{B}^s_{p1}(\mathbb{R}^n)$. Full statements for Besov spaces can be found in many books. See [126, 147, 148, 149].

Theorem 2.2.6. *Let* $1 \le p < \infty$ *and* $0 < s \le \frac{n}{p}$. *Define* $q \in (p, \infty)$ *by*

$$-\frac{n}{q} = s - \frac{n}{p}, \quad \text{that is } q = n\left(\frac{n}{p} - s\right)^{-1}.$$

Then, the Besov space $\dot{B}^s_{p1}(\mathbb{R}^n)$ *coincides with the set of all* $f \in L^q(\mathbb{R}^n)$ *for which it can be expressed:*

$$f = \sum_{j=-\infty}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right)$$

in $L^q(\mathbb{R}^n)$ *for some complex sequence* $\Lambda \equiv \{\lambda_{jm}\}_{j\in\mathbb{Z}, m\in\mathbb{Z}^n}$ *and some sequence* $A \equiv \{a_{jm}\}_{j\in\mathbb{Z}, m\in\mathbb{Z}^n}$ *of* C^∞*-functions satisfying*

$$\sum_{j=-\infty}^{\infty} 2^{js} \left\| \sum_{m\in\mathbb{Z}^n} \lambda_{jm} \chi_{Q_{jm}} \right\|_{L^p} < \infty, \quad |\partial^\alpha a_{jm}| \le 2^{j|\alpha|} \chi_{3Q_{jm}}$$

for all $(j, m) \in \mathbb{Z} \times \mathbb{Z}^n$ *and multiindices* α *with* $|\alpha| \le [s+1]$. *Moreover the Besov norm* $\|f\|_{\dot{B}^s_{p1}}$ *is equivalent to the infimum of*

$$\begin{aligned} \sum_{j=-\infty}^{\infty} 2^{js} \left\| \sum_{m\in\mathbb{Z}^n} \lambda_{jm} \chi_{Q_{jm}} \right\|_{L^p} &= \sum_{j=-\infty}^{\infty} 2^{js - j\frac{n}{p}} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p \right)^{\frac{1}{p}} \\ &= \sum_{j=-\infty}^{\infty} 2^{-j\frac{n}{q}} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p \right)^{\frac{1}{p}}, \end{aligned}$$

where $\Lambda = \{\lambda_{jm}\}_{j\in\mathbb{Z}, m\in\mathbb{Z}^n}$ *moves over all possible expressions.*

Remark that we use a different definition of sequence spaces to make a_{jm} behave almost similarly to $\chi_{Q_{jm}}$. See [148] and [149, §13.1].

As is easily seen from Proposition 2.2.5, $\dot{B}^s_{p1}(\mathbb{R}^n)$ is continuously embedded into $L^q(\mathbb{R}^n)$. We remark that Theorem 2.2.6 is an expression of homogeneous Besov spaces that differs from the one in [126, 147, 148, 149]. See [126, 148, 149] for the fact that these two definitions are actually the same.

Finally, we give an example of the elements of $\dot{B}^s_{p1}(\mathbb{R}^n)$. This example will use to prove Theorem 4.1.5.

Example 2.2.7. Let $1 < p < \infty$ and $0 < s \leq \dfrac{n}{p}$. Fix $j \in \mathbb{Z}$. Then, by using the partition of unity subordinate to the covering $\{3Q_{jm}\}_{m\in\mathbb{Z}^n}$, we can show that $\exp(-|2^j \cdot -m|^2) \in \dot{B}^s_{p1}(\mathbb{R}^n)$ satisfies $\|\exp(-|2^j \cdot -m|^2)\|_{\dot{B}^s_{p1}} \lesssim 2^{js-j\frac{n}{p}}$ for all $m \in \mathbb{Z}^n$.

2.2.2 Microlocal Besov spaces

We recall the generalized 2-microlocal Besov space $B^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, w)$. The idea of 2-microlocal analysis is due to Bony in 1984 [16]. It is an appropriate instrument to describe the local reguraliry and the oscillatory behavior of functions near singularities. Later many authors investigated the function spaces introduced this idea. In particular, Moritoh and Yamada introduced this idea into Besov spaces and characterized these spaces in 2004 [96]. After that Kempka defined and investigated the generalized 2-microlocal Besov space $B^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, w)$ [74].

To define 2-microlocal Besov spaces, we recall the class $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$:

Definition 2.2.8 (Weight class $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$)**.** Let $\alpha_1, \alpha_2, \alpha_3 \in [0, \infty)$. The *class* $\mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ of weights is defined as the set of all the sequences of the measurable functions $w = \{w_j\}_{j=-\infty}^{\infty}$ satisfying the following conditions:

1. There exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,
$$0 < w_j(x) \leq C w_j(y)(1 + 2^j|x - y|)^{\alpha_3}.$$
2. For all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,
$$2^{-\alpha_1} w_j(x) \leq w_{j+1}(x) \leq 2^{\alpha_2} w_j(x).$$

Such a sequence $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$ is called an admissible weight sequence.

The (homogeneous) generalized 2-microlocal Besov spaces are usually defined by the use of the Fourier multipliers as follows.

Definition 2.2.9 (Generalized 2-microlocal Besov spaces)**.** Let $w \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$. Let φ satisfy (2.17) and define $\varphi_j(x) = \varphi(2^{-j}x)$. Let $0 < p, q \leq \infty$. Then for $f \in \mathcal{S}'(\mathbb{R}^n)$ define

$$\|f\|_{\dot{B}^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n,w)} \equiv \left(\sum_{j=-\infty}^{\infty} 2^{js} \left\| w_j \mathcal{F}^{-1}[\varphi_j \mathcal{F} f] \right\|_p^q \right)^{\frac{1}{q}}.$$

The *generalized 2-microlocal Besov space* $\dot{B}^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, w)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{\dot{B}^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n,w)}$ is finite.

Note that if we take $w_j(x) = 1$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, we obtain classical Besov spaces. Also remark that the norm $\|\cdot\|_{\dot{B}^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n,w)}$ is independent of the choice of φ. We refer to [74].

Kempka characterized the spaces $B^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, w)$ via the atomic decomposition in [74].

Theorem 2.2.10. *Let $w = \{w_j\}_{j=-\infty}^{\infty} \in \mathcal{W}^{\alpha_3}_{\alpha_1,\alpha_2}$, $s \in \mathbb{R}$, and $0 < p, q \le \infty$. Furthermore, let $K, L \in \mathbb{N}_0$ with*

$$K > s + \alpha_2, \quad \text{and} \quad L > \max\left\{0, n\left(\frac{1}{p} - 1\right)\right\} - s + \alpha_1.$$

For each $f \in \dot{B}^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, w)$, there exist sequences

$$\{\lambda_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n} \subset \mathbb{C},\ \{a_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n} \subset C^\infty(\mathbb{R}^n) \text{ and } \{Q_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n} \subset \mathcal{D}$$

such that the representation

$$f = \sum_{j=-\infty}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm},$$

holds, where the triplet $\{\lambda_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n}$, $\{a_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n}$ and $\{Q_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n}$ satisfies

$$|\partial^\alpha a_{jm}| \le 2^{j|\alpha|}\chi_{3Q_{jm}}, \quad \int_{\mathbb{R}^n} x^\beta a_{jm} \mathrm{d}x = 0, \quad \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \left\| \sum_{m\in\mathbb{Z}^n} w_j \lambda_{jm} \chi_{Q_{jm}} \right\|_p^q \right)^{\frac{1}{q}} < \infty$$

for all multiindices α and β with $|\alpha| \le K$ and $|\beta| \le L$. Here the convergence is in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, there exists a constant $C > 0$ such that

$$\left(\sum_{j\in\mathbb{Z}} 2^{jsq} \left\| \sum_{m\in\mathbb{Z}^n} w_j \lambda_{jm} \chi_{Q_{jm}} \right\|_p^q \right)^{\frac{1}{q}} \le C\|f\|_{\dot{B}^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n,w)}$$

for all $f \in \dot{B}^{s,\mathrm{mloc}}_{pq}(\mathbb{R}^n, w)$.

Note that this characterization will be used in Section 4, see Definition 4.1.2.

2.2.3 Lorentz spaces

Next, we turn to Lorentz spaces. Lorentz spaces were introduced by Lorentz in [88, 89]. A general treatment of this space was given in the article of Hunt [61]. The boundedness of the classical results on Lorentz spaces are investigated in [9, 136]. Here, we recall the definitions and elementary facts needed in Subsection 4.3.2. For more details, we refer to [41, Section 1.4] or [143, Chapter V].

To define Lorentz spaces, we prepare some notation. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function. Then the *distribution function* $\lambda_f : [0, \infty) \to [0, \infty]$ is a function defined by

$$\lambda_f(t) \equiv |\{x \in \mathbb{R}^n : |f(x)| > t\}| \quad (t \ge 0).$$

Definition 2.2.11. Let $f \in L^0(\mathbb{R}^n)$. Then its *decreasing rearrangement* f^* is the function defined on $(0, \infty)$ by

$$f^*(t) \equiv \inf(\{s \in [0, \infty) \,:\, \lambda_f(s) \le t\} \cup \{\infty\}) \quad (t > 0).$$

Definition 2.2.12.

1. Let $0 < p < \infty$ and $0 < q < \infty$. Then the *Lorentz space* $L^{p,q}(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{L^{p,q}} \equiv \left\{ \int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{q}}$$

   is finite.

2. The *Lorentz space* $L^{\infty,\infty}(\mathbb{R}^n)$ stands for $L^\infty(\mathbb{R}^n)$.

3. If $0 < p < \infty$, then the *Lorentz space* $L^{p,\infty}(\mathbb{R}^n)$ denotes the weak L^p-space: $L^{p,\infty}(\mathbb{R}^n) = \mathrm{W}L^p(\mathbb{R}^n)$.

By virtue of the definition, we easily show that $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $0 < p \le \infty$. Remark that the space $L^{p,q}(\mathbb{R}^n)$ is complete under the above quasi-norm $\| \cdot \|_{L^{p,q}}$, that is, $L^{p,q}(\mathbb{R}^n)$ is a quasi-Banach space. Meanwhile, we can show that the Lorentz space $L^{p,q}(\mathbb{R}^n)$ is normable when $p, q > 1$. To see this, we shall define the function f^{**}. The idea to use this function is due to Calderón [17]. We now set the maximal function f^{**} by

$$f^{**}(t) \equiv \frac{1}{t} \int_0^t f^*(s)\mathrm{d}s, \quad 0 < t \le \infty.$$

Then define the norm

$$\|f\|^*_{L^{p,q}} \equiv \begin{cases} \left\{ \displaystyle\int_0^\infty (t^{\frac{1}{p}} f^{**}(t))^q \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{q}}, & 1 < p < \infty,\ 1 \le q < \infty, \\ \displaystyle\sup_{t>0} t^{\frac{1}{p}} f^{**}(t), & 1 < p \le \infty,\ q = \infty. \end{cases}$$

Thanks to the Hardy inequality, if $1 < p \le \infty$ and $1 \le q \le \infty$, then $\|\cdot\|^*_{L^{p,q}} \sim \|\cdot\|_{L^{p,q}}$ holds. Furthermore, $L^{p,q}(\mathbb{R}^n)$ is a Banach space with the norm $\| \cdot \|^*_{L^{p,q}}$.

The following result shows that the the scale of Lorentz spaces is monotone for the parameter q increases for any fixed p.

Proposition 2.2.13. *Suppose that $0 < p \le \infty$ and $0 < q < r \le \infty$. Then, the embedding*

$$L^{p,q}(\mathbb{R}^n) \hookrightarrow L^{p,r}(\mathbb{R}^n)$$

holds.

At last, we consider the quasi-triangle inequality for the infinite sum. The following lemma is somehow well known. But it seems that its proof is missing in the literature. So, we give a proof.

Lemma 2.2.14. *Let $0 < p \le q \le \infty$. If we decompose a measurable function f by $f = \sum_{j=1}^{\infty} f_j$ such that $\{\mathrm{supp}(f_j)\}_{j=1}^{\infty}$ are pairwise disjoint, then we have*

$$\|f\|_{L^{p,q}}^p \lesssim \sum_{j=1}^{\infty} \|f_j\|_{L^{p,q}}^p .$$

Proof. We will use the property of the Lorentz norm (see [41, Proposition 1.4.9]):

$$\|f\|_{L^{p,q}} \sim \begin{cases} \left\{ \int_0^\infty (t\lambda_f(t)^{\frac{1}{p}})^q \frac{\mathrm{d}t}{t} \right\}^{\frac{1}{q}}, & q < \infty, \\ \sup_{t>0} t\lambda_f(t)^{\frac{1}{p}}, & q = \infty. \end{cases}$$

Note that we calculate

$$\lambda_f(t) = \left| \bigsqcup_{j=1}^{\infty} \{x \in \mathbb{R}^n : |f_j(x)| > t\} \right| = \sum_{j=1}^{\infty} \lambda_{f_j}(t).$$

Then by using the triangle inequality, we have

$$\|f\|_{L^{p,q}}^p \sim \left\| \sum_{j=1}^{\infty} \lambda_{f_j} \right\|_{L^{\frac{q}{p}}(t^{q-1}\mathrm{d}t)} \le \sum_{j=1}^{\infty} \|\lambda_{f_j}\|_{L^{\frac{q}{p}}(t^{q-1}\mathrm{d}t)} = \sum_{j=1}^{\infty} \|f_j\|_{L^{p,q}}^p,$$

where $\|\cdot\|_{L^{\frac{q}{p}}(t^{q-1}\mathrm{d}t)}$ denotes the $L^{\frac{q}{p}}$-norm with respect to the measure $t^{q-1}\mathrm{d}t$. This is a desired result. ∎

2.2.4 Orlicz spaces

Orlicz spaces initially appeared in 1930's. Birnbaum–Orlicz [14], Orlicz [109, 110, 111], and Nakano [105] investigated Orlicz spaces. Kita investigated the boundedness property of the Hardy–Littlewood maximal operator on Orlicz spaces in [75, 76, 77]. In this thesis, we only recall fundamental facts used in Subsection 4.3.1. For the proof and more details of Orlicz spaces, we refer to [12, 90, 119].

To define Orlicz spaces, we recall the definition of Young functions. A function $\Phi : [0, \infty) \to [0, \infty)$ is a *Young function*, if it satisfies the following conditions:

1. $\Phi(0) = 0$.
2. Φ is convex. That is, $\Phi((1-\theta)t_1 + \theta t_2) \le (1-\theta)\Phi(t_1) + \theta\Phi(t_2)$ for all $t_1, t_2 \in (0, \infty)$ and $0 < \theta < 1$.
3. $\lim_{t\to 0} \Phi(t) = \Phi(0)$, $\quad \lim_{t\to\infty} \Phi(t) = \infty$.

So, we define the Orlicz space $L^\Phi(\mathbb{R}^n)$.

Definition 2.2.15 (Orlicz space)**.** Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function. Then define the *Luxemburg–Nakano norm* $\|\cdot\|_{L^\Phi}$ by

$$\|f\|_{L^\Phi} \equiv \inf\left(\left\{\lambda \in (0,\infty) \, : \, \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \le 1\right\} \cup \{\infty\}\right)$$

for $f \in L^0(\mathbb{R}^n)$. The *Orlicz space* $L^\Phi(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{L^\Phi}$ is finite.

Orlicz spaces extend Lebesgue spaces in the following sense:

Example 2.2.16. If $\Phi(t) = t^p$ for $1 \le p < \infty$, then $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with coincidence of norms.

Here we content ourselves with the completeness of $L^\Phi(\mathbb{R}^n)$ without the proof.

Theorem 2.2.17. *Let Φ be a Young function. Then, the Orlicz space $L^\Phi(\mathbb{R}^n)$ is a Banach space with respect to $\|\cdot\|_{L^\Phi}$.*

If we suppose bijective for Young functions Φ, we can calculate $\|\chi_E\|_{L^\Phi}$.

Lemma 2.2.18. *Assume that Φ is a Young function. Also let Φ be bijective. Then, for all measurable set E with $0 < |E| < \infty$,*

$$\|\chi_E\|_{L^\Phi} = \left\{\Phi^{-1}\left(\frac{1}{|E|}\right)\right\}^{-1}.$$

Proof. We write the norm $\|\chi_E\|_{L^\Phi}$ in full:

$$\|\chi_E\|_{L^\Phi} \equiv \inf\left\{\lambda \in (0,\infty) \, : \, \int_{\mathbb{R}^n} \Phi\left(\frac{\chi_E(x)}{\lambda}\right) \mathrm{d}x \le 1\right\}.$$

Since $\Phi(0) = 0$, we have

$$1 \ge \int_{\mathbb{R}^n} \Phi\left(\frac{\chi_E}{\lambda}\right) \mathrm{d}x = \int_E \Phi\left(\frac{1}{\lambda}\right) \mathrm{d}x = \Phi\left(\frac{1}{\lambda}\right) \times |E|.$$

By virtue of the bijection of Φ,

$$\lambda \ge \left\{\Phi^{-1}\left(\frac{1}{|E|}\right)\right\}^{-1}.$$

Thus,

$$\|\chi_E\|_{L^\Phi} = \left\{\Phi^{-1}\left(\frac{1}{|E|}\right)\right\}^{-1}.$$

∎

Chapter 3

Boundedness of commutators of fractional integral operators on mixed Morrey spaces

3.1 Introduction and theorems

In this section we consider the necessary and sufficient conditions for the boundedness of commutators generated by BMO functions and the fractional integral operator I_α on mixed Morrey spaces.

First, we look back on the background and classical results. The idea of commutators for functions and operators appeared first in Coifman-Rocheberg-Weiss's paper [20]. In this paper, they gave the necessary and sufficient condition for the boundedness of commutators generated by functions and singular integral operators on Lebesgue spaces. Meanwhile, Chanillo obtained the following boundedness results for $[b, I_\alpha]$ [18].

Theorem 3.1.1. *Let $1 < p, q < \infty$ and $0 < \alpha < n$. Assume that*

$$\frac{1}{p} = \frac{\alpha}{n} + \frac{1}{q}.$$

If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then the commutator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Conversely, if $n - \alpha$ is even and $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then we have $b \in \mathrm{BMO}(\mathbb{R}^n)$.

These results were extended to Morrey spaces. In 1991, Di Fazio and Ragusa gave the necessary and sufficient condition for the boundedness of commutator $[b, I_\alpha]$ on Morrey spaces [24].

Theorem 3.1.2. *Let $0 < \alpha < n, 1 < q \le p < \dfrac{n}{\alpha}$. Assume that*

$$\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{s}{r}.$$

If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then

$$[b, I_\alpha] : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathcal{M}^r_s(\mathbb{R}^n).$$

Conversely, if $n - \alpha$ is even and $[b, I_\alpha] : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathcal{M}^r_s(\mathbb{R}^n)$, then $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Shirai removed the condition for $n - \alpha$ in [140].

Theorem 3.1.3. *Let $0 < \alpha < n, 1 < q \le p < \dfrac{n}{\alpha}$. Also, assume that*

$$\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{s}{r}.$$

Then, the following conditions are equivalent:

(a) $b \in \mathrm{BMO}(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ *is bounded from* $\mathcal{M}^p_q(\mathbb{R}^n)$ *to* $\mathcal{M}^r_s(\mathbb{R}^n)$.

Our main theorem extends these results to mixed Morrey spaces. We state our result (This is a same one as Theorem 1.2.1).

Theorem 3.1.4. *Let $0 < \alpha < n, 1 < p < \dfrac{n}{\alpha}, \dfrac{n}{p} \le \displaystyle\sum_{j=1}^{n} \frac{1}{q_j}$, and $\dfrac{n}{r} \le \displaystyle\sum_{j=1}^{n} \frac{1}{s_j}$. Also, assume that*

$$\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q_j}{p} = \frac{s_j}{r} \quad (j = 1, \ldots, n).$$

Then, the following conditions are equivalent:

(a) $b \in \mathrm{BMO}(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ *is bounded from* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ *to* $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$.

(c) $[b, I_\alpha]$ *is bounded from* $\widetilde{\mathcal{M}}^p_{\vec{q}}(\mathbb{R}^n)$ *to* $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$.

(d) $[b, I_\alpha]$ *is bounded from* $\widetilde{\mathcal{M}}^p_{\vec{q}}(\mathbb{R}^n)$ *to* $\mathcal{M}^r_1(\mathbb{R}^n)$.

Here, $\widetilde{\mathcal{M}}^p_{\vec{q}}(\mathbb{R}^n)$ is the $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$-closure of $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

The remaining parts of this chapter are as follows. In Section 3.2, we establish the sharp maximal inequality on mixed Morrey spaces (Theorem 3.2.6). In Subsection 3.2.1, we prepare some ingredients to show the sharp maximal inequality. Its proof is given in Subsection 3.2.2. Additionally, we check the action of the commutator $[b, I_\alpha]$ on mixed Morrey spaces in Subsection 3.2.3. To prove Theorem 3.1.4, Lemma 3.2.15 is significant. Finally, we prove the main theorem in Section 3.3.

3.2 Sharp maximal inequality

We next consider the sharp maximal inequality on mixed Morrey spaces and the relation of the sharp maximal operator and commutators on mixed Morrey spaces to prove the main theorem.

3.2.1 Preliminaries

As we said in Subsection 1.2.1, we cannot apply the layer cake formula and good-λ inequality for mixed norm setting. So instead of these tools, we employ the dyadic local maximal operator $M^{\#,d}_{\lambda;Q_0}$ and the concept of the sparse family. We follow the definition in [84].

Definition 3.2.1. Let $f \in L^0(\mathbb{R}^n)$ and $Q \in \mathcal{Q}$.

1. The *decreasing rearrangement* of f on $\mathbb{R}^n$ is defined by

$$f^*(t) \equiv |\{\rho > 0 \, : \, \mu_f(\rho) > t\}| \quad (0 < t < \infty),$$

   where μ_f is a distribution of f. That is, $\mu_f(\rho) = |\{x \in \mathbb{R}^n \, : \, |f(x)| > \rho\}|$.

2. The *local mean oscillation* of f on Q is defined by

$$\omega_\lambda(f\,;\,Q) \equiv \inf_{c \in \mathbb{C}} \left((f-c)\chi_Q\right)^* (\lambda|Q|) \quad (0 < \lambda < 2^{-1}).$$

3. Assume that the function f is real-valued. Then, the *median* of f over Q, which is denoted by $m_f(Q)$, is a real number satisfying

$$|\{x \in Q \, : \, |f(x)| > m_f(Q)\}|\,, \quad |\{x \in Q \, : \, |f(x)| < m_f(Q)\}| \le \frac{1}{2}|Q|.$$

   Note that the median $m_f(Q)$ is possibly non-unique.

The symbol $\mathcal{D}(Q_0)$ denotes a set of all cubes with respect to the cube Q_0, that is, $\mathcal{D}(Q_0)$ is the set of the form

$$\prod_{j=1}^n \left[x_j + \frac{(m_j-1)\ell(Q_0)}{2^k}, x_j + \frac{m_j\ell(Q_0)}{2^k}\right)$$

for all $k \in \mathbb{N}_0$ and $m_j = 1, \ldots, 2^k$ $(j = 1, \ldots, n)$, where $(x_1, \ldots, x_n)$ denotes the left corner of the cube Q_0. For $0 < \lambda < 2^{-1}$ and $Q_0 \in \mathcal{Q}$, the *dyadic local sharp maximal operator* $M^{\#,d}_{\lambda;Q_0}$ is defined by

$$M^{\#,d}_{\lambda;Q_0} f(x) \equiv \sup_{Q \in \mathcal{D}(Q_0)} \omega_\lambda(f\,;\,Q)\chi_Q(x) \quad \left(x \in \mathbb{R}^n, f \in L^0(\mathbb{R}^n)\right).$$

Moreover, we use the following sharp maximal operator

$$M_\lambda^{\#,d} f(x) \equiv \sup_{Q_0 \in \mathcal{Q}} \sup_{Q \in \mathcal{D}(Q_0)} \omega_\lambda(f\,;\,Q)\chi_Q(x) \quad \left(x \in \mathbb{R}^n, f \in L^0(\mathbb{R}^n)\right).$$

Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$. The *Fefferman–Stein sharp maximal operator* is defined by

$$f^{\#,\eta}(x) \equiv \sup_{x \in Q} \left(\frac{1}{|Q|}\int_Q |f(y) - f_Q|^\eta {\rm d}y\right)^{\frac{1}{\eta}} \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ containing x. When $\eta = 1$, $f^{\#,\eta}$ equals to $f^\#$:

$$f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|{\rm d}y \quad (x \in \mathbb{R}^n).$$

Jawerth and Torchinsky proved a pointwise equivalence between these two types of the sharp maximal operators in [71] :

$$M^{(\eta)}\left[M_\lambda^{\#,d} f\right](x) \sim f^{\#,\eta}(x) \quad (x \in \mathbb{R}^n) \tag{3.1}$$

for sufficiently small λ, where $M^{(\eta)}$ denotes the *powered Hardy–Littlewood maximal operator* defined by

$$M^{(\eta)} f(x) \equiv \sup_{x \in Q} \left(\frac{1}{|Q|}\int_Q |f(y)|^\eta {\rm d}y\right)^{\frac{1}{\eta}} \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ containing x.

The *fractional maximal operator* M_α is defined by

$$M_\alpha f(x) = \sup_{x \in Q} \frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q |f(y)|{\rm d}y \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ containing x. Note that the pointwise inequality $M_\alpha f(x) \lesssim I_\alpha(|f|)(x), x \in \mathbb{R}^n$ holds.

Moreover we also employ the following ingredient.

Definition 3.2.2. We say that the family of dyadic cubes $\{Q_j^k\}_{k \in \mathbb{N}_0, j \in J_k}$ is a sparse family if the following properties hold:

1. For each fixed $k \in \mathbb{N}_0$, the cubes $\{Q_j^k\}_{j \in J_k}$ are disjoint;
2. If $\Omega_k \equiv \bigcup_{j \in J_k} Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$;
3. $|\Omega_{k+1} \cap Q_j^k| \leq \dfrac{1}{2}|Q_j^k|$ for all $j \in J_k$.

Remark 3.2.3. Recently, we substitute the definition of the sparse for the above: Let $0 < \eta < 1$. Then, $\mathcal{S} \subset \mathcal{Q}$ is η-sparse if for each $Q \in \mathcal{Q}$, there exist pairwise disjoint measurable subsets $\{E_Q\}_{Q \in \mathcal{S}}$ such that $E_Q \subset Q$ and $|E_Q| \geq \eta|Q|$. For more details, we refer to [85].

The following lemma is used in the estimate of the dyadic local maximal operator.

Lemma 3.2.4. *Let $\{Q_j^k\}_{k\in\mathbb{N}_0,\, j\in J_k}$ be a sparse family. Then,*

$$|Q_j^k| \le 2|Q_j^k \cap \Omega_{k+1}^c|$$

holds for all $k \in \mathbb{N}_0$ and $j \in J_k$.

Proof. For $k \in \mathbb{N}_0$ and $j \in J_k$,

$$|Q_j^k| \le |Q_j^k \cap \Omega_{k+1}^c| + |Q_j^k \cap \Omega_{k+1}| \le |Q_j^k \cap \Omega_{k+1}^c| + \frac{1}{2}|Q_j^k|$$

by the condition 3 in Definition 3.2.2. Thus, we obtain the result. ∎

Note that, thanks to this lemma, we can see that the sparse family $\{Q_j^k\}_{k\in\mathbb{N}_0,\, j\in J_k}$ in Definition 3.2.2 is $\frac{1}{2}$-sparse. Namely, the idea of η-sparse is generalization of Definition 3.2.2.

To prove the Theorems 3.2.7 and 3.2.8, we invoke the following inequality.

Theorem 3.2.5 ([84])**.** *Let $f \in L^0(\mathbb{R}^n)$ and $Q_0 \in \mathcal{Q}$. Then, there exists a sparse family of $\{Q_j^k\}_{k\in\mathbb{N}_0,\, j\in J_k} \subset \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$,*

$$|f(x) - m_f(Q_0)| \le 4M^{\#,d}_{\lambda_n;Q_0} f(x) + 2\sum_{k\in\mathbb{N}_0}\sum_{j\in J_k} \omega_{\lambda_n}(f; Q_j^k)\chi_{Q_j^k}(x).$$

Here, $\lambda_n \equiv 2^{-n-2}$.

3.2.2 Sharp maximal inequality on mixed Morrey spaces

Our aim in this subsection is to show the following sharp maximal inequality for mixed Morrey spaces.

Theorem 3.2.6. *Let $0 < \vec{q} < \infty$ and $0 < p < \infty$ satisfy*

$$\frac{n}{p} \le \sum_{j=1}^n \frac{1}{q_j}.$$

Then, for any $f \in L^0(\mathbb{R}^n)$ satisfying $Mf \in \mathcal{M}^{p_0}_{\vec{q_0}}(\mathbb{R}^n)$ for some $0 < p_0 < \infty$ and $\vec{q_0} = (q_{0,1}, \ldots, q_{0,n}) \in (0,\infty)^n$ with

$$\frac{n}{p_0} \le \sum_{j=1}^n \frac{1}{q_{0,j}},$$

we have

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \sim \left\|M^{\#,d}_{\lambda} f\right\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \left\|f^{\#}\right\|_{\mathcal{M}^p_{\vec{q}}}. \tag{3.2}$$

To obtain the above theorem, we have to consider the following norm equivalence similar to [104, 132].

Theorem 3.2.7. *Let $0 < \vec{q} < \infty$ and $0 < p, s < \infty$ satisfy*

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad s \le \min(q_1, \ldots, q_n, p).$$

For all $f \in L^0(\mathbb{R}^n)$, it holds that

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \sim \left\| M^{\#,d}_{\lambda} f \right\|_{\mathcal{M}^p_{\vec{q}}} + \|f\|_{\mathcal{M}^p_s}.$$

The term $\|f\|_{\mathcal{M}^p_s}$ in Theorem 3.2.7 is an auxiliary one although this explains how Morrey spaces can be used to control operators acting on Lebesgue spaces. We can remove this term under a reasonable condition using the idea by Fujii [38].

Theorem 3.2.8. *Let $0 < \vec{s} \le \vec{q} < \infty$ and $0 < p < \infty$ satisfy*

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}.$$

Assume that $f \in L^0(\mathbb{R}^n)$ satisfies

$$m_f(2^\ell Q) \to 0$$

as $\ell \to \infty$ for any $Q \in \mathcal{Q}$ and for some medians $\{m_f(2^\ell Q)\}_{\ell \in \mathbb{N}_0}$. Then we have

$$\|f\|_{\mathcal{M}^p_{\vec{s}}} \lesssim \left\| M^{\#,d}_{\lambda} f \right\|_{\mathcal{M}^p_{\vec{s}}} \le \left\| M^{\#,d}_{\lambda} f \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

Meanwhile, the condition proposed by Fujii [38] can be verified as follows.

Lemma 3.2.9. *Let $f \in L^0(\mathbb{R}^n)$. Assume that $Mf \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ for some $0 < \vec{q} < \infty$ and $0 < p < \infty$ satisfying*

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}.$$

For any $Q \in \mathcal{Q}$ and any medians $\{m_f(2^\ell Q)\}_{\ell \in \mathbb{N}_0}$, it holds that

$$\lim_{\ell \to \infty} m_f(2^\ell Q) = 0.$$

Proof. For $0 < \lambda < 2^{-1}$, we have

$$|m_f(2^\ell Q)| \le (f \cdot \chi_{2^\ell Q})^*(\lambda |2^\ell Q|) \le \frac{1}{\lambda} \inf_{x \in 2^\ell Q} Mf(x).$$

Then,

$$|m_f(2^\ell Q)| \lesssim |2^\ell Q|^{-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}} \|Mf\chi_{2^\ell Q}\|_{\vec{q}} \le |2^\ell Q|^{-\frac{1}{p}} \|Mf\|_{\mathcal{M}^p_{\vec{q}}}.$$

Thus, we obtain $\lim_{\ell \to \infty} m_f(2^\ell Q) = 0$. ∎

At first, we show Theorem 3.2.6 applying Theorems 3.2.7 and 3.2.8.

Proof of Theorem 3.2.6. By Theorem 3.2.7, we easily know that

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \sim \left\|M^{\#,d}_\lambda f\right\|_{\mathcal{M}^p_{\vec{q}}} + \|f\|_{\mathcal{M}^p_s} \geq \left\|M^{\#,d}_\lambda f\right\|_{\mathcal{M}^p_{\vec{q}}}.$$

Since $Mf \in \mathcal{M}^{p_0}_{\vec{q_0}}(\mathbb{R}^n)$ for some $0 < p_0 < \infty$ and $\vec{q_0} \in (0,\infty)^n$, we have $\lim_{\ell\to\infty} m_f(2^\ell Q) = 0$. Thus, we are in position to use Theorem 3.2.8. Since $s \leq \min(q_1, \ldots, q_n, p)$, combining Theorems 3.2.7 and 3.2.8, we have

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \sim \left\|M^{\#,d}_\lambda f\right\|_{\mathcal{M}^p_{\vec{q}}} + \|f\|_{\mathcal{M}^p_s} = \left\|M^{\#,d}_\lambda f\right\|_{\mathcal{M}^p_{\vec{q}}} + \|f\|_{\mathcal{M}^p_{(\underbrace{s,\ldots,s}_{n \text{ times}})}} \lesssim \left\|M^{\#,d}_\lambda f\right\|_{\mathcal{M}^p_{\vec{q}}}.$$

Hence we obtain the left equivalence in (3.2). Meanwhile, the right inequality in (3.2) follows from the pointwise estimate (3.1). ∎

We move on to the proofs of Theorems 3.2.7 and 3.2.8. First, we prepare some lemmas to show Theorem 3.2.7 and give its proof. After that, we prove Theorem 3.2.8.

The following estimates are significant for the proof of Theorem 3.2.7. First, we show that the dyadic local sharp maximal operator is bounded on mixed Morrey spaces.

Proposition 3.2.10. *Let $0 < \vec{q} < \infty$ and $0 < p < \infty$ satisfy*

$$\frac{n}{p} \leq \sum_{j=1}^n \frac{1}{q_j},$$

and $0 < \lambda < 2^{-1}$. Then, for $f \in L^0(\mathbb{R}^n)$, we have

$$\left\|M^{\#,d}_\lambda f\right\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

Proof. In [104], it is known that

$$M^{\#,d}_\lambda f(x) \lesssim M^{(\eta)} f(x) \quad (x \in \mathbb{R}^n)$$

for any $\eta > 0$. By virtue of Theorem 2.1.21, taking $\eta < \min(q_1, \ldots, q_n, p)$, we have

$$\left\|M^{\#,d}_\lambda f\right\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \left\|M^{(\eta)} f\right\|_{\mathcal{M}^p_{\vec{q}}} = \|M[|f|^\eta]\|^{\frac{1}{\eta}}_{\mathcal{M}^{\frac{p}{\eta}}_{\frac{\vec{q}}{\eta}}} \lesssim \||f|^\eta\|^{\frac{1}{\eta}}_{\mathcal{M}^{\frac{p}{\eta}}_{\frac{\vec{q}}{\eta}}} = \|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

∎

To prove "$\lesssim$" part of Theorem 3.2.7, we evaluate $\|f\chi_Q\|_{\vec{q}}$ using the dyadic local sharp maximal operator. Here and below, let $\lambda_n = 2^{-n-2}$.

Theorem 3.2.11. *Let $0 < \vec{q} < \infty$and $0 < p, r < \infty$ satisfy*

$$\frac{n}{p} \leq \sum_{j=1}^{n} \frac{1}{q_j},$$

and $Q_0 \in \mathcal{Q}$. Then, for all $f \in L^0(\mathbb{R}^n)$, we have

$$\|f\chi_{Q_0}\|_{\vec{q}} \lesssim \left\| \left(M^{\#,d}_{\lambda_n;Q_0} f \right) \chi_{Q_0} \right\|_{\vec{q}} + |Q_0|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}} \left(\frac{1}{|Q_0|} \int_{Q_0} |f(x)|^r \mathrm{d}x \right)^{\frac{1}{r}}.$$

Proof. We take a median $m_f(Q_0)$ and use the quasi-triangle inequality to get

$$\|f\chi_{Q_0}\|_{\vec{q}} \lesssim \|[f - m_f(Q_0)]\chi_{Q_0}\|_{\vec{q}} + |Q_0|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}} |m_f(Q_0)|.$$

First, we estimate the first term. Applying Theorem 3.2.5, we obtain

$$\|[f - m_f(Q_0)]\chi_{Q_0}\|_{\vec{q}} \lesssim \left\| \left(M^{\#,d}_{\lambda_n;Q_0} f \right) \chi_{Q_0} \right\|_{\vec{q}} + \left\| \left(\sum_{k\in\mathbb{N}_0} \sum_{j\in J_k} \omega_{\lambda_n}(f; Q^k_j)\chi_{Q^k_j} \right) \chi_{Q_0} \right\|_{\vec{q}}.$$

Since $\{Q^k_j\}_{k\in\mathbb{N}_0,\, j\in J_k}$ is a sparse family, we have $|Q^k_j| \leq 2|Q^k_j \cap \Omega^c_{k+1}|$ by Lemma 3.2.4. This implies a pointwise estimate

$$\chi_{Q^k_j}(x) \leq 2M[\chi_{Q^k_j \cap \Omega^c_{k+1}}](x).$$

Let $\eta > \max(1, q_1^{-1}, \ldots, q_n^{-1})$. By Proposition 2.1.24, it follows that

$$\begin{aligned}
\left\| \left(\sum_{k\in\mathbb{N}_0} \sum_{j\in J_k} \omega_{\lambda_n}(f; Q^k_j)\chi_{Q^k_j} \right) \chi_{Q_0} \right\|_{\vec{q}} &\lesssim \left\| \left(\sum_{k\in\mathbb{N}_0} \sum_{j\in J_k} \omega_{\lambda_n}(f; Q^k_j) M[\chi_{Q^k_j \cap \Omega^c_{k+1}}]^{\eta} \right) \chi_{Q_0} \right\|_{\vec{q}} \\
&\lesssim \left\| \sum_{k\in\mathbb{N}_0} \sum_{j\in J_k} \omega_{\lambda_n}(f; Q^k_j)\chi_{Q^k_j \cap \Omega^c_{k+1}} \right\|_{\vec{q}}.
\end{aligned}$$

We deduce from the disjointness of $\{Q^k_j \cap \Omega^c_{k+1}\}_{k\in\mathbb{N}_0, j\in J_k}$ and the definition of ω_λ that

$$\omega_{\lambda_n}(f; Q^k_j)\chi_{Q^k_j} \leq M^{\#,d}_{\lambda_n;Q_0} f, \quad \sum_{k\in\mathbb{N}_0} \sum_{j\in J_k} \chi_{Q^k_j \cap \Omega^c_{k+1}} \leq \chi_{Q_0}.$$

Then we obtain

$$\left\| \left(\sum_{k\in\mathbb{N}_0} \sum_{j\in J_k} \omega_{\lambda_n}(f; Q^k_j)\chi_{Q^k_j} \right) \chi_{Q_0} \right\|_{\vec{q}} \lesssim \left\| \left(M^{\#,d}_{\lambda_n;Q_0} f \right) \chi_{Q_0} \right\|_{\vec{q}}.$$

Next, we evaluate the second term. For, $0 < \lambda < 2^{-1}$, we have

$$|m_f(Q_0)| \le (f \cdot \chi_{Q_0})^*(\lambda|Q_0|) \le \left(\frac{1}{\lambda|Q_0|}\int_0^{\lambda|Q_0|}(f\cdot\chi_{Q_0})^*(t)^r \mathrm{d}t\right)^{\frac{1}{r}}$$
$$\lesssim \left(\frac{1}{|Q_0|}\int_{Q_0}|f(x)|^r \mathrm{d}x\right)^{\frac{1}{r}}.$$

Thus, combining the two estimates gives the desired result. ∎

By virtue of Theorem 3.2.11, we have the following norm estimate.

Corollary 3.2.12. *Let $0 < \vec{q} < \infty$ and $0 < s < \infty$. Moreover, let $0 < p < \infty$ satisfy*

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad 0 < s \le p.$$

Then, for $f \in L^0(\mathbb{R}^n)$, we have

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \left\|M^{\#,d}_{\lambda_n} f\right\|_{\mathcal{M}^p_{\vec{q}}} + \|f\|_{\mathcal{M}^p_s}.$$

So we turn to the proofs of Theorems 3.2.7 and 3.2.8.

Proof of Theorem 3.2.7. First, Since $0 < s \le \min\{q_1, \ldots, q_n, p\}$, we have

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \left\|M^{\#,d}_{\lambda_n} f\right\|_{\mathcal{M}^p_{\vec{q}}} + \|f\|_{\mathcal{M}^p_s}$$

by Corollary 3.2.12. Conversely, combining Proposition 3.2.10, and embedding

$$\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_s(\mathbb{R}^n),$$

we obtain

$$\left\|M^{\#,d}_{\lambda_n} f\right\|_{\mathcal{M}^p_{\vec{q}}} + \|f\|_{\mathcal{M}^p_s} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}} + \|f\|_{\mathcal{M}^p_s} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}},$$

as desired. ∎

Next, we give the proof of Theorem 3.2.8.

Proof of Theorem 3.2.8. Fix any $Q_0 \in \mathcal{Q}$. Then,

$$|Q_0|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{s_j}\right)} \|f\chi_{Q_0}\|_{\vec{s}} \lesssim |Q_0|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{s_j}\right)} \left\|[f - m_f(2^\ell Q_0)]\chi_{Q_0}\right\|_{\vec{s}}$$
$$+ |Q_0|^{\frac{1}{p}} |m_f(2^\ell Q_0)|.$$

By the assumption, it follows that

$$|Q_0|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{s_j}\right)}\|f\chi_{Q_0}\|_{\vec{s}} \lesssim \limsup_{\ell\to\infty} |Q_0|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{s_j}\right)} \left\|[f-m_f(2^\ell Q_0)]\chi_{Q_0}\right\|_{\vec{s}}. \quad (3.3)$$

We use Theorem 3.2.5, and obtain

$$\begin{aligned}&\left\|[f-m_f(2^\ell Q_0)]\chi_{Q_0}\right\|_{\vec{s}}\\ &\lesssim \left\|\left(M_\lambda^{\#,d} f\right)\chi_{Q_0}\right\|_{\vec{s}} + \left\|\left(\sum_{k\in\mathbb{N}_0}\sum_{j\in J_k}\omega_\lambda(f;Q_j^k)\chi_{Q_j^k}\right)\chi_{Q_0}\right\|_{\vec{s}}. \quad (3.4)\end{aligned}$$

Here, we remark that the family $\{Q_j^k\}_{k\in\mathbb{N}_0, j\in J_k} \subset \mathcal{D}(2^\ell Q_0)$ is a sparse family generated by $2^\ell Q_0$. To evaluate the second term of (3.4), we have only to calculate the following two terms:

$$\mathrm{I} = \left\|\left(\sum_{k\in\mathbb{N}_0}\sum_{j\in J_k: Q_j^k\subset Q_0}\omega_\lambda(f;Q_j^k)\chi_{Q_j^k}\right)\chi_{Q_0}\right\|_{\vec{s}},$$

$$\mathrm{II} = \left\|\left(\sum_{k\in\mathbb{N}_0}\sum_{j\in J_k: Q_j^k\supsetneq Q_0}\omega_\lambda(f;Q_j^k)\chi_{Q_j^k}\right)\chi_{Q_0}\right\|_{\vec{s}}.$$

We can handle this with a similar argument to the proof of Theorem 3.2.11, that is, due to Lemma 3.2.4 and the boundedness of the Hardy–Littlewood maximal operator. Thus, we have

$$\mathrm{I} \lesssim \left\|\left(\sum_{k\in\mathbb{N}_0}\sum_{j\in J_k}\omega_\lambda(f;Q_j^k)\chi_{Q_j^k\cap\Omega_{k+1}^c}\right)\chi_{Q_0}\right\|_{\vec{s}}.$$

Note that the summation is taken over the cubes contained in Q_0. Then, we have

$$\mathrm{I} \lesssim \left\|\left(M_\lambda^{\#,d} f\right)\chi_{Q_0}\right\|_{\vec{s}},$$

thanks to the disjointness of $\{Q_j^k \cap \Omega_{k+1}^c\}_{k\in\mathbb{N}_0, j\in J_k}$.

Meanwhile, by recalling that $Q_j^k \subset 2^\ell Q_0$ and the dyadic property, we can rewrite the summation of II as follows.

$$\mathrm{II} \le \left\|\left(\sum_{m=1}^\infty \omega_\lambda(f;Q_0^{(m)})\chi_{Q_0^{(m)}}\right)\chi_{Q_0}\right\|_{\vec{s}}.$$

Here, $Q_0^{(m)}$ denotes the dyadic m-th ancestor of Q_0. Namely, $Q_0^{(m)}$ is a unique dyadic cube with respect to $2^\ell Q_0$ whose side length is $2^m\ell(Q_0)$ and containing Q_0. Then, by Example 2.1.32, we see that

$$\begin{aligned}\mathrm{II} &\le \|\chi_{Q_0}\|_{\vec{s}}\sum_{m=1}^\infty \omega_\lambda(f;Q_0^{(m)})\\ &= |Q_0|^{\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{s_j}\right)}\sum_{m=1}^\infty \left|Q_0^{(m)}\right|^{-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{s_j}\right)}\left\|\left(\omega_\lambda(f;Q_0^{(m)})\right)\chi_{Q_0^{(m)}}\right\|_{\vec{s}}.\end{aligned}$$

By virtue of the definition of the local sharp maximal operator $M^{\#,d}_{\lambda}$, we have

$$\begin{aligned}\mathrm{II} &\le |Q_0|^{\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{s_j}\right)} \sum_{m=1}^{\infty} \left|Q_0^{(m)}\right|^{-\frac{1}{p}} \left\|M^{\#,d}_{\lambda} f\right\|_{\mathcal{M}^p_{\vec{s}}} \\ &= \left(\sum_{m=1}^{\infty} 2^{-\frac{m}{p}}\right) |Q_0|^{-\frac{1}{p}+\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{s_j}\right)} \left\|M^{\#,d}_{\lambda} f\right\|_{\mathcal{M}^p_{\vec{s}}} \sim |Q_0|^{-\frac{1}{p}+\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{s_j}\right)} \left\|M^{\#,d}_{\lambda} f\right\|_{\mathcal{M}^p_{\vec{s}}}.\end{aligned}$$

Thus, combining estimates I and II, we obtain

$$\left\|[f - m_f(2^{\ell}Q_0)]\chi_{Q_0}\right\|_{\vec{s}} \lesssim \left\|\left(M^{\#,d}_{\lambda} f\right)\chi_{Q_0}\right\|_{\vec{s}} + |Q_0|^{-\frac{1}{p}+\frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{s_j}\right)} \left\|M^{\#,d}_{\lambda} f\right\|_{\mathcal{M}^p_{\vec{s}}}. \quad (3.5)$$

Therefore, by (3.3) and (3.5), we obtain the desired result. ∎

3.2.3 The relation to commutators and the sharp maximal function on mixed Morrey spaces

First, we check that the commutator $[b, I_{\alpha}]f$ is well defined for any $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Lemma 3.2.13. *Let* $0 < \alpha < n, 1 < \vec{q} < \infty, 1 < p < \dfrac{n}{\alpha}$, $1 < r < \infty$, *and* $\dfrac{n}{p} \le \displaystyle\sum_{j=1}^{n} \frac{1}{q_j}$. *Also, assume that*

$$\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}.$$

For any $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ *and* $Q \in \mathcal{Q}$ *containing* x, *we have*

$$\int_{\mathbb{R}^n\setminus 2Q} \frac{|b(x)-b(y)|}{|x-y|^{n-\alpha}} |f(y)|\mathrm{d}y \lesssim |Q|^{-\frac{1}{r}} \|f\|_{\mathcal{M}^p_{\vec{q}}} \left(\|b\|_{\mathrm{BMO}} + |b(x) - b_Q|\right).$$

Proof. By the triangle inequality, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus 2Q} \frac{|b(x)-b(y)|}{|x-y|^{n-\alpha}} |f(y)|\mathrm{d}y \\ &\le \int_{\mathbb{R}^n\setminus 2Q} \frac{|b(x)-b_Q|}{|x-y|^{n-\alpha}} |f(y)|\mathrm{d}y + \int_{\mathbb{R}^n\setminus 2Q} \frac{|b(y)-b_Q|}{|x-y|^{n-\alpha}} |f(y)|\mathrm{d}y \\ &\equiv \mathrm{I} + \mathrm{II}.\end{aligned}$$

Note that if $x \in Q$ and $y \in 2^{j+1}Q \setminus 2^jQ$ for each $j \ge 1$, then we have $|x-y| \lesssim 2^{j+1}\ell(Q)$.

We shall estimate I. By the above observation, we estimate

$$
\begin{aligned}
\mathrm{I} &= |b(x)-b_Q| \sum_{j=1}^{\infty} \int_{2^{j+1}Q\setminus 2^jQ} \frac{1}{|x-y|^{n-\alpha}} |f(y)|\mathrm{d}y \\
&\lesssim |b(x)-b_Q| \sum_{j=1}^{\infty} \int_{2^{j+1}Q\setminus 2^jQ} \frac{1}{(2^{j+1}\ell(Q))^{n-\alpha}} |f(y)|\mathrm{d}y \\
&\le |b(x)-b_Q| \sum_{j=1}^{\infty} |2^{j+1}Q|^{-1+\frac{\alpha}{n}} \int_{2^{j+1}Q} |f(y)|\mathrm{d}y.
\end{aligned}
$$

Since $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_1(\mathbb{R}^n)$, we obtain

$$
\begin{aligned}
\mathrm{I} &\le |b(x)-b_Q| \sum_{j=1}^{\infty} |2^{j+1}Q|^{-1+\frac{\alpha}{n}} |2^{j+1}\ell(Q)|^{1-\frac{1}{p}} \|f\|_{\mathcal{M}^p_1} \\
&= \left(\sum_{j=1}^{\infty} 2^{(j+1)(-\frac{1}{r})} \right) |b(x)-b_Q| \cdot |Q|^{-\frac{1}{r}} \|f\|_{\mathcal{M}^p_{\vec{q}}} \\
&\lesssim |b(x)-b_Q| \cdot |Q|^{-\frac{1}{r}} \|f\|_{\mathcal{M}^p_{\vec{q}}}.
\end{aligned}
$$

Here we use the convergence of the series $\sum_{j=1}^{\infty} 2^{(j+1)(-\frac{1}{r})}$.

Next, we consider the second term II. Using Hölder's inequality, we have

$$
\begin{aligned}
\mathrm{II} &= \sum_{j=1}^{\infty} \int_{2^{j+1}Q\setminus 2^jQ} \frac{|b(y)-b_Q|}{|x-y|^{n-\alpha}} |f(y)|\mathrm{d}y \\
&\lesssim \sum_{j=1}^{\infty} |2^{j+1}Q|^{-1+\frac{\alpha}{n}} \int_{2^{j+1}Q} |b(y)-b_Q||f(y)|\mathrm{d}y \\
&\le \sum_{j=1}^{\infty} |2^{j+1}Q|^{\frac{\alpha}{n}} \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |b(y)-b_Q|^{s'}\mathrm{d}y \right)^{\frac{1}{s'}} \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |f(y)|^{s}\mathrm{d}y \right)^{\frac{1}{s}}
\end{aligned}
$$

for some $s>1$. Since $|b_{2^{j+1}Q}-b_Q| \lesssim j\|b\|_{\mathrm{BMO}}$, we get

$$
\begin{aligned}
&\left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |b(y)-b_Q|^{s'}\mathrm{d}y \right)^{\frac{1}{s'}} \\
&\le \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |b(y)-b_{2^{j+1}Q}|^{s'}\mathrm{d}y \right)^{\frac{1}{s'}} + |b_{2^{j+1}Q}-b_Q| \\
&\lesssim (1+j)\|b\|_{\mathrm{BMO}}. \qquad (3.6)
\end{aligned}
$$

Furthermore, since $M^{(s)}$ is bounded on $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, we have

$$\begin{aligned}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|f(y)|^s \mathrm{d}y\right)^{\frac{1}{s}} &\le \frac{\left\|\chi_{2^{j+1}Q}\right\|_{\mathcal{M}^p_{\vec{q}}}}{\left\|\chi_{2^{j+1}Q}\right\|_{\mathcal{M}^p_{\vec{q}}}} \inf_{x\in 2^{j+1}Q} M^{(s)}f(x) \\ &\le \left\|\chi_{2^{j+1}Q}\right\|^{-1}_{\mathcal{M}^p_{\vec{q}}} \left\|M^{(s)}f\right\|_{\mathcal{M}^p_{\vec{q}}} \\ &\lesssim |2^{j+1}Q|^{-\frac{1}{p}}\|f\|_{\mathcal{M}^p_{\vec{q}}}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{II} &\lesssim \sum_{j=1}^{\infty}|2^{j+1}Q|^{\frac{\alpha}{n}-\frac{1}{p}}(1+j)\|b\|_{\mathrm{BMO}}\|f\|_{\mathcal{M}^p_{\vec{q}}} \\ &= \left(\sum_{j=1}^{\infty}2^{-\frac{1}{r}(j+1)}(1+j)\right)|Q|^{-\frac{1}{r}}\|b\|_{\mathrm{BMO}}\|f\|_{\mathcal{M}^p_{\vec{q}}}.\end{aligned}$$

Since $\sum_{j=1}^{\infty}2^{-\frac{1}{r}(j+1)}(1+j)$ converges, we obtain

$$\mathrm{II} \lesssim |Q|^{-\frac{1}{r}}\|b\|_{\mathrm{BMO}}\|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

Combining these two estimates, we conclude

$$\int_{\mathbb{R}^n\setminus 2Q}\frac{|b(x)-b(y)|}{|x-y|^{n-\alpha}}|f(y)|\mathrm{d}y \lesssim \left(|b(x)-b_Q|+\|b\|_{\mathrm{BMO}}\right)|Q|^{-\frac{1}{r}}\|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

∎

Remark 3.2.14. Let $f\in\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, $b\in\mathrm{BMO}(\mathbb{R}^n)$, $x\in\mathbb{R}^n$, and $Q\in\mathcal{Q}$ containing x. Then, we can show that $f\chi_Q\in L^{\vec{q}}(\mathbb{R}^n)$ and $(bf)\chi_Q\in L^{\vec{r}}(\mathbb{R}^n)$ for any $1<\vec{r}<\vec{q}$. In fact, by the definition of the mixed Morrey norm, it is clear that $f\chi_Q\in L^{\vec{q}}(\mathbb{R}^n)$. On the other hand,

$$\|(bf)\chi_Q\|_{\vec{r}} \le \|(b-b_Q)f\chi_Q\|_{\vec{r}} + \|b_Q f\chi_Q\|_{\vec{r}}.$$

By Hölder's inequality, for $\vec{s}$ satisfying $\frac{1}{\vec{s}}+\frac{1}{\vec{q}}=\frac{1}{\vec{r}}$,

$$\begin{aligned}\|(b-b_Q)f\chi_Q\|_{\vec{r}} &\le \|(b-b_Q)\chi_Q\|_{\vec{s}}\|f\chi_Q\|_{\vec{q}} \\ &= \frac{\|\chi_Q\|_{\vec{s}}}{\|\chi_Q\|_{\vec{s}}}\|(b-b_Q)\chi_Q\|_{\vec{s}}\|f\chi_Q\|_{\vec{q}} \\ &\le \|b\|_{\mathrm{BMO}_{\vec{s}}}|Q|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{s_j}} \times \|f\|_{\mathcal{M}^p_{\vec{q}}}|Q|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{1}{p}} \\ &\sim |Q|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{r_j}-\frac{1}{p}}\|b\|_{\mathrm{BMO}}\|f\|_{\mathcal{M}^p_{\vec{q}}},\end{aligned}$$

where

$$\|b\|_{\mathrm{BMO}_{\vec{s}}} \equiv \sup_{Q\in\mathcal{Q}} \frac{1}{\|\chi_Q\|_{\vec{s}}} \|(b-b_Q)\chi_Q\|_{\vec{s}}$$

and since mixed Lebesgue space is a ball Banach function space, we use the characterization of BMO via ball Banach function spaces [68]:

$$\|b\|_{\mathrm{BMO}_{\vec{s}}} \sim \|b\|_{\mathrm{BMO}}.$$

Note that other characterizations of BMO can be found in [67, 69] and [59, Theorem 4.11]. Furthermore,

$$\|b_Q f\chi_Q\|_{\vec{r}} \le |Q|^{\frac{1}{n}\sum_{j=1}^n \frac{1}{r_j}-\frac{1}{p}} |b_Q| \|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

Thus, $(bf)\chi_Q \in L^{\vec{r}}(\mathbb{R}^n)$. Since $|x-y|^{\alpha-n}$ is integrable on $2Q$, we see that

$$|I_\alpha(f\chi_{2Q})(x)| \le \int_{\mathbb{R}^n} \frac{|f(y)|\chi_{2Q}}{|x-y|^{n-\alpha}}\mathrm{d}y \le \|f\chi_{2Q}\|_{\vec{q}} \left\| |x-\cdot|^{\alpha-n}\chi_{2Q}\right\|_{\vec{q}'} < \infty$$

and

$$|I_\alpha(bf\chi_{2Q})| \le \int_{\mathbb{R}^n} \frac{|b(y)f(y)|\chi_{2Q}}{|x-y|^{n-\alpha}}\mathrm{d}y \le \|bf\chi_{2Q}\|_{\vec{r}} \left\| |x-\cdot|^{\alpha-n}\chi_{2Q}\right\|_{\vec{r}'} < \infty.$$

Hence, $I_\alpha(f\chi_{2Q})$ and $I_\alpha(bf\chi_{2Q})$ are well defined.

Using this fact and Lemma 3.2.13, we can justify the definition of $[b, I_\alpha]f$.

Finally, we evaluate the sharp maximal function of the commutator $[b, I_\alpha]f$. The following estimate is also important to show the main theorem.

Lemma 3.2.15. *Let $0<\alpha<n$ and $1<\eta<\infty$. Then,*

$$([b, I_\alpha]f)^\#(x) \lesssim \|b\|_{\mathrm{BMO}} \left(M^{(\eta)}[I_\alpha f](x) + M^{(\eta)}_{\eta\alpha} f(x) \right)$$

for all $b\in \mathrm{BMO}(\mathbb{R}^n)$, $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$.

Note that similar estimates to Lemma 3.2.15 were proved in [8, 140]. Shirai showed this estimate for $f\in C_c^\infty(\mathbb{R}^n)$ [140, Lemma 4.2], while Arai and Nakai showed a similar estimate for the element of generalized Campanato spaces and generalized Morrey spaces [8, Proposition 5.2].

Proof of Lemma 3.2.15. Note that since

$$h^\#(x) \sim \sup_{x\in Q} \inf_{c\in\mathbb{C}} \frac{1}{|Q|}\int_Q |h(y)-c|\mathrm{d}y,$$

for any locally integrable function h in general, to prove this lemma, we only show that

$$\frac{1}{|Q|}\int_Q |[b, I_\alpha]f(y) - c|\,\mathrm{d}y \lesssim \|b\|_{\mathrm{BMO}} \left(M^{(\eta)}[I_\alpha f](x) + M^{(\eta)}_{\eta\alpha} f(x) \right)$$

for some $c \in \mathbb{C}$. Let $Q \equiv Q(x, \ell(Q))$. First, we decompose

$$f = f_1 + f_2 \equiv f\chi_{2Q} + f\chi_{\mathbb{R}^n \setminus 2Q}.$$

For, $y \in Q$, we define

$$\begin{aligned} F_1(y) &\equiv (b(y) - b_{2Q}) I_\alpha f(y), \\ F_2(y) &\equiv I_\alpha[(b - b_{2Q}) f_1](y), \\ F_3(y) &\equiv I_\alpha[(b - b_{2Q}) f_2](y) - C_Q, \end{aligned}$$

where $C_Q \equiv I_\alpha[(b - b_{2Q}) f_2](x)$. Then,

$$[b, I_\alpha] f + C_Q = [(b - b_{2Q}), I_\alpha] f + C_Q = F_1 - F_2 - F_3.$$

Thus, we should evaluate

$$\frac{1}{|Q|} \int_Q |F_i(y)| \mathrm{d}y \quad (i = 1, 2, 3).$$

First, we estimate F_1. By Hölder's inequality, we obtain

$$\begin{aligned} \frac{1}{|Q|} \int_Q |F_1(y)| \mathrm{d}y &\lesssim \left(\frac{1}{|2Q|} \int_{2Q} |b(y) - b_{2Q}|^{\eta'} \mathrm{d}y \right)^{\frac{1}{\eta'}} \left(\frac{1}{|2Q|} \int_{2Q} |I_\alpha f(y)|^\eta \mathrm{d}y \right)^{\frac{1}{\eta}} \\ &\leq \|b\|_{\mathrm{BMO}} M^{(\eta)}(I_\alpha f)(x). \end{aligned} \tag{3.7}$$

Next, we estimate F_2. By Hölder's inequality and the boundedness of the fractional integral operator, we get

$$\frac{1}{|Q|} \int_Q |F_2(y)| \mathrm{d}y \leq \frac{1}{|Q|} \|\chi_Q\|_{r'} \|F_2\|_r \lesssim |Q|^{-1 + \frac{1}{r'}} \|(b - b_{2Q}) f_1\|_v,$$

where $1 < v < \eta$ satisfies $\dfrac{1}{r} = \dfrac{1}{v} - \dfrac{\alpha}{n}$. Let $\dfrac{1}{v} = \dfrac{1}{u} + \dfrac{1}{\eta}$. By virtue of Hölder's inequality again, we have

$$\begin{aligned} \frac{1}{|Q|} \int_Q |F_2(y)| \mathrm{d}y &\lesssim |2Q|^{\frac{1}{v} - \frac{1}{r}} \left(\frac{1}{|2Q|} \int_{2Q} |b(y) - b_{2Q}|^u \mathrm{d}y \right)^{\frac{1}{u}} \left(\frac{1}{|2Q|} \int_{2Q} |f(y)|^\eta \mathrm{d}y \right)^{\frac{1}{\eta}} \\ &\leq \|b\|_{\mathrm{BMO}} \left(\frac{1}{|2Q|^{1 - \frac{\eta\alpha}{n}}} \int_{2Q} |f(y)|^\eta \mathrm{d}y \right)^{\frac{1}{\eta}} \leq \|b\|_{\mathrm{BMO}} M^{(\eta)}_{\eta\alpha} f(x). \end{aligned} \tag{3.8}$$

Finally, we estimate F_3. If we write out fully the definition of I_α,

$$\begin{aligned} F_3(y) &= I_\alpha[(b - b_{2Q}) f_2](y) - I_\alpha[(b - b_{2Q}) f_2](x) \\ &= \int_{\mathbb{R}^n} \left(\frac{1}{|y - z|^{n-\alpha}} - \frac{1}{|x - z|^{n-\alpha}} \right) (b(z) - b_{2Q}) f_2(z) \mathrm{d}z. \end{aligned}$$

Since $z \in \mathbb{R}^n \backslash 2Q$ and $y \in Q$, we get $|x-z| \lesssim |y-z|$. Hence,

$$\left|\frac{1}{|y-z|^{n-\alpha}} - \frac{1}{|x-z|^{n-\alpha}}\right| = (n-\alpha)\left|\int_{|x-z|}^{|y-z|} \frac{1}{t^{n+1-\alpha}}\mathrm{d}t\right|$$
$$\leq \min(|y-z|, |x-z|)^{-n-1+\alpha}|x-y| \lesssim \frac{|x-y|}{|x-z|^{n+1-\alpha}}.$$

Therefore,

$$\begin{aligned}
|F_3(y)| &\lesssim \int_{\mathbb{R}^n\backslash 2Q} \frac{|x-y|}{|x-z|^{n+1-\alpha}}|b(z)-b_{2Q}||f(z)|\mathrm{d}z \\
&= \sum_{j=1}^{\infty} \int_{2^{j+1}Q\backslash 2^jQ} \frac{|x-y|}{|x-z|^{n+1-\alpha}}|b(z)-b_{2Q}||f(z)|\mathrm{d}z \\
&\lesssim \sum_{j=1}^{\infty} \frac{\ell(Q)}{(2^{j+1}\ell(Q))^{n-\alpha+1}} \int_{2^{j+1}Q\backslash 2^jQ} |b(z)-b_{2Q}||f(z)|\mathrm{d}z.
\end{aligned}$$

Next, Hölder's inequality yields

$$\begin{aligned}
&\frac{\ell(Q)}{(2^{j+1}\ell(Q))^{n-\alpha+1}} \int_{2^{j+1}Q\backslash 2^jQ} |b(z)-b_{2Q}||f(z)|\mathrm{d}z \\
&\leq \frac{\ell(Q)}{(2^{j+1}\ell(Q))^{n-\alpha+1}} \left(\int_{2^{j+1}Q} |b(z)-b_{2Q}|^{\eta'}\mathrm{d}z\right)^{\frac{1}{\eta'}} \left(\int_{2^{j+1}Q} |f(z)|^{\eta}\mathrm{d}z\right)^{\frac{1}{\eta}} \\
&= \frac{1}{2^{j+1}} \left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q} |b(z)-b_{2Q}|^{\eta'}\mathrm{d}z\right)^{\frac{1}{\eta'}} \left(\frac{1}{|2^{j+1}Q|^{1-\frac{\eta\alpha}{n}}}\int_{2^{j+1}Q} |f(z)|^{\eta}\mathrm{d}z\right)^{\frac{1}{\eta}}.
\end{aligned}$$

By the definition of M_α, we have

$$\left(\frac{1}{|2^{j+1}Q|^{1-\frac{\eta\alpha}{n}}}\int_{2^{j+1}Q} |f(z)|^{\eta}\mathrm{d}z\right)^{\frac{1}{\eta}} \leq M_{\eta\alpha}^{(\eta)}f(x).$$

Moreover, in the same way as (3.6), we obtain

$$\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q} |b(z)-b_{2Q}|^{\eta'}\mathrm{d}z\right)^{\frac{1}{\eta'}} \lesssim (1+j)\|b\|_{\mathrm{BMO}}.$$

Since $\displaystyle\sum_{j=1}^{\infty}\frac{1+j}{2^{j+1}}$ converges, we conclude

$$|F_3(y)| \lesssim \|b\|_{\mathrm{BMO}} \cdot M_{\eta\alpha}^{(\eta)}f(x). \tag{3.9}$$

Since estimate (3.9) is independent of y, we have

$$\frac{1}{|Q|}\int_Q |F_3(y)|\mathrm{d}y \lesssim \|b\|_{\mathrm{BMO}} \cdot M_{\eta\alpha}^{(\eta)}f(x). \tag{3.10}$$

Combining estimates (3.7), (3.8), and (3.10) provides the desired result. ∎

3.3 Proof of the main theorem (Theorem 3.1.4)

Let us show Theorem 3.1.4.

Proof. (a) $\Rightarrow$ (b): Let $1 < \eta < \min(s_1, \ldots, s_n, r)$ and $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. Put $s_0 \equiv \min(s_1, \ldots, s_n)$ and $q_0 \equiv \min(q_1, \ldots, q_n)$. Since $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_0}(\mathbb{R}^n)$, $[b, I_\alpha]f \in \mathcal{M}^r_{s_0}(\mathbb{R}^n) = \mathcal{M}^r_{\underbrace{(s_0, \ldots, s_0)}_{n \text{ times}}}(\mathbb{R}^n)$ by the result for classical Morrey spaces. Then, we see that $M([b, I_\alpha]f) \in \mathcal{M}^r_{\underbrace{(s_0, \ldots, s_0)}_{n \text{ times}}}(\mathbb{R}^n)$. Thus, the assumption of Corollary 3.2.6 is satisfied. By virtue of Corollary 3.2.6 and Lemma 3.2.15, we have

$$\begin{aligned}
\|[b, I_\alpha](f)\|_{\mathcal{M}^r_{\vec{s}}} \lesssim \left\|([b, I_\alpha]f)^{\#}\right\|_{\mathcal{M}^r_{\vec{s}}} &\lesssim \|b\|_{\mathrm{BMO}} \left\| M^{(\eta)}[I_\alpha f] + M^{(\eta)}_{\eta\alpha} f \right\|_{\mathcal{M}^r_{\vec{s}}} \\
&\le \|b\|_{\mathrm{BMO}} \left\{ \|M^{(\eta)}[I_\alpha f]\|_{\mathcal{M}^r_{\vec{s}}} + \left\| M^{(\eta)}_{\eta\alpha} f \right\|_{\mathcal{M}^r_{\vec{s}}} \right\} \\
&\lesssim \|b\|_{\mathrm{BMO}} \left\{ \|I_\alpha(|f|)\|_{\mathcal{M}^r_{\vec{s}}} + \|I_{\eta\alpha}(|f|^\eta)\|^{\frac{1}{\eta}}_{\mathcal{M}^{\frac{r}{\eta}}_{\frac{\vec{s}}{\eta}}} \right\}.
\end{aligned}$$

Using Theorem 2.1.22, we conclude

$$\|[b, I_\alpha](f)\|_{\mathcal{M}^r_{\vec{s}}} \lesssim \|b\|_{\mathrm{BMO}} \left\{ \|f\|_{\mathcal{M}^p_{\vec{q}}} + \||f|^\eta\|^{\frac{1}{\eta}}_{\mathcal{M}^{\frac{p}{\eta}}_{\frac{\vec{q}}{\eta}}} \right\} = \|b\|_{\mathrm{BMO}} \|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

(b) $\Rightarrow$ (c): It is clear since only the domain is restricted.

(c) $\Rightarrow$ (d): Using the embedding $\mathcal{M}^p_{\vec{s}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_1(\mathbb{R}^n)$ and (c), we have

$$\|[b, I_\alpha]f\|_{\mathcal{M}^p_1} \le \|[b, I_\alpha]f\|_{\mathcal{M}^p_{\vec{s}}} \lesssim \|f\|_{\widetilde{\mathcal{M}}^p_{\vec{q}}}$$

for $f \in \widetilde{\mathcal{M}}^p_{\vec{q}}(\mathbb{R}^n)$. Thus we obtain (d).

(d) $\Rightarrow$ (a): We use the same method as Janson [70]. Choose $z_0 \in \mathbb{R}^n$ such that $|z_0| = 5$. Since $0 \notin Q(z_0, 2)$, $|x|^{n-\alpha} \in C^\infty(Q(z_0, 2))$ for $x \in Q(z_0, 2)$. Hence, we choose a function $\varphi \in C^\infty(\mathbb{R}^n)$ with a π periodicity and satisfying $\varphi(x) = |x|^{n-\alpha}$ for all $x \in Q(z_0, 2)$. Then, we can expand this function into the absolutely convergent Fourier series on $Q(z_0, 2)$;

$$|x|^{n-\alpha} \chi_{Q(z_0,2)}(x) = \sum_{m \in \mathbb{Z}^n} a_m e^{2im\cdot x} \chi_{Q(z_0,2)}(x), \tag{3.11}$$

with $\sum_{m\in\mathbb{Z}^n} |a_m| < \infty$. For any $x_0 \in \mathbb{R}^n$ and $t > 0$, let $Q \equiv Q(x_0, t)$ and $Q' \equiv Q(x_0 + z_0 t, t)$. Let

$$s(x) \equiv \operatorname{sgn}\left(\int_{Q'} (b(x) - b(y))\mathrm{d}y\right).$$

If $x \in Q$ and $y \in Q'$, then $\frac{y-x}{t} \in Q(z_0, 2)$. Hence, we have

$$\begin{aligned}\int_Q |b(x) - b_{Q'}|\mathrm{d}x &= \int_Q (b(x) - b_{Q'})\overline{s(x)}\mathrm{d}x \\ &= \frac{1}{|Q'|}\int_Q \overline{s(x)}\left(\int_{Q'} (b(x) - b(y))\mathrm{d}y\right)\mathrm{d}x \\ &= \frac{t^{n-\alpha}}{t^n}\int_Q \overline{s(x)}\left(\int_{Q'} (b(x) - b(y))|x-y|^{-n+\alpha}\left|\frac{x-y}{t}\right|^{n-\alpha}\mathrm{d}y\right)\mathrm{d}x.\end{aligned}$$

By (3.11) and the triangle inequality, we get

$$\begin{aligned}&\int_Q |b(x) - b_{Q'}|\mathrm{d}x \\ &= t^{-\alpha}\sum_{m\in\mathbb{Z}^n}\int_Q \overline{s(x)}\left(\int_{Q'} (b(x) - b(y))|x-y|^{-n+\alpha}a_m e^{2im\cdot\frac{y}{t}}\mathrm{d}y\right)e^{-2im\cdot\frac{x}{t}}\mathrm{d}x \\ &\le t^{-\alpha}\sum_{m\in\mathbb{Z}^n}\left|a_m\int_{\mathbb{R}^n}\overline{s(x)}[b, I_\alpha](e^{2im\cdot\frac{\cdot}{t}}\chi_{Q'})(x)\chi_Q(x)e^{-2im\cdot\frac{x}{t}}\mathrm{d}x\right| \\ &\le t^{-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\int_Q\left|[b, I_\alpha](e^{2im\cdot\frac{\cdot}{t}}\chi_{Q'})(x)\right|\mathrm{d}x \\ &\le t^{-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\,|Q|^{-\frac{1}{r}+1}\left\|[b, I_\alpha](e^{2im\cdot\frac{\cdot}{t}}\chi_{Q'})\right\|_{\mathcal{M}^r_1} \\ &\le t^{-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\,\|[b, I_\alpha]\|_{\tilde{\mathcal{M}}^p_{\vec{q}}\to\mathcal{M}^r_1}\|\chi_{Q'}\|_{\mathcal{M}^p_{\vec{q}}}|Q|^{-\frac{1}{r}+1} \\ &\le t^{-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\,\|[b, I_\alpha]\|_{\tilde{\mathcal{M}}^p_{\vec{q}}\to\mathcal{M}^r_1}t^{\frac{n}{p}}\cdot t^{\frac{n}{r'}} \sim t^n\,\|[b, I_\alpha]\|_{\tilde{\mathcal{M}}^p_{\vec{q}}\to\mathcal{M}^r_1}.\end{aligned}$$

Thus, we have

$$\frac{1}{|Q|}\int_Q |b(x) - b_Q|\mathrm{d}x \le \frac{2}{|Q|}\int_Q |b(x) - b_{Q'}|\mathrm{d}x \lesssim \|[b, I_\alpha]\|_{\tilde{\mathcal{M}}^p_{\vec{q}}\to\mathcal{M}^r_1}.$$

This implies that $b \in \mathrm{BMO}(\mathbb{R}^n)$ since Q is an arbitrary cube. ∎

Chapter 4

A characterization of Morrey spaces associated with Banach lattice in terms of pointwise multiplier spaces

4.1 Introduction and theorems

In [58], Ho defined vector-valued Morrey spaces. In this chapter, inspired these ideas, we will define Morrey spaces associated to general Banach lattices and establish that these spaces arise naturally as multiplier spaces from microlocal Besov spaces to Banach lattices. Recall that a *Banach* (*function*) *lattice* on $\mathbb{R}^n$ is a Banach space $(E, \|\cdot\|_E)$ contained in $L^0(\mathbb{R}^n)$, the linear space of all measurable functions, such that, for all $f, g \in E$, the implication "$|f| \le |g| \Rightarrow \|f\|_E \le \|g\|_E$" holds.

To define Morrey spaces associated to general Banach lattices, we assume the following:

Assumption 4.1.1. The Banach lattice $E(\mathbb{R}^n)$ is translation invariant, that is, let $E(\mathbb{R}^n)$ be a Banach lattice be such that

$$\|f(\cdot - x)\|_E = \|f\|_E$$

for all $f \in E(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

A direct consequence of the translation invariance is that $\|\chi_{3Q}\|_E \lesssim \|\chi_Q\|_E$ for all cubes Q, where $3Q$ denotes the triple of Q, that is, $3Q$ is a cube which is concentric to Q and has volume $3^n|Q|$. With a natural modification we can include the case of variable exponents but here for the sake of simplicity we do not do this. These assumptions are postulated so as to simplify matters. Nevertheless, as our examples show, we have translation invariant many function spaces.

Based on [58, Definition 2.6], we define $\mathcal{M}^p_E(\mathbb{R}^n)$ to be the set of all measurable functions f for which

$$\|f\|_{\mathcal{M}^p_E} \equiv \sup_Q |Q|^{\frac{1}{p}} \left(\frac{1}{\|\chi_Q\|_E} \|f\chi_Q\|_E \right)$$

is finite, where Q moves over all cubes whose edges are parallel to the coordinate axes. As we will show in Section 4.3, Morrey spaces associated to general Banach lattices realize mixed Morrey spaces [106], Morrey–Lorentz spaces [118] and Orlicz–Morrey spaces (of the third kind) [26].

From Assumption 4.1.1, we learn that the following dyadic Morrey norm is equivalent to the original norm

$$\|f\|_{\mathcal{M}^p_E} \sim \sup_{Q\in\mathcal{D}} |Q|^{\frac{1}{p}} \left(\frac{1}{\|\chi_Q\|_E} \|f\chi_Q\|_E \right),$$

where $\mathcal{D}$ denotes the set of all dyadic cubes (see Section 1.3 (6)).

With the definition of dyadic cubes in mind, we formulate the definition of microlocal Besov spaces. Although we gave it in Section 2.2.2, we will adopt Theorem 2.2.10 ([74, Theorem 1]) with $p = q = 1$, $s = 0$, $K = [\alpha_2 + 1]$ and $L = [\alpha_1 + 1]$ as a definition of the microlocal Besov space (we write $\dot{B}^w_{11}(\mathbb{R}^n)$) in this chapter.

Definition 4.1.2. Let $w \equiv \{w_j\}_{j=-\infty}^{\infty} \subset \mathbb{R}$ be a positive sequence, and let $F(\mathbb{R}^n)$ be a Banach function lattice satisfying

$$\|\chi_{Q_{jm}}\|_F \lesssim w_j |Q_{j0}| \quad (j \in \mathbb{Z}, m \in \mathbb{Z}^n)$$

and

$$2^{-\alpha_1 j} w_j \le w_{j+1} \le 2^{\alpha_2 j} w_j \quad (j \in \mathbb{Z}),$$

where α_1 and α_2 are fixed parameters. One defines the microlocal homogeneous Besov space $\dot{B}^w_{11}(\mathbb{R}^n)$ by the set of all $f \in F(\mathbb{R}^n)$ for which it can be written as

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm}(x)$$

for almost every $x \in \mathbb{R}^n$, where for all $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$, we have a collection $\{a_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n}$ of C^∞-functions and a collection $\{\lambda_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n}$ of complex constants satisfying

$$|\partial^\alpha a_{jm}| \le 2^{j|\alpha|} \chi_{3Q_{jm}}, \qquad \int_{\mathbb{R}^n} x^\beta a_{jm}(x) \mathrm{d}x = 0, \qquad \sum_{j=-\infty}^{\infty} 2^{-jn} w_j \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}| \right) < \infty$$

for all multiindices α with $|\alpha| \le [\alpha_2 + 1]$ and for all multiindices β with $|\beta| \le [\alpha_1 + 1]$. Then the *microlocal Besov norm* $\|f\|_{\dot{B}^w_{11}}$ is defined as the infimum of

$$\sum_{j=-\infty}^{\infty} 2^{-jn} w_j \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}| \right)$$

where $\Lambda = \{\lambda_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n}$ moves over all possible expressions.

In the above, the convergence in $\mathcal{S}'(\mathbb{R}^n)$ is guaranteed thanks to the condition on Λ. The functions a_{jm} are called atoms. Note also that in terms of Definition 2.2.9, $\dot{B}^w_{11}(\mathbb{R}^n) = \dot{B}^{0,\mathrm{mloc}}_{11}(\mathbb{R}^n, w)$.

Finally, to state our main results, we turn to the definition of pointwise multipliers spaces. Given Banach spaces $E_1(\mathbb{R}^n)$ and $E_2(\mathbb{R}^n)$ of measurable functions defined on $\mathbb{R}^n$, we define $\mathrm{PWM}(E_1(\mathbb{R}^n), E_2(\mathbb{R}^n))$ as follows: A measurable function g is a *pointwise multiplier* from $E_1(\mathbb{R}^n)$ to $E_2(\mathbb{R}^n)$ if the pointwise product $f \cdot g$ belongs to $E_2(\mathbb{R}^n)$ for each $f \in E_1(\mathbb{R}^n)$ and there exists a constant $M > 0$ such that

$$\|f \cdot g\|_{E_2(\mathbb{R}^n)} \le M \|f\|_{E_1(\mathbb{R}^n)}. \tag{4.1}$$

One defines a norm on $\mathrm{PWM}(E_1(\mathbb{R}^n), E_2(\mathbb{R}^n))$ by

$$\|g\|_{\mathrm{PWM}(E_1,E_2)} \equiv \inf\{M > 0 \,:\, (4.1) \text{ holds for all } f \in E_1(\mathbb{R}^n)\}$$

for $g \in \mathrm{PWM}(E_1(\mathbb{R}^n), E_2(\mathbb{R}^n))$.

A simple example is the case of Lebesgue spaces.

Example 4.1.3. Let $1 \le p_1, p_2, p_3 \le \infty$ satisfy $\dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{p_3}$. Then, by Hölder's inequality, we have

$$\mathrm{PWM}(L^{p_1}(\mathbb{R}^n), L^{p_3}(\mathbb{R}^n)) = L^{p_2}(\mathbb{R}^n).$$

We also refer to [83, 99] for the case where $E_1(\mathbb{R}^n)$ and $E_2(\mathbb{R}^n)$ are Morrey spaces.

Our main result in this chapter is the following assertion (This is the same one as Theorem 1.2.5).

Theorem 4.1.4. *Let $E(\mathbb{R}^n)$ and $F(\mathbb{R}^n)$ be Banach lattices such that*

$$\|\chi_{Q_{jm}}\|_F \lesssim \|\chi_{Q_{jm}}\|_E |Q_{j0}|^{-\frac{1}{p}} \quad (j \in \mathbb{Z}, m \in \mathbb{Z}^n).$$

Set

$$w_j \equiv \|\chi_{Q_{j0}}\|_E |Q_{j0}|^{-1-\frac{1}{p}} \quad (j \in \mathbb{Z}).$$

Then $\dot{B}^w_{11}(\mathbb{R}^n)$ is continuously embedded into $F(\mathbb{R}^n)$ and

$$\mathrm{PWM}(\dot{B}^w_{11}(\mathbb{R}^n), E(\mathbb{R}^n)) \approx \mathcal{M}^p_E(\mathbb{R}^n)$$

with equivalence of norms.

In connection with Theorem 4.1.4, we consider the result by Lemarié-Rieusset. Lemarié-Rieusset showed that Morrey spaces arises naturally when we consider the pointwise multipliers from $\dot{B}^s_{p1}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with $0 < s \le \dfrac{n}{p}$.

Theorem 4.1.5 (cf. [82])**.** *Let $1 \le p < \infty$ and $0 < s \le \dfrac{n}{p}$. Then*

$$\mathrm{PWM}(\dot{B}^s_{p1}(\mathbb{R}^n), L^p(\mathbb{R}^n)) \approx \mathcal{M}^{\frac{n}{s}}_p(\mathbb{R}^n)$$

with equivalence of norms.

Lemarié-Rieusset obtained Theorem 4.1.5 for $n = 3$ and $p = 2$ [82, Lemma 6]. A passage to the general case is a minor modification. Here for the sake of completeness, we give a proof in Subsection 4.2.1.

Furthermore, we can generalize the result of Lemarié-Rieusset (Theorem 4.1.5) to multiplier spaces from homogeneous Besov spaces to Morrey spaces. This proof is also given in Subsection 4.2.1.

Theorem 4.1.6. *Let $1 \le q \le p < \infty$ and $0 < s \le \frac{n}{q}$ satisfy*

$$0 < \frac{s}{n} - \frac{1}{q} + \frac{1}{p} \le \frac{1}{p}$$

Let

$$\frac{1}{\sigma} \equiv \frac{s}{n} - \frac{1}{q} + \frac{1}{p}.$$

Then with equivalence norms

$$\mathrm{PWM}\left(\dot{B}^s_{q1}(\mathbb{R}^n), \mathcal{M}^p_q(\mathbb{R}^n)\right) \approx \mathcal{M}^\sigma_{\mathcal{M}^p_q}(\mathbb{R}^n) = \mathcal{M}^\sigma_q(\mathbb{R}^n).$$

The structure of Chapter 4 is as follows: In Section 4.2, we prove our results. Subsection 4.2.1 is devoted to the proof of results for generalized 2-microlocal Besov spaces (Theorem 4.1.4). In Subsection 4.2.2, we prove the results for the pointwise multiplier spaces from classical Besov spaces to Lebesgue spaces (Theorem 4.1.5) and from Besov spaces to Morrey spaces (Theorem 4.1.6). Finally, we give the examples for our results in Section 4.3. We apply our results to Orlicz spaces (Subsection 4.3.1), Lorentz spaces (Subsection 4.3.2), mixed Lebesgue spaces (Subsection 4.3.3), and mixed Morrey spaces (Subsection 4.3.4), respectively.

4.2 Proofs of the main theorems

4.2.1 Pointwise multipliers from generalized 2-microlocal Besov spaces to Banach lattices (Theorem 4.1.4)

We will show the former half of Theorem 4.1.4 to check that $\mathrm{PWM}(\dot{B}^w_{11}(\mathbb{R}^n), E(\mathbb{R}^n))$ is well defined.

Lemma 4.2.1. *Let $F(\mathbb{R}^n)$ be a Banach function lattice satisfying*

$$\|\chi_{Q_{jm}}\|_F \lesssim 2^{-jn} w_j \quad (j \in \mathbb{Z}, m \in \mathbb{Z}^n).$$

Then $\dot{B}^w_{11}(\mathbb{R}^n) \hookrightarrow F(\mathbb{R}^n)$.

Proof. Let $f \in \dot{B}^w_{11}(\mathbb{R}^n)$ be a decomposition as in Definition 4.1.2. By the triangle inequality, we have

$$\begin{aligned}\left\|\sum_{j=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{jm}a_{jm}\right\|_F &\le \sum_{j=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\|\lambda_{jm}a_{jm}\|_F\\ &\lesssim \sum_{j=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\left\|\lambda_{jm}\chi_{3Q_{jm}}\right\|_F\\ &\lesssim \sum_{j=-\infty}^{\infty}2^{-jn}w_j\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|\right).\end{aligned}$$

∎

We prove the latter half of Theorem 4.1.4. Let $f \in \mathcal{M}^p_E(\mathbb{R}^n)$. Let also $g \in \dot{B}^w_{11}(\mathbb{R}^n)$, so that there exist a collection $\{a_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n}$ of C^∞-functions and a collection $\{\lambda_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n}$ of complex constants satisfying

$$g = \sum_{j=-\infty}^{\infty}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{jm}a_{jm}\right), \quad \sum_{j=-\infty}^{\infty}2^{-jn}w_j\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|\right) < \infty$$

and $|\partial^\alpha a_{jm}| \le 2^{j|\alpha|}\chi_{3Q_{jm}}$ for all $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then

$$\begin{aligned}\|fa_{jm}\|_E &\le \|f\chi_{3Q_{jm}}\|_E\\ &= |Q_{jm}|^{\frac{1}{p}}\frac{1}{\|\chi_{3Q_{jm}}\|_E}\|f\chi_{3Q_{jm}}\|_E\|\chi_{3Q_{j0}}\|_E|Q_{jm}|^{-\frac{1}{p}}\\ &\lesssim \|f\|_{\mathcal{M}^p_E}\|\chi_{Q_{j0}}\|_E|Q_{j0}|^{-\frac{1}{p}}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|f\cdot g\|_E &\le \sum_{j=-\infty}^{\infty}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|\|fa_{jm}\|_E\right)\\ &\lesssim \|f\|_{\mathcal{M}^p_E}\sum_{j=-\infty}^{\infty}\|\chi_{Q_{j0}}\|_E|Q_{j0}|^{-\frac{1}{p}}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|\right).\end{aligned}$$

If we take the infimum over all possible expressions of g, we obtain

$$\|f\cdot g\|_E \lesssim \|f\|_{\mathcal{M}^p_E}\|g\|_{\dot{B}^w_{11}}.$$

Thus, $f \in \mathrm{PWM}(\dot{B}^w_{11}(\mathbb{R}^n), E(\mathbb{R}^n))$ and $\|f\|_{\mathrm{PWM}(\dot{B}^w_{11},E)} \lesssim \|f\|_{\mathcal{M}^p_E}$.

Conversely, we let $f \in \mathrm{PWM}(\dot{B}^w_{11}(\mathbb{R}^n), E(\mathbb{R}^n))$. Choose a smooth function $\kappa \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ so that $\kappa(x) = 0$ for any $x \in \mathbb{R}^n \setminus 3Q_{00}$, that $\kappa(x) = 1$ for any $x \in Q_{00}$ and that

$$\int_{\mathbb{R}^n} x^\beta\kappa(x)\mathrm{d}x = 0$$

for any β with $|\beta| \le [\alpha_1+1]$. Define $\kappa_{jm}(x) \equiv \kappa\left(2^j x - m\right)$ for each $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then since $2^{-jn} w_j \|f\|_{\mathrm{PWM}(\dot{B}^w_{11},E)} = \|\chi_{Q_{j0}}\|_E |Q_{j0}|^{-\frac{1}{p}} \|f\|_{\mathrm{PWM}(\dot{B}^w_{11},E)}$, we have

$$\|f\chi_{Q_{jm}}\|_E \le \|f\kappa_{jm}\|_E \le \|f\|_{\mathrm{PWM}(\dot{B}^w_{11},E)} \|\kappa_{jm}\|_{\dot{B}^w_{11}} \lesssim \|\chi_{Q_{j0}}\|_E |Q_{j0}|^{-\frac{1}{p}} \|f\|_{\mathrm{PWM}(\dot{B}^w_{11},E)}. \tag{4.2}$$

Thus, $f \in \mathcal{M}^p_E(\mathbb{R}^n)$.

4.2.2 Pointwise multipliers from Besov spaces to Lebesgue spaces and Morrey spaces (Theorems 4.1.5 and 4.1.6)

Let us prove Theorem 4.1.5.

Proof of Theorem 4.1.5. Let $f \in \mathrm{PWM}(\dot{B}^s_{p1}(\mathbb{R}^n), L^p(\mathbb{R}^n))$. Then define κ_{jm} as before for all $m \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$. Thus, by Example 2.2.7 and the same argument of (4.2),

$$\|f\|_{L^p(Q_{jm})} \lesssim 2^{js - j\frac{n}{p}} \|f\|_{\mathrm{PWM}(\dot{B}^s_{p1},L^p)}.$$

Consequently, $f \in \mathcal{M}^{\frac{n}{s}}_p(\mathbb{R}^n)$.

To show the opposite estimate, by Theorem 2.2.6, it suffices to show that

$$\left\| f \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm} \right\|_{L^p} \lesssim 2^{js} \|f\|_{\mathcal{M}^{\frac{n}{s}}_p} \left\| \sum_{m\in\mathbb{Z}^n} \lambda_{jm} \chi_{Q_{jm}} \right\|_{L^p}$$

for all sequences $\{a_{jm}\}_{j\in\mathbb{Z}, m\in\mathbb{Z}^n}$ of C^∞-functions satisfying $|\partial^\alpha a_{jm}| \le 2^{j|\alpha|}\chi_{3Q_{jm}}$ with $|\alpha| \le [s+1]$; once this is achieved, we have only to add this estimate over $j \in \mathbb{Z}$.

We calculate

$$\begin{aligned}
\left\| f \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm} \right\|_{L^p} &\lesssim \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p \|f\chi_{3Q_{jm}}\|_{L^p}{}^p \right)^{\frac{1}{p}} \\
&\le \sup_{m\in\mathbb{Z}^n} \|f\chi_{3Q_{jm}}\|_{L^p} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p \right)^{\frac{1}{p}} \\
&\lesssim 2^{js} \|f\|_{\mathcal{M}^{\frac{n}{s}}_p} \left\| \sum_{m\in\mathbb{Z}^n} \lambda_{jm} \chi_{Q_{jm}} \right\|_{L^p}.
\end{aligned}$$

Thus, the proof of Theorem 4.1.5 is complete.
∎

We move on to the proof of Theorem 4.1.6.

Proof of Theorem 4.1.6. If we write out the norm of $\mathcal{M}^{\sigma}_{\mathcal{M}^p_q}(\mathbb{R}^n)$ using the dyadic Morrey norm, then we obtain

$$\|f\|_{\mathcal{M}^{\sigma}_{\mathcal{M}^p_q}} \equiv \sup_{Q\in\mathcal{D}} \sup_{R\in\mathcal{D}} |Q|^{\frac{1}{\sigma}-\frac{1}{p}}|R|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_{Q\cap R}\|_{L^q}.$$

Let Q, R be dyadic cubes. In order that $|Q|^{\frac{1}{\sigma}-\frac{1}{p}}|R|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_{Q\cap R}\|_{L^q}$ is not zero, Q and R must intersect. Since

$$|Q|^{\frac{1}{\sigma}-\frac{1}{p}}|R|^{\frac{1}{p}-\frac{1}{q}} \le |Q\cap R|^{\frac{1}{\sigma}-\frac{1}{p}}|Q\cap R|^{\frac{1}{p}-\frac{1}{q}},$$

it follows that

$$\|f\|_{\mathcal{M}^{\sigma}_{\mathcal{M}^p_q}} = \sup_{Q\in\mathcal{D}} |Q|^{\frac{1}{\sigma}-\frac{1}{p}}|Q|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_Q\|_{L^q} = \sup_{Q\in\mathcal{D}} |Q|^{\frac{1}{\sigma}-\frac{1}{q}}\|f\chi_Q\|_{L^q} = \|f\|_{\mathcal{M}^{\sigma}_q}.$$

Let $f \in \mathrm{PWM}\left(\dot{B}^s_{q1}(\mathbb{R}^n), \mathcal{M}^p_q(\mathbb{R}^n)\right)$. As in the proof of Theorem 4.1.5, we obtain

$$\|f\chi_{Q_{jm}}\|_{\mathcal{M}^p_q} \lesssim 2^{js-j\frac{n}{q}}\|f\|_{\mathrm{PWM}(\dot{B}^s_{q1},\mathcal{M}^p_q)}.$$

Thus, $f \in \mathcal{M}^{\sigma}_{\mathcal{M}^p_q}(\mathbb{R}^n)$.

To prove the opposite inclusion, it suffices to show that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left\|\left(f\sum_{m\in\mathbb{Z}^n}\lambda_{jm}a_{jm}\right)\chi_Q\right\|_{L^q} \lesssim 2^{js}\|f\|_{\mathcal{M}^{\sigma}_{\mathcal{M}^p_q}}\left\|\sum_{m\in\mathbb{Z}^n}\lambda_{jm}\chi_{Q_{jm}}\right\|_{L^q}$$

for all cubes Q and sequences $\{a_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n}$ of C^∞-functions satisfying $|\partial^\alpha a_{jm}| \le 2^{j|\alpha|}\chi_{3Q_{jm}}$ with $|\alpha| \le [s+1]$; once this is achieved, we have only to take the supremum over all cubes and to add this estimate over $j \in \mathbb{Z}$.

We calculate

$$\begin{aligned}
|Q|^{\frac{1}{p}-\frac{1}{q}}\left\|\left(f\sum_{m\in\mathbb{Z}^n}\lambda_{jm}a_{jm}\right)\chi_Q\right\|_{L^q} &\lesssim |Q|^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|^q\left\|\left(f\chi_{3Q_{jm}}\right)\chi_Q\right\|_{L^q}{}^q\right)^{\frac{1}{q}}\\
&\le \sup_{m\in\mathbb{Z}^n}|Q|^{\frac{1}{p}-\frac{1}{q}}\left\|\left(f\chi_{3Q_{jm}}\right)\chi_Q\right\|_{L^q}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|^q\right)^{\frac{1}{q}}\\
&\le \sup_{m\in\mathbb{Z}^n}\|f\chi_{3Q_{jm}}\|_{\mathcal{M}^p_q}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|^q\right)^{\frac{1}{q}}\\
&\lesssim 2^{js}\|f\|_{\mathcal{M}^{\sigma}_{\mathcal{M}^p_q}}\left\|\sum_{m\in\mathbb{Z}^n}\lambda_{jm}\chi_{Q_{jm}}\right\|_{L^q}.
\end{aligned}$$

Thus, we obtain the desired result. ∎

4.3 Examples

We will consider various special cases. We suppose that $E = L^\Phi(\mathbb{R}^n)$, the Orlicz space in Section 4.3.1, $E = L^{p,q}(\mathbb{R}^n)$, the Lorentz space in Section 4.3.2, $E = L^{\vec{q}}(\mathbb{R}^n)$, the mixed Lebesgue space in Section 4.3.3 and $E = \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, the mixed Morrey space in Section 4.3.4.

4.3.1 Pointwise multipliers from Besov spaces to Orlicz spaces

As an example of $E(\mathbb{R}^n)$, we first take up the case of $E(\mathbb{R}^n) = L^\Phi(\mathbb{R}^n)$. Then, it seems that the spaces $\mathcal{M}^p_E(\mathbb{R}^n)$ coincide Orlicz–Morrey spaces. However, there are at least three kinds of generalized Orlicz–Morrey spaces. In this time, we consider Orlicz–Morrey spaces of the third kind defined in [50].

Definition 4.3.1. Let $1 \le p < \infty$. Suppose that Φ is a Young function. Also let Φ be bijective. Then, the Orlicz–Morrey space $\mathcal{M}_{p,\Phi}(\mathbb{R}^n)$ of the third kind is defined as the set of all measurable functions f for which the norm

$$\|f\|_{\mathcal{M}_{p,\Phi}} \equiv \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}} \Phi^{-1}\left(\frac{1}{|Q|}\right) \|f\chi_Q\|_{L^\Phi}$$

is finite.

Generalized Orlicz–Morrey spaces of the first kind date back to 2004; see the paper [100] by Nakai, while the one of the second kind date back to 2012; see the paper [135] by Sawano, Sugano and Tanaka. See [26] for the definition of generalized Orlicz–Morrey spaces. In [101], Nakai showed the boundedness of the Hardy–Littlewood maximal operator for the Orlicz–Morrey spaces. As was shown in [39, Theorems 1.4 and 1.6], generalized Orlicz–Morrey spaces of the first kind and the one of the second kind are different. Deringoz, Guliyev, Hasanov, Noi, Samko and Sawano investigated the decomposition of Orlicz–Morrey spaces of the third kind [50]. See [27, 47] for vanishing generalized Orlicz–Morrey spaces of the third kind. We refer to [25] for the weighted Orlicz–Morrey spaces of the third kind. See [25] for the maximal operator and its commutator generated by BMO for the weighted Orlicz–Morrey spaces of the third kind. Finally, see [46] for the fractional integral operators together with commutators for the Orlicz–Morrey spaces of the third kind.

Thanks to Lemma 2.2.18, we can check the coincidence of $\mathcal{M}^p_{L^\Phi}(\mathbb{R}^n)$ and $\mathcal{M}_{p,\Phi}(\mathbb{R}^n)$. Indeed,

$$\|f\|_{\mathcal{M}^p_{L^\Phi}} = \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}} \frac{1}{\|\chi_Q\|_{L^\Phi}} \|f\chi_Q\|_{L^\Phi} = \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}} \Phi^{-1}\left(\frac{1}{|Q|}\right) \|f\chi_Q\|_{L^\Phi} = \|f\|_{\mathcal{M}_{p,\Phi}}.$$

From these observations, we can apply Theorem 4.1.4 to Orlicz spaces and obtain the following result.

Theorem 4.3.2. *Let $1 \le p < \infty$ and Φ be as in Definition 4.3.1. Set*

$$w_j = \Phi^{-1}\left(\frac{1}{|Q_{j0}|}\right)|Q_{j0}|^{-1-\frac{1}{p}} \quad (j \in \mathbb{Z}).$$

Then

$$\mathrm{PWM}(\dot{B}^w_{11}(\mathbb{R}^n), L^\Phi(\mathbb{R}^n)) \approx \mathcal{M}_{p,\Phi}(\mathbb{R}^n)$$

with equivalence of norms.

4.3.2 Pointwise multipliers from Besov spaces to Lorentz spaces

As an example of $E(\mathbb{R}^n)$, we take up the case of $E(\mathbb{R}^n) = L^{p,q}(\mathbb{R}^n)$. Then, we see that the space $\mathcal{M}^p_E(\mathbb{R}^n)$ coincides the Morrey–Lorentz space $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$. We define

$$\|f\|_{\mathcal{M}^p_{q,r}} \equiv \sup_Q |Q|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_Q\|_{L^{q,r}}$$

for $0 < q \le p < \infty$, $0 < r \le \infty$. The Morrey–Lorentz space $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ denotes the set of all $f \in L^0(\mathbb{R}^n)$ for which $\|f\|_{\mathcal{M}^p_{q,r}}$ is finite. This space is introduced by Ragusa in [118]. Later, this space of characterizations and applications are investigated in [5, 36]. If $1 < q \le p < \infty$ and $1 \le r \le \infty$, then $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ is a Banach space since above argument. The weak Morrey space $\mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$ denotes $\mathcal{M}^p_{q,\infty}(\mathbb{R}^n)$. We note that $\|f\|_{\mathrm{W}\mathcal{M}^p_q} = \sup_{\lambda>0} \lambda\|\chi_{\{|f|>\lambda\}}\|_{\mathcal{M}^p_q}$ for any $f \in L^0(\mathbb{R}^n)$.

When we replace E and F by $L^{q,r}(\mathbb{R}^n)$ and $L^{u,v}(\mathbb{R}^n)$ respectively in Theorem 4.1.4, we obtain the following result by using the embeddings of Lorentz spaces (Theorem 2.2.13).

Theorem 4.3.3. *Let $1 < q \le p < \infty$, $1 \le r, v \le \infty$, $1 < u \le \infty$. Assume that*

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{u}.$$

Then $\dot{B}^{\frac{n}{p}+\frac{n}{q'}}_{11}(\mathbb{R}^n)$ is continuously embedded into $L^{u,v}(\mathbb{R}^n)$ and

$$\mathrm{PWM}(\dot{B}^{\frac{n}{p}+\frac{n}{q'}}_{11}(\mathbb{R}^n), L^{q,r}(\mathbb{R}^n)) \approx \mathcal{M}^p_{q,r}(\mathbb{R}^n)$$

with equivalence of norms. In particular,

$$\mathrm{PWM}(\dot{B}^{\frac{n}{p}+\frac{n}{q'}}_{11}(\mathbb{R}^n), \mathrm{W}L^q(\mathbb{R}^n)) \approx \mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$$

with equivalence of norms.

We now extend Theorem 4.1.5 to Morrey-Lorentz spaces as follows.

Theorem 4.3.4. *Let $1 < p < \infty$, $1 \le q \le \infty$ and $0 < s \le \dfrac{n}{\min(p,q)}$. Assume*

$$0 < \frac{s}{n} - \frac{1}{\min(p,q)} + \frac{1}{p} \le \frac{1}{p}.$$

Define $\sigma > 0$ by

$$\frac{1}{\sigma} \equiv \frac{s}{n} - \frac{1}{\min(p,q)} + \frac{1}{p}.$$

Then

$$\mathrm{PWM}(\dot{B}^s_{\min(p,q)1}(\mathbb{R}^n), L^{p,q}(\mathbb{R}^n)) \approx \mathcal{M}^\sigma_{p,q}(\mathbb{R}^n)$$

with equivalence of norms.

Theorem 4.3.5. *Let $1 < q \le p < \infty$, $1 \le r \le \infty$ and $0 < s \le \dfrac{n}{\min(q,r)}$. Assume*

$$0 < \frac{s}{n} - \frac{1}{\min(q,r)} + \frac{1}{p} \le \frac{1}{p}.$$

Define σ by

$$\frac{1}{\sigma} \equiv \frac{s}{n} - \frac{1}{\min(q,r)} + \frac{1}{p}.$$

Then

$$\mathrm{PWM}(\dot{B}^s_{\min(q,r)1}(\mathbb{R}^n), \mathcal{M}^p_{q,r}(\mathbb{R}^n)) \approx \mathcal{M}^\sigma_{\mathcal{M}^p_{q,r}}(\mathbb{R}^n) = \mathcal{M}^\sigma_{q,r}(\mathbb{R}^n)$$

with equivalence of norms.

Theorem 4.3.4 is included in Theorem 4.3.5. So we prove Theorem 4.3.5.

Proof of Theorem 4.3.5. As in the proof of Theorem 4.1.6, we obtain

$$\mathcal{M}^\sigma_{\mathcal{M}^p_{q,r}}(\mathbb{R}^n) = \mathcal{M}^\sigma_{q,r}(\mathbb{R}^n).$$

Let $f \in \mathrm{PWM}(\dot{B}^s_{\min(q,r)1}(\mathbb{R}^n), \mathcal{M}^p_{q,r}(\mathbb{R}^n))$. Then define κ_{jm} as before. Similar to the proof of Theorem 4.1.5, then we have

$$\|f\|_{\mathcal{M}^\sigma_{\mathcal{M}^p_{q,r}}} \lesssim \|f\|_{\mathrm{PWM}(\dot{B}^s_{\min(q,r)1}, \mathcal{M}^p_{q,r})}.$$

Consequently, $f \in \mathcal{M}^\sigma_{\mathcal{M}^p_{q,r}}(\mathbb{R}^n)$.

To show the opposite estimate, it suffices to show that

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \left\| \left(f \sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm} \right) \chi_Q \right\|_{L^{q,r}} \lesssim 2^{js} \|f\|_{\mathcal{M}^\sigma_{\mathcal{M}^p_{q,r}}} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{jm} \chi_{Q_{jm}} \right\|_{L^{\min(q,r)}}$$

for all sequences $\{a_{jm}\}_{j \in \mathbb{Z}, m \in \mathbb{Z}^n}$ of C^∞-functions satisfying $|\partial^\alpha a_{jm}| \le 2^{j|\alpha|} \chi_{3Q_{jm}}$ with $|\alpha| \le [s+1]$; once this is achieved, we have only to add this estimate over $j \in \mathbb{Z}$.

Let $r < q$. By using Minkowski's inequality,

$$\begin{aligned}
|Q|^{\frac{1}{p}-\frac{1}{q}} \left\| \left(f \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm} \right) \chi_Q \right\|_{L^{q,r}} &\lesssim |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^r \|(f\chi_{3Q_{jm}})\chi_Q\|_{L^{q,r}}{}^r \right)^{\frac{1}{r}} \\
&\leq \sup_{m\in\mathbb{Z}^n} \|f\chi_{3Q_{jm}}\|_{\mathcal{M}^p_{q,r}} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^r \right)^{\frac{1}{r}} \\
&\lesssim 2^{js} \|f\|_{\mathcal{M}^{\sigma}_{\mathcal{M}^p_{q,r}}} \left\| \sum_{m\in\mathbb{Z}^n} \lambda_{jm} \chi_{Q_{jm}} \right\|_{L^r}.
\end{aligned}$$

Conversely, let $q \leq r$. Note that, each cube Q_{jl} intersects 3^n cubes $3Q_{jm_{l,k}}, k = 1, 2, \ldots, 3^n$. Therefore, using Lemma 2.2.14, we have

$$\begin{aligned}
&|Q|^{\frac{q}{p}-1} \left\| \left(f \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm} \right) \chi_Q \right\|_{L^{q,r}}^q \\
&\lesssim |Q|^{\frac{q}{p}-1} \sum_{l\in\mathbb{Z}^n} \left\| \left(f \sum_{k=1}^{3^n} \lambda_{jm_{l,k}} a_{jm_{l,k}} \right) \chi_Q \right\|_{L^{q,r}(Q_{jl})}^q \\
&\lesssim |Q|^{\frac{q}{p}-1} \sum_{l\in\mathbb{Z}^n} \sum_{k=1}^{3^n} |\lambda_{jm_{l,k}}|^q \left\| \left(|f| \sum_{k=1}^{3^n} |\chi_{3Q_{jm_{l,k}}}| \right) \chi_Q \right\|_{L^{q,r}(Q_{jl})}^q \\
&\lesssim \sup_{l\in\mathbb{Z}^n} \|f\chi_{Q_{jl}}\|_{\mathcal{M}^p_{q,r}}^q \left(\sum_{l\in\mathbb{Z}^n} |\lambda_{jl}|^q \right)
\end{aligned}$$

Hence

$$\left\| f \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm} \right\|_{\mathcal{M}^p_{q,r}} \lesssim 2^{js} \|f\|_{\mathcal{M}^{\sigma}_{\mathcal{M}^p_{q,r}}} \left\| \sum_{l\in\mathbb{Z}^n} \lambda_{jl} \chi_{Q_{jl}} \right\|_{L^q}$$

Thus, f is a pointwise multiplier from $\dot{B}^s_{\min(q,r)1}(\mathbb{R}^n)$ to $\mathcal{M}^{\sigma}_{q,r}(\mathbb{R}^n)$. ■

4.3.3 Pointwise multipliers from Besov spaces to mixed Lebesgue spaces

Applying Theorem 4.1.4 for mixed Lebesgue spaces, we obtain the following result:

Theorem 4.3.6. *Let $1 \leq q_1, \ldots, q_n < \infty$, and suppose that p satisfies*

$$\frac{1}{p} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1}{q_k}.$$

Define r by

$$\frac{1}{r} \equiv \frac{1}{n} \sum_{k=1}^{n} \frac{1}{q_k} - \frac{1}{p}.$$

Then $\dot{B}^{\frac{n}{r'}}_{11}(\mathbb{R}^n)$ *is continuously embedded into* $L^r(\mathbb{R}^n)$ *and*

$$\mathrm{PWM}\left(\dot{B}_{11}^{\sum_{k=1}^{n}\frac{1}{r_k}}(\mathbb{R}^n), L^{\vec{q}}(\mathbb{R}^n)\right) = \mathcal{M}^p_{L^{\vec{q}}}(\mathbb{R}^n) = \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n).$$

Proof. Simply observe that $E(\mathbb{R}^n) = L^{\vec{q}}(\mathbb{R}^n)$ and $F(\mathbb{R}^n) = L^r(\mathbb{R}^n)$ satisfy the condition of Theorem 4.1.4. ∎

4.3.4 Pointwise multipliers from Besov spaces to mixed Morrey spaces

As an example of $E(\mathbb{R}^n)$, we take up the case of $E(\mathbb{R}^n) = \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

Remark 4.3.7. In the definition of $\mathcal{M}^p_E$, let $E(\mathbb{R}^n) = \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. Then

$$\|f\|_{\mathcal{M}^p_{\mathcal{M}^p_{\vec{q}}}} = \sup_Q |Q|^{\frac{1}{p}} \left(\frac{1}{\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}}}\|f\chi_Q\|_{\mathcal{M}^p_{\vec{q}}}\right) = \sup_Q |Q|^{\frac{1}{p}} \left(\frac{1}{|Q|^{\frac{1}{p}}}\|f\chi_Q\|_{\mathcal{M}^p_{\vec{q}}}\right) = \|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

Thus, we see that

$$\mathcal{M}^p_{\mathcal{M}^p_{\vec{q}}}(\mathbb{R}^n) = \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n).$$

Keeping this remark in mind, we apply Theorem 4.1.4 for this space.

Theorem 4.3.8. *Let* $1 \le p < \infty$ *and* $1 \le \vec{q} < \infty$ *satisfy*

$$\sum_{j=1}^{n} \frac{1}{q_j} \ge \frac{n}{p}.$$

Then

$$\mathrm{PWM}\left(\dot{B}^n_{11}(\mathbb{R}^n), \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)\right) = \mathcal{M}^p_{\mathcal{M}^p_{\vec{q}}}(\mathbb{R}^n) = \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n).$$

Also we extend Theorem 4.1.5 to mixed Morrey spaces.

Theorem 4.3.9. *Let* $\vec{q} = (q_1, \ldots, q_n) \in [1,\infty)^n, 1 \le p < \infty$ *and* $0 < s \le \frac{n}{q_n}$*. Suppose that* σ *is given by*

$$\frac{1}{\sigma} \equiv \frac{s}{n} - \frac{1}{q_n} + \frac{1}{p}.$$

Then with equivalence norms

$$\mathrm{PWM}\left(\dot{B}^s_{q_n 1}(\mathbb{R}^n), \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)\right) \approx \mathcal{M}^\sigma_{\mathcal{M}^p_{\vec{q}}}(\mathbb{R}^n) = \mathcal{M}^\sigma_{\vec{q}}(\mathbb{R}^n).$$

Proof. Let $f \in \mathrm{PWM}\left(\dot{B}^s_{q_n 1}(\mathbb{R}^n), \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)\right)$. As in the proof of Theorem 4.1.5, we obtain

$$\|f\chi_{Q_{jm}}\|_{\mathcal{M}^p_{\vec{q}}} \lesssim 2^{js - j\frac{n}{q_n}} \|f\|_{\mathrm{PWM}(\dot{B}^s_{q_n 1}, \mathcal{M}^p_{\vec{q}})}.$$

Thus, $f \in \mathcal{M}^{\sigma}_{\mathcal{M}^p_{\vec{q}}}(\mathbb{R}^n)$.

To establish the opposite inclusion, it suffices to show that

$$|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{q_k}}\left\|\left(f\sum_{m\in\mathbb{Z}^n}\lambda_{jm}a_{jm}\right)\chi_Q\right\|_{L^{\vec{q}}} \lesssim 2^{js}\|f\|_{\mathcal{M}^{\sigma}_{\mathcal{M}^p_{\vec{q}}}}\left\|\sum_{m\in\mathbb{Z}^n}\lambda_{jm}\chi_{Q_{jm}}\right\|_{L^{q_n}}$$

for all cubes Q and sequences $\{a_{jm}\}_{j\in\mathbb{Z},m\in\mathbb{Z}^n}$ of C^∞-functions satisfying $|\partial^\alpha a_{jm}| \le 2^{j|\alpha|}\chi_{3Q_{jm}}$ with $|\alpha| \le [s+1]$; once this is achieved, again we have only to take the supremum over all cubes and to add this estimate over $j \in \mathbb{Z}$.

We calculate

$$\begin{aligned}
&|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{q_k}}\left\|\left(f\sum_{m\in\mathbb{Z}^n}\lambda_{jm}a_{jm}\right)\chi_Q\right\|_{L^{\vec{q}}}\\
&\lesssim |Q|^{\frac{1}{p}-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{q_k}}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|^{q_n}\left\|\left(f\chi_{3Q_{jm}}\right)\chi_Q\right\|_{L^{\mathbf{q}}}{}^{q_n}\right)^{\frac{1}{q_n}}\\
&\le \sup_{m\in\mathbb{Z}^n}|Q|^{\frac{1}{p}-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{q_k}}\left\|\left(f\chi_{3Q_{jm}}\right)\chi_Q\right\|_{L^{\vec{q}}}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|^{q_n}\right)^{\frac{1}{q_n}}\\
&\le \sup_{m\in\mathbb{Z}^n}\|f\chi_{3Q_{jm}}\|_{\mathcal{M}^p_{\vec{q}}}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|^{q_n}\right)^{\frac{1}{q_n}}\\
&\lesssim 2^{js}\|f\|_{\mathcal{M}^{\sigma}_{\mathcal{M}^p_{\vec{q}}}}\left\|\sum_{m\in\mathbb{Z}^n}\lambda_{jm}\chi_{Q_{jm}}\right\|_{L^{q_n}}.
\end{aligned}$$

Thus, f is a pointwise multiplier from $\dot{B}^s_{q_n 1}(\mathbb{R}^n)$ to $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. ∎

Chapter 5

Atomic decomposition for mixed Morrey spaces

5.1 Introduction and theorems

One of the characterizations of function spaces is to decompose an element of function spaces into good functions. The most important example is the Fourier series for L^2 functions. In addition, there are the Littlewood–Paley decomposition (cf. Subsection 2.2.1), the molecular decomposition, and the wavelet decomposition. In this thesis, we focus our interest on atomic decompositions, which were introduced by Coifman in 1974 to characterize the functions belonging to Hardy spaces. Here, an atom is a function which has a support on a cube (support condition), a norm estimate with respect to the cube (size condition), and moment condition. This decomposition is applied to characterize many function spaces.

The aim of this chapter is to develop a theory of decompositions for mixed Morrey spaces. Furthermore, we can extend these results to Hardy-mixed Morrey spaces. Our results are extension of the results for classical Morrey spaces and Hardy-Morrey spaces in [66].

First, we will prove the following boundedness of the maximal operator on mixed Lebesgue spaces.

Theorem 5.1.1. *Assume that*

$$1 \leq t_k < \min\{q_1, \ldots, q_k\} \leq \infty \quad (k = 1, \ldots, n).$$

Define

$$M^{(\vec{t})} f(x) = \sup_{Q \in \mathcal{Q}} \frac{\chi_Q(x)}{\|\chi_Q\|_{L^{\vec{t}}}} \|f \chi_Q\|_{L^{\vec{t}}}$$

for a measurable function f. Then for all measurable functions f

$$\|M^{(\vec{t})} f\|_{L^{\vec{q}}} \lesssim \|f\|_{L^{\vec{q}}}.$$

Based on this boundedness, we prove the following construction theorem. This is the same one as Theorem 1.2.6 in Chapter 1.

Theorem 5.1.2. *Suppose that the parameters $p, \vec{q}, s, \vec{t}$ satisfy*

$$1 < p < s < \infty, \quad 1 < \max\{q_1, \dots, q_k\} < t_k < \infty \quad (k = 1, \dots, n),$$

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n}{s} \le \sum_{j=1}^{n} \frac{1}{t_j}.$$

Assume that $\{a_j\}_{j=1}^{\infty} \subset \mathcal{M}^s_{\vec{t}}(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$, and $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ fulfill

$$\|a_j\|_{\mathcal{M}^s_{\vec{t}}} \le |Q_j|^{\frac{1}{s}}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\vec{q}}} < \infty.$$

Then $f = \sum_{j=1}^{\infty} \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \le C_{p,\vec{q},s,\vec{t}} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

In connection with Theorem 5.1.2, we refer to [2, 40, 107] for more recent characterizations of the predual spaces of classical and mixed Morrey spaces.

Before we state the next results, we recall the some notation. For $d \ge 0$, denote by $\mathcal{P}_d(\mathbb{R}^n)$ the set of all polynomial functions with degree less than or equal to d, so that $\mathcal{P}(\mathbb{R}^n) \equiv \bigcup_{d=0}^{\infty} \mathcal{P}_d(\mathbb{R}^n)$. It is clear that $\mathcal{P}_{-1}(\mathbb{R}^n) = \{0\}$. The set $\mathcal{P}_K(\mathbb{R}^n)^{\perp}$ denotes the set of measurable function f for which $\langle \cdot \rangle^K f \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} x^{\alpha} f(x) \mathrm{d}x = 0$ for any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le K$, where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. Such a function f is said to satisfy the *moment condition of order* K. In this case, one also writes $f \perp \mathcal{P}_K(\mathbb{R}^n)$.

The next assertion concerns the decomposition of functions in $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. This theorem is the same one as Theorem 1.2.7 in Chapter 1.

Theorem 5.1.3 (cf. [60])**.** *Suppose that the real parameters $p, \vec{q}, K$ satisfy*

$$1 < p < \infty, \quad 1 < \vec{q} < \infty, \quad \frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad K \in \mathbb{N}_0 \cap \left(\frac{n}{q_0} - n - 1, \infty \right),$$

where $q_0 = \min(q_1, \dots, q_n)$. Let $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. Then there exists a triplet $\{a_j\}_{j=1}^{\infty} \subset L^{\infty}(\mathbb{R}^n) \cap \mathcal{P}_K^{\perp}(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$, and $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ in

$\mathcal{S}'(\mathbb{R}^n)$ and that, for any $v > 0$

$$|a_j| \le \chi_{Q_j}, \quad \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{\vec{q}}} \le C_v \|f\|_{\mathcal{M}^p_{\vec{q}}}. \tag{5.1}$$

Here the constant $C_v > 0$ is independent of f.

We rephrase Theorems 5.1.2 and 5.1.3 in the case of mixed Lebesgue spaces.

Corollary 5.1.4. *Suppose that the parameters $\vec{q}, \vec{t}$ satisfy*

$$1 < \max\{q_1, \ldots, q_k\} < t_k < \infty \quad (k = 1, \ldots, n).$$

Assume that $\{a_j\}_{j=1}^{\infty} \subset L^{\vec{t}}(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$, and $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ fulfill

$$\|a_j\|_{L^{\vec{t}}} \le |Q_j|^{\frac{1}{n}\sum_{k=1}^{n}\frac{1}{t_k}}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^{\vec{q}}} < \infty.$$

Then $f = \sum_{j=1}^{\infty} \lambda_j a_j$ converges in $L^{\vec{q}}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{L^{\vec{q}}} \le C_{p,q,s,t} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^{\vec{q}}}.$$

Corollary 5.1.5 (cf. [60])**.** *Let $1 < \vec{q} < \infty$ and $K \in \mathbb{N}_0$. Let $f \in L^{\vec{q}}(\mathbb{R}^n)$. Then there exists a triplet $\{a_j\}_{j=1}^{\infty} \subset L^{\infty}(\mathbb{R}^n) \cap \mathcal{P}_K^{\perp}(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$, and $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$ such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ in $L^{\vec{q}}(\mathbb{R}^n)$ and that, for any $v > 0$*

$$|a_j| \le \chi_{Q_j}, \quad \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{L^{\vec{q}}} \le C_v \|f\|_{L^{\vec{q}}}.$$

Here the constant $C_v > 0$ is independent of f.

Next, we generalize Theorems 5.1.2 and 5.1.3. Based on [126], we define Hardy-mixed Morrey spaces.

Definition 5.1.6. For $0 < \vec{q}, p < \infty$ satisfying $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$, the Hardy-mixed Morrey space $H\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ is defined as the set of any $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{H\mathcal{M}^p_{\vec{q}}} = \left\| \sup_{t>0} |e^{t\Delta} f| \right\|_{\mathcal{M}^p_{\vec{q}}}$$

is finite, where $e^{t\Delta}f$ stands for the heat extension of f;

$$e^{t\Delta}f(x) = \left\langle \frac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\frac{|x-\cdot|^2}{4t}\right), f \right\rangle \quad (t>0,\ x\in\mathbb{R}^n).$$

See [150] for the equivalent norms of Hardy–Morrey spaces. We rephrase Theorem 5.1.2 and 5.1.3 in full generality in terms of Hardy-mixed Morrey spaces.

Theorem 5.1.7. *Suppose that the parameters $p, \vec{q}, s, \vec{t}$ satisfy*

$$1<p<s<\infty, \quad 0<\max\{1,q_1,\ldots,q_k\}<t_k<\infty \quad (k=1,\ldots,n),$$
$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n}{s} \le \sum_{j=1}^{n} \frac{1}{t_j}.$$

Write $v(\vec{q}) \equiv \min\{1,q_1,\ldots,q_n\}$ and $d_q = \left[n\left(\frac{1}{v(\vec{q})}-1\right)\right]$. Assume that a triple

$$(\{a_j\}_{j=1}^\infty, \{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty) \in (\mathcal{M}^s_{\vec{t}}(\mathbb{R}^n) \cap \mathcal{P}^\perp_{d_q}(\mathbb{R}^n)) \times [0,\infty) \times \mathcal{Q}(\mathbb{R}^n)$$

fulfills

$$\|a_j\|_{\mathcal{M}^s_{\vec{t}}} \le |Q_j|^{\frac{1}{s}}, \quad \operatorname{supp}(a_j)\subset Q_j, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^{v(\vec{q})} \right)^{\frac{1}{v(\vec{q})}} \right\|_{\mathcal{M}^p_{\vec{q}}} < \infty.$$

Then $f = \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and satisfies

$$\|f\|_{H\mathcal{M}^p_{\vec{q}}} \le C_{p,\vec{q},s,\vec{t}} \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^{v(\vec{q})} \right)^{\frac{1}{v(\vec{q})}} \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

Theorem 5.1.3 has the following counterpart.

Theorem 5.1.8 (cf. [60])**.** *Suppose that the real parameters $p, \vec{q}, K$ satisfy*

$$0<p<\infty, \quad 0<\vec{q}<\infty, \quad \frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad K \in \mathbb{N}_0 \cap \left(\frac{n}{q_0}-n-1, \infty\right),$$

where $q_0 = \min(1,q_1,\ldots,q_n)$. Let $f \in H\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. Then there exists a triplet $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n) \cap \mathcal{P}^\perp_K(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, and $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and that, for any $v>0$,

$$|a_j| \le \chi_{Q_j}, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^{v} \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{\vec{q}}} \le C_v \|f\|_{H\mathcal{M}^p_{\vec{q}}}. \tag{5.2}$$

Here the constant C_v is a constant that is independent on v but not on f.

As we said in Subsection 1.2.3, although Theorems 5.1.3 and 5.1.8 are given as corollaries of the results in [60], we prove these theorems directly without using Hertz spaces which were used in [60].

Finally, we survey the classical results of this section. In fact, atomic decompositions are roughly classified into two types. One is the decomposition by smooth atoms, and the other is the decomposition by non-smooth atoms. Our results correspond to the non-smooth case. Concerning non-smooth results for Morrey spaces, Jia and Wang considered non-smooth atomic decompositions for Hardy-Morrey spaces in [72]. After that in [66], Iida, Sawano, and Tanaka investigated non-smooth atomic decompositions for Morrey spaces, which include results of Jia and Wang. So, our results for mixed Morrey spaces are extension of these results. At last, with respect to smooth results for Morrey spaces, we refer to [92, 131].

The remaining parts of this chapter is as follows. In Section 5.2, we establish the boundedness result for the maximal operator to prove Theorem 5.1.2. In Section 5.3, we give characterizations of Hardy-Morrey spaces. Section 5.4 is devoted to showing the main theorems. In Subsections 5.4.1 and 5.4.2, we consider the reconstruction theorems for mixed Morrey spaces (Theorem 5.1.2) and Hardy-mixed Morrey spaces (Theorem 5.1.7), respectively. Meanwhile in Subsection 5.4.3, we prove the decomposition theorems for mixed Morrey spaces (Theorem 5.1.3) and Hardy-mixed Morrey spaces (Theorem 5.1.8). As an application, we concern Olsen's inequality for mixed Morrey spaces in Section 5.5.

5.2 The boundedness of the maximal operator $M^{(\vec{t})}$ (Theorem 5.1.1)

In this section, we prove the boundedness property of the maximal operator $M^{(\vec{t})}$ on mixed Morrey spaces. This theorem is applied to the proof of Theorem 5.1.2. We invoke a result due to Bagby [10].

Lemma 5.2.1. *Let $1 < q_1, \ldots, q_m < \infty$ and $1 < p < \infty$. For $i = 1, 2 \ldots, m$, let (Ω_i, μ_i) be σ-finite measure spaces, and $\Omega = \Omega_1 \times \cdots \times \Omega_m$. For $f \in L^0(\mathbb{R}^n \times \Omega)$,*

$$\int_{\mathbb{R}^n} \|Mf(x,\cdot)\|^p_{L^{(q_1,\ldots,q_m)}} \,\mathrm{d}x \lesssim \int_{\mathbb{R}^n} \|f(x,\cdot)\|^p_{L^{(q_1,\ldots,q_m)}} \,\mathrm{d}x.$$

The following lemma is used in the induction step.

Lemma 5.2.2. *Let $\vec{q} = (q_1, q_2, \ldots, q_n) \in (1,\infty)^n$ and let*

$$t_n \in [1, \min\{q_1, q_2, \ldots, q_n\}).$$

Then

$$\left\| M_n^{(t_n)} f \right\|_{L^{\vec{q}}} \lesssim \|f\|_{L^{\vec{q}}}$$

for all $f \in L^{\vec{q}}(\mathbb{R}^n)$.

For the proof we use the following notation for $h \in L^0(\mathbb{R}^n)$:

$$\|h\|_{L^{(q_1,\ldots,q_m)}}(x_{m+1},\ldots,x_n) \equiv \left\|\left[\|h\|_{L^{(q_1,\ldots,q_{m-1})}}\right]\right\|_{L^{(q_m)}}(x_{m+1},\ldots,x_n)$$

and when $m=1$, we define

$$\|h\|_{L^{(q_1)}}(x_2,\ldots,x_n) \equiv \left(\int_{\mathbb{R}} |h(x_1,\ldots,x_n)|^{q_1} \mathrm{d}x_1\right)^{\frac{1}{q_1}}.$$

Proof. Thanks to Lemma 5.2.1, we obtain

$$\begin{aligned}
\left\|M_n^{(t_n)} f\right\|_{L^{\vec{q}}}^{q_n} &= \int_{\mathbb{R}} \left\|M_n^{(t_n)} f(\cdot, x_n)\right\|_{L^{(q_1,\ldots,q_{n-1})}}^{q_n} \mathrm{d}x_n \\
&= \int_{\mathbb{R}} \left\|M_n[|f|^{t_n}](\cdot, x_n)\right\|_{L^{\left(\frac{q_1}{t_n},\ldots,\frac{q_{n-1}}{t_n}\right)}}^{\frac{q_n}{t_n}} \mathrm{d}x_n \\
&\lesssim \int_{\mathbb{R}} \left\|[|f(\cdot, x_n)|^{t_n}]\right\|_{L^{\left(\frac{q_1}{t_n},\ldots,\frac{q_{n-1}}{t_n}\right)}}^{\frac{q_n}{t_n}} \mathrm{d}x_n = \|f\|_{L^{\vec{q}}}^{q_n}.
\end{aligned}$$

Thus, we obtain the desired result. ∎

Proof of Theorem 5.1.1. We start with a preliminary observation for maximal operators. Let $x \in \mathbb{R}^n$. Let $Q = I_1 \times \cdots \times I_n$ where each I_j is an interval in $\mathbb{R}$ with the same length. Then,

$$\begin{aligned}
&\frac{\chi_Q(x)}{\|\chi_Q\|_{L^{\vec{t}}}}\|f\chi_Q\|_{L^{\vec{t}}} = \frac{\bigotimes_{j=1}^n \chi_{I_j}(x)}{\prod\limits_{j=1}^n |I_j|^{\frac{1}{t_j}}} \left\|f\chi_{\prod\limits_{j=1}^n I_j}\right\|_{L^{\vec{t}}} \\
&= \frac{\bigotimes_{j=2}^n \chi_{I_j}(x_2,\ldots,x_n)}{\prod\limits_{j=2}^n |I_j|^{\frac{1}{t_j}}} \\
&\quad\times \left\|\left[\left(\frac{\chi_{I_1}(x_1)}{|I_1|}\int |f(y)|^{t_1}\chi_{I_1}(y_1)\mathrm{d}y_1\right)^{\frac{1}{t_1}}\right]\chi_{\prod\limits_{j=2}^n I_j}\right\|_{L^{(t_2,\ldots,t_n)}} \\
&\le \frac{\bigotimes_{j=2}^n \chi_{I_j}(x_2,\ldots,x_n)}{\prod\limits_{j=2}^n |I_j|^{\frac{1}{t_j}}} \left\|\left[M^{(t_1)}f\right]\chi_{\prod\limits_{j=2}^n I_j}\right\|_{L^{(t_2,\ldots,t_n)}}.
\end{aligned}$$

Continuing this procedure, we have

$$\frac{\chi_Q(x)}{\|\chi_Q\|_{L^{\vec{t}}}}\|f\chi_Q\|_{L^{\vec{t}}} \le M_n^{(t_n)}\cdots M_1^{(t_1)}(f)(x).$$

Thus, it follows that

$$M^{(\vec{t})}f(x) \le M_n^{(t_n)}\cdots M_1^{(t_1)}(f)(x).$$

Therefore, it suffices to show that

$$\left\| M_n^{(t_n)} \cdots M_1^{(t_1)}(f) \right\|_{L^{\vec{q}}} \lesssim \|f\|_{L^{\vec{q}}}. \tag{5.3}$$

We proceed by induction on n. For $n = 1$, the result follows by the classical case of the boundedness of the Hardy–Littlewood maximal operator.
Suppose that the result holds for $n = m-1$ with $m > 1$ in $\mathbb{N}$: assume that

$$\|M_{m-1}^{(t_{m-1})} \cdots M_1^{(t_1)} h\|_{L^{(q_1,\dots,q_{m-1})}} \lesssim \|h\|_{L^{(q_1,\dots,q_{m-1})}}$$

for $1 < t_k < \min\{q_1, \dots, q_k\} < \infty$ for each $k = 1, \dots, m-1$, and for $h \in L^0(\mathbb{R}^{m-1})$. Since $t_m < \min\{q_1, \dots, q_m\}$, for $g \in L^0(\mathbb{R}^m)$, we have

$$\begin{aligned}
\left\| M_m^{(t_m)} g \right\|_{L^{(q_1,\dots,q_m)}} &= \left\| \left[\left\| M_m^{(t_m)} g \right\|_{L^{(q_1,\dots,q_{m-1})}} \right] \right\|_{L^{(q_m)}} \\
&= \left\| \left[\left\| M_m[|g|^{t_m}] \right\|_{L^{\left(\frac{q_1}{t_m},\dots,\frac{q_{m-1}}{t_m}\right)}} \right] \right\|_{L^{\left(\frac{q_m}{t_m}\right)}}^{\frac{1}{t_m}} \\
&\lesssim \left\| \left[\|g\|_{L^{(q_1,\dots,q_{m-1})}} \right] \right\|_{L^{(q_m)}} = \|g\|_{L^{(q_1,\dots,q_m)}}.
\end{aligned}$$

Thus, by the induction assumption, letting $g = M_{m-1}^{(t_{m-1})} \cdots M_1^{(t_1)}(f)$, we obtain

$$\begin{aligned}
\left\| M_m^{(t_m)} \cdots M_1^{(t_1)}(f) \right\|_{L^{(q_1,\dots,q_m)}} &= \left\| M_m^{(t_m)} [M_{m-1}^{(t_{m-1})} \cdots M_1^{(t_1)}(f)] \right\|_{L^{(q_1,\dots,q_m)}} \\
&\lesssim \left\| M_{m-1}^{(t_{m-1})} \cdots M_1^{(t_1)}(f) \right\|_{L^{(q_1,\dots,q_m)}} \\
&= \left\| \left\| M_{m-1}^{(t_{m-1})} \cdots M_1^{(t_1)}(f) \right\|_{L^{(q_1,\dots,q_{m-1})}} \right\|_{L^{(q_m)}} \\
&\lesssim \left\| \|f\|_{L^{(q_1,\dots,q_{m-1})}} \right\|_{L^{(q_m)}} \lesssim \|f\|_{L^{(q_1,\dots,q_m)}}.
\end{aligned}$$

Hence, inequality (5.3) holds for any dimension n. We obtain the desired result.
∎

One can show that the condition

$$t_k < \min\{q_1, q_2, \dots, q_k\}$$

is sharp.

Proposition 5.2.3. *In Theorem 5.1.1, for each $k = 1, 2, \dots, n$, the condition $t_k < \min\{q_1, q_2, \dots, q_k\}$ can not be removed.*

Proof. We induct on n. The base case $n = 1$ is clear since the Hardy–Littlewood maximal operator is bounded on $L^p(\mathbb{R})$ if and only if $p > 1$. Assume that the conclusion of Proposition 5.2.3 is true for $n = m-1$ and that $M^{(t_1,t_2,\dots,t_m)}$ is bounded on $L^{(q_1,q_2,\dots,q_m)}(\mathbb{R}^m)$. Let $h \in L^{(t_1,t_2,\dots,t_{m-1})}(\mathbb{R}^{m-1})$ and $N \in \mathbb{N}$. Then

$$\begin{aligned}
&\chi_{[-N,N]^m} \left(M^{(t_1,t_2,\dots,t_{m-1})} \left[\chi_{[-N,N]^{m-1}} h \right] \otimes \chi_{[-N,N]} \right) \\
&\le M^{(t_1,t_2,\dots,t_m)} \left[(\chi_{[-N,N]^{m-1}} h) \otimes \chi_{[-N,N]} \right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
&(2N)^{\frac{1}{q_m}} \left\| \chi_{[-N,N]^{m-1}} M^{(t_1,t_2,\dots,t_{m-1})} \left[\chi_{[-N,N]^{m-1}} h\right] \right\|_{L^{(q_1,q_2,\dots,q_{m-1})}} \\
&= \left\| \chi_{[-N,N]^m} M^{(t_1,t_2,\dots,t_{m-1})} \left[\chi_{[-N,N]^{m-1}} h\right] \otimes \chi_{[-N,N]} \right\|_{L^{(q_1,q_2,\dots,q_m)}} \\
&\le \left\| M^{(t_1,t_2,\dots,t_{m-1})} \left[\chi_{[-N,N]^{m-1}} h\right] \otimes \chi_{[-N,N]} \right\|_{L^{(q_1,q_2,\dots,q_m)}} \\
&\le C \left\| \left(\chi_{[-N,N]^{m-1}} h\right) \otimes \chi_{[-N,N]} \right\|_{L^{(q_1,q_2,\dots,q_m)}} \\
&\le C(2N)^{\frac{1}{q_m}} \|h\|_{L^{(q_1,q_2,\dots,q_{m-1})}}.
\end{aligned}$$

So, we are led to

$$\left\| \chi_{[-N,N]^{m-1}} M^{(t_1,t_2,\dots,t_{m-1})} [\chi_{[-N,N]^{m-1}} h] \right\|_{L^{(q_1,q_2,\dots,q_{m-1})}} \le C \|h\|_{L^{(q_1,q_2,\dots,q_{m-1})}}.$$

Letting $N \to \infty$, we obtain

$$\left\| M^{(t_1,t_2,\dots,t_{m-1})} h \right\|_{L^{(q_1,q_2,\dots,q_{m-1})}} \le C \|h\|_{L^{(q_1,q_2,\dots,q_{m-1})}}.$$

By the induction assumption, we have $t_k < \min\{q_1, q_2, \dots, q_k\}$ for all $k = 1, 2, \dots, m-1$. If we start from the inequality

$$\begin{aligned}
&\chi_{[-N,N]^m} \left(\chi_{[-N,N]} \otimes M^{(t_1,t_2,\dots,t_{m-1})} \left[\chi_{[-N,N]^{m-1}} h\right] \right) \\
&\le M^{(t_1,t_2,\dots,t_m)} \left[\chi_{[-N,N]} \otimes \left(\chi_{[-N,N]^{m-1}} h\right) \right],
\end{aligned}$$

and argue similarly, we obtain

$$\left\| M^{(t_2,t_3,\dots,t_m)} h \right\|_{L^{(q_2,q_3,\dots,q_m)}} \le C \|h\|_{L^{(q_2,q_3,\dots,q_m)}}.$$

Thus $t_m < \min(q_2, q_3, \dots, q_m)$ by the induction assumption. It remains to show that $t_m < q_1$. To this end, we consider the function of the form:

$$f(x_1, x_2, \dots, x_m) = \sum_{j=-\infty}^{\infty} \chi_{([jN,(j+1)N] \times [-N,N]^{m-1})}(x_1, x_2, \dots, x_m) h_j(x_m),$$

where $h_j \in L^{q_m}(\mathbb{R})$. Then for all $(x_1, x_2, \dots, x_m)$

$$\begin{aligned}
&\chi_{(\mathbb{R} \times [-N,N]^{m-1})}(x_1, x_2, \dots, x_m) M^{(\vec{t})} f(x_1, x_2, \dots, x_m) \\
&\ge \sum_{j=-\infty}^{\infty} \chi_{([jN,(j+1)N] \times [-N,N]^{m-1})}(x_1, x_2, \dots, x_m) M^{(t_m)} \left[\chi_{[-N,N]} h_j\right](x_m).
\end{aligned}$$

We abbreviate

$$H_m(x_m) \equiv M^{(t_m)} \left[\chi_{[-N,N]} h_j\right](x_m).$$

Hence, we obtain

$$\begin{aligned}
&\left\| \sum_{j=-\infty}^{\infty} \chi_{([jN,(j+1)N]\times[-N,N]^{m-1})} M^{(t_m)} \left[\chi_{[-N,N]} h_j\right] \right\|_{L^{\vec{q}}} \\
&= \left\| \left(\sum_{j=-\infty}^{\infty} \int_{\mathbb{R}} \chi_{[jN,(j+1)N]} \left(H_m(\cdot_m)\right)^{q_1} \mathrm{d}x_1 \right)^{\frac{1}{q_1}} \chi_{[-N,N]^{m-1}} \right\|_{L^{(q_2,\dots,q_m)}} \\
&\sim (2N)^{\frac{1}{q_1}+\cdots+\frac{1}{q_{m-1}}} \left\| \left(\sum_{j=-\infty}^{\infty} \left(H_m(\cdot_m)\right)^{q_1} \right)^{\frac{1}{q_1}} \right\|_{L^{q_m}}.
\end{aligned}$$

In the same way, we deduce

$$\begin{aligned}
&\left\| \chi_{(\mathbb{R}\times[-N,N]^{m-1})} M^{(\vec{t})} f \right\|_{L^{\vec{q}}} \\
&\lesssim (2N)^{\frac{1}{q_1}+\cdots+\frac{1}{q_{m-1}}} \left\| \left(\sum_{j=-\infty}^{\infty} \left(|\chi_{[-N,N]} h_j(\cdot_m)| \right)^{q_1} \right)^{\frac{1}{q_1}} \right\|_{L^{q_m}},
\end{aligned}$$

since $M^{(\vec{t})}$ is bounded. Thus, letting $N \to \infty$, we obtain

$$\left\| \{M^{(t_m)} h_j\}_{j=-\infty}^{\infty} \right\|_{L^{q_m}(\ell^{q_1})} \le \left\| \{h_j\}_{j=-\infty}^{\infty} \right\|_{L^{q_m}(\ell^{q_1})}.$$

This forces $q_1 > t_m$. ∎

5.3 Some observations of Hardy-mixed Morrey spaces

In this section, we consider the characterizations of Hardy-Morrey spaces. Concerning $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ and $H\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ when $\vec{q} > 1$, we have the following assertion:

Proposition 5.3.1. *Let $1 < p < \infty$ and $1 < \vec{q} < \infty$ satisfy*

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}.$$

1. *If $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, then $f \in H\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.*
2. *If $f \in H\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, then f can be represented by a locally integrable function and the representative belongs to $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.*

Proposition 5.3.1 was investigated in [66, 72] when $q_j = q$ for all $j = 1, \dots, n$. To prove Proposition 5.3.1, we need the description of the predual spaces of mixed Morrey spaces (see Subsection 2.1.4).

Proof of Proposition 5.3.1.

1. Denote by $B(R) = \{x \in \mathbb{R}^n : |x| < R\}$ for $R > 0$. Since

$$\|f\|_{L^1(B(R))} \le C R^{-\frac{n}{p}+n} \|f\|_{\mathcal{M}^p_{\vec{q}}},$$

   we have $f \in \mathcal{S}'(\mathbb{R}^n)$. As is described in [29], we have a pointwise estimate $|e^{t\Delta} f| \le Mf$, where M denotes the Hardy–Littlewood maximal operator. Since M is shown to be bounded in [19], we have $f \in H\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

2. Let $f \in H\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. Then $\{e^{t\Delta} f\}_{t>0}$ is a bounded set of $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, which admits a separable predual as we have seen in Lemma 2.1.30. Therefore, there exists a sequence $\{t_j\}_{j=1}^\infty$ decreasing to 0 such that $\{e^{t_j\Delta} f\}_{j=1}^\infty$ converges to a function g in the weak-* topology of $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. Meanwhile, it can be shown that $\lim_{t\downarrow 0} e^{t\Delta} f = f$ in the topology of $\mathcal{S}'(\mathbb{R}^n)$ [126]. Since the weak-* topology of $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ is stronger than the topology of $\mathcal{S}'(\mathbb{R}^n)$, it follows that $f = g \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

∎

Furthermore, Hardy-mixed Morrey spaces admit a characterization by using the grand maximal operator. To formulate the result, we recall the following two fundamental notions.

1. Topologize $\mathcal{S}(\mathbb{R}^n)$ by norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) \equiv \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)|$$

   for each $N \in \mathbb{N}$. Define $\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}$.

2. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. The grand maximal operator $\mathcal{M}f$ is given by

$$\mathcal{M}f(x) \equiv \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(x)| : t > 0, \psi \in \mathcal{F}_N\} \quad (x \in \mathbb{R}^n), \tag{5.4}$$

   where we choose and fix a large integer N.

The following proposition can be proved.

Proposition 5.3.2. *Let $0 < \vec{q} < \infty, 0 < p < \infty$, and $\frac{n}{p} \le \sum_{j=1}^n \frac{1}{q_j}$. Then*

$$\|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}} \sim \|f\|_{H\mathcal{M}^p_{\vec{q}}}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

When $p \le 1$ and $q_1 = q_2 = \cdots = q_n$, this proposition represents the result for classical Morrey spaces which is in [72].

Meanwhile the proof of Proposition 5.3.2 is similar to Hardy spaces with variable exponents [21, 102]. We content ourselves with stating two fundamental estimates (5.5) and (5.6).

We define the (discrete) maximal function with respect to $e^{t\Delta}$ by

$$M_{\rm heat}f(x) \equiv \sup_{j \in \mathbb{Z}} |e^{2^j\Delta}f(x)| \quad (x \in \mathbb{R}^n).$$

Suppose that we are given an integer $K \gg 1$. We write

$$M^*_{\rm heat}f(x) \equiv \sup_{j \in \mathbb{Z}} \left(\sup_{y \in \mathbb{R}^n} \frac{|e^{2^j\Delta}f(y)|}{(1+4^j|x-y|^2)^K} \right) \quad (x \in \mathbb{R}^n).$$

The next lemma connects $M^*_{\rm heat}$ with $M_{\rm heat}$ in terms of the usual Hardy–Littlewood maximal operator M.

Lemma 5.3.3 ([102, Lemma 3.2], [125, §4]). *For $0 < \theta < 1$, there exists K_θ so that for all $K \ge K_\theta$, we have*

$$M^*_{\rm heat}f(x) \le CM^{(\theta)}[M_{\rm heat}f](x) = CM\left[\sup_{k \in \mathbb{Z}} |e^{2^k\Delta}f|^\theta\right](x)^{\frac{1}{\theta}} \quad (x \in \mathbb{R}^n) \tag{5.5}$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$, where $M^{(\theta)}$ is the powered maximal operator given by

$$M^{(\theta)}g(x) \equiv M[|g|^\theta](x)^{\frac{1}{\theta}} \quad (x \in \mathbb{R}^n)$$

for measurable functions g.

In the course of the proof of [102, Theorem 3.3], it can be shown that

$$\mathcal{M}f(x) \sim \sup_{\tau \in \mathcal{F}_N,\, j \in \mathbb{Z}} |\tau^j * f(x)| \lesssim M^*_{\rm heat}f(x) \tag{5.6}$$

once we fix an integer $K \gg 1$ and $N \gg 1$.

With the fundamental pointwise estimates (5.5) and (5.6), Proposition 5.3.2 can be proved with ease.

Proof of Proposition 5.3.2. Take $\theta \in (0,1)$ with $\theta < \min\{q_1, \ldots, q_n, p\}$. Then, using the pointwise estimates (5.5) and (5.6) and the boundedness of the Hardy–Littlewood maximal operator, we have

$$\|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|M^*_{\rm heat}f\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|M^{(\theta)}[M_{\rm heat}f]\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|M_{\rm heat}f\|_{\mathcal{M}^p_{\vec{q}}}.$$

Since it is known that $\sup_{t>0} |e^{t\Delta}f(x)| \lesssim \mathcal{M}f(x)$ (see [141, p. 98]), we obtain

$$\|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|M_{\rm heat}f\|_{\mathcal{M}^p_{\vec{q}}} \le \left\| \sup_{t>0} |e^{t\Delta}f| \right\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}}.$$

This is the desired result. ∎

5.4 Proofs of the main theorems (Theorems 5.1.2, 5.1.3, 5.1.7, and 5.1.8)

5.4.1 Reconstruction of mixed Morrey spaces by atoms (Theorem 5.1.2)

Before we show the main theorems, we observe the relation to the maximal operator $M^{(\vec{t}')}$ and blocks.

Example 5.4.1. Suppose that $1 \le t'_k < \min(q'_1, q'_2, \ldots, q'_k) < \infty$. If we let κ be the operator norm of the maximal operator $M^{(\vec{t}')}$ on $L^{\vec{q}'}(\mathbb{R}^n)$, whose finiteness is guaranteed by Theorem 5.1.1, then we obtain $\kappa^{-1}\chi_Q M^{(\vec{t}')}g$ is a $(p', \vec{q}')$-block modulo a multiplicative constant for any $(p', \vec{q}')$-block g. Indeed, it is supported on a cube Q and it satisfies

$$\left\|\kappa^{-1}\chi_Q M^{(\vec{t}')}g\right\|_{L^{\vec{q}'}} \le \|\chi_Q g\|_{L^{\vec{q}'}} = \|g\|_{L^{\vec{q}'}} \le |Q|^{\frac{1}{n}\sum\limits_{j=1}^{n}\frac{1}{q'_j}-\frac{1}{p'}}.$$

Keeping in mind this observation, we turn to the proof.

Proof of Theorem 5.1.2. By decomposing Q_j suitably, we may suppose each Q_j is dyadic.

To prove this theorem, we resort to the duality. For the time being, we assume that there exists $N \in \mathbb{N}$ such that $\lambda_j = 0$ whenever $j \ge N$. Let us assume in addition that a_j are non-negative. Fix a non-negative $(p', \vec{q}')$-block $g \in \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$ with the associated cube Q.

Assume first that each Q_j contains Q as a proper subset. If we group j's such that Q_j are identical, we can assume that Q_j is the j-th dyadic parent of Q for each $j \in \mathbb{N}$. Then by the Hölder inequality [11]

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x = \sum_{j=1}^{\infty}\lambda_j\int_Q a_j(x)g(x)\,\mathrm{d}x \le \sum_{j=1}^{\infty}\lambda_j\|a_j\|_{L^{\vec{q}}(Q)}\|g\|_{L^{\vec{q}'}(Q)}$$

from $f = \sum\limits_{j=1}^{\infty}\lambda_j a_j$. Due to the size condition of a_j and g, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x &\le \sum_{j=1}^{\infty}\lambda_j|Q|^{\frac{1}{n}\sum\limits_{j=1}^{n}\frac{1}{q_j}-\frac{1}{s}}|Q_j|^{\frac{1}{s}}|Q|^{\frac{1}{n}\sum\limits_{j=1}^{n}\frac{1}{q'_j}-\frac{1}{p'}}\\ &\le \sum_{j=1}^{\infty}\lambda_j|Q|^{\frac{1}{p}-\frac{1}{s}}|Q_j|^{\frac{1}{s}}.\end{aligned}$$

Note that

$$\left\|\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right\|_{\mathcal{M}^p_{\vec{q}}} \ge \left\|\lambda_{j_0}\chi_{Q_{j_0}}\right\|_{\mathcal{M}^p_{\vec{q}}} = |Q_{j_0}|^{\frac{1}{p}}\lambda_{j_0}$$

for each j_0. Consequently, it follows from the condition $p < s$ that

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \le \sum_{j=1}^{\infty} |Q|^{\frac{1}{p}-\frac{1}{s}} |Q_j|^{\frac{1}{s}-\frac{1}{p}} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\vec{q}}} \le C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

Conversely assume that Q contains each Q_j. Then by the Hölder inequality

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x = \sum_{j=1}^{\infty} \lambda_j \int_{Q_j} a_j(x)g(x)\,\mathrm{d}x \le \sum_{j=1}^{\infty} \lambda_j \|a_j\|_{L^{\vec{t}}(Q_j)} \|g\|_{L^{\vec{t}'}(Q_j)}.$$

Thanks to the condition of a_j, we obtain

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \le \sum_{j=1}^{\infty} \lambda_j |Q_j|^{\frac{1}{n}\sum\limits_{j=1}^{n}\frac{1}{t_j}-\frac{1}{s}} |Q_j|^{\frac{1}{s}} \|g\|_{L^{\vec{t}'}(Q_j)}.$$

Thus, in terms of the maximal operator $M^{(\vec{t'})}$ defined in Theorem 5.1.1, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x &\le \sum_{j=1}^{\infty} \lambda_j |Q_j| \times \inf_{y\in Q_j} M^{(\vec{t}')}g(y) \\
&\le \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}(y) \right) M^{(\vec{t}')}g(y)\,\mathrm{d}y \\
&\le \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}(y) \right) \chi_Q(y) M^{(\vec{t}')}g(y)\,\mathrm{d}y.
\end{aligned}$$

Hence, by Example 5.4.1, we obtain

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \le \kappa \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

This is the desired result. Finally, we can remove the assumption that $\lambda_j = 0$ for large j. Thus, the proof is complete. ∎

5.4.2 Reconstruction of Hardy-mixed Morrey spaces by atoms (Theorem 5.1.7)

Before we move on the proof of Theorem 5.1.7, we prepare the estimate on the grand maximal operator $\mathcal{M}$. Recall again that the grand maximal operator $\mathcal{M}$ was given by

$$\mathcal{M}f(x) = \sup\{|\varphi_t * f(x)| \,:\, \varphi \in \mathcal{F}_N,\, t > 0\} \quad (x \in \mathbb{R}^n).$$

Then we know that

$$\mathcal{M}a_j(x) \lesssim \chi_{3Q_j}(x) M a_j(x) + (M\chi_{Q_j}(x))^{\frac{n+d_q+1}{n}}, \tag{5.7}$$

where $d_q = \left[n\left(\frac{1}{v(\vec{q})}-1\right)\right]$ and $v(\vec{q}) = \min(1, q_1, \ldots, q_n)$. See [102, (5.2)] for more details. The first term can be controlled by an argument similar to Theorem 5.1.2. The second term can be handled by using the Fefferman–Stein maximal inequality for mixed Morrey spaces (see Proposition 2.1.24).

Let us show Theorem 5.1.7. Using Proposition 5.3.2 and (5.7), we have

$$\begin{aligned}
\|f\|_{H\mathcal{M}^p_{\vec{q}}} &\sim \|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}} \\
&\le \left\|\sum_{j=1}^{\infty} \lambda_j \mathcal{M}a_j\right\|_{\mathcal{M}^p_{\vec{q}}} \\
&\lesssim \left\|\sum_{j=1}^{\infty} \lambda_j \left(\chi_{3Q_j} Ma_j + (M\chi_{Q_j})^{\frac{n+d_q+1}{n}}\right)\right\|_{\mathcal{M}^p_{\vec{q}}} \\
&\lesssim \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{3Q_j} Ma_j\right\|_{\mathcal{M}^p_{\vec{q}}} + \left\|\sum_{j=1}^{\infty} \lambda_j (M\chi_{Q_j})^{\frac{n+d_q+1}{n}}\right\|_{\mathcal{M}^p_{\vec{q}}} \equiv I_1 + I_2.
\end{aligned}$$

First, we consider I_1. The proof is similar to Theorem 5.1.2. For the sake of completeness, we supply the proof. Thanks to decomposing Q_j suitably, we may suppose each Q_j is dyadic. We will use duality again. We assume that there exists $N \in \mathbb{N}$ such that $\lambda_j = 0$ whenever $j \ge N$. Let $r = \frac{p}{v(\vec{q})}$ and $\vec{w} = \frac{\vec{q}}{v(\vec{q})}$, so that $r, \vec{w} > 1$. Then,

$$\begin{aligned}
\left\|\sum_{j=1}^{\infty} \lambda_j \chi_{3Q_j} Ma_j\right\|_{\mathcal{M}^p_{\vec{q}}} &\le \left\|\left(\sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j} Ma_j\right]^{v(\vec{q})}\right)^{\frac{1}{v(\vec{q})}}\right\|_{\mathcal{M}^p_{\vec{q}}} \\
&= \left\|\sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j} Ma_j\right]^{v(\vec{q})}\right\|_{\mathcal{M}^r_{\vec{w}}}^{\frac{1}{v(\vec{q})}}.
\end{aligned}$$

Fix a non-negative $(r', \vec{w}\,')$-block $g \in \mathcal{H}^{r'}_{\vec{w}\,'}(\mathbb{R}^n)$ with the associated cube Q. Assume first that each Q_j contains Q as a proper subset. If we group j's such that Q_j are identical, we can assume that Q_j is the j-th dyadic parent of Q for each $j \in \mathbb{N}$. Then, we calculate

$$\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) Ma_j(x)\right]^{v(\vec{q})} g(x)\mathrm{d}x = \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} \int_Q \left[Ma_j(x)\right]^{v(\vec{q})} g(x)\mathrm{d}x.$$

Using the Hölder inequality twice, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\vec{q})} g(x)\mathrm{d}x \\
&\quad \le \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} \left\| [M a_j]^{v(\vec{q})} \right\|_{L^{\vec{w}}(Q)} \|g\|_{L^{\vec{w}'}(Q)} \\
&\quad \le \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} \|M a_j\|_{L^{\vec{q}}(Q)}^{v(\vec{q})} \|g\|_{L^{\vec{w}'}(Q)} \\
&\quad \le \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} \left[\|M a_j\|_{L^{\vec{t}}(Q)} |Q|^{\frac{1}{n} \sum_{j=1}^{n} \left(\frac{1}{q_j} - \frac{1}{t_j}\right)} \right]^{v(\vec{q})} \|g\|_{L^{\vec{w}'}(Q)}.
\end{aligned}$$

Using the boundedness of the Hardy–Littlewood maximal operator on $\mathcal{M}_{\vec{t}}^{s}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\vec{q})} g(x)\mathrm{d}x \\
&\quad \le \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} \left[\|M a_j\|_{L^{\vec{t}}(Q)} |Q|^{\frac{1}{n} \sum_{j=1}^{n} \left(\frac{1}{q_j} - \frac{1}{t_j}\right)} \right]^{v(\vec{q})} \|g\|_{L^{\vec{w}'}(Q)} \\
&\quad \le \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} |Q|^{\frac{1}{n} \sum_{j=1}^{n} \frac{1}{w_j}} \left[|Q|^{-\frac{1}{s}} \|M a_j\|_{\mathcal{M}_{\vec{t}}^{s}} \right]^{v(\vec{q})} \|g\|_{L^{\vec{w}'}(Q)} \\
&\quad \lesssim \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} |Q|^{\frac{1}{n} \sum_{j=1}^{n} \frac{1}{w_j}} \left[|Q|^{-\frac{1}{s}} \|a_j\|_{\mathcal{M}_{\vec{t}}^{s}} \right]^{v(\vec{q})} \|g\|_{L^{\vec{w}'}(Q)}.
\end{aligned}$$

Thus, using the size condition of a_j and g, we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\vec{q})} g(x)\mathrm{d}x \right)^{\frac{1}{v(\vec{q})}} \\
&\lesssim \left(\sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} |Q|^{\frac{1}{n} \sum_{j=1}^{n} \frac{1}{w_j}} \left[|Q|^{-\frac{1}{s}} |Q_j|^{\frac{1}{s}} \right]^{v(\vec{q})} |Q|^{\frac{1}{n} \sum_{j=1}^{n} \frac{1}{w_j'} - \frac{1}{r'}} \right)^{\frac{1}{v(\vec{q})}} \\
&= |Q|^{\frac{1}{p} - \frac{1}{s}} \left(\sum_{j=1}^{\infty} \left[\lambda_j |Q_j|^{\frac{1}{s}} \right]^{v(\vec{q})} \right)^{\frac{1}{v(\vec{q})}}.
\end{aligned}$$

Note that

$$\left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{v(\vec{q})} \right)^{\frac{1}{v(\vec{q})}} \right\|_{\mathcal{M}_{\vec{q}}^{p}} \ge \left\| \lambda_{j_0} \chi_{Q_{j_0}} \right\|_{\mathcal{M}_{\vec{q}}^{p}} = \lambda_{j_0} |Q_{j_0}|^{\frac{1}{p}}$$

for each $j_0 \in \mathbb{N}$. Thus,

$$\left(\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\vec{q})} g(x) \mathrm{d}x\right)^{\frac{1}{v(\vec{q})}}$$

$$\lesssim \sum_{k=1}^{\infty} |Q|^{\frac{1}{p}-\frac{1}{s}} |Q_k|^{\frac{1}{s}-\frac{1}{p}} \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{v(\vec{q})}\right)^{\frac{1}{v(\vec{q})}} \right\|_{\mathcal{M}^p_{\vec{q}}}$$

$$\sim \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{v(\vec{q})}\right)^{\frac{1}{v(\vec{q})}} \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

Conversely assume that Q contains each Q_j. Then by the Hölder inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\vec{q})} g(x) \mathrm{d}x \\
&\quad = \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} \int_{3Q_j} [M a_j(x)]^{v(\vec{q})} g(x) \mathrm{d}x \\
&\quad \leq \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} \left\| [M a_j]^{v(\vec{q})} \right\|_{L^{\vec{\tau}}(3Q_j)} \|g\|_{L^{\vec{\tau}\,'}(3Q_j)} \qquad \left(\vec{\tau} = \frac{\vec{t}}{v(\vec{q})}\right) \\
&\quad \leq \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} \|M a_j\|_{L^{\vec{t}}(3Q_j)}^{v(\vec{q})} \|g\|_{L^{\vec{\tau}\,'}(3Q_j)}.
\end{aligned}$$

Additionally, by virtue of the boundedness of the Hardy–Littlewood maximal operator on $L^{\vec{t}}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\vec{q})} g(x) \mathrm{d}x \lesssim \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} \|a_j\|_{L^{\vec{t}}}^{v(\vec{q})} \|g\|_{L^{\vec{\tau}\,'}(3Q_j)}.$$

Considering the condition of a_j, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\vec{q})} g(x) \mathrm{d}x \\
&\lesssim \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} \left[|Q_j|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{t_j}-\frac{1}{s}} \|a_j\|_{\mathcal{M}^s_{\vec{t}}} \right]^{v(\vec{q})} \|g\|_{L^{\vec{\tau}\,'}(3Q_j)} \\
&\leq \sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} \left[|Q_j|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{t_j}} \right]^{v(\vec{q})} \|g\|_{L^{\vec{\tau}\,'}(3Q_j)}.
\end{aligned}$$

Thus, in terms of the maximal operator $M^{(\vec{t}')}$ defined in Theorem 5.1.1, we obtain

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\vec{q})} g(x) \mathrm{d}x\right)^{\frac{1}{v(\vec{q})}} \\
&\le \left(\sum_{j=1}^{\infty} \lambda_j^{v(\vec{q})} |Q_j| \times \inf_{y \in Q_j} M^{(\vec{\tau}')} g(y)\right)^{\frac{1}{v(\vec{q})}} \\
&\le \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j}(y)\right)^{v(\vec{q})}\right) M^{(\vec{\tau}')} g(y) \, \mathrm{d}y\right)^{\frac{1}{v(\vec{q})}} \\
&\le \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \left(\lambda_j \chi_{Q_j}(y)\right)^{v(\vec{q})}\right) \chi_Q(y) M^{(\vec{\tau}')} g(y) \, \mathrm{d}y\right)^{\frac{1}{v(\vec{q})}}.
\end{aligned}
$$

As in Example 5.4.1, $\kappa^{-1} \chi_Q M^{(\vec{\tau}')} g$ is a $(r', \vec{w}\,')$-block as long as κ is the operator norm of $M^{(\vec{\tau}')}$ on $L^{\vec{q}\,'}(\mathbb{R}^n)$. Hence, we obtain

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \left[\lambda_j \chi_{3Q_j}(x) M a_j(x)\right]^{v(\vec{q})} g(x) \mathrm{d}x\right)^{\frac{1}{v(\vec{q})}} \\
&\lesssim \kappa^{\frac{1}{v(\vec{q})}} \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{v(\vec{q})}\right) \right\|_{\mathcal{M}^r_{\vec{w}}}^{\frac{1}{v(\vec{q})}} = \kappa^{\frac{1}{v(\vec{q})}} \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{v(\vec{q})}\right)^{\frac{1}{v(\vec{q})}} \right\|_{\mathcal{M}^p_{\vec{q}}}.
\end{aligned}
$$

Next, we consider I_2. Put

$$
u = \frac{n + d_q + 1}{n} p, \quad \vec{v} = \frac{n + d_q + 1}{n} \vec{q}.
$$

Then, by Proposition 2.1.24 and the embedding $\ell^{v(\vec{q})} \hookrightarrow \ell^1$, we have

$$
\begin{aligned}
I_2 = \left\| \left[\sum_{j=1}^{\infty} \lambda_j (M \chi_{Q_j})^{\frac{n+d_q+1}{n}}\right]^{\frac{n}{n+d_q+1}} \right\|_{\mathcal{M}^u_{\vec{v}}}^{\frac{n+d_q+1}{n}} &\lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\vec{q}}} \\
&\le \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{v(\vec{q})}\right)^{\frac{1}{v(\vec{q})}} \right\|_{\mathcal{M}^p_{\vec{q}}}.
\end{aligned}
$$

Thus, we obtain the desired result.

5.4.3 Atomic decomposition for mixed Morrey spaces and Hardy-mixed Morrey spaces (Theorems 5.1.3 and 5.1.8)

Theorem 5.1.3 is included in Theorem 5.1.8 so that we concentrate in the proof of Theorem 5.1.8.

We invoke the following lemma. We refer to [66, Lemma 3.2] and [141, p.101–105].

Lemma 5.4.2. *Let $f \in \mathcal{S}'(\mathbb{R}^n), K \in \{0,1,2,\ldots\}$ and $j \in \mathbb{Z}$. Then, there are collections of cubes $\{Q^*_{j,k}\}_{k \in K_j}$ and functions $\{\eta_{j,k}\}_{k \in K_j} \subset C^\infty_c(\mathbb{R}^n)$, which are all indexed be a set K_j for every j, and a decomposition*

$$f = g_j + b_j, \quad b_j = \sum_{k \in K_j} b_{j,k},$$

such that

(i) Define $\mathcal{O}_j \equiv \{y \in \mathbb{R}^n : \mathcal{M}f(y) > 2^j\}$ and consider its Whitney decomposition. Then, the cubes $\{200Q^*_{j,k}\}_{k \in K_j}$ have the bounded intersection property, and

$$\mathcal{O}_j = \bigcup_{k \in K_j} Q^*_{j,k} = \bigcup_{k \in K_j} 200Q^*_{j,k}. \tag{5.8}$$

(ii) Consider the partition of unity with respect to the collections of cubes $\{Q^*_{j,k}\}_{k \in K_j}$. Denote it by $\{\eta_{j,k}\}_{k \in K_j}$. Then each function $\eta_{j,k}$ is supported in $Q^*_{j,k}$ and

$$\sum_{k \in K_j} \eta_{j,k} = \chi_{\{y \in \mathbb{R}^n : \mathcal{M}f(y) > 2^j\}}, \quad 0 \le \eta_{j,k} \le 1.$$

(iii) The distribution g_j satisfies the inequality:

$$\mathcal{M}g_j(x) \le C \left(\mathcal{M}f(x)\chi_{\mathcal{O}_j^c}(x) + 2^j \sum_{k \in K_j} \frac{\ell_{j,k}^{n+K+1}}{(\ell_{j,k} + |x - x_{j,k}|)^{n+K+1}} \right)$$

for all $x \in \mathbb{R}^n$.

(iv) Each distribution $b_{j,k}$ is given by $b_{j,k} = (f - c_{j,k})\eta_{j,k}$ with a certain polynomial $c_{j,k} \in \mathcal{P}_K(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} b_{j,k}(x) q(x) \mathrm{d}x = 0$$

for all $q \in \mathcal{P}_K(\mathbb{R}^n)$, and

$$\mathcal{M}b_{j,k}(x) \le C \left(\mathcal{M}f(x)\chi_{Q^*_{j,k}}(x) + 2^j \cdot \frac{\ell_{j,k}^{n+K+1}}{|x - x_{j,k}|^{n+K+1}} \chi_{\mathbb{R}^n \setminus Q^*_{j,k}} \right) \tag{5.9}$$

for all $x \in \mathbb{R}^n$.

In the above $x_{j,k}$ and $\ell_{j,k}$ denote the center and the side-length of $Q^*_{j,k}$, respectively, and the implicit constants are dependent only on n.

Observe that (5.8) together with the bounded overlapping property, that is, every point is contained in at most a fixed number (we denote it by N) of the $\{Q^*_{j,k}\}$, yields

$$\chi_{\mathcal{O}_j}(x) \le \sum_{k \in K_j} \chi_{Q^*_{j,k}}(x) \le \sum_{k \in K_j} \chi_{200Q^*_{j,k}}(x) \le N\chi_{\mathcal{O}_j}(x) \quad (x \in \mathbb{R}^n). \tag{5.10}$$

For the proof of Theorem 5.1.8, we need the following embedding.

Lemma 5.4.3. *Let $0 < \vec{q} < \infty, 0 < p < \infty, \dfrac{n}{p} \le \sum_{j=1}^{n} \dfrac{1}{q_j}$ and $q_0 = \min(q_1, \ldots, q_n)$. Choose τ to be $1 - \dfrac{q_0}{p} < \tau < 1$. Then*

$$\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_0}(\mathbb{R}^n) \hookrightarrow L^{q_0}(M\chi_{B(1)}{}^{\tau})(\mathbb{R}^n),$$

where $L^q(w)$ denotes the weighted Lebesgue space with respect to the measure $w \cdot \mathrm{d}x$ for a non-negative measurable function w.

Proof. The first embedding follows from Proposition 2.1.17. We shall show the second embedding. Let $f \in \mathcal{M}^p_{q_0}(\mathbb{R}^n)$. Then, we have

$$\begin{aligned}
\|f\|_{L^{q_0}(M\chi_{B(1)}{}^{\tau})} &\le \|f\|_{L^{q_0}\left(M\chi_{B(1)}{}^{\tau}\cdot\chi_{B(1)}\right)} + \sum_{k=1}^{\infty} \|f\|_{L^{q_0}\left(M\chi_{B(1)}{}^{\tau}\cdot\chi_{B(2^k)\setminus B(2^{k-1})}\right)} \\
&\lesssim \sum_{k=0}^{\infty} 2^{-\frac{nk\tau}{q_0}} \|f\|_{L^{q_0}(B(2^k))} \\
&\lesssim \sum_{k=0}^{\infty} 2^{-\frac{nk\tau}{q_0}+\frac{nk}{q_0}-\frac{nk}{p}} \|f\|_{\mathcal{M}^p_{q_0}}.
\end{aligned}$$

Since $1 - \dfrac{q_0}{p} < \tau < 1$, $-\dfrac{n\tau}{q_0} + \dfrac{n}{q_0} - \dfrac{n}{p} < 0$. Thus, $f \in L^{q_0}(M\chi_{B(1)}{}^{\tau})(\mathbb{R}^n)$. ∎

Recall that a non-negative measurable function w is A_1-weight if w satisfies

$$Mw(x) \le Cw(x) \quad (x \in \mathbb{R}^n).$$

Lemma 5.4.4. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Keep to the same notation as Lemmas 5.4.2 and 5.4.3. Then we have*

$$|\langle b_j, \varphi\rangle| \le C\|\chi_{\mathcal{O}_j}\mathcal{M}f\|_{L^{q_0}\left(M\chi_{B(1)}{}^{\tau}\right)} \tag{5.11}$$

and

$$|\langle g_j, \varphi\rangle| \le C\left\|\min(\mathcal{M}f, 2^j)\right\|_{L^{q_0}\left(M\chi_{B(1)}{}^{\tau}\right)}, \tag{5.12}$$

where the constants in (5.11) and (5.12) depend on φ but not on j or k.

Proof. By the subadditivity of $\mathcal{M}$ given by (5.4), we have

$$|\langle b_j, \varphi\rangle| \le C \inf_{x\in B(1)} \mathcal{M}b_j(x) \le C \inf_{x\in B(1)} \sum_{k\in K_j} \mathcal{M}b_{j,k}(x).$$

Observe also that

$$CM\chi_Q(x) \ge \frac{|Q|}{|Q|+|x-x_Q|^n},$$

if Q is a cube centered at x_Q. Take a and θ satisfying $q_0 > a$ and $(\theta-\tau)\left(\frac{q_0}{a}\right)' > 1$. It follows from (5.9) and Hölder's inequality that

$$\begin{aligned}
&\left(\inf_{x\in B(1)} \sum_{k\in K_j} \mathcal{M}b_{j,k}(x)\right)^a \\
&\lesssim \int_{\mathbb{R}^n} \left(\sum_{k\in K_j} \mathcal{M}b_{j,k}(x)\right)^a \chi_{B(1)}(x)\mathrm{d}x \\
&\lesssim \int_{\mathbb{R}^n} \left(\mathcal{M}f(x)\chi_{\mathcal{O}_j}(x) + 2^j \sum_{k\in K_j} M\chi_{Q^*_{j,k}}(x)^{\frac{n+K+1}{n}}\right)^a M\chi_{B(1)}(x)^\theta \mathrm{d}x \\
&\lesssim \left\| \left((\mathcal{M}f)\chi_{\mathcal{O}_j} + 2^j \sum_{k\in K_j} M\chi_{Q^*_{j,k}}{}^{\frac{n+K+1}{n}}\right)^a \right\|_{L^{\frac{q_0}{a}}(M\chi_{B(1)}{}^\tau)} \left\| M\chi_{B(1)}{}^{\theta-\tau}\right\|_{L^{(\frac{q_0}{a})'}}.
\end{aligned}$$

Using the triangle inequality and the Fefferman–Stein weighted vector-valued inequality for A_1-weights (see [6]), we obtain

$$\begin{aligned}
&\left(\inf_{x\in B(1)} \sum_{k\in K_j} \mathcal{M}b_{j,k}(x)\right)^a \\
&\lesssim \left\|\left((\mathcal{M}f)\chi_{\mathcal{O}_j}\right)^a\right\|_{L^{\frac{q_0}{a}}(M\chi_{B(1)}{}^\tau)} + \left\| \left(2^j \sum_{k\in K_j} M\chi_{Q^*_{j,k}}{}^{\frac{n+K+1}{n}}\right)^a \right\|_{L^{\frac{q_0}{a}}(M\chi_{B(1)}{}^\tau)} \\
&\lesssim \left\|\left((\mathcal{M}f)\chi_{\mathcal{O}_j}\right)^a\right\|_{L^{\frac{q_0}{a}}(M\chi_{B(1)}{}^\tau)} + \left\| \left(2^j \sum_{k\in K_j} \chi_{Q^*_{j,k}}\right)^a \right\|_{L^{\frac{q_0}{a}}(M\chi_{B(1)}{}^\tau)} \\
&\lesssim \left\|(\mathcal{M}f)\chi_{\mathcal{O}_j}\right\|^a_{L^{q_0}(M\chi_{B(1)}{}^\tau)}.
\end{aligned}$$

Thus, (5.11) is proved.

In the same way we can prove (5.12). In fact, using the Fefferman–Stein inequality for A_1-weighted Lebesgue spaces, we obtain

$$\begin{aligned}
&\left(\inf_{x\in B(1)} \mathcal{M}g_j(x)\right)^a \\
&\lesssim \left\| \left((\mathcal{M}f)\,\chi_{\mathcal{O}_j^c} + 2^j \sum_{k\in K_j} \frac{\ell_{j,k}^{n+K+1}}{(\ell_{j,k}+|\cdot - x_{j,k}|)^{n+d+1}} \right)^a \right\|_{L^{\frac{q_0}{a}}(M\chi_{B(1)}{}^{\tau})} \\
&\lesssim \left\| \left((\mathcal{M}f)\chi_{\mathcal{O}_j^c} \right)^a \right\|_{L^{\frac{q_0}{a}}(M\chi_{B(1)}{}^{\tau})} + \left\| \left(2^j \sum_{k\in K_j} M\chi_{Q_{j,k}^*}{}^{\frac{n+K+1}{n}} \right)^a \right\|_{L^{\frac{q_0}{a}}(M\chi_{B(1)}{}^{\tau})} \\
&\lesssim \left\| \left((\mathcal{M}f)\chi_{\mathcal{O}_j^c} \right)^a \right\|_{L^{\frac{q_0}{a}}(M\chi_{B(1)}{}^{\tau})} + \left\| (2^j \chi_{\mathcal{O}_j})^a \right\|_{L^{\frac{q_0}{a}}(M\chi_{B(1)}{}^{\tau})} \\
&\lesssim \left\| (\mathcal{M}f)\chi_{\mathcal{O}_j^c} + 2^j\chi_{\mathcal{O}_j} \right\|^a_{L^{q_0}(M\chi_{B(1)}{}^{\tau})} \\
&= \left\| \min(\mathcal{M}f, 2^j) \right\|^a_{L^{q_0}\left(M\chi_{B(1)}{}^{\tau}\right)}.
\end{aligned}$$

Thus, (5.12) is proven. ■

Lemma 5.4.5. *In the notation of Lemmas 5.4.2 and 5.4.3, in the topology of* $\mathcal{S}'(\mathbb{R}^n)$*, we have* $g_j \to 0$ *as* $j \to -\infty$ *and* $b_j \to 0$ *as* $j \to \infty$*. In particular,*

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j)$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$*.*

Proof. By Lemma 5.4.3, since

$$\|\mathcal{M}f\|_{L^{q_0}(M\chi_{B(1)}{}^{\tau})} \lesssim \|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}} \sim \|f\|_{H\mathcal{M}^p_{\vec{q}}} < \infty$$

for $f \in H\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, we have $\mathcal{M}f \in L^{q_0}(M\chi_{B(1)}{}^{\tau})$.

Let us show that $b_j \to 0$ as $j \to \infty$. Once this is proved, then we have $f = \lim_{j\to\infty} g_j$ in $\mathcal{S}'(\mathbb{R}^n)$. Let us choose a test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then by (5.11) we have

$$|\langle b_j, \varphi\rangle| \lesssim \|\chi_{\mathcal{O}_j}\mathcal{M}f\|_{L^{q_0}\left(M\chi_{B(1)}{}^{\tau}\right))}.$$

Hence it follows that $b_j \to 0$ as $j \to \infty$. Likewise by (5.12),

$$|\langle g_j, \varphi\rangle| \lesssim \|\min(\mathcal{M}f, 2^j)\|_{L^{q_0}\left(M\chi_{B(1)}{}^{\tau}\right))}.$$

Thus, $g_j \to 0$ as $j \to -\infty$. Consequently, we have $f = \lim_{j\to\infty}(g_j - g_{-j}) = \sum_{j=-\infty}^{\infty}(g_{j+1} - g_j)$ in $\mathcal{S}'(\mathbb{R}^n)$. ■

To prove Theorem 5.1.8, we first assume $f \in L^1_{\rm loc}(\mathbb{R}^n)$. After that, we show this theorem for all $f \in H\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

Lemma 5.4.6. *Theorem 5.1.8 holds for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. Assume that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For each $j \in \mathbb{Z}$, consider the level set $\mathcal{O}_j$ as in Lemma 5.4.2. Then it follows immediately from the definition that

$$\mathcal{O}_{j+1} \subset \mathcal{O}_j.$$

If we invoke Lemma 5.4.2, then f can be decomposed;

$$f = g_j + b_j, \quad b_j = \sum_k b_{j,k}, \quad b_{j,k} = (f - c_{j,k})\eta_{j,k}$$

where each $b_{j,k}$ is supported in a cube $Q^*_{j,k}$ as is described in Lemma 5.4.2.

We know that

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j),$$

in $\mathcal{S}'(\mathbb{R}^n)$. Here, going through the same argument as the one in [141, p.108–109], we have an expression;

$$f = \sum_{j,k} A_{j,k}, \quad g_{j+1} - g_j = \sum_k A_{j,k} \quad (j \in \mathbb{Z})$$

in the sense of distributions, where each $A_{j,k}$ satisfies the pointwise estimate $|A_{j,k}(x)| \le C_0 2^j \chi_{Q^*_{j,k}}(x)$, and belongs to $\mathcal{P}_K(\mathbb{R}^n)^\perp$. With these observations in mind, let us set

$$a_{j,k} \equiv \frac{A_{j,k}}{C_0 2^j}, \quad \kappa_{j,k} \equiv C_0 2^j.$$

Then we automatically obtain that each $a_{j,k}$ belongs to $\mathcal{P}_K(\mathbb{R}^n)^\perp$ and satisfies that

$$|a_{j,k}| \le \chi_{Q^*_{j,k}},$$

and that $f = \sum_{j,k} \kappa_{j,k} a_{j,k}$ in the topology of $\mathcal{S}'(\mathbb{R}^n)$, once we prove the estimate of coefficients. Put $\Lambda = \{(j,k) : j \in \mathbb{Z}, k \in K_j\}$ and fix a bijection $\mu : \Lambda \to \mathbb{N}$. Then, we put

$$Q_{\mu(j,k)} = Q^*_{j,k}, \quad \lambda_{\mu(j,k)} = \lambda^*_{j,k}, \quad a_{\mu(j,k)} = a^*_{j,k}.$$

To establish (5.2) we need to estimate

$$\alpha \equiv \left\| \left(\sum_{(j,k)\in\Lambda}^{\infty} |\lambda_{\mu(j,k)} \chi_{Q_{\mu(j,k)}}|^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

Since $\{(\kappa_{j,k}, Q^*_{j,k})\}_{j,k} = \{(\lambda_{\mu(j,k)}, Q_{\mu(j,k)})\}_{(j,k)\in\Lambda}$, we have

$$\alpha = \left\| \left(\sum_{j=-\infty}^{\infty} \sum_{k\in K_j} |\kappa_{j,k} \chi_{Q^*_{j,k}}|^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

If we insert the definition of $\kappa_{j,k}$ into the definition of α, then we have

$$\alpha = C_0 \left\| \left(\sum_{j=-\infty}^{\infty} \sum_{k \in K_j} |2^j \chi_{Q^*_{j,k}}|^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{\vec{q}}} = C_0 \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jv} \sum_{k \in K_j} \chi_{Q^*_{j,k}} \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

Thus, by (5.10), we obtain

$$\alpha \le C \left\| \left(\sum_{j=-\infty}^{\infty} \left(2^j \chi_{\mathcal{O}_j}\right)^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

Since $\mathcal{O}_j \supset \mathcal{O}_{j+1}$ for each $j \in \mathbb{Z}$, for $x \in \mathbb{R}^n$ we have

$$\sum_{j=-\infty}^{\infty} \left(2^j \chi_{\mathcal{O}_j}(x)\right)^v \le \sum_{j=-\infty}^{[\log_2 \mathcal{M}f(x)]+1} \left(2^j \chi_{\mathcal{O}_j}(x)\right)^v \sim 2^{[\log_2 \mathcal{M}f(x)]+1} \sim \mathcal{M}f(x).$$

Hence, we conclude $\alpha \le C\|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}} \sim \|f\|_{H\mathcal{M}^p_{\vec{q}}}$. This is the desired result. ■

Note that by Lemma 5.4.6, for $f \in H\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$, there exists a decomposition:

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where $a_j \in \mathcal{P}_K^{\perp}(\mathbb{R}^n)$, $\lambda_j \ge 0$ and

$$|a_j| \le \chi_{Q_j}, \quad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}}.$$

By replacing each Q_j by a dyadic cube suitably, we have a decomposition:

$$f = \sum_{Q' \in \mathcal{D}} \lambda_{Q'} a_{Q'},$$

where $a_{Q'} \in \mathcal{P}_K^{\perp}(\mathbb{R}^n)$, $\lambda_{Q'} \ge 0$ and

$$|a_{Q'}| \le \chi_{3Q'}, \quad \left\| \sum_{Q' \in \mathcal{D}}^{\infty} \lambda_{Q'} \chi_{3Q'} \right\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}}.$$

Here, $\mathcal{D}$ denotes the set of all dyadic cubes in $\mathbb{R}^n$.

Let us prove Theorem 5.1.8.

Proof of Theorem 5.1.8. Let $f \in H\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. Then we consider the decomposition:

$$e^{t\Delta} f = \sum_{Q \in \mathcal{D}} \lambda^t_Q a^t_Q$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$, where $a^t_Q \in \mathcal{P}^{\perp}_K(\mathbb{R}^n)$, $\lambda^t_Q \geq 0$ and

$$|a^t_Q| \leq \chi_{3Q}, \quad \left\| \sum_{Q \in \mathcal{D}} \lambda^t_Q \chi_{3Q} \right\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|\mathcal{M}[e^{t\Delta} f]\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|\mathcal{M} f\|_{\mathcal{M}^p_{\vec{q}}}.$$

Due to the weak-* compactness of the unit ball of $L^\infty(\mathbb{R}^n)$, there exists a sequence $\{t_l\}_{l=1}^\infty$ that converges to 0 such that

$$\lambda_Q = \lim_{l \to \infty} \lambda^{t_l}_Q, \quad a_Q = \lim_{l \to \infty} a^{t_l}_Q$$

exist for all $Q \in \mathcal{D}$ in the sense that

$$\lim_{l \to \infty} \int_{\mathbb{R}^n} a^{t_l}_Q(x) \varphi(x) \mathrm{d}x = \int_{\mathbb{R}^n} a_Q(x) \varphi(x) \mathrm{d}x$$

for all $\varphi \in L^1(\mathbb{R}^n)$. We claim

$$f = \sum_{Q \in \mathcal{D}} \lambda_Q a_Q$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a test function. Then we have

$$\langle f, \varphi \rangle = \lim_{l \to \infty} \langle e^{t_l \Delta} f, \varphi \rangle = \lim_{l \to \infty} \sum_{Q \in \mathcal{D}} \lambda^{t_l}_Q \int_{\mathbb{R}^n} a^{t_l}_Q(x) \varphi(x) \mathrm{d}x$$

from the definition of the convergence in $\mathcal{S}'(\mathbb{R}^n)$. Once we fix m, we have

$$|\lambda^{t_l}_Q| \lesssim \frac{\|\mathcal{M} f\|_{\mathcal{M}^p_{\vec{q}}}}{\|\chi_{[0,2^{-m})^n}\|_{\mathcal{M}^p_{\vec{q}}}} \tag{5.13}$$

and

$$\left| \int_{\mathbb{R}^n} a^{t_l}_Q(x) \varphi(x) \mathrm{d}x \right| \leq \int_{3Q} |\varphi(x)| \mathrm{d}x.$$

Since

$$\sum_{Q \in \mathcal{D}_m} \frac{\|\mathcal{M} f\|_{\mathcal{M}^p_{\vec{q}}}}{\|\chi_{[0,2^{-m})^n}\|_{\mathcal{M}^p_{\vec{q}}}} \int_{3Q} |\varphi(x)| \mathrm{d}x = 3^n \frac{\|\mathcal{M} f\|_{\mathcal{M}^p_{\vec{q}}}}{\|\chi_{[0,2^{-m})^n}\|_{\mathcal{M}^p_{\vec{q}}}} \|\varphi\|_{L^1} < \infty,$$

we are in the position of using the Fubini theorem to have

$$\sum_{m \in \mathbb{Z}} \int_{\mathbb{R}^n} \left(\sum_{Q \in \mathcal{D}_m} \lambda^{t_l}_Q a^{t_l}_Q(x) \right) \varphi(x) \mathrm{d}x = \sum_{m \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_m} \lambda^{t_l}_Q \int_{\mathbb{R}^n} a^{t_l}_Q(x) \varphi(x) \mathrm{d}x.$$

With this in mind, let us set

$$a_{m,l} \equiv \sum_{Q \in \mathcal{D}_m} \lambda_Q^{t_l} \int_{\mathbb{R}^n} a_Q^{t_l}(x)\varphi(x)\mathrm{d}x$$

for each $m \in \mathbb{Z}$ and $l \in \mathbb{N}$. Then we have

$$|a_{m,l}| \le C2^{\frac{nm}{p}} \|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}} \|\varphi\|_1 \quad (m \in \mathbb{Z}) \tag{5.14}$$

thanks to (5.13).

Let $m \in \mathbb{Z}$. Then we have

$$\begin{aligned} a_{m,l} &= \sum_{Q \in \mathcal{D}_m} \lambda_Q^{t_l} \int_{3Q} a_Q^{t_l}(x)\varphi(x)\mathrm{d}x \\ &= \sum_{Q \in \mathcal{D}_m} \lambda_Q^{t_l} \int_{3Q} a_Q^{t_l}(x) \left(\varphi(x) - \sum_{|\beta| \le K} \frac{1}{\beta!} \partial^\beta \varphi(c(Q))(x - c(Q))^\beta \right) \mathrm{d}x \end{aligned}$$

since $a_Q^{t_l} \in \mathcal{P}_K^\perp(\mathbb{R}^n)$. Thus, by the mean-value theorem, we have

$$|a_{m,l}| \le C(\varphi) \sum_{Q \in \mathcal{D}_m} |\lambda_Q^{t_l}| \ell(Q)^{n+K+1} \sup_{y \in 3Q} \frac{1}{1 + |y|^{n+1}}. \tag{5.15}$$

Here $C(\varphi)$ is a constant depending on φ.

Meanwhile, for each $\tilde{m} \in \mathbb{Z}^n$, we have

$$\left\| \sum_{Q \in \mathcal{D}_m, |c(Q) - \tilde{m}| \le n} \lambda_Q^{t_l} \chi_Q \right\|_{\mathcal{M}^p_{q_0}} \lesssim \|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}},$$

which implies

$$\left\| \sum_{Q \in \mathcal{D}_m, |c(Q) - \tilde{m}| \le n} \lambda_Q^{t_l} \chi_Q \right\|_{L^{q_0}} \lesssim \|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}}$$

or equivalently

$$\left(\sum_{Q \in \mathcal{D}_m, |c(Q) - \tilde{m}| \le n} 2^{-mn} |\lambda_Q^{t_l}|^{q_0} \right)^{\frac{1}{q_0}} \lesssim \|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}}.$$

Since $\ell^{q_0}(\mathbb{Z}^n) \hookrightarrow \ell^1(\mathbb{Z}^n)$,

$$\sum_{Q \in \mathcal{D}_m, |c(Q) - \tilde{m}| \le n} |\lambda_Q^{t_l}| \lesssim 2^{\frac{mn}{q_0}} \|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}}.$$

Combining this estimate with (5.15), we obtain

$$\begin{aligned}|a_{m,l}| &\lesssim \sum_{\tilde{m}\in\mathbb{Z}^n}\sum_{Q\in\mathcal{D}_m,|c(Q)-\tilde{m}|\le n}|\lambda_Q^{t_l}|\ell(Q)^{n+K+1}\sup_{y\in 3Q}\frac{1}{1+|y|^{n+1}}\\ &\sim \sum_{\tilde{m}\in\mathbb{Z}^n}\sum_{Q\in\mathcal{D}_m,|c(Q)-\tilde{m}|\le n}\frac{|\lambda_Q^{t_l}|\ell(Q)^{n+K+1}}{1+|\tilde{m}|^{n+1}}\\ &\lesssim 2^{\frac{mn}{q_0}-(n+K+1)m}\|\mathcal{M}f\|_{\mathcal{M}^p_{\vec{q}}}. \qquad (5.16)\end{aligned}$$

Since $K+1>n\left(\dfrac{1}{q_0}-1\right)$, we obtain

$$n+K+1>\frac{n}{q_0}.$$

Thus by (5.14) and (5.16), we obtain

$$|a_{m,l}|\lesssim \min(2^{\frac{mn}{q_0}-(n+K+1)m},2^{\frac{mn}{p}}).$$

Since

$$\sum_{m=-\infty}^{\infty}\min(2^{\frac{mn}{q_0}-(n+K+1)m},2^{\frac{mn}{p}})\lesssim 1,$$

we are in the position of using the Lebesgue convergence theorem to have

$$\lim_{l\to\infty}\sum_{m=-\infty}^{\infty}a_{m,l}=\sum_{m=-\infty}^{\infty}\left(\lim_{l\to\infty}a_{m,l}\right).$$

That is,

$$\langle f,\varphi\rangle=\lim_{l\to\infty}\langle e^{t_l\Delta}f,\varphi\rangle=\sum_{m=-\infty}^{\infty}\left(\lim_{l\to\infty}\sum_{Q\in\mathcal{D}_m}\lambda_Q^{t_l}\int_{\mathbb{R}^n}a_Q^{t_l}(x)\varphi(x)\mathrm{d}x\right).$$

Hence, using Fubini's theorem again, we obtain

$$\begin{aligned}\langle f,\varphi\rangle &= \sum_{m=-\infty}^{\infty}\left(\lim_{l\to\infty}\int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{D}_m}\lambda_Q^{t_l}a_Q^{t_l}(x)\right)\varphi(x)\mathrm{d}x\right)\\ &= \sum_{m=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_m}\lim_{l\to\infty}\left(\int_{\mathbb{R}^n}\lambda_Q^{t_l}a_Q^{t_l}(x)\varphi(x)\mathrm{d}x\right)\\ &= \sum_{m=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_m}\int_{\mathbb{R}^n}\lambda_Q a_Q(x)\varphi(x)\mathrm{d}x=\left\langle\sum_{Q\in\mathcal{D}}\lambda_Q a_Q,\varphi\right\rangle.\end{aligned}$$

Consequently, we obtain the desired result. ∎

5.5 Application : Olsen's inequality on mixed Morrey spaces

As an application of Theorem 5.1.2, we can prove the following Olsen inequality about the fractional integral operator I_α. Based upon Proposition 2.1.22, we can prove the following result.

Theorem 5.5.1. *Suppose that the parameters $\alpha, p, \vec{q}, p^*, \vec{q}^*, s, \vec{t}$ satisfy*

$$1 < p, p^*, s < \infty, \quad 1 < \vec{q}, \vec{q}^*, \vec{t} < \infty,$$

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n}{p^*} \le \sum_{j=1}^{n} \frac{1}{q_j^*}, \quad \frac{n}{s} \le \sum_{j=1}^{n} \frac{1}{t_j},$$

$$\max\{t_1, \dots, t_j\} < q_j^*, \quad \frac{1}{p} > \frac{\alpha}{n}, \quad \frac{1}{p^*} \le \frac{\alpha}{n},$$

for each $j = 1, 2, \dots, n$, and that

$$\frac{1}{s} = \frac{1}{p^*} + \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t_j}{s} = \frac{q_j}{p} \quad (j = 1, 2, \dots, n).$$

Then for all $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ and $g \in \mathcal{M}^{p^*}_{\vec{q}^*}(\mathbb{R}^n)$

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^s_{\vec{t}}} \le C \|g\|_{\mathcal{M}^{p^*}_{\vec{q}^*}} \cdot \|f\|_{\mathcal{M}^p_{\vec{q}}},$$

where the constant C is independent of f and g.

This result recaptures [134, Proposition 1.8]. Note that a detailed calculation in [133, p.6] shows that Theorem 5.5.1 is not just a combination of Proposition 2.1.22 and Lemma 5.5.2.

Lemma 5.5.2. *Suppose that the parameters $p, \vec{q}, p^*, \vec{q}^*, s, \vec{t}$ satisfy*

$$1 < p, p^*, s < \infty, \quad 1 < \vec{q}, \vec{q}^*, \vec{t} < \infty,$$

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n}{p^*} \le \sum_{j=1}^{n} \frac{1}{q_j^*}, \quad \frac{n}{s} \le \sum_{j=1}^{n} \frac{1}{t_j}.$$

Assume

$$\frac{1}{s} = \frac{1}{p^*} + \frac{1}{p}, \quad \frac{1}{t_j} = \frac{1}{q_j^*} + \frac{1}{q_j}.$$

Then

$$\|f \cdot g\|_{\mathcal{M}^s_{\vec{t}}} \le \|f\|_{\mathcal{M}^p_{\vec{q}}} \|g\|_{\mathcal{M}^{p^*}_{\vec{q}^*}} \quad (f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n), g \in \mathcal{M}^{p^*}_{\vec{q}^*}(\mathbb{R}^n)).$$

We can prove this lemma easily by using Hölder's inequality. So we omit the proof. We write $\infty' = 1$ and $s' = \dfrac{s}{s-1}$ for $1 < s < \infty$. We have the following proposition:

Proposition 5.5.3. *In addition to the assumption in Theorem 5.5.1, suppose that $u \in (1, \infty]$ satisfies $u' < \min\{q_1, q_2, \ldots, q_n, p\}$. Let $\Omega \in L^s(\mathbb{S}^{n-1})$ be homogeneous of degree zero, that is, Ω satisfies, for any $\lambda > 0$, $\Omega(\lambda x) = \Omega(x)$. Then,*

$$\|g \cdot I_{\Omega,\alpha}(f)\|_{\mathcal{M}^{s}_{\vec{t}}} \le C \|g\|_{\mathcal{M}^{p^*}_{\vec{q}^*}} \|\Omega\|_{L^u(\mathbb{S}^{n-1})} \|f\|_{\mathcal{M}^{p}_{\vec{q}}},$$

where

$$I_{\Omega,\alpha} f(x) \equiv \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) \mathrm{d}y.$$

Proposition 5.5.3 is a direct consequence of Theorem 5.5.1, the next lemma and the boundedness of the Hardy–Littlewood maximal operator M.

Lemma 5.5.4. [64] *If $1 < u \le \infty$, then we have*

$$|I_{\Omega,\alpha} f(x)| \le C \|\Omega\|_{L^u(\mathbb{S}^{n-1})} |I_\alpha F(x)|,$$

where $F(x) \equiv M\left(|f|^{u'}\right)(x)^{\frac{1}{u'}}$.

First, we prove two lemmas. We invoke an estimate from [30, Lemma 2.2] and [31, Lemma 2.1].

Lemma 5.5.5. *There exists a constant depending only on n and α such that, for every cube Q, we have $I_\alpha \chi_Q(x) \ge C \ell(Q)^\alpha \chi_Q(x)$ for all $x \in Q$.*

To prove the next estimate, we use Proposition 2.1.22. We invoke another estimate from [66, Lemma 4.2].

Lemma 5.5.6. *Let $K = 0, 1, 2, \ldots$. Suppose that A is an $L^\infty(\mathbb{R}^n) \cap \mathcal{P}_K(\mathbb{R}^n)^\perp$-function supported on a cube Q. Then,*

$$|I_\alpha A(x)| \le C_{\alpha,K} \|A\|_{L^\infty} \ell(Q)^\alpha \sum_{k=1}^{\infty} \frac{1}{2^{k(n+K+1-\alpha)}} \chi_{2^k Q}(x) \quad (x \in \mathbb{R}^n).$$

Now we prove Theorem 5.5.1. We may assume that $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is a positive measurable function in view of the positivity of the integral kernel. We decompose f according to Theorem 5.1.3 with $K > \alpha - \dfrac{n}{p^*} - 1$; $f = \sum\limits_{j=1}^{\infty} \lambda_j a_j$, where $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n) \cap \mathcal{P}_K(\mathbb{R}^n)^\perp$ and $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$ fulfill (5.1). Then by Lemma 5.5.6, we obtain

$$|g(x) I_\alpha f(x)| \le C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\lambda_j}{2^{k(n+K+1-\alpha)}} \left(\ell(Q_j)^\alpha |g(x)| \chi_{2^k Q_j}(x)\right).$$

Therefore, we conclude

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{s}_{\vec{t}}} \le C \|g\|_{\mathcal{M}^{p^*}_{\vec{q}^*}} \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\lambda_j \ell(2^k Q_j)^{\alpha - \frac{n}{p^*}}}{2^{k(n+K+1)}} \cdot \frac{\ell(2^k Q_j)^{\frac{n}{p^*}}}{\|g\|_{\mathcal{M}^{p^*}_{\vec{q}^*}}} |g| \chi_{2^k Q_j} \right\|_{\mathcal{M}^{s}_{\vec{t}}}.$$

For each $(j,k) \in \mathbb{N} \times \mathbb{N}$, write

$$\kappa_{jk} \equiv \frac{\lambda_j \ell(2^k Q_j)^{\alpha - \frac{n}{p^*}}}{2^{k(n+K+1)}}, \quad b_{jk} \equiv \frac{\ell(2^k Q_j)^{\frac{n}{p^*}}}{\|g\|_{\mathcal{M}^{p^*}_{\vec{q}^*}}} |g| \chi_{2^k Q_j}.$$

Then,

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\lambda_j \ell(2^k Q_j)^{\alpha - \frac{n}{p^*}}}{2^{k(n+K+1)}} \cdot \frac{\ell(2^k Q_j)^{\frac{n}{p^*}}}{\|g\|_{\mathcal{M}^{p^*}_{\vec{q}^*}}} |g| \chi_{2^k Q_j} = \sum_{j,k=1}^{\infty} \kappa_{jk} b_{jk},$$

each b_{jk} is supported on a cube $2^k Q_j$ and

$$\|b_{jk}\|_{\mathcal{M}^{p^*}_{\vec{q}^*}} \le \ell(2^k Q_j)^{\frac{n}{p^*}}.$$

Observe that $\chi_{2^k Q_j} \le 2^{kn} M \chi_{Q_j}$. Hence, if we choose $1 < \theta$ so that

$$K > \alpha - \frac{n}{p^*} - 1 + \theta n - n,$$

then we have

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \kappa_{jk} \chi_{2^k Q_j} \right\|_{\mathcal{M}^s_{\vec{t}}} &= \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\lambda_j \ell(2^k Q_j)^{\alpha - \frac{n}{p^*}}}{2^{k(n+K+1)}} \chi_{2^k Q_j} \right\|_{\mathcal{M}^s_{\vec{t}}} \\
&= \left\| \sum_{j=1}^{\infty} \lambda_j \ell(Q_j)^{\alpha - \frac{n}{p^*}} (M \chi_{Q_j})^{\theta} \right\|_{\mathcal{M}^s_{\vec{t}}} \\
&\le C \left\| \sum_{j=1}^{\infty} \left(M \left[\lambda_j^{\frac{1}{\theta}} \ell(Q_j)^{\frac{1}{\theta}(\alpha - \frac{n}{p^*})} \chi_{Q_j} \right] \right)^{\theta} \right\|_{\mathcal{M}^s_{\vec{t}}} \\
&\le C \left(\left\| \left\{ \sum_{j=1}^{\infty} \left(M \left[\lambda_j^{\frac{1}{\theta}} \ell(Q_j)^{\frac{1}{\theta}(\alpha - \frac{n}{p^*})} \chi_{Q_j} \right] \right)^{\theta} \right\}^{\frac{1}{\theta}} \right\|_{\mathcal{M}^{\theta s}_{\theta \vec{q}}} \right)^{\theta}.
\end{aligned}$$

Thanks to Proposition 2.1.24 with

$$f_j = \lambda_j^{\frac{1}{\theta}} \ell(Q_j)^{\frac{1}{\theta}(\alpha - \frac{n}{p^*})} \chi_{Q_j},$$

we can remove the maximal operator and we obtain

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^s_{\vec{t}}} \le C \|g\|_{\mathcal{M}^{p^*}_{\vec{q}^*}} \left\| \sum_{j=1}^{\infty} \lambda_j \ell(Q_j)^{\alpha - \frac{n}{p^*}} \chi_{Q_j} \right\|_{\mathcal{M}^s_{\vec{t}}}.$$

We distinguish two cases here.

1. If $\alpha = \dfrac{n}{p^*}$, then $p = s$ and $\vec{q} = \vec{t}$. Thus, we can use (5.1).

2. If $\alpha > \dfrac{n}{p^*}$, then, by Proposition 2.1.22 and Lemma 5.5.5, we obtain

$$\left\|\sum_{j=1}^{\infty}\lambda_j \ell(Q_j)^{\alpha-\frac{n}{p^*}}\chi_{Q_j}\right\|_{\mathcal{M}^s_{\vec{t}}} \le C\left\|I_{\alpha-\frac{n}{p^*}}\left[\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right]\right\|_{\mathcal{M}^s_{\vec{t}}} \le C\left\|\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}\right\|_{\mathcal{M}^p_{\vec{q}}}.$$

Thus, we are still in the position of using (5.1).

Consequently, we obtain

$$\left\|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\kappa_{jk}\chi_{2^kQ_j}\right\|_{\mathcal{M}^s_{\vec{t}}} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}} < \infty. \tag{5.17}$$

Observe also that $p^* > s$ and that $\vec{q}^* > \vec{t}$. Thus, by Theorem 5.1.2 and (5.17), it follows that

$$\|g\cdot I_\alpha f\|_{\mathcal{M}^s_{\vec{t}}} \le C\|g\|_{\mathcal{M}^{p^*}_{\vec{q}^*}}\left\|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\kappa_{jk}\chi_{2^kQ_j}\right\|_{\mathcal{M}^s_{\vec{t}}} \le C\|g\|_{\mathcal{M}^{p^*}_{\vec{q}^*}}\|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

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