On submanifolds in pseudo-Riemannian space forms

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Contents

Abstract

We study some submanifolds in pseudo-Riemannian space forms in terms of the degeneracy, which means that metrics on manifolds are degenerate. More precisely, via d-minimal surface theory, we classify spacelike flat zero mean curvature surfaces and visualize a deformation of zero mean curvature surfaces in Minkowski 4-space, and via lightlike geometry, we obtain a complete list of totally umbilical submanifolds in non-flat space forms, which are pseudo-spheres or pseudo-hyperbolic spaces with arbitrary index.

Part I

Introductions

Let M^m be an $m(\geq 2)$-dimensional manifold, and g a $(0,2)$-type symmetric tensor field on M. For each point $x \in M^m$, the signature (p,q,r) of g_x is determined, where p,q,r are the numbers of negative, positive or zero eigenvalues, respectively. We assume that the signature (p,q,r) is constant in M. Then, we call p the *index* and r the *nullity* of the metric g, respectively, and we call the pair (M,g) and to study it as follows:

A pseudo-Riemannian manifold	Pseudo-Riemannian geometry	when $r = 0$
A Riemannian manifold	Riemannian geometry	when $r = 0, p = 0$
A Lorentzian manifold	Lorentzian geometry	when $r = 0, p = 1$
An r-lightlike manifold	Lightlike geometry	when $r \geq 1$

Table 1: Kinds of geometries

In Riemannian geometry, geometers actively have studied submanifolds in Riemannian space forms, not only locally but also globally in detail. In pseudo-Riemannian geometry, submanifolds are studied as well. Some studies are done by using analogical methods in Riemannian geometry, others detect phenomena which never occur in Riemannian geometry. It is pointed out that Lorentzian, or lightlike geometries are related to relativity and electromagnetism [5]. For example, a lightcone in a Minkowski space and an event horizon, which are boundaries of black holes, are known to be lightlike manifolds.

This doctoral thesis is derived as follows. In Part II, we recall fundamental and necessary notions in pseudo-Riemannian geometry, lightlike geometry and submanifold theory.

Part III proves main results in this thesis. In Chapter 3 of Part III, we investigate surfaces in a three-dimensional singular pseudo-Euclidean space with the signature $(0,2,1)$. The history of surface theory is very long, and there is a lot of researches. Minimal surfaces attain stationary values for the volume functional of surfaces. We have many results of the research for minimal surfaces. In particular, they are characterized by having the mean curvature vector field which vanishes identically. Recently, Umehara and Yamada et al. [21, 22, 50] studied the zero mean curvature surfaces in a three-dimensional Minkowski space actively. For such surfaces, they showed that singularities appear generically, and relate to the topology of surfaces. On the other hand, the author [40] classified ruled minimal surfaces in pseudo-Euclidean spaces. As a consequence, it was obtained that certain surfaces are included in a three-dimensional subspaces whose metrics are degenerate forms. Inspired by this fact, in this work we study a differential geometry, which allows to have degenerate metrics. In particular, we establish a surface theory. We introduce a degenerate metric $dx^2 + dy^2 + 0dz^2$ to a three-dimensional vector space $\mathbb{R}^3$ with the canonical coordinates (x,y,z). We call the pair $\mathbb{E}^{0,2,1} := (\mathbb{R}^3, dx^2 + dy^2 + 0dz^2)$ a singular pseudo-Euclidean space. This is denoted by $\mathbb{E}^{0,2,1}$. Let M^2 be a surface in $\mathbb{E}^{0,2,1}$. We assume that the induced metric of M^2 is non-degenerate. Actually, this geometry is equivalent to simply isotropic geometry, which is one of the Cayley–Klein geometries (See [34]). For isotropic geometry, the well-known reference is [38]. In terms of the affine geometry with metrics and connections. we reformulate geometrical objects of surface theory such as induced connections and second fundamental forms.

In Chapter 4 of Part III, we investigate totally umbilical submanifolds in pseudo-Riemannian space forms. A totally umbilical submanifold in a pseudo-Riemannian manifold is a fundamental notion. For example, a complete non-totally geodesic, totally umbilical submanifold in a Euclidean space is a round sphere. A submanifold is called totally umbilical if the second fundamental form is proportional to the metric on the submanifold. In the case of Riemannian geometry, there are researches of totally umbilical submanifolds in various ambient spaces [8, 11, 29, 35, 48].

Part II

Preliminaries

Chapter 1

Pseudo-Riemannian geometry

In this chapter, we explain fundamental properties for pseudo-Riemannian manifolds and their non-degenerate submanifolds.

1.1 Pseudo-Riemannian manifolds

Let M^m be an m-dimensional connected manifold, and g a $(0,2)$-type non-degenerate symmetric tensor field on M. Since M is connected, the signature (p,q) of g is determined on M, where p,q are the numbers of negative and positive eigenvalues of g, respectively. We call the pair (M,g) a *pseudo-Riemannian manifold*, and the number p the *index* of (M,g). When $p=0$, the pair (M,g) is nothing but a Riemannian manifold. Namely, the notion of pseudo-Riemannian manifolds is a generalization of that of Riemannian manifolds. When $p=1$, we also call (M,g) a *Lorentzian manifold*. It is well known that Lorentzian manifolds play important roles in Relativity.

Let (M,g) be a pseudo-Riemannian manifold. For each $x \in M$ and a tangent vector $X \in T_xM$, we call X

$$\begin{aligned} \textit{spacelike} \quad &\text{if} \quad g(X,X) > 0 \text{ or } X = 0, \\ \textit{timelike} \quad &\text{if} \quad g(X,X) < 0, \\ \textit{lightlike} \text{ (or } \textit{null}) \quad &\text{if} \quad g(X,X) = 0 \text{ and } X \neq 0. \end{aligned}$$

These are called *causal properties* of tangent vectors [33]. As in the case of Riemannian manifolds, there exists a unique torsion-free and metric connection ∇ for a pseudo-Riemannian manifold. We call ∇ the *Levi-Civita connection* of (M,g). Hereinafter, a connection for pseudo-Riemannian manifolds is the Levi-Civita connection.

We define the *curvature tensor field* R of a pseudo-Riemannian manifold (M,g) as

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad (X,Y,Z \in \Gamma(TM)),$$

where we denote the set consisting of smooth vector fields on M by $\Gamma(TM)$. Next, for each $x \in M$, let P be a two-dimensional non-degenerate subspace of the tangent vector space T_xM, and let $\{X,Y\}$ be a basis of P. We define the *sectional curvature* $K(P)$ of P as

$$K(P) := \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

where a subspace $P \subset T_xM$ is called *non-degenerate* if the restriction on P of g is a non-degenerate form and it is called *degenerate* if otherwise. In particular, when the dimension of

M is two, the sectional curvature are also called the *Gaussian curvature*. We denote the set consisting of smooth functions on M by $C^\infty(M)$. For each $u \in C^\infty(M)$, we define the *gradient vector field* grad$u$ of u as

$$g(\mathrm{grad}u, X) = du(X) \quad (X \in \Gamma(TM)),$$

where du denotes the exterior derivative of u. Next, for each $X \in \Gamma(TM)$, we define the *divergence* div$X$ of X as

$$\mathrm{div}X := \mathrm{tr}((X_1, X_2) \mapsto g(\nabla_{X_1} X, X_2)) \quad (X_1, X_2 \in \Gamma(TM)).$$

For each $u \in C^\infty(M)$, we define the *Laplacian* $\Delta_g u$ of u with respect to g as

$$\Delta_g u := \mathrm{div}(\mathrm{grad}u).$$

When $\Delta_g u \equiv 0$, we say that u is a *harmonic* function.

Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of (M, g). The gradient vector field, the divergence and the Laplacian, respectively have the following local expressions

$$\begin{aligned} \mathrm{grad}u &= \sum_{i=1}^{n} \epsilon_i du(e_i) e_i, \\ \mathrm{div}X &= \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} X, e_i), \\ \Delta_g u &= \sum_{i,j=1}^{n} \epsilon_i \epsilon_j g(\nabla_{e_i} du(e_j) e_j, e_i) \end{aligned}$$

where $\epsilon_i = g(e_i, e_i) = \pm 1$.

1.2 Non-degenerate submanifolds

Let m and n be positive integers. Let M^m be an m-dimensional manifold, and let $\bar{M}^n$ be an n-dimensional pseudo-Riemannian manifold with index p. Here, when L is a manifold, L^m_s denotes an m-dimensional pseudo-Riemannian manifold with index s. The notation $\cong_{pRm}$ means the existence of an isometric isomorphism between pseudo-Riemannian manifolds. Here, we assume that a C^∞-mapping $\phi : M \to \bar{M}$ is an immersion. Then, we call $\phi(M)$ an immersed submanifold in $\bar{M}$. In particular, when ϕ is injective, and M is homeomorphic to the image $\phi(M)$ as a subspace of $\bar{M}$, $\phi(M)$ is said to be an *embedded submanifold* in $\bar{M}$. In pseudo-Riemannian geometry, we remark that the induced metric is not always non-degenerate on M even if ϕ is an immersion. When the induced metric is non-degenerate, we call $\phi(M)$ a *non-degenerate submanifold*, or a *pseudo-Riemannian submanifold* in $\bar{M}^n_p$.

As another situation, let $M^m_s, \bar{M}^n_p$ be pseudo-Riemannian manifolds, and $g, \bar{g}$ denote pseudo-Riemannian metrics of $M, \bar{M}$, respectively. When ϕ is an isometric immersion from M^m_s into $\bar{M}^n_p$, i.e. $\phi^* \bar{g} = g$, we also call $\phi(M)$ a non-degenerate submanifold in $\bar{M}^n_p$.

Hereinafter, when we describe submanifolds, we consider immersed, non-degenerate submanifolds unless otherwise stated. For each $x \in M$, a *normal vector space* $T^\perp_x M$ is defined as

$$T^\perp_x M := \{v \in T_{\phi(x)} \bar{M} \mid \bar{g}((\phi_*)_x(w), v) = 0, \text{ for all } w \in T_x M\}.$$

Then, we obtain a vector bundle $T^\perp M = \bigcup_{x \in M} T^\perp_x M$ of rank $(n - m)$ over M. This is called a *normal bundle* over M. By definition for each $x \in M$, we have an orthogonal direct sum decomposition

$$T_{\phi(x)} \bar{M} = T_x M \perp T^\perp_x M,$$

where $\perp$ stands for the orthogonal direct sum, and we identify $(\phi_*)_x(T_xM)$ with T_xM. In particular, we see that, as the orthogonal direct sum of vector bundles, it holds

$$\phi^* T\bar{M} = TM \perp T^{\perp}M, \tag{1.1}$$

where $\phi^*T\bar{M}$ is the pull-back bundle over M by ϕ. We denote the Levi-Civita connection of $(\bar{M}, \bar{g})$ and that of (M, g) by $\bar{\nabla}$ and ∇, respectively. We define $\Gamma(T^{\perp}M)$ as the set consisting of smooth sections of the normal bundle $T^{\perp}M$, and we call $\xi \in \Gamma(T^{\perp}M)$ a *normal vector field.*

Let $X, Y, \cdots, \xi, \eta, \cdots$, be tangent and normal vector fields on M, respectively. By using the orthogonal direct sum decomposition given above (1.1), we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.2}$$
$$\bar{\nabla}_X \xi = -A_{\xi} X + \nabla^{\perp}_X \xi, \tag{1.3}$$

where given objects h, A_{ξ} and $\nabla^{\perp}$ are called the *second fundamental form*, the *shape operator* with respect to ξ and the *normal connection* of ϕ, respectively. We call the formula (1.2) and (1.3) the *Gauss formula* and the *Weingarten formula* of an isometric immersion ϕ, respectively.

1.3 Pseudo-Riemannian space forms

We define an n-dimensional *singular pseudo-Euclidean space* with the signature (p, q, r) as

$$\mathbb{E}^{p,q,r} := \left(\mathbb{R}^n, (\cdot,\cdot) = -\sum_{i=1}^{p} dx_i^2 + \sum_{j=p+1}^{p+q} dx_j^2 + \sum_{k=p+q+1}^{n} 0dx_k^2 \right), \tag{1.4}$$

where $n = p + q + r$ and $(x_1, \cdots, x_n)$ expresses the canonical coordinates on $\mathbb{R}^n$ [49]. We use the following notations:

- When $r = 0$, $\mathbb{E}^{p,q,0}$ is called a *pseudo-Euclidean space*, and we denote $\mathbb{E}^n_p := \mathbb{E}^{p,q,0}$ and $\langle\cdot,\cdot\rangle_p := (\cdot,\cdot)$, respectively.
- When $r = 0, p = 1$, $\mathbb{E}^{1,n-1,0} = \mathbb{E}^n_1$ is called a *Minkowski n-space* and we denote it by $\mathbb{L}^n$.
- When $p = r = 0$, $\mathbb{E}^{0,n,0} = \mathbb{E}^n_0$ is nothing but a Euclidean n-space $\mathbb{E}^n$.

These spaces define flat pseudo-Riemannian space forms if $r = 0$. We remark that $r \geq 1$ if and only if the metric $(\cdot,\cdot)$ is degenerate. In the context of isotropic geometry, the notation $\mathbb{E}^{0,n-1,1}$ would be denoted by $\mathbb{I}^n$ [38].

Next, we define non-flat pseudo-Riemannian space forms with index p as

$$\mathbb{S}^n_p(r^2) := \left\{ x \in \mathbb{E}^{n+1}_p \mid \langle x, x\rangle_p = r^2 \right\}, \quad \mathbb{H}^n_p(-r^2) := \left\{ x \in \mathbb{E}^{n+1}_{p+1} \mid \langle x, x\rangle_{p+1} = -r^2 \right\},$$

where $r > 0$. We call $\mathbb{S}^n_p(r^2)$ (resp. $\mathbb{H}^n_p(-r^2)$) an n-dimensional *pseudo-sphere* (resp. *pseudo-hyperbolic space*). When $p = 0$, $\mathbb{S}^n_0(1)$ and $\mathbb{H}^n_0(-1) \cap \{x_1 > 0\}$ are simply a standard sphere $\mathbb{S}^n(1)$ and a hyperbolic space $\mathbb{H}^n(-1)$, respectively. When $p = 1$, $\mathbb{S}^n_1(1)$ and $\mathbb{H}^n_1(-1)$ are called a *de Sitter n-spacetime* and an *anti-de Sitter n-spacetime*, denoted by $d\mathbb{S}^n(1), Ad\mathbb{S}^n(-1)$, respectively. For $\varepsilon = \pm 1, 0$, we define for brevity

$$\mathbb{M}^n_p(\varepsilon) := \begin{cases} \mathbb{E}^n_p & (\varepsilon = 0), \\ \mathbb{S}^n_p(1) \subset \mathbb{E}^{n+1}_p & (\varepsilon = 1), \\ \mathbb{H}^n_p(-1) \subset \mathbb{E}^{n+1}_{p+1} & (\varepsilon = -1). \end{cases}$$

Let $M_s^m, \bar{M}_p^n$ be pseudo-Riemannian manifolds, and $g, \bar{g}$ denote pseudo-Riemannian metrics of $M, \bar{M}$, respectively. Let $\phi : M_s^m \to \bar{M}_p^n$ be an isometric immersion, and h, H the second fundamental form and mean curvature vector field of ϕ, respectively. Here, we define the *mean curvature vector field* H of ϕ by

$$H = \frac{1}{m}\text{trace}_g h \in \Gamma(T^\perp M).$$

Then, we call ϕ *totally geodesic* if h identically vanishes, and call ϕ *totally umbilical* if, for all $X, Y \in \Gamma(TM)$, it holds

$$h(X, Y) = g(X, Y)H.$$

We call ϕ *minimal* if H identically vanishes. As an easy observation, we see that ϕ is totally geodesic if and only if ϕ is totally umbilical and minimal. In addition to these notions, ϕ is called to be *marginally trapped* if $H \neq 0$ and $\bar{g}(H, H) = 0$. Finally, when an isometric immersion $\phi : M_s^m \to \bar{M}_p^n$ is totally geodesic, totally umbilical, minimal or marginally trapped, we call the image $\phi(M)$ a totally geodesic, totally umbilical, minimal or marginally trapped submanifold in $\bar{M}$, respectively.

In the following, the ambient $\bar{M}_p^n$ will be a pseudo-Riemannian space form $\mathbb{M}_p^n(\varepsilon)$. Let $\phi : M_s^m \to \mathbb{M}_p^n(\varepsilon)$ be an isometric immersion. Let $X, Y, Z, W \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^\perp M)$. The *Gauss equation*, *Codazzi equation* and *Ricci equation* of ϕ are given by the following

$$\begin{aligned}\langle R(X, Y)Z, W\rangle &= \varepsilon(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)) \\ &\quad + \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle, \quad (1.5)\\ (\tilde{\nabla}_X h)(Y, Z) &= (\tilde{\nabla}_Y h)(X, Z), \quad (1.6)\\ \langle R^\perp(X, Y)\xi, \eta\rangle &= \langle [A_\xi, A_\eta]X, Y\rangle, \quad (1.7)\end{aligned}$$

where R and $R^\perp$ are curvature tensor fields with respect to connections ∇ and $\nabla^\perp$, respectively, and $\nabla_X h$ is the covariant derivative of the second fundamental form h for the tangent vector field X, i.e. it is defined by

$$(\tilde{\nabla}_X h)(Y, Z) := \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Moreover, the normal bundle $T^\perp M$ of M is called *flat* if $R^\perp \equiv 0$.

Here, we define two additional classes of submanifolds. Let M and $\bar{M}$ be pseudo-Riemannian manifolds, and $\phi : M \to \bar{M}$ an isometric immersion. We call the immersion ϕ *parallel* if the covariant derivative of the second fundamental form h of ϕ vanishes identically. We call the immersion ϕ *symmetric* if, for each $x \in M$, there exist isometries $\Phi_x \in \text{Isom}(M), \Psi_x \in \text{Isom}(\bar{M})$ such that

$$\Phi_x(x) = x, \quad \Psi_x \circ \phi = \phi \circ \Phi_x, \quad (\Psi_x)_*\phi_* X = -\phi_* X, \quad (\Psi_x)_*(\xi) = \xi,$$

where $X \in T_x M$ and $\xi \in T_x^\perp M$. We may consider the local version of the above conditions. Namely, for each $x \in M$, Φ_x and Ψ_x are local isometries, we call ϕ *locally symmetric*. When an isometric immersion $\phi : M \to \bar{M}$ is parallel or (locally) symmetric, we call the image $\phi(M)$ a parallel or (locally) symmetric submanifold in $\bar{M}$, respectively.

Two isometric immersions ϕ_1 and ϕ_2 given by

$$\phi_i : M_s^m \to \bar{M}_p^n \ (i = 1, 2)$$

are said to be *congruent* if there exists an isometry Ψ of $\bar{M}$ such that $\phi_2 = \Psi \circ \phi_1$. The congruency defines an equivalence relation on the set consisting of submanifolds.

Chapter 2

Lightlike geometry

There exist some generalizations of the notion of Riemannian manifolds. In the previous chapter, we saw that pseudo-Riemannian manifolds are one of generalizations. In addition, Finsler manifolds or sub-Riemannian manifolds are also well known as generalizations. In this chapter, as another generalization, we consider lightlike manifolds whose metrics are degenerate everywhere.

2.1 Reinhart lightlike manifolds

Let V be a real n-dimensional vector space, and B a (possibly degenerate) symmetric bilinear form on V. We define the *radical subspace* of (V, B) as

$$\mathrm{Rad}V := \{v \in V \mid B(v, w) = 0, \text{ for all } w \in V\}.$$

Obviously, B is non-degenerate if and only if it holds $\mathrm{Rad}V = \{0\}$.

Let M be an n-dimensional manifold, and g a symmetric $(0, 2)$-type tensor field on M. For the pair (M, g), if the mapping

$$M \ni x \longmapsto \mathrm{Rad}T_xM \subset T_xM$$

defines a smooth distribution of constant rank $r \geq 0$, we call

$$\mathrm{Rad}TM := \bigcup_{x \in M} \mathrm{Rad}T_xM$$

the *radical distribution* on M. Then, the metric g is called *r-lightlike*, and (M, g) is called an *r-lightlike manifold*. In particular, we simply call (M, g) a lightlike manifold when we need not detect the number r. As an easy observation, 0-lightlike manifolds are pseudo-Riemannian manifolds.

For an r-lightlike manifold, we assume the radical distribution $\mathrm{Rad}TM$ is integrable. Then, an r-dimensional foliation structure is defined on M. We call an r-lightlike manifold (M, g) *Reinhart* if for any foliated charts $\{U; (x^1, \cdots, x^n)\}$ of M, it holds conditions

$$\frac{\partial g_{ij}}{\partial x^\alpha}(x^1, \cdots, x^n) = 0 \quad \text{for all } i, j \in \{r + 1, \cdots, n\},\ \alpha \in \{1, \cdots, r\},$$

where $\{x^\alpha\}_{1 \leq \alpha \leq r}$ are local coordinates of leaves of $\mathrm{Rad}TM$ and $g_{ij} := g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$. These conditions do not depend on the choice of foliated charts.

Let (M, g) be an r-lightlike manifold. A vector field $X \in \Gamma(TM)$ is called a *Killing vector field* if it satisfies $\mathcal{L}_X g = 0$, where $\mathcal{L}_X$ is the Lie derivative with respect to X defined by

$$(\mathcal{L}_X g)(Y, Z) := Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) \quad Y, Z \in \Gamma(TM).$$

A smooth distribution D of M is called a *Killing distribution* if any vector fields belonging to D are Killing.

The following theorem gives characterizations of Reinhart lightlike manifolds.

Theorem 2.1 ([5, Theorem 5.1, Chapter 2])**.** Let (M, g) be an r-lightlike manifold. The following are equivalent to each other:

(i) (M, g) is Reinhart.

(ii) $\mathrm{Rad}TM$ is a Killing distribution.

(iii) There exists a torsion-free connection ∇ on M such that the connection is metric parallel with respect to g, i.e. $\nabla g = 0$.

Remark 2.2. Regarding Theorem 2.1, for Reinhart lightlike manifolds, there exists a connection which is quite close to the Levi-Civita connection, but the uniqueness does not hold. We call such connections *quasi Levi-Civita connections* in this thesis. For two distinct quasi Levi-Civita connections ∇, ∇', let L be the difference, i.e.

$$L(X, Y) := \nabla_X Y - \nabla'_X Y \quad (X, Y \in \Gamma(TM)).$$

L is a $\mathrm{Rad}TM$-valued $(0, 2)$-type symmetric tensor field on M. Conversely, when we set ∇ as a quasi Levi-Civita connection and L as a $\mathrm{Rad}TM$-valued $(0, 2)$-type symmetric tensor field on M, we see that $\nabla + L$ is a quasi Levi-Civita connection. In particular, in the case of pseudo-Riemannian manifolds, i.e. $r = 0$, quasi Levi-Civita connections are unique since $\mathrm{Rad}TM = \{0\}$. Namely, the unique one is just Levi-Civita connection.

From now on, for a Reinhart r-lightlike manifold (M, g), we fix a quasi Levi-Civita connection ∇, and call the triplet (M, g, ∇) a Reinhart r-lightlike manifold equipped with ∇. Then, we define its automorphism group

$$\mathrm{Aut}(M, g, \nabla) := \{F \in \mathrm{Diff}(M) \mid F^*g = g,\ F^*\nabla = \nabla\}.$$

Remark 2.3. The automorphism group $\mathrm{Aut}(M, g, \nabla)$ is finite dimensional Lie group, and acts on M effectively. When $r = 0$, it simply coincides with the isometry group $\mathrm{Isom}(M, g)$ since (M, g, ∇) is a pseudo-Riemannian manifold equipped with the Levi–Civita connection.

Let $(M, g, \nabla), (\bar{M}, \bar{g}, \bar{\nabla})$ be two Reinhart r-lightlike manifolds. We define (M, g, ∇) is *isomorphic* to $(\bar{M}, \bar{g}, \bar{\nabla})$ if there exists a diffeomorphism $f : M \to \bar{M}$ such that

$$f^*\bar{g} = g, \quad f^*\bar{\nabla} = \nabla.$$

We call such f an *isomorphism* between Reinhart lightlike manifolds. When we have such isomorphism, we denote $(M, g, \nabla) \cong_{\mathrm{Rlm}} (\bar{M}, \bar{g}, \bar{\nabla})$. In the case that f is locally isomorphic, we call f a *local isomorphism* between Reinhart lightlike manifolds, and (M, g, ∇) is *locally isomorphic* to $(\bar{M}, \bar{g}, \bar{\nabla})$. By definition, if (M, g, ∇) is isomorphic to $(\bar{M}, \bar{g}, \bar{\nabla})$, we see that

$$\dim M = \dim \bar{M}, \quad \mathrm{rk}\ \mathrm{Rad}TM = \mathrm{rk}\ \mathrm{Rad}T\bar{M} = r.$$

When $r = 0$, f is an isometry, and $M \cong_{\mathrm{Rlm}} \bar{M}$ expresses M is isometric to $\bar{M}$, i.e. $\cong_{\mathrm{Rlm}}$ coincides with $\cong_{\mathrm{pRm}}$.

Proposition 2.4. If $f : (M, g, \nabla) \to (\bar{M}, \bar{g}, \bar{\nabla})$ is an isomorphism between Reinhart r-lightlike manifolds, and $\gamma : I \to M$ a geodesic on M with respect to ∇, then the composition $f \circ \gamma : I \to \bar{M}$ is also a geodesic on $\bar{M}$ with respect to $\bar{\nabla}$. In particular, if (M, g, ∇) is geodesically complete, then so is $(\bar{M}, \bar{g}, \bar{\nabla})$.

Proof. By definition, it is obvious that geodesics in M are mapped onto geodesics in $\bar{M}$ by f. Thus, the proof is completed. □

Proposition 2.5 ([10, Lemma 1.5, Chapter 1]). If $\phi, \psi : (M, g, \nabla) \to (\bar{M}, \bar{g}, \bar{\nabla})$ are two local isomorphisms between Reinhart r-lightlike manifolds, then, for some point $x \in M$, if $d\phi_x = d\psi_x$, then $\phi = \psi$.

A motivation of investigation of Reinhart r-lightlike manifolds is that some submanifolds in a pseudo-Riemannian manifold are r-lightlike manifolds. Among them, Reinhart r-lightlike submanifolds are remarkable in terms of admitting connections which have the same properties as Levi-Civita connections. In addition, it would be the first step to expose pseudo-Riemannian manifold and their degenerate submanifolds.

2.2 Lightlike submanifolds

When we consider a submanifold in a pseudo-Riemannian manifold, the induced metric is not always non-degenerate. In this section, we deal with submanifolds with a degenerate metric, say *lightlike submanifolds*. Bejancu–Duggal [5] and Kupeli [27] constructed a fundamental theory of lightlike submanifolds. However, there are many unsolvable fundamental problems such as existence problems and classification problems. Therefore, we would say these studies only just started.

Incidentally, there exist submanifolds which have both of non-degenerate parts and degenerate parts for the induced metric. These submanifolds are called *mixed type*. There exist some researches for mixed type surfaces. For example, refer to [21]. We explain this section based on Bejancu–Duggal's lightlike submanifold theory.

Let $(\bar{M}, \bar{g})$ be an $(m+n)$-dimensional pseudo-Riemannian manifold, M an m-dimensional manifold, and $f : M \to \bar{M}$ an immersion. We denote the induced metric on M by $g = f^*\bar{g}$. For each $x \in M$, we define

$$T_x^{\perp}M := \{v \in T_x\bar{M} \mid \bar{g}_x((f_*)_x(v), w) = 0, \text{ for all } w \in T_xM\}.$$

If $\bar{g}_x$ is degenerate on T_xM, then it is also degenerate on $T_x^{\perp}M$ vice versa. Then, there exists a non-trivial intersection

$$\mathrm{Rad}T_xM = \mathrm{Rad}T_x^{\perp}M = T_xM \cap T_x^{\perp}M \supsetneqq \{0\}.$$

However, the dimension of $\mathrm{Rad}T_xM$ may depend on points $x \in M$.

Let $\bar{M}$ be a pseudo-Riemannian manifold, and M a manifold. An immersion $f : M \to \bar{M}$ is called *r-lightlike* if the pair (M, g) itself is an r-lightlike manifold. In the above situation, we also call the pair (M, g) an r-lightlike submanifold of $(\bar{M}, \bar{g})$. In particular, we simply call M a lightlike submanifold when we need not detect the metric g and the number r.

Theorem 2.6 ([5, Theorem 1.1, Chapter 5]). Let $f : M \to \bar{M}$ be an immersion from a manifold M into a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$, and g the induced metric by f. The following are equivalent to each other:

(i) (M, g) is r-lightlike.

(ii) For each coordinate neighborhood $U \subset M$, the mapping $U \ni x \mapsto \mathrm{Rad}T_xU$ defines a smooth distribution of rank r on U.

(iii) For each coordinate neighborhood $U \subset M$, the metric g has constant rank $m - r$ on U.

Here, r satisfies $1 \leq r \leq m$.

When (M, g) is an r-lightlike submanifold, we can construct vector bundles $TM, T^{\perp}M, \mathrm{Rad}TM$ of rank m, n, r, respectively. More precisely, we define the following objects.

(a) $S(TM)$ is called to be a *screen distribution* if it is a subbundle of TM over M, and it holds an orthogonal direct sum

$$TM = S(TM) \perp \mathrm{Rad}TM.$$

It is a non-degenerate vector bundle of rank $(m - r)$, that is, the induced fibre metric is non-degenerate.

(b) $S(T^{\perp}M)$ is called to be a *screen transversally vector bundle* if it is a subbundle of $T^{\perp}M$ over M, and it holds an orthogonal direct sum

$$T^{\perp}M = S(T^{\perp}M) \perp \mathrm{Rad}TM.$$

It is a non-degenerate vector bundle of rank $(n - r)$.

(c) $\mathrm{ltr}(TM)$ is called to be a *lightlike transversally vector bundle* if it is a subbundle of $f^*T\bar{M}$ over M, and, for each coordinate neighborhood $U \subset M$ and local frames $\{\xi_i\}_{i=1}^r$ of $\mathrm{Rad}TM|_U$, there exist sections $\{N_i\}_{i=1}^r$ belonging to $\mathrm{ltr}(TM)|_U$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0 \quad (i, j \in \{1, \cdots, r\}).$$

It is a vector bundle of rank r.

(d) B is called to be a *complementary screen vector bundle* if it is a subbundle of $f^*T\bar{M}$ over M, and it is a non-degenerate vector bundle such that

$$\mathrm{Rad}TM \subset B.$$

It is a vector bundle of rank $2r$.

We should remark that the existence of the above subbundles is ensured but the uniqueness is not so. Namely, when we do the study of lightlike submanifolds in pseudo-Riemannian manifolds, we have to see through geometric notions and properties independent of the choice of these vector bundles.

Next, we classify classes of r-lightlike submanifolds M. We call M

(Case I) : a *proper r-lightlike submanifold* if $0 < r < \min\{m, n\}$,

(Case II) : a *coisotropic submanifold* if $0 < r = n < m$,

(Case III) : a *isotropic submanifold* if $0 < r = m < n$,

(Case IV) : a *totally lightlike submanifold* if $0 < r = m = n$.

By definition, when M is a lightlike curve, i.e. $m = 1$, it is a isotropic submanifold. When M is a lightlike hypersurface, i.e. $n = 1$, it is a coisotropic submanifold. We promise a rank zero subbundle of $f^*T\bar{M}$ as a vector bundle which consists of zero section $\{0\}$ of $f^*T\bar{M}$. Then, the following holds:

Proposition 2.7 ([5, Theorem 1.3, Chapter 5])**.** If (M, g) is an r-lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$, then, between the set of the pairs $(S(TM), S(T^{\perp}M))$ and the set of complementary screen vector bundles B, there exists a one-to-one corresponding such that

$$f^*T\bar{M} = S(TM) \perp S(T^{\perp}M) \perp B$$

and moreover the following holds

$$\begin{aligned}
&M : \text{coisotropic} \Longleftrightarrow S(T^{\perp}M) = \{0\} \Longleftrightarrow f^*T\bar{M} = S(TM) \perp B, \\
&M : \text{isotropic} \Longleftrightarrow S(TM) = \{0\} \Longleftrightarrow f^*T\bar{M} = S(T^{\perp}M) \perp B, \\
&M : \text{totally lightlike} \Longleftrightarrow S(TM) = S(T^{\perp}M) = \{0\} \Longleftrightarrow f^*T\bar{M} = B.
\end{aligned}$$

For an r-lightlike submanifold M, we define a vector bundle over M as

$$\mathrm{tr}(TM) := S(T^{\perp}M) \perp \mathrm{ltr}(TM).$$

Then, it holds

$$f^*T\bar{M} = TM \oplus \mathrm{tr}(TM),$$

where the symbol $\oplus$ simply expresses a direct sum. We call $\mathrm{tr}(TM)$ a *transversally vector bundle* over M, and *local quasi-orthonormal frames* of $\bar{M}$ along M are given by

$$\{\xi_i, N_i, X_a, W_\alpha\}_{i,a,\alpha},$$

where $1 \le i \le r,\ r+1 \le a \le m,\ r+1 \le \alpha \le n,\ \{\xi_i\}_{1 \le i \le r}, \{N_i\}_{1 \le i \le r}$ are, respectively local frames of $\mathrm{Rad}TM, \mathrm{ltr}(TM)$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0 \quad (1 \le i, j \le r),$$

and $\{X_a\}_{r+1 \le a \le m}, \{W_\alpha\}_{r+1 \le \alpha \le n}$ are, respectively local orthonormal frames of $S(TM), S(T^{\perp}M)$.

Remark 2.8. In the case of coisotropic or totally lightlike submanifolds, we define a transversally vector bundle $\mathrm{tr}(TM)$ of M as

$$\mathrm{tr}(TM) := \mathrm{ltr}(TM)$$

and we see that

$$f^*T\bar{M} := TM \oplus \mathrm{ltr}(TM).$$

In the case of isotropic submanifolds, we define a transversally vector bundle $\mathrm{tr}(TM)$ of M as

$$\mathrm{tr}(TM) := S(T^{\perp}M) \perp \mathrm{ltr}(TM)$$

and we see that

$$f^*T\bar{M} = TM \oplus \mathrm{tr}(TM).$$

Proposition 2.9 ([5, Theorem 1.2, 1.4, Chapter 5])**.** If (M, g) is a 1-lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$, then, between the set of the pairs $(S(TM), S(T^{\perp}M))$ and the set of $\mathrm{ltr}(TM)$, there exists a one-to-one corresponding such that

$$f^*T\bar{M} = S(TM) \perp S(T^{\perp}M) \perp (\mathrm{Rad}TM \oplus \mathrm{ltr}(TM)).$$

In the case $r > 1$, this result is still valid except for the uniqueness of lightlike transversally vector bundles.

In any case, for r-lightlike submanifolds, since we obtain a direct sum as vector bundles

$$f^*T\bar{M} = TM \oplus \mathrm{tr}(TM),$$

we will define some geometric objects by using this decomposition.

Let $\bar{\nabla}$ be the Levi–Civita connection of a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$. For any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(\mathrm{tr}(TM))$, we can decompose as follows

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h(X, Y) \in TM \oplus \mathrm{tr}(TM), \\ \bar{\nabla}_X V &= -A(V, X) + \nabla^t_X V \in TM \oplus \mathrm{tr}(TM). \end{aligned}$$

These fomulas are called the *Gauss formula* and the *Weingarten formula* of a lightlike submanifold M, respectively. We call ∇, ∇^t the *induced connection* and the *transversal connection* of M, respectively, and the $\Gamma(\mathrm{tr}(TM))$-valued symmetric bilinear form h the *second fundamental form* of M. Moreover, for each $V \in \Gamma(\mathrm{tr}(TM))$, we define a linear operator as

$$A_V : \Gamma(TM) \to \Gamma(TM) \ ; \ A_V(X) := A(V, X)$$

and call it the *shape operator* with respect to V of M.

From now on, we assume M is proper or isotropic. Then, from $\mathrm{tr}(TM) = \mathrm{ltr}(TM) \oplus S(T^\perp M)$, the Gauss formula and the Weingarten formula of M are expressed by

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h^l(X, Y) + h^s(X, Y) \in TM \oplus \mathrm{ltr}(TM) \oplus S(T^\perp M), \\ \bar{\nabla}_X V &= -A(V, X) + D^l_X V + D^s_X V \in TM \oplus \mathrm{ltr}(TM) \oplus S(T^\perp M) \end{aligned}$$

respectively. Here, we call h^l, h^s the *lightlike second fundamental form* and the *screen second fundamental form*, respectively. However we remark both D^l and D^s are not linear connections of $\mathrm{tr}(TM)$. So, we set L, S as projective bundle morphisms onto $\mathrm{ltr}(TM), S(T^\perp M)$ for $\mathrm{tr}(TM)$. For any $X \in \Gamma(TM)$, when we define differential operators as

$$\begin{aligned} \nabla^l_X &: \Gamma(\mathrm{ltr}(TM)) \to \Gamma(\mathrm{ltr}(TM)) \ ; \ \nabla^l_X(LV) := D^l_X(LV), \\ \nabla^s_X &: \Gamma(S(T^\perp M)) \to \Gamma(S(T^\perp M)) \ ; \ \nabla^s_X(SV) := D^s_X(SV), \end{aligned}$$

∇^l, ∇^s are linear connections of $\mathrm{ltr}(TM), S(T^\perp M)$, respectively. We call them the *lightlike transversal connection* and *screen transversal connection*, respectively.

Let M be an r-lightlike submanifold, and $\{\xi_i, N_i, X_a, W_\alpha\}_{1 \le i \le r, r+1 \le a \le m, r+1 \le \alpha \le n}$ local quasi-orthonormal frames along M. In the case that M is proper or isotropic, we can locally express

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h^l_i(X, Y) N_i + \sum_{\alpha=r+1}^{n} h^s_\alpha(X, Y) W_\alpha.$$

In the same way, in the case that M is coisotropic or totally lightlike, we can locally express

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{n} h^l_i(X, Y) N_i.$$

We call family of these symmetric bilinear forms $\{h^l_i\}_{1 \le i \le r}, \{h^s_\alpha\}_{r+1 \le \alpha \le n}$ *locally lightlike second fundamental forms* and *locally screen seconf fundamental forms* of M, respectively.

Theorem 2.10 ([5, Theorem 2.1, Chapter 5])**.** If (M, g) is an r-lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$, then locally lightlike second fundamental forms of M do not depend on the choice of screen distributions, screen transversally vector bundles and lightlike transversally vector bundles. However, we remark that they depend on the choice of local frames $\{\xi_i\}_{1 \le i \le r}$ of the radical distribution $\mathrm{Rad}TM$.

Since we want to consider a situation that the rank of a screen distribution is not zero, we deal with the case of proper r-lightlike or coisotropic submanifolds. Then, since we know

$$TM = S(TM) \perp \mathrm{Rad}TM,$$

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(\mathrm{Rad}TM)$, we can decompose

$$\nabla_X PY = \overset{*}{\nabla}_X PY + h^*(X, PY) \in S(TM) \perp \mathrm{Rad}TM,$$
$$\nabla_X \xi = -\overset{*}{A}(\xi, X) + \overset{*}{\nabla}{}^t{}_X \xi \in S(TM) \perp \mathrm{Rad}TM,$$

where P is the projection bundle morphism $P : TM \to S(TM)$. Then, $\overset{*}{\nabla}$ and $\overset{*}{\nabla}{}^t$ are metric connection of $S(TM)$ and $\mathrm{Rad}TM$, respectively, and we call them *induced connections* of $S(TM)$ and $\mathrm{Rad}TM$, respectively. h^* is a $\Gamma(\mathrm{Rad}TM)$-valued smooth bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$ and we call it the *second fundamental form* of $S(TM)$. $\overset{*}{A}$ is a $\Gamma(S(TM))$-valued smooth bilinear form on $\Gamma(\mathrm{Rad}TM) \times \Gamma(TM)$ and, for any $\xi \in \Gamma(\mathrm{Rad}TM)$, it defines a smooth linear operator

$$\overset{*}{A}_\xi : \Gamma(TM) \to \Gamma(S(TM)) \ ; \ \overset{*}{A}_\xi(X) = \overset{*}{A}(\xi, X) \quad (X \in \Gamma(TM)),$$

and we call it the *shape operator* with respect to ξ of $S(TM)$.

We can show the following relation. For any $X, Y \in \Gamma(TM), \xi \in \Gamma(\mathrm{Rad}TM)$, it holds

$$\bar{g}(h^l(X, PY), \xi) = g(\overset{*}{A}_\xi X, PY).$$

Namely, the shape operator $\overset{*}{A}$ and the restriction of h^l on $\Gamma(TM) \times \Gamma(S(TM))$ are equivalent each other. For any $X, Y \in \Gamma(TM), N \in \Gamma(\mathrm{ltr}(TM))$, it holds

$$\bar{g}(h^*(X, PY), N) = \bar{g}(A_N X, PY).$$

Namely, the shape operator A and the second fundamental form h^* are equivalent each other.

Let (M, g) be an r-lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$, and ∇ the induced connection of M. We call M *totally geodesic* if arbitrary geodesics in M with respect to ∇ are geodesics in $\bar{M}$ with respect to $\bar{\nabla}$.

Theorem 2.11 ([5, Theorem 2.8, Chapter 5]). Let (M, g) be an r-lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$. When we select vector bundles $S(TM), S(T^\perp M)$ and $\mathrm{ltr}(TM)$, the following are equivalent to each other:

(i) M is totally geodesic.

(ii) $h^l \equiv 0, h^s \equiv 0$ on M, that is, $h \equiv 0$ on M.

(iii) For any $\xi \in \Gamma(\mathrm{Rad}TM)$, $\overset{*}{A}_\xi \equiv 0$ on M and, for any $W \in \Gamma(S(TM))$, A_W is a $\Gamma(\mathrm{Rad}TM)$-valued operator such that $D^l(X, SV) = 0$ for any $X \in \Gamma(TM), V \in \Gamma(\mathrm{tr}(TM))$.

For r-lightlike submanifolds, the notion of to being totally geodesic is a geometric condition independent of the choice of vector bundles $S(TM), S(T^\perp M)$ and $\mathrm{ltr}(TM)$ because of Theorems 2.10 and 2.11.

Corollary 2.12 ([5, Corollary 2.2, Chapter 5]). The induced connection of a totally geodesic r-lightlike submanifold is a metric connection and coincides with the restriction on M of the Levi–Civita connection $\bar{\nabla}$.

Corollary 2.13 ([5, Corollary 2.3, Chapter 5])**.** Let (M, g) be a coisotropic submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$. The following are equivalent to each other:

(i) M is totally geodesic.

(ii) $h^l \equiv 0$ on M.

(iii) For any $\xi \in \Gamma(T^\perp M)$, $\overset{*}{A}_\xi \equiv 0$ on M.

(iv) The induced connection ∇ is a metric connection.

Corollary 2.14 ([5, Corollary 2.4, Chapter 5])**.** Let (M, g) be an isotropic submanifold in a pseudo-Riemannain manifold $(\bar{M}, \bar{g})$. The following are equivalent to each other:

(i) M is totally geodesic.

(ii) $h^s \equiv 0$ on M.

(iii) For any $X \in \Gamma(TM), V \in \Gamma(\mathrm{tr}(TM))$, $D^l(X, SV) = 0$.

Corollary 2.15 ([5, Corollary 2.5, Chapter 5])**.** Any totally lightlike submanifolds are totally geodesic.

Let W be an m-dimensional degenerate subspace in $(m+n)$-dimensional pseudo-Euclidean space $\mathbb{E}^{m+n}_p$, and $v \in \mathbb{E}^{m+n}_p$. We call $v + W$ a *lightlike m-plane* in $\mathbb{E}^{m+n}_p$, and we define a *canonical r-lightlike m-plane* in $\mathbb{E}^n_p$ with signature (s, t, r) as follows

$$\Pi^m_{s,t,r} := \{(\underbrace{z_1, \cdots, z_r, x_1, \cdots, x_s, 0, \cdots, 0}_{p}, \underbrace{0, \cdots, 0, y_1, \cdots, y_t, z_r, \cdots, z_1}_{n-p}) \in \mathbb{E}^n_p\}. \tag{2.1}$$

Then, we easily can verify that a canonical r-lightlike m-plane $\Pi^m_{s,t,r}$ is a totally geodesic r-lightlike submanifold in pseudo-Euclidean space and is isometric to $\mathbb{E}^{s,t,r}$. It holds the converse.

Theorem 2.16. Let $m \geq 2, n \geq 1, 1 \leq p \leq \left[\frac{m+n}{2}\right]$. If M a connected, m-dimensional totally geodesic r-lightlike submanifold of $\mathbb{E}^{m+n}_p$, then, up to isometry of $\mathbb{E}^{m+n}_p$, M is an open subset of $\Pi^m_{s,t,r}$. In particular, if M is simply-connected and geodesically complete, it coincides with $\Pi^m_{s,t,r}$. Moreover, M is isomorphic to $\mathbb{E}^{s,t,r}$ as a Reinhart r-lightlike manifold.

Proof. The claim follows directly from the fact that any geodesics in pseudo-Euclidean space are lines. □

Let $\varepsilon := \pm 1$ and $\bar{p} := p + \frac{1-\varepsilon}{2}$. We construct a hypersurface $N(\varepsilon)$ in $\mathbb{M}^{m+1}(\varepsilon)$ as follows. Let $v := (1, 0, \cdots, 0, 1) \in \mathbb{E}^{m+2}_{\bar{p}}$. A space consisting of vectors which are orthogonal to v

$$W := \{x \in \mathbb{E}^{m+2}_{\bar{p}} \mid \langle x, v\rangle_{\bar{p}} = 0\}$$

is a 1-lightlike $(m+1)$-plane. So, we define

$$N(\varepsilon) := \mathbb{M}^{m+1}_p(\varepsilon) \cap W.$$

Proposition 2.17. $N(\varepsilon)$ is a totally geodesic 1-lightlike hypersurface in $\mathbb{M}^{m+1}_p(\varepsilon)$.

Proof. By direct calculations, we see that

$$\mathrm{Rad}TN(\varepsilon) = \mathrm{Span}\{\xi := (1, 0, \cdots, 0, 1)\}.$$

Thus, we compute

$$\langle h(X,Y), \xi\rangle_{\bar{p}} = X\langle Y, \xi\rangle_{\bar{p}} - \langle Y, d_X\xi\rangle_{\bar{p}} = 0,$$

where $X, Y \in \Gamma(TN(\varepsilon))$ and d is the canonical connection of $\mathbb{E}^{m+2}_{\bar{p}}$. Since $N(\varepsilon)$ is a 1-lightlike hypersurface in $\mathbb{M}^{m+1}_p(\varepsilon)$, we have $h(X,Y) = 0$. Namely, $N(\varepsilon)$ is a totally geodesic 1-lightlike hypersurface in $\mathbb{M}^{m+1}_p(\varepsilon)$. □

Moreover, the converse is also true.

Theorem 2.18. Let $m \geq 2, 1 \leq p \leq [\frac{m+1}{2}], \varepsilon = \pm 1$, and $M \subset \mathbb{M}^{m+1}_p(\varepsilon)$ an r-lightlike hypersurface. If M is a totally geodesic lightlike hypersurface in $\mathbb{M}^{m+1}_p(\varepsilon)$, then, up to isometry of $\mathbb{M}^{m+1}_p(\varepsilon)$, M is an open subset of $N(\varepsilon)$.

Proof. Since lightlike hypersurfaces are coisotropic, by using Corollary 2.12 and 2.13 and elementary computations, we can show taking a frame field of the radical distribution $\mathrm{Rad}TM$ as a constant vector, so that M is contained in the complementary space of the space spanned by the constant vector. Thus, the proof is completed. □

Let (M, g) be an r-lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$. We call (M, g) *totally umbilical* if there exists $\mathcal{H} \in \mathrm{tr}(TM)$ such that

$$h(X,Y) = g(X,Y)\mathcal{H}$$

for all $X, Y \in \Gamma(TM)$. This notion is independent of the choice of transversally vector bundles as well as totally geodesic lightlike submanifolds (See [16]).

Here, we describe relations between lightlike submanifolds and Reinhart lightlike manifolds.

Proposition 2.19 ([5, Theorem 2.4, 2.8, Chapter 5])**.** Let (M, g) be an r-lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$. If (M, g) is totally geodesic, then it is Reinhart.

Here, an r-lightlike submanifold (M, g) is *Reinhart* in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$ if (M, g) is Reinhart itself.

Proposition 2.20 ([5, Corollary 2.2, 2.3, Chapter 5])**.** Let (M, g) be an r-lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$. If (M, g) is Reinhart, then any induced metrics on M are quasi Levi-Civita connections.

Proposition 2.21 ([5, Theorem 2.2, Chapter 5])**.** Let (M, g) be an r-lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$. The r-lightlike submanifold (M, g) is Reinhart if and only if the lightlike second fundamental form h^l vanishes identically on M.

Remark that there are infinitely many Reinhart r-lightlike surfaces which are not congruent to each other in four-dimensional Minkowski space $\mathbb{L}^4_1$ [5]. From Proposition 2.21 and Corollary 2.13, lightlike hypersurfaces are Reinhart if and only if they are totally geodesic. In the case of higher co-dimension, the converse is not true, i.e. there is an example which is Reinhart but not totally geodesic. Kupeli [27] pointed out that a manifold does not necessarily admit a metric with arbitrary signature (p, q, r). Namely, there exists a topological obstruction.

A Reinhart r-lightlike manifold (M, g) is *trivial* if $r = m$, i.e. $\mathrm{Rad}TM = TM$. Equivalently, the metric satisfies $g = 0$. In intrinsic geometry, trivial Reinhart lightlike manifolds have no information on metrics, however, in extrinsic geometry, there exist trivial Reinhart lightlike submanifolds in pseudo-Riemannian manifolds. Actually, if (M, g) is a isotropic, or totally lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$, then the lightlike second fundamental form h^l of M vanishes identically. Therefore, (M, g) is a trivial Reinhart lightlike manifold.

Corollary 2.22 ([5, Theorem 2.4, Chapter 5])**.** If (M, g) is a Reinhart r-lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$, then arbitrary induced connections of M are quasi Levi-Civita ones.

In summary, we can observe for r-lightlike submanifolds in a pseudo-Riemannian manifold. Let (M, g) be an r-lightlike submanifold in a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$, and ∇ an induced connection of M. The following are equivalent to each other:

(i) ∇ is metric parallel, i.e. it holds $\nabla g \equiv 0$.

(ii) For any $\xi \in \Gamma(\mathrm{Rad}TM)$, it holds $A^*_\xi \equiv 0$ on M.

(iii) The radical distribution RadTM is a Killing distribution.

(iv) The radical distribution RadTM is a parallel distribution with respect to ∇.

(v) The lightlike second fundamental form h^l of M vanishes identically on M.

As a conclusion to this chapter, we can summarize it as follows:

- An isotropic submanifold is trivial Reinhart.
- A totally lightlike submanifold is totally geodesic and trivial Reinhart.
- A totally geodesic lightlike submanifold is Reinhart.
- An r-lightlike curve, i.e. $m = 1$, is Reinhart and $r = 1$.
- An r-lightlike hypersurface, i.e. $n = 1$, is coisotropic and $r = 1$.
- For a coisotropic submanifold, to be totally geodesic if and only if to be Reinhart.
- There exist examples which are Reinhart but not totally geodesic in the case proper r-lightlike or isotropic submanifolds.
- There exist examples which are totally geodesic, but not trivial Reinhart lightlike submanifolds.

Remark 2.23. For non-flat pseudo-Riemannian space forms, the classification problem of totally geodesic submanifolds is open. In the case of Lorentzian space forms, there exists a classification theorem via Riemannian submersions (See [20]). In connection with totally geodesic lightlike submanifolds, the notion of minimal lightlike submanifolds is given by [39].

Part III

Main results

Chapter 3

Surface theory in a singular pseudo-Euclidean space

In this chapter, we consider a three-dimensional singular pseudo-Euclidean space with the signature $(0, 2, 1)$, whose canonical coordinates are (x, y, z), and study its surfaces.

Here, we remark how to use the terms. First, in the canonical three-dimensional Euclidean space $\mathbb{E}^3$, a surface whose mean curvature vanishes identically gives a stationary value for the volume functional. In a certain situation, its value is minimal, but not minimum in general. Historically, we call such surfaces *minimal*.

Next, in a three-dimensional Minkowski space $\mathbb{E}^3_1$, surfaces with vanishing mean curvature have different properties depending on the causal character of the induced metrics. When the induced metric is spacelike, i.e. Riemannian, we call such surfaces *maximal*. This means that, when we consider the volume functional analytically, such surfaces always give maximal values unlike the Euclidean case. On the other hand, when timelike, i.e. Lorentzian, we simply call such surfaces *minimal*. We should remark that timelike minimal surfaces give stationary values for the volume functional, but give neither minimal nor maximal values. We can refer these facts in Remark 32 and Theorem 37 of Chapter 6 of [3]. When connected surfaces have the part of spacelike maximal surfaces and that of timelike minimal surfaces, we call such surfaces *mixed type* [21].

In a four-dimensional Minkowski space $\mathbb{E}^4_1$, a surface whose mean curvature vector field vanishes identically is more complicated. Therefore, in order to treat uniformly, we call all such surfaces *zero mean curvature* when the ambient space is $\mathbb{E}^4_1$. This is why we have to pay attention to the terminology.

In Section 3.1, we establish settings to investigate surfaces in a singular pseudo-Euclidean space. We prepare the Gauss and Weingarten formulas of such surfaces.

In Section 3.2, we define non-degenerate surfaces in $\mathbb{E}^{0,2,1}$ and study their properties in detail. In addition, we calculate some examples.

In Section 3.3, we consider d-minimal surfaces which we define are analogue objects to classical minimal surfaces. They are called *isotropic minimal surfaces* in terms of simply isotropic geometry [38]. In addition, we show a representation formula of Weierstrass type for d-minimal surfaces (Theorem 3.15), and claim that d-minimal surfaces allow to have isolated singularities. Moreover, we see that spacelike flat zero mean curvature (ZMC) surfaces in $\mathbb{E}^4_1$ are contained in a three-dimensional subspace endowed with a degenerate induced metric (Theorem 3.24). In [4] and [30], some representation formulas are known. However, we should remark that singularities do not appear. Actually, since the regularity condition are assumed on surfaces, the possibility of singularities appearing is omitted in the obtained representation formula.

In Section 3.4, we give two applications. Firstly, we prove that d-minimal surfaces and spacelike flat ZMC surfaces in four-dimensional Minkowski space are in one-to-one correspondence (Corollary 3.26). In particular, we see that there exist infinitely many spacelike flat ZMC surfaces in $\mathbb{E}^4_1$ which are not congruent to each other. Secondly, we give a visualization of a deformation of zero mean curvature surfaces in a four-dimensional Minkowski space.

From Table 3.1, we see that d-minimal surfaces in $\mathbb{E}^{0,2,1}$ have intermediate properties between minimal surfaces in $\mathbb{E}^3$ and maximal surfaces in $\mathbb{E}^3_1$. Regarding singularities, they do not appear on minimal surfaces. However, on maximal surfaces, cuspidal edges, swallowtails and cuspidal crosscaps appear in generic case. Refer to [22] in detail. On the other hand, for d-minimal surfaces, isolated singularities are allowed. However, in this thesis, these singularities will be not classified. This chapter is based on [41].

3.1 Preparations in Chapter 3

Let M be a two-dimensional manifold, $f : M \to \mathbb{E}^{0,2,1}$ a C^∞-immersion, and g the induced metric by f. We assume that the metric g is positive definite, and we call f a *non-degenerate immersion* or a *non-degenerate surface*. Then, for each $x \in M$, a *normal vector space* $T^\perp_x M$ is defined by

$$T^\perp_x M := \{\xi \in \mathbb{R}^3 \mid (df_x(v), \xi) = 0, \text{ for all } v \in T_x M\} = \mathrm{span}_\mathbb{R}\{(0,0,1)\},$$

where $(\cdot,\cdot)$ is the degenerate inner product of $\mathbb{E}^{0,2,1}$ defined by (1.4). So, we have a vector bundle of rank one over M

$$T^\perp M := \bigcup_{x\in M} T^\perp_x M.$$

Therefore, we obtain an orthogonal direct sum decomposition

$$T_{f(x)}\mathbb{E}^{0,2,1} = T_x M \perp T^\perp_x M$$

for each $x \in M$. In particular, we see, as a vector bundle decomposition,

$$f^*T\mathbb{E}^{0,2,1} = TM \perp T^\perp M,$$

where TM is the tangent bundle over M and $f^*T\mathbb{R}^3$ is the pull-back bundle by f over M.

Proposition 3.1. We get an isomorphism as vector bundle

$$T^\perp M \cong M \times \mathbb{R}.$$

Proof. We can take $\xi = (0,0,1) \in \Gamma(T^\perp M)$ as a non-vanishing global section. So, the claim holds from the existence of the non-vanishing global section. □

Remark 3.2. For three-dimensional singular pseudo-Euclidean space with the signature (p,q,r), where $p+q+r=3,\ r\geq 1,\ p\leq q$, we can define non-degenerate surfaces when $r=1$, i.e.

$$(p,q,r) = (0,2,1),\ (1,1,1).$$

When $r \geq 2$, the metric induced on surfaces is degenerate. We remark that $\mathbb{E}^{1,1,1}$ is equivalent to the pseudo-isotropic 3-space $\mathbb{I}^3_1$ (Refer to [3, 44, 43]). As the notation, we define

$$|v| := \sqrt{(v,v)} = \sqrt{v_1^2 + v_2^2}$$

for a vector $v = (v_1, v_2, v_3) \in \mathbb{E}^{0,2,1}$.

On the other hand, how to control null vectors of $\mathbb{E}^{0,2,1}$ is untouched. Since every null vector is proportional to $\xi = (0,0,1)$, it is natural to introduce the *co-metric* $\langle\langle \cdot, \cdot \rangle\rangle$ on the set of null vectors as below

$$\langle\langle (0,0,\alpha), (0,0,\beta) \rangle\rangle := \alpha\beta \in \mathbb{R}.$$

In addition, $\mathbb{E}^{0,2,1}$ can lead to either doubly isotropic $\mathbb{I}^3_{(2)}$, Galilean $\mathbb{G}^3$, or pseudo-Galilean $\mathbb{G}^3_1$ geometries depending on how we deal with null vectors. For example, refer to [18]. On the other hand, in the case $r \geq 2$, this problem is no longer trivial.

Next, we recall affine differential geometry [31]. Let $(\mathbb{R}^{n+1}, d)$ be $(n+1)$-dimensional Euclidean space with the canonical connection d, and M an n-dimensional manifold. A C^∞-immersion $f : M \to \mathbb{R}^{n+1}$ is an *affine immersion* if for any $x \in M$ there exists a neighborhood U at x and a non-vanishing vector field ξ on U over $\mathbb{R}^{n+1}$ such that

$$T_{f(y)}\mathbb{R}^{n+1} = T_yM \oplus \mathbb{R}\xi_y \quad (y \in U),$$

where $\oplus$ stands for the direct sum. In particular, when there exists ξ globally on M, it is called a *transversally vector field* on M. Then, a torsion-free connection ∇ is induced on M, and it satisfies

$$d_XY = \nabla_XY + h(X,Y)\xi$$

for any $X, Y \in \Gamma(TM)$. This implies that h is a $(0,2)$-type symmetric tensor field over M, and we call h an *affine fundamental form* (with respect to ξ). In affine differential geometry, we often assume that h is non-degenerate. Moreover, let $f : M \to \mathbb{R}^{n+1}$ be an affine immersion, and let ξ be its transversally vector field. We call ξ *equiaffine* when, for all $X \in \Gamma(TM)$, it holds

$$d_X\xi \in \Gamma(TM).$$

Then, f is called an *equiaffine immersion*.

In terms of affine differential geometry, we see the following proposition.

Proposition 3.3. Let M be a two-dimensional manifold. A non-degenerate immersion $f : M \to \mathbb{E}^{0,2,1}$ is an equiaffine immersion whose transversally vector field over M is $\xi \equiv (0,0,1)$.

Proof. By using the orthogonal direct sum $f^*T\mathbb{R}^3 = TM \perp T^\perp M$, we can show $d_X\xi = 0$ for all $X \in \Gamma(TM)$. Thus, the proof is completed. □

Hereinafter, let ξ be the constant vector field $\xi = (0,0,1)$, and let d be the canonical connection as a linear connection, i.e. for all $X, Y \in \Gamma(T\mathbb{E}^{0,2,1})$, identifying Y with the vector-valued function $Y = (Y_1, Y_2, Y_3)$,

$$d_XY := dX(Y) = (X(Y_1), X(Y_2), X(Y_3)).$$

The connection d is torsion-free and preserves the degenerate metric $(\cdot\,,\cdot\,)$. Thus, the connection d plays the role of the Levi-Civita connection.

We define the automorphism group $\mathrm{Aut}(\mathbb{E}^{0,2,1}, d)$ with respect to $\mathbb{E}^{0,2,1}$ and d as

$$\begin{aligned} \mathrm{Aut}(\mathbb{E}^{0,2,1}, d) &:= \{A \in \mathrm{Diff}(\mathbb{R}^3) \mid A^*d = d,\ A^*(\cdot,\cdot) = (\cdot,\cdot)\} \\ &= O(0,2,1) \ltimes \mathbb{R}^3, \end{aligned}$$

where $\mathrm{Diff}(\mathbb{R}^3)$ is the diffeomorphism group of $\mathbb{R}^3$ and

$$O(0,2,1) := \left\{ \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{T}} & 0 \\ & & 0 \\ a & b & c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R},\ c \neq 0,\ T \in O(2) \right\}.$$

We call $\mathrm{Aut}(\mathbb{E}^{0,2,1}, d)$ an *affine isometry group*. In particular, $\mathrm{Aut}(\mathbb{E}^{0,2,1}, d)$ is a seven-dimensional Lie group. From the view of Cayley–Klein geometry, this automorphism group is nothing but the simply isotropic rigid motion group [44]. Da Silva studied invariant surfaces generated by subgroups of $O(0, 2, 1)$ [45].

By using the decomposition $f^*T\mathbb{R}^3 = TM \perp T^\perp M$, for each $X, Y \in \Gamma(TM)$, $\alpha\xi \in \Gamma(T^\perp M)$ ($\alpha \in C^\infty(M)$), we have

$$\begin{aligned} d_X Y &= \nabla_X Y + h(X, Y)\xi, \\ d_X(\alpha\xi) &= X(\alpha)\xi. \end{aligned}$$

Then, we see that the connection ∇ is the Levi-Civita connection with respect to the induced metric g on M. We call the given affine fundamental form h a *second fundamental form* of the non-degenerate immersion f.

For all $X, Y, Z \in \Gamma(TM)$, since the connection d is flat, we obtain

$$0 = {}^dR(X, Y)Z = {}^\nabla R(X, Y)Z + \{(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z)\}\xi,$$

where ${}^dR$ and ${}^\nabla R$ are the curvature tensor fields for d and ∇, respectively, and we define $(\nabla_X h)(Y, Z) := X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$. Therefore, we get

$$\begin{aligned} {}^\nabla R &\equiv 0, && (3.1) \\ (\nabla_X h)(Y, Z) &= (\nabla_Y h)(X, Z). && (3.2) \end{aligned}$$

The formula (3.1) implies that the non-degenerate surface is always flat, and we call the formula (3.2) *Gauss-Codazzi equation* of the non-degenerate surface. These formulas (3.1) and (3.2) were obtained by Sachs in [38].

Let $f : M \to \mathbb{E}^{0,2,1}$ be a non-degenerate immersion. The image of f is locally expressed by the form of a graph surface $\{(u, v, F(u, v)) \in \mathbb{E}^{0,2,1} \mid (u, v) \in U\}$, where F is a smooth function on an open subset $U \subset \mathbb{R}^2$. So, we call $U\mathbb{R}^2$ a *flat coordinate neighborhood*. In addition, let (M^2, g) be a two-dimensional Riemannian manifold. It is well-known that, for each $x \in M^2$, there exists a coordinate neighborhood $\{(x_1, x_2)\}$ at x such that

$$g_{11} = g_{22} > 0, \quad g_{12} = 0,$$

where

$$g_{ij} := g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Such coordinate neighborhoods are called *isothermal*. In particular, flat coordinates neighborhoods are isothermal.

3.2 Properties of d-minimal surfaces

We define the *mean curvature* $\mathcal{H}$ of a non-degenerate surface as follows

$$\mathcal{H} := \frac{1}{2}\mathrm{trace}_g h = \frac{1}{2} g^{ij} h_{ij},$$

where g^{ij} is the components of the inverse matrix of $(g_{ij})_{1 \le i,j \le 2}$ and h_{ij} are the coefficients of the second fundamental form h. So, we define some classes of non-degenerate surfaces. Namely,

(i) *d-totally geodesic surface* if the second fundamental form satisfies $h \equiv 0$,

(ii) *d-totally umbilical surface* if there exists $\lambda \in C^\infty(M)$ such that $h = \lambda g$,

(iii) *d-minimal surface* if the mean curvature satisfies $\mathcal{H} = 0$.

For (ii), we remark that (ii) is equivalent to (i) when $\lambda = 0$.

Proposition 3.4. If M is a two-dimensional manifold, and $f : M \to \mathbb{E}^{0,2,1}$ is connected, not d-totally geodesic and d-totally umbilical surface, that is, there exists a function $\lambda \in C^\infty(M)$ such that $h = \lambda g$ and $\lambda \neq 0$, then λ is a constant function, and the image of f is an open subset of a paraboloid of revolution

$$\left\{ \left(u, v, \frac{\lambda}{2}(u^2 + v^2) + Au + Bv + C \right) \in \mathbb{R}^3 \;\middle|\; (u, v) \in \mathbb{R}^2 \right\},$$

where $A, B, C \in \mathbb{R}$ are constants. In particular, it is, up to affine isometry, an open subset of

$$\left\{ (u, v, u^2 + v^2) \in \mathbb{R}^3 \;\middle|\; (u, v) \in \mathbb{R}^2 \right\}.$$

Proof. Since non-degenerate surfaces satisfy Gauss-codazzi equation (3.2), the function λ is a constant. Let g be the induced metric by f, and h its second fundamental form. From the assumption, there exists a non-zero constant number $\lambda \in \mathbb{R}$ such that $h = \lambda g$. Since f is the non-degenerate immersion, for each point of M, there exists a coordinate neighborhood $\{U; (u, v)\}$ such that

$$f(u, v) = (u, v, \varphi(u, v)) \in \mathbb{E}^{0,2,1},$$

where φ is a C^∞-function on U. Then, we get

$$h_{11} = \varphi_{uu}, \;\; h_{12} = \varphi_{uv}, \;\; h_{22} = \varphi_{vv}.$$

Therefore, since we have

$$\varphi_{uu} = \lambda g_{11} = \lambda, \;\; \varphi_{uv} = \lambda g_{12} = 0, \;\; \varphi_{vv} = \lambda g_{22} = \lambda,$$

there exist constant numbers $A, B, C \in \mathbb{R}$ such that

$$\varphi(u, v) = \frac{\lambda}{2}(u^2 + v^2) + Au + Bv + C.$$

Finally, gluing these pieces of surface in the whole of M, we obtain the consequence. □

In the context of isotropic geometry, d-totally umbilical surfaces are known as spheres of parabolic type. See [43] in detail.

Here, we define a *relative Gaussian curvature* $\mathcal{K}$ which is introduced in [38] as

$$\mathcal{K} := \frac{\det h}{\det g} \in C^\infty(M).$$

This quantity expresses the shape of the non-degenerate surface when we look from the ambient space $\mathbb{E}^3$. However, the canonical Gaussian curvature, i.e. the sectional curvature of two-dimensional Riemannian manifolds with respect to the induced metric, identically vanishes.

Proposition 3.5 ([38, Definition 8.11])**.** Let M be a two-dimensional manifold, and $f : M \to \mathbb{E}^{0,2,1}$ a non-degenerate immersion. Moreover, let $\mathcal{K}$ be its relative Gaussian curvature. In $\mathbb{E}^{0,2,1}$, we define for each $x \in M$,

$$\begin{aligned} x &: \textit{elliptic point} \quad \text{if} \quad \mathcal{K}(x) > 0, \\ x &: \textit{hyperbolic point} \quad \text{if} \quad \mathcal{K}(x) < 0, \\ x &: \textit{parabolic point} \quad \text{if} \quad \mathcal{K}(x) = 0. \end{aligned}$$

If we consider f as an immersion to Euclidean space $\mathbb{E}^3$, then the Euclidean Gaussian curvature does not correspond to the relative Gaussian curvature in general, however the two curvatures have the same sign.

Proof. Since f is a non-degenerate immersion, for each point of M, there exists a coordinate neighborhood $\{U; (u, v)\}$ such that

$$f(u, v) = (u, v, \varphi(u, v)) \in \mathbb{E}^{0,2,1},$$

where φ is a C^∞-function on U. When we consider f as an immersion to $\mathbb{E}^3$, the Euclidean Gaussian curvature K_G is expressed by

$$K_G = \frac{\varphi_{uu}\varphi_{vv} - \varphi_{uv}^2}{(1 + \varphi_u^2 + \varphi_v^2)^2}$$

on U. On the other hand, the relative Gaussian curvature $\mathcal{K}$ is expressed by

$$\mathcal{K} = \varphi_{uu}\varphi_{vv} - \varphi_{uv}^2$$

on U. Therefore, K_G does not correspond to $\mathcal{K}$ in general, but the signs are the same. □

The notion of elliptic, hyperbolic and parabolic points in Euclidean ($\mathbb{E}^3$) and singular pseudo-Euclidean ($\mathbb{E}^{0,2,1}$) geometry are equivalent.

Remark 3.6. We consider the sign of the relative Gaussian curvature for some surfaces. First, for d-totally geodesic surfaces, since we have $h = 0$ by definition, it holds

$$\mathcal{K} = \frac{\det h}{\det g} \equiv 0.$$

Next, for d-totally umbilical surfaces, we have, by definition and Proposition 3.4, there exists a constant number $\lambda \in \mathbb{R}$ such that $h = \lambda g$. We assume $\lambda \neq 0$. Then, we obtain

$$\mathcal{K} = \frac{\det h}{\det g} = \frac{\lambda^2 \det g}{\det g} = \lambda^2 > 0,$$

that is, all points are elliptic. Finally, for d-minimal surfaces, we make use of isothermal coordinates, that is, we choose the coordinates in which the coefficients of the induced metric hold

$$g_{11} = g_{22} > 0, \quad g_{12} = 0.$$

Then, since the mean curvature identically vanishes, we have

$$2\mathcal{H} = \mathrm{trace}_g h = \frac{g_{22}h_{11} + g_{11}h_{22}}{g_{11}g_{22}} = \frac{h_{11} + h_{22}}{g_{11}} \equiv 0.$$

Moreover, by using $h_{22} = -h_{11}$, we obtain

$$\mathcal{K} = \frac{\det h}{\det g} = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22}} = -\frac{h_{11}^2 + h_{12}^2}{g_{11}^2} \leq 0,$$

that is, almost all points are hyperbolic. Actually, we immediately see that $h = 0$ if and only if $\mathcal{K} = 0$. From Theorem 3.24, umbilic points are isolated.

Here, we give some descriptions for curves in $\mathbb{E}^{0,2,1}$. For a connected open interval $I \subset \mathbb{R}$, let c be a C^∞-map $c : I \to \mathbb{E}^{0,2,1}$. We call c a *curve* in $\mathbb{E}^{0,2,1}$. Moreover, we call c a *regular* curve if it holds, for all $t \in I$,

$$c'(t) \neq 0.$$

We call a parameter s of a curve $c = c(s)$ *arc-length* if it holds

$$|c'(s)| \equiv 1.$$

Next, let π be the projection to xy-plane, i.e.

$$\pi : \mathbb{E}^{0,2,1} \ni (x, y, z) \mapsto (x, y) \in \mathbb{R}^2.$$

From the view of isotropic geometry, π is said to be the *top view* of (x, y, z) [44], [43]. Then, a direct calculation provides

Proposition 3.7. Let $c = c(t)$ $(t \in I)$ be a regular curve in $\mathbb{E}^{0,2,1}$. The following are equivalent to each other:

(i) The curve $c = c(t)$ admits an arc-length parameter.

(ii) For all $t \in I$, it holds $|c'(t)| > 0$.

(iii) The mapping $\pi \circ c$ is regular as a planar curve in $\mathbb{E}^2$.

Let $c = c(t)$ be a regular curve in $\mathbb{E}^{0,2,1}$. We call c *null* if it holds

$$|c'(t)| \equiv 0.$$

Moreover, we call c a *spacial line* if the image is a line segment in $\mathbb{R}^3$.

Proposition 3.8. A regular curve $c : I \to \mathbb{E}^{0,2,1}$ is null if and only if it is a spacial line which is parallel with the z-axis.

Proof. The claim is proved by easy calculations. □

Proposition 3.9 ([38, Theorem 9.4 and Equation (9.31)])**.** For any connected surfaces in $\mathbb{E}^{0,2,1}$,

(0) d-totally geodesic surfaces in $\mathbb{E}^{0,2,1}$ are open subsets of non-degenerate planes.

(1) a graph surface in $\mathbb{E}^{0,2,1}$

$$\{(u, v, f(u, v)) \in \mathbb{E}^{0,2,1} \mid (u, v) \in U \subset \mathbb{R}^2\}$$

is d-minimal if and only if f is a harmonic function on U.

(2) non-planar, ruled d-minimal surfaces in $\mathbb{E}^{0,2,1}$ are locally, up to affine isometry, open subset of one of

(a) $f(u, v) = (v \cos u, v \sin u, u)$ (refer to Figure 3.1),

(b) $f(u, v) = (u, v, uv)$ (refer to Figure 3.1),

where $(u, v) \in \mathbb{R}^2$ (See [40, Theorem 6]).

(3) non-planar, d-minimal rotational surfaces in $\mathbb{E}^{0,2,1}$ are locally, up to affine isometry, open subset of one of

$$f(u,v) = (e^u \cos v, e^u \sin v, u)$$

(refer to Figure 3.1), where rotational surfaces are the surfaces invariant by the group of rotations around the z-axis, which acts on the xy-plane as Euclidean rotations, i.e. they are $SO(2)$-invariant surfaces.

Proof. (0) and (1) are proved by easy calculations. In case of (2), we apply the method of classification described by [40] since $\mathbb{E}^{0,2,1}$ is isometrically embedded in $\mathbb{E}^4_1$ as a *totally geodesic lightlike submanifold* by the natural way [5]. In fact, the following mapping

$$\iota : \mathbb{E}^{0,2,1} \ni (x,y,z) \mapsto (x,y,z,z) \in \mathbb{E}^4_1 := (\mathbb{R}^4, dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2) \tag{3.3}$$

is an isometric embedding. We should remark that causal characters in $\mathbb{E}^{0,2,1}$ are ones in $\mathbb{E}^4_1$ because of Eq. (3.3). From the classification of Theorem 6 in [40], non-planar ruled minimal surfaces in the degenerate subspace $\iota(\mathbb{E}^{0,2,1}) \subset \mathbb{E}^4_1$ are locally contained in one of the following:

(a) *An elliptic helicoid of the second kind*

$$f(s,t) = (\cos se_1 + \sin se_2)t + se_3,$$

where $e_1 = (1,0,0,0), e_2 = (0,1,0,0)$ and $e_3 = (0,0,1,1)$.

(b) *A minimal hyperbolic paraboloid*

$$f(s,t) = ste_1 + se_2 + te_3,$$

where $e_1 = (0,0,1,1), e_2 = (1,0,0,0)$ and $e_3 = (0,1,0,0)$.

These lead to the consequence of the case (2).

In case of (3), we explain the meaning of $SO(2)$-invariant firstly. It is well-known that

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; \theta \in \mathbb{R} \right\}.$$

We realize $SO(2)$ as a subgroup of $\mathrm{Aut}(\mathbb{E}^{0,2,1}, d)$ as below.

$$H := \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Aut}(\mathbb{E}^{0,2,1}, d) \;\middle|\; \theta \in \mathbb{R} \right\}.$$

Then, the group H is isomorphic to $SO(2)$ as a Lie group. We simply denote H as $SO(2)$. A surface in $\mathbb{E}^{0,2,1}$ is said to be $SO(2)$-invariant if it is invariant under the action of this group. Such surfaces are locally parametrized by

$$f(u,v) = (x(u)\cos v, x(u)\sin v, y(u)) \in \mathbb{E}^{0,2,1},$$

where x, y are real variable functions satisfying $x > 0,\ (x')^2 + (y')^2 = 1$. Then, we have

$$f_u = (x'\cos v, x'\sin v, y'), \quad f_v = (-x\sin v, x\cos v, 0).$$

Thus, we compute

$$g_{11} = (x')^2, \quad g_{12} = 0, \quad g_{22} = x^2.$$

The non-degeneracy implies $x' \neq 0$. Moreover, since $\xi = (0,0,1)$ and we compute

$$\begin{aligned} f_{uu} &= (x'' \cos v, x'' \sin v, y'') = \frac{x''}{x'} f_u + \left(-\frac{x''}{x'} y' + y'' \right) \xi, \\ f_{uv} &= (-x' \sin v, x' \cos v, 0) = \frac{x'}{x} f_v, \\ f_{vv} &= (-x \cos v, -x \sin v, 0) = -\frac{x}{x'} f_u + \frac{x}{x'} y' \xi, \end{aligned}$$

the coefficients of second fundamental form h hold

$$h_{11} = -\frac{x''}{x'} y' + y'', \quad h_{12} = 0, \quad h_{22} = \frac{x}{x'} y'.$$

Therefore, we compute that the mean curvature of $SO(2)$-invariant d-minimal surfaces is

$$2\mathcal{H} = g^{ij} h_{ij} = \frac{1}{(x')^3} (-x'' y' + x' y'') + \frac{y'}{x x'} \equiv 0. \tag{3.4}$$

Since $x' \neq 0$, by the coordinate transformation, we can represent y as a function with respect to x. Then, the equation (3.4) is equal to the following equation

$$\frac{d^2 y}{dx^2} = -\frac{1}{x} \frac{dy}{dx}.$$

By solving the ordinary differential equation, we have

$$y(x) = C_1 \log x + C_2 \quad (C_1, C_2 \in \mathbb{R} : \text{constants}).$$

Again, when we replace the parameter x with $x(w) = e^w$, we get $y(w) = C_1 w + C_2$. In particular, if $C_1 = 0$, then it is a plane. So, if it is not a plane, by an affine isometry, we obtain

$$f(u, v) = (e^u \cos v, e^u \sin v, u).$$

The proof is completed. □

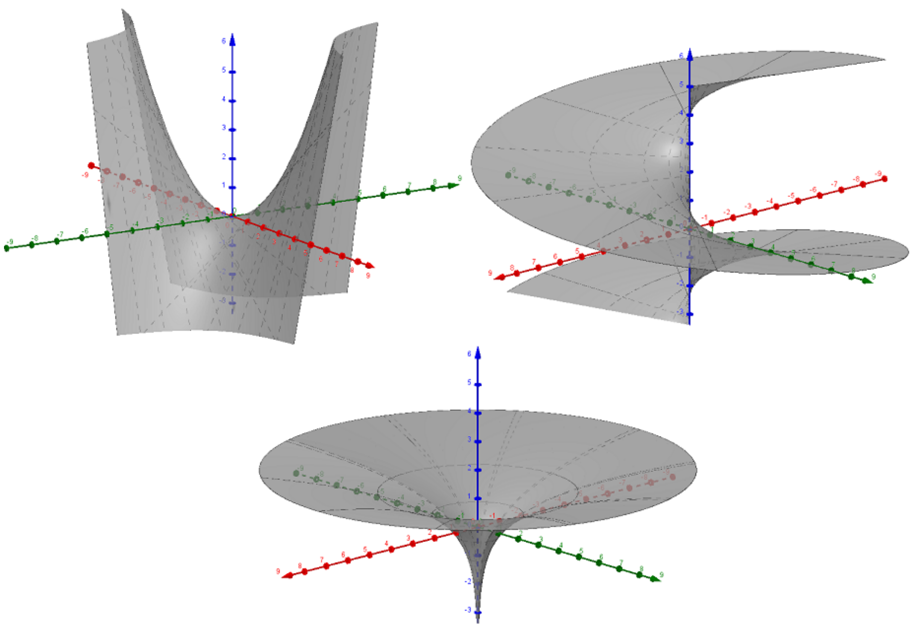

Figure 3.1: Upper-left: the minimal hyperbolic paraboloid. Upper-right: the elliptic helicoid of the second kind. Lower-middle: the d-minimal rotational surface.

Regarding the surfaces (b) in Proposition 3.9 (2), it is also known as a warped translation surface with a generating curve. Regarding Proposition 3.9 (3), the obtained revolution surfaces are special instances of invariant surfaces. Da Silva classified all invariant minimal simply isotropic surfaces, that is, invariant d-minimal surfaces. See [45] in detail.

Remark 3.10. We recall that non-degenerate surfaces are locally expressed by graph surfaces. However, (a) of Proposition 3.9 (2) is an example which can not be entirely expressed as a graph surface.

We consider the canonical connection d as a linear connection for $\mathbb{E}^{0,2,1}$. This connection d is a torsion-free connection which is parallel with respect to the degenerate metric $(\cdot,\cdot)$, i.e. d plays the role of Levi-Civita connection. However, since the metric is degenerate, connections having such properties are not unique. Vogel [51] characterized linear connections compatible with a degenerate metric. For example, let $\lambda \in \mathbb{R}$ be a real parameter, and we define a tensor field $L_\lambda \in \Gamma(S^2T^*\mathbb{R}^3)$ as

$$L_\lambda(X,Y) := \lambda \sum_{i,j} X_i Y_j,$$

where the set $\Gamma(S^2T^*\mathbb{R}^3)$ expresses the whole of $(0,2)$-type symmetric tensor fields over $\mathbb{R}^3$ and X, Y are vector fields over $\mathbb{R}^3$, and we regard X and Y, respectively as vector-valued functions

$$X = (X_1, X_2, X_3), \quad Y = (Y_1, Y_2, Y_3).$$

When we put $d^\lambda := d + L_\lambda \xi$, d^λ is a flat connection over $\mathbb{E}^{0,2,1}$ which has the same properties of Levi-Civita connections. In particular, when $\lambda = 0$, d^λ coincides with the canonical connection d. $(\mathbb{E}^{0,2,1}, d)$ is a geodesically complete Reinhart 1-lightlike manifold. However, $(\mathbb{E}^{0,2,1}, d^\lambda)$ is not geodesically com plete, but Reinhart 1-lightlike if $\lambda \neq 0$. Actually, in $(\mathbb{E}^{0,2,1}, d^\lambda)$ we calculate the geodesic $\gamma(t)$ with the initial data $\gamma(0) = (0,0,0), \gamma'(0) = (1,0,0)$ as

$$\gamma(t) = \left(t, 0, \frac{1}{\lambda}\log|\lambda t + 1| - t\right).$$

Namely, the domain of parameters of the curve γ is not defined in the whole of real numbers $\mathbb{R}$. For $\mathbb{E}^{0,2,1}$, it would be interesting to consider the geometric meaning of torsion-free, metric connections. In [43], the issue is taken into account by Da Silva as well. For example, we can raise the problem of whether a complete connection is limited to d.

3.3 Representation formula of Weierstrass type for d-minimal surfaces

Let $f : M \to \mathbb{E}^{0,2,1}$ be a non-degenerate immersion. When we set $f = (f_1, f_2, f_3)$, we define the Laplacian $\Delta_g f$ of f with respect to the induced metric g as the Laplacians of each coordinate functions f_i $(i = 1, 2, 3)$, i.e.

$$\Delta_g f := (\Delta_g f_1, \Delta_g f_2, \Delta_g f_3).$$

Proposition 3.11. If $\mathcal{H}$ is the mean curvature of a non-degenerate immersion f, then $2\mathcal{H}\xi \in \Gamma(T^\perp M)$ is equal to the Laplacian $\Delta_g f$ of f with respect to the induced metric g. In particular, the non-degenerate surface is d-minimal if and only if all coordinate functions of f are harmonic with respect to g.

Proof. Since f is non-degenerate, with respect to a flat coordinate neighborhood $U \subset \mathbb{R}^2$, f is locally expressed by

$$f(u,v) = (u, v, F(u,v)) \in \mathbb{E}^{0,2,1},$$

where F is a function on U. Thus, we have

$$2\mathcal{H}\xi = (0, 0, F_{uu} + F_{vv}) = \Delta_g f.$$

The proof is completed. □

In case of graph surfaces, Proposition 3.11 is equivalent to the formula (8) in [36].
Next, we prepare some simple lemmas.

Lemma 3.12. For a real two variable function $f(u, v)$, we define a complex function $F(w)$ with respect to the complex variable $w = u + iv$ as

$$F(w) := \frac{\partial f}{\partial u}(u, v) - i\frac{\partial f}{\partial v}(u, v).$$

Then, F is a holomorphic function if and only if $f(u, v)$ is a harmonic function.

Proof. By using Cauchy–Riemann's equations, the claim holds. □

Lemma 3.13. In $\mathbb{E}^{0,2,1}$, we consider a surface determined by

$$f(u, v) := (x(u, v), y(u, v), z(u, v)) \in \mathbb{E}^{0,2,1} \quad ((u, v) \in U)\,,$$

where U is a open subset in $\mathbb{R}^2$. We define complex functions φ, ψ with respect to the complex variable $w = u + iv$ as

$$\varphi(w) := \frac{\partial x}{\partial u}(u, v) - i\frac{\partial x}{\partial v}(u, v), \quad \psi(w) := \frac{\partial y}{\partial u}(u, v) - i\frac{\partial y}{\partial v}(u, v).$$

Then, the coordinates (u, v) are isothermal if and only if it holds

$$\varphi^2 + \psi^2 \equiv 0.$$

Proof. By direct calculations, we have

$$\varphi^2 + \psi^2 \;\; = \;\; |f_u|^2 - |f_v|^2 - 2i(f_u, f_v).$$

This completes the proof. □

See Remark 3.2 in this thesis for the definition of the norm $|\cdot|$.

Theorem 3.14. Let U be an open subset of uv-plane. In $\mathbb{E}^{0,2,1}$, let f be an immersion on U which is parametrized by $f(u, v) = (x(u, v), y(u, v), z(u, v))$. We assume that (u, v) are isothermal coordinates and f is d-minimal. Then, complex functions $\varphi_1, \varphi_2, \varphi_3$ with respect to the complex variable $w = u + iv$ defined by

$$\varphi_1(w) = \frac{\partial x}{\partial u} - i\frac{\partial x}{\partial v}, \quad \varphi_2(w) = \frac{\partial y}{\partial u} - i\frac{\partial y}{\partial v}, \quad \varphi_3(w) = \frac{\partial z}{\partial u} - i\frac{\partial z}{\partial v} \tag{3.5}$$

are all holomorphic, and it holds

$$|\varphi_1|^2 + |\varphi_2|^2 > 0, \quad \varphi_1^2 + \varphi_2^2 = 0. \tag{3.6}$$

Moreover, it holds

$$(f_u, f_u) = (f_v, f_v) = \frac{1}{2}(|\varphi_1|^2 + |\varphi_2|^2).$$

Conversely, let U be a simply-connected domain on $\mathbb{C}$, and we assume that holomorphic functions $\varphi_1(w), \varphi_2(w), \varphi_3(w)$ satisfy the formula (3.6). When we set $w = u + iv \in U$, there exists a d-minimal surface satisfying the formula (3.5) such that, for the parametrized expression

$$\begin{aligned} f(u,v) &= (x(u,v), y(u,v), z(u,v)) \\ &= \left(\mathrm{Re}\int_w \varphi_1(w)dw, \mathrm{Re}\int_w \varphi_2(w)dw, \mathrm{Re}\int_w \varphi_3(w)dw\right) \end{aligned}$$

the coordinates (u,v) are isothermal.

Proof. Since f is d-minimal, each coordinate function is harmonic from Proposition 3.11. Thus, by using Lemma 3.12, each φ_i is holomorphic. Since (u,v) are isothermal coordinates, it holds $\varphi_1^2 + \varphi_2^2 \equiv 0$ from Lemma 3.13. Next, we compute

$$|\varphi_1|^2 + |\varphi_2|^2 = x_u^2 + y_u^2 + x_v^2 + y_v^2 = |f_u|^2 + |f_v|^2 = 2|f_u|^2 = 2|f_v|^2.$$

Since f is an immersion, we have that $|f_u|^2 = |f_v|^2 \neq 0$. Thus, the former of the claim holds. For the latter, we assume that holomorphic functions $\varphi_1, \varphi_2, \varphi_3$ on a simply-connected domain U satisfy the formula (3.6). We fix a point $w_0 \in U$ and define a real function $x = x(u,v)$ as

$$x(u,v) := \mathrm{Re}\int_{w_0}^{w} \varphi_1(w)dw \quad (w = u + iv \in U).$$

This is well-defined since U is simply-connected. When we act on this relation by the differential operator

$$\frac{\partial}{\partial u} - i\frac{\partial}{\partial v} = 2\frac{\partial}{\partial w},$$

we have

$$\frac{\partial x}{\partial u} - i\frac{\partial x}{\partial v} = 2\frac{\partial}{\partial w}\mathrm{Re}\int_{w_0}^{w} \varphi_1(w)dw = \varphi_1(w).$$

As above, when we define $y = y(u,v)$ and $z = z(u,v)$, we have

$$\frac{\partial y}{\partial u} - i\frac{\partial y}{\partial v} = \varphi_2(w), \quad \frac{\partial z}{\partial u} - i\frac{\partial z}{\partial v} = \varphi_3(w).$$

From Lemma 3.12 again, we see that $x(u,v), y(u,v), z(u,v)$ are harmonic functions on U. Next, we prove that the mapping $f(u,v) = (x(u,v), y(u,v), z(u,v))$ gives a surface, i.e. a two-dimensional manifold. For the purpose of that, we prove that the Jacobi matrix

$$\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}$$

has rank two for any point $w \in U$. We prove by using contradiction, i.e. we assume that there is a point $w' \in U$ such that the rank of its Jacobi matrix is less than two. Since we have

$$0 < |\varphi_1|^2 + |\varphi_2|^2 = (x_u)^2 + (x_v)^2 + (y_u)^2 + (y_v)^2,$$

at the point w', we see that either of column vectors

$$\begin{pmatrix} x_u \\ x_v \end{pmatrix}, \begin{pmatrix} y_u \\ y_v \end{pmatrix}$$

is not the zero vector. So, we suppose that the former is not the zero vector. Since we can set that there exists $\lambda \in \mathbb{R}$ such that

$$\begin{pmatrix} y_u \\ y_v \end{pmatrix} = \lambda \begin{pmatrix} x_u \\ x_v \end{pmatrix},$$

we compute, by using $\varphi_2 = \lambda\varphi_1$,

$$\{\varphi_1(w')\}^2 + \{\varphi_2(w')\}^2 = (1+\lambda^2)\{\varphi_1(w')\}^2 \neq 0$$

at w'. This contradicts the formula (3.6). Thus, since f is a C^∞-immersion, $f(u,v) = (x(u,v), y(u,v), z(u,v))$ gives a surface in $\mathbb{E}^{0,2,1}$, and $(u,v) \in U$ are isothermal coordinates from the condition (3.6). In particular, f is a d-minimal surface satisfying the formula (3.5). □

In Theorem 3.14, the function φ_3 seems not to play any role in the characterization of isothermal coordinates because of Eq. (3.6). Does this *apparent independence* bring any kind of symmetry or freedom to construct d-minimal surfaces? It would be interesting to research the geometrical interpretation of such a symmetry, or freedom.

Corollary 3.15 (Weierstrass-type representation formula for d-minimal surfaces)**.** Let $U \subset \mathbb{C}$ be a simply-connected domain. If F, G are holomorphic and meromorphic functions on U, respectively such that F does not have zero points on U and FG is a holomorphic function on U, then a mapping

$$f(u,v) = \mathrm{Re}\int_w (F, iF, 2FG)dw \quad (w := u+iv \in U)$$

gives a d-minimal surface in $\mathbb{E}^{0,2,1}$, and the coordinates $(u,v) \in U$ are isothermal. Moreover, it holds

$$(f_u, f_u) = (f_v, f_v) = |F|^2.$$

Conversely, a d-minimal surface in $\mathbb{E}^{0,2,1}$ locally has the expression as above.

Proof. For the former of the claims, when we set $\varphi_1 := F, \varphi_2 := iF, \varphi_3 := 2FG$, it immediately holds from Theorem 3.14. For the latter of the claims, given a d-minimal surface, it is locally considered on a simply-connected domain. From Theorem 3.14 again, we have the parametrized expression

$$f(u,v) = \mathrm{Re}\int(\varphi_1, \varphi_2, \varphi_3)dw.$$

Since it satisfies

$$|\varphi_1|^2 + |\varphi_2|^2 > 0, \ \varphi_1^2 + \varphi_2^2 = 0,$$

setting $F := \varphi_1, G := \dfrac{\varphi_3}{2F}$, we obtain the expression which we want. □

Regarding the function F in Corollary 3.15, zero points of F correspond to singular points of d-minimal surfaces. For example, there exist cross-caps on d-minimal surfaces. We remark that there exist singularities of other types besides cross-caps. Here, we recall the definition of singular points. Let M, N be manifolds, and f an immersion from M into N. A point $x \in M$ is a *singular* of f if the differential map df_x is not injective. For a d-minimal surface $f : M \to \mathbb{E}^{0,2,1}$ and a point $x \in M$, we see that F has a zero point at x if and only if f has a singular point at x by easy calculation. We describe other types in the next section.

At the end of this section, for Weierstrass type expression formula for d-minimal surfaces

$$f(u,v) = \mathrm{Re}\int_w (F, iF, 2FG)dw \quad (w := u+iv \in U),$$

the function F expresses the induced metric g, i.e. it holds

$$g = |F|^2(du^2 + dv^2).$$

On the other hand, the function G is concerned with the second fundamental form h by the following proposition.

Proposition 3.16 ([46, Lemma 1])**.** Under the situation stated above, it holds

$$h = 2\mathrm{Re}(FG')(du^2 - dv^2) - 4\mathrm{Im}(FG')dudv,$$

where f' represents the derivative of a holomorphic function f.

Proof. By direct calculations, we have

$$f_u = \mathrm{Re}(F, iF, 2FG), \quad f_v = -\mathrm{Im}(F, iF, 2FG)$$

and

$$f_{uu} = \mathrm{Re}(F', iF', 2F'G'),\ f_{uv} = -\mathrm{Im}(F', iF', 2F'G'),\ f_{vv} = -\mathrm{Re}(F', iF', 2F'G').$$

Then, the coefficients of the second fundamental form are

$$h_{11} = 2\mathrm{Re}(FG'),\ h_{12} = -4\mathrm{Im}(FG'),\ h_{22} = -2\mathrm{Re}(FG').$$

Thus, the proof is completed. □

Remark 3.17. The pair (F, G) is called a *Weierstrass data*. For any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$,

$$f_\theta(s,t) = \cos\theta \left(\mathrm{Re} \int (F, iF, 2FG)dw\right) + \sin\theta \left(\mathrm{Im} \int (F, iF, 2FG)dw\right) \tag{3.7}$$

is a d-minimal surface in $\mathbb{E}^{0,2,1}$. As a remark, it follows

$$\mathrm{Re} \int_{w_0}^{w} (-iF, F, -2iFG)dw = \mathrm{Im} \int_{w_0}^{w} (F, iF, 2FG)dw.$$

Thus, d-minimal surfaces defined by the Weierstrass data $(-iF, -iG)$ corresponds to the imaginary part of the formulas defined by the Weierstrass data (F, G). For $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, when we consider the d-minimal surface whose Weierstrass data is $(e^{-i\theta}F, e^{-i\theta}G)$, the given immersion is called an *associated family* and, when we denote f_θ, we have the S^1-family of mappings. Moreover, we see that

$$\begin{aligned} f_\theta(u,v) &= \mathrm{Re} \int_{w_0}^{w} (e^{-i\theta}F, ie^{-i\theta}F, 2e^{-i\theta}FG)dw \\ &= \cos\theta \left(\mathrm{Re} \int_{w_0}^{w} (F, iF, 2FG)dw\right) + \sin\theta \left(\mathrm{Im} \int_{w_0}^{w} (F, iF, 2FG)dw\right). \end{aligned}$$

In particular, when $\theta = 0, \pi/2$, they correspond to the d-minimal surfaces given by the real part and imaginary part from (F, G), respectively. Moreover, for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, since the induced metric of f_θ satisfies

$$((f_\theta)_u, (f_\theta)_u) = ((f_\theta)_v, (f_\theta)_v) = |e^{-i\theta}F|^2 = |F|^2,\ ((f_\theta)_u, (f_\theta)_v) = 0,$$

the mapping (3.7) gives an isometric deformation between $f = f_0$ and f_θ. We call $f_{\pi/2}$ a *conjugate surface* of f_0.

Example.

(0) When $(F, G) = (\alpha, \beta)$ $(\alpha, \beta \in \mathbb{C}, \alpha \neq 0)$, a non-degenerate plane appears.

(1) When $(F, G) = (w, 1/w)$, we have

$$f_0(u, v) = \left(\frac{1}{2}(u^2 - v^2), -uv, 2u\right), \quad f_{\frac{\pi}{2}}(u, v) = \left(uv, \frac{1}{2}(u^2 - v^2), 2v\right).$$

These are surfaces which have self-intersections and both have singularities called as cross-caps at $(u, v) = (0, 0)$ (refer to (a) of Figure 3.2).

(2) When $(F, G) = (e^w, e^{-w})$, we have

$$f_0(u, v) = (e^u \cos v, -e^u \sin v, 2u), \quad f_{\frac{\pi}{2}}(u, v) = (e^u \sin v, e^u \cos v, 2v).$$

f_0 is the d-minimal rotational surface given by Proposition 3.9 (3), and $f_{\frac{\pi}{2}}$ is the elliptic helicoid of the second kind (refer to Figure 3.1).

(3) When $(F, G) = (1, w)$, we have

$$f_0(u, v) = \left(u, -v, u^2 - v^2\right), \quad f_{\frac{\pi}{2}}(u, v) = (u, v, 2uv).$$

These both are minimal hyperbolic paraboloids (refer to Figure 3.1).

Remark 3.18. The above Weierstrass-type representation formula contains the ones known in [4] or [30]. However, the formulas stated in [4] or [30] do not give singularities on surfaces. In this sense, Theorem 3.15 is more complete. On the other hand, we can see that examples of isotropic minimal surfaces which have isolated singularities in [36].

Here, we recall some Weierstrass(-type) representation formulas [3]. For simplicity, let F, G be holomorphic functions in a simply-connected domain of the complex plane.

- Case of $\mathbb{E}^3 = (\mathbb{R}^3, dx^2 + dy^2 + dz^2)$, i.e. minimal surfaces.

$$f_{\mathbb{E}^3} = \mathrm{Re} \int_w (F(1 - G^2), iF(1 + G^2), 2FG) dw.$$

- Case of $\mathbb{E}^3_1 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$, i.e. maximal surfaces.

$$f_{\mathbb{E}^3_1} = \mathrm{Re} \int_w (F(1 + G^2), iF(1 - G^2), 2FG) dw.$$

Thus, among minimal surfaces $f_{\mathbb{E}^3}$, maximal surfaces $f_{\mathbb{E}^3_1}$ and d-minimal surfaces $f_{\mathbb{E}^{0,2,1}}$, we obtain the relation

$$f_{\mathbb{E}^{0,2,1}} = \frac{1}{2}\left(f_{\mathbb{E}^3} + f_{\mathbb{E}^3_1}\right),$$

where $f_{\mathbb{E}^{0,2,1}}$ is the mapping given by Theorem 3.15.

3.4 Applications

Theorem 3.19. If (M, g) is a connected, two-dimensional complete Riemannian manifold, and $f : (M, g) \to \mathbb{E}^{0,2,1}$ is an isometric immersion, then (M, g) is isometric to the canonical two-dimensional Euclidean space $\mathbb{E}^2$, and the image of f corresponds to an entire graph

$$\{(u, v, F(u, v)) \in \mathbb{E}^{0,2,1} \mid (u, v) \in \mathbb{R}^2\},$$

where F is a C^∞-function on $\mathbb{R}^2$.

Proof. Let α, β, γ be the coordinate functions of f, i.e.

$$f(x) = (\alpha(x), \beta(x), \gamma(x)) \quad (x \in M).$$

We assume that $\mathbb{E}^2$ is the canonical Euclidean space which treats (u, v) as the coordinates, and define a C^∞-map $f_0 : (M, g) \to \mathbb{E}^2$ as

$$f_0(x) := (\alpha(x), \beta(x)) \quad (x \in M).$$

f_0 is an isometric immersion. We prove that f_0 is an isometry. We remark that $\dim M = \dim \mathbb{E}^2 = 2$ and, from the inverse function theorem, f_0 is a local diffeomorphism. Thus, in order to prove that f_0 is an isometry, it is sufficient to prove that f_0 is bijective.

For the surjectivity, since f_0 is a locally homeomorphism, f_0 is an open mapping. Thus, $\mathrm{Im} f_0$ is an open subset of $\mathbb{E}^2$. Next, since isometric mappings preserve the geodesic completeness, from Hopf-Rinow's theorem, $(\mathrm{Im} f_0, du^2 + dv^2) \subset \mathbb{E}^2$ is complete, where we consider $\mathrm{Im} f_0$ as the metric subspace of $\mathbb{E}^2$ naturally. Thus, $\mathrm{Im} f_0$ is a closed subset of $\mathbb{E}^2$. Therefore, since $\mathrm{Im} f_0$ is an open and closed subset of $\mathbb{E}^2$, it holds $\mathrm{Im} f_0 = \mathbb{E}^2$, i.e. $f_0 : M \to \mathbb{E}^2$ is surjective.

For the injectivity, we denote the Riemannian distance function with respect to the metric g by d_M. For distinct points $x, y \in M$, since (M, g) is complete, there exists a shortest geodesic $\delta : [0, 1] \to M$ such that $\delta(0) = x, \delta(1) = y$. Moreover, since f_0 is isometric, $f_0 \circ \delta : [0, 1] \to \mathbb{E}^2$ is a geodesic in $\mathbb{E}^2$ which connects $f_0(x)$ and $f_0(y)$. For a curve c, when we denote the length of c by $L(c)$, we see that

$$0 < d_M(x, y) = L(\delta) = L(f_0 \circ \delta) = |f_0(x) - f_0(y)|_{\mathbb{E}^2}.$$

This implies $f_0(x) \neq f_0(y)$, i.e. $f_0 : M \to \mathbb{E}^2$ is injective. As a consequence, we use the fact that geodesics in $\mathbb{E}^2$ are straight lines for the last equation above.

In summary, since we obtain that $f_0 : M \to \mathbb{E}^2$ is a local isometry and bijection, it is simply an isometry, that is, (M, g) is isometric to the canonical two-dimensional Euclidean space $\mathbb{E}^2$. We denote the inverse of f_0 by $\phi : \mathbb{E}^2 \to M$. For any $(u, v) \in \mathbb{R}^2$, we have

$$\begin{aligned} f(\phi(u, v)) &= (\alpha(\phi(u, v)), \beta(\phi(u, v)), \gamma(\phi(u, v))) \\ &= ((f_0 \circ \phi)(u, v), (\gamma \circ \phi)(u, v)) = (u, v, F(u, v)), \end{aligned}$$

where $F := \gamma \circ \phi$ is a C^∞-function on $\mathbb{R}^2$. Therefore, the image of f is the entire graph expressed by a function F on $\mathbb{R}^2$. □

Corollary 3.20. If $f : M^2 \to \mathbb{E}^{0,2,1}$ is a connected, complete d-minimal surface, then the image of f corresponds to the entire graph

$$\{(u, v, \psi(u, v)) \in \mathbb{E}^{0,2,1} \mid (u, v) \in \mathbb{R}^2\},$$

where ψ is a harmonic function on $\mathbb{R}^2$.

Proof. From Proposition 3.9 (1), it follows immediately. □

Corollary 3.21. If M is a connected, compact two-dimensional manifold without boundary, i.e. a connected closed surface, then there exists no non-degenerate immersion $f : M \to \mathbb{E}^{0,2,1}$.

Proof. We prove the corollary by contradiction. We assume that there exists a non-degenerate immersion $f : M \to \mathbb{E}^{0,2,1}$. When we denote the induced metric by g, (M, g) is a connected, compact Riemannian manifold. In particular, it is complete. From Theorem 3.19, as we have a homeomorphism $M \cong \mathbb{R}^2$, this contradicts the compactness of M. □

Let $f : M \to \mathbb{E}^{0,2,1}$ be a non-degenerate immersion, and h its second fundamental form. We recall that the Gauss-Codazzi equation of the non-degenerate immersion is given by the formula (3.2), i.e.

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) \quad (X, Y, Z \in \Gamma(TM)).$$

By using the flat local coordinates (u, v), the formula (3.2) is equivalent to

$$(h_{11})_v = (h_{12})_u, \quad (h_{22})_u = (h_{12})_v, \tag{3.8}$$

where h_{ij} are coefficients of h.

Theorem 3.22 (The fundamental theorem of non-degenerate surfaces, [38, Theorem 8.8])**.** If $U \subset \mathbb{R}^2$ is a simply-connected domain with uv-coordinates, and h_{11}, h_{12} and h_{22} are C^∞-functions on U, then there exists, up to affine isometry, a non-degenerate immersion whose induced metric and second fundamental form are

$$du^2 + dv^2 \quad \text{and} \quad h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2$$

respectively, if and only if the functions h_{ij} satisfy the Gauss-Codazzi equation (3.8) of the non-degenerate surface.

From now on, we consider four-dimensional Minkowski space $\mathbb{E}^4_1$ equipped with the Lorentzian metric

$$\langle \cdot, \cdot \rangle_1 := dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) is the canonical coordinates of $\mathbb{R}^4$. We only deal with spacelike surfaces, i.e. we require that the induced metric of surfaces is positive definite.

A surface M in $\mathbb{E}^4_1$ is called *zero mean curvature* if it holds $H \equiv 0$, where H is the mean curvature vector field of M, and a surface M is called *flat* if it holds $K \equiv 0$, where K is the Gaussian curvature of M. We abbreviate zero mean curvature to ZMC.

Remark 3.23. We give one of the motivations of studying flat and ZMC surfaces. We firstly remark that flat minimal submanifolds in n-dimensional Euclidean space $\mathbb{E}^n$ and spacelike flat ZMC surfaces in three dimensonal Minkowski space $\mathbb{E}^3_1$ are totally geodesic. On the other hand, there exist timelike flat ZMC surfaces in $\mathbb{E}^3_1$ (See [40]). Thus, we are interested in the question of whether spacelike flat ZMC surfaces in $\mathbb{E}^4_1$ should be trivial, that is, totally geodesic surfaces.

Next, spacelike flat ZMC surfaces in four-dimensional Minkowski space $\mathbb{E}^4_1$ are not always planes. We also remark that spacelike flat ZMC surfaces in $\mathbb{E}^4_2$ are totally geodesic (See [23]).

Theorem 3.24. Let $f : M^2 \to \mathbb{E}^4_1$ be an immersion which gives a non-totally geodesic, connected spacelike flat ZMC surface, and let h be the second fundamental form of M. We define a subset E of M as

$$E := \{x \in M \mid h_x = 0\}.$$

Then, it holds the following statements:

(1) $M \setminus E$ is an open dense subset of M, and it is connected.

(2) The normal bundle of M is flat, i.e. the normal curvature $R^\perp \equiv 0$.

(3) M is, by an isometry of $\mathbb{E}^4_1$, immersed in $\mathbb{E}^{0,2,1} \subset \mathbb{E}^4_1$, and it is a d-minimal surface.

Proof. For the claim (1), it is easily proved that E is a closed subset of M. Let U be the flat coordinate neighborhood of M. We define a $\mathbb{C}^4$-valued mapping $\varphi = \varphi(w)$ for a complex variable $w = u + iv$ ($(u, v) \in U$) as

$$\varphi(w) := f_{uu}(u, v) - if_{uv}(u, v).$$

Then, by using that f is smooth and harmonic, we compute

$$\frac{\partial \varphi}{\partial \bar{w}} = \frac{1}{2}(f_{uuu} + f_{uvv}) + \frac{i}{2}(f_{uuv} - f_{uvu}) = 0,$$

and φ is a holomorphic mapping on U. Since M is not totally geodesic, we obtain the interior of E is empty. Moreover, we see that zero points of φ correspond to elements of E. Therefore, since the set of zero points for a holomorphic function is discrete, the set E is a discrete subset of M which is made of isolated points. Since M is connected and E is discrete, it is proved for $M \setminus E$ to be connected.

For (2), see Corollary 1.2 in [4]. The claim (3) is proved by using Proposition 3.11 and Proposition 3.5 in [4] □

Remark 3.25. The set E is a discrete subset of M consisting of isolated points. As an example which satisfies $E \neq \emptyset$, when we define a C^∞-immersion $f : \mathbb{R}^2 \to \mathbb{E}^{0,2,1} \subset \mathbb{E}^4_1$ as

$$f(u, v) := (u, v, u^3 - 3uv^2, u^3 - 3uv^2),$$

it is a spacelike flat ZMC surface which satisfies $h = 0$ only at the origin $(0, 0)$.

Let $f : M \to \mathbb{E}^{0,2,1}$ be a d-minimal surface. By the isometric embedding ι given in Eq. (3.3), we see that M is a spacelike flat ZMC surface in $\mathbb{R}^4_1$. M is a spacelike flat surface since ι is an isometric embedding. To show that M is ZMC, we directly calculate the mean curvature vector field of M. By using a harmonic function φ, since we can locally express f by

$$f(u, v) = (u, v, \varphi(u, v)),$$

from the composition of ι, we have

$$(\iota \circ f)(u, v) = (u, v, \varphi(u, v), \varphi(u, v)).$$

Thus, we compute that the mean curvature vector field H is

$$2H = (\iota \circ f)_{uu} + (\iota \circ f)_{vv} = (\varphi_{uu} + \varphi_{vv})(0, 0, 1, 1) \equiv 0.$$

Therefore, we obtain the following corollary.

Corollary 3.26. Let X be the set of congruent classes of spacelike flat ZMC surfaces in $\mathbb{E}^4_1$, and Y the set of equivalence classes of d-minimal surfaces in $\mathbb{E}^{0,2,1}$ by a subgroup

$$K := \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c}{\multirow{2}{*}{T}} & 0 \\ & & 0 \\ 0 & 0 & c \end{array} \right) \middle| \; c \neq 0,\ T \in O(2) \right\} \ltimes \mathbb{R}^3 \subset \mathrm{Aut}(\mathbb{E}^{0,2,1}, d).$$

Except for planes, we have that X and Y are in one-to-one correspondence.

Proof. It is obvious as long as we remark that this subgroup K corresponds to the subgroup of isometries of $\mathbb{E}^4_1$ which preserves the degenerate subspace $\mathbb{E}^{0,2,1} \subset \mathbb{E}^4_1$. Thus, it follows immediately from Theorem 3.24. □

Notice that all 1-parameter subgroups of simply isotropic isometries have been already described by [38] and [45].

Regarding minimal surfaces in $\mathbb{E}^3$, maximal surfaces in $\mathbb{E}^3_1$ and d-minimal surfaces in $\mathbb{E}^{0,2,1}$, we have

$$\left\{ \begin{array}{c} \text{minimal, maximal,} \\ \text{and } d\text{-minimal surfaces} \end{array} \right\} \subsetneq \{\text{spacelike ZMC surfaces in } \mathbb{E}^4_1\}.$$

In fact, for the spaces $\mathbb{E}^3$ and $\mathbb{E}^3_1$, there exist isometric embeddings defined by

$$\mathbb{E}^3 \ni (x, y, z) \mapsto (x, y, z, 0) \in \mathbb{E}^4_1, \tag{3.9}$$
$$\mathbb{E}^3_1 \ni (x, y, z) \mapsto (x, y, 0, z) \in \mathbb{E}^4_1 \tag{3.10}$$

respectively. Since minimal surfaces in $\mathbb{E}^3$ and maximal surfaces in $\mathbb{E}^3_1$ are ZMC surfaces in $\mathbb{E}^4_1$ via the above embeddings, we see that there quite fruitfully exist ZMC surfaces in $\mathbb{E}^4_1$. On the other hand, there exist spacelike ZMC surfaces in $\mathbb{E}^4_1$ which are neither minimal, maximal nor d-minimal. For example, see Section 4 in [4].

In general, singularity points appear in d-minimal surfaces. Refer to the figures from (a) to (d) in Figure 3.2 as such examples. From the Whitney's criterion, a cross-cap appears in (a), and from the Saji's criterion [37], a D_4^--type singularity appears in (c). Other singularities have been not identified and classified. In summary, we give Table 3.1 which compares properties among each surfaces. We assume the connectedness of surfaces;

	min.	max.	d-min.
Compact	$\nexists$	$\nexists$	$\nexists$ (Cor. 3.21)
Entire graph	Planes only	Planes only	$\exists$ (Prop. 3.9 (2) (b))
Singularity	$\nexists$	$\exists$ ([22])	$\exists$
Complete	$\exists$	Planes only	$\exists$ (Thm. 3.19)
Gaussian curvature	≤ 0	≥ 0	$\equiv 0$

Table 3.1: In terms of singularity, the symbol $\exists$ expresses that singularities appear, and in terms of otherwise, $\exists$ expresses that there exist such surfaces which are not planes. In addition, the abbreviations min., max. and d-min. are minimal surfaces in $\mathbb{E}^3$, maximal surfaces in $\mathbb{E}^3_1$ and d-minimal surfaces in $\mathbb{E}^{0,2,1}$, respectively.

At the end of this section, we point out that there may exist the relation among minimal surfaces in $\mathbb{E}^3$, maximal surfaces in $\mathbb{E}^3_1$ and d-minimal surfaces in $\mathbb{E}^{0,2,1}$.

Theorem 3.27 ([19, p.168])**.** Let $n \geq 3$. If (M, g) is a two-dimensional oriented Riemannian manifold, and $f : (M, g) \to \mathbb{E}^n_1 = (\mathbb{R}^n,\ dx_1^2 + \cdots + dx_{n-1}^2 - dx_n^2)$ is a ZMC isometric immersion, then f is locally expressed by the following:

$$f = \mathrm{Re} \int_w (\phi_1, \cdots, \phi_{n-1}, \phi_n) dw,$$

where $\phi_1, \cdots, \phi_n$ are holomorphic functions which satisfy

$$\phi_1^2 + \cdots + \phi_{n-1}^2 - \phi_n^2 = 0, \quad |\phi_1|^2 + \cdots + |\phi_{n-1}|^2 - |\phi_n|^2 > 0.$$

From Theorem 3.27, we can define an immersion with S^1-parameter as

$$\tilde{f}_\theta(u, v) = \mathrm{Re} \int_w (F(1 - \cos 2\theta G^2), iF(1 + \cos 2\theta G^2), 2\cos\theta FG, 2\sin\theta FG) dw.$$

$\tilde{f}_\theta$ implies a spacelike ZMC surface in four-dimensional Minkowski space for arbitrary $\theta \in S^1$. In particular, through embeddings of formulas (3.3), (3.9) and (3.10), $\tilde{f}_0, \tilde{f}_{\pi/2}$ and $\tilde{f}_{\pi/4}$ coincide with the Weierstrass representation formulas of minimal, maximal and d-minimal surfaces in $\mathbb{E}^3, \mathbb{E}^3_1$ and $\mathbb{E}^{0,2,1}$, respectively. See also Remark 3.18. As a remark, we compute the induced metric g_θ of $\tilde{f}_\theta$ as

$$g_\theta = (1 + \cos 2\theta |G|^2)^2 |F|^2 (du^2 + dv^2).$$

Thus, we should note that this deformation of surfaces is not isometric. However, there may be applications in the study of singularities of d-minimal surfaces since a classification of singular points that appear on maximal surfaces is completed in generic situation (See [22]).

In the end of this section, we give a visualization of ZMC surfaces in four-dimensional Minkowski space.

Let $\theta \in [0, \pi/2]$. We define

$$\mathbb{R}^3(\theta) := (\mathbb{R}^3, dx^2 + dy^2 + \cos 2\theta dz^2).$$

In particular, we see that

$$\begin{aligned} \mathbb{R}^3(0) &= (\mathbb{R}^3, dx^2 + dy^2 + dz^2) = \mathbb{E}^3, \\ \mathbb{R}^3\left(\frac{\pi}{2}\right) &= (\mathbb{R}^3, dx^2 + dy^2 - dz^2) = \mathbb{E}^3_1, \\ \mathbb{R}^3\left(\frac{\pi}{4}\right) &= (\mathbb{R}^3, dx^2 + dy^2 + 0dz^2) = \mathbb{E}^{0,2,1}. \end{aligned}$$

We define a three-dimensional subspace in $\mathbb{E}^4_1$ as follows

$$V(\theta) := \langle (0, 0, \sin\theta, \cos\theta) \rangle_{\mathbb{R}}^{\perp} \subset \mathbb{E}^4_1.$$

Proposition 3.28. As linear spaces with metrics, we have an isometric isomorphism

$$V(\theta) \cong \mathbb{R}^3(\theta).$$

Proof. The following map

$$\mathbb{R}^3(\theta) \ni (x, y, z) \mapsto (x, y, \cos\theta z, \sin\theta z) \in V(\theta)$$

is an isometric isomorphism between $\mathbb{R}^3(\theta)$ and $V(\theta)$. Thus, the claim holds. □

From Proposition 3.28, we can prove the following.

Corollary 3.29. Fix $\theta \in [0, \pi/2]$. If $U \subset \mathbb{C}$ is a simply-connected domain and F, G are holomorphic functions on U. Then, for any $w = u + iv \in U$, the mapping

$$f_\theta(u, v) = \mathrm{Re} \int_w (F(1 - \cos 2\theta G^2), iF(1 + \cos 2\theta G^2), 2FG) dw$$

gives a conformal zero mean curvature surface in $\mathbb{R}^3(\theta)$.

From Corollary 3.29, we realize a visualization of the deformation of ZMC surfaces in $\mathbb{E}^4_1$ as follows

$$\tilde{f}_\theta(u, v) = \mathrm{Re} \int_w (F(1 - \cos 2\theta G^2), iF(1 + \cos 2\theta G^2), 2\cos\theta FG, 2\sin\theta FG) dw$$

since $\mathbb{R}^3(\theta)$ is a three-dimensional vector space. Here, we see that the mapping $\tilde{f}_\theta$ is ZMC because of Theorem 3.27.

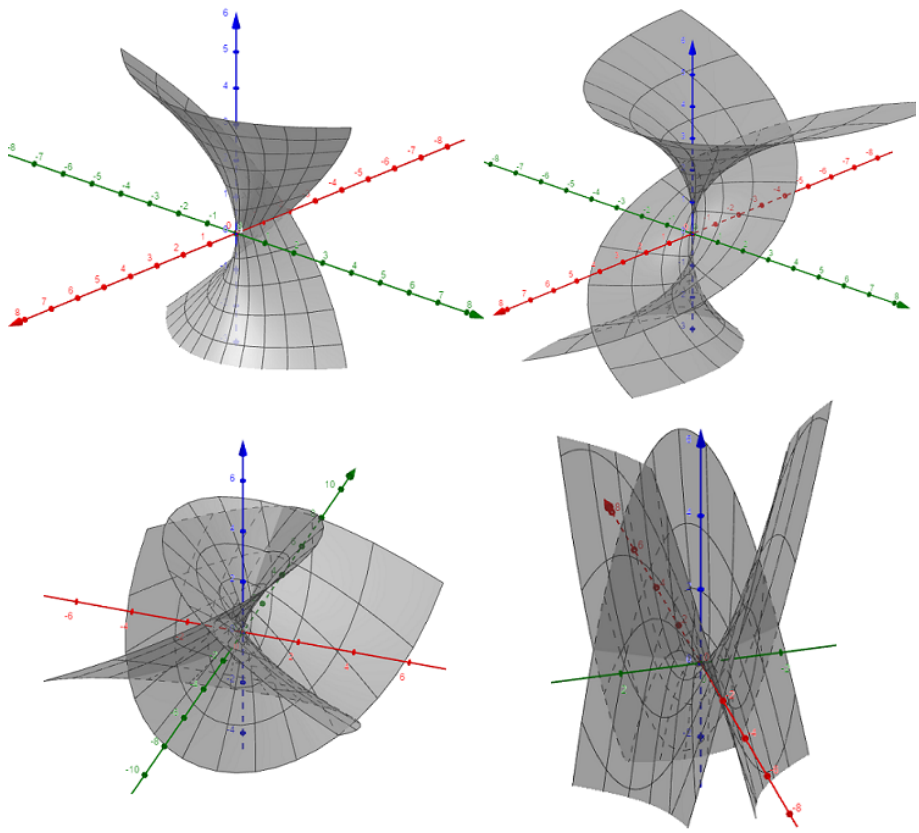

Figure 3.2: (a) Upper-left: $(F, G) = (w, 1/w)$. (b) Upper-right: $(F, G) = (w^2, 1/w^2)$. (c) Lower-left: $(F, G) = (w^2, 1/w)$. (d) Lower-right: $(F, G) = (w, w)$. Singularities appear at the origin $w = 0$. The rank of the Jacobi matrix is one for upper figures and that is zero for lower figures. The red, green and blue axis correspond to x-axis, y-axis and z-axis, respectively.

Chapter 4

Totally umbilical submanifolds in pseudo-Riemannian space forms

In this chapter, we give a complete classification of totally umbilical submanifolds in pseudo-spheres or pseudo-hyperbolic spaces, that is, we classify congruent classes of full totally umbilical submanifolds.

There exist researches which characterizes the totally umbilicity of submanifolds in pseudo-Riemannian space forms [2, 12, 13, 47, 52]. Ahn, Kim and Kim [1] gave a complete classification of totally umbilical submanifolds in pseudo-Euclidean spaces, which are flat space forms. For non-flat cases, there exist some recent researches [10] by Chen. In [10, Propositions 3.7, 3.8, Chapter 3], the following is mentioned: If $\phi : M_s^m \to \mathbb{S}_p^n(1)$ is a totally umbilical isometric immersion, then it is congruent to an open portion of one of the following submanifolds:

- $\mathbb{S}_s^m(r^2) \to \mathbb{S}_s^{m+1}(1)\ ;\ x \mapsto (x, \sqrt{1-r^2})\ (0 < r \leq 1)$,
- $\mathbb{S}_s^m(r^2) \to \mathbb{S}_{s+1}^{m+1}(1)\ ;\ x \mapsto (\sqrt{r^2-1}, x)\ (r \geq 1)$,
- $\mathbb{H}_s^m(-r^2) \to \mathbb{S}_s^{m+1}(1)\ ;\ x \mapsto (x, \sqrt{1+r^2})\ (r > 0)$,
- $\mathbb{E}_s^m \to \mathbb{S}_{s+1}^{m+2}(1)\ ;\ x \mapsto \left(r\langle x,x\rangle_s + rb - \frac{r}{4}, rx, \sqrt{1+br^2}, r\langle x,x\rangle_s - rb + \frac{r}{4}\right)\ (r > 0, br^2 \geq -1)$,
- $\mathbb{E}_s^m \to \mathbb{S}_{s+2}^{m+2}(1)\ ;\ x \mapsto \left(r\langle x,x\rangle_s + rb - \frac{r}{4}, \sqrt{br^2-1}, rx, r\langle x,x\rangle_s + rb + \frac{r}{4}\right)\ (r > 0, br^2 \geq 1)$.

This classification of totally umbilical submanifolds in $\mathbb{S}_p^n(1)$ is insufficient. In fact, the following example is not contained in the above list

$$\psi : \mathbb{S}_s^m(1) \to \mathbb{S}_{s+1}^{m+2}(1)\ ;\ x \mapsto (1, x, 1). \tag{4.1}$$

When we compute the mean curvature vector field H of ψ, we get

$$H = (1, 0, \cdots, 0, 1) \in \mathbb{E}_{s+1}^{m+3}.$$

Thus, H is a non-zero lightlike vector field. Hence ψ is non-totally geodesic. Moreover, there are some observations for this example (See Section 4.4). It is obvious that the co-dimension is two and the co-index is one.

Dajczer and Fornari in [14] showed that let $\phi : \mathbb{S}_s^m(1) \to \mathbb{S}_s^{m+n}(1)$ be an isometric immersion, then ϕ is totally geodesic, where $m \geq 2$ and $1 \leq n \leq m-s-1$. In addition, Dajczer and Rodriguez in [15] showed the following rigidity theorem: If $\phi : \mathbb{S}_s^m(1) \to \mathbb{S}_{s+1}^{m+2}(1)$ is an isometric immersion with $m - s \geq 4$. If the set of totally geodesic points does not disconnect $\mathbb{S}_s^m(1)$, then ϕ is congruent to an isometric immersion of the following type;

$$\mathbb{S}_s^m(1) \ni x \mapsto (f(x), x, f(x)) \in \mathbb{S}_{s+1}^{m+2}(1), \tag{4.2}$$

where $f : \mathbb{S}^m_s(1) \to \mathbb{R}$ is a smooth function. For any $a \in \mathbb{R} \setminus \{0\}$, the following isometric immersions

$$\psi_a : \mathbb{S}^m_s(1) \to \mathbb{S}^{m+2}_{s+1}(1) \ ; \ x \mapsto (a, x, a)$$

are congruent to the above $\psi = \psi_1$. Namely, co-dimension two totally umbilical immersions ψ_a are in a special case that f is a non-zero constant function for the mapping (4.2). Moreover, the set of totally geodesic points is empty since ψ_a is not totally geodesic but totally umbilical.

Moreover, we consider moduli spaces of totally umbilical submanifolds. As a consequence, we show that some moduli spaces of isometric immersions between space forms which are of the same constant curvature are non-Hausdorff.

As applications, we obtain some totally umbilical lightlike submanifolds in non-flat pseudo-Riemannian space forms. A classification of totally umbilical lightlike submanifolds in pseudo-Riemannian space forms is an open problem.

At the end of this thesis, we devote in Section 4.6 to the study of parallel isometric immersions. As a consequence, we see that the existence of marginally trapped parallel isometric immersions from an indefinite symmetric R-space into a pseudo-sphere or a pseudo-hyperbolic space. An isometric immersion between pseudo-Riemannian manifolds is marginally trapped if the mean curvature vector field is not zero at arbitrary point, but its norm vanishes identically. This never occurs in the Riemannian geometry. This chapter is based on [42].

4.1 Preparations in Chapter 4

The followings are well-known results in pseudo-Riemannian geometry.

Theorem 4.1 ([3, Proposition 4, Chapter 1])**.** Any non-degenerate affine subspace in the pseudo-Euclidean space $\mathbb{E}^n_p$ is a totally geodesic submanifold. Conversely, any connected non-degenerate totally geodesic submanifold in $\mathbb{E}^n_p$ is an open subset of a non-degenerate affine subspace.

Remark 4.2. Let $\Pi^m_{s,t,r}$ be a canonical r-lightlike m-plane in $\mathbb{E}^n_p$ with signature (s, t, r) given by the formula (2.1). From Theorem 4.1, arbitrary subspaces $V \subset \mathbb{E}^n_p$ are congruent to non-degenerate subspaces $\mathbb{E}^m_s$, or degenerate subspaces $\Pi^m_{s,t,r}$ up to isometry of $\mathbb{E}^n_p$.

We define an n-dimensional *lightcone* with index p in $\mathbb{E}^{n+1}_{p+1}$ as follows

$$\Lambda^n_p := \{x \in \mathbb{E}^{n+1}_{p+1} \setminus \{0\} \mid \langle x, x\rangle_{p+1} = 0\}.$$

The lightcone Λ^n_p is a totally umbilical 1-lightlike hypersurface in $\mathbb{E}^{n+1}_{p+1}$, that is, the induced metric on Λ^n_p is degenerate (See [5]).

We recall totally geodesic submanifolds in non-flat pseudo-Riemannian space forms. We define a *pseudo m-subsphere* of $\mathbb{S}^n_p(r^2)$ by

$$\left\{(x_1, \cdots, x_s, 0 \cdots, 0, x_{s+1}, \cdots, x_{m+1}) \in \mathbb{S}^n_p(r^2)\right\} \cong_{\mathrm{pRm}} \mathbb{S}^m_s(r^2).$$

Analogously, we define a *pseudo-hyperbolic m-subspace* of $\mathbb{H}^n_p(-r^2)$ by

$$\left\{(x_1, \cdots, x_{s+1}, 0 \cdots, 0, x_{s+1}, \cdots, x_{m+1}) \in \mathbb{H}^n_p(-r^2)\right\} \cong_{\mathrm{pRm}} \mathbb{H}^m_s(-r^2).$$

Then, the following holds:

Theorem 4.3 ([10, Proposition 3.3, 3.4, Chapter 3])**.** Up to isometry, an m-dimensional non-degenerate totally geodesic submanifold of an n-dimensional pseudo-sphere $\mathbb{S}^n_p(r^2)$ is an open portion of a pseudo m-subsphere. Up to isometry, an m-dimensional non-degenerate totally geodesic submanifold of an n-dimensional pseudo-hyperbolic space $\mathbb{H}^n_p(-r^2)$ is an open portion of a pseudo-hyperbolic m-subspace.

Here, we refer the classification of totally umbilical submanifolds in pseudo-Euclidean spaces.

Theorem 4.4 ([1, Proposition 3.1], [28, Theorem 1.4])**.** If $\phi : M_s^m \to \mathbb{E}_p^n$ is a totally umbilical isometric immersion, and H is its mean curvature vector field, then the image is congruent to an open portion of one of the following submanifolds:

(1) a totally geodesic pseudo-Euclidean subspace $\mathbb{E}_s^m \subset \mathbb{E}_p^n$ $(H = 0)$,

(2) a pseudo m-sphere $\mathbb{S}_s^m(r^2) \hookrightarrow \mathbb{E}_s^{m+1} \subset \mathbb{E}_p^n$ $(\langle H, H\rangle_p > 0)$,

(3) a pseudo-hyperbolic m-space $\mathbb{H}_s^m(-r^2) \hookrightarrow \mathbb{E}_{s+1}^{m+1} \subset \mathbb{E}_p^n$ $(\langle H, H\rangle_p < 0)$,

(4) a flat marginally trapped submanifold $\mathbb{U}_s^m$ defined by

$$\mathbb{E}_s^m \to \mathbb{E}_{s+1}^{m+2} \subset \mathbb{E}_p^n \ ; \ x \mapsto \left(\langle x, x\rangle_s + \frac{1}{4}, x, \langle x, x\rangle_s - \frac{1}{4}\right) \ (H \neq 0,\ \langle H, H\rangle_p = 0).$$

An isometric immersion $\phi : M_s^m \to \mathbb{M}_p^n(\varepsilon)$ is called *full* if the image $\phi(M)$ is not contained in any non-degenerate totally geodesic hypersurface in $\mathbb{M}_p^n(\varepsilon)$.

Lemma 4.5 (Erbacher–Magid Reduction Theorem, [17, Theorem], [28, Theorem 0.2])**.** Let $\phi : M_s^m \to \mathbb{E}_p^n$ be an isometric immersion. For each $x \in M_s^m$, we define

$$N^0(x) := \{\xi \in T_x^\perp M \mid A_\xi = 0\}$$

and define a *first normal space* as the orthogonal subspace of $N^0(x)$ in $T_x^\perp M$, i.e.

$$N^1(x) = (N^0(x))^\perp.$$

If a normal subbundle $N^1 = \bigcup_{x\in M} N^1(x) \subset T^\perp M$ is parallel with respect to the normal connection, then there exists a geodesically complete $(m + k)$-dimensional (possibly lightlike) totally geodesic submanifold $E^* \subset \mathbb{E}_p^n$ such that $\phi(M) \subset E^*$, where $k = \text{rank } N^1$.

Remark 4.6. If M^m is a geodesically complete m-dimensional (possibly lightlike) totally geodesic submanifold in a pseudo-Euclidean space $\mathbb{E}_p^n$, then, up to isometric translation, M^m coincides with a subspace of $\mathbb{E}_p^n$ because of the completeness. This claim holds by the fact that any geodesic of M^m is a geodesic of $\mathbb{E}_p^n$, i.e. a line segment.

Lemma 4.7 ([10, Corollary 3.1, Chapter 3])**.** Let $\phi : M_s^m \to \mathbb{S}_p^n(1)$ (resp. $\mathbb{H}_p^n(1)$) be an isometric immersion, and $\iota : \mathbb{S}_p^n(1) \to \mathbb{E}_p^{n+1}$ (resp. $\mathbb{E}_{p+1}^{n+1}$) the canonical inclusion map. When we set a mapping $f = \iota \circ \phi$, the followings hold:

(1) ϕ has parallel mean curvature vector if and only if f has parallel mean curvature vector,

(2) ϕ is parallel if and only if f is parallel,

(3) ϕ is totally umbilical if and only if f is totally umbilical.

See Section 1.3 in this thesis for the definition of parallel isometric immersions. The followings are main results in this chapter.

Theorem 4.8. If $\phi : M_s^m \to \mathbb{S}_p^n(1)$ is a full totally umbilical isometric immersion, $\bar{g}$ is the metric of $\mathbb{S}_p^n(1)$, and H is its mean curvature vector field, then, up to isometry, the image is congruent to an open portion of one of the followings:

(1) $\mathbb{S}_s^m(1) \to \mathbb{S}_s^{m+1}(1) \subset \mathbb{S}_p^n(1)$; $x \mapsto (x, 0)$ (totally geodesic, $H = 0$),

(2) $\mathbb{S}^m_s(1) \to \mathbb{S}^{m+1}_{s+1}(1) \subset \mathbb{S}^n_p(1)$; $x \mapsto (0, x)$ (totally geodesic, $H = 0$),

(3) $\mathbb{S}^m_s\left(r^2\right) \to \mathbb{S}^{m+1}_s(1) \subset \mathbb{S}^n_p(1)$; $x \mapsto (x, \sqrt{1-r^2})$ $(0 < r < 1,\ \bar{g}(H,H) > 0)$,

(4) $\mathbb{S}^m_s\left(r^2\right) \to \mathbb{S}^{m+1}_{s+1}(1) \subset \mathbb{S}^n_p(1)$; $x \mapsto (\sqrt{r^2-1}, x)$ $(r > 1,\ -1 < \bar{g}(H,H) < 0)$,

(5) $\mathbb{S}^m_s(1) \to \mathbb{S}^{m+2}_{s+1}(1) \subset \mathbb{S}^n_p(1)$; $x \mapsto (1, x, 1)$ $(H \neq 0,\ \bar{g}(H,H) = 0)$,

(6) $\mathbb{H}^m_s\left(-r^2\right) \to \mathbb{S}^{m+1}_{s+1}(1) \subset \mathbb{S}^n_p(1)$; $x \mapsto (x, \sqrt{1+r^2})$ $(r > 0,\ \bar{g}(H,H) < -1)$,

(7) $\mathbb{E}^m_s \to \mathbb{S}^{m+1}_{s+1}(1) \subset \mathbb{S}^n_p(1)$; $x \mapsto \left(\langle x,x\rangle_s - \dfrac{3}{4}, x, \langle x,x\rangle_s - \dfrac{5}{4}\right)$ $(\bar{g}(H,H) = -1)$.

Moreover, when M^m_s is geodesically complete, the image globally coincides with one of the above list (1)–(7).

Theorem 4.9. If $\phi : M^m_s \to \mathbb{H}^n_p(-1)$ is a full totally umbilical isometric immersion, $\bar{g}$ is the metric of $\mathbb{S}^n_p(1)$, and H is its mean curvature vector field, then, up to isometry, the image is congruent to an open portion of one of the followings:

(1) $\mathbb{H}^m_s(-1) \to \mathbb{H}^{m+1}_s(-1) \subset \mathbb{H}^n_p(-1)$; $x \mapsto (x, 0)$ (totally geodesic, $H = 0$),

(2) $\mathbb{H}^m_s(-1) \to \mathbb{H}^{m+1}_{s+1}(-1) \subset \mathbb{H}^n_p(-1)$; $x \mapsto (0, x)$ (totally geodesic, $H = 0$),

(3) $\mathbb{H}^m_s\left(-r^2\right) \to \mathbb{H}^{m+1}_{s+1}(-1) \subset \mathbb{H}^n_p(-1)$; $x \mapsto (\sqrt{1-r^2}, x)$ $(0 < r < 1,\ \bar{g}(H,H) < 0)$,

(4) $\mathbb{H}^m_s\left(-r^2\right) \to \mathbb{H}^{m+1}_s(-1) \subset \mathbb{H}^n_p(-1)$; $x \mapsto (x, \sqrt{r^2-1})$ $(r > 1,\ 0 < \bar{g}(H,H) < 1)$,

(5) $\mathbb{H}^m_s(-1) \to \mathbb{H}^{m+2}_{s+1}(-1) \subset \mathbb{H}^n_p(-1)$; $x \mapsto (1, x, 1)$ $(H \neq 0,\ \bar{g}(H,H) = 0)$,

(6) $\mathbb{S}^m_s\left(r^2\right) \to \mathbb{H}^{m+1}_s(-1) \subset \mathbb{H}^n_p(-1)$; $x \mapsto (\sqrt{1+r^2}, x)$ $(r > 0,\ \bar{g}(H,H) > 1)$,

(7) $\mathbb{E}^m_s \to \mathbb{H}^{m+1}_s(-1) \subset \mathbb{H}^n_p(-1)$; $x \mapsto \left(\langle x,x\rangle_s + \dfrac{5}{4}, x, \langle x,x\rangle_s + \dfrac{3}{4}\right)$ $(\bar{g}(H,H) = 1)$.

Moreover, when M^m_s is geodesically complete, the image globally coincides with one of the above list (1)–(7).

4.2 Proof of Theorem 4.8 and 4.9

Since the argument is parallel, we only give a proof in the case of pseudo-spheres. We assume that $\phi : M^m_s \to \mathbb{S}^n_p(1)$ is a totally umbilical isometric immersion, and $\iota : \mathbb{S}^n_p(1) \hookrightarrow \mathbb{E}^{n+1}_p$ is the inclusion. Then, $f := \iota \circ \phi : M^m_s \to \mathbb{E}^{n+1}_p$ is totally umbilical because of Lemma 4.7 (3). When we set $\tilde{h}$ and $\tilde{H}$ as the second fundamental form and the mean curvature vector field of f, respectively, we have, for any $X, Y \in \Gamma(TM)$,

$$\tilde{h}(X,Y) = \langle X, Y\rangle_p \tilde{H}.$$

In other words, when we set $\tilde{A}$ as the shape operator of f, we have

$$\tilde{A}_\xi(X) = \langle \xi, \tilde{H}\rangle_p X$$

for any $\xi \in \Gamma(T^\perp M), X \in \Gamma(TM)$. Therefore, we have

$$N^1 = \mathrm{Span}\{\tilde{H}\}.$$

In addition, if ϕ is totally umbilical, then ϕ has parallel mean curvature from the Codazzi equation (1.6) of ϕ. By using Lemma 4.5, there exists an $(m+1)$-dimensional complete totally geodesic submanifold $E^* \subset \mathbb{E}^{n+1}_p$ such that $f(M) \subset E^*$. We see that $\phi(M) \subset \mathbb{S}^n_p(1) \cap E^*$. By Theorems 2.16 and 4.1, the subspace E^* is congruent to one of $\mathbb{E}^{m+1}_s, \mathbb{E}^{m+1}_{s+1}, \Pi^{m+1}_{s,m-s,1}$. Thus, it suffices to check these three cases of E^*. Cohesively we use the formal notation $\bar{s} \in \{s, s+1\}$. Then, we have only to check the two possibilities $\mathbb{E}^{m+1}_{\bar{s}}$ and $\Pi^{m+1}_{s,m-s,1}$.

Here, if we should take translations of E^* into account, the direction of translations has to be transverse to E^*. When E^* is congruent to $\mathbb{E}^{m+1}_{\bar{s}}$, i.e. in the non-degenerate case, taking a vector $v \in (\mathbb{E}^{m+1}_{\bar{s}})^{\perp}$, we have only to consider $E^* = \mathbb{E}^{m+1}_{\bar{s}} + v$. When E^* is congruent to $\Pi^{m+1}_{s,m-s,1}$, i.e. in the degenerate case, $(\Pi^{m+1}_{s,m-s,1})^{\perp}$ is no longer a complementary of $\Pi^{m+1}_{s,m-s,1}$. From the viewpoint of lightlike geometry in Chapter 2, by regarding $\Pi^{m+1}_{s,m-s,1}$ as a 1-lightlike submanifold in $\mathbb{E}^n_p$, we have decompositions

$$\begin{aligned} \Pi^{m+1}_{s,m-s,1} &= \mathrm{Rad}\Pi^{m+1}_{s,m-s,1} \oplus \mathbb{E}^m_s, \\ (\Pi^{m+1}_{s,m-s,1})^{\perp} &= \mathrm{tr}\Pi^{m+1}_{s,m-s,1} \oplus \mathbb{E}^{n-m}_{p-s-1}, \\ \mathbb{E}^{n+1}_p &= \Pi^{m+1}_{s,m-s,1} \oplus \mathbb{E}^{n-m}_{p-s-1} \oplus \mathrm{tr}\Pi^{m+1}_{s,m-s,1}, \end{aligned}$$

where we define

$$\begin{aligned} \mathrm{Rad}\Pi^{m+1}_{s,m-s,1} &:= \mathrm{Span}_{\mathbb{R}}\{\xi := (1, 0, \cdots, 0, 1) \in \Pi^{m+1}_{s,m-s,1}\}, \\ \mathrm{tr}\Pi^{m+1}_{s,m-s,1} &:= \mathrm{Span}_{\mathbb{R}}\left\{ N := \frac{1}{2}(-1, 0, \cdots, 0, 1) \in (\Pi^{m+1}_{s,m-s,1})^{\perp} \right\} \end{aligned}$$

and ξ, N satisfy $\langle \xi, \xi \rangle_p = \langle N, N \rangle_p = 0, \langle \xi, N \rangle_p = 1$. In the case, taking a vector $v \in \mathbb{E}^{n-m}_{p-s-1} \oplus \mathrm{tr}\Pi^{m+1}_{s,m-s,1}$, we have only to consider $E^* = \Pi^{m+1}_{s,m-s,1} + v$.

Under the above preparation and by isometry of $\mathbb{S}^n_p(1)$, we consider sequences of subspaces of $\mathbb{E}^{n+1}_p$;

$$\begin{array}{ccccccc} & & \mathbb{E}^{m+1}_{\bar{s}} + v_S & \subset & \mathbb{E}^{m+2}_{\bar{s}} & \subset & \\ & \rotatebox{30}{$=$} & & & & & \\ E^* & = & \mathbb{E}^{m+1}_{\bar{s}} + v_T & \subset & \mathbb{E}^{m+2}_{\bar{s}+1} & \subset & \mathbb{E}^{n+1}_p, \\ & \rotatebox{-30}{$=$} & & & & & \\ & & \mathbb{E}^{m+1}_{\bar{s}} + v_L & \subset & \mathbb{E}^{m+3}_{\bar{s}+1} & \subset & \end{array} \tag{4.3}$$

$$\begin{array}{ccccccc} & & \Pi^{m+1}_{s,m-s,1} + v_S & \subset & \mathbb{E}^{m+3}_{s+1} & \subset & \\ & = & \Pi^{m+1}_{s,m-s,1} + v_T & \subset & \mathbb{E}^{m+3}_{s+2} & \subset & \\ E^* & & & & & & \mathbb{E}^{n+1}_p, \\ & = & \Pi^{m+1}_{s,m-s,1} + v_L & \subset & \mathbb{E}^{m+4}_{s+2} & \subset & \\ & = & \Pi^{m+1}_{s,m-s,1} + N & \subset & \mathbb{E}^{m+2}_{s+1} & \subset & \end{array} \tag{4.4}$$

where, by isometry of $\mathbb{S}^n_p(1)$, v_S, v_T and v_L are given by

$$v_S = (\underbrace{0, \cdots, 0}_{\bar{s}}, 0, \cdots, 0, \rho) \in \mathbb{E}^{m+2}_{\bar{s}} \quad (\rho \geq 0), \tag{4.5}$$

$$v_T = (\underbrace{\rho, 0, \cdots, 0}_{\bar{s}+1}, 0, \cdots, 0) \in \mathbb{E}^{m+2}_{\bar{s}+1} \quad (\rho \geq 0), \tag{4.6}$$

$$v_L = (\underbrace{1, 0, \cdots, 0}_{\bar{s}+1}, 0, \cdots, 0, 1) \in \mathbb{E}^{m+3}_{\bar{s}+1} \tag{4.7}$$

in the non-degenerate case (4.3),

$$v_S = (0, \underbrace{0, \cdots, 0}_{s}, 0, \cdots, 0, \rho, 0) \in \mathbb{E}^{m+3}_{s+1} \quad (\rho \geq 0), \tag{4.8}$$

$$v_T = (0, \underbrace{\rho, 0, \cdots, 0}_{s+1}, 0, \cdots, 0, 0) \in \mathbb{E}^{m+3}_{s+2} \quad (\rho \geq 0), \tag{4.9}$$

$$v_L = (0, \underbrace{1, 0, \cdots, 0}_{s+1}, 0, \cdots, 0, 1, 0) \in \mathbb{E}^{m+4}_{s+2} \tag{4.10}$$

in the degenerate case (4.4). The proof is completed by checking the intersection of $\mathbb{S}^n_p(1)$ and E^* in each case (4.5)–(4.10). □

4.3 Observation 1 : Riemannian or Lorentzian cases

We first restore the classification of totally umbilical submanifolds in spheres and hyperbolic spaces, i.e. Riemannian case $p = 0$.

- Totally umbilical submanifolds of $\mathbb{S}^n(1)$:
  (1) $\mathbb{S}^m(1) \to \mathbb{S}^{m+1}(1)$; $x \mapsto (x, 0)$ (totally geodesic);
  (2) $\mathbb{S}^m\left(r^2\right) \to \mathbb{S}^{m+1}(1)$; $x \mapsto (x, \sqrt{1-r^2}) \quad (0 < r < 1)$.
- Totally umbilical submanifolds of $\mathbb{H}^n(-1)$:
  (1) $\mathbb{H}^m(-1) \to \mathbb{H}^{m+1}(-1)$; $x \mapsto (x, 0)$ (totally geodesic);
  (2) $\mathbb{H}^m\left(-r^2\right) \to \mathbb{H}^{m+1}(-1)$; $x \mapsto (x, \sqrt{r^2-1}) \quad (r > 1)$;
  (3) $\mathbb{S}^m\left(r^2\right) \to \mathbb{H}^{m+1}(-1)$; $x \mapsto (\sqrt{1+r^2}, x) \quad (r > 0)$;
  (4) $\mathbb{E}^m \to \mathbb{H}^{m+1}(-1)$; $x \mapsto \left(||x||^2 + \frac{5}{4}, x, ||x||^2 + \frac{3}{4}\right)$.

In de Sitter and anti-de Sitter spacetimes, i.e. Lorentzian case $p = 1$, we obtain the followings:

- Totally umbilical submanifolds of $d\mathbb{S}^n(1)$:
  (1) $d\mathbb{S}^m(1) \to d\mathbb{S}^{m+1}(1)$; $x \mapsto (x, 0)$ (totally geodesic);
  (2) $\mathbb{S}^m(1) \to d\mathbb{S}^{m+1}(1)$; $x \mapsto (0, x)$ (totally geodesic);
  (3) $d\mathbb{S}^m\left(r^2\right) \to d\mathbb{S}^{m+1}(1)$; $x \mapsto (x, \sqrt{1-r^2}) \quad (0 < r < 1)$;
  (4) $\mathbb{S}^m\left(r^2\right) \to d\mathbb{S}^{m+1}(1)$; $x \mapsto (\sqrt{r^2-1}, x) \quad (r > 1)$;
  (5) $\mathbb{S}^m(1) \to d\mathbb{S}^{m+2}(1)$; $x \mapsto (1, x, 1)$;
  (6) $\mathbb{H}^m\left(-r^2\right) \to d\mathbb{S}^{m+1}(1)$; $x \mapsto (x, \sqrt{1+r^2}) \quad (r > 0)$;
  (7) $\mathbb{E}^m \to d\mathbb{S}^{m+1}(1)$; $x \mapsto \left(||x||^2 - \frac{3}{4}, x, ||x||^2 - \frac{5}{4}\right)$.
- Totally umbilical submanifolds of $Ad\mathbb{S}^n(-1)$:
  (1) $\mathbb{H}^m(-1) \to Ad\mathbb{S}^{m+1}(-1)$; $x \mapsto (x, 0)$ (totally geodesic);
  (2) $Ad\mathbb{S}^m(-1) \to Ad\mathbb{S}^{m+1}(-1)$; $x \mapsto (0, x)$ (totally geodesic);
  (3) $\mathbb{H}^m\left(-r^2\right) \to Ad\mathbb{S}^{m+1}(-1)$; $x \mapsto (\sqrt{1-r^2}, x) \quad (0 < r < 1)$;

(4) $Ad\mathbb{S}^m(-r^2) \to Ad\mathbb{S}^{m+1}(-1)$; $x \mapsto (x, \sqrt{r^2-1}) \quad (r > 1)$;

(5) $\mathbb{H}^m(-1) \to Ad\mathbb{S}^{m+2}(-1)$; $x \mapsto (1, x, 1)$;

(6) $d\mathbb{S}^m\left(r^2\right) \to Ad\mathbb{S}^{m+1}(-1)$; $x \mapsto (\sqrt{1+r^2}, x) \quad (r > 0)$;

(7) $\mathbb{L}^m \to Ad\mathbb{S}^{m+1}(-1)$; $x \mapsto \left(\langle x, x\rangle_1 + \frac{5}{4}, x, \langle x, x\rangle_1 + \frac{3}{4}\right)$.

Remark that $||\cdot||$ is the canonical Euclidean norm of $\mathbb{E}^m$, i.e. $||x||^2 := \langle x, x\rangle_0$.

4.4 Observation 2 : Totally umbilical lightlike submanifolds

As a by-product of the proof of Theorem 4.8 and 4.9, we obtain the following lightlike submanifolds in non-flat space forms.

Proposition 4.10. Let $m \geq 2$. The followings are full totally umbilical lightlike submanifolds in a pseudo-sphere $\mathbb{S}^n_p(1)$.

(1) $\mathbb{S}^{m-1}_s(1) \times \mathbb{E}^{0,0,1} \to \mathbb{S}^{m+1}_{s+1}(1)$; $(x, t) \mapsto (t, x, t)$ (totally geodesic),

(2) $\mathbb{S}^{m-1}_s(r^2) \times \mathbb{E}^{0,0,1} \to \mathbb{S}^{m+2}_{s+1}(1)$; $(x, t) \mapsto (t, x, \sqrt{1-r^2}, t) \quad (0 < r < 1)$,

(3) $\mathbb{S}^{m-1}_s(r^2) \times \mathbb{E}^{0,0,1} \to \mathbb{S}^{m+2}_{s+2}(1)$; $(x, t) \mapsto (t, \sqrt{r^2-1}, x, t) \quad (r > 1)$,

(4) $\mathbb{H}^{m-1}_s(-r^2) \times \mathbb{E}^{0,0,1} \to \mathbb{S}^{m+2}_{s+1}(1)$; $(x, t) \mapsto (t, x, \sqrt{1+r^2}, t) \quad (r > 0)$,

(5) $\Lambda^m_s \to \mathbb{S}^{m+1}_{s+1}(1)$; $x \mapsto (x, 1)$,

(6) $\Lambda^m_s \times \mathbb{E}^{0,0,1} \to \mathbb{S}^{m+2}_{s+2}(1)$; $(x, t) \mapsto (t, x, 1, t)$,

(7) $\mathbb{S}^{m-1}_s(1) \times \mathbb{E}^{0,0,1} \to \mathbb{S}^{m+3}_{s+2}(1)$; $(x, t) \mapsto (t, 1, x, 1, t)$.

We remark that the above (6) is 2-lightlike, and others are 1-lightlike.

Proposition 4.11. Let $m \geq 2$. The followings are full totally umbilical lightlike submanifolds in a pseudo-hyperbolic space $\mathbb{H}^n_p(-1)$.

(1) $\mathbb{H}^{m-1}_s(-1) \times \mathbb{E}^{0,0,1} \to \mathbb{H}^{m+1}_{s+1}(-1)$; $(x, t) \mapsto (t, x, t)$ (totally geodesic),

(2) $\mathbb{H}^{m-1}_s(-r^2) \times \mathbb{E}^{0,0,1} \to \mathbb{H}^{m+2}_{s+1}(-1)$; $(x, t) \mapsto (t, \sqrt{1-r^2}, x, t) \quad (0 < r < 1)$,

(3) $\mathbb{H}^{m-1}_s(-r^2) \times \mathbb{E}^{0,0,1} \to \mathbb{H}^{m+2}_{s+2}(-1)$; $(x, t) \mapsto (t, x, \sqrt{r^2-1}, t) \quad (r > 1)$,

(4) $\mathbb{S}^{m-1}_s(r^2) \times \mathbb{E}^{0,0,1} \to \mathbb{H}^{m+2}_{s+1}(-1)$; $(x, t) \mapsto (t, \sqrt{1+r^2}, x, t) \quad (r > 0)$,

(5) $\Lambda^m_s \to \mathbb{H}^{m+1}_{s+1}(-1)$; $x \mapsto (1, x)$,

(6) $\Lambda^m_s \times \mathbb{E}^{0,0,1} \to \mathbb{H}^{m+2}_{s+2}(-1)$; $(x, t) \mapsto (t, 1, x, t)$,

(7) $\mathbb{H}^{m-1}_s(1) \times \mathbb{E}^{0,0,1} \to \mathbb{H}^{m+3}_{s+2}(-1)$; $(x, t) \mapsto (t, 1, x, 1, t)$.

We remark that the above (6) is 2-lightlike, and others are 1-lightlike.

A classification problem of totally umbilical lightlike submanifolds in pseudo-Riemannin space forms is open. It is hard to classify totally umbilical submanifolds in pseudo-Riemannian space forms. We state one of the evidences below. Let s be a non-negative integers. We consider the following example.

$$\begin{aligned} E^* &:= \Pi^{m+2}_{s,m-s,2} + N \subset \mathbb{E}^{m+4}_{s+2} \\ &= \left\{ \left(w - \frac{1}{2}, t, u_1, \cdots, u_s, v_1, \cdots, v_t, t, w + \frac{1}{2} \right) \,\middle|\, t, u_i, v_j, w \in \mathbb{R} \right\}, \end{aligned}$$

where $N = \frac{1}{2}(-1, 0, \cdots, 0, 1) \in \mathbb{E}^{m+4}_{s+2}$. Then, we see that

$$\begin{aligned} \mathcal{S}^{m+1} := \mathbb{S}^{m+3}_{s+2}(1) \cap E^* &= \left\{ \left(w - \frac{1}{2}, x, w + \frac{1}{2} \right) \,\middle|\, -\left(w - \frac{1}{2} \right)^2 + \langle x, x \rangle_{s+1} + \left(w + \frac{1}{2} \right)^2 = 1 \right\} \\ &= \left\{ \left(-\frac{1}{2}\langle x, x \rangle_{s+1}, x, 1 - \frac{1}{2}\langle x, x \rangle_{s+1} \right) \,\middle|\, x \in \Pi^{m+1}_{s,m-s,1} \right\} \cong_{\mathrm{Rlm}} \mathbb{E}^{s,m-s,1}, \end{aligned}$$

where we set $x = (t, u_1, \cdots, u_s, v_1, \cdots, v_t, t) \in \Pi^{m+1}_{s,m-s,1} \subset \mathbb{E}^{m+2}_{s+1}$. Moreover, it holds

$$\begin{aligned} \mathcal{S}^{m+1} \subset \mathcal{H}^{m+2} &:= \left\{ \left(-\frac{1}{2}\langle \tilde{x}, \tilde{x} \rangle_{s+1}, \tilde{x}, 1 - \frac{1}{2}\langle \tilde{x}, \tilde{x} \rangle_{s+1} \right) \,\middle|\, \tilde{x} \in \mathbb{E}^{m+2}_{s+1} \right\} \cong_{\mathrm{pRm}} \mathbb{E}^{m+2}_{s+1} \\ &\subset \mathbb{S}^{m+3}_{s+2}(1), \end{aligned}$$

where we set $\tilde{x} = (u_0, u_1, \cdots, u_s, v_1, \cdots, v_t, v_0) \in \mathbb{E}^{m+2}_{s+1}$. Since $\Pi^{m+1}_{s,m-s,1}$ is a flat totally geodesic 1-lightlike hypersurface in $\mathbb{E}^{m+2}_{s+1}$, and $\mathcal{H}^{m+2}$ is a flat totally umbilical hypersurface in $\mathbb{S}^{m+2}_{s+2}(1)$, we can claim that $\mathcal{S}^{m+1}$ is a flat totally umbilical 2-lightlike submanifold in $\mathbb{S}^{m+3}_{s+2}(1)$. We remark that $\mathcal{H}^{m+2}$ is congruent to (7) in Theorem 4.8. The submanifold M^{m+1} is an example which cannot be obtained by the proof of Theorem 4.8 and 4.9. If we consider the higher co-dimensional case, we can construct a 1-parameter family of flat totally umbilical lightlike submanifolds

$$\begin{aligned} &\mathbb{E}^{0,m,1} \to \mathbb{S}^{m+3}_2(1) \\ &\quad \cup \qquad\quad \cup \\ &(x, r) \mapsto \Bigg(-\frac{1}{2}||x||^2 \cos\theta - (r - \theta)\sin\theta, -\frac{1}{2}||x||^2 \sin\theta + (r - \theta)\cos\theta, x \\ &\qquad\qquad \frac{1}{2}(2 - ||x||^2)\sin\theta + (r - \theta)\cos\theta, \frac{1}{2}(2 - ||x||^2)\cos\theta - (r - \theta)\sin\theta \Bigg), \end{aligned}$$

where $\theta \in \mathbb{R}$ and we set $(x, r) \in \mathbb{E}^{0,m,1} = \mathbb{E}^m \oplus \mathbb{E}^{0,0,1}$. When $\theta = 0$, we obtain the case of $\mathcal{S}^{m+1}$. Therefore, it is expected that there are many other examples which are given by the intersection a pseudo-sphere and an affine subspace and a higher dimension in a pseudo-Euclidean space. On the other hand, the following result is known:

Proposition 4.12 ([5, Proposition 5.3, Chapter 4])**.** Any lightlike surface M^2 of a three-dimensional Lorentzian manifold is either totally umbilical or totally geodesic.

This is the reason why a classification of totally umbilical lightlike submanifolds is much more complicated than that of totally umbilical non-degenerate submanifolds.

We will observe co-dimension two and co-index one totally umbilical submanifolds in Theorems 4.4, 4.8 and 4.9. When we define the hyperplane in $\mathbb{E}^{m+2}_{s+1}$ by

$$N^{m+1}(0) := \left\{ \left(t + \frac{1}{4}, x, t - \frac{1}{4} \right) \in \mathbb{E}^{m+2}_{s+1} \,\middle|\, t \in \mathbb{R}, x \in \mathbb{E}^m_s \right\} \cong_{\mathrm{Rlm}} \mathbb{E}^{s,m-s,1}.$$

This is a totally geodesic 1-lightlike hypersurface in $\mathbb{E}^{m+2}_{s+1}$. We can regard a flat marginally trapped submanifold $\mathbb{U}^m_s$ in Theorem 4.4 as a hypersurface in $N^{m+1}(0)$ by

$$\mathbb{E}^m_s \ni x \mapsto \left(\langle x, x \rangle_s + \frac{1}{4}, x, \langle x, x \rangle_s - \frac{1}{4} \right) \in N^{m+1}(0) \subset \mathbb{E}^{m+2}_{s+1}. \tag{4.11}$$

In addition, we can also regard as a hypersurface in the lightcone Λ^{m+1}_s by

$$\mathbb{E}^m_s \ni x \mapsto \left(\langle x, x \rangle_s + \frac{1}{4}, x, \langle x, x \rangle_s - \frac{1}{4} \right) \in \Lambda^{m+1}_s \subset \mathbb{E}^{m+2}_{s+1}.$$

For $\varepsilon = \pm 1$, the hypersurface in $\mathbb{M}^{m+2}_{s+1}(\varepsilon)$ defined by

$$N^{m+1}(\varepsilon) := \{(t, x, t) \in \mathbb{M}^{m+2}_{s+1}(\varepsilon) \mid t \in \mathbb{R}, x \in \mathbb{M}^m_s(\varepsilon)\} \cong_{\mathrm{Rlm}} \mathbb{M}^m_s(\varepsilon) \times \mathbb{E}^{0,0,1}$$

is a totally geodesic 1-lightlike hypersurface in $\mathbb{M}^{m+2}_{s+1}(\varepsilon)$. For the co-dimension two and co-index one totally umbilical isometric embedding in given (4.1), say ψ, in Theorem 4.8 and 4.9, we can regard ψ as a hypersurface in $N^{m+1}(1)$, i.e.

$$\psi : \mathbb{M}^m_s(\varepsilon) \ni x \mapsto (1, x, 1) \in N^{m+1}(\varepsilon) \subset \mathbb{M}^{m+2}_{s+1}(\varepsilon).$$

This is an analogue of the consideration of the mapping (4.11).

From Proposition 4.10 and 4.11, isometric embeddings of Λ^{m+1}_s into $\mathbb{M}^{m+2}_{s+1}(\varepsilon)$

$$\chi : \Lambda^{m+1}_s \to \mathbb{M}^{m+2}_{s+1}(\varepsilon) \ ; \ \begin{cases} x \mapsto (x, 1) & (\varepsilon = 1), \\ x \mapsto (1, x) & (\varepsilon = -1) \end{cases}$$

are totally umbilical 1-lightlike hypersurfaces. On the other hand, non-flat space forms $\mathbb{M}^m_s(\varepsilon)$ are isometrically embedded in the lightcone Λ^{m+1}_s as follows

$$\rho : \mathbb{M}^m_s(\varepsilon) \to \Lambda^{m+1}_s \ ; \ \begin{cases} x \mapsto (1, x) & (\varepsilon = 1), \\ x \mapsto (x, 1) & (\varepsilon = -1). \end{cases}$$

For ψ, we can see that there exists a nested structure of space forms via the lightcone

$$\psi = \chi \circ \rho : \mathbb{M}^m_s(\varepsilon) \overset{\rho}{\hookrightarrow} \Lambda^{m+1}_s \overset{\chi}{\hookrightarrow} \mathbb{M}^{m+2}_{s+1}(\varepsilon) \ ; \ x \mapsto (1, x, 1).$$

In summary, we can find out the following relation among pseudo-Riemannian space forms and lightcones:

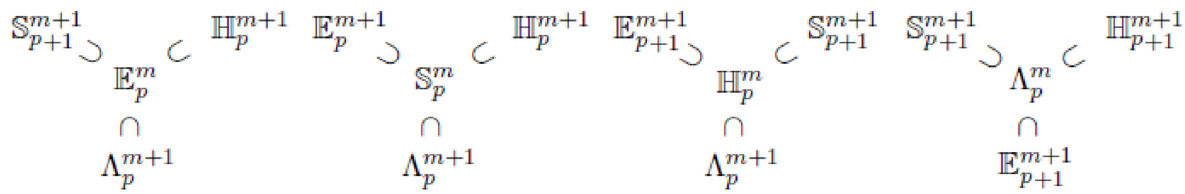

Figure 4.1: Totally umbilical inclusion relations.

4.5 Application 1 : The moduli space of isometric immersions

Let $M_s^m, \bar{M}_p^n$ be pseudo-Riemannian manifolds, and $g, \bar{g}$ their pseudo-Riemannian metrics, respectively. We define a mapping space

$$\{\phi \in C^\infty\left(M_s^m, \bar{M}_p^n\right) \mid \phi^* \bar{g} = g\}, \tag{4.12}$$

where $C^\infty\left(M_s^m, \bar{M}_p^n\right)$ denotes the set of all smooth mapping from M_s^m into $\bar{M}_p^n$. We introduce the compact open C^∞-topology in the set and consider it as a topological space. The isometric group of $\bar{M}_p^n$ naturally acts on this mapping space (4.12). We call the quotient space by the action the *moduli space* for isometric immersions $\phi : M_s^m \to \bar{M}_p^n$, denoted by $\mathcal{M}(M_s^m, \bar{M}_p^n)$. We shall denote the moduli space for totally umbilical isometric immersions by $\mathcal{M}_{umb}(M_s^m, \bar{M}_p^n)$. It is obvious that $\mathcal{M}_{umb}(M_s^m, \bar{M}_p^n) \subset \mathcal{M}(M_s^m, \bar{M}_p^n)$ as a subspace.

Proposition 4.13. If $\varepsilon = 0, \pm 1$, then it holds

$$\mathcal{M}_{umb}\left(\mathbb{M}_s^m(\varepsilon), \mathbb{M}_p^n(\varepsilon)\right) \overset{\text{homeo}}{\cong} \begin{cases} \{*\} & (n = m+1, p = s, s+1, \text{or } n = m+2, p = s), \\ (X, \mathcal{O}_X) & (n \geq m+2,\ p \geq s+1), \end{cases}$$

where $\{*\}$ is the one-point space, and a topological space $(X, \mathcal{O}_X)$ is defined by

$$X := \{g, u\}, \quad \mathcal{O}_X := \{\emptyset, \{u\}, X\}.$$

Here the elements g and u of X express the congruent classes of totally geodesic isometric immersions and non-totally geodesic, totally umbilical isometric immersions, respectively. Moreover, the space $(X, \mathcal{O}_X)$ is connected, non-Hausdorff.

Proof. From Theorem 4.4, 4.8 and 4.9, the set $\mathcal{M}_{umb}\left(\mathbb{M}_s^m(\varepsilon), \mathbb{M}_p^n(\varepsilon)\right)$ is a one-point set when $n = m+1, p = s, s+1$ or $n = m+2, p = s$, and a two-point set when $n \geq m+2,\ p \geq s+1$. In the case $n \geq m+2,\ p \geq s+1$, we consider the following totally umbilical isometric immersions, for each $a \in \mathbb{R}$,

$$\begin{aligned} &\psi_a : \mathbb{M}_s^m(\varepsilon) \to \mathbb{M}_{s+1}^{m+2}(\varepsilon)\ ;\ x \mapsto (a, x, a) \quad (\varepsilon = \pm 1), \\ &\psi_a : \mathbb{M}_s^m(\varepsilon) \to \mathbb{M}_{s+1}^{m+2}(\varepsilon)\ ;\ x \mapsto (a\langle x, x\rangle_s, x, a\langle x, x\rangle_s) \quad (\varepsilon = 0). \end{aligned}$$

They are congruent to ψ_1 if $a \neq 0$, and a totally geodesic isometric immersion ψ_0 if $a = 0$. On the other hand, it is obvious that

$$\lim_{a \to 0} \psi_a = \psi_0.$$

Therefore, $\mathcal{M}_{umb}\left(\mathbb{M}_s^m(\varepsilon), \mathbb{M}_p^n(\varepsilon)\right)$ is not a discrete space. Since mean curvature vector fields H_0, H_1 of ψ_0, ψ_1 entirely satisfy $H_0 = 0$ (closed condition) and $H_1 \neq 0$ (open condition), respectively, we obtain the conclusion. □

Corollary 4.14. If $n \geq m+2,\ p \geq s+1,\ \varepsilon = 0, \pm 1$, then the moduli space

$$\mathcal{M}\left(\mathbb{M}_s^m(\varepsilon), \mathbb{M}_p^n(\varepsilon)\right)$$

is a non-Hausdorff space.

4.6 Application 2 : Parallel submanifolds

An isometric immersion $\phi : M_s^m \to \mathbb{M}_p^n(\varepsilon)$ is called *substantial* if it is not contained in any non-degenerate totally umbilical submanifold in $\mathbb{M}_p^n(\varepsilon)$. By definition, if ϕ is substantial, then ϕ is full.

When an ambient space $\bar{M}$ is a pseudo-Riemannian space form, a submanifold M of $\bar{M}$ is parallel if and only if it is locally symmetric. Moreover, M is complete and parallel if and only if it is symmetric [7].

Lemma 4.15. Let M' and $\bar{M}$ be pseudo-Riemannian space forms, and let $\phi : M \to M'$ and $\psi : M' \to \bar{M}$ isometric immersions. We assume that ψ is totally umbilical, that is, M' is embedded as a totally umbilical submanifold in $\bar{M}$. Then, ϕ is parallel if and only if $\psi \circ \phi$ is parallel.

Proof. In Riemannian case, refer [6, Lemma 3.7.5]. The proof is done by the same argument since totally umbilical submanifolds in pseudo-Riemannian space forms are of parallel mean curvature vector fields from Lemma 4.7. □

Proposition 4.16. If $\phi : M_s^m \to \mathbb{S}_p^n(1)$ is a substantial isometric immersion, and $\psi : \mathbb{S}_p^n(1) \to \mathbb{S}_{p+1}^{n+2}(1)$ is a totally umbilical isometric immersion defined by

$$\psi(x) = (1, x, 1) \in \mathbb{S}_{p+1}^{n+2}(1) \quad (x \in \mathbb{S}_p^n(1)), \tag{4.13}$$

then the composition $\psi \circ \phi : M_s^m \to \mathbb{S}_{p+1}^{n+2}(1)$ is a full parallel isometric immersion. Moreover, if ι be the totally geodesic inclusion

$$\iota(x) = (0, x, 0) \in \mathbb{S}_{p+1}^{n+2}(1) \quad (x \in \mathbb{S}_p^n(1)),$$

then $\psi \circ \phi$ is not congruent to $\iota \circ \phi$ in $\mathbb{S}_{p+1}^{n+2}(1)$.

Proof. We prove by contradiction. Assume that $\iota \circ \phi$ is congruent to $\psi \circ \phi$, that is, there exists an isometry $\Psi \in \mathrm{Isom}(\mathbb{S}_{p+1}^{n+2}(1))$ such that

$$\Psi \circ \iota \circ \phi = \psi \circ \phi. \tag{4.14}$$

For any $x \in M$, we denote $\phi(x)$ by

$$\phi(x) = (f_1(x), \cdots, f_{n+1}(x)) \in \mathbb{S}_p^n(1) \subset \mathbb{E}_p^{n+1}.$$

Then, we see, for any $x \in M$,

$$(\iota \circ \phi)(x) = (0, f_1(x), \cdots, f_{n+1}(x), 0), \ (\psi \circ \phi)(x) = (1, f_1(x), \cdots, f_{n+1}(x), 1).$$

From the assumption of contradiction, when we denote Ψ by

$$\Psi = [x_{ij}]_{0 \le i,j \le n+2} \in O(p+1, n-p+2),$$

the formula (4.14) implies that

$$x_{01} f_1(x) + \cdots + x_{0n+1} f_{n+1}(x) = 1 \quad (x \in M).$$

When we set a non-zero vector $w \in \mathbb{E}_p^{n+1}$ as

$$w = (-x_{01}, \cdots, -x_{0p}, x_{0p+1}, \cdots, x_{0n+1}),$$

the formula (4.6) is equivalent to

$$\langle w, \phi(x)\rangle_p = 1 \quad (x \in M).$$

Therefore, when we set

$$W = \{v \in \mathbb{E}^{n+1}_p \mid \langle w, v\rangle_p = 1\},$$

we see that the image $\phi(M)$ is contained the intersection $\mathbb{S}^n_p(1) \cap W$. This contradicts that ϕ is substantial. □

Remark 4.17. Proposition 4.16 is also valid when the ambient space is a pseudo-hyperbolic space $\mathbb{H}^n_p(-1)$. In case of Riemannian parallel surfaces, this construction is known in [9, Theorem 9.1 (C) and Theorem 10.1 (C)].

Let G and K be a Lie group and its closed Lie subgroup, respectively, and G/K an irreducible indefinite symmetric R-space such as indefinite Grassmann manifolds, indefinite orthogonal groups and complex spheres etc [7, 32]. If $f : G/K \to \mathbb{E}^{n+1}_p$ is a standard embedding, then, by scaling of the metric, $f(G/K)$ is a minimal submanifold of $\mathbb{S}^n_p(1)$ or $\mathbb{H}^n_{p-1}(-1)$. When ψ is the co-dimension two and co-index one totally umbilical isometric embedding (4.13), considering the composition $\psi \circ f$, we obtain a full complete, parallel isometric embedding in $\mathbb{S}^{n+2}_{p+1}(1)$ or $\mathbb{H}^{n+2}_p(-1)$ by using Proposition 4.16. However, its mean curvature vector field H of $\psi \circ f$ is non-zero and satisfies $\langle H, H\rangle_{p+1} = 0$. Namely, $\psi \circ f$ is a marginally trapped isometric immersion.

On the other hand, a full parallel, minimal isometric immersion of an irreducible Riemannian symmetric R-space into a unit sphere is rigid, i.e. it is congruent to a standard embedding. However, in the indefinite case, there exist full parallel, marginally trapped isometric immersions of irreducible indefinite symmetric R-spaces into unit pseudo-spheres which are not congruent to standard embeddings. See also Blomstrom's rigidity theorem [7, Theorem 3].

B. Y. Chen et al. classified Riemannian and Lorentzian parallel surfaces in pseudo-Riemannian space forms. In [10], he commented that the explicit classifications of parallel submanifolds in pseudo-Riemannian space forms are much more complicated than that of Riemannian situations. In fact, it is known that there exist 24 families and 53 families of parallel Lorentzian surfaces in neutral space forms $\mathbb{S}^4_2(1)$ and $\mathbb{H}^4_2(-1)$, respectively. Some of these surfaces are full but not substantial. Regarding Riemannian parallel surfaces in $\mathbb{S}^n_p(1)$, we see that in [11] the following complete, parallel and flat surfaces

$$f : \mathbb{E}^2 \ni (u, v) \mapsto \left(v^2 + a^2 - \frac{3}{4}, a\cos u, a\sin u, v, v^2 + a^2 - \frac{5}{4}\right) \ (a > 0).$$

This parallel surface is full but not substantial. In fact, we set

$$\phi : \mathbb{E}^3 \ni (x, y, z) \mapsto \left(x^2 + y^2 + z^2 - \frac{3}{4}, x, y, z, x^2 + y^2 + z^2 - \frac{5}{4}\right) \in \mathbb{S}^4_1(1), \tag{4.15}$$
$$\psi : \mathbb{E}^2 \ni (u, v) \mapsto (a\cos u, a\sin u, v) \in \mathbb{E}^3 \ (a > 0).$$

Then, by direct calculation, we see that $\phi \circ \psi = f$. Since the hypersurface (4.15) is totally umbilical, f is full but not substantial. Via totally umbilical isometric immersions

$$\mathbb{E}^n_p \ni x \mapsto \left(\langle x, x\rangle_p - \frac{3}{4}, x, \langle x, x\rangle_p - \frac{5}{4}\right) \in \mathbb{S}^{n+1}_{p+1}(1),$$
$$\mathbb{S}^n_p(1) \ni x \mapsto (1, x, 1) \in \mathbb{S}^{n+2}_{p+1}(1),$$

substantial parallel submanifolds in $\mathbb{E}^n_p$ or $\mathbb{S}^n_p(1)$ induce full parallel ones in $\mathbb{S}^{n+1}_{p+1}(1)$ and $\mathbb{S}^{n+2}_{p+1}(1)$, respectively. A classification of full parallel submanifolds may be difficult, but a classification of substantial complete ones may be possible. As further references, see also [7, 24, 25, 26, 32].

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