

On the twisted Alexander polynomials
for some hyperbolic knots

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Chapter 1

Introduction

Twisted Alexander polynomial is a generalization of Alexander polynomial, which is one of the classical invariants of knots, and is defined for a knot together with a representation of the knot group i.e. the fundamental group of the knot complement. Twisted Alexander polynomial was introduced by Lin [L] for knot groups, and Wada defined it for arbitrary finitely presentable groups and its representations [W] in 1990's. Wada showed that the twisted Alexander polynomial can distinguish Kinoshita-Terasaka knot and Conway's 11 crossing knot, whose Alexander polynomials are trivial.

It is known that there are relations between twisted Alexander polynomials and the properties of knots, e.g. the genus and the fiberedness of knots. More precisely, for a knot K and a nonabelian $\mathrm{SL}(2, \mathbb{F})$ -representation $\rho : G(K) \rightarrow \mathrm{SL}(2, \mathbb{F})$ of the knot group $G(K)$, the degree of the twisted Alexander polynomial $\Delta_{K, \rho}(t)$ (i.e. the difference of the the highest degree and the lowest degree of $\Delta_{K, \rho}(t)$) is less than or equal to the number obtained from the genus of K , and if K is fibered $\Delta_{K, \rho}(t)$ is a monic polynomial [KM].

We say that a knot is hyperbolic if the knot complement admits a complete hyperbolic metric of finite volume. For a hyperbolic knot K , there is a canonical representation of the knot group $G(K)$, called the holonomy representation of K , and Dunfield–Friedl–Jackson [DFJ] conjectured that the genus and the fiberedness of K are determined by the twisted Alexander polynomial associated to the holonomy representation of K .

In Chapter 3, for a $(-2, 3, 2n + 1)$ -pretzel knot K and a family of representations of $G(K)$ which contains the holonomy representation of K , we computed the twisted Alexander polynomials of K associated to each representation in the family, and we proved that Dunfield–Friedl–Jackson's conjecture is true for $(-2, 3, 2n + 1)$ -pretzel knots. Moreover, in Chapter 4, we studied the twisted Alexander polynomials of all Montesinos knots with tunnel number one which contains $(-2, 3, 2n + 1)$ -pretzel knots and two-bridge knots. More precisely, for a family, which contains $(-2, 3, 2n + 1)$ -pretzel knots and two-bridge knots, we computed the degree and the leading coefficient of their twisted Alexander polynomials associated to any nonabelian $\mathrm{SL}(2, \mathbb{C})$ -representations, and then we reduced Dunfield–Friedl–Jackson's conjecture to a certain condition of the holonomy representations. For other Montesinos knots with tunnel number one, in a similar way as in Chapter 3, we computed twisted Alexander polynomials associated to any nonabelian $\mathrm{SL}(2, \mathbb{C})$ -representations, and we proved that Dunfield–Friedl–Jackson's conjecture is true in this case.

Another application of the twisted Alexander polynomial is some relations to the hyperbolic volume. For a cusped hyperbolic 3-manifold, Menal-Ferrer–Porti showed that the hyperbolic volume appears in the asymptotic behavior of Reidemeister torsion [MP], and Kitano [K] and Yamaguchi [Ya] showed some relations between twisted Alexander polynomials of knots and the Reidemeister torsions. By using these results, Goda [Go] proved that for a hyperbolic knot K , the hyperbolic volume $\text{Vol}(S^3 \setminus K)$ of $S^3 \setminus K$ appears in the asymptotic behavior of the twisted Alexander polynomials associated to certain $\text{SL}(n, \mathbb{C})$ -representations ρ_n , where ρ_n is induced from the holonomy representation of K . Furthermore, Park gave a generalization of the formula of the hyperbolic volume with the Reidemeister torsion, and he conjectured that the complex volume is obtained by a complexification of his results [P]. Here the complex volume $\text{cv}(M)$ of a hyperbolic manifold M is defined to be the complex number $\text{Vol}(M) + 2\pi^2 \text{CS}(M) \sqrt{-1}$ whose real part is the hyperbolic volume $\text{Vol}(M)$ of M and the imaginary part is a multiple of the Chern-Simons invariant $\text{CS}(M)$ of M .

In Chapter 5, to obtain a complexification of Goda’s formula in [Go], for some hyperbolic knots K of 6 crossings or fewer, we studied the asymptotic behavior of the twisted Alexander polynomials of K associated to ρ_n , and we conjectured the equality

$$\lim_{n \rightarrow \infty} \frac{4\pi \log \Delta_{K, \rho_n}(1)}{n^2} = \text{cv}(S^3 \setminus K).$$

In fact, we observed that the left hand side approaches to $\text{cv}(S^3 \setminus K)$ as n gets bigger.

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Chapter 2

Preliminaries

The Alexander polynomial, introduced by Alexander in 1928, is determined by the fundamental group of a knot complement called knot group [A1]. Fox defined Fox derivative [F1], and he gave an alternative definition of the Alexander polynomial by using Fox derivative [F2], which is given as follows.

Definition 2.1. Let $F_n = \langle x_1, \dots, x_n \rangle$ be the free group of rank n , and $d : \mathbb{Z}F_n \rightarrow DF_n$ be a \mathbb{Z} -linear map defined by $d\gamma = \gamma - 1$ ($\gamma \in F_n$), where DF_n is a left ideal of $\mathbb{Z}F_n$ generated by elements $\gamma - 1$ ($\gamma \in F_n$). Then, for any $w \in F_n$, we can express dw as

$$dw = \sum_{i=1}^n \frac{\partial w}{\partial x_i} dx_i,$$

and we call each coefficient $\frac{\partial w}{\partial x_i}$ of dw the Fox derivative of w with respect to x_i .

Remark 2.2. $d : \mathbb{Z}F_n \rightarrow DF_n$ satisfies

$$\begin{aligned} d(\gamma\gamma') &= d(\gamma) + \gamma d(\gamma'), \\ d(\gamma^{-1}) &= -\gamma^{-1}d(\gamma), \end{aligned}$$

for any $\gamma, \gamma' \in F_n$.

Then the Alexander polynomial is defined as follows.

Definition 2.3. Let K be a knot in S^3 and

$$G(K) = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$$

the knot group of K . Then we put

$$a_{i,j} = \mathbf{a} \circ \phi \left(\frac{\partial r_i}{\partial x_j} \right) \in \mathbb{Z}[t^{\pm 1}],$$

where $\mathbf{a} : \mathbb{Z}G(K) \rightarrow \mathbb{Z}[t^{\pm 1}]$ is the abelianization and $\phi : \mathbb{Z}F_n \rightarrow \mathbb{Z}G(K)$ is the natural ring homomorphism of the free group F_n generated by x_1, \dots, x_n . Then, the Alexander

polynomial $\Delta_K(t)$ of K is defined by

$$\Delta_K(t) = \begin{vmatrix} a_{1,1} & \cdots & a_{1,k-1} & a_{1,k+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n} \end{vmatrix}$$

up to $\pm t^i$.

As a generalization of the Alexander polynomial, the twisted Alexander polynomial was introduced by Lin for knot groups and their representations [L] and Wada for finitely presented groups and their representations [W] in 1990s. In this thesis, we use the following definition due to Wada.

Definition 2.4. Let K be a knot in S^3 and

$$G(K) = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$$

the knot group of K . Then we put

$$A_{i,j} = (\rho \otimes \mathbf{a}) \circ \phi \left(\frac{\partial r_i}{\partial x_j} \right) \in M_d(\mathbb{C}[t^{\pm 1}]),$$

where $\mathbf{a} : \mathbb{Z}G(K) \rightarrow \mathbb{Z}[t^{\pm 1}]$ is the abelianization and $\phi : \mathbb{Z}F_n \rightarrow \mathbb{Z}G(K)$ is the natural ring homomorphism of the free group F_n generated by x_1, \dots, x_n . Then the twisted Alexander polynomial of K associated to a representation $\rho : G(K) \rightarrow GL_d(\mathbb{C})$ of $G(K)$ is defined by

$$\Delta_{K,\rho}(t) = \frac{\det M_{\rho,k}}{\det[(\rho \otimes \mathbf{a}) \circ \phi(x_k - 1)]}$$

up to $\pm t^i$, where $M_{\rho,k}$ stands for the $d(n-1) \times d(n-1)$ matrix

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,k-1} & A_{1,k+1} & \cdots & A_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n-1,1} & \cdots & A_{n-1,k-1} & A_{n-1,k+1} & \cdots & A_{n-1,n} \end{pmatrix}.$$

Remark 2.5. By definition, for the trivial representation $\mathbf{1} : G(K) \rightarrow SL_1(\mathbb{C})$, we have

$$\Delta_{K,\mathbf{1}}(t) = \frac{\Delta_K(t)}{t-1}.$$

By Kitano and Morifuji [KM], it is known that Wada's twisted Alexander polynomials of the knot groups for any nonabelian representations into $SL_2(\mathbb{F})$ over a field \mathbb{F} are polynomials. As a corollary, they also showed that if K is a fibered knot of genus g , i.e. the exterior of K fibers over the circle, then its twisted Alexander polynomials are monic polynomials of degree $4g - 2$ for any nonabelian $SL_2(\mathbb{F})$ -representations. The converse does not hold, in other words, there exists an example of a nonfibered knot which has an $SL_2(\mathbb{C})$ -representation such that the twisted Alexander polynomial of the representation is monic (see [GoMo]).

If K is hyperbolic, i.e. the complement $S^3 \setminus K$ of K admits a complete hyperbolic metric of finite volume, the most important representation is its holonomy representation into $SL_2(\mathbb{C})$ which is a lift of the representation into the group of orientation-preserving isometries of the hyperbolic 3-space \mathbb{H}^3 .

Dunfield, Friedl and Jackson [DFJ] conjectured that the twisted Alexander polynomials of hyperbolic knots associated to their holonomy representations (so-called hyperbolic torsion polynomials) determine the genus and fiberedness of the knots. In fact, they computed the twisted Alexander polynomials of all hyperbolic knots up to 15 crossings associated to their holonomy representations, and confirmed that the conjecture is true for these hyperbolic knots. Recently, the twisted Alexander polynomials of some infinite families of knots, twist knots and genus one two-bridge knots associated to their holonomy representations, are computed by Morifuji [Mo1] and Tran [T1], and genus one two-bridge knots associated to the adjoint representations of their holonomy representations is also computed by Tran [T2]. These examples are also supporting evidences of the conjecture. They also gave some questions and open problems. For example, the second highest coefficients of the hyperbolic torsion polynomials are often real for fibered knots.

Another application of the twisted Alexander polynomial is the relations to the volume of a hyperbolic knot complement. In fact, Menal-Ferrer and Porti showed that the volume is related to the higher dimensional Reidemeister torsion of a cusped hyperbolic 3-manifold [MP], and Kitano [K] and Yamaguchi [Ya] showed that the twisted Alexander polynomial can be regarded as a Reidemeister torsion.

Recently, Goda proved that the volume is determined by the asymptotic behavior of the twisted Alexander polynomials of a hyperbolic knot.

Theorem 2.6 (Goda [Go]). *Let K be a hyperbolic knot in S^3 . Then we have*

$$\lim_{n \rightarrow \infty} \frac{4\pi \log |\mathcal{A}_{K,n}(1)|}{n^2} = \text{Vol}(S^3 \setminus K),$$

where

$$\mathcal{A}_{K,2k}(t) = \frac{\Delta_{K,\rho_{2k}}(t)}{\Delta_{K,\rho_2}(t)}, \quad \mathcal{A}_{K,2k+1}(t) = \frac{\Delta_{K,\rho_{2k+1}}(t)}{\Delta_{K,\rho_3}(t)},$$

and ρ_n is an n -dimensional representation obtained from the holonomy representation.

On the other hand, Park generalized the result of Menal-Ferrer and Porti.

Theorem 2.7 (Park [P]). *Let M be a complete hyperbolic 3-manifold of finite volume with h cusps. For $n \geq 3$,*

$$\left| \tau(M, \rho_{2(n-1)}, \{m_i\}) \prod_{i=1}^h \eta(\tau_i)^2 \right|^{-1} = \left| \exp \left(\frac{1}{\pi} (n^2 - n + \frac{1}{6}) \text{Vol}(M) \right) F_n(M) \right|,$$

where $\eta(\tau_i)$ denotes the Dedekind eta function of complex numbers τ_i associated to i -th cusp of M , and $F_n(M)$ denotes Zagier infinite product.

Furthermore, he conjectured that the complex volume are obtained by complexification of his results.

Conjecture 2.8 (Park [P]). *Let M be a complete hyperbolic 3-manifold of finite volume with h cusps. For $n \geq 3$,*

$$\begin{aligned} & \tau(M, \rho_{2(n-1)}, \{m_i\})^{-12} \prod_{i=1}^h \eta(\tau_i)^{-24} \\ &= c_{M,n} \exp \left(\frac{2}{\pi} (6n^2 - 6n + 1) (\text{Vol}(M) + i2\pi^2 \text{CS}(M)) \right) F_n(M)^{12}, \end{aligned}$$

where $\text{CS}(M)$ denotes the Chern-Simons invariant of M .

Chapter 3

$(-2, 3, 2n + 1)$ -pretzel knots

Let K_n be a knot depicted in Figure 1, called $(-2, 3, 2n + 1)$ -pretzel knot. Note that $(-2, 3, 2n + 1)$ -pretzel knot is a hyperbolic knot for $n \neq 0, 1, 2$. In this chapter, we compute the twisted Alexander polynomials of K_n associated to the family of their $SL_2(\mathbb{C})$ -representations which contains their holonomy representations given in the following section for $n \neq 0, 1, 2$. The twisted Alexander polynomials of K_n are monic polynomials of degree $4(|n + 1| + 1) - 2$, where $n \leq -2$ or $2 < n$, and the twisted Alexander polynomial of K_{-1} is a non-monic polynomial of degree 2. We can observe that K_n is fibered if $n \neq -1$ and the genus of K_n is $|n + 1| + 1$ (see [HM, Mu1, O] for more details). Hence Dunfield, Friedl and Jackson's conjecture holds for $(-2, 3, 2n + 1)$ -pretzel knots. Furthermore, the second highest coefficients of the twisted Alexander polynomials of K_n associated to their holonomy representations are 0 for $n > 2$ and are -2 for $n \leq -2$, i.e. the second highest coefficients are real when K_n is fibered. This result coincide with the question of Dunfield, Friedl and Jackson.

On the other hand, $(-2, 3, 2n + 1)$ -pretzel knot is an infinite family of knots which contains the Fintushel-Stern knot i.e. $(-2, 3, 7)$ -pretzel knot. It plays an important role in studying of exceptional surgeries of knots [Ma] and the A-polynomials of $(-2, 3, 2n + 1)$ -pretzel knot are computed by Tamura-Yokota [TY] and Garoufalidis-Mattman [GaMa].

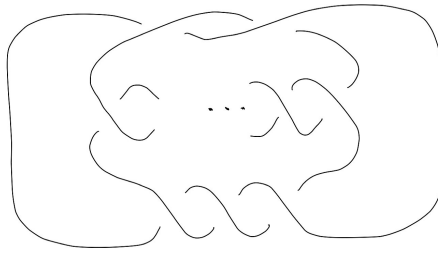


Figure 3.1: $(-2, 3, 2n + 1)$ -pretzel knot

3.1 Presentations and holonomy representations

In this section, we give a presentation of the knot group $G(K_n)$ and its holonomy representation $\rho : G(K_n) \rightarrow SL_2(\mathbb{C})$.

Let L be the link depicted in Figure 2 and $E = S^3 \setminus L$. Then, the Wirtinger presentation (see [CF]) of $\pi_1(E)$ is given by

$$\langle a, b, x \mid \{axba(xb)^{-1}\}^{-1}x = xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb, [x, axba(xb)^{-1}] = 1 \rangle,$$

where a, b and x are Wirtinger generators assigned to the corresponding pass depicted in Figure 2. Note that $E_n := S^3 \setminus K_n$ is obtained from L by $(-\frac{1}{n})$ -surgery along the trivial component, that is, removing the tubular neighborhood of the trivial component and regluing the solid torus again after twisting $-n$ times along the longitude. Therefore, by the van Kampen theorem, we have

$$\pi_1(E_n) = \langle a, b, x \mid \{axba(xb)^{-1}\}^{-1}x = xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb, x = \{axba(xb)^{-1}\}^n \rangle.$$

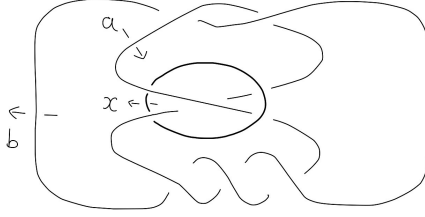


Figure 3.2: Link L

Proposition 3.1. *For a non-zero complex number m , there exists a representation $\rho : \pi_1(E_n) \rightarrow SL_2(\mathbb{C})$ such that*

$$\rho(a) = \begin{pmatrix} m & -\frac{(m^2 - s)(s^{2n+1} + 1)}{m(s+1)} \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(b) = \frac{1}{s\alpha} \begin{pmatrix} \beta & -\frac{(s\alpha - m\beta)(ms\alpha - \beta)}{m\beta} \\ \beta & \frac{m(ms\alpha - \beta) + s\alpha}{m} \end{pmatrix},$$

and

$$\rho(x) = \begin{pmatrix} s^n & 0 \\ \frac{s^n - s^{-n}}{s^{2n+1} + 1} & s^{-n} \end{pmatrix},$$

where s is a solution to

$$\begin{aligned}
0 = & m^8(s-1)(s+1)^2(s^{2n}-s^2)s^{2n+2} \\
& -m^6\{s^{6n+3}+(2s^6+s^5-4s^4+s^3+s^2-s-1)s^{4n+1} \\
& \quad -(s^6+s^5-s^4-s^3+4s^2-s-2)s^{2n+2}+s^6\} \\
& +m^4\{(s^2+1)s^{6n+2}+(s^6+2s^5-3s^4-2s^3+6s^2-4s-2)s^{4n+3} \\
& \quad -(2s^6+4s^5-6s^4+2s^3+3s^2-2s-1)s^{2n}+(s^2+1)s^5\} \\
& -m^2\{s^{6n+3}+(2s^6+s^5-4s^4+s^3+s^2-s-1)s^{4n+1} \\
& \quad -(s^6+s^5-s^4-s^3+4s^2-s-2)s^{2n+2}+s^6\} \\
& +(s-1)(s+1)^2(s^{2n}-s^2)s^{2n+2}
\end{aligned} \tag{3.1}$$

and α, β are given by

$$\begin{aligned}
\alpha = & (s^2-1)s^{2n}\{-m^6(s-1)s^2(s^{2n+1}+1)+m^4(s^{2n+2}(s^4-2s^2+3s-1)+s^4-3s^3+2s^2-1) \\
& -m^2s(s^{2n}(2s^3-s^2+1)-s(s^3-s+2))+s^2(s^{2n}-s^2)\}, \\
\beta = & m^7s^{2n+2}(s^2-1)(s^3+1) \\
& -m^5s^3\{s^{4n}(s^3-s^2+1)+s^{2n-2}(s-1)(s^3+s+1)(s^3+s^2+1)-(s^3-s+1)\} \\
& +m^3s^2(s^3+1)(s^{2n}-1)(s^{2n}+s^2)-ms^3(s^{2n}-s^2)(s^{2n}+s).
\end{aligned}$$

In what follows, for simplicity, we denote the right hand side of (1) by r_0 .

Proof. For simplicity, put $A = \rho(a)$, $B = \rho(b)$, $X = \rho(x)$. By the aid of Mathematica, we have

$$\begin{aligned}
& AXBA(XB)^{-1} \\
= & \begin{pmatrix} \frac{s}{s^2-1} & 0 \\ \frac{1}{s^{2n+1}+1} & \frac{1}{s} \end{pmatrix} + r_1 \begin{pmatrix} \frac{1}{m^3s(s^{2n+1}+1)\alpha^2} & -\frac{1}{m^3s(s+1)\alpha^2} \\ \frac{s+1}{m^3s^2(s^{2n+1}+1)^2\alpha^2} & -\frac{1}{m^3s^2(s^{2n+1}+1)\alpha^2} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
r_1 = & -\alpha^2ms(m^2s^{2n+2}-m^2-s^{2n+1}+s)+\alpha\beta(m^2-1)(m^2+1)s^{2n+1}(s+1) \\
& +\beta^2ms^{2n}(m^2s^{2n+1}-m^2s-s^{2n+2}+1) \equiv 0 \pmod{r_0}.
\end{aligned}$$

Therefore, by (1), we have $X = \{AXBA(XB)^{-1}\}^n$, that is, $\rho(x) = \rho(\{axba(xb)^{-1}\}^n)$.

On the other hand, we can observe

$$AXB\{AXBA(XB)^{-1}\} \equiv XBX^{-1}\{AXBA(XB)^{-1}\}XB \pmod{r_0}$$

and so $AXB\{AXBA(XB)^{-1}\} = XBX^{-1}\{AXBA(XB)^{-1}\}XB$ by (1). Further more, we obtain

$$\begin{aligned}
XB\{AXBA(XB)^{-1}\}^{-1}(AXB)^{-1}XB &= XB(AXB\{AXBA(XB)^{-1}\})^{-1}XB \\
&= XB(XBX^{-1}\{AXBA(XB)^{-1}\}XB)^{-1}XB \\
&= \{AXBA(XB)^{-1}\}^{-1}X
\end{aligned}$$

that is, $\rho(\{axba(xb)^{-1}\}^{-1}x) = \rho(xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb)$. This completes the proof. \square

Remark 3.2. Since the representation ρ comes from the holonomy representation obtained from the ideal triangulation of E given in [TY], the holonomy representation of $G(K_n)$ is given by the solution to (1) which maximizes the hyperbolic volume of $S^3 \setminus K_n$.

3.2 Calculation of the twisted Alexander polynomial

For simplicity, we put

$$\begin{aligned} H &= 1 - m^2 s + m^2 s^{2n+1} - s^{2n+2}, \\ \eta_1 &= m\alpha - ms^{2n+1}\alpha + s^{2n}\beta + m^2 s^{2n}\beta, \\ \eta_2 &= -ms\alpha + ms^{2n+1}\alpha - s^{2n}\beta - s^{2n+1}\beta. \end{aligned}$$

The following is the main result of this chapter.

Theorem 3.3. *The twisted Alexander polynomial of K_n associated to ρ is given by the following.*

(i) $n > 2$

$$\Delta_{K_n, \rho}(t) = 1 + \sum_{i=0}^{2n-1} \lambda_i (t^{i+3} + t^{4n-i+3}) + t^{4n+6},$$

where

$$\lambda_i = \begin{cases} \frac{s(m^2+1)}{m} \left(s^{\frac{i}{2}} - \frac{\eta_1 + \eta_2}{H\beta} (s^{\frac{i}{2}+1} - s^{-\frac{i}{2}-1}) \right) & \text{if } 0 \leq i \leq 2n-2 \text{ and } i \text{ is even,} \\ \frac{s^{\frac{i-1}{2}} - s^{-\frac{i-1}{2}}}{s - s^{-1}} & \text{if } 0 \leq i \leq 2n-2 \text{ and } i \text{ is odd,} \\ \frac{s^{n-1} - s^{-(n-1)}}{s - s^{-1}} - \frac{(s^2-1)\eta_1}{Hs^n\beta} & \text{if } i = 2n-1 \end{cases}$$

(ii) $n = -1$

$$\Delta_{K_{-1}, \rho}(t) = \frac{ms(s-1)\alpha + (3m^2+1)\beta}{m^2\beta} - \frac{2(m^2+1)}{m}t + \frac{ms(s-1)\alpha + (3m^2+1)\beta}{m^2\beta}t^2$$

(iii) $n = -2$

$$\begin{aligned} \Delta_{K_{-2}, \rho}(t) &= 1 + t^6 - \frac{m^2+1}{m}(t+t^5) \\ &\quad + \left(\frac{ms^2(s^3-1)\alpha + (m^2+1)s\beta}{(m^2-s+m^2s^2)\beta} + \frac{s^2+s+1}{s} \right) (t^2+t^4) \\ &\quad - \frac{2(m^2+1)s((s-1)s^2\alpha + m\beta)}{(m^2-s+m^2s^2)\beta} t^3 \end{aligned}$$

(iv) $n < -2$

$$\Delta_{K_n, \rho}(t) = \sum_{i=0}^{-2n-1} \lambda_i(t^i + t^{-4n-2-i}),$$

where

$$\lambda_i = \begin{cases} \frac{s^{\frac{i}{2}+1} - s^{-\frac{i}{2}-1}}{s - s^{-1}} & \text{if } i \text{ is even and } i \neq 2, -2n-2, \\ \frac{1 + s + s^2}{s} & \text{if } i = 2, \\ -\frac{s^{-n-1}(Hs\beta(s^{2n}-1) + (s^2-1)^2\eta_1)}{H(s-s^{-1})\beta} & \text{if } i = -2n-2, \\ \frac{s(m^2+1)}{m} \left(\frac{\eta_1 + \eta_2}{H\beta} (s^{-\frac{i-1}{2}} - s^{\frac{i-1}{2}}) - s^{-\frac{i-1}{2}-1} \right) & \text{if } i \text{ is odd} \end{cases}$$

By Theorem 3.3, we can observe the following.

Corollary 3.4. *The coefficient of the second highest degree of the twisted Alexander polynomial Δ_{K_n, ρ_1} is real if K_n is a fibered knot i.e. the second coefficients are*

$$\begin{cases} 0 & \text{if } n > 2, \\ -2 & \text{if } n = -2, \\ -2 & \text{if } n < -2, \end{cases}$$

when $m = 1$, which corresponds to their holonomy representations.

Proof. By Theorem 3.3, the second coefficients are written by

$$\begin{cases} \frac{s(m^2+1)}{m} \left(s^{\frac{0}{2}} - \frac{\eta_1 + \eta_2}{H\beta} (s^{\frac{0}{2}+1} - s^{-\frac{0}{2}-1}) \right) & \text{if } n > 2, \\ -\frac{m^2+1}{m} & \text{if } n = -2, \\ \frac{s(m^2+1)}{m} \left(\frac{\eta_1 + \eta_2}{H\beta} (s^{-\frac{1-1}{2}} - s^{\frac{1-1}{2}}) - s^{-\frac{1-1}{2}-1} \right) & \text{if } n < -2. \end{cases}$$

The proof is done by setting $m = 1$. □

To prove Theorem 3.3, it suffices to show

Proposition 3.5. *For simplicity, we put $S = s^n$ and $T = t^n$. The twisted Alexander polynomial $\Delta_{K_n, \rho}(t)$ is given by*

$$\begin{aligned} & \frac{S - T^2}{s - t^2} \frac{s}{S} \left(\frac{mst - mStT^2 + (1+m^2)(1-s^2)St^2T^2}{m(1-s^2)t^3} + \frac{(1+m^2)(1-sSt^2T^2)(\eta_1 + \eta_2)}{Hmt^3\beta} \right) \\ & + \frac{1 - ST^2}{1 - st^2} \frac{s}{S} \left(\frac{(1+m^2)(1-s^2)S - mSt + mstT^2}{m(1-s^2)t^3} - \frac{(1+m^2)(sS - t^2T^2)(\eta_1 + \eta_2)}{Hmt^3\beta} \right) \\ & + \frac{1}{t^6} + T^4 + \frac{(1-s^2)(1+t^2)T^2\eta_1}{HSt^4\beta}. \end{aligned}$$

In fact, by using

$$\frac{S - T^2}{s - t^2} = \frac{|n|}{n} s^{\frac{1}{2}(|n|+n)-1} t^{n-|n|} \sum_{i=0}^{|n|-1} \frac{t^{2i}}{s^i}, \quad \frac{ST^2 - 1}{st^2 - 1} = \frac{|n|}{n} s^{\frac{1}{2}(n-|n|)} t^{n-|n|} \sum_{i=0}^{|n|-1} s^i t^{2i},$$

for the formula of Proposition 3.5, we obtain

$$\begin{aligned} \frac{1}{2} \sum_{i=0}^{2|n|-1} \left\{ \left(1 + \frac{|n|}{n} (-1)^i \right) \kappa_i + \left(1 - \frac{|n|}{n} (-1)^i \right) \lambda_i \right\} \left(t^i + t^{\frac{|n|}{n}(4n+1)-1-i} \right) \\ + t^{\frac{|n|}{2n}(4n+1)-\frac{1}{2}} \left(t^{-2n-3} - \frac{(s^2 - 1)\eta_1}{Hs^n\beta} (t^{-1} + t) + t^{2n+3} \right) \end{aligned}$$

up to $t^{-\frac{3}{2n}(n+|n|)}$, where

$$\begin{aligned} \kappa_i &= \frac{s(m^2 + 1)}{m} \left\{ \frac{\eta_1 + \eta_2}{H\beta} \left(s^{-\frac{1}{4}(2i + \frac{3|n|}{n} + 1)} - s^{\frac{1}{4}(2i + \frac{3|n|}{n} + 1)} \right) + \frac{|n|}{n} s^{\frac{|n|}{4n}(2i+1)-\frac{1}{4}} \right\}, \\ \lambda_i &= \frac{1}{s - s^{-1}} \left(s^{\frac{1}{4}(2i - \frac{3|n|}{n} + 1)} - s^{-\frac{1}{4}(2i - \frac{3|n|}{n} + 1)} \right). \end{aligned}$$

Then, by multiplying $t^{2(|n|-n+1)+\frac{1}{2n}(|n|+n)}$ and rearranging with respect to t , we obtain Theorem 3.3.

3.3 Proof of Proposition 3.5

Recall that

$$\begin{aligned} \pi_1(E_n) &= \langle a, b, x \mid \{axba(xb)^{-1}\}^{-1}x = xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb, \ x = \{axba(xb)^{-1}\}^n \rangle \\ &= \langle a, c \mid (acac^{-1})^{n-1} = c(acac^{-1})^{-1}(ac)^{-1}c \rangle. \end{aligned}$$

First of all, we suppose $(n > 2)$. Then the twisted Alexander polynomial of K_n is given by

$$\Delta_{K_n, \rho}(t) = \frac{\left| (\rho \otimes \mathbf{a}) \circ \phi \left(\frac{\partial}{\partial a} (acac^{-1})^{n-1} - \frac{\partial}{\partial a} c(acac^{-1})^{-1}(ac)^{-1}c \right) \right|}{|(\rho \otimes \mathbf{a}) \circ \phi(c - 1)|},$$

where

$$\begin{aligned} &(\rho \otimes \mathbf{a}) \circ \phi \left(\frac{\partial}{\partial a} (acac^{-1})^{n-1} - \frac{\partial}{\partial a} c(acac^{-1})^{-1}(ac)^{-1}c \right) \\ &= \sum_{i=1}^{n-1} t^{2(i-1)} \rho \left(\{axba(xb)^{-1}\}^{i-1} \right) \left\{ \rho(1) + t^{2(n+1)} \rho(axb) \right\} + t^{4n+1} \rho(xbxa^{-1}) \\ &+ t^{2n-1} \rho \left(xb \{axba(xb)^{-1}\}^{-1} \right) + t^{-3} \rho \left(xb \{axba(xb)^{-1}\} (axb)^{-1} \right). \end{aligned} \tag{3.2}$$

For simplicity, we put

$$\gamma_1 = s\alpha - m\beta, \quad \gamma_2 = ms\alpha - \beta, \quad \gamma_3 = m^2s(sS^2 + 1)\alpha.$$

By the aid of Mathematica, the first term of the right hand side of (2) is given by

$$\begin{aligned} & \sum_{i=1}^{n-1} t^{2(i-1)} (AXBA(XB)^{-1})^{i-1} (E + t^{2(n+1)} AXB) \\ &= \begin{pmatrix} \frac{(ST^2 - st^2)(St^2\beta T^2 + m\alpha)}{mst^2(st^2 - 1)\alpha} & -\frac{T^2(ST^2 - st^2)(\gamma_1\eta_2 + (m\alpha - \beta)\gamma_3)}{m^2s(s+1)S(st^2 - 1)\alpha\beta} \\ \frac{mC_1\alpha - St^2T^2C_2\beta}{msS(sS^2 + 1)t^2(s - t^2)(st^2 - 1)\alpha} & \frac{C_3t^4T^4 + C_4t^2T^4 + C_5t^6T^2 + C_6t^4T^2 + C_7}{(s+1)S^2t^2(s - t^2)(st^2 - 1)\gamma_3\beta} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} C_1 &= -t^4s(s^2 - 1)S - T^2\{t^2(S^2 - s^4) - s(S^2 - s^2)\}, \\ C_2 &= -t^2(t^2 - 1)s(s+1)S + T^2\{t^2(S^2 + s^3) + s(S^2 - s)\}, \\ C_3 &= (s^3 + S^2)\gamma_1\eta_2 - \{s^3(ms\alpha + \beta) - S^2(m\alpha - \beta)\}\gamma_3, \\ C_4 &= -s(s + S^2)\gamma_1\eta_2 + s\{s(ms\alpha + \beta) - S^2(m\alpha - \beta)\}\gamma_3, \\ C_5 &= -s(s+1)S\{\gamma_1\eta_2 + (\eta_1 + \eta_2 - (1 + m^2S^2 - sS^2)\beta)\gamma_3\}, \\ C_6 &= s(s+1)S\{s\alpha\eta_2 - m(s+1)S^2\beta\gamma_2\}, \\ C_7 &= s(s+1)S(st^2 - 1)(St^2 - sT^2)\beta\gamma_3. \end{aligned}$$

Similarly, the coefficient of the second term of the right hand side of (2) is given by

$$XBXA^{-1} = \begin{pmatrix} \frac{S^2D_1}{\gamma_3\alpha} & \frac{msD_1D_2 - (sS^2 + 1)(sS^2D_1 + m\gamma_3\alpha)\beta^2}{(s+1)\gamma_3\alpha\beta^2} \\ \frac{(s+1)D_2}{(sS^2 + 1)\gamma_3\alpha} & \frac{msS^2D_1D_2 + s(sS^2 + 1)(m^2s\alpha^2 - S^2\beta^2)D_2}{S^2(sS^2 + 1)\gamma_3\alpha\beta^2} - m \end{pmatrix},$$

where

$$\begin{aligned} D_1 &= -(s+1)\alpha\gamma_2 + m(\eta_1 + \gamma_2 + mS^2\gamma_1)\beta, \\ D_2 &= -\alpha\eta_2 + mS^2(\eta_1 + mS^2\gamma_1 + \gamma_2)\beta, \end{aligned}$$

the coefficient of the third term of the right hand side of (2) is given by

$$XB\{AXBA(XB)^{-1}\}^{-1} = \begin{pmatrix} \frac{SE_1}{ms(sS^2 + 1)\alpha\beta} & -\frac{S\gamma_1\gamma_2}{m\alpha\beta} \\ \frac{(s+1)E_2}{msS(sS^2 + 1)^2\alpha\beta} & \frac{E_3}{mS(sS^2 + 1)\alpha\beta} \end{pmatrix},$$

where

$$\begin{aligned} E_1 &= (s^2 - 1)\alpha\gamma_2 + m(\eta_1 + mS^2\gamma_1 - s\gamma_2)\beta, \\ E_2 &= (s - 1)\alpha\eta_2 + mS^2(\eta_1 + mS^2\gamma_1 - s\gamma_2)\beta, \\ E_3 &= -s\alpha\eta_2 + m(s+1)S^2\beta\gamma_2, \end{aligned}$$

and the coefficient of the fourth term of the right hand side of (2) is given by

$$XB(AXBAXBA(XB)^{-1})^{-1} = \begin{pmatrix} \frac{mF_3}{\gamma_3^2\beta^2} & \frac{F_4}{m(s+1)\gamma_3\alpha\beta^2} \\ \frac{m(s^2-1)F_1F_2}{S^2(sS^2+1)\gamma_3^2\beta^2} & \frac{mF_5}{S^2\gamma_3^2\beta^2} \end{pmatrix},$$

where

$$\begin{aligned} F_1 &= m(s+1)S^2(\eta_1 + mS^2\gamma_1)\beta - \eta_2\alpha, \\ F_2 &= m(s+1)S^2(sS^2+1)\beta^2 - sF_1, \\ F_3 &= -\{m\beta(\eta_1 + mS^2\gamma_1) + s\gamma_1\gamma_2 - \gamma_2\alpha\}F_2 + ms(s+1)S^2(sS^2+1)\gamma_1\gamma_2\beta^2, \\ F_4 &= (s^2-1)\{m(\eta_1 + mS^2\gamma_1)\beta - \gamma_2\alpha\}F_2 \\ &\quad + \gamma_3\{m\gamma_2\alpha - (m^2\eta_1 + s^2\eta_2 + m^3S^2\gamma_1 - s^2(S^2-1)\gamma_2)\beta - ms\gamma_1\gamma_2\}\alpha, \\ F_5 &= (s-1)(sF_1 - m\gamma_3\alpha)F_2 - m^2S^2(sS^2+1)\gamma_3\alpha\beta^2. \end{aligned}$$

Therefore, the determinant of the right hand side of (2) is written as

$$\frac{\sum_{i,j} U_{i,j} t^i T^j}{m^3 S^2 t^6 (s-t^2)(st^2-1)\beta^2 \iota},$$

where

$$\begin{aligned} U_{0,0} &= U_{4,0} = U_{6,0} = U_{2,4} = U_{10,4} = U_{6,8} = U_{8,8} = U_{12,8} = -m^3 s S^2 \beta^2 \iota, \\ U_{2,0} &= U_{10,8} = m^3 (s^2+1) S^2 \beta^2 \iota, \\ HU_{3,0} &\equiv HU_{9,8} \equiv -m^2 (m^2+1) s S^2 \beta (Hs\beta - (s^2-1)(\eta_1 + \eta_2)) \iota \pmod{r_0}, \\ U_{5,0} &\equiv U_{7,8} \equiv m^2 (m^2+1) s S^2 \beta^2 \iota \pmod{r_0}, \\ HU_{1,2} &\equiv HU_{11,6} \equiv m^2 (m^2+1) (s-1) s S \beta \eta_2 \iota \pmod{r_0}, \\ HU_{2,2} &= HU_{6,2} = HU_{8,2} = HU_{4,6} \equiv HU_{6,6} = HU_{10,6} \equiv m^3 (s^2-1) s S \beta \eta_1 \iota \pmod{r_1}, \\ HU_{3,2} &\equiv HU_{9,6} \equiv m^2 (m^2+1) (s-1) S \beta \{HsS^2\beta - s(sS^2+1)\eta_1 - (s^2S^2+s^2+1)\eta_2\} \iota \pmod{r_0}, \\ H^2U_{4,2} &\equiv H^2U_{8,6} \\ &\equiv m(s-1) s S \{H^2m^3\alpha\beta + H(m^2+1)(m^2s+s+1)\beta\eta_2 - (m^2+1)^2(s^2-1)\eta_2(\eta_1 + \eta_2)\} \iota \\ &\quad \pmod{r_0}, \\ HU_{5,2} &\equiv HU_{7,6} \equiv -m^2 (m^2+1) (s-1) s S \beta \eta_2 \iota \pmod{r_0}, \\ HU_{7,2} &\equiv HU_{5,6} \equiv m^2 (m^2+1) (s-1) s S \beta (HS^2\beta - (sS^2+1)\eta_1 - (sS^2-1)\eta_2) \iota \pmod{r_1}, \\ H^2U_{3,4} &\equiv H^2U_{9,4} \equiv -m^2 (m^2+1) (s-1)^2 s (s+1) \eta_1 \eta_2 \iota \pmod{r_0}, \\ H^2U_{4,4} &= H^2U_{8,4} \\ &\equiv m\{H^2m^2(s^2-s+1)S^2\beta^2 + (m^2+1)^2(s-1)^2 s \eta_2 (-HS^2\beta + (sS^2+1)\eta_1 + sS^2\eta_2)\} \iota \\ &\quad \pmod{r_1}, \\ H^2U_{5,4} &\equiv H^2U_{7,4} \\ &\equiv -(m^2+1) (s-1) s \{(s-1)\eta_2(m^3H\alpha + (m^2+1)\eta_2) + m^2S^2H\beta(H\beta - (s+1)(\eta_1 + \eta_2))\} \iota \\ &\quad \pmod{r_0}, \\ H^2U_{6,4} &\equiv -2ms(HmS\beta - (m^2+1)(s-1)\eta_2)(HmS\beta + (m^2+1)(s-1)\eta_2) \iota \pmod{r_0}, \end{aligned}$$

where we put $\iota = m^2 s^2 (s+1) S (sS^2 + 1)^3 \alpha^3 \beta$, and the other $U_{i,j}$'s are 0.

On the other hand, by the aid of Mathematica,

$$\begin{aligned} |(\rho \otimes \mathbf{a}) \circ \phi(c-1)| &= \left| t^{2n+1} \rho(xb) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| \\ &= \frac{mSH\beta + mSHt^2T^4\beta - (m^2+1)(s-1)tT^2\eta_2}{mSH\beta} - \frac{(S^2-1)tT^2}{mS(sS^2+1)H\alpha\beta} r_1 \\ &= \frac{mSH\beta + mSHt^2T^4\beta - (m^2+1)(s-1)tT^2\eta_2}{mSH\beta}. \end{aligned}$$

Consequently, we have

$$\Delta_{K_n, \rho}(t) = \frac{\sum_{i,j} V_{i,j} t^i T^j}{Hm^2 S t^6 (s-t^2)(st^2-1)\beta}, \quad (3.3)$$

where

$$\begin{aligned} V_{0,0} &= V_{4,0} = V_{6,0} = V_{4,4} = V_{6,4} = V_{10,4} = -Hm^2 s S \beta, \\ V_{2,0} &= V_{8,4} = Hm^2 (s^2+1) S \beta, \\ V_{3,0} &= V_{7,4} = m(m^2+1) s S \{ (s^2-1)(\eta_1+\eta_2) - Hs\beta \}, \\ V_{5,0} &= V_{5,4} = Hm(m^2+1) s S \beta, \\ V_{2,2} &= V_{8,2} = m^2 s (s^2-1) \eta_1, \\ V_{3,2} &= V_{7,2} = m(m^2+1) (s-1) s \{ (s+1)\eta_1 + \eta_2 \} \\ V_{4,2} &= V_{6,2} = (s-1) s \{ (m^2+1)\eta_2 + Hm^3 \alpha \}, \\ V_{5,2} &= -2m(m^2+1) (s-1) s \eta_2, \end{aligned}$$

and the other $V_{i,j}$'s are 0. By the aid of Mathematica, the difference between the right hand side of (3) and the formula in Proposition 3.5 is equal to

$$\frac{s\zeta_1 + t\zeta_2 - 2t^2\zeta_1 + t^3\zeta_2 + st^4\zeta_1}{Hm^2 S t^3 (s+1)(s-t^2)(st^2-1)\beta} T^2,$$

where

$$\begin{aligned} \zeta_1 &= m(m^2+1) s (s+1) (HS^2\beta - s(S^2-1)\eta_1 - (sS^2-1)\eta_2), \\ \zeta_2 &= Hm^2 s (m\alpha - ms^2\alpha + s\beta + S^2\beta) - (s^2-1)(m^2\eta_1 + m^2s^3\eta_1 + s\eta_2 + m^2s\eta_2). \end{aligned}$$

Note that $\zeta_1 = 0$ by the definition of H, η_1 and η_2 and that

$$\zeta_2 = m \{ (m^2(s^2-s+1) - s)(s^3S^2+1) - Hs(s-1) \} r_0 = 0.$$

This completes the proof of Proposition 3.5 for $n > 2$.

Second, we suppose $n < 0$. Since the knot group $G(K_n)$ is presented by

$$\begin{aligned} \pi_1(E_n) &= \langle a, c \mid (acac^{-1})^{n-1} = c(acac^{-1})^{-1}(ac)^{-1}c \rangle \\ &= \langle a, c \mid (acac^{-1})^{-n+1} = c^{-1}(ac)(acac^{-1})c^{-1} \rangle, \end{aligned}$$

the twisted Alexander polynomial of K_n is given by

$$\Delta_{K_n, \rho}(t) = \frac{\left| (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial}{\partial a} (acac^{-1})^{-n+1} - \frac{\partial}{\partial a} c^{-1}(ac)(acac^{-1})c^{-1} \right) \right|}{|(\rho \otimes \mathfrak{a}) \circ \phi(c-1)|},$$

where

$$\begin{aligned} & (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial}{\partial a} (acac^{-1})^{-n+1} - \frac{\partial}{\partial a} c^{-1}(ac)(acac^{-1})c^{-1} \right) \\ &= \sum_{i=0}^{-n} t^{2i} \rho \left(\{ axba(xb)^{-1} \}^i \right) \left\{ \rho(1) + t^{2(n+1)} \rho(axb) \right\} - t^{-2n-1} \rho((xb)^{-1}) \\ & \quad - t \rho((xb)^{-1} axb) - t^{2n+3} \rho((xb)^{-1} axbaxb). \end{aligned}$$

By a similar calculation as in case of $n > 2$ and multiplying t^{4n-4} , we can obtain the formula of Proposition 3.5.

Chapter 4

Tunnel number one Montesinos knots

A rational tangle of slope β/α is a tangle as in Figure 4.1. Then, a rational number β/α has continued fraction expansions, i.e.

$$\frac{\beta}{\alpha} = c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \frac{1}{\ddots + \frac{1}{c_k}}}}$$

denoted by $[c_0, c_1, \dots, c_k]$, where c_0, c_1, \dots, c_k are integers. Each integer c_i corresponds to the number of twists on a tangle obtained from the rational tangle depicted in Figure 4.1. A rational number β/α has some continued fraction expansions, however, they has a continued fraction expansion $[2c_0, 2c_1, \dots, 2c_k]$, where either α or β is even.

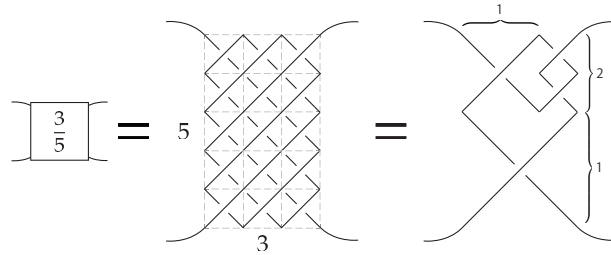


Figure 4.1: A continued fractional expansion of the tangle $\frac{3}{5} = \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}$

A Montesinos knot $M(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_r, \beta_r))$ is the knot depicted in Figure 4.2, where the box β_i/α_i (with $\gcd(\alpha_i, \beta_i) = 1$ for each i) represents a rational tangle of the slope β_i/α_i and each crossing of the twists are opposite if the integer b is negative.

The Montesinos knots is an important class of knots which contains famous families of knots, for example, two-bridge knots and pretzel knots. The genus of all Montesinos knots are completely determined by Hirasawa and Murasugi [HM].

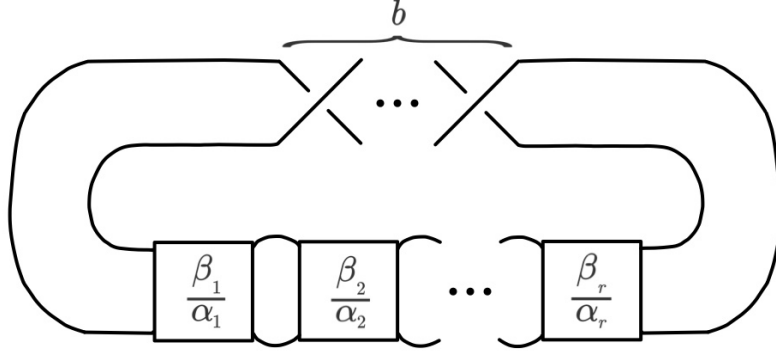


Figure 4.2: Montesinos knot $M(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_r, \beta_r))$

In general, for a knot in S^3 , the tunnel number $\tau(K)$ of K is the minimal number of mutually disjoint properly embedded arcs, say τ_i in the complement of K , such that the complement of an open regular neighborhood of $K \cup (\cup \tau_i)$ is a handle body.

In this chapter, we compute the twisted Alexander polynomials of tunnel number one Montesinos knots. To this end, we use the following theorem.

Theorem 4.1 (Klimenko–Sakuma [KS]). *A Montesinos knot $M(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_r, \beta_r))$ has tunnel number one if and only if one of the following conditions holds up to cyclic permutation of the indices:*

- (1) $r = 2$
- (2) $r = 3$, $\alpha_1 = 2$, and $\alpha_2 \equiv \alpha_3 \equiv 1 \pmod{2}$
- (3) $r = 3$, $\beta_2/\alpha_2 \equiv \beta_3/\alpha_3 \equiv \pm 1/3 \in \mathbb{Q}/\mathbb{Z}$, and $e(K) = b - \sum_{i=1}^r \beta_i/\alpha_i = \pm 1/(3\alpha_1)$

In Section 4.1, we consider the case (1), that is, two-bridge knots. It is a special class of knots which contains twist knots and has many interesting properties [BZ, H]. They are alternating and algebraic knots, and have been completely classified [S]. There are two kinds of famous projections of two-bridge knots and links, i.e. the Schubert presentations and the Conway presentations. In this paper, we use the Conway presentations $C(2m_0, -2m_1, \dots, 2m_{k-1}, -2m_k)$, where k is odd; these presentation contains all two-bridge knots, and if k is even, then they are two-bridge links [Mu3]. Since two-bridge knots and links are alternating, their genus are obtained from the degree of their Alexander polynomials [C, Mu1, Mu2]. For a two-bridge link (knot) $L = C(2c_1, 2c_2, \dots, 2c_l)$, it is known that the genus $g(L)$ and the leading coefficient $\gamma(L)$ of the Alexander polynomial are given by

$$g(L) = \frac{1}{2}(l - \mu + 1),$$

$$\gamma(L) = \prod_{i=1}^l |c_i|,$$

where μ is the number of the components of L [BZ]. On the other hand, it is known that the twisted Alexander polynomials of two-bridge knots associated to their parabolic

representations are nontrivial [SW]. The twisted Alexander polynomials of some class of two-bridge knots were computed, for example, genus one two-bridge knots which contains twist knots [T1, MT]. We compute the twisted Alexander polynomials of all two-bridge knots associated to their $SL_2(\mathbb{C})$ -representations and obtain their leading coefficients and their degree explicitly. We also get the Alexander polynomials and make sure that the result satisfies above equations.

In Section 4.2, we consider the case (2). The family of knots which satisfies the condition (2) is a huge family which contains all $(-2, 2n+1, 2m+1)$ -pretzel knots. This case is quite important because the fiberedness and the genus of knots are not determined by their Alexander polynomials. It is known that there exist nonfibered knots with monic Alexander polynomials, and since knots of the case (2) are not alternating, the degree of the Alexander polynomial is not always $2g$. As in the case (1), we compute the twisted Alexander polynomials of all knots which satisfy the condition (2) associated to their $SL_2(\mathbb{C})$ -representations and explicitly obtain their leading coefficients and their degree. As a corollary, we show that the twisted Alexander polynomials of the exceptional cases i.e. nonfibered knots with monic Alexander polynomials, may have degree $4g-2$ and be non-monic polynomials.

In Section 4.3, we consider the case (3). Knots which satisfy the condition (3) are denoted by

$$K_n = M(0; (3n+2, -2n-1), (3, 1), (3, 1)),$$

where n is an integer [MSY]. In this case, the fiberedness of knots are determined by their Alexander polynomials. Recall that K_n is not a hyperbolic knot when $n = -1, 0$, and so we consider each case for odd $n \neq -1$ and for even $n \neq 0$, and compute the twisted Alexander polynomials of each case associated to their $SL_2(\mathbb{C})$ -representations. We obtain the degree and all the coefficients in the case (3).

4.1 The case (1)

Throughout this section, let K be a two-bridge knot depicted in Figure 4.3, where k is odd and $2m_0, -2m_1, \dots, -2m_k$ denotes the numbers of half twists in each of boxes.

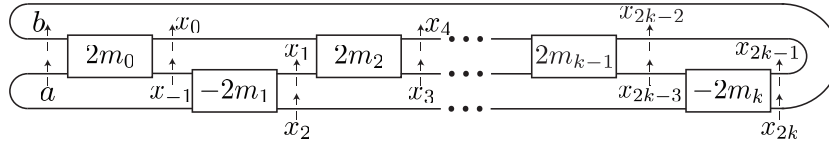


Figure 4.3: Two-bridge knots

We choose generators $a, b, x_{-1}, \dots, x_{2k}$ of $G(K)$ depicted in Figure 4.3. Then, the relators are given by

$$\begin{aligned} r_{2i-1} : x_{2i-1} &= (x_{2i-3}^{(-1)^i} x_{2i-4}^{(-1)^i})^{m_i} x_{2i-3} (x_{2i-3}^{(-1)^i} x_{2i-4}^{(-1)^i})^{-m_i}, \\ r_{2i} : x_{2i} &= (x_{2i-3}^{(-1)^i} x_{2i-4}^{(-1)^i})^{m_i} x_{2i-4} (x_{2i-3}^{(-1)^i} x_{2i-4}^{(-1)^i})^{-m_i}, \end{aligned}$$

where we consider $x_{-4} = b$, $x_{-3} = a$, $x_{-2} = a^{-1}$, together with

$$\begin{aligned} r_{2k-1} : x_{2k-1} &= x_{2k-2}^{-1}, \\ r_{2k} : x_{2k} &= b, \end{aligned}$$

The following is the main result in this section.

Theorem 4.2. *Let K be a knot depicted in Figure 4.3. Then, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ of K associated to a nonabelian representation $\rho : G(K) \rightarrow SL_2(\mathbb{C})$ is given by*

$$\Delta_{K,\rho}(t) = \prod_{i=0}^k \left| \sum_{j=1}^{|m_i|} (\rho(x_{2i-3}^{(-1)^i} x_{2i-4}^{(-1)^i}))^j \right| (1 + t^{2k}) + \sum_{i=1}^k \lambda_i(t^i + t^{2k-i}).$$

Note that

$$G(K) = \langle b, a, x_{-1}, \dots, x_{2k} \mid r_{-1}, \dots, r_{2k}, r_{2k+2} \rangle.$$

Then we put

$$\begin{aligned} R_{-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{-1}}{\partial a} \right), \quad R_0 = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_0}{\partial a} \right), \\ R_1 &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_1}{\partial a} \right), \quad R'_1 = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_1}{\partial x_{-1}} \right), \\ R_2 &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_2}{\partial a} \right), \quad R'_2 = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_2}{\partial x_{-1}} \right), \end{aligned}$$

and

$$\begin{aligned} R_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-4}} \right), \quad R'_{2i-1} = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-3}} \right), \\ R_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-4}} \right), \quad R'_{2i} = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-3}} \right), \end{aligned}$$

for $2 \leq i \leq k$.

We prepare two lemmas to prove Theorem 4.2.

Lemma 4.3. *The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ of K associated to a representation $\rho : G(K) \rightarrow SL_2(\mathbb{C})$ is given by*

$$\Delta_{K,\rho}(t) = \frac{|N_{2k}|}{t^2 - \text{tr} \rho(b)t + 1},$$

where the matrix N_{2k} is defined by

$$\begin{aligned} N_0 &= R_0, \\ N_1 &= R_1 + (-R'_1)R_{-1}, \\ N_2 &= R_2 + (-R'_2)R_{-1}, \end{aligned}$$

and

$$\begin{aligned} N_{2i-1} &= (-R_{2i-1})N_{2i-4} + (-R'_{2i-1})N_{2i-3}, \\ N_{2i} &= (-R_{2i})N_{2i-4} + (-R'_{2i})N_{2i-3}, \end{aligned}$$

for $2 \leq i \leq k$.

Proof. By the definition of twisted Alexander polynomials, we have

$$\Delta_{K,\rho}(t) = \frac{|M_{\rho,1}|}{|(\rho \otimes \mathfrak{a}) \circ \phi(b-1)|} = \frac{|M_{\rho,1}|}{t^2 - \text{tr}\rho(b)t + 1}.$$

By simple calculation, we have

$$\begin{aligned} |M_{\rho,1}| &= \begin{vmatrix} R_{-1} & E & O & O & \dots & O & O & O & O & O \\ R_0 & O & E & O & \dots & O & O & O & O & O \\ R_1 & R'_1 & O & E & & O & O & O & O & O \\ R_2 & R'_2 & O & O & \ddots & O & O & O & O & O \\ O & O & R_3 & R'_3 & & O & O & O & O & O \\ O & O & R_4 & R'_4 & & O & O & O & O & O \\ \vdots & \vdots & & & \ddots & & & \ddots & & \vdots \\ O & O & O & O & & R_{2k-1} & R'_{2k-1} & O & E & O \\ O & O & O & O & & R_{2k} & R'_{2k} & O & O & E \\ O & O & O & O & \dots & O & O & O & O & E \end{vmatrix} \\ &= \begin{vmatrix} R_{-1} & E & O & O & \dots & O & O & O & O & O \\ N_0 & O & E & O & \dots & O & O & O & O & O \\ N_1 & O & O & E & & O & O & O & O & O \\ N_2 & O & O & O & \ddots & O & O & O & O & O \\ N_3 & O & O & O & & O & O & O & O & O \\ N_4 & O & O & O & & O & O & O & O & O \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & & \vdots \\ N_{2k-1} & O & O & O & \dots & O & O & O & E & O \\ N_{2k} & O & O & O & \dots & O & O & O & O & E \\ O & O & O & O & \dots & O & O & O & O & E \end{vmatrix} \\ &= |N_{2k}|, \end{aligned}$$

where

$$\begin{aligned} N_0 &= R_0, \\ N_1 &= R_1 + (-R'_1)R_{-1}, \\ N_2 &= R_2 + (-R'_2)R_{-1}, \end{aligned}$$

and

$$\begin{aligned} N_{2i-1} &= (-R_{2i-1})N_{2i-4} + (-R'_{2i-1})N_{2i-3}, \\ N_{2i} &= (-R_{2i})N_{2i-4} + (-R'_{2i})N_{2i-3}, \end{aligned}$$

for $2 \leq i \leq k$. □

Now, we compute the matrix N_{2k} . The following gives the highest and the lowest degree of each N_{2i-1} and N_{2i} .

Lemma 4.4. *There exist $N_{2i-1}^j, N_{2i}^j \in M_2(\mathbb{C})$ such that*

$$N_{2i-1} = \sum_{j=-\frac{i}{2}-1}^{\frac{i}{2}} t^j N_{2i-1}^j, \quad N_{2i} = \sum_{j=-\frac{i}{2}}^{\frac{i}{2}+1} t^j N_{2i}^j,$$

if i is even, and

$$N_{2i-1} = \sum_{j=-\frac{i+1}{2}}^{\frac{i+1}{2}} t^j N_{2i-1}^j, \quad N_{2i} = \sum_{j=-\frac{i+1}{2}+1}^{\frac{i+1}{2}+1} t^j N_{2i}^j.$$

if i is odd.

Proof. We put $A = \rho(a)$, $B = \rho(b)$ and $X_i = \rho(x_i)$ for $i = -1, \dots, 2k$. By Appendix A, we can put

$$\begin{aligned} R_0 &= t^0 R_0^0 + t^1 R_0^1, \\ R_{-1} &= t^{-1} R_{-1}^{-1} + t^0 R_{-1}^0, \\ R_1 &= t^0 R_1^0 + t^1 R_1^1, \\ -R'_1 &= t^0 R'_1{}^0 + t^1 R'_1{}^1, \\ R_2 &= t^1 R_2^1 + t^2 R_2^2, \\ -R'_2 &= t^1 R'_2{}^1 + t^2 R'_2{}^2, \end{aligned}$$

and

$$\begin{aligned} -R_{2i-1} &= t^{-2} R_{2i-1}^{-2} + t^{-1} R_{2i-1}^{-1}, \\ -R'_{2i-1} &= t^{-1} R'_{2i-1}{}^{-1} + t^0 R'_{2i-1}{}^0, \\ -R_{2i} &= t^{-1} R_{2i}^{-1} + t^0 R_{2i}^0, \\ -R'_{2i} &= t^0 R'_{2i}{}^0 + t^1 R'_{2i}{}^1, \end{aligned}$$

if i is even and $2 \leq i \leq k$,

$$\begin{aligned} -R_{2i-1} &= t^{-1} R_{2i-1}^{-1} + t^0 R_{2i-1}^0, \\ -R'_{2i-1} &= t^0 R'_{2i-1}{}^0 + t^1 R'_{2i-1}{}^1, \\ -R_{2i} &= t^0 R_{2i}^0 + t^1 R_{2i}^1, \\ -R'_{2i} &= t^1 R'_{2i}{}^1 + t^2 R'_{2i}{}^2, \end{aligned}$$

if i is odd and $2 \leq i \leq k$, where $R_i^j, R'_i{}^j \in M_2(\mathbb{C})$.

Then we show the existence of $N_{2i-1}^j, N_{2i}^j \in M_2(\mathbb{C})$ by using the induction on i . First, by direct calculation, we get

$$\begin{aligned} N_0 &= R_0 = t^0 R_0^0 + t^1 R_0^1, \\ N_1 &= R_1 + (-R'_1)R_{-1} = t^{-1} \left(R_1^0 R_{-1}^{-1} \right) + t^0 \left(R_1^0 + R_1^0 R_{-1}^0 + R_1^1 R_{-1}^{-1} \right) + t^1 \left(R_1^1 + R_1^1 R_{-1}^0 \right), \\ N_2 &= R_2 + (-R'_2)R_{-1} = t^0 \left(R_2^1 R_{-1}^{-1} \right) + t^1 \left(R_2^1 + R_2^1 R_{-1}^0 + R_2^2 R_{-1}^{-1} \right) + t^2 \left(R_2^2 + R_2^2 R_{-1}^0 \right). \end{aligned}$$

Hence the existence is shown for $i = 0$ and $i = 1$.

Second, let i be an even integer and we assume that there exists $N_{2(i-2)}^j, N_{2(i-1)-1}^j \in M_2(\mathbb{C})$, such that

$$N_{2(i-2)} = \sum_{j=-\frac{i-2}{2}}^{\frac{i-2}{2}+1} t^j N_{2(i-2)}^j, \quad N_{2(i-1)-1} = \sum_{j=-\frac{(i-1)+1}{2}}^{\frac{(i-1)+1}{2}} t^j N_{2(i-1)-1}^j.$$

Then we have

$$\begin{aligned} N_{2i-1} &= (-R_{2i-1})N_{2i-4} + (-R'_{2i-1})N_{2i-3} \\ &= (t^{-2}R_{2i-1}^{-2} + t^{-1}R_{2i-1}^{-1}) N_{2(i-2)} + \left(t^{-1}R'_{2i-1}{}^{-1} + t^0R'_{2i-1}{}^0\right) N_{2(i-1)-1} \\ &= (t^{-2}R_{2i-1}^{-2} + t^{-1}R_{2i-1}^{-1}) \sum_{j=-\frac{i-2}{2}}^{\frac{i-2}{2}+1} t^j N_{2(i-2)}^j + \left(t^{-1}R'_{2i-1}{}^{-1} + t^0R'_{2i-1}{}^0\right) \sum_{j=-\frac{i-1}{2}-1}^{\frac{i-1}{2}} t^j N_{2(i-1)-1}^j \\ &= t^{-\frac{i}{2}-1} \left(R_{2i-1}^{-2} N_{2(i-2)}^{-\frac{i}{2}+1} + R'_{2i-1}{}^{-1} N_{2(i-1)-1}^{-\frac{i}{2}}\right) + \cdots + t^{\frac{i}{2}} \left(R'_{2i-1}{}^0 N_{2(i-1)-1}^{\frac{i}{2}}\right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} N_{2i} &= (-R_{2i})N_{2i-4} + (-R'_{2i})N_{2i-3} \\ &= t^{-\frac{i}{2}} \left(R_{2i}^{-1} N_{2(i-2)}^{-\frac{i}{2}+1} + R'_{2i}{}^0 N_{2(i-1)-1}^{-\frac{i}{2}}\right) + \cdots + t^{\frac{i}{2}+1} \left(R'_{2i}{}^1 N_{2(i-1)-1}^{\frac{i}{2}}\right). \end{aligned}$$

Lastly, let i be an odd integer and we assume that there exists $N_{2(i-2)}^j, N_{2(i-1)-1}^j \in M_2(\mathbb{C})$, such that

$$N_{2(i-2)} = \sum_{j=-\frac{(i-2)+1}{2}}^{\frac{(i-2)+1}{2}+1} t^j N_{2(i-2)}^j, \quad N_{2(i-1)-1} = \sum_{j=-\frac{i-1}{2}-1}^{\frac{i-1}{2}} t^j N_{2(i-1)-1}^j.$$

Then we have

$$\begin{aligned} N_{2i-1} &= (-R_{2i-1})N_{2i-4} + (-R'_{2i-1})N_{2i-3} \\ &= (t^{-1}R_{2i-1}^{-1} + t^0R_{2i-1}^0) N_{2(i-2)} + \left(t^0R'_{2i-1}{}^0 + t^1R'_{2i-1}{}^1\right) N_{2(i-1)-1} \\ &= (t^{-1}R_{2i-1}^{-1} + t^0R_{2i-1}^0) \sum_{j=-\frac{(i-2)+1}{2}}^{\frac{(i-2)+1}{2}+1} t^j N_{2(i-2)}^j + \left(t^0R'_{2i-1}{}^0 + t^1R'_{2i-1}{}^1\right) \sum_{j=-\frac{i-1}{2}-1}^{\frac{i-1}{2}} t^j N_{2(i-1)-1}^j \\ &= t^{-\frac{i+1}{2}} \left(R'_{2i-1}{}^0 N_{2(i-1)-1}^{-\frac{i+1}{2}}\right) + \cdots + t^{\frac{i+1}{2}} \left(R_{2i-1}^0 N_{2(i-2)}^{\frac{i+1}{2}} + R'_{2i-1}{}^1 N_{2(i-1)-1}^{\frac{i-1}{2}}\right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} N_{2i} &= (-R_{2i})N_{2i-4} + (-R'_{2i})N_{2i-3} \\ &= t^{-\frac{i+1}{2}+1} \left(R'_{2i}{}^1 N_{2(i-1)-1}^{-\frac{i+1}{2}}\right) + \cdots + t^{\frac{i+1}{2}+1} \left(R_{2i}^1 N_{2(i-2)}^{\frac{i+1}{2}} + R'_{2i}{}^2 N_{2(i-1)-1}^{\frac{i-1}{2}}\right). \end{aligned}$$

□

Proof of Theorem 4.2. Since k is odd, we have

$$|N_{2k}| = \left| \sum_{j=-\frac{k+1}{2}+1}^{\frac{k+1}{2}+1} t^j N_{2k}^j \right| = t^{-k+1} \left| N_{2k}^{-\frac{k+1}{2}+1} \right| + \cdots + t^{k+3} \left| N_{2k}^{\frac{k+1}{2}+1} \right|,$$

by Lemma 4.4. Then, we obtain

$$\Delta_{K,\rho}(t) = \frac{|N_{2k}|}{t^2 - \text{tr}\rho(b)t + 1} = t^0 \left| N_{2k}^{-\frac{k+1}{2}+1} \right| + \cdots + t^{2k} \left| N_{2k}^{\frac{k+1}{2}+1} \right|.$$

Now we compute the leading coefficient $\left| N_{2k}^{\frac{k+1}{2}+1} \right|$. We suppose i is odd. From the proof of Lemma 4.4, we have

$$N_{2i-1}^{\frac{i+1}{2}} = R_{2i-1}^0 N_{2(i-2)}^{\frac{i+1}{2}} + R_{2i-1}'^1 N_{2(i-1)-1}^{\frac{i-1}{2}}, \quad N_{2i}^{\frac{i+1}{2}+1} = R_{2i}^1 N_{2(i-2)}^{\frac{i+1}{2}} + R_{2i}'^2 N_{2(i-1)-1}^{\frac{i-1}{2}}.$$

By Appendix A, we have

$$R_{2i}^1 = -X_{2i} R_{2i-1}^0, \quad R_{2i}'^2 = -X_{2i} R_{2i-1}'^1.$$

Then, we have

$$\begin{aligned} N_{2i}^{\frac{i+1}{2}+1} &= R_{2i}^1 N_{2(i-2)}^{\frac{i+1}{2}} + R_{2i}'^2 N_{2(i-1)-1}^{\frac{i-1}{2}} \\ &= -X_{2i} R_{2i-1}^0 N_{2(i-2)}^{\frac{i+1}{2}} - X_{2i} R_{2i-1}'^1 N_{2(i-1)-1}^{\frac{i-1}{2}} \\ &= -X_{2i} \left(R_{2i-1}^0 N_{2(i-2)}^{\frac{i+1}{2}} + R_{2i-1}'^1 N_{2(i-1)-1}^{\frac{i-1}{2}} \right) \\ &= -X_{2i} N_{2i-1}^{\frac{i+1}{2}}. \end{aligned}$$

Furthermore, from the proof of Lemma 4.4, we have

$$N_{2(i-1)-1}^{\frac{i-1}{2}} = R_{2(i-1)-1}'^0 N_{2(i-2)-1}^{\frac{(i-2)+1}{2}}.$$

By using above relations and the relation $R_{2i-1}'^1 = R_{2i-1}^0 X_{2i-4}$ obtained by Appendix A, we get

$$\begin{aligned} N_{2i-1}^{\frac{i+1}{2}} &= R_{2i-1}^0 N_{2(i-2)}^{\frac{i+1}{2}} + R_{2i-1}'^1 N_{2(i-1)-1}^{\frac{i-1}{2}} \\ &= R_{2i-1}^0 \left(-X_{2(i-2)} N_{2(i-2)-1}^{\frac{(i-2)+1}{2}} \right) + (R_{2i-1}^0 X_{2i-4}) \left(R_{2(i-1)-1}'^0 N_{2(i-2)-1}^{\frac{(i-2)+1}{2}} \right) \\ &= R_{2i-1}^0 X_{2i-4} \left(-E + R_{2(i-1)-1}'^0 \right) N_{2(i-2)-1}^{\frac{(i-2)+1}{2}}. \end{aligned}$$

Hence we have

$$\begin{aligned}
N_{2k}^{\frac{k+1}{2}+1} &= -X_{2k} N_{2k-1}^{\frac{k+1}{2}} \\
&= -X_{2k} \left\{ R_{2k-1}^0 X_{2k-4} \left(-E + R_{2(k-1)-1}'^0 \right) \right\} N_{2(k-2)-1}^{\frac{(k-2)+1}{2}} \\
&\quad \vdots \\
&= -X_{2k} \left\{ R_{2k-1}^0 X_{2k-4} \left(-E + R_{2(k-1)-1}'^0 \right) \right\} \cdots \left\{ R_5^0 X_2 \left(-E + R_3'^0 \right) \right\} N_1^1 \\
&= -X_{2k} \left\{ R_{2k-1}^0 X_{2k-4} \left(-E + R_{2(k-1)-1}'^0 \right) \right\} \cdots \left\{ R_5^0 X_2 \left(-E + R_3'^0 \right) \right\} \left(R_1^1 + R_1'^1 R_{-1}^0 \right).
\end{aligned}$$

Then, since

$$\begin{aligned}
R_{2i-1}^0 &= \begin{cases} -\sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j & \text{if } m_i > 0 \\ 0 & \\ \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j & \text{if } m_i < 0 \end{cases}, \\
R_{2(i-1)-1}'^0 &= \begin{cases} E + \sum_{j=1}^{m_{i-1}} (X_{2(i-1)-3} X_{2(i-1)-4})^j & \text{if } m_{i-1} > 0 \\ 0 & \\ E - \sum_{j=m_{i-1}+1}^0 (X_{2(i-1)-3} X_{2(i-1)-4})^j & \text{if } m_{i-1} < 0 \end{cases}, \\
R_1^1 = R_1'^1 &= \begin{cases} -\sum_{j=1}^{m_1} (X_{-1}^{-1} A)^j A^{-1} & \text{if } m_1 > 0 \\ 0 & \\ \sum_{j=m_1+1}^0 (X_{-1}^{-1} A)^j A^{-1} & \text{if } m_1 < 0 \end{cases}, \\
R_{-1}^0 &= \begin{cases} -E - \sum_{j=1}^{m_0} (AB)^j & \text{if } m_0 > 0 \\ 0 & \\ -E + \sum_{j=m_0+1}^0 (AB)^j & \text{if } m_0 < 0 \end{cases},
\end{aligned}$$

by Appendix A, we have

$$\begin{aligned}
|R_{2i-1}^0| &= \left| \sum_{j=1}^{|m_i|} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j \right|, \\
|-E + R_{2(i-1)-1}'^0| &= \left| \sum_{j=1}^{|m_{i-1}|} (X_{2(i-1)-3} X_{2(i-1)-4})^j \right|, \\
|R_1^1 + R_1'^1 R_{-1}^0| &= |R_1^1| |E + R_{-1}^0| = \left| \sum_{j=1}^{|m_1|} (X_{-1}^{-1} A)^j \right| \left| \sum_{j=1}^{|m_0|} (AB)^j \right|.
\end{aligned}$$

Note that $X_{-4} = B$, $X_{-3} = A$, $X_{-2} = A^{-1}$ and $X_i \in SL_2(\mathbb{C})$ for all i , we have

$$\begin{aligned} & \left| N_{2k}^{-\frac{k+1}{2}+1} \right| \\ &= |X_{2k}| \left\{ \left| R_{2k-1}^0 \right| |X_{2k-4}| \left| -E + R_{2(k-1)-1}'^0 \right| \right\} \cdots \left\{ \left| R_5^0 \right| |X_2| \left| -E + R_3'^0 \right| \right\} \left| R_1^1 + R_1'^1 R_{-1}^0 \right| \\ &= \left\{ \left| R_{2k-1}^0 \right| \left| -E + R_{2(k-1)-1}'^0 \right| \right\} \cdots \left\{ \left| R_5^0 \right| \left| -E + R_3'^0 \right| \right\} \left| R_1^1 + R_1'^1 R_{-1}^0 \right| \\ &= \prod_{i=0}^k \left| \sum_{j=1}^{|m_i|} (X_{2i-3}^{(-1)^i} X_{2i-4}^{(-1)^i})^j \right|. \end{aligned}$$

Similarly, we compute $\left| N_{2k}^{-\frac{k+1}{2}+1} \right|$. We suppose i is even. Then from the proof of Lemma 4.4, we have

$$N_{2i-1}^{-\frac{i}{2}-1} = R_{2i-1}^{-2} N_{2(i-2)}^{-\frac{i}{2}+1} + R_{2i-1}'^{-1} N_{2(i-1)-1}^{-\frac{i}{2}}, \quad N_{2i}^{-\frac{i}{2}} = R_{2i}^{-1} N_{2(i-2)}^{-\frac{i}{2}+1} + R_{2i}'^0 N_{2(i-1)-1}^{-\frac{i}{2}}.$$

By Appendix A, we have

$$R_{2i-1}^{-2} = -X_{2i-1} R_{2i}^{-1}, \quad R_{2i-1}'^{-1} = -X_{2i-1} R_{2i}'^0.$$

Hence we have

$$N_{2i-1}^{-\frac{i}{2}-1} = -X_{2i-1} N_{2i}^{-\frac{i}{2}}.$$

Furthermore, from the proof of Lemma 4.4, we have

$$N_{2(i-1)-1}^{-\frac{i}{2}} = R_{2(i-1)-1}'^0 N_{2(i-2)-1}^{-\frac{i}{2}}.$$

Since we have $R_{2i-1}^{-2} = R_{2i-1}'^{-1} X_{2i-3}$ by Appendix A, we get

$$\begin{aligned} N_{2i-1}^{-\frac{i}{2}-1} &= R_{2i-1}^{-2} N_{2(i-2)}^{-\frac{i}{2}+1} + R_{2i-1}'^{-1} N_{2(i-1)-1}^{-\frac{i}{2}} \\ &= \left(R_{2i-1}'^{-1} X_{2i-3} \right) \left(-X_{2(i-2)-1}^{-1} N_{2(i-2)-1}^{-\frac{i}{2}-1} \right) + R_{2i-1}'^{-1} \left(R_{2(i-1)-1}'^0 N_{2(i-2)-1}^{-\frac{i}{2}} \right) \\ &= R_{2i-1}'^{-1} \left(-X_{2i-3} X_{2i-5}^{-1} + R_{2(i-1)-1}'^0 \right) N_{2(i-2)-1}^{-\frac{i}{2}-1}. \end{aligned}$$

Since k is odd, from the proof of Lemma 4.4, we have

$$\begin{aligned} N_{2k}^{-\frac{k+1}{2}+1} &= R_{2k}^1 N_{2(k-1)-1}^{-\frac{(k-1)}{2}-1} \\ &= R_{2k}^1 \left\{ R_{2(k-1)-1}'^{-1} \left(-X_{2(k-1)-3}^{-1} X_{2(k-1)-5}^{-1} + R_{2((k-1)-1)-1}'^0 \right) \right\} N_{2((k-1)-2)-1}^{-\frac{(k-1)-2}{2}-1} \\ &\quad \vdots \\ &= R_{2k}^1 \left\{ R_{2k-3}'^{-1} \left(-X_{2k-5} X_{2k-7}^{-1} + R_{2k-5}'^0 \right) \right\} \cdots \left\{ R_7'^{-1} \left(-X_5 X_3^{-1} + R_5'^0 \right) \right\} N_3^{-2} \\ &= R_{2k}^1 \left\{ R_{2k-3}'^{-1} \left(-X_{2k-5} X_{2k-7}^{-1} + R_{2k-5}'^0 \right) \right\} \cdots \left\{ R_7'^{-1} \left(-X_5 X_3^{-1} + R_5'^0 \right) \right\} \\ &\quad \left(R_3^{-2} N_0^0 + R_3'^{-1} N_1^{-1} \right). \end{aligned}$$

Hence we obtain

$$\begin{aligned}
\left| N_{2k}^{-\frac{k+1}{2}+1} \right| &= \left| R'_{2k} \right| \left\{ \left| R'_{2k-3} \right| \left| -X_{2k-5} X_{2k-7}^{-1} + R'_{2k-5} \right| \right\} \cdots \left\{ \left| R'_{7^{-1}} \right| \left| -X_5 X_3^{-1} + R'_5 \right| \right\} \\
&\quad \left| R_3^{-2} N_0^0 + R_3'^{-1} N_1^{-1} \right| \\
&= \prod_{i=0}^k \left| \sum_{j=1}^{|m_i|} (X_{2i-3}^{(-1)^i} X_{2i-4}^{(-1)^i})^j \right|.
\end{aligned}$$

This completes the proof of Theorem 4.2. \square

Remark 4.5. If we denote the Alexander polynomial of K by $\Delta_K(t)$, then we have

$$\begin{aligned}
\Delta_K(t) &= |N_{2k}| = \left| \sum_{j=-\frac{k+1}{2}+1}^{\frac{k+1}{2}+1} t^j N_{2k}^j \right| \\
&= t^{-\frac{k+1}{2}+1} N_{2k}^{-\frac{k+1}{2}+1} + \cdots + t^{\frac{k+1}{2}+1} N_{2k}^{\frac{k+1}{2}+1}.
\end{aligned}$$

In this case, each N_{2k}^j is scalar and all X_i are 1. Then, as above we have

$$\begin{aligned}
N_{2k}^{\frac{k+1}{2}+1} &= (-1)^{\frac{k+1}{2}} X_{2k} \frac{|m_k m_{k-1} \cdots m_0|}{m_k m_{k-1} \cdots m_0} \\
&\quad \left\{ \sum_j (X_{2k-3}^{-1} X_{2k-4}^{-1})^j X_{2k-4} \sum_j (X_{2k-5} X_{2k-6})^j \right\} \cdots \left\{ \sum_j (X_3^{-1} X_2^{-1})^j X_2 \sum_j (X_1 X_0)^j \right\} \\
&\quad \left\{ \sum_j (X_{-1}^{-1} A)^j A^{-1} \sum_j (AB)^j \right\} \\
&= (-1)^{\frac{k+1}{2}} \frac{|m_k m_{k-1} \cdots m_0|}{m_k m_{k-1} \cdots m_0} |m_k| |m_{k-1}| \cdots |m_0| \\
&= (-1)^{\frac{k+1}{2}} m_k m_{k-1} \cdots m_0
\end{aligned}$$

and

$$\begin{aligned}
N_{2k}^{-\frac{k+1}{2}+1} &= (-1)^{\frac{k+1}{2}} \frac{|m_k \cdots m_0|}{m_k \cdots m_0} \left(\sum_j (X_{2k-3}^{-1} X_{2k-4}^{-1})^j \right) \\
&\quad \left\{ \left(\sum_j (X_{2k-5} X_{2k-6})^j \right) X_{2k-6}^{-1} \left(\sum_j (X_{2k-7}^{-1} X_{2k-8}^{-1})^j \right) \right\} \\
&\quad \vdots \\
&\quad \left\{ \left(\sum_j (X_1 X_0)^j \right) X_0^{-1} \left(\sum_j (X_{-1}^{-1} A)^j \right) \right\} \\
&\quad \left(\sum_j (AB)^j \right)
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{\frac{k+1}{2}} \frac{|m_k m_{k-1} \cdots m_0|}{m_k m_{k-1} \cdots m_0} |m_k| |m_{k-1}| \cdots |m_0| \\
&= (-1)^{\frac{k+1}{2}} m_k m_{k-1} \cdots m_0.
\end{aligned}$$

Hence we can write

$$\Delta_K(t) = |m_k m_{k-1} \cdots m_0| (t^0 + t^{k+1}) + \sum_{i=1}^k t^i \kappa_i,$$

where $\kappa_i \in \mathbb{Z}$. Thus, since the genus of K is given by $\frac{k+1}{2}$, we have

$$\deg(\Delta_K(t)) = 2g(K).$$

These results coincide with the results of [BZ]. Furthermore, we have

$$\deg(\Delta_{K,\rho}(t)) = 4g(K) - 2,$$

if the representation $\rho : G(K) \rightarrow SL_2(\mathbb{C})$ satisfies

$$\left| \sum_{j=1}^{|m_i|} (X_{2i-3}^{(-1)^i} X_{2i-4}^{(-1)^i})^j \right| \neq 0,$$

for $i = 0, 1, \dots, k$. This condition means that any eigenvalue λ of 2×2 matrices $X_{2i-3}^{(-1)^i} X_{2i-4}^{(-1)^i}$ satisfies $\lambda^{m_i} \neq 1$.

4.2 The case (2)

Throughout this section, we suppose K satisfies condition (2) of Theorem 4.1. In this section, we give the presentation of $G(K)$ and compute their twisted Alexander polynomials associated to their $SL_2(\mathbb{C})$ representations. To this end, we prepare two lemmas.

Lemma 4.6. *For non-zero integers α and β , we have the continued fraction expansions*

$$\frac{\beta}{\alpha} = \begin{cases} 2m_0 + \frac{1}{2m_1 + \frac{1}{\ddots + \frac{1}{2m_{2l}}}} & \text{if } \alpha \text{ is odd and } \beta \text{ is even,} \\ 2m_0 + \frac{1}{2m_1 + \frac{1}{\ddots + \frac{1}{2m_{2l+1}}}} & \text{if } \alpha \text{ is even and } \beta \text{ is odd,} \\ 2m_0 + \frac{1}{2m_1 + \frac{1}{\ddots + \frac{1}{2m_l + 1}}} & \text{if } \alpha \text{ and } \beta \text{ are odd,} \end{cases}$$

where $m_i \in \mathbb{Z}$.

Proof. There exist $m_0 \in \mathbb{Z}$ and $\gamma_0 \in \mathbb{Z}$ with $|\gamma_0| < |\alpha|$ such that $\beta = 2m_0\alpha + \gamma_0$. Then we have

$$\frac{\beta}{\alpha} = 2m_0 + \frac{1}{\frac{\alpha}{\gamma_0}}.$$

Similarly, there exist $m_1 \in \mathbb{Z}$ and $\gamma_1 \in \mathbb{Z}$ with $|\gamma_1| < |\gamma_0|$ such that

$$\frac{\beta}{\alpha} = 2m_0 + \frac{1}{2m_1 + \frac{1}{\frac{\gamma_0}{\gamma_1}}}$$

Since $|\gamma_i|$ decrease, by repeating this process, we obtain integers $m_0, \dots, m_k, m' \in \mathbb{Z}$ such that

$$\frac{\beta}{\alpha} = 2m_0 + \frac{1}{2m_1 + \frac{1}{\ddots + \frac{1}{2m_k + \frac{1}{m'}}}}.$$

Consider integers $a, b, c, d, e \in \mathbb{Z}$ such that

$$\frac{a}{b} = 2c + \frac{1}{\frac{d}{e}}.$$

Then we have the following:

- a and b are odd if and only if d and e are odd.
- a is odd and b is even if and only if d is even and e is odd.
- a is even and b is odd if and only if d is odd and e is even.

This completes the proof. □

Lemma 4.7. *The knot K is equivalent to a Montesinos knot $M(0; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ whose three rational tangles β_1/α_1 , β_2/α_2 and β_3/α_3 are written by*

$$\begin{aligned} \frac{\beta_1}{\alpha_1} &= \pm \frac{1}{2}, \\ \frac{\beta_2}{\alpha_2} &= 2m_0 + \frac{1}{2m_1 + \frac{1}{2m_2 + \frac{1}{\ddots + \frac{1}{2m_k}}}}, \\ \frac{\beta_3}{\alpha_3} &= \frac{1}{2n_1 + \frac{1}{2n_2 + \frac{1}{\ddots + \frac{1}{2n_l}}}}, \end{aligned}$$

where both k and l are even and $m_i, n_i \in \mathbb{Z}$.

Proof. For a Montesinos knot $K = M(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$, we can move b half twists into a tangle β_1/α_1 , i.e. we may suppose $b = 0$. Then, since $\alpha_1 = 2$, we have

$$\frac{\beta_1}{\alpha_1} = \frac{2m+1}{2} = m + \frac{1}{2},$$

and we can move m half twists into tangle β_2/α_2 . Hence we put

$$\frac{\beta'_1}{\alpha_1} = \frac{\beta_1}{\alpha_1} - m = \frac{1}{2}.$$

Similarly, by moving some twists from the tangle β_3/α_3 into the tangle β_2/α_2 if necessary, we suppose $-\alpha_3 < \beta_3 < \alpha_3$.

The numbers α and β of the rational tangle β/α corresponds to the number of lines of the tangle, i.e. α means the number of horizontal lines and β means the number of the vertical lines of the tangle. Then, we have the following three cases:

- (i) both β_2 and β_3 are even.
- (ii) one of β_2 and β_3 is odd and the other is even.
- (iii) both β_2 and β_3 are odd.

In the case (i), our assertion follows from Lemma 4.6.

In the case (ii), if β_2 is odd and β_3 is even, as in in Figure 4.4, we add two half twists and denote the tangle in the dotted box by $\frac{\beta'_2}{\alpha_2}$, where β'_2 is even because

$$\frac{\beta'_2}{\alpha_2} = \frac{\beta_2}{\alpha_2} + 1 = \frac{\beta_2 + \alpha_2}{\alpha_2}.$$

Then we obtain new tangle

$$\frac{\beta''_1}{\alpha_1} = -\frac{1}{2},$$

and our assertion follows from Lemma 4.6. Similary, if β_2 is even and β_3 is odd, we can obtain two tangles

$$\frac{\beta''_1}{\alpha_1} = -\frac{1}{2}, \quad \frac{\beta'_3}{\alpha_3} = \frac{\beta_3}{\alpha_3} + 1 = \frac{\beta_3 + \alpha_3}{\alpha_3},$$

and our assertion follows from Lemma 4.6.

In the case (iii), as in in Figure 4.5, we add two half twists and denote the tangle in the dotted boxes by $\frac{\beta'_2}{\alpha_2}$ and $\frac{\beta'_3}{\alpha_3}$, where β'_2 and β'_3 are even because

$$\begin{aligned} \frac{\beta'_2}{\alpha_2} &= \frac{\beta_2}{\alpha_2} + \frac{|\beta_3|}{\beta_3}, \\ \frac{\beta'_3}{\alpha_3} &= \frac{\beta_3}{\alpha_3} - \frac{|\beta_3|}{\beta_3}, \end{aligned}$$

and our assertion follows from Lemma 4.6.

□

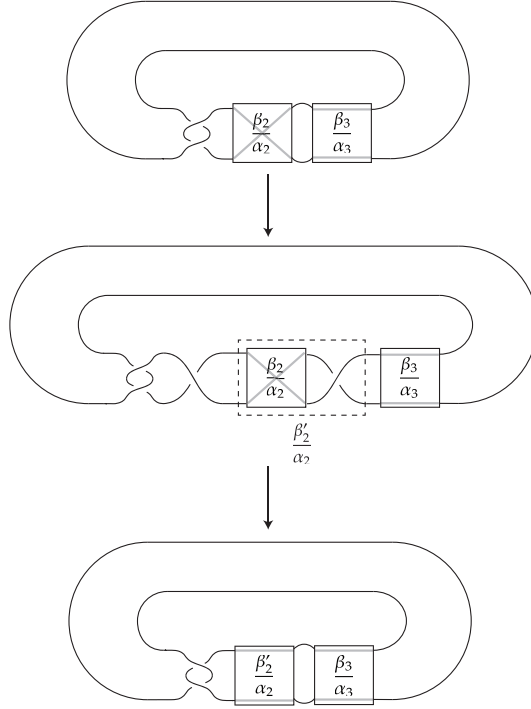


Figure 4.4: The case of either β_2 or β_3 is odd

We consider the knot depicted in Figure 4.6. By Lemma 4.7, two rational tangles are written by

$$\frac{\beta_2}{\alpha_2} = 2m_0 + \frac{1}{2m_1 + \frac{1}{2m_2 + \frac{1}{\ddots + \frac{1}{2m_k}}}},$$

$$\frac{\beta_3}{\alpha_3} = \frac{1}{2n_1 + \frac{1}{2n_2 + \frac{1}{\ddots + \frac{1}{2n_l}}}},$$

where both k and l are even. Then, we put

$$x_{-4} = c^{-1}a^{-1}c, \quad x_{-3} = a, \quad x_{-2} = b,$$

$$y_{-2} = b, \quad y_{-1} = c, \quad y_0 = c^{-1}ac^{-1}a^{-1}c.$$

As in the case (1), by extending these, we can take generators $\{x_{-4}, \dots, x_{2k}, y_{-2}, \dots, y_{2l}\}$

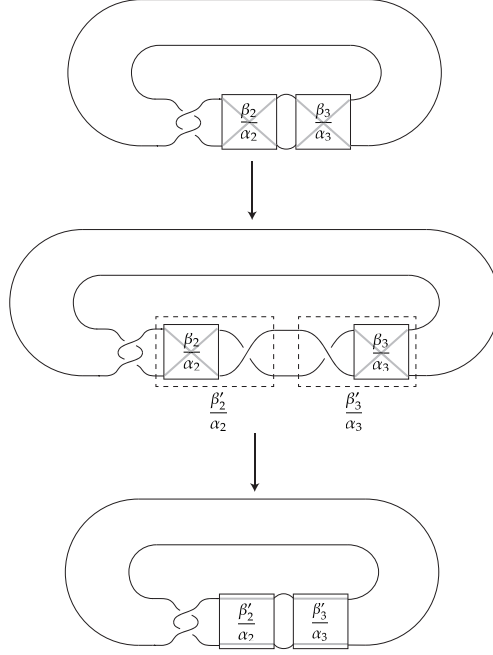


Figure 4.5: The case of both β_2 and β_3 are odd

of the knot group $G(K)$ such that the following relations hold:

$$\begin{aligned}
r_{2i-1} : \quad x_{2i-1} &= (x_{2i-3}^{(-1)^i} x_{2i-4}^{(-1)^i})^{m_i} x_{2i-3} (x_{2i-3}^{(-1)^i} x_{2i-4}^{(-1)^i})^{-m_i}, \\
r_{2i} : \quad x_{2i} &= (x_{2i-3}^{(-1)^i} x_{2i-4}^{(-1)^i})^{m_i} x_{2i-4} (x_{2i-3}^{(-1)^i} x_{2i-4}^{(-1)^i})^{-m_i}, \\
s_{2i-1} : \quad y_{2i-1} &= (y_{2i-3}^{(-1)^i} y_{2i-4}^{(-1)^i})^{-n_i} y_{2i-3} (y_{2i-3}^{(-1)^i} y_{2i-4}^{(-1)^i})^{n_i}, \\
s_{2i} : \quad y_{2i} &= (y_{2i-3}^{(-1)^i} y_{2i-4}^{(-1)^i})^{-n_i} y_{2i-4} (y_{2i-3}^{(-1)^i} y_{2i-4}^{(-1)^i})^{n_i},
\end{aligned}$$

and

$$\begin{aligned}
r_{2k+1} : \quad x_{2k-1} &= x_{2k-2}^{-1}, \\
s_{2l+1} : \quad y_{2l-1} &= y_{2l-2}^{-1}, \\
rs : \quad x_{2k} &= y_{2l}.
\end{aligned}$$

Let $\rho : G(K) \rightarrow SL_2(\mathbb{C})$ be a representation of the knot group $G(K)$ and put

$$\rho(a) := A, \quad \rho(b) := B, \quad \rho(c) := C, \quad \rho(x_i) := X_i, \quad \rho(y_i) := Y_i.$$

Then, the following is the main result in this section.

Theorem 4.8. *Let K be a knot depicted in Figure 4.6. Then, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ of K associated to a nonabelian representation $\rho : G(K) \rightarrow SL_2(\mathbb{C})$ is*

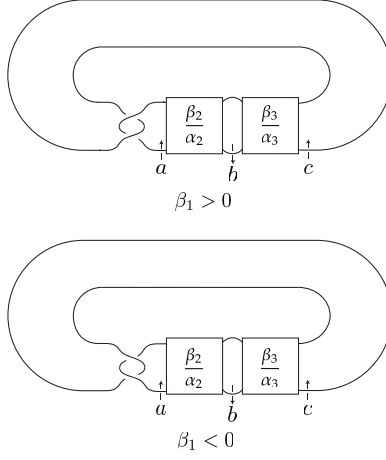


Figure 4.6: The knot of case (2)

given by

$$\left\{ \begin{array}{l} \prod_{i=0}^k \left| \sum_{j=1}^{|m_i|} (X_{2i-3}^{(-1)^i} X_{2i-4}^{(-1)^i})^j \right| \prod_{i=1}^l \left| \sum_{j=1}^{|n_i|} (Y_{2i-3}^{(-1)^i} Y_{2i-4}^{(-1)^i})^j \right| (t^0 + t^{2(k+l+1)}) \\ \quad + \sum_{i=1}^{k+l+1} \kappa_i (t^i + t^{2(k+l+1)-i}) \quad \text{if } m_0 \neq 0, \\ \prod_{i=2}^k \left| \sum_{j=1}^{|m_i|} (X_{2i-3}^{(-1)^i} X_{2i-4}^{(-1)^i})^j \right| \prod_{i=2}^l \left| \sum_{j=1}^{|n_i|} (Y_{2i-3}^{(-1)^i} Y_{2i-4}^{(-1)^i})^j \right| \lambda (t^0 + t^{2(k+l-1)}) \\ \quad + \sum_{i=1}^{k+l-1} \lambda_i (t^i + t^{2(k+l-1)-i}) \quad \text{if } m_0 = 0, \end{array} \right.$$

where

$$\lambda = \begin{cases} \left| \frac{|m_1|}{m_1} \left(\sum_j (BA)^j \right) + \frac{|n_1|}{n_1} A^{-1} C \left(\sum_j (BC)^j \right) + \frac{|m_1 n_1|}{m_1 n_1} \left(\sum_j (BA)^j \right) \left(\sum_j (BC)^j \right) BC \right| & \text{if } \beta_1 > 0, \\ \left| \frac{|m_1|}{m_1} \left(\sum_j (BA)^j \right) + \frac{|n_1|}{n_1} AC^{-1} \left(\sum_j (BC)^j \right) - \frac{|m_1 n_1|}{m_1 n_1} \left(\sum_j (BA)^j \right) \left(\sum_j (BC)^j \right) \right| & \text{if } \beta_1 < 0. \end{cases}$$

Note that

$$G(K) = \langle a, b, c, x_{-4}, \dots, x_{2k}, y_{-2}, \dots, y_{2l} \mid r_{-4}, \dots, r_{2k+1}, s^{-2}, \dots, s^{2l+1} \rangle.$$

Then we put

$$R_{-4} = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{-4}}{\partial a} \right)$$

and

$$\begin{aligned} R_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-4}} \right), \quad R'_{2i-1} = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-3}} \right), \\ R_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-4}} \right), \quad R'_{2i} = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-3}} \right), \end{aligned}$$

for $0 \leq i \leq k$,

$$\begin{aligned} S_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i-1}}{\partial y_{2i-4}} \right), \quad S'_{2i-1} = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i-1}}{\partial y_{2i-3}} \right), \\ S_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i}}{\partial y_{2i-4}} \right), \quad S'_{2i} = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i}}{\partial y_{2i-3}} \right), \end{aligned}$$

for $1 \leq i \leq l$.

We prepare three lemmas to prove Theorem 4.8.

Lemma 4.9. *We have*

$$\Delta_{K,\rho}(t) = \frac{\begin{vmatrix} M_{2k+1} & M'_{2k+1} \\ N_{2l+1} & N'_{2l+1} \end{vmatrix}}{t^2 - \text{tr}\rho(c)t + 1},$$

where the matrices $M_{2k+1}, M'_{2k+1}, N_{2l+1}, N'_{2l+1}$ are computed as in the case (1).

Proof. By the definition of twisted Alexander polynomials, we have

$$\Delta_{K,\rho}(t) = \frac{|M_{\rho,3}|}{|(\rho \otimes \mathfrak{a}) \circ \phi(c-1)|} = \frac{|M_{\rho,3}|}{t^2 - \text{tr}\rho(c)t + 1}.$$

By simple calculation, we have

$$= \frac{|M_{\rho,3}|}{\begin{vmatrix} R_{-4} & O & E & O & \cdots & O & O & O & O & O & O & \cdots & O & O & O & O & O \\ -E & O & O & E & & O & O & O & O & O & O & \cdots & O & O & O & O & O \\ O & -E & O & O & \ddots & O & O & O & O & O & O & \cdots & O & O & O & O & O \\ O & O & R_{-1} & R'_{-1} & & O & O & O & O & O & O & \cdots & O & O & O & O & O \\ O & O & R_0 & R'_0 & & O & O & O & O & O & O & \cdots & O & O & O & O & O \\ \vdots & \vdots & & & \ddots & & & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & O & & R_{2k-1} & R'_{2k-1} & O & E & O & O & O & \cdots & O & O & O & O & O \\ O & O & O & O & & R_{2k} & R'_{2k} & O & O & E & O & O & \cdots & O & O & O & O & O \\ O & O & O & O & \cdots & O & O & tX_{2k-1} & E & O & O & O & \cdots & O & O & O & O & O \\ O & -E & O & O & \cdots & O & O & O & O & O & E & O & \cdots & O & O & O & O & O \\ O & O & O & O & \cdots & O & O & O & O & O & O & E & & O & O & O & O & O \\ S_0 & O & O & O & \cdots & O & O & O & O & O & O & O & \ddots & O & O & O & O & O \\ O & O & O & O & \cdots & O & O & O & O & O & S_1 & S'_1 & & O & O & O & O & O \\ O & O & O & O & \cdots & O & O & O & O & O & S_2 & S'_2 & & O & O & O & O & O \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \ddots & & \vdots & \vdots & \vdots \\ O & O & O & O & \cdots & O & O & O & O & O & O & O & S_{2l-1} & S'_{2l-1} & O & E & O & O \\ O & O & O & O & \cdots & O & O & O & O & O & O & O & S_{2l} & S'_{2l} & O & O & E & O \\ O & O & O & O & \cdots & O & O & O & O & O & O & O & \cdots & O & O & tY_{2l-1} & E & O \end{vmatrix}}$$

$$\begin{aligned}
&= \begin{vmatrix}
M_{-4} & M'_{-4} & E & O & \cdots & O & O & O & O & O & O & O & \cdots & O & O & O & O & O \\
M_{-3} & M'_{-3} & O & E & & O & O & O & O & O & O & O & \cdots & O & O & O & O & O \\
M_{-2} & M'_{-2} & O & O & \ddots & O & O & O & O & O & O & O & \cdots & O & O & O & O & O \\
M_{-1} & M'_{-1} & O & O & & O & O & O & O & O & O & O & \cdots & O & O & O & O & O \\
M_0 & M'_0 & O & O & & O & O & O & O & O & O & O & \cdots & O & O & O & O & O \\
\vdots & \vdots & & & & & & \ddots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & \vdots \\
M_{2k-1} & M'_{2k-1} & O & O & \cdots & O & O & O & E & O & O & O & \cdots & O & O & O & O & O \\
M_{2k} & M'_{2k} & O & O & \cdots & O & O & O & O & E & O & O & \cdots & O & O & O & O & O \\
M_{2k+1} & M'_{2k+1} & O & O & \cdots & O & O & O & O & O & O & O & \cdots & O & O & O & O & O \\
N_{-2} & N'_{-2} & O & O & \cdots & O & O & O & O & O & E & O & \cdots & O & O & O & O & O \\
N_{-1} & N'_{-1} & O & O & \cdots & O & O & O & O & O & O & E & & O & O & O & O & O \\
N_0 & N'_0 & O & O & \cdots & O & O & O & O & O & O & O & \ddots & O & O & O & O & O \\
N_1 & N'_1 & O & O & \cdots & O & O & O & O & O & O & O & & O & O & O & O & O \\
N_2 & N'_2 & O & O & \cdots & O & O & O & O & O & O & O & & O & O & O & O & O \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & & \vdots \\
N_{2l-1} & N'_{2l-1} & O & O & \cdots & O & O & O & O & O & O & O & \cdots & O & O & O & E & O \\
N_{2l} & N'_{2l} & O & O & \cdots & O & O & O & O & O & O & O & \cdots & O & O & O & O & E \\
N_{2l+1} & N'_{2l+1} & O & O & \cdots & O & O & O & O & O & O & O & \cdots & O & O & O & O & O
\end{vmatrix} \\
&= \begin{vmatrix}
M_{2k+1} & M'_{2k+1} \\
N_{2l+1} & N'_{2l+1}
\end{vmatrix},
\end{aligned}$$

where the matrix M_{2k+1} , M'_{2k+1} , N_{2l+1} and N'_{2l+1} are defined as follows:
We consider the sequences $\{M_i\}_{-4}^{2k+1}$ and $\{M'_i\}_{-4}^{2k+1}$ which are defined by

$$\begin{aligned}
M_{-4} &= R_{-4}, \quad M_{-3} = -E, \quad M_{-2} = O, \\
M'_{-4} &= O, \quad M'_{-3} = O, \quad M'_{-2} = -E,
\end{aligned}$$

and

$$\begin{aligned}
M_{2i-1} &= (-R_{2i-1})M_{2i-4} + (-R'_{2i-1})M_{2i-3}, \\
M_{2i} &= (-R_{2i})M_{2i-4} + (-R'_{2i})M_{2i-3}, \\
M'_{2i-1} &= (-R_{2i-1})M'_{2i-4} + (-R'_{2i-1})M'_{2i-3}, \\
M'_{2i} &= (-R_{2i})M'_{2i-4} + (-R'_{2i})M'_{2i-3},
\end{aligned}$$

for $0 \leq i \leq k$. Then we put

$$\begin{aligned}
M_{2k+1} &= (-tX_{2k-1})M_{2k-2} + (-E)M_{2k-1}, \\
M'_{2k+1} &= (-tX_{2k-1})M'_{2k-2} + (-E)M'_{2k-1}.
\end{aligned}$$

Similarly, we consider the sequences $\{N_i\}_{-2}^{2l+1}$ and $\{N'_i\}_{-2}^{2l+1}$ which are defined by

$$\begin{aligned}
N_{-2} &= O, \quad N_{-1} = O, \quad N_0 = S_0, \\
N'_{-2} &= -E, \quad N'_{-1} = O, \quad N'_0 = O,
\end{aligned}$$

and

$$\begin{aligned} N_{2i-1} &= (-S_{2i-1})N_{2i-4} + (-S'_{2i-1})N_{2i-3}, \\ N_{2i} &= (-S_{2i})N_{2i-4} + (-S'_{2i})N_{2i-3}, \\ N'_{2i-1} &= (-S_{2i-1})N'_{2i-4} + (-S'_{2i-1})N'_{2i-3}, \\ N_{2i} &= (-S_{2i})N'_{2i-4} + (-S'_{2i})N'_{2i-3}, \end{aligned}$$

for $1 \leq i \leq l$. Then we set

$$\begin{aligned} N_{2l+1} &= (-tY_{2l-1})N_{2l-2} + (-E)N_{2l-1}, \\ N'_{2l+1} &= (-tY_{2l-1})N'_{2l-2} + (-E)N'_{2l-1}. \end{aligned}$$

This complete the proof of Lemma 4.9. □

Lemma 4.10. *If $m_0 \neq 0$, then the matrix $\begin{vmatrix} M_{2k+1} & M'_{2k+1} \\ N_{2l+1} & N'_{2l+1} \end{vmatrix}$ is given by*

$$\begin{vmatrix} M_{2k+1} & M'_{2k+1} \\ N_{2l+1} & N'_{2l+1} \end{vmatrix} = t^{-(k+l)} \left| M_{2k+1}^{-\frac{k}{2}-1} \right| \left| N'_{2l+1}^{-\frac{l}{2}+1} \right| + \cdots + t^{k+l+4} \left| M_{2k+1}^{\frac{k}{2}+1} \right| \left| N'_{2l+1}^{\frac{l}{2}+1} \right|,$$

where

$$\begin{aligned} \left| M_{2k+1}^{\frac{k}{2}+1} \right| &= \left| M_{2k+1}^{-\frac{k}{2}-1} \right| = \prod_{i=0}^k \left| \sum_{j=1}^{|m_i|} (X_{2i-3}^{(-1)^i} X_{2i-4}^{(-1)^i})^j \right|, \\ \left| N'_{2l+1}^{\frac{l}{2}+1} \right| &= \left| N'_{2l+1}^{-\frac{l}{2}+1} \right| = \prod_{i=1}^l \left| \sum_{j=1}^{|n_i|} (Y_{2i-3}^{(-1)^i} Y_{2i-4}^{(-1)^i})^j \right|. \end{aligned}$$

Proof. The matrices M_{2k+1} and M'_{2k+1} are computed as in the case (1), that is, we have

$$\begin{aligned} M_{2k+1} &= t^{-\frac{k}{2}-1} M_{2k+1}^{-\frac{k}{2}-1} + \cdots + t^{\frac{k}{2}+1} M_{2k+1}^{\frac{k}{2}+1}, \\ M'_{2k+1} &= t^{-\frac{k}{2}+1} M'_{2k+1}^{-\frac{k}{2}+1} + \cdots + t^{\frac{k}{2}+1} M'_{2k+1}^{\frac{k}{2}+1}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} N_{2l+1} &= t^{-\frac{l}{2}} N_{2l+1}^{-\frac{l}{2}} + \cdots + t^{\frac{l}{2}} N_{2l+1}^{\frac{l}{2}}, \\ N'_{2l+1} &= t^{-\frac{l}{2}+1} N'_{2l+1}^{-\frac{l}{2}+1} + \cdots + t^{\frac{l}{2}+1} N'_{2l+1}^{\frac{l}{2}+1}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \begin{vmatrix} M_{2k+1} & M'_{2k+1} \\ N_{2l+1} & N'_{2l+1} \end{vmatrix} &= \begin{vmatrix} t^{-\frac{k}{2}-1} M_{2k+1}^{-\frac{k}{2}-1} + \cdots + t^{\frac{k}{2}+1} M_{2k+1}^{\frac{k}{2}+1} & t^{-\frac{k}{2}+1} M'_{2k+1}^{-\frac{k}{2}+1} + \cdots + t^{\frac{k}{2}+1} M'_{2k+1}^{\frac{k}{2}+1} \\ t^{-\frac{l}{2}} N_{2l+1}^{-\frac{l}{2}} + \cdots + t^{\frac{l}{2}} N_{2l+1}^{\frac{l}{2}} & t^{-\frac{l}{2}+1} N'_{2l+1}^{-\frac{l}{2}+1} + \cdots + t^{\frac{l}{2}+1} N'_{2l+1}^{\frac{l}{2}+1} \end{vmatrix} \\ &= t^{-(k+l)} \left| M_{2k+1}^{-\frac{k}{2}-1} \right| \left| N'_{2l+1}^{-\frac{l}{2}+1} \right| + \cdots + t^{k+l+4} \left| M_{2k+1}^{\frac{k}{2}+1} \right| \left| N'_{2l+1}^{\frac{l}{2}+1} \right|. \end{aligned}$$

Then, since we have

$$\begin{aligned}
M_{2k+1}^{\frac{k}{2}+1} &= (-1)^{\frac{k}{2}+1} \frac{|m_k m_{k-1} \cdots m_1|}{m_k m_{k-1} \cdots m_1} X_{2k-1} \\
&\quad \left\{ \sum_j (X_{2k-3} X_{2k-4})^j X_{2k-4}^{-1} \sum_j (X_{2k-5}^{-1} X_{2k-6}^{-1})^j \right\} \cdots \left\{ \sum_j (X_1 X_0)^j X_0^{-1} \sum_j (X_{-1}^{-1} X_{-2}^{-1})^j \right\} X_{-1}^{-1} M_{-1}^{\max}, \\
M_{2k+1}^{-\frac{k}{2}-1} &= (-1)^{\frac{k}{2}+1} \frac{|m_k m_{k-1} \cdots m_1|}{m_k m_{k-1} \cdots m_1} \sum_j (X_{2k-3} X_{2k-4})^j \\
&\quad \left\{ \sum_j (X_{2k-5}^{-1} X_{2k-6}^{-1})^j X_{2k-6} \sum_j (X_{2k-7} X_{2k-8})^j \right\} \cdots \left\{ \sum_j (X_3^{-1} X_2^{-1})^j X_2 \sum_j (X_1 X_0)^j X_0^{-1} \right\} \\
&\quad \sum_j (X_{-1}^{-1} X_{-2}^{-1})^j X_{-2} M_{-1}^{\min},
\end{aligned}$$

where

$$\begin{aligned}
M_{-1}^{\max} &= \begin{cases} \frac{|m_0|}{m_0} X_{-1} \sum_j (X_{-3} X_{-4})^j X_{-1}^{-1} A^{-1} & \text{if } \beta_1 > 0, \\ -\frac{|m_0|}{m_0} X_{-1} \sum_j (X_{-3} X_{-4})^j X_{-1}^{-1} A^{-1} C A^{-1} & \text{if } \beta_1 < 0, \end{cases} \\
M_{-1}^{\min} &= \begin{cases} \frac{|m_0|}{m_0} \sum_j (X_{-3} X_{-4})^j C^{-1} & \text{if } \beta_1 > 0, \\ -\frac{|m_0|}{m_0} \sum_j (X_{-3} X_{-4})^j A^{-1} & \text{if } \beta_1 < 0, \end{cases}
\end{aligned}$$

we get

$$\left| M_{2k+1}^{\frac{k}{2}+1} \right| = \left| M_{2k+1}^{-\frac{k}{2}-1} \right| = \prod_{i=0}^k \left| \sum_{j=1}^{|m_i|} (X_{2i-3}^{(-1)^i} X_{2i-4}^{(-1)^i})^j \right|.$$

Similarly, since we have

$$\begin{aligned}
N'_{2l+1}^{\frac{l}{2}+1} &= (-1)^{\frac{l}{2}} \frac{|n_l n_{l-1} \cdots n_1|}{n_l n_{l-1} \cdots n_1} Y_{2l-1} \\
&\quad \left\{ \sum_j (Y_{2l-3} Y_{2l-4})^j Y_{2l-4}^{-1} \sum_j (Y_{2l-5}^{-1} Y_{2l-6}^{-1})^j \right\} \cdots \left\{ \sum_j (Y_1 Y_0)^j Y_0^{-1} \sum_j (Y_{-1}^{-1} Y_{-2}^{-1})^j \right\}, \\
N'_{2l+1}^{-\frac{l}{2}+1} &= (-1)^{\frac{l}{2}} \frac{|n_l n_{l-1} \cdots n_1|}{n_l n_{l-1} \cdots n_1} \sum_j (Y_{2l-3} Y_{2l-4})^j \\
&\quad \left\{ \sum_j (Y_{2l-5}^{-1} Y_{2l-6}^{-1})^j Y_{2l-6} \sum_j (Y_{2l-7} Y_{2l-8})^j \right\} \cdots \left\{ \sum_j (Y_3^{-1} Y_2^{-1})^j Y_2 \sum_j (Y_1 Y_0)^j Y_0^{-1} \right\} \\
&\quad \sum_j (Y_{-1}^{-1} Y_{-2}^{-1})^j,
\end{aligned}$$

we obtain

$$\left| N'_{2l+1}^{\frac{l}{2}+1} \right| = \left| N'_{2l+1}^{-\frac{l}{2}+1} \right| = \prod_{i=1}^l \left| \sum_{j=1}^{|n_i|} (Y_{2i-3}^{(-1)^i} Y_{2i-4}^{(-1)^i})^j \right|.$$

This complete the proof of Lemma 4.10. \square

Lemma 4.11. *If $m_0 = 0$, then $\begin{vmatrix} M_{2k+1} & M'_{2k+1} \\ N_{2l+1} & N'_{2l+1} \end{vmatrix}$ is given by*

$$t^{-(k+l)+2} \begin{vmatrix} M_{2k+1}^{-\frac{k}{2}} & M'_{2k+1}^{-\frac{k}{2}+1} \\ N_{2l+1}^{-\frac{l}{2}} & N'_{2l+1}^{-\frac{l}{2}+1} \end{vmatrix} + \dots + t^{k+l+2} \begin{vmatrix} M_{2k+1}^{\frac{k}{2}} & M'_{2k+1}^{\frac{k}{2}+1} \\ N_{2l+1}^{\frac{l}{2}} & N'_{2l+1}^{\frac{l}{2}+1} \end{vmatrix},$$

where $\begin{vmatrix} M_{2k+1}^{\frac{k}{2}} & M'_{2k+1}^{\frac{k}{2}+1} \\ N_{2l+1}^{\frac{l}{2}} & N'_{2l+1}^{\frac{l}{2}+1} \end{vmatrix}$ and $\begin{vmatrix} M_{2k+1}^{-\frac{k}{2}} & M'_{2k+1}^{-\frac{k}{2}+1} \\ N_{2l+1}^{-\frac{l}{2}} & N'_{2l+1}^{-\frac{l}{2}+1} \end{vmatrix}$ are given by

$$\prod_{i=2}^k \left| \sum_{j=1}^{|m_i|} (X_{2i-3}^{(-1)^i} X_{2i-4}^{(-1)^i})^j \right| \prod_{i=2}^l \left| \sum_{j=1}^{|n_i|} (Y_{2i-3}^{(-1)^i} Y_{2i-4}^{(-1)^i})^j \right| \lambda,$$

and

$$\lambda = \begin{cases} \left| \frac{|m_1|}{m_1} \left(\sum_j (BA)^j \right) + \frac{|n_1|}{n_1} A^{-1} C \left(\sum_j (BC)^j \right) + \frac{|m_1 n_1|}{m_1 n_1} \left(\sum_j (BA)^j \right) \left(\sum_j (BC)^j \right) BC \right| & \text{if } \beta_1 > 0, \\ \left| \frac{|m_1|}{m_1} \left(\sum_j (BA)^j \right) + \frac{|n_1|}{n_1} AC^{-1} \left(\sum_j (BC)^j \right) - \frac{|m_1 n_1|}{m_1 n_1} \left(\sum_j (BA)^j \right) \left(\sum_j (BC)^j \right) \right| & \text{if } \beta_1 < 0. \end{cases}$$

Proof. If $m_0 = 0$, since we have

$$\begin{aligned} x_{-1} &= x_{-3}, \\ x_0 &= x_{-4}, \end{aligned}$$

then we get

$$R_{-1} = O, \quad R'_{-1} = -E, \quad R_0 = -E, \quad R'_0 = O.$$

Hence, we can compute as in the case of $m_0 \neq 0$ and obtain

$$\begin{aligned} M_{2k+1} &= t^{-\frac{k}{2}} M_{2k+1}^{-\frac{k}{2}} + \dots + t^{\frac{k}{2}} M_{2k+1}^{\frac{k}{2}}, \\ M'_{2k+1} &= t^{-\frac{k}{2}+1} M'_{2k+1}^{-\frac{k}{2}+1} + \dots + t^{\frac{k}{2}+1} M'_{2k+1}^{\frac{k}{2}+1}. \end{aligned}$$

Then, since N_{2l+1} and N'_{2l+1} are same as in the case of $m_0 \neq 0$, we have

$$\begin{aligned} \begin{vmatrix} M_{2k+1} & M'_{2k+1} \\ N_{2l+1} & N'_{2l+1} \end{vmatrix} &= \begin{vmatrix} t^{-\frac{k}{2}} M_{2k+1}^{-\frac{k}{2}} + \dots + t^{\frac{k}{2}} M_{2k+1}^{\frac{k}{2}} & t^{-\frac{k}{2}+1} M'_{2k+1}^{-\frac{k}{2}+1} + \dots + t^{\frac{k}{2}+1} M'_{2k+1}^{\frac{k}{2}+1} \\ t^{-\frac{l}{2}} N_{2l+1}^{-\frac{l}{2}} + \dots + t^{\frac{l}{2}} N_{2l+1}^{\frac{l}{2}} & t^{-\frac{l}{2}+1} N'_{2l+1}^{-\frac{l}{2}+1} + \dots + t^{\frac{l}{2}+1} N'_{2l+1}^{\frac{l}{2}+1} \end{vmatrix} \\ &= t^{-(k+l)+2} \begin{vmatrix} M_{2k+1}^{-\frac{k}{2}} & M'_{2k+1}^{-\frac{k}{2}+1} \\ N_{2l+1}^{-\frac{l}{2}} & N'_{2l+1}^{-\frac{l}{2}+1} \end{vmatrix} + \dots + t^{k+l+2} \begin{vmatrix} M_{2k+1}^{\frac{k}{2}} & M'_{2k+1}^{\frac{k}{2}+1} \\ N_{2l+1}^{\frac{l}{2}} & N'_{2l+1}^{\frac{l}{2}+1} \end{vmatrix}. \end{aligned}$$

If we put

$$\begin{aligned}
M &= (-1)^{\frac{k}{2}} \frac{|m_k m_{k-1} \cdots m_2|}{m_k m_{k-1} \cdots m_2} X_{2k-1} \\
&\quad \left\{ \sum_j (X_{2k-3} X_{2k-4})^j X_{2k-4}^{-1} \sum_j (X_{2k-5}^{-1} X_{2k-6}^{-1})^j \right\} \cdots \left\{ \sum_j (X_5 X_4)^j X_4^{-1} \sum_j (X_3^{-1} X_2^{-1})^j \right\} \\
&\quad \sum_j (X_1 X_0)^j X_0^{-1}, \\
N &= (-1)^{\frac{l}{2}} \frac{|n_l n_{l-1} \cdots n_2|}{n_l n_{l-1} \cdots n_2} Y_{2l-1} \\
&\quad \left\{ \sum_j (Y_{2l-3} Y_{2l-4})^j Y_{2l-4}^{-1} \sum_j (Y_{2l-5}^{-1} Y_{2l-6}^{-1})^j \right\} \cdots \left\{ \sum_j (Y_5 Y_4)^j Y_4^{-1} \sum_j (Y_3^{-1} Y_2^{-1})^j \right\} \\
&\quad \sum_j (Y_1 Y_0)^j Y_0^{-1},
\end{aligned}$$

then we have

$$\begin{aligned}
\begin{vmatrix} M_{2k+1}^{\frac{k}{2}} & M_{2k+1}^{\frac{k}{2}+1} \\ N_{2l+1}^{\frac{l}{2}} & N_{2l+1}^{\frac{l}{2}+1} \end{vmatrix} &= \begin{vmatrix} -M(M_0^{-1} - X_1^{-1} R_1^0) & \frac{|m_1|}{m_1} M \sum_j (X_{-1}^{-1} X_{-2}^{-1})^j \\ N S_0^{-1} & \frac{|n_1|}{n_1} N \sum_j (Y_{-1}^{-1} Y_{-2}^{-1})^j \end{vmatrix} \\
&= \begin{vmatrix} M & O \\ O & N \end{vmatrix} \begin{vmatrix} -(M_0^{-1} - X_1^{-1} R_1^0) & \frac{|m_1|}{m_1} \sum_j (X_{-1}^{-1} X_{-2}^{-1})^j \\ S_0^{-1} & \frac{|n_1|}{n_1} \sum_j (Y_{-1}^{-1} Y_{-2}^{-1})^j \end{vmatrix} \\
&= |M||N| \begin{vmatrix} -(M_0^{-1} - X_1^{-1} R_1^0) & \frac{|m_1|}{m_1} \sum_j (X_{-1}^{-1} X_{-2}^{-1})^j \\ S_0^{-1} & \frac{|n_1|}{n_1} \sum_j (Y_{-1}^{-1} Y_{-2}^{-1})^j \end{vmatrix}.
\end{aligned}$$

Since we have

$$\begin{aligned}
-(M_0^{-1} - X_1^{-1} R_1^0) &= \begin{cases} \{(X_{-1}^{-1} X_{-2}^{-1})^{m_1+1} + \frac{|m_1|}{m_1} \sum_j (X_{-1}^{-1} X_{-2}^{-1})^j\} X_{-2} & \text{if } \beta_1 > 0, \\ \{-A^{-1} C + \frac{|m_1|}{m_1} \sum_j (X_{-1}^{-1} X_{-2}^{-1})^j\} X_{-1}^{-1} & \text{if } \beta_1 < 0, \end{cases} \\
S_0^{-1} &= \begin{cases} C^{-1} & \text{if } \beta_1 > 0, \\ A^{-1} & \text{if } \beta_1 < 0, \end{cases}
\end{aligned}$$

we obtain

$$\begin{aligned}
&\begin{vmatrix} -(M_0^{-1} - X_1^{-1} R_1^0) & \frac{|m_1|}{m_1} \sum_j (X_{-1}^{-1} X_{-2}^{-1})^j \\ S_0^{-1} & \frac{|n_1|}{n_1} \sum_j (Y_{-1}^{-1} Y_{-2}^{-1})^j \end{vmatrix} \\
&= \begin{cases} \left| \frac{|m_1|}{m_1} \left(\sum_j (BA)^j \right) + \frac{|n_1|}{n_1} A^{-1} C \left(\sum_j (BC)^j \right) + \frac{|m_1 n_1|}{m_1 n_1} \left(\sum_j (BA)^j \right) \left(\sum_j (BC)^j \right) BC \right| & \text{if } \beta_1 > 0, \\ \left| \frac{|m_1|}{m_1} \left(\sum_j (BA)^j \right) + \frac{|n_1|}{n_1} AC^{-1} \left(\sum_j (BC)^j \right) - \frac{|m_1 n_1|}{m_1 n_1} \left(\sum_j (BA)^j \right) \left(\sum_j (BC)^j \right) \right| & \text{if } \beta_1 < 0. \end{cases}
\end{aligned}$$

We also have

$$|M||N| = \prod_{i=2}^k \left| \sum_{j=1}^{|m_i|} (X_{2i-3}^{(-1)^i} X_{2i-4}^{(-1)^i})^j \right| \prod_{i=2}^l \left| \sum_{j=1}^{|n_i|} (Y_{2i-3}^{(-1)^i} Y_{2i-4}^{(-1)^i})^j \right|.$$

Hence we obtain the formula of the statement. Similarly, we can compute $\begin{vmatrix} M_{2k+1}^{-\frac{k}{2}} & M'_{2k+1}^{-\frac{k}{2}+1} \\ N_{2l+1}^{-\frac{l}{2}} & N'_{2l+1}^{-\frac{l}{2}+1} \end{vmatrix}$.

This complete the proof of Lemma 4.11. \square

It follows from Lemma 4.9, 4.10, 4.11 that

$$\begin{aligned} \Delta_{K,\rho}(t) &= \frac{\begin{vmatrix} M_{2k+1} & M'_{2k+1} \\ N_{2l+1} & N'_{2l+1} \end{vmatrix}}{t^2 - \text{tr}\rho(c)t + 1} \\ &\doteq \begin{cases} \kappa_0 t^0 + \dots + \kappa_0 t^{2(k+l+1)} & \text{if } m_0 \neq 0, \\ \lambda_0 t^0 + \dots + \lambda_0 t^{2(k+l)-2} & \text{if } m_0 = 0. \end{cases} \end{aligned}$$

This completes the proof of Theorem 4.8.

4.2.1 Examples

Remark 4.12. If we denote the Alexander polynomial of K by $\Delta_K(t)$, then we have

$$\Delta_K(t) = \begin{cases} \kappa_0 t^0 + \dots + \kappa_0 t^{k+l+2} & \text{if } m_0 \neq 0, \\ \lambda_0 t^0 + \dots + \lambda_0 t^{k+l} & \text{if } m_0 = 0, \end{cases}$$

where

$$\begin{aligned} \kappa_0 &= |m_k \cdots m_0| |n_l \cdots n_1|, \\ \lambda_0 &= \begin{cases} |m_k \cdots m_2| |n_l \cdots n_2| |m_1 + n_1 + m_1 n_1| & \text{if } \beta_1 > 0, \\ |m_k \cdots m_2| |n_l \cdots n_2| |m_1 + n_1 - m_1 n_1| & \text{if } \beta_1 < 0. \end{cases} \end{aligned}$$

Then by [HM], it is known that the genus of K is given by

$$2g(K) = \begin{cases} k + l + 2 & \text{if } m_0 \neq 0, \\ k + l & \text{if } m_0 = 0. \end{cases}$$

It is known that

$$\begin{aligned} \deg(\Delta_K(t)) &\leq 2g(K), \\ \deg(\Delta_{K,\rho}(t)) &\leq 4g(K) - 2, \end{aligned}$$

for any knot K . Furthermore, if K is fibered, then

$$\begin{aligned} \deg(\Delta_K(t)) &= 2g(K), \\ \deg(\Delta_{K,\rho}(t)) &= 4g(K) - 2, \end{aligned}$$

and both $\Delta_K(t)$ and $\Delta_{K,\rho}(t)$ are monic.

In the following examples, we assume that $\beta_1 > 0$, $m_0 = 0$ and

$$|m_2| = \dots = |m_k| = |n_2| = \dots = |n_l| = 1.$$

Then their leading coefficients of their Alexander polynomials are

$$\lambda_0 = |m_1 + n_1 + m_1 n_1|$$

Example 4.13. If $(m_1, n_1) = (-2, -2)$, then $\deg(\Delta_K(t))$ is less than $2g(K)$. On the other hand, the leading coefficient of $\Delta_{K,\rho}(t)$ is

$$\lambda_0 = |(BA)^{-1}(BC)^{-1} - E| = 2 - \text{tr}BABC.$$

Thus, if there exists a representation ρ which gives $\text{tr}BABC \neq 2$, then

$$\deg(\Delta_{K,\rho}(t)) = 4g(K) - 2.$$

Example 4.14. If $(m_1, n_1) = (-2, -3)$, then $\Delta_K(t)$ is monic but K is not fibered (see [HM]). On the other hand, the leading coefficient of $\Delta_{K,\rho}(t)$ is

$$\lambda_0 = |(BA)^{-1}(BC)^{-1} - E| = 1 - \text{tr}(BABC - E)(BC + E).$$

Thus, if there exists a representation ρ which gives $\text{tr}(BABC - E)(BC + E) \neq 0$, then $\Delta_{K,\rho}(t)$ can not be monic.

4.3 The case (3)

In this case, we give the presentation of knot groups of knots $K_n = M(0; (3n+2, -2n-1), (3, 1), (3, 1))$ depicted in Figure 4.7, by using links whose surgery along the trivial component gives these knots. With the presentation, we compute their twisted Alexander polynomials associated to their $SL_2(\mathbb{C})$ -representations.

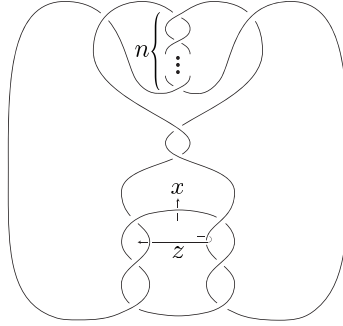


Figure 4.7: The knot K_n

For simplicity, we put $X = \rho(x)$, $Z = \rho(z)$ and

$$W = \begin{cases} \rho(x^{-1}[x, z][x^{-1}, z^{-1}]x) & \text{if } n \text{ is even,} \\ \rho([z, x^{-1}][z^{-1}, x]) & \text{if } n \text{ is odd,} \end{cases}$$

$$A = \begin{cases} E & \text{if } n > 0, \\ -W^{\lfloor \frac{n}{2} \rfloor} & \text{if } n < -1, \end{cases}$$

where E denotes the identity matrix.

Theorem 4.15. *Let $\rho : G(K_n) \rightarrow SL_2(\mathbb{C})$ be a non-abelian representation of $G(K_n)$. If n is even, we have*

$$\Delta_{K_n, \rho}(t) = \kappa_0(t^0 + t^{14}) + \kappa_1(t^1 + t^{13}) + \kappa_2(t^2 + t^{12}) + \kappa_3(t^3 + t^{11}) \\ + \kappa_4(t^4 + t^{10}) + \kappa_5(t^5 + t^9) + \kappa_6(t^6 + t^8) + \kappa_7 t^7,$$

where

$$\begin{aligned} \kappa_0 &= \left| \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} W^i \right|, \kappa_1 = - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \text{tr} A W^i X^{-1} [Z^{-1}, X^{-1}] - \left| \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} W^i \right| \text{tr} X, \kappa_2 = 1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \text{tr} A W^i - \left| \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} W^i \right|, \\ \kappa_3 &= \left| \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} W^i \right| \left\{ \text{tr} X Z + \text{tr} W X Z X Z^{-1} W^{-\frac{n}{2}} Z X Z^{-1} W^{\frac{n}{2}} \right\}, \\ \kappa_4 &= - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left\{ \text{tr} X Z \text{tr} A W^{i-1} Z X^{-1} Z^{-1} + \text{tr} A W^i X Z X Z^{-1} \right\} - \left| \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} W^i \right| \left\{ \text{tr} Z + 2 \text{tr} X^2 Z + \text{tr} X Z X Z^{-1} \right\}, \\ \kappa_5 &= \text{tr} X Z + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left\{ \text{tr} X Z \text{tr} A W^i - \text{tr} X \text{tr} A W^{i-\frac{n}{2}} X^{-1} Z^{-1} X^{-1} + 2 \text{tr} A W^{i+1} X^{-1} Z^{-1} X^{-1} W^{-\frac{n}{2}} Z X Z^{-1} \right\} \\ &\quad + \left| \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} W^i \right| \left\{ \text{tr} X Z + \text{tr} W^2 X^{-1} Z^{-1} X^{-1} W^{-\frac{n}{2}} Z X Z^{-1} W^{\frac{n}{2}} \right\}, \\ \kappa_6 &= \text{tr} W^{\frac{n}{2}} X Z X + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left\{ \text{tr} A W^{i-\frac{n}{2}} (X^{-1} Z^{-1})^2 - \text{tr} A W^i (X^{-1} Z^{-1})^2 + \text{tr} A W^{i-\frac{n}{2}} X^{-1} Z^{-1} X^{-1} \right\} \\ &\quad + \left| \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} W^i \right| \left\{ 2 + \text{tr} X Z \text{tr} W X Z X Z^{-1} W^{-\frac{n}{2}} Z X Z^{-1} W^{\frac{n}{2}} \right\}, \\ \kappa_7 &= - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left\{ \text{tr} X Z (\text{tr} A W^{i-1} Z + \text{tr} A W^i X Z X Z^{-1}) + \text{tr} A W^{i-1} Z X^2 Z X - \text{tr} A W^i X^{-1} Z^2 \right\} \\ &\quad - \left| \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} W^i \right| \left\{ \text{tr} X + \text{tr} X Z (\text{tr} X^2 Z + \text{tr} X Z X Z^{-1}) + \text{tr} W Z^{-1} W^{-\frac{n}{2}} Z X Z^{-1} W^{\frac{n}{2}} \right\}. \end{aligned}$$

If n is odd, we have

$$\Delta_{K_n, \rho}(t) = \lambda_0(t^0 + t^6) + \lambda_1(t^1 + t^5) + \lambda_2(t^2 + t^4) + \lambda_3 t^3,$$

where

$$\begin{aligned} \lambda_0 &= \left| \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} W^i \right|, \\ \lambda_1 &= \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \text{tr} A W^{i-1} Z X^{-1} Z^{-1} + \left| \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} W^i \right| \left\{ \text{tr} X Z^{-1} - \text{tr} X + \text{tr} W X^2 Z^{-1} W^{\frac{n+1}{2}} Z X Z^{-1} W^{-\frac{n+1}{2}} \right\}, \end{aligned}$$

$$\begin{aligned}
\lambda_2 = & 1 + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \{ \text{tr} A W^i X^2 Z^{-1} - \text{tr} A W^{i-1} + \text{tr} A W^{i-1} Z X^{-2} \} \\
& + \left| \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} W^i \right| \left\{ 3 - \text{tr} X \text{tr} W X^2 Z^{-1} W^{\frac{n+1}{2}} Z X Z^{-1} W^{-\frac{n+1}{2}} \right. \\
& \left. + \text{tr} X Z^{-1} (\text{tr} W X^2 Z^{-1} W^{\frac{n+1}{2}} Z X Z^{-1} W^{-\frac{n+1}{2}} - \text{tr} X) \right\}, \\
\lambda_3 = & \text{tr} X Z^{-1} - \text{tr} X \\
& + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \left\{ \text{tr} A W^{i-1} (Z X Z^{-1})^{-1} W (X Z X^{-1})^{-1} - \text{tr} A W^{i-\frac{n+1}{2}} (X Z X^{-1}) W^{\frac{n+1}{2}-1} (Z X Z^{-1})^{-1} \right. \\
& \quad - \text{tr} A W^i (X Z X^{-1})^{-1} (Z X Z^{-1}) + \text{tr} A W^{i-\frac{n+1}{2}} (X Z X^{-1})^{-1} W^{\frac{n+1}{2}} (Z X Z^{-1}) \\
& \quad \left. + (\text{tr} X Z^{-1} - \text{tr} X) \text{tr} A W^i X^2 Z^{-1} \right\} \\
& + \left| \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} W^i \right| \left\{ 2(\text{tr} X Z^{-1} - \text{tr} X + \text{tr} W X^2 Z^{-1} W^{\frac{n+1}{2}} Z X Z^{-1} W^{-\frac{n+1}{2}}) \right. \\
& \quad \left. - \text{tr} X Z^{-1} \text{tr} X \text{tr} W X^2 Z^{-1} W^{\frac{n+1}{2}} Z X Z^{-1} W^{-\frac{n+1}{2}} \right\}.
\end{aligned}$$

The rest of this section is devoted to the proof of Theorem 4.15.

4.3.1 The case when n is even

In this case, K_n is obtained by $-\frac{2}{n}$ -surgery along the trivial component of the link L_0 depicted in Figure 4.8.

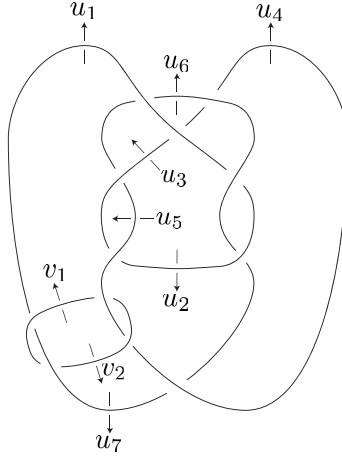


Figure 4.8: The link L_0

Let $u_1, u_2, u_3, u_4, u_5, u_6, u_7, v_1, v_2$ be the Wirtinger generators of $G(L_0)$ depicted in

Figure 4.8. Then, Wirtinger generators of $G(L_0)$ are given by

$$u_1 u_6 u_2^{-1} u_6^{-1} = 1 \quad (4.1)$$

$$u_3 u_5^{-1} u_2^{-1} u_5 = 1 \quad (4.2)$$

$$u_3 u_1 u_6 u_4^{-1} u_6^{-1} u_1^{-1} = 1 \quad (4.3)$$

$$v_2 u_4^{-1} v_2^{-1} u_5 = 1 \quad (4.4)$$

$$v_1 u_5^{-1} v_2^{-1} u_5 = 1 \quad (4.5)$$

$$u_3 u_1 u_6 u_1^{-1} u_3^{-1} u_5^{-1} = 1 \quad (4.6)$$

$$u_6 u_2 u_4 u_7^{-1} u_4^{-1} u_2^{-1} = 1 \quad (4.7)$$

$$v_1 u_7^{-1} v_2^{-1} u_7 = 1 \quad (4.8)$$

$$v_1 u_1 v_1^{-1} u_7^{-1} = 1 \quad (4.9)$$

From the relations (4.1), (4.3), (4.4), (4.5), (4.6) and (4.7), we can write

$$\begin{aligned} u_1 &= (u_3 u_1) u_6 (u_3 u_1) u_6 (u_3 u_1)^{-1} v_1^{-1} (u_3 u_1) u_6^{-1} (u_3 u_1)^{-1} v_1 (u_3 u_1) u_6^{-1} (u_3 u_1)^{-1} u_6^{-1}, \\ u_2 &= u_6^{-1} (u_3 u_1) u_6 (u_3 u_1) u_6 (u_3 u_1)^{-1} v_1^{-1} (u_3 u_1) u_6^{-1} (u_3 u_1)^{-1} v_1 (u_3 u_1) u_6^{-1} (u_3 u_1)^{-1}, \\ u_3 &= (u_3 u_1) u_6 (u_3 u_1) u_6 (u_3 u_1)^{-1} v_1^{-1} (u_3 u_1) u_6 (u_3 u_1)^{-1} v_1 (u_3 u_1) u_6^{-1} (u_3 u_1)^{-1} u_6^{-1} (u_3 u_1)^{-1}, \\ u_4 &= (u_3 u_1) u_6 (u_3 u_1)^{-1} v_1^{-1} (u_3 u_1) u_6 (u_3 u_1)^{-1} v_1 (u_3 u_1) u_6^{-1} (u_3 u_1)^{-1}, \\ u_5 &= (u_3 u_1) u_6 (u_3 u_1)^{-1}, \\ u_7 &= u_6^{-1} (u_3 u_1)^{-1} u_6 (u_3 u_1) u_6, \\ v_2 &= (u_3 u_1) u_6 (u_3 u_1)^{-1} v_1 (u_3 u_1) u_6^{-1} (u_3 u_1)^{-1}, \end{aligned}$$

with generators $u_3 u_1, u_6$ and v_1 . Since we can obtain relation (4.8) from relations (4.2) and (4.9), we have

$$G(L_0) = \langle u_6, u_3 u_1, v_1 \mid r_1 = 1, r_2 = 1 \rangle,$$

where

$$\begin{aligned} r_1 &= ((u_3 u_1) u_6)^2 (u_3 u_1)^{-1} v_1^{-1} (u_3 u_1) u_6 (u_3 u_1)^{-1} v_1 (u_3 u_1) ((u_3 u_1) u_6)^{-2} \\ &\quad v_1^{-1} (u_3 u_1) u_6 (u_3 u_1)^{-1} v_1 (u_3 u_1) ((u_3 u_1) u_6)^{-2} u_6 (u_3 u_1) u_6 (u_3 u_1)^{-1}, \\ r_2 &= v_1 ((u_3 u_1) u_6)^2 (u_3 u_1)^{-1} v_1^{-1} (u_3 u_1) u_6^{-1} (u_3 u_1)^{-1} v_1 (u_3 u_1) u_6^{-1} (u_3 u_1)^{-1} \\ &\quad u_6^{-1} v_1^{-1} u_6^{-1} (u_3 u_1)^{-1} u_6^{-1} (u_3 u_1) u_6. \end{aligned}$$

Since K_n is obtained from L_0 by $-\frac{2}{n}$ -surgery, $G(K_n)$ is obtained from $G(L_0)$ by adding the relator

$$v_1 = u_6^{-1} \{ [u_6, (u_3 u_1)] [u_6^{-1}, (u_3 u_1)^{-1}] \}^{\frac{n}{2}} u_6$$

which obtained from the relation $v_1 = (u_5^{-1} u_7)^{\frac{n}{2}}$ along the trivial component. Then, we can reduce r_1 and get

$$G(K_n) = \langle x, z \mid [x^{-1}, z^{-1}] x y x (z x z^{-1}) y^{-1} = y z x (z x z^{-1}) y^{-1} (z x z^{-1})^{-1} \rangle,$$

where we put $x = u_6, z = u_3u_1$ and $y = x^{-1}([x, z][x^{-1}, z^{-1}])^{\frac{n}{2}}x$. The Fox derivative of the single relation, say r , with respect to x is given by

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}[x^{-1}, z^{-1}]xyx(zzz^{-1})y^{-1} - \frac{\partial}{\partial x}yzx(zzz^{-1})y^{-1}(zzz^{-1})^{-1} \\ &= x^{-1}z^{-1} - x^{-1} + [x^{-1}, z^{-1}] + [x^{-1}, z^{-1}]xy - yz + [x^{-1}, z^{-1}]xyxz - yzxx + yzx(zzz^{-1})y^{-1}zx^{-1} \\ &\quad + \left\{-1 + [x^{-1}, z^{-1}]x - [x^{-1}, z^{-1}]xyx(zzz^{-1})y^{-1} + yzx(zzz^{-1})y^{-1}\right\}\frac{\partial y}{\partial x}.\end{aligned}$$

We now compute $(\rho \otimes \mathfrak{a}) \circ \phi\left(\frac{\partial r}{\partial x}\right)$. For simplicity, we put $Y = \rho(y)$ and

$$B = \sum_{i=1}^{\lfloor n/2 \rfloor} W^i.$$

Then, since $\mathfrak{a}(x) = t$ and $\mathfrak{a}(z) = t^2$, we can write

$$\begin{aligned}(\rho \otimes \mathfrak{a}) \circ \phi\left(\frac{\partial y}{\partial x}\right) &= AB(t^{-4}X^{-1}Z^{-1}X^{-1} - t^{-2}X^{-1}Z^{-1}X^{-1}Z + t^{-1}X^{-1} - tX^{-1}[Z^{-1}, X^{-1}]Z) \\ &= t^{-4}M + t^{-3}N + t^{-2}O + t^{-1}P + t^0Q + t^1R + t^2S + t^3T + t^4U + t^5V,\end{aligned}$$

where

$$\begin{aligned}M &= ABX^{-1}Z^{-1}X^{-1}, \\ N &= X^{-1}Z^{-1} + [X^{-1}, Z^{-1}]XABX^{-1}Z^{-1}X^{-1}, \\ O &= ABX^{-1}Z^{-1}X^{-1}Z, \\ P &= -X^{-1} - ABX^{-1} - [X^{-1}, Z^{-1}]XABX^{-1}Z^{-1}X^{-1}Z \\ &\quad - [X^{-1}, Z^{-1}]XYX(ZXZ^{-1})Y^{-1}ABX^{-1}Z^{-1}X^{-1}, \\ Q &= [X^{-1}, Z^{-1}] + [X^{-1}, Z^{-1}]XABX^{-1} + YZX(ZXZ^{-1})Y^{-1}ABX^{-1}Z^{-1}X^{-1}, \\ R &= [X^{-1}, Z^{-1}]XY + ABX^{-1}Z^{-1}X^{-1}ZXZ + [X^{-1}, Z^{-1}]XYX(ZXZ^{-1})Y^{-1}ABX^{-1}Z^{-1}X^{-1}Z, \\ S &= -YZ - YZX(ZXZ^{-1})Y^{-1}ABX^{-1}Z^{-1}X^{-1}Z - [X^{-1}, Z^{-1}]XABX^{-1}[Z^{-1}, X^{-1}]Z \\ &\quad - [X^{-1}, Z^{-1}]XYX(ZXZ^{-1})Y^{-1}ABX^{-1}, \\ T &= YZX(ZXZ^{-1})Y^{-1}ABX^{-1}, \\ U &= [X^{-1}, Z^{-1}]XYXZ + [X^{-1}, Z^{-1}]XYX(ZXZ^{-1})Y^{-1}ABX^{-1}[Z^{-1}, X^{-1}]Z, \\ V &= -YZXZ + YZX(ZXZ^{-1})Y^{-1}ZX^{-1} - YZX(ZXZ^{-1})Y^{-1}ABX^{-1}Z^{-1}X^{-1}ZXZ.\end{aligned}$$

Then, by using the identity

$$|A + B| = |A| + |B| + \text{tr}AB^*,$$

where $A, B \in M_2(\mathbb{C})$ and B^* is the cofactor matrix of B , we have

$$\begin{aligned}
& \left| (\rho \otimes \mathbf{a}) \circ \phi \left(\frac{\partial r}{\partial x} \right) \right| \\
&= |M|t^{-8} + (\text{tr}MN^*)t^{-7} + (|N| + \text{tr}MO^*)t^{-6} + (\text{tr}MP^* + \text{tr}NO^*)t^{-5} \\
&\quad + (|O| + \text{tr}MQ^* + \text{tr}NP^*)t^{-4} + (\text{tr}MR^* + \text{tr}NQ^* + \text{tr}OP^*)t^{-3} \\
&\quad + (|P| + \text{tr}MS^* + \text{tr}NR^* + \text{tr}OQ^*)t^{-2} + (\text{tr}MT^* + \text{tr}NS^* + \text{tr}OR^* + \text{tr}PQ^*)t^{-1} \\
&\quad + (|Q| + \text{tr}MU^* + \text{tr}NT^* + \text{tr}OS^* + \text{tr}PR^*)t^0 \\
&\quad + (\text{tr}MV^* + \text{tr}NU^* + \text{tr}OT^* + \text{tr}PS^* + \text{tr}QR^*)t^1 \\
&\quad + (|R| + \text{tr}NV^* + \text{tr}OU^* + \text{tr}PT^* + \text{tr}QS^*)t^2 \\
&\quad + (\text{tr}OV^* + \text{tr}PU^* + \text{tr}QT^* + \text{tr}RS^*)t^3 + (|S| + \text{tr}PV^* + \text{tr}QU^* + \text{tr}RT^*)t^4 \\
&\quad + (\text{tr}QV^* + \text{tr}RU^* + \text{tr}ST^*)t^5 + (|T| + \text{tr}RV^* + \text{tr}SU^*)t^6 + (\text{tr}SV^* + \text{tr}TU^*)t^7 \\
&\quad + (|U| + \text{tr}TV^*)t^8 + (\text{tr}UV^*)t^9 + |V|t^{10}.
\end{aligned}$$

Since we have

$$|(\rho \otimes \mathbf{a}) \circ \phi(z-1)| = |t^2Z - E| = t^0 - (\text{tr}Z)t^2 + t^4,$$

we obtain

$$\Delta_{K_n, \rho}(t) = \frac{\left| (\rho \otimes \mathbf{a}) \circ \phi \left(\frac{\partial r}{\partial x} \right) \right|}{|(\rho \otimes \mathbf{a}) \circ \phi(z-1)|} \doteq \sum_{i=0}^6 \kappa_i(t^i + t^{14-i}) + \kappa_7 t^7,$$

where

$$\begin{aligned}
\kappa_0 &= |M| = |V|, \\
\kappa_1 &= \text{tr}MN^* = \text{tr}UV^*, \\
\kappa_2 &= (|N| + \text{tr}MO^*) + \text{tr}Z\kappa_0 = (|U| + \text{tr}TV^*) + \text{tr}Z\kappa_0, \\
\kappa_3 &= (\text{tr}MP^* + \text{tr}NO^*) + (\text{tr}Z)\kappa_1 = (\text{tr}SV^* + \text{tr}TU^*) + (\text{tr}Z)\kappa_1, \\
\kappa_4 &= (|O| + \text{tr}MQ^* + \text{tr}NP^*) - \kappa_0 + (\text{tr}Z)\kappa_2 = (|T| + \text{tr}RV^* + \text{tr}SU^*) - \kappa_0 + (\text{tr}Z)\kappa_2, \\
\kappa_5 &= (\text{tr}MR^* + \text{tr}NQ^* + \text{tr}OP^*) - \kappa_1 + (\text{tr}Z)\kappa_3 \\
&= (\text{tr}QV^* + \text{tr}RU^* + \text{tr}ST^*) - \kappa_1 + (\text{tr}Z)\kappa_3, \\
\kappa_6 &= (|P| + \text{tr}MS^* + \text{tr}NR^* + \text{tr}OQ^*) - \kappa_2 + (\text{tr}Z)\kappa_4 \\
&= (|S| + \text{tr}PV^* + \text{tr}QU^* + \text{tr}RT^*) - \kappa_2 + (\text{tr}Z)\kappa_4, \\
\kappa_7 &= (\text{tr}MT^* + \text{tr}NS^* + \text{tr}OR^* + \text{tr}PQ^*) - \kappa_3 + (\text{tr}Z)\kappa_5 \\
&= (\text{tr}OV^* + \text{tr}PU^* + \text{tr}QT^* + \text{tr}RS^*) - \kappa_3 + (\text{tr}Z)\kappa_5.
\end{aligned}$$

This proves Theorem 4.15 when n is even.

4.3.2 The case when n is odd

In this case, K_n is obtained by $-\frac{2}{n+1}$ -surgery along the trivial component of the link L_1 depicted in Figure 4.9.

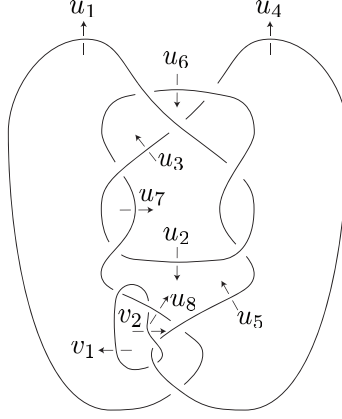


Figure 4.9: The link L_1

Let $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, v_1, v_2$ be the Wirtinger generators of $G(L_1)$ depicted in Figure 4.9. Then, Wirtinger generators of $G(L_1)$ are given by

$$u_2 u_6 u_1^{-1} u_6^{-1} = 1 \quad (4.10)$$

$$u_6 u_2 u_5^{-1} u_2^{-1} = 1 \quad (4.11)$$

$$u_7 u_2 u_7^{-1} u_3^{-1} = 1 \quad (4.12)$$

$$u_3 u_1 u_6^{-1} u_4^{-1} u_6 u_1^{-1} = 1 \quad (4.13)$$

$$v_2 u_4 v_1^{-1} u_4^{-1} = 1 \quad (4.14)$$

$$u_4 v_2 u_5^{-1} v_2^{-1} = 1 \quad (4.15)$$

$$u_7 u_3 u_1 u_6^{-1} u_1^{-1} u_3^{-1} = 1 \quad (4.16)$$

$$v_1 u_7 v_1^{-1} u_8^{-1} = 1 \quad (4.17)$$

$$v_2 u_8 v_1^{-1} u_8^{-1} = 1 \quad (4.18)$$

$$u_5 u_8 u_5^{-1} u_4 u_1^{-1} u_4^{-1} = 1 \quad (4.19)$$

From the relations (4.10), (4.11), (4.14), (4.15), (4.16), (4.17), (4.18) and (4.19), we can write

$$\begin{aligned} u_1 &= (u_3 u_1) u_6 (u_3 u_1)^{-1} [v_1, (u_3 u_1) u_6 (u_3 u_1)^{-1}], \\ u_2 &= u_6 (u_3 u_1) u_6 (u_3 u_1)^{-1} [v_1, (u_3 u_1) u_6 (u_3 u_1)^{-1}] u_6^{-1}, \\ u_3 &= (u_3 u_1) [(u_3 u_1) u_6 (u_3 u_1)^{-1}, v_1] (u_3 u_1) u_6^{-1} (u_3 u_1)^{-1}, \\ u_4 &= v_1 [(u_3 u_1) u_6 (u_3 u_1)^{-1}, v_1] u_6 [(u_3 u_1) u_6 (u_3 u_1)^{-1}, v_1] [(u_3 u_1), u_6^{-1}] u_6 \\ &\quad [u_6^{-1}, (u_3 u_1)] [v_1, (u_3 u_1) u_6 (u_3 u_1)^{-1}] u_6^{-1} [v_1, (u_3 u_1) u_6 (u_3 u_1)^{-1}] v_1^{-1}, \\ u_5 &= u_6 [(u_3 u_1) u_6 (u_3 u_1)^{-1}, v_1] [(u_3 u_1), u_6^{-1}] u_6 [u_6^{-1}, (u_3 u_1)] [v_1, (u_3 u_1) u_6 (u_3 u_1)^{-1}] u_6^{-1}, \\ u_7 &= (u_3 u_1) u_6 (u_3 u_1)^{-1}, \\ u_8 &= v_1 (u_3 u_1) u_6 (u_3 u_1)^{-1} v_1^{-1}, \\ v_2 &= v_1 [(u_3 u_1) u_6 (u_3 u_1)^{-1}, v_1], \end{aligned}$$

with generators u_3u_1, u_6 and v_1 . Since we can get relation (4.13) from relations (4.12) and (4.14), we have

$$G(L_1) = \langle u_6, u_3u_1, v_1 \mid r_1 = 1, r_2 = 1 \rangle,$$

where

$$\begin{aligned} r_1 &= (u_3u_1)u_6(u_3u_1)^{-1}u_6(u_3u_1)u_6(u_3u_1)^{-1}[v_1, (u_3u_1)u_6(u_3u_1)^{-1}]u_6^{-1} \\ &\quad [v_1, (u_3u_1)u_6(u_3u_1)^{-1}](u_3u_1)^{-1}, \\ r_2 &= v_1(u_3u_1)u_6(u_3u_1)^{-1}v_1^2(u_3u_1)u_6^{-1}(u_3u_1)^{-1}v_1^{-1}u_6[(u_3u_1)u_6(u_3u_1)^{-1}, v_1] \\ &\quad [(u_3u_1), u_6^{-1}]u_6[u_6^{-1}, (u_3u_1)][v_1, (u_3u_1)u_6(u_3u_1)^{-1}]u_6^{-1}[v_1, (u_3u_1)u_6(u_3u_1)^{-1}]v_1^{-1} \\ &\quad [(u_3u_1)u_6(u_3u_1)^{-1}, v_1]u_6[(u_3u_1)u_6(u_3u_1)^{-1}, v_1][(u_3u_1), u_6^{-1}]u_6^{-1}[u_6^{-1}, (u_3u_1)] \\ &\quad [v_1, (u_3u_1)u_6(u_3u_1)^{-1}]u_6^{-1}[v_1, (u_3u_1)u_6(u_3u_1)^{-1}]v_1^{-1}. \end{aligned}$$

Since K_n is obtained from L_1 by $-\frac{2}{n+1}$ -surgery, $G(K_n)$ is obtained from $G(L_1)$ by adding the relator

$$v_1 = \{[(u_3u_1), u_6^{-1}][(u_3u_1)^{-1}, u_6]\}^{\frac{n+1}{2}},$$

which obtained from the relation $v_1 = (u_8^{-1}u_4)^{\frac{n+1}{2}}$ along the trivial component. Then, we can reduce r_2 and get

$$G(K_n) = \langle x, z \mid xzxz^{-1}[y, zxz^{-1}] = zx^{-1}[zzx^{-1}, y]x \rangle,$$

where we put $x = u_6, z = u_3u_1$ and $y = \{[z, x^{-1}][z^{-1}, x]\}^{(n+1)/2}$. The Fox derivative of the single relation, say r , with respect to x is given by

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{\partial}{\partial x} xzxz^{-1}[y, zxz^{-1}] - \frac{\partial}{\partial x} zx^{-1}[zzx^{-1}, y]x \\ &= 1 + zx^{-1} - zx^{-1}[zzx^{-1}, y] + xz - zx^{-1}z + zx^{-1}(zzx^{-1})yzx^{-1} + x(zzx^{-1})yz \\ &\quad - x(zzx^{-1})y(zzx^{-1})y^{-1}zx^{-1} \\ &\quad + \{zx^{-1}[zzx^{-1}, y] + (x - zx^{-1})(zzx^{-1}) - x(zzx^{-1})y(zzx^{-1})y^{-1}\} \frac{\partial y}{\partial x}. \end{aligned}$$

We now compute $(\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r}{\partial x} \right)$. For simplicity, we put $Y = \rho(y)$ and

$$B = \sum_{i=1}^{|(n+1)/2|} W^i.$$

Then, since $\mathfrak{a}(x) = t$ and $\mathfrak{a}(z) = t^2$, we can write

$$\begin{aligned} (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial y}{\partial x} \right) &= AB (t^{-2}XZ^{-1}X^{-1} + t^{-1}W^{-1}(ZXZ^{-1})^{-1} - E - tW^{-1}ZX^{-1}) \\ &= t^{-1}P + t^0Q + t^1R + t^2S + t^3T + t^4U, \end{aligned}$$

where

$$\begin{aligned}
P &= (XZX^{-1})W^{-1}Y(ZXZ^{-1})^{-1}Y^{-1}AB(XZX^{-1})^{-1}, \\
Q &= E + (XZX^{-1})W^{-1}Y(ZXZ^{-1})^{-1}Y^{-1}ABW^{-1}(ZXZ^{-1})^{-1} + X(ZXZ^{-1})AB(XZX^{-1})^{-1} \\
&\quad - (XZX^{-1})W^{-1}AB(XZX^{-1})^{-1}, \\
R &= ZX^{-1} - ZX^{-1}[ZXZ^{-1}, Y] - (XZX^{-1})W^{-1}Y(ZXZ^{-1})^{-1}Y^{-1}AB \\
&\quad + X(ZXZ^{-1})ABW^{-1}(ZXZ^{-1})^{-1} - (XZX^{-1})W^{-1}ABW^{-1}(ZXZ^{-1})^{-1} \\
&\quad - X(ZXZ^{-1})Y(ZXZ^{-1})Y^{-1}AB(XZX^{-1})^{-1}, \\
S &= - (XZX^{-1})W^{-1}Y(ZXZ^{-1})^{-1}Y^{-1}ABW^{-1}ZX^{-1} - X(ZXZ^{-1})AB \\
&\quad + (XZX^{-1})W^{-1}AB - X(ZXZ^{-1})Y(ZXZ^{-1})Y^{-1}ABW^{-1}(ZXZ^{-1})^{-1}, \\
T &= XZ - ZX^{-1}Z + (XZX^{-1})W^{-1}YZX^{-1} - X(ZXZ^{-1})ABW^{-1}ZX^{-1} \\
&\quad + (XZX^{-1})W^{-1}ABW^{-1}ZX^{-1} + X(ZXZ^{-1})Y(ZXZ^{-1})Y^{-1}AB, \\
U &= X(ZXZ^{-1})YZ - X(ZXZ^{-1})Y(ZXZ^{-1})Y^{-1}ZX^{-1} \\
&\quad + X(ZXZ^{-1})Y(ZXZ^{-1})Y^{-1}ABW^{-1}ZX^{-1}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\left| (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r}{\partial x} \right) \right| \\
&= |P|t^{-2} + (\text{tr}PQ^*)t^{-1} + (|Q| + \text{tr}PR^*)t^0 + (\text{tr}PS^* + \text{tr}QR^*)t^1 + (|R| + \text{tr}PT^* + \text{tr}QS^*)t^2 \\
&\quad + (\text{tr}PU^* + \text{tr}QT^* + \text{tr}RS^*)t^3 + (|S| + \text{tr}QU^* + \text{tr}RT^*)t^4 + (\text{tr}RU^* + \text{tr}ST^*)t^5 \\
&\quad + (|T| + \text{tr}SU^*)t^6 + (\text{tr}TU^*)t^7 + |U|t^8,
\end{aligned}$$

and

$$|(\rho \otimes \mathfrak{a}) \circ \phi(z - 1)| = t^0 - (\text{tr}Z)t^2 + t^4.$$

Hence we obtain

$$\Delta_{K_n, \rho}(t) = \frac{\left| (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r}{\partial x} \right) \right|}{|(\rho \otimes \mathfrak{a}) \circ \phi(z - 1)|} \doteq \sum_{i=0}^2 \lambda_i(t^i + t^{6-i}) + \lambda_3 t^3,$$

where

$$\begin{aligned}
\lambda_0 &= |P| = |U|, \\
\lambda_1 &= \text{tr}PQ^* = \text{tr}TU^*, \\
\lambda_2 &= (|Q| + \text{tr}PR^*) + (\text{tr}Z)\lambda_0 = (|T| + \text{tr}SU^*) + (\text{tr}Z)\lambda_0, \\
\lambda_3 &= (\text{tr}PS^* + \text{tr}QR^*) + (\text{tr}Z)\lambda_1 = (\text{tr}RU^* + \text{tr}ST^*) + (\text{tr}Z)\lambda_1.
\end{aligned}$$

This proves Theorem 4.15 when n is odd.

Chapter 5

Asymptotics

First of all, we review Goda's theorem. It is known that the pair of the vector space

$$V_n = \text{span}_{\mathbb{C}} \langle x^{n-1}, x^{n-2}y, \dots, xy^{n-2}, y^{n-1} \rangle \subset \mathbb{C}[x, y]$$

and the action of $A \in SL_2(\mathbb{C})$ on V_n defined by

$$A \cdot p \begin{pmatrix} x \\ y \end{pmatrix} = p \left(A^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

gives an n -dimensional irreducible representation $\sigma_n : SL_2(\mathbb{C}) \rightarrow SL_n(\mathbb{C})$, where $p \begin{pmatrix} x \\ y \end{pmatrix} \in V_n$ is a homogeneous polynomial. Then, composing the holonomy representation ρ with σ_n , we obtain the representation $\rho_n : G(K) \rightarrow SL_n(\mathbb{C})$.

Theorem 5.1 (Goda [Go]). *Let K be a hyperbolic knot in S^3 . Then we have*

$$\lim_{n \rightarrow \infty} \frac{4\pi \log |\mathcal{A}_{K,n}(1)|}{n^2} = \text{Vol}(S^3 \setminus K),$$

where

$$\mathcal{A}_{K,2k}(t) = \frac{\Delta_{K,\rho_{2k}}(t)}{\Delta_{K,\rho_2}(t)}, \quad \mathcal{A}_{K,2k+1}(t) = \frac{\Delta_{K,\rho_{2k+1}}(t)}{\Delta_{K,\rho_3}(t)}.$$

Park's conjecture suggests the complexification of Goda's theorem.

Conjecture 5.2. *For any hyperbolic knot K , we have*

$$\lim_{n \rightarrow \infty} \frac{4\pi \log \mathcal{A}_{K,n}(1)}{n^2} = \text{Vol}(S^3 \setminus K) + i2\pi^2 \text{CS}(S^3 \setminus K).$$

To confirm that Conjecture 5.2 is true, we observe the asymptotic behavior of

$$\frac{4\pi \log \mathcal{A}_{K,n}(1)}{n^2}.$$

If we have

$$\lim_{n \rightarrow \infty} \frac{4\pi \log \mathcal{A}_{K,n}(1)}{n^2} = \text{Vol}(S^3 \setminus K) + i2\pi^2 \text{CS}(S^3 \setminus K),$$

then for large number $n \gg 0$, we have

$$\mathcal{A}_{K,n}(1) \sim \exp \left(\frac{n^2}{4\pi} \{ \text{Vol}(S^3 \setminus K) + i2\pi^2 \text{CS}(S^3 \setminus K) \} \right).$$

Then we should have

$$\frac{\mathcal{A}_{K,n-2}(1)\mathcal{A}_{K,n+2}(1)}{(\mathcal{A}_{K,n}(1))^2} \sim \exp \left(\frac{2}{\pi} \{ \text{Vol}(S^3 \setminus K) + i2\pi^2 \text{CS}(S^3 \setminus K) \} \right),$$

and then we have

$$\lim_{n \rightarrow \infty} \frac{\pi}{2} \log \frac{\mathcal{A}_{K,n-2}(1)\mathcal{A}_{K,n+2}(1)}{(\mathcal{A}_{K,n}(1))^2} = \text{Vol}(S^3 \setminus K) + i2\pi^2 \text{CS}(S^3 \setminus K).$$

On the other hand, by the definition of $\mathcal{A}_{K,n}(t)$, we have

$$\frac{\mathcal{A}_{K,n-2}(1)\mathcal{A}_{K,n+2}(1)}{(\mathcal{A}_{K,n}(1))^2} = \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2}.$$

In this chapter, we compute

$$\frac{\pi}{2} \log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2}$$

for some hyperbolic knots and natural numbers n .

Remark 5.3. Since the imaginary parts of $\log \mathcal{A}_{K,n}(1)$ are given in $[-\pi, \pi]$ by Mathematica, the computation

$$\lim_{n \rightarrow \infty} \frac{4\pi \log \mathcal{A}_{K,n}(1)}{n^2}$$

by Mathematica is not obtain the information for the imaginary parts.

For any $SL_2(\mathbb{C})$ representation, we have the following.

Lemma 5.4. For a representation $\rho : G(K) \rightarrow SL_2(\mathbb{C})$, suppose $\rho(s)^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then

$$\rho_n(s) = \left[\sum_k \frac{(n-j)!(j-1)!a^k b^{n-j-k} c^{n-i-k} d^{i+j-n+k-1}}{k!(n-j-k)!(n-i-k)!(i+j-n+k-1)!} \right],$$

where

$$\begin{cases} 0 \leq k \leq n-j & \text{if } i \leq j, n+1 \leq i+j, \\ 0 \leq k \leq n-i & \text{if } j \leq i, n+1 \leq i+j, \\ n-i-j+1 \leq k \leq n-j & \text{if } i \leq j, i+j \leq n+1, \\ n-i-j+1 \leq k \leq n-i & \text{if } j \leq i, i+j \leq n+1. \end{cases}$$

5.1 The knot 5_2

In this section, let K be the knot 5_2 depicted in Figure 5.1. Then their knot group $G(K) = \pi_1(S^3 \setminus K)$ is given by

$$\langle a, b \mid b[a, b][a, b] = [a, b]a \rangle,$$

where generators a and b are elements of the fundamental group corresponds to meridians depicted in the figure. Their holonomy representation $\rho : G(K) \rightarrow SL(2; \mathbb{C})$ is given by

$$\rho(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & -\frac{1}{x^2} \\ 0 & 1 \end{pmatrix},$$

where x is a solution of $x^3 - x - 1 = 0$. This representation is obtained by the picture of the ideal triangulation of the knot complement (see [Yo]).

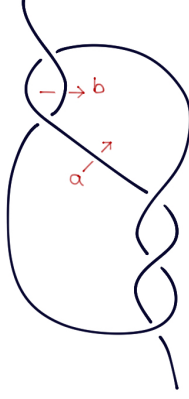


Figure 5.1: The knot 5_2

Since $\rho(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\rho_n(a) = [a_{ij}]$ is given by

$$a_{ij} = \begin{cases} \frac{(j-1)!}{(i-1)!(j-i)!} (-1)^{j-i} & \text{if } i \leq j, \\ 0 & \text{if } i > j. \end{cases}$$

Similarly, since $\rho(b) = \begin{pmatrix} 1 & -\frac{1}{x^2} \\ 0 & 1 \end{pmatrix}$, $\rho_n(b) = [b_{ij}]$ is given by

$$b_{ij} = \begin{cases} 0 & \text{if } i < j, \\ \frac{(n-j)!}{(n-i)!(i-j)!} x^{-2(i-j)} & \text{if } i \geq j. \end{cases}$$

For simplicity, we put

$$A_n = \rho(a), \quad B_n = \rho(b), \quad C_n = A_n B_n A_n^{-1} B_n^{-1}.$$

Then we have

$$\Delta_{K,\rho_n}(t) = \frac{N_n(t)}{D_n(t)},$$

where

$$\begin{aligned} N_n(t) &= |-t^{-1}B_n^{-1}(C_n + E) + t^0(B_n^{-1}C_nB_n + E + C_n) - t^1(C_n + E)C_nB_n|, \\ D_n(t) &= |tA_n - E| = (t - 1)^n. \end{aligned}$$

Hence we have

$$\log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2} = \log \frac{N_{n-2}(t)N_{n+2}(t)}{(N_n(t))^2},$$

and we obtain the following.

n	$\frac{\pi}{2} \log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2}$	n	$\frac{\pi}{2} \log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2}$
4	$3.90921... + (1.21314...)i$	5	$2.06464... + (2.28856...)i$
6	$2.00009... + (3.60568...)i$	7	$3.12694... + (3.86417...)i$
8	$3.52256... + (2.85486...)i$	9	$2.7451... + (2.46852...)i$
10	$2.41642... + (3.03596...)i$	11	$2.79327... + (3.31223...)i$
12	$3.03141... + (3.09112...)i$	13	$2.90802... + (2.88221...)i$
14	$2.73082... + (2.94575...)i$	15	$2.75758... + (3.08777...)i$
16	$2.86697... + (3.08451...)i$	17	$2.87796... + (3.00243...)i$
18	$2.81791... + (2.98433...)i$	19	$2.79732... + (3.02695...)i$
20	$2.82645... + (3.04734...)i$	21	$2.84518... + (3.02826...)i$
22	$2.83331... + (3.01192...)i$	23	$2.81966... + (3.01876...)i$
24	$2.82309... + (3.02978...)i$	25	$2.83171... + (3.02857...)i$
26	$2.83186... + (3.02199...)i$	27	$2.82699... + (3.02108...)i$
28	$2.82571... + (3.02459...)i$		

Since

$$\begin{aligned} \text{Vol}(S^3 \setminus K) &= 2.82812\dots, \\ 2\pi^2 \text{CS}(S^3 \setminus K) &= 3.02413\dots \pmod{\pi^2}, \end{aligned}$$

we conjecture that

$$\lim_{n \rightarrow \infty} \frac{\pi}{2} \log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2} = 2.82812\dots + (3.02413\dots)i.$$

5.2 The knot 6_1

In this section, let K be the knot 6_1 depicted in Figure 5.2. Their knot group $G(K)$ is given by

$$\langle a, b \mid [a, b][a, b] = b[a, b][a, b]a \rangle,$$

where generators a and b corresponds to meridians depicted in the figure. Then their holonomy representation $\rho : G(K) \rightarrow SL_2(\mathbb{C})$ is given by

$$\rho(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & -\frac{x-1}{x} \\ 0 & 1 \end{pmatrix},$$

where x is a solution of $2x^4 - 5x^3 + 6x^2 - 3x + 1 = 0$.

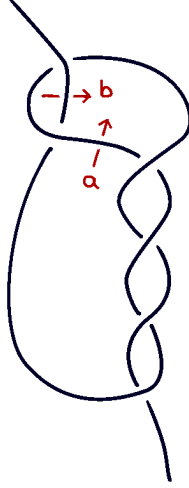


Figure 5.2: The knot 6_1

For simplicity, we put

$$A_n = \rho_n(a), \quad B_n = \rho_n(b), \quad C_n = A_n B_n A_n^{-1} B_n^{-1}.$$

Then we have

$$\Delta_{K, \rho_n}(t) = \frac{N_n(t)}{D_n(t)},$$

where

$$N_n = |(E - tB_n)(E + C_n)(E - tC_n B_n) - tB_n C_n C_n|,$$

$$D_n = |tA_n - E| = (t - 1)^n.$$

Hence we have

$$\log \frac{\Delta_{K, \rho_{n-2}}(1) \Delta_{K, \rho_{n+2}}(1)}{(\Delta_{K, \rho_n}(1))^2} = \log \frac{N_{n-2}(t) N_{n+2}(t)}{(N_n(t))^2},$$

and we obtain the following.

n	$\frac{\pi}{2} \log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2}$	n	$\frac{\pi}{2} \log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2}$
4	$1.10035\dots + (4.78702\dots)i$	5	$1.84117\dots - (1.11143\dots)i$
6	$5.56439\dots - (1.99856\dots)i$	7	$3.43741\dots - (4.81277\dots)i$
8	$1.4012\dots - (3.29514\dots)i$	9	$3.50659\dots - (1.63977\dots)i$
10	$4.19526\dots - (3.53126\dots)i$	11	$2.66498\dots - (3.95452\dots)i$
12	$2.53582\dots - (2.44756\dots)i$	13	$3.771\dots - (2.68371\dots)i$
14	$3.40126\dots - (3.63418\dots)i$	15	$2.64798\dots - (3.19445\dots)i$
16	$3.16741\dots - (2.62951\dots)i$	17	$3.52798\dots - (3.15493\dots)i$
18	$3.04638\dots - (3.37271\dots)i$	19	$2.93679\dots - (2.92699\dots)i$
20	$3.32811\dots - (2.9191\dots)i$	21	$3.27008\dots - (3.23953\dots)i$
22	$3.01150\dots - (3.14172\dots)i$	23	$3.13841\dots - (2.9411\dots)i$
24	$3.28178\dots - (3.08068\dots)i$	25	$3.14344\dots - (3.17666\dots)i$
26	$3.08572\dots - (3.04515\dots)i$	27	$3.20519\dots - (3.01979\dots)i$
28	$3.20615\dots - (3.12272\dots)i$		

Since we have

$$\begin{aligned}\text{Vol}(S^3 \setminus K) &= 3.16396\dots, \\ 2\pi^2 \text{CS}(S^3 \setminus K) &= -3.07886\dots \pmod{\pi^2},\end{aligned}$$

we conjecture that

$$\lim_{n \rightarrow \infty} \frac{\pi}{2} \log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2} = 3.16396\dots - (3.07886\dots)i.$$

5.3 The knot 6_2

In this section, let K be the knot 6_2 depicted in Figure 5.3. The knot group $G(K)$ is given by

$$\langle a, b \mid (ab\bar{a})(ba\bar{b})(ab\bar{a})b = (ba\bar{b})(ab\bar{a})(ba\bar{b})(ab\bar{a}) \rangle,$$

where generators a and b corresponds to meridians depicted in the figure. Then their holonomy representation $\rho : G(K) \rightarrow SL_2(\mathbb{C})$ is given by

$$\rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ \frac{x^2}{(-1+x)(1+x)^2} & 1 \end{pmatrix},$$

where x is a solution of $1 + 2x - x^2 - 2x^3 + x^5 = 0$.

For simplicity, we put

$$A_n = \rho_n(a), \quad B_n = \rho_n(b), \quad X_n = A_n B_n A_n^{-1}, \quad Y_n = B_n A_n B_n^{-1}.$$

Then we have

$$\Delta_{K,\rho_n}(t) = \frac{N_n(t)}{D_n(t)},$$

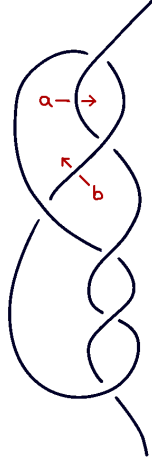


Figure 5.3: The knot 6_2

where

$$N_n = |(E - tY_n)(E + t^2X_nY_n)(E + B_nA_n^{-1} - tX_n) - B_nA_n^{-1}|,$$

$$D_n = |tA_n - E| = (t - 1)^n.$$

Hence we have

$$\log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2} = \log \frac{N_{n-2}(t)N_{n+2}(t)}{(N_n(t))^2},$$

and we obtain the following.

n	$\frac{\pi}{2} \log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2}$	n	$\frac{\pi}{2} \log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2}$
4	$4.68241... + (3.26794...)i$	5	$3.07457... - (4.44537...)i$
6	$4.04742... - (3.09489...)i$	7	$5.0858... - (3.76851...)i$
8	$4.5338... - (4.5069...)i$	9	$4.01741... - (4.06763...)i$
10	$4.36261... - (3.71779...)i$	11	$4.60593... - (3.96618...)i$
12	$4.42781... - (4.15257...)i$	13	$4.28296... - (4.01862...)i$
14	$4.38351... - (3.90932...)i$	15	$4.4656... - (3.98257...)i$
16	$4.41281... - (4.04489...)i$	17	$4.36561... - (4.00665...)i$
18	$4.39322... - (3.97128...)i$		
20	$4.40554... - (4.01077...)i$		

Since we have

$$\text{Vol}(S^3 \setminus K) = 4.40083 \dots,$$

$$2\pi^2 \text{CS}(S^3 \setminus K) = -3.99704 \dots \pmod{\pi^2},$$

we conjecture that

$$\lim_{n \rightarrow \infty} \frac{\pi}{2} \log \frac{\Delta_{K,\rho_{n-2}}(1)\Delta_{K,\rho_{n+2}}(1)}{(\Delta_{K,\rho_n}(1))^2} = 4.40083 \dots - (3.99704 \dots)i.$$

Appendix A

Calculations of R_i and R'_i defined in Section 4.2

By definition, we calculate $R_{-1}, R_0, \dots, R_{2k}$ and $R'_1, R'_2, \dots, R'_{2k}$ in Section 4.2 as follows.

$$\begin{aligned}
R_{-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{-1}}{\partial a} \right) \\
&= \begin{cases} t^{-1} \sum_{j=0}^{m_0-1} X_{-1}(AB)^j - t^0 \left(E + \sum_{j=1}^{m_0} (AB)^j \right) & \text{if } m_0 > 0, \\ -t^{-1} \sum_{j=m_0}^{-1} X_{-1}(AB)^j + t^0 \left(-E + \sum_{j=m_0+1}^0 (AB)^j \right) & \text{if } m_0 < 0, \end{cases} \\
R_0 &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_0}{\partial a} \right) = \begin{cases} -t^0 \sum_{j=0}^{m_0-1} (AB)^j + t^1 \sum_{j=0}^{m_0-1} X_0(AB)^j & \text{if } m_0 > 0, \\ t^0 \sum_{j=m_0}^{-1} (AB)^j - t^1 \sum_{j=m_0}^{-1} X_0(AB)^j & \text{if } m_0 < 0, \end{cases} \\
R_1 &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_1}{\partial a} \right) \\
&= \begin{cases} t^0 X_1 \sum_{j=1}^{m_1} (X_{-1}^{-1} A)^j A^{-1} - t^1 \sum_{j=1}^{m_1} (X_{-1}^{-1} A)^j A^{-1} & \text{if } m_1 > 0, \\ -t^0 X_1 \sum_{j=m_1+1}^0 (X_{-1}^{-1} A)^j A^{-1} + t^1 \sum_{j=m_1+1}^0 (X_{-1}^{-1} A)^j A^{-1} & \text{if } m_1 < 0, \end{cases}
\end{aligned}$$

$$\begin{aligned}
R_2 &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_2}{\partial a} \right) \\
&= \begin{cases} -t^1 \left(-E + \sum_{j=0}^{m_1-1} (X_{-1}^{-1} A)^j \right) A^{-1} + t^2 X_2 \sum_{j=1}^{m_1} (X_{-1}^{-1} A)^j A^{-1} & \text{if } m_1 > 0, \\ t^1 \left(E + \sum_{j=m_1}^{-1} (X_{-1}^{-1} A)^j \right) A^{-1} - t^2 X_2 \sum_{j=m_1+1}^0 (X_{-1}^{-1} A)^j A^{-1} & \text{if } m_1 < 0, \end{cases} \\
-R'_1 &= -(\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_1}{\partial x_{-1}} \right) \\
&= \begin{cases} t^0 \left(E + \sum_{j=1}^{m_1} (X_{-1}^{-1} A)^j \right) - t^1 \sum_{j=1}^{m_1} (X_{-1}^{-1} A)^j A^{-1} & \text{if } m_1 > 0, \\ -t^0 \left(-E + \sum_{j=m_1+1}^0 (X_{-1}^{-1} A)^j \right) + t^1 \sum_{j=m_1+1}^0 (X_{-1}^{-1} A)^j A^{-1} & \text{if } m_1 < 0, \end{cases} \\
-R'_2 &= -(\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_2}{\partial x_{-1}} \right) \\
&= \begin{cases} -t^1 \sum_{j=1}^{m_1} (X_{-1}^{-1} A)^j A^{-1} + t^2 X_2 \sum_{j=1}^{m_1} (X_{-1}^{-1} A)^j A^{-1} & \text{if } m_1 > 0, \\ t^1 \sum_{j=m_1+1}^0 (X_{-1}^{-1} A)^j A^{-1} - t^2 X_2 \sum_{j=m_1+1}^0 (X_{-1}^{-1} A)^j A^{-1} & \text{if } m_1 < 0, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
-R_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-4}} \right) \\
&= \begin{cases} -t^{-2} X_{2i-1} \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j X_{2i-3} + t^{-1} \sum_{j=1}^{m_i} (X_{2i-3} X_{2i-4})^j X_{2i-4}^{-1} & \text{if } m_i > 0, \\ t^{-2} X_{2i-1} \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j X_{2i-3} - t^{-1} \sum_{j=m_i+1}^0 (X_{2i-3} X_{2i-4})^j X_{2i-4}^{-1} & \text{if } m_i < 0, \end{cases} \\
-R'_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-3}} \right) \\
&= \begin{cases} -t^{-1} X_{2i-1} \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j + t^0 \left(E + \sum_{j=1}^{m_i} (X_{2i-3} X_{2i-4})^j \right) & \text{if } m_i > 0, \\ t^{-1} X_{2i-1} \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j + t^0 \left(E - \sum_{j=m_i+1}^0 (X_{2i-3} X_{2i-4})^j \right) & \text{if } m_i < 0, \end{cases}
\end{aligned}$$

$$\begin{aligned}
-R_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-4}} \right), \\
&= \begin{cases} t^{-1} \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j X_{2i-3} + t^0 X_{2i} \left(E - \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j \right) X_{2i-4}^{-1} & \text{if } m_i > 0, \\ -t^{-1} \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j X_{2i-3} + t^0 X_{2i} \left(E + \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j \right) X_{2i-4}^{-1} & \text{if } m_i < 0, \end{cases} \\
-R'_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-3}} \right) \\
&= \begin{cases} t^0 \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j - t^1 X_{2i} \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j & \text{if } m_i > 0, \\ -t^0 \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j + t^1 X_{2i} \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j & \text{if } m_i < 0, \end{cases}
\end{aligned}$$

if i is even and $2 \leq i \leq k$,

$$\begin{aligned}
-R_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-4}} \right) \\
&= \begin{cases} t^{-1} X_{2i-1} \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j - t^0 \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j & \text{if } m_i > 0, \\ -t^{-1} X_{2i-1} \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j + t^0 \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j & \text{if } m_i < 0, \end{cases} \\
-R'_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-3}} \right) \\
&= \begin{cases} t^0 X_{2i-1} X_{2i-3}^{-1} \left(E + \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j \right) - t^1 \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} & \text{if } m_i > 0, \\ t^0 X_{2i-1} X_{2i-3}^{-1} \left(E - \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j \right) + t^1 \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} & \text{if } m_i < 0, \end{cases} \\
-R_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-4}} \right) \\
&= \begin{cases} t^0 \left(E - \sum_{j=0}^{m_i-1} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j \right) + t^1 X_{2i} \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j & \text{if } m_i > 0, \\ t^0 \left(E + \sum_{j=m_i}^{-1} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j \right) - t^1 X_{2i} \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j & \text{if } m_i < 0, \end{cases}
\end{aligned}$$

$$\begin{aligned}
-R'_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-3}} \right) \\
&= \begin{cases} -t^1 \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} + t^2 X_{2i} \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} & \text{if } m_i > 0, \\ t^1 \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} - t^2 X_{2i} \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} & \text{if } m_i < 0, \end{cases}
\end{aligned}$$

if i is odd and $3 \leq i \leq k$.

Appendix B

Calculations of R_i, R'_i and S_i, S'_i defined in Section 4.3

By definition, we calculate $R_{-4}, R_{-1}, R_0, \dots, R_{2k}, R'_{-1}, R'_0, \dots, R'_{2k}, S_1, S_2, \dots, S_{2l}$ and $S'_1, S'_2, \dots, S'_{2l}$ in Section 4.3 as follows.

$$R_{-4} = (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{-4}}{\partial a} \right) = \begin{cases} t^{-2} X_{-4} C^{-1} & \text{if } \beta_1 = 1, \\ -t^{-2} X_{-4} A^{-1} + t^{-1} A^{-1} (E + C A^{-1}) & \text{if } \beta_1 = -1, \end{cases}$$

and

$$\begin{aligned} -R_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-4}} \right) \\ &= \begin{cases} t^1 \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j X_{2i-3} - t^2 X_{2i-1} \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j X_{2i-3} & \text{if } m_i > 0, \\ -t^1 \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j X_{2i-3} + t^2 X_{2i-1} \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j X_{2i-3} & \text{if } m_i < 0, \end{cases} \\ -R'_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-3}} \right) \\ &= \begin{cases} t^0 \left(E + \sum_{j=1}^{m_i} (X_{2i-3} X_{2i-4})^j \right) - t^1 X_{2i-1} \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j & \text{if } m_i > 0, \\ t^0 \left(E - \sum_{j=m_i+1}^0 (X_{2i-3} X_{2i-4})^j \right) + t^1 X_{2i-1} \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j & \text{if } m_i < 0, \end{cases} \end{aligned}$$

$$\begin{aligned}
-R_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-4}} \right) \\
&= \begin{cases} t^0 X_{2i} \left(E - \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j \right) X_{2i-4}^{-1} + t^1 \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j X_{2i-3} & \text{if } m_i > 0, \\ t^0 X_{2i} \left(E + \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j \right) X_{2i-4}^{-1} - t^1 \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j X_{2i-3} & \text{if } m_i < 0, \end{cases} \\
-R'_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-3}} \right) \\
&= \begin{cases} -t^{-1} X_{2i} \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j + t^0 \sum_{j=0}^{m_i-1} (X_{2i-3} X_{2i-4})^j & \text{if } m_i > 0, \\ t^{-1} X_{2i} \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j - t^0 \sum_{j=m_i}^{-1} (X_{2i-3} X_{2i-4})^j & \text{if } m_i < 0, \end{cases}
\end{aligned}$$

if i is even and $0 \leq i \leq k$,

$$\begin{aligned}
-R_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-4}} \right) \\
&= \begin{cases} -t^0 \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j + t^1 X_{2i-1} \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j & \text{if } m_i > 0, \\ t^0 \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j - t^1 X_{2i-1} \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j & \text{if } m_i < 0, \end{cases} \\
-R'_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i-1}}{\partial x_{2i-3}} \right) \\
&= \begin{cases} -t^{-1} \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} + t^0 X_{2i-1} X_{2i-3}^{-1} \left(E + \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j \right) & \text{if } m_i > 0, \\ t^{-1} \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} + t^0 X_{2i-1} X_{2i-3}^{-1} \left(E - \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j \right) & \text{if } m_i < 0, \end{cases} \\
-R_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-4}} \right) \\
&= \begin{cases} t^{-1} X_{2i} \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j + t^0 \left(E - \sum_{j=0}^{m_i-1} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j \right) & \text{if } m_i > 0, \\ -t^{-1} X_{2i} \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j + t^0 \left(E + \sum_{j=m_i}^{-1} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j \right) & \text{if } m_i < 0, \end{cases}
\end{aligned}$$

$$\begin{aligned}
-R'_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial r_{2i}}{\partial x_{2i-3}} \right) \\
&= \begin{cases} t^{-2} X_{2i} \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} - t^{-1} \sum_{j=1}^{m_i} (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} & \text{if } m_i > 0, \\ -t^{-2} X_{2i} \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} + t^{-1} \sum_{j=m_i+1}^0 (X_{2i-3}^{-1} X_{2i-4}^{-1})^j X_{2i-4} & \text{if } m_i < 0, \end{cases}
\end{aligned}$$

if i is odd and $0 \leq i \leq k$.

$$\begin{aligned}
-S_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i-1}}{\partial y_{2i-4}} \right) \\
&= \begin{cases} -t^1 \sum_{j=-n_i}^{-1} (Y_{2i-3} Y_{2i-4})^j Y_{2i-3} + t^2 Y_{2i-1} \sum_{j=-n_i}^{-1} (Y_{2i-3} Y_{2i-4})^j Y_{2i-3} & \text{if } n_i > 0, \\ t^1 \sum_{j=0}^{-n_i-1} (Y_{2i-3} Y_{2i-4})^j Y_{2i-3} - t^2 Y_{2i-1} \sum_{j=0}^{-n_i-1} (Y_{2i-3} Y_{2i-4})^j Y_{2i-3} & \text{if } n_i < 0, \end{cases}
\end{aligned}$$

$$\begin{aligned}
-S'_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i-1}}{\partial y_{2i-3}} \right) \\
&= \begin{cases} t^0 \left(E - \sum_{j=-n_i+1}^0 (Y_{2i-3} Y_{2i-4})^j \right) + t^1 Y_{2i-1} \sum_{j=-n_i}^{-1} (Y_{2i-3} Y_{2i-4})^j & \text{if } n_i > 0, \\ t^0 \left(E + \sum_{j=1}^{-n_i} (Y_{2i-3} Y_{2i-4})^j \right) - t^1 Y_{2i-1} \sum_{j=0}^{-n_i-1} (Y_{2i-3} Y_{2i-4})^j & \text{if } n_i < 0, \end{cases} \\
-S_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i}}{\partial y_{2i-4}} \right) \\
&= \begin{cases} t^0 Y_{2i} \left(E + \sum_{j=-n_i}^{-1} (Y_{2i-3} Y_{2i-4})^j \right) Y_{2i-4}^{-1} - t^1 \sum_{j=-n_i}^{-1} (Y_{2i-3} Y_{2i-4})^j Y_{2i-3} & \text{if } n_i > 0, \\ t^0 Y_{2i} \left(E - \sum_{j=0}^{-n_i-1} (Y_{2i-3} Y_{2i-4})^j \right) Y_{2i-4}^{-1} + t^1 \sum_{j=0}^{-n_i-1} (Y_{2i-3} Y_{2i-4})^j Y_{2i-3} & \text{if } n_i < 0, \end{cases} \\
-S'_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i}}{\partial y_{2i-3}} \right) \\
&= \begin{cases} t^{-1} Y_{2i} \sum_{j=-n_i}^{-1} (Y_{2i-3} Y_{2i-4})^j - t^0 \sum_{j=-n_i}^{-1} (Y_{2i-3} Y_{2i-4})^j & \text{if } n_i > 0, \\ -t^{-1} Y_{2i} \sum_{j=0}^{-n_i-1} (Y_{2i-3} Y_{2i-4})^j + t^0 \sum_{j=0}^{-n_i-1} (Y_{2i-3} Y_{2i-4})^j & \text{if } n_i < 0, \end{cases}
\end{aligned}$$

if i is even and $1 \leq i \leq l$,

$$\begin{aligned}
-S_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i-1}}{\partial y_{2i-4}} \right) \\
&= \begin{cases} t^0 \sum_{j=-n_i+1}^0 (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j - t^1 Y_{2i-1} \sum_{j=-n_i+1}^0 (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j & \text{if } n_i > 0, \\ -t^0 \sum_{j=1}^{-n_i} (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j + t^1 Y_{2i-1} \sum_{j=1}^{-n_i} (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j & \text{if } n_i < 0, \end{cases} \\
-S'_{2i-1} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i-1}}{\partial y_{2i-3}} \right) \\
&= \begin{cases} t^{-1} \sum_{j=-n_i+1}^0 (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j Y_{2i-4} + t^0 Y_{2i-1} \left(E - \sum_{j=-n_i+1}^0 (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j \right) Y_{2i-3}^{-1} & \text{if } n_i > 0, \\ -t^{-1} \sum_{j=1}^{-n_i} (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j Y_{2i-4} + t^0 Y_{2i-1} \left(E + \sum_{j=1}^{-n_i} (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j \right) Y_{2i-3}^{-1} & \text{if } n_i < 0, \end{cases} \\
-S_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i}}{\partial y_{2i-4}} \right) \\
&= \begin{cases} -t^{-1} Y_{2i} \sum_{j=-n_i+1}^0 (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j + t^0 \left(E + \sum_{j=-n_i}^{-1} (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j \right) & \text{if } n_i > 0, \\ t^{-1} Y_{2i} \sum_{j=1}^{-n_i} (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j + t^0 \left(E - \sum_{j=0}^{-n_i-1} (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j \right) & \text{if } n_i < 0, \end{cases} \\
-S'_{2i} &= (\rho \otimes \mathfrak{a}) \circ \phi \left(\frac{\partial s_{2i}}{\partial y_{2i-3}} \right) \\
&= \begin{cases} -t^{-2} Y_{2i} \sum_{j=-n_i+1}^0 (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j Y_{2i-4} + t^{-1} \sum_{j=-n_i+1}^0 (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j Y_{2i-4} & \text{if } n_i > 0, \\ t^{-2} Y_{2i} \sum_{j=1}^{-n_i} (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j Y_{2i-4} - t^{-1} \sum_{j=1}^{-n_i} (Y_{2i-3}^{-1} Y_{2i-4}^{-1})^j Y_{2i-4} & \text{if } n_i < 0, \end{cases}
\end{aligned}$$

if i is odd and $1 \leq i \leq l$.

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