Doctoral Dissertation

Study on Design and Maintenance Policies for Linear Consecutive-k-out-of-n:G Systems

線形連続k-out-of-n:G システムの設計と保全方策に関する研究

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Abstract

System failure is an unavoidable event, and the consequences of such failures could significantly impact our lives. Therefore, in reliability engineering, optimization plays an important role in ensuring high reliability and availability of a system. In general, optimization problems include design problems in design phase and maintenance problems in operation phase.

In practice, a system exists such that a series of working components causes system work, which can be modeled as a consecutive-k-out-of-n:G system. Given the theoretical development and practical applications in the reliability field, much effort has been devoted to studying the properties of this system. Many of these studies have been solved the component arrangement problem for improving the system reliability. But though this system is important in terms of its application to the general system, there are no studies on the above optimization problems for this system which has significance in reliability engineering. Therefore, the main objective of this thesis is to consider two types of optimization problems and obtain the optimal results for consecutive-k-out-of-n:G systems. First, in system design phase, the size of the system is considered, and the optimal number of components is obtained. Second, in operation phase, the age replacement is considered, and the optimal replacement time is obtained.

This thesis consists of six chapters. Chapter 1 briefly gives the background of this study, and introduces the concept of consecutive-k-out-of-n systems and optimization problems. In addition, literature reviews related to this thesis are detailed and systematically classified.

Chapter 2 focuses on system reliability evaluation, which includes system reliability and mean time to failure (MTTF). System reliability evaluation is a fundamental step in system performance evaluation. Furthermore, to build mathematical models of optimization problems, system reliability and MTTF are needed. We propose a closed form of the system reliability. In detail, we consider two cases: $k = 2$ and $k \geq 3$. When $k = 2$, system reliability and MTTF can be easily derived using a simple expression, which could reduce the complexity of the calculation. Furthermore, when $k \geq 3$, we propose the general expressions of the system reliability and MTTF.

Chapter 3 deals with the general formulas for calculating the expected number of failed components of a coherent system. A coherent system is the one in which every component is relevant, and the improvement of components cannot lead to

a deterioration in system performance. The consecutive-k-out-of-n:G system is an example of coherent systems. The number of failed or working components in a working or failed system is considered to be useful for understanding the behavior of the system, and it gives an idea that how many spare components should be available to replace failed components. The purpose of this chapter is to propose the general formulas for calculating the expected number of failed components of any coherent system. Furthermore, some examples are given, which include a bridge-structure system and a consecutive-k-out-of-n:F system, and the results of the expected number of failed components are given.

In Chapter 4, we consider two optimization problems for the consecutive-k-out-of-n:G system. In system design phase, although the system reliability increases with n, the large number of components will cause the wastage of resources. Therefore, we focus on system configuration, e.g., number of components, and the optimal number of components is discussed. In operation phase, we consider the age replacement, and the optimal replacement time is discussed. Under the assumption that all components are replaced, we build the models of the expected cost rates of these two problems and obtain the corresponding optimal policies by minimizing the expected cost rates. To investigate the proposed optimal policies, we perform the numerical experiments and analyze the results.

In Chapter 5, under the assumption that all components follow the exponential lifetime distribution, we consider that only failed components are replaced and working components are maintained from an economical view. First, we focus on the number of components in system design phase. We give the expected number of failed components at the time of system failure for the consecutive-k-out-of-n:G system. By building the model of the expected cost rate and minimizing it, we derive the results of the optimal number of components. Second, we focus on the replacement time in system operation phase. By considering the number of failed components at the time of replacement, we first obtain the expected number of failed components at a particular time t. Then, we build the model of the expected cost rate and judge the existence of the optimal replacement time by numerical experiments. Finally, we perform numerical experiments to evaluate the efficiency of the proposed improved optimal policies.

Chapter 6 summarizes the contributions of this thesis and discusses various future perspectives. The optimal policies proposed in this thesis will be useful for the improvement of reliability and availability of the consecutive-k-out-of-n:G system.

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Acronyms and Notation

The following list the acronyms and that are used in this thesis. Additional notations are given if necessary when they are introduced for the first time.

Acronyms

- Con/2/n:G system: consecutive-2-out-of-n:G system
- Con/2/N:G system: consecutive-2-out-of-N:G system
- Con/k/n:F system: consecutive-k-out-of-n:F system
- Con/k/n:G system: consecutive-k-out-of-n:G system
- i.i.d.: independent and identically distributed
- MTTF: mean time to failure
- IFR: increasing failure rate
- DFR: decreasing failure rate
- CFR: constant failure rate
- CM: corrective maintenance
- PM: preventive maintenance
- TBM: time-based maintenance
- CBM: condition-based maintenance
- ACO: ant colony optimization
- GA: genetic algorithm

Notation

- X_j: state of the jth component $(j = 1, 2, \cdots, n)$

$$X_j = \begin{cases} 0 & \text{if the } j\text{th component failed,} \\ 1 & \text{if the } j\text{th component works.} \end{cases}$$

- $\phi_{k|n:F}(X_1, \cdots, X_n)$: structure function of a Con/k/n:F system

- $\phi_{k|n:G}(X_1, \cdots, X_n)$: structure function of a Con/k/n:G system

- $F(t)$: cumulative density function of a component for a Con/k/n:G system with i.i.d. components

- $\bar{F}(t)$: reliability of a component for a Con/k/n:G system with i.i.d. components

- $f(t)$: probability density function of a component for a Con/k/n:G system with i.i.d. components

- $h(t)$: failure rate/hazard rate of a component for a Con/k/n:G system with i.i.d. components

- $\lfloor a \rfloor$: the largest integer less than or equal to a

- $R_G(2, n; t)$: reliability of the Con/2/n:G system with i.i.d. components

- N: random variable of the number of components of a Con/2/N:G system

- $p_n \equiv \Pr\{N = n\}$ $(n = 2, 3, \cdots)$: probability that the number of components is n

- θ: parameter of the Possion distribution

- $R_G(2, \theta; t)$: reliability of the Con/2/N:G system when the number of components is a random variable with a truncated Poisson distribution

- $\mu_{n,2}$: MTTF of a Con/2/n:G system

- $\tilde{\mu}_{n,2}$: approximate value of the MTTF of a Con/2/n:G system

- $\mu_{\theta,2}$: MTTF of a Con/2/N:G system when the number of components is a random variable with a truncated Poisson distribution

- $\tilde{\mu}_{\theta,2}$: approximate value of the MTTF of a Con/2/N:G system when the number of components is a random variable with a truncated Poisson distribution

- $R_G(k, n; a_i)$: reliability of a Con/k/n:G system with independent components, which have reliabilities $a_1, \cdots, a_n$

- $R_F(k, n; b_i)$: reliability of a Con/k/n:F system with independent components, which have reliabilities $b_1, \cdots, b_n$

- $F_f(t)$: cumulative density function of a component for a Con/k/n:F system with i.i.d. components

- $\bar{F}_f(t)$: reliability of a component for a Con/k/n:F system with i.i.d. components

- $N_F(j, k, n)$: number of combinations to arrange j failed components in a line such that no k or more failed components are consecutive for a Con/k/n:F system, that is, the number of path sets of a Con/k/n:F system with exactly j failed components

- $R_F(k, n; t)$: reliability of the Con/k/n:F system with i.i.d. components

- $N_G(j, k, n)$: number of combinations to arrange j working components in a line such that at least k working components are consecutive for a Con/k/n:G system, that is, the number of path sets of a Con/k/n:G system with exactly j working components

- $R_G(k, n; t)$: reliability of the Con/k/n:G system with i.i.d. components

- μ_n: MTTF of a Con/k/n:G system

- T: lifetime of the system

- T_i: lifetime of the component i $(i = 1, 2, \cdots, n)$

- $T_{[i]}$: the ith order statistic (that is, the ith smallest value) among $T_1, T_2, \cdots, T_n$ for $i = 1, 2, \cdots, n$

- X: number of failed components at the time of system failure

- $\boldsymbol{s} = (s_1, \cdots, s_n)$: system signature

- N_j: number of path sets of a system with exactly j working components

- t: planned replacement time

- $X(t)$: number of failed components when system fails before time t

- $B_i = \{T_{[i]} \leq t < T_{[i+1]}\}$: event that there are i failed components until time t

- $\Pr\{B_i\}$: probability of event B_i

- $S(t)$: number of failed components at time t while system is working

- P: number of paths for a Con/k/n:F system

- π_j: probability that the system failure follows path j for a Con/k/n:F system

- W_j: number of steps until system failure in path j for a Con/k/n:F system

- α_{ji}: sum of failure rates of working components in step i in path j

- C_R: replacement cost for a failed system

- C_1: acquisition cost for a component

- $C_1(n;2)$: expected cost rate for a failed Con/2/n:G system

- n^*: optimal number of components which minimizes $C_1(n;2)$

- $\tilde{n}^*$: approximate optimal number of components which minimizes $C_1(n;2)$

- $E[\min(t,T)]$: expected replacement time of one renewal cycle

- $C_1(t;2,n)$: expected cost rate in one renewal cycle for a Con/2/n:G system

- $H(t;2,n)$: system failure rate of a Con/2/n:G system

- t_a^*: optimal replacement time which minimizes $C_1(t;2,n)$

- $C_1(t;2,\theta)$: expected cost rate in one renewal cycle for a Con/2/N:G system, where N is a random variable with parameter θ

- $H(t;2,\theta)$: system failure rate of a Con/2/N:G system, where N is a random variable with parameter θ

- t_b^*: optimal replacement time which minimizes $C_1(t;2,\theta)$

- $C_1(n;k)$: expected cost rate for a failed Con/k/n:G system

- n_1^*: optimal number of components which minimizes $C_1(n;k)$

- $C_1(t;k,n)$: expected cost rate in one renewal cycle for a Con/k/n:G system

- $H(t;k,n)$: system failure rate of a Con/k/n:G system

- t_1^*: optimal replacement time which minimizes $C_1(t;k,n)$

- X_n: number of failed components at the time of system failure for a Con/k/n:G system

- $C_2(n;k)$: expected cost rate for a failed Con/k/n:G system, where only failed components are replaced

- n_2^*: optimal number of components which minimizes $C_2(n; k)$

- $(X(T)|T \leq t)$: number of failed components at the time of system failure, under the condition that system fails before time t

- $(S(t)|T > t)$: number of failed components at time t, under the condition that system is working

- $C_2(t; k, n)$: expected cost rate in one renewal cycle for a Con/k/n:G system, where only failed components are replaced

- t_2^*: optimal replacement time which minimizes $C_2(t; k, n)$

Chapter 1

Introduction

1.1   Background

As most of industrial systems become more complex and multiple-function oriented, such as aircrafts, submarines, military systems, and nuclear systems, it is extremely important to prevent accidents and reduce the causes of failure, which can be dangerous or disastrous [89]. As a result, monitoring and evaluating the performance of the system is essential to ensure the normal operation. Reliability, or the probability of survival, is a critical performance metric of a component or a system, and is defined as the probability that a component or a system will perform its required function under given conditions for a stated time interval [8]. Other measures of performance include failure rate, percentile of system life, mean time to failure, mean time between failures, availability, mean time between repairs, and maintainability.

We present an overview of historial development in reliability engineering. The theory of reliability engineering has its roots in the research of the performance of various military electronic systems in the World War II. These systems became less reliable because they were placed in a severe environment, and the complexity of the systems had increased. For this reason, a severe situation occurred in which many of the electronic systems could not perform their functions due to frequent failures. To overcome this problem, a new approach using probability theory and statistics has been introduced, which is called reliability engineering today. Over the years, many developments in reliability engineering have studied. The reliability engineering is summarized in Birolini [8], and the basic concepts used in the reliability engineering were introduced in Barlow and Proschan [5]. The primary goal of reliability engineering is to evaluate the reliability of a component or a system and to enhance the reliability during the design and operation phases. According to Kuo *et al.* [58], to realize that, there are several general options:

(1) keeping the system as simple as possible while meeting performance requirements;

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(2) increasing the reliability of the components in the system;

(3) using parallel redundancy for the less reliable components;

(4) using standby redundancy, which is switched to active components when failure occurs;

(5) optimizing system reliability in the design phase;

(6) optimizing system reliability in the operation phase.

A system should satisfy the performance requirements desired by the customer. As described in Lad *et al.* [60], reliability has become a mandatory requirement for customer satisfaction and is playing an increasing role in determining the competitiveness of products (systems). Hence, it is necessary to design systems with high reliability, leading to the study of reliability optimization. Therefore, this thesis focuses on (5), optimal systems design.

Another important research area in reliability engineering is the study of various maintenance policies during system operation phase in order to prevent the occurrence of system failure and improve system availability at lowest possible maintenance costs. Over the last few decades, maintenance functions have drastically evolved with the growth of technology [23, 25]. Therefore, this thesis also focuses on (6), optimal system maintenance.

1.2 System Reliability Modeling

In realistic, systems are large and complicated, yet they often have characteristic features and structures. In study of these practical systems, we often simplify system models as particular types of coherent systems based on these characteristic features and structures, where a coherent system is one in which every component is relevant for the system and the lifetime is non-decreasing function of components lifetimes.

1.2.1 Consecutive-k-out-of-n Systems

In reliability theory, literature has focused on different types of coherent systems. Two typical coherent systems are series systems and parallel systems; a series system fails if and only if at least one component fails, whereas a parallel system fails if and only if all components fail. Besides, k-out-of-n:F/G system fails/works if and only if at least k of the n components fail/work. Of course, k-out-of-n systems include series and parallel systems. For example, when $k = 1$, a k-out-of-n:F system becomes a series system, and a k-out-of-n:G system becomes a parallel system. On the other hand, when $k = n$, a k-out-of-n:F system becomes a parallel system, and a k-out-of-n:G system becomes a series system. These systems are

relatively simple and quite general, but can be applied to a variety of problems. The state of k-out-of-n systems depends only on the number of failed or working components, but not related to the position of failed or working components. However, in a practical situation, a system exists such that a cluster of failed/working components causes system failure/working, and the positions of failed/working components are necessary to be considered. In 1980, Kontoleon [52] first studied such a system where a cluster of failed components causes system failure, and subsequently, Chiang and Niu [16] formally named it "consecutive-k-out-of-n:F system", where the system consists of n components and it fails if and only if at least k consecutive components fail. On contrast, Tong [101] first introduced the consecutive-k-out-of-n:G system, where the system consists of n components and it works if and only if at least k consecutive components work. Kuo *et al.* [54] well explained the relationship between the consecutive-k-out-of-n:F system and the consecutive-k-out-of-n:G system. The consecutive-k-out-of-n systems also include the series and the parallel systems as special cases, similarly k-out-of-n systems. Over the past four decades, because of theoretical development and practical applications in the reliability field, much effort has been devoted to studying the reliability of these systems. Furthermore, these systems can be classified into linear-type and circular-type systems. The linear-type consecutive-k-out-of-n systems consist of n components which are arranged in a line; and the circular-type consecutive-k-out-of-n systems consists of n components which are arranged in a circle. Throughout this thesis, we consider the linear-type consecutive-k-out-of-n systems, and the circular-type consecutive-k-out-of-n systems will be discussed in the future.

For the applications of these systems, we give some practical examples.

Example 1. Telecommunication System (Chiang and Niu [16])

Consider a telecommunication system with n relay stations (either satelites or ground stations). We will name stations consecutively from 1 to n. Suppose a signal emitted from station 1 can be received by both stations 2 and 3, and a signal relayed from station 2 can be received by both stations 3 and 4, etc. Thus, when station 2 is failed, the telecommunication system still is able to transmit a signal from station 1 to station n. However, if both stations 2 and 3 are failed, since a signal cannot transmit from station 1 to station 4, the system fails. Similarly, if any two consecutive stations in the system fail, the system fails. This is an example of the linear consecutive-2-out-of-n:F system. A configuration of the system is given in Fig. 1.1.

Example 2. Oil Pipeline System (Chiang and Niu [16])

Consider a pipeline system for transporting oil from point A to point B by n pump stations. Pump stations are equally spaced between points A and B. Assume that each pump station is able to transport oil to a distance including k ($k > 1$) other pump stations. If one pump station is down, the flow of oil is not interrupted because the following $k-1$ stations can still carry the load. However, when at least
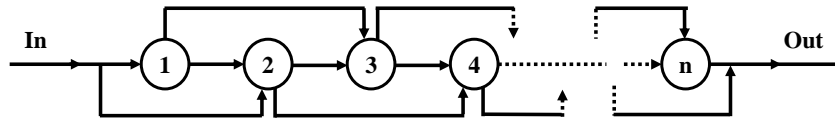
Figure 1.1: Telecommunication system.
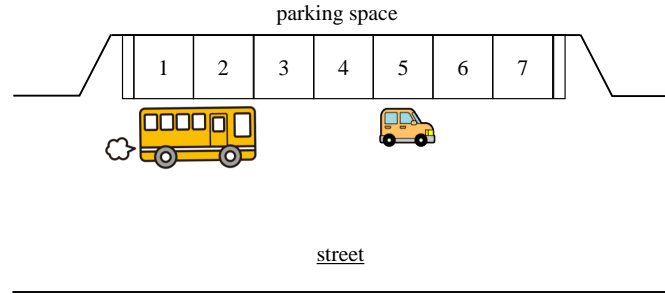
Figure 1.2: Street parking system.

k consecutive pump stations fail, the oil flow is interrupted and the system fails. Such an oil pipeline system can be modeled as a linear consecutive-k-out-of-n:F system.

Example 3. Production Monitoring System (Zhao *et al.* [117])

Consider a production monitoring system, which has n monitors. Each monitor can observe k components, and they are equally spaced along the production line. If one monitor fails, the neighboring monitors can observe the components that the failed monitor cannot observe. However, if k consecutive monitors fail, the monitoring system will have a blind area, which means that the whole system is in failure. Such a production monitoring system can be modeled as a linear consecutive-k-out-of-n:F system.

Example 4. Street Parking System (Zhang *et al.* [113])

Consider a street parking system. Suppose that there are seven parking spaces on a street. Each space is suitable for one car, and every parking space has a probability that it is not occupied.. If a bus parks on the street, it will take two spaces, that is, the bus can park if and only if at least two consecutive parking spaces on the street are empty. Such a street parking system can be modeled as a linear consecutive-2-out-of-7:G system. A configuration of the system is given in Fig. 1.2.

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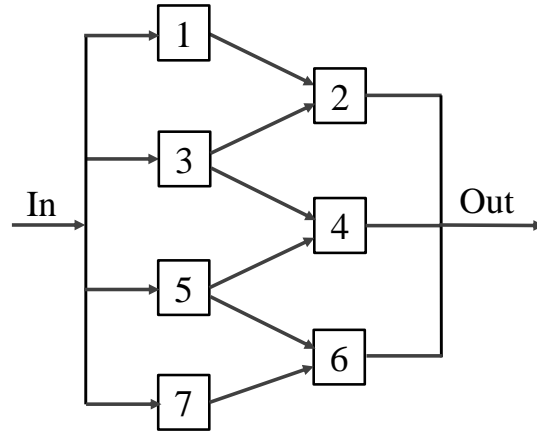
Figure 1.3: Signal transmission system.
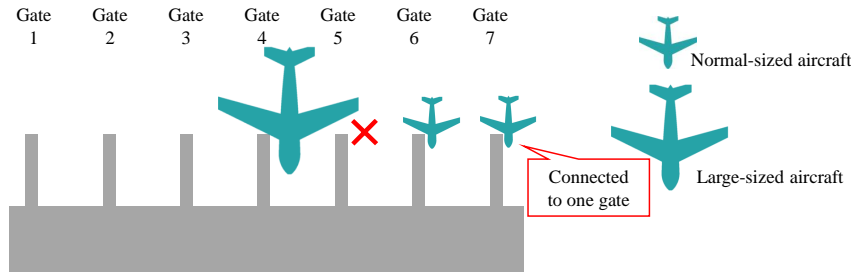
Figure 1.4: Airport gate system.

Example 5. Signal Transmission System

Consider a signal transmission system with seven transmitters. Assume that at least two consecutive transmitters are working, the signal can be transmitted successfully. Such a signal transmission system can be modeled as a linear consecutive-2-out-of-7:G system. A configuration of the system is given in Fig. 1.3.

Example 6. Airport Gate System at the Terminal

Consider an airport gate system at the terminal. Refer to Fig. 1.4, suppose that there are seven gate bridges at the terminal. Each gate bridge is suitable for one normal-sized aircraft. When a large-sized aircraft like A380 breaks in, one adjacent gate bridge becomes unusable. That is, in order to guarantee the parking of the large-sized aircraft, we should ensure that there are two or more consecutive gates are empty and can be worked. Such an airport gate system can be modeled as a linear consecutive-2-out-of-7:G system.
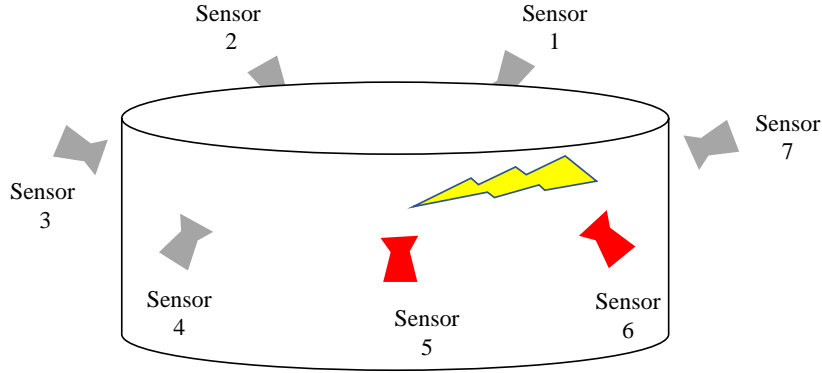
Figure 1.5: Photographing of a nuclear accelerator system.

Example 7. Photographing of a Nuclear Accelerator (Kuo and Zuo [59])

In analysis of the acceleration activities that occur in a nuclear accelerator, highspeed cameras are used to take pictures of the activities. Because of the high speed of the activities and the high cost involved in implementing such an experiment, the photographing system must be very reliable and accurate. A set of n cameras are installed around the accelerator. If and only if at least k consecutive cameras work properly, the photographing system will work. Such a photographing of a nuclear accelerator can be modeled as a circular consecutive-k-out-of-n:G system. A configuration of the system is given in Fig. 1.5.

For a comprehensive survey on reliability properties for consecutive systems, see $[18, 28, 30, 33, 53, 83, 119]$.

Before describing the linear-type systems by mathematical models, we define some common notations. Define the random variable X_j be the state of the jth component during the total n components ($j = 1, 2, \cdots, n$), then

$$X_j = \begin{cases} 0 & \text{if the } j\text{th component failed} \\ 1 & \text{if the } j\text{th component works} \end{cases} \tag{1.1}$$

For a linear consecutive-k-out-of-n:F system, n components are arranged in a line and it fails if and only if at least k consecutive components fail. Figure 1.6 depicts the linear consecutive-k-out-of-n:F system. Let $\phi_{k|n:F}(X_1, \cdots, X_n)$ denote the structure function of the linear consecutive-k-out-of-n:F system, then we have:

$$\phi_{k|n:F}(X_1, \cdots, X_n) = \prod_{j=1}^{n-k+1} \{1 - \prod_{i=j}^{j+k-1} (1 - X_i)\}.$$

For a linear consecutive-k-out-of-n:G system, n components are arranged in a line and it works if and only if at least k consecutive components work. Figure 1.7
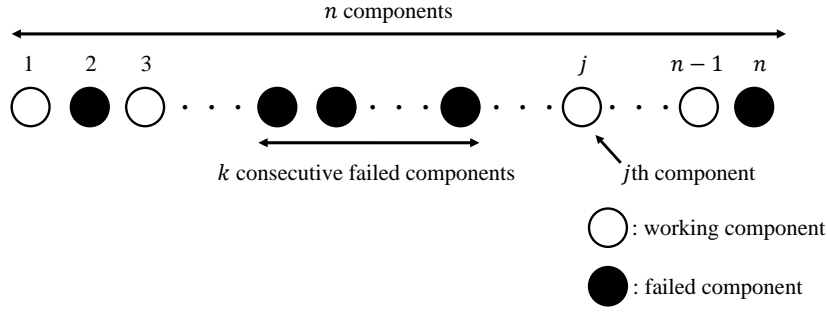
Figure 1.6: A linear consecutive-k-out-of-n:F system.
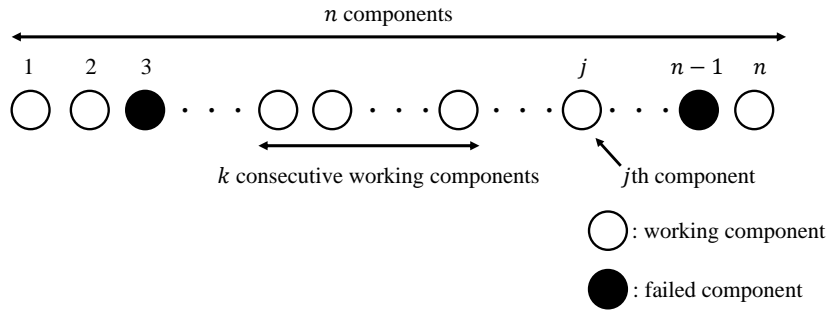
Figure 1.7: A linear consecutive-k-out-of-n:G system.

depicts the linear consecutive-k-out-of-n:G system. To make it easy to understand, we consider an example of a consecutive-2-out-of-4:G system. As shown in Fig. 1.8, the all system status are listed. Fig. 1.8 (a) gives the all working status of the consecutive-2-out-of-4:G system, and Fig. 1.8 (b) gives the all failed status of the consecutive-2-out-of-4:G system. Let $\phi_{k|n:G}(X_1, \cdots, X_n)$ denote the structure function of the linear consecutive-k-out-of-n:G system. Then, using the relation between the structure functions of a system and its dual system, we have

$$\phi_{k|n:G}(X_1, \cdots, X_n) = 1 - \phi_{k|n:F}(1 - X_1, \cdots, 1 - X_n),$$
$$= 1 - \prod_{j=1}^{n-k+1} \{1 - \prod_{i=j}^{j+k-1} X_i\}.$$

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Figure 1.8: Status of a consecutive-2-out-of-4:G system.

1.2.2 Other Consecutive-k-out-of-n Systems

Various types of consecutive-k-out-of-n systems have been developed. Some of the representative systems are listed as follows:

Redundant consecutive-k-out-of-n:F systems [44]

This system has n components and component i is actually a subsystem with r_i subcomponents connected in parallel ($1 \leq i \leq n$). Component i fails if and only if all of its r_i subcomponents are failed, and the system fails if and only if at least k components are failed. If $r_i = r$ for $1 \leq i \leq n$, such a redundant system is equivalent to a two-dimensional $(r, k)/(r, n)$:F system. This system consists of rn components arranged into a rectangular pattern with r rows and n columns. The system fails if and only if at least one failure pattern with r rows and k columns in which all components fail.

Linear and circular m-consecutive-k-out-of-n model [39]

If an n-component system fails if and only if there exist at least m non-overlapping runs of exactly k consecutive component failures ($1 \leq m \leq n/k$), then the system is called an m-consecutive-k-out-of-n system. A run of sk consecutive component failures is considered to be s runs of exactly k consecutive component failures ($s > 1$).

The k-within-consecutive-m-out-of-n systems

Another generalization of the consecutive-k-out-of-n system is the k-within-consecutive-m-out-of-n:F system, which consists of n components $1, 2, \cdots, n$. The system is failed if, among any consecutive m components, there are at least k ($k \leq m$) failed components. In the case of $k = m$, the system becomes a consecutive-k-out-of-n:F system.

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Any other types of consecutive systems can refer to [59].

1.3 Optimization Problems

As mentioned in Section 1.1, the system reliability and availability can be enhanced during the system design phase and operation phase. We will introduce optimization problems during these two phases in detail.

1.3.1 Design Problems

The primary objective of the optimization problems in system design phase is to enhance system reliability. There are several ways to realize [66]: (i) increasing the reliability of each component in the system; (ii) providing redundant components in parallel; (iii) using a combination of enhanced component reliability and redundant components provisioned in parallel; (iv) reassigning the exchangeable components; (v) determining the system configurations. First, increasing the component reliability is achieved by the internal technology of each field and cannot be conducted by reliability engineering. The second and third options are called the redundancy allocation problem (RAP) and the reliability-redundancy allocation problem (RRAP), respectively. The RAP aims to determine a system configuration by either maximizing system reliability under budget constraints or by minimizing system cost under constraints on the system reliability. The RRAP is the problem of maximizing system reliability through component reliability choices and component redundancy. Relevant work could be seen in Kuo and Prasad [57] and Coit and Zio [17]. Furthermore, the fourth approach aims to find the optimal arrangement of exchangeable components that maximizes system reliability, namely, the so-called optimal arrangement. Optimal arrangements are classified into two types: invariant and variant optimal arrangements. An invariant optimal arrangement is an optimal arrangement that depends only on the ordering of the values of component reliabilities, and a variant optimal arrangement is an optimal arrangement that depends on the numerical values of component reliabilities. Another approach is to adjust the system configurations, including a parameter for determining the system size. This approach will be explained later.

1.3.2 Maintenance Problems

In reliability engineering, maintenance (e.g., inspection, overhaul, repair or replacement) plays an important role during the operation phase. The main objective of this process is to determine the optimal maintenance policies that aim to provide maximal system reliability or availability and safety performance at the lowest possible maintenance cost. In the past several decades, maintenance problems have been extensively discussed in the literature. Pham and Wang [89]

classified the maintenance according to the degree to which the operating conditions of the system is repaired by maintenance in the following way:

(a) Perfect maintenance: a maintenance action which repairs the system operating condition to as good as new. That is, upon perfect maintenance, a system has the same lifetime distribution and failure rate function as a brand new one. Generally, replacement of a failed system by a brand new one is a perfect maintenance.

(b) Minimal maintenance: a maintenance action which repairs the system to the state when it failed. After minimal maintenance, the system operation state is often called as bad as old. Minimal maintenance is first studied by Barlow and Proshan [5].

(c) Imperfect maintenance [15]: a maintenance action does not make a system like as good as new, but younger. Usually, it is assumed that imperfect maintenance repairs the system operating state to somewhere between as good as new and as bad as old. Clearly, imperfect maintenance is general maintenance which can include two extreme cases: minimal and perfect maintenance.

(d) Worse maintenance: a maintenance action which makes the system failure rate or actual age increases but the system does not break down.

(e) Worst maintenance: a maintenance action which undeliberately makes the system fail or break down.

On the other hand, maintenance can be classified by two major categories: corrective maintenance (CM) and preventive maintenance (PM), according to the status of the system when the maintenance occurs. Corrective maintenance (CM) is a strategy that is used to maintain (repair or replace) the system to its required function after it has failed [9]. This strategy leads to high levels of system downtime (production loss) and maintenance (repair or replacement) costs due to sudden failure [103]. Preventive maintenance (PM) means all actions performed in an attempt to retain the system in specified condition by providing systematic inspection, detection, and prevention of incipient failures [15,67]. The main objective of PM is to determine the optimal maintenance policies during system operation phase and avoid the loss caused by system failure. The scientific methods to perform PM include statistics, mathematical programming, artificial intelligence, etc.

Furthermore, based on scientific approach, PM can be classified into time-based maintenance (TBM) and condition-based maintenance (CBM). TBM, also known as periodic-based maintenance is a traditional maintenance technique [109]. In TBM, maintenance decisions (e.g., preventive maintenance times/intervals) are determined based on failure time analysis. TBM assumes that the failure behaviour

of the system is predictable. Let T be the random variable representing the lifetime of a system. Its lifetime distribution can be described by its probability density function denoted as $w(t)$, and system reliability function is denoted as $R(t)$. Then, the *failure rate function*, or the *hazard function*, denoted by $H(t)$, is defined to be the limit of the probability that a failure occurs per unit time interval Δt given that no failure has occurred before time t, which can be expressed as [59]

$$
\begin{aligned}
H(t) &= \lim_{\Delta t \to 0} \frac{\Pr\{T \le t + \Delta t | T > t\}}{\Delta t}, \\
&= \lim_{\Delta t \to 0} \frac{(R(t) - R(t + \Delta t))/\Delta t}{R(t)}, \\
&= \frac{w(t)}{R(t)}.
\end{aligned}
\tag{1.2}
$$

The failure rate functions of many devices exhibit the "bathtub" curve shown in Fig. 1.9, which has been divided into three sections: burn-in, useful life, and wear-out. In the interval $(0, t_1)$, which is usually short, a *decreasing-failure-rate* (DFR) function is observed. This is often referred to as the early-failure period. The failures that occur in this interval are called early failures or burn-in failures. They are mainly due to manufacturing defects and can be screened out using burn-in techniques. In the interval (t_1, t_2), the failure rate function is fairly constant. This section is often referred to as the useful life of the system or the constant-failure-rate period. The failures that occur in this interval are called chance failures or random failures. In the interval (t_2, ∞), the failure rate function is increasing. This interval is often called the *increasing-failure-rate* (IFR) period or the wear-out failure period. The life of the device is close to its end once entering this period, unless there is preventive maintenance or major overhauls to revitalize the system.

It should be noted that the shapes of bathtub curves of different systems may be dramatically different. For example, electronic devices have a very long useful life period. Computer softwares generally have a decreasing failure rate. Mechanical devices have a long wear-out period where preventive maintenance measures are used to extend the lives of these devices. For further discussions, refer to Kuo *et al.* [55] and Kuo and Kim [56].

After the failure time analysis, maintenance can be determined. For non-repairable types of systems, the replacement policy is considered, which include age replacement, block replacement, random replacement and others [5]. The most popular decision model under replacement policy in the literature is the age replacement model [3,41]. Under this policy, a system is always replaced at time t or failure, whichever occurs first, and the replaced system assumed to be as new as one. In the finite time span replacement models we try to minimize expected cost $C(t)$ experienced during $[0, t]$. For an infinite time span, an appropriate objective
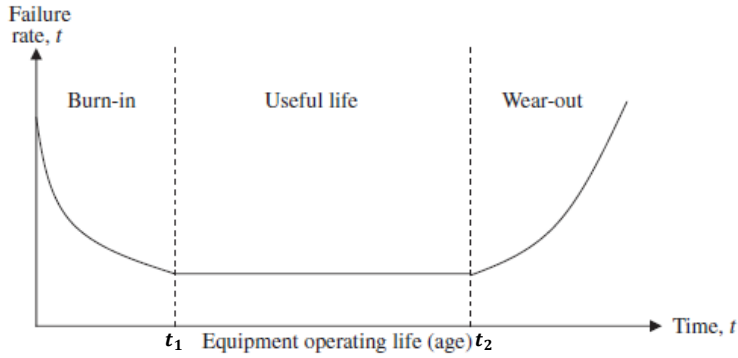
Figure 1.9: Bathtub curve.

function is expected cost per unit of time, expressed as

$$\lim_{t \to \infty} \frac{C(t)}{t}. \tag{1.3}$$

Condition-based maintenance (CBM) is the most modern and popular mainte-
nance technique discussed in the literature [40,73]. According to Jardine *et al.* [47],
CBM is a maintenance program that recommends maintenance based on the infor-
mation collected through condition monitoring process. In CBM, the lifetime of a
system is monitored through its operating condition, which can be measured based
on various monitoring parameters, such as vibration, temperature, lubricating oil,
contaminants, and noise levels. The goal of CBM is to reduce unnecessary main-
tenance actions and eliminate the risks associated with preventive maintenance
actions. Rapid development of computer based monitoring technologies(e.g., ad-
vanced sensors) has further facilitated CBM practices.

In this thesis, we consider the most popular replacement model, i.e., age re-
placement model in PM.

1.4 Overview of the Optimization Problems for Different Types of Systems

In this section, we briefly take an overview of the optimization problems in design
phase and operation phase for different types of systems. The existing discussed
systems include one-unit system, series/parallel system, series-parallel/parallel-
series system, k-out-of-n system and consecutive-k-out-of-n systems.

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1.4.1 One-unit System

The optimization problems for a one-unit system, which is the most simple system, have been well studied in the literature. Mine and Kawai [72] discussed an optimal inspection and replacement policy for a one-unit system which assumes any one of several Markov states. The policy evaluation function is the expected cost per unit time over an infinite time span. The problem is formulated as a semi-Markov decision process with a modified policy-improvement routine. Nakagawa [76] considered an inspection policy for a standby unit by taking a standby electric generator as an example. The expected cost by the time of the electric power supply failure was derived and the optimal checking time which minimizes the expected cost. For the further studies, we refer to [1,4,77,80]. Recently, Wang *et al.* [104] and Wang *et al.* [105] proposed the preventive maintenance for a single component system by considering two-phase inspection (minor inspection and major inspection) based on a three-stage failure process (good, minor defective and severe defective stages).

1.4.2 Series/Parallel System

As mentioned before, series/parallel systems have been applied to various problems, and the component assignment problem plays an important role to maximize system reliability. During the system design phase, Derman *et al.* [20] discussed the optimal component arrangement into series systems and they obtained the optimal arrangement which is only dependent on the ordering of the component reliabilities. Such an optimal arrangement is called an invariant optimal arrangement/assignment/design. Furthermore, Derman *et al.* [21] discussed the same optimal arrangement problem for the parallel system. Other results for the optimal arrangement of series/parallel systems can be found in [64].

On the other hand, during the system operation phase, maintenance policies were well studied. Laggoune *et al.* [61] proposed a preventive maintenance policy for a series system subjected to random failures. The goal of the maintenance was to minimize the expected cost per unit time under general lifetime distribution. A solution procedure based on Monte Carlo simulations with informative search method was also proposed. Mine and Kawai [71] first discussed the maintenance policy for the parallel system with two units. Furthermore, Nakagawa [74, 75] proposed the replacement policy for the parallel system in random environment. In detail, the system was exchanged when the total number of failed components was more than k. On the other hand, the system would be replaced if all components have failed. Then the optimal number of k which minimized the expected cost was obtained. Nakagawa [78] also considered another optimal system design problem for the parallel system. The optimal number of components n^* was determined by minimizing the expected cost by a unique solution of equations. Furthermore, the optimal design problem and the optimal replacement problem were considered

simultaneously and the optimal n^* and optimal replacement time before system failure T^* were derived. Nakagawa and Zhao [81] extended the problem in [78] by considering the parallel system with a random number of components with Poisson distribution. The optimal number of components and replacement time which minimiz the expected cost rates were derived analytically. For the recent results of the maintenance problems, we can refer to [35, 114, 116].

1.4.3 Parallel-Series/Series-Parallel System

In reliability engineering, the structure of parallel-series system or series-parallel system are well applied in various fields. A parallel-series system consists of k subsystems connected in parallel while subsystem j ($1 \leq j \leq k$) has n_j components connected in series, whereas a series-parallel system consists of k subsystems connected in series while subsystem j ($1 \leq j \leq k$) has n_j components connected in parallel.

The optimal arrangement problem during system design phase has been discussed in several papers. El-Neweihi *et al.* [26] discussed the optimal allocation of components to parallel-series and series-parallel systems by maximizing the reliability of the system. In addition, in series-parallel system, El-Neweihi *et al.* [26] obtained that the optimal assignment cannot be determined if only the ordering of the component reliabilities is known, that is, the optimal assignment depends on not only the ordering but also the actual values of the component reliabilities. Prasad *et al.* [90] extended the optimal allocation problem of parallel-series and series-parallel systems to a more general case wherein each position has a probability of being shock free. Each component was assumed that can be assigned to any position of the system and the reliability of component j is $r_{ij}p_j$ if it is assigned to position i, where r_{ij} is the probability that component j is assigned to position i. An algorithm was developed to obtain the optimal assignment problem. Any other results for the component arrangement can be found in [7, 63, 108].

During operation phase, maintenance problems have been discussed by considering the method of simulation. Bris *et al.* [14] proposed a new method to minimize the preventive maintenance cost for a series-paralle system. A special ratio-criterion was used to generate the ordered sequence of first inspection times. Basic system availability calculations of the paper were done by using simulation approach with parallel simulation algorithm based on direct Monte Carlo technique for availability analysis. A genetic algorithm optimization technique was used to solve the problem of finding the best maintenance policy with a given restriction. Based on Bris *et al.* [14], Samrout *et al.* [95] proposed to improve the optimal results by developing a new method based on the Ant Colony Optimization (ACO). The resolution consists in determining the solution vector of system component inspection periods, by using the programming tool Matlab. Furthermore, Lin and Wang [65] presented a hybrid genetic algorithm (GA) to optimize the preventive maintenance model in a series-parallel system. The properties of the system, in-

cluding the structure of system, maintenance priority of components, and their maintenance periods, are considered in developing the proposed hybrid GA. The optimal maintenance periods were then determined to minimize total maintenance cost by using the GA search mechanism.

1.4.4 k-out-of-n:G System

In a k-out-of-n:G system, at least k components need to work for the system to work. For a fixed k value, the higher the system size n, the higher the reliability of the system. The difference between n and k represents the degree of redundancy built into the k-out-of-n:G system. However, as n increases, there is a diminishing benefit for each additional component. As a result, the optimal value of n is necessary to consider. On the other hand, to improve the system availability, the maintenance policies during system operation phase is also well studied.

Nakagawa [79] assumed that the reliability of each component is a random variable but has a constant failure rate. Then, the optimal number of components n in system design phase, and the optimal replacement time before system failure during operation phase was derived respectively, by minimizing the expected cost per unit time. Pham [87,88] proposed another model to obtain the optimal system size n. He considered the components in system have the same reliability with a fixed value p, and the optimal n was found by minimizing the expected total cost. Sheu and Kuo [97, 98] improved the maintenance policies for k-out-of-n systems and introduced a new replacement policy which incorporates minimal repair, planned and unplanned replacements, and general random repair costs. The expected cost per unit time was also considered as the objective function and the optimal maintenance time was derived by minimizing the expected cost per unit time. Furthermore, Eryilmaz [32] considered a more practical condition that the k-out-of-n systems consists of independent and multi-type components. When the total number of components was constant, the optimal number of each type of components was obtained. Furthermore, the optimal replacement time during operation stage was also discussed. Recently, Ito *et al.* [46] extended the k-out-of-n:G system where the number of k is a stochastic parameter. They then discussed the optimization problems which are the same in [79]. Other studies can be found in [2, 99].

1.4.5 Consecutive-k-out-of-n:F System

Optimization problems for consecutive systems have gained more attention in recent decades as the quick development of complex systems. Firstly, the optimal design problem for the consecutive-k-out-of-n:F system was well studied. The optimal design problem can be divided into invariant optimal design and variant optimal design. Invariant optimal design means that the optimal arrangement depends only on the ordering of component reliability but not on their actual

values, whereas variant optimal design means that the optimal arrangement of components depends on the values of the reliabilities of these components.

The invariant optimal design problem of a linear consecutive-k-out-of-n:F system was first studied by Derman *et al.* [22] when $k = 2$, which was well proved by Wei *et al.* [106], Malon [68], and Du and Hwang [24]. Furthermore, Malon [69] studied the optimal design of the linear consecutive-k-out-of-n:F system for all possible k values, and found out that the invariant optimal design exists if and only if $k \in \{1, 2, n-2, n-1, n\}$. For the circular systems, Hwang [42] gave the conjecture of the optimal design when $k = 2$, and was proved by Malon [68] and Du and Hwang [24]. Tong [102] discovered that when $k = n - 1$ or $k = n$, any arrangement of components for such systems is an optimal design. In addition, Hwang [45] identifies all invariant optimal designs of the circular systems. For the details of the variant optimal design problems, we refer to Kuo *et al.* [54] and Zuo and Kuo [119].

Furthermore, the optimal number of k or n for the consecutive-k-out-of-n:F system was also discussed. Yun *et al.* [110] focused on a circular consecutive-k-out-of-n:F system with $(k-1)$-step Markov dependence and obtained a near-optimal k, which minimizes the expected cost per unit time. Yun *et al.* [111] considered the consecutive-k-out-of-n:F system again with another type of dependent components, where all components are load-sharing. Under this situation, they obtained the optimal number of components by using a full search method.

During the operation phase, the studies for the maintenance policies of such systems is few. In [111], the replacement during operation phase was discussed and the optimal replacement time was obtained by minimizing the expected cost per unit time. Furthermore, Endharta *et al.* [27] proposed a condition-based replacement policy for such system. The all possible paths to system failure were listed and the optimal replacement time was obtained by using the same method in [111].

1.4.6 Consecutive-k-out-of-n:G System

Similar to consecutive-k-out-of-n:F systems, the optimal arrangement of components with different reliabilities of consecutive-k-out-of-n:G systems is also well studied in the literature. On invariant optimal design, Kuo *et al.* [54] gave the invariant optimal design of a linear consecutive-k-out-of-n:G system under the condition that $n \leq 2k$, and stated that for a circular consecutive-k-out-of-n:G system with $k = n - 1$, any possible arrangement has the same system reliability. Zuo and Kuo [119] gave the optimal design of a circular consecutive-k-out-of-n:G system under the condition that $n \leq 2k + 1$. They also proved that for a linear system with $n > 2k$ or a circular system with $n > 2k + 1$, the invariant optimal designs do not exist. For the systems without invariant optimal designs, Kuo *et al.* [54] summarized the results that would narrow down the choices of possible optimal designs. Zuo and Kuo [119] also applied the heuristic algorithm to the

optimal design of consecutive-k-out-of-n:G systems.

1.5   Research Scope and Objective

In practice, a system exists such that a cluster of working components causes system work, which can be modeled as a consecutive-k-out-of-n:G system. Because of theoretical development and practical applications in the reliability field, much effort has been devoted to studying the calculation of the reliability of this system. On the other hand, only the component arrangement problem has been studied to improve the system reliability and availability. Therefore, the general objective of this thesis is to propose optimum policies, including the optimal number of components in system design phase and the optimal replacement time during system operation phase for the consecutive-k-out-of-n:G system.

Figure 1.10 summarizes the studies of optimization problems for several well applied system models in practical. These systems include one-unit system, parallel/series system, parallel-series/series-parallel system, k-out-of-n system, and consecutive-k-out-of-n:F/G system. In the second row, system design problem include the number of components and the component arrangement problem. From the third row to the fifth row, the maintenance policies are divided into inspection, repair and replacement. The symbol "—" implies that the topic has not been studied, "Ch.4" and "Ch.5" means that the optimization problems which include the design problem and maintenance problem of the consecutive-k-out-of-n:G system will be discussed in Chapter 4 and 5.

According to the Fig. 1.10, several research has been devoted to studying the optimal component arrangement design of consecutive-k-out-of-n:G systems, whereas no study has focused on the other type of optimization problems for this system. As mentioned in Section 1.1, the optimization problems play an important role to keep the high reliability and availability of the system, especially such complex consecutive-k-out-of-n:G systems. Therefore, we propose two optimization problems for the consecutive-k-out-of-n:G systems. In detail, at system design phase, we focus on the optimal number of components. We refer the model of the expected cost rate proposed by Nakagawa [79] and propose the objective function for the consecutive-k-out-of-n:G system. By minimizing the expected cost rate, the optimal number of components will be derived. During system operation phase, we consider the most typical type of maintenance policy, that is the age replacement, and system is replaced at a planned time or the time of system failure, which occurs first. The model of the expected cost rate will also be proposed, and the optimal replacement time will be derived by minimizing this expected cost rate.

Furthermore, the model of the expected cost rate proposed above is based on the assumption that all components will be replaced when maintenance occurs, whether they are failed or still working. It would be reasonable because the life

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of the working components at the time of replacement may be damaged and will not be as new as ones at the next operation cycle. From an economical view, it is possible to replace failed components only. In particular, if the failure rates of the components are approximately constant, then the working components will be as new ones at the beginning of the next operation without any life damage. As a result, under the condition that the lifetimes of components follow the exponential distribution where the failure rate is constant, we improve the policies that only failed components will be replaced by new ones at the time of maintenance.

Throughout this thesis, we make the following assumptions unless specified otherwise:

(a) each component and the system can have only two states: either working or failed;

(b) all components are mutually statistically independent and identical.

Figure 1.10: Summary of the studies for the optimization problems in different types of systems.

Optimization Problems		(1)	(2)	(3)	(4)	(5)	(6) 6-1	(6) 6-2
A	A-1	—	[20,21,64]	[7,26,63,90,108]	—	[22,42,68,69,106] [24,45,54,102,119]	[54,119]	
	A-2	—	[78,81]	—	[46,79,87,88] [2,32,99]	[110,111]	Ch.4	Ch.5
B	B-1	[1,72,76,77] [4,80,105]	[96,115]	[118]	[100,107]	—	—	—
	B-2	[1,77,104]	[61,71,114,115]	[14,65,95,118]	[97,98,107]	—	—	—
	B-3	[4,72,80]	[35,74,75,81,116]	—	[32,46,79,100]	[27,111]	Ch.4	Ch.5

A System Design.
A-1 Components Arrangement.
A-2 Number of Components.
B System Maintenance.
B-1 Inspection.
B-2 Repair.
B-3 Replacement.
(1) one-unit system.
(2) series/parallel system.
(3) series-parallel/parallel-series system.
(4) k-out-of-n:G system.
(5) consecutive-k-out-of-n:F system.
(6) consecutive-k-out-of-n:G system.
6-1 consider all components.
6-2 consider failed components.

1.6 Organization of the Thesis

This thesis considers two types of optimization problems for linear consecutive-k-out-of-n:G systems: design problem in system design phase, and maintenance problem in system operation phase. As mentioned in Subsection 1.4, we consider the expected cost rates as objective functions in this thesis, where the system reliability analysis is required. On the other hand, to consider the improved optimum policies, the expected number of failed components is also necessary. As a result, this thesis consists of six chapters.

- Chapter 1: Introduction

- Chapter 2: Reliability Analysis of Linear Consecutive Systems

- Chapter 3: Number of Failed Components for Coherent Systems

- Chapter 4: System Design and Maintenance Policies

- Chapter 5: System Design and Maintenance Policies by Considering Number of Failed Components

- Chapter 6: Conclusions

Also, Fig. 1.11 shows the organization of this thesis.

Chapter 1 briefly explains the background and introduces the different types of optimization problems. In addition, the literature reviews related to this thesis are introduced in detail, and the works in this thesis are explained. Chapter 2 gives the reliability analysis for consecutive-k-out-of-n:G systems which include the system reliability and mean time to failure (MTTF). They are needed to build mathematical models of optimization problems. Chapter 3 focuses on the general calculation formulas of the expected number of failed components in a working or failed system for any coherent system. A consecutive-k-out-of-n:G system is an example of the coherent system, and the results in this chapter provides the reference when we obtain the expected number of failed components for the consecutive-k-out-of-n:G system in Chapter 5. Chapter 4 focuses on the optimal system design policy and the optimal maintenance policy for the consecutive-k-out-of-n:G system. The expected cost rate models of consecutive-k-out-of-n:G systems are based on the models of k-out-of-n:G systems proposed by Nakagawa [79]. In Chapter 5, we improve the optimization problems discussed in Chapter 4 by considering to replace failed components only. The expected number of failed components can be easily derived by using the results in Chapter 3. The comparison of the optimal policies between the replacement of all components and the replacement of failed components is also given. The contributions of this thesis are summarized in Chapter 6, and various future perspectives are discussed.

Chapter 1: Introduction

1.1 Background
1.2 System Reliability Modeling
1.3 Optimization Problems
1.4 Overview of the Optimization Problems for Different Types of Systems
1.5 Research Scope and Objective
1.6 Organization of the Thesis

Chapter 2: Reliability Analysis of
Linear Consecutive Systems

2.1 Reliability Analysis of a Con/2/n:G system

2.2 Reliability Analysis of a Con/k/n:G system

Chapter 3: Number of Failed
Components for Coherent Systems

3.1 Number of Failed Components at
the Time of System Failure

3.2 Number of Failed Components
at Time t

3.3 Illustrative Examples

Chapter 4: Design and
Maintenance Policies

4.1 Consecutive-2-out-of-n:G system

4.2 Consecutive-k-out-of-n:G system

Chapter 5: Design and Maintenance Policies by
Considering Number of Failed Components

5.1 System Design

5.2 Maintenance Policy

5.3 Optimal Number of Components
and Replacement Time

5.4 Comparison of the Optimization Problems
Between Replacing All Components and
Replacing Failed Components

Chapter 6: Conclusion

6.1 Summary

6.2 Future work

Figure 1.11: Organizaiton of this thesis.

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Chapter 2

Reliability Analysis of Linear Consecutive Systems

The aim of this chapter is to give reliability analysis for consecutive-k-out-of-n:G systems. The system reliability analysis is the foundation in system performance evaluation. Furthermore, as mentioned in Section 1.4, we will consider the expected cost rates as objective functions in this thesis, where system reliability is necessary to build the total expected costs and the expected replacement time. On the other hand, under a special case that $k = 2$, the system reliability can be easily obtained and the expression of reliability is simple. As a result, we first give system reliability analysis of a simple consecutive-2-out-of-n:G system in Section 2.1. We then expand the value of k and give reliability analysis of a consecutive-k-out-of-n:G system with the general cases of k. Finally, we summarize the contributions of this chapter.

2.1 Reliability Analysis of a Consecutive-2-out-of-n:G System

As defined in Section 1.1, a consecutive-k-out-of-n:G system consists of n components which are arranged in a line and the system works if and only if at least k consecutive components work. A street parking system is a good example of a linear consecutive system [113]. Suppose that there are seven parking spaces on a street. Each space is suitable for one car. If a bus parks on the street, it will take two spaces. Every parking space has a probability that it is not occupied. A problem of interest is to find the probability that the bus can park on this street. Precisely, it is a reliability problem of a linear consecutive-2-out-of-7:G system. The bus can park if and only if at least two consecutive parking spaces on the street are empty. In addition, it might be encountered with the case where we could not know the exact number of components because of the complexity of the objective system. Therefore, such system with a random number of components

is potentially useful in various real-life situations. We consider a street parking system again. Suppose that there are several parking lots near a tourist spot and each parking lot has a different number of parking spaces. Each space is suitable for one car and has a probability that it is not occupied. We do not know the number of parking spaces in each parking lot. When a bus is coming, it should need two consecutive spaces to park and it is interesting to find the probability that the bus can park in a parking lot which the number of parking spaces is a random variable.

As a result, we give the reliability analysis for a consecutive-2-out-of-n:G system when the number of n is fixed or random.

2.1.1 System Reliability

Before we present the reliability of the consecutive-2-out-of-n:G system, some assumptions are given:

- Each component is either working or failed.

- The system is either working or failed.

- All component lifetimes are independent and identically distributed (i.i.d.).

- The consecutive-2-out-of-n:G system is working if and only if at least 2 consecutive components are working.

We first consider the case that n is fixed. Denote that $\lfloor a \rceil$ is the largest integer less than or equal to a, $\bar{F}(t)$ is the reliability of a component at time t, $F(t)$ is the unreliability of a component, and $R_G(2, n; t)$ is the reliability of a consecutive-2-out-of-n:G system. Then,

$$R_G(2, n; t) = \Pr\{\text{system is working}\},$$
$$= \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \Pr\{\text{system is working when there are } j \text{ working components}\}$$
$$+ \sum_{j=\lfloor (n+3)/2 \rfloor}^{n} \Pr\{\text{system is working when there are } j \text{ working components}\}.$$

$$(2.1)$$

If the number of working components is greater than $\lfloor (n+1)/2 \rfloor$, then there exists two consecutive working components, i.e., the system is working. Hence, we

have

$$\sum_{j=\lfloor (n+3)/2 \rfloor}^{n} \Pr\{\text{system is working when there are } j \text{ working components}\}$$

$$= \sum_{j=\lfloor (n+3)/2 \rfloor}^{n} \binom{n}{j} \bar{F}(t)^j F(t)^{n-j}. \tag{2.2}$$

Furthermore, if the number of working components is less than or equal to $\lfloor (n+1)/2 \rfloor$, the system has the possibility of failure, in which there exists at least one failed component between every two working components. As shown in Fig. 2.1, consider a consecutive-2-out-of-n:G system with j $(j \leq \lfloor (n+1)/2 \rfloor)$ working components, and assume that there is a lattice between each failed component. Then there exists $(n-j+1)$ lattices for j working components to arrange in random, and the number of such arrangement is $\binom{n-j+1}{j}$. As a result, the number of arrangement that such j working components causes system working is $\binom{n}{j} - \binom{n-j+1}{j}$. Then we have

$$\sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \Pr\{\text{system is working when there are } j \text{ working components}\}$$

$$= \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \left[\binom{n}{j} - \binom{n-j+1}{j} \right] \bar{F}(t)^j F(t)^{n-j}. \tag{2.3}$$

Therefore, the reliability of a consecutive-2-out-of-n:G system is

$$R_G(2,n;t)$$

$$= \sum_{j=\lfloor (n+3)/2 \rfloor}^{n} \binom{n}{j} \bar{F}(t)^j F(t)^{n-j} + \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \left[\binom{n}{j} - \binom{n-j+1}{j} \right] \bar{F}(t)^j F(t)^{n-j},$$

$$= \sum_{j=0}^{n} \binom{n}{j} \bar{F}(t)^j F(t)^{n-j} - \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} \bar{F}(t)^j F(t)^{n-j},$$

$$= 1 - \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} \bar{F}(t)^j F(t)^{n-j}. \tag{2.4}$$

We then consider a consecutive-2-out-of-N:G system with $N(N \geq 2)$ components where N is a random variable which holds a function of probability and $p_n \equiv \Pr\{N = n\}$ $(n = 2,3,\cdots)$. In this thesis, we assume that the random variable N holds as truncated Poisson distribution with parameter θ, which is the average number of events (the number of components in a system), i.e., the event rate. Then

$$p_n = \frac{\theta^n/n!}{\sum_{k=2}^{\infty} \theta^k/k!} = \frac{\theta^n/n!}{e^\theta - 1 - \theta} \quad (n = 2,3,\cdots), \tag{2.5}$$

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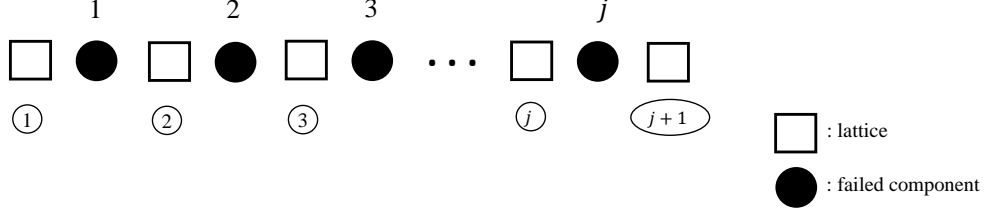
Figure 2.1: No connected working components in a consecutive-2-out-of-n:G system.

with its mean

$$E[N] = \sum_{n=2}^{\infty} n \cdot p_n = \frac{\theta(\mathrm{e}^\theta - 1)}{\mathrm{e}^\theta - 1 - \theta}. \tag{2.6}$$

Then, the reliability of the system is

$$
\begin{aligned}
R_G(2, \theta; t) &= \sum_{n=2}^{\infty} p_n \cdot R_G(2, n; t), \\
&= \frac{\sum_{n=2}^{\infty}(\theta^n/n!)\left[1 - \sum_{j=0}^{\lfloor (n+1)/2 \rfloor}\binom{n-j+1}{j}\bar{F}(t)^j F(t)^{n-j}\right]}{\mathrm{e}^\theta - 1 - \theta}, \\
&= 1 - \frac{\sum_{n=2}^{\infty}(\theta^n/n!)\sum_{j=0}^{\lfloor (n+1)/2 \rfloor}\binom{n-j+1}{j}\bar{F}(t)^j F(t)^{n-j}}{\mathrm{e}^\theta - 1 - \theta}.
\end{aligned} \tag{2.7}
$$

2.1.2 MTTF

Mean Time To Failure (MTTF) is an important measure of system reliability analysis. For a consecutive-2-out-of-n:G system, we have the following result.

Theorem 2.1. The MTTF of a consecutive-2-out-of-n:G system is

$$
\begin{aligned}
\mu_{n,2} &= \int_0^\infty R_G(2, n; t)\mathrm{d}t, \\
&= \int_0^\infty \left[1 - \sum_{j=0}^{\lfloor (n+1)/2 \rfloor}\binom{n-j+1}{j}\bar{F}(t)^j F(t)^{n-j}\right]\mathrm{d}t,
\end{aligned} \tag{2.8}
$$

for $n = 2, 3, \cdots$, and the value of $\mu_{n,2}$ increases strictly with n to ∞.

In particular, when all components have the same exponential lifetime distribution, we obtain a more concise form of the MTTF.

Corollary 2.1. The MTTF of a consecutive-2-out-of-n:G system when $F(t) = 1 - \exp(-\lambda t)$ is given by

$$\mu_{n,2} = \frac{1}{\lambda} \sum_{j=2}^{n} \frac{1}{j} - \frac{1}{\lambda} \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \frac{1}{j} \prod_{l=0}^{j-2} \left(1 - \frac{j}{n-l} \right). \tag{2.9}$$

Proof. We rewrite the reliability of the consecutive-2-out-of-n:G system as:

$$R_G(2, n; t) = 1 - \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} \bar{F}(t)^j F(t)^{n-j},$$

$$= \sum_{j=0}^{n} \binom{n}{j} \bar{F}(t)^j F(t)^{n-j} - \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} \bar{F}(t)^j F(t)^{n-j},$$

$$= \sum_{j=2}^{n} \binom{n}{j} \bar{F}(t)^j F(t)^{n-j} - \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} \bar{F}(t)^j F(t)^{n-j}. \tag{2.10}$$

When $F(t) = 1 - \exp(-\lambda t)$, from Equations (2.8) and (2.10), we have

$\mu_{n,2}$

$$= \sum_{j=2}^{n} \binom{n}{j} \int_0^{\infty} e^{-j\lambda t} (1 - e^{-\lambda t})^{n-j} dt - \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} \int_0^{\infty} e^{-j\lambda t} (1 - e^{-\lambda t})^{n-j} dt,$$

$$= \frac{1}{\lambda} \sum_{j=2}^{n} \binom{n}{j} \int_0^1 x^{j-1} (1-x)^{n-j} dx - \frac{1}{\lambda} \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} \int_0^1 x^{j-1} (1-x)^{n-j} dx,$$

$$= \frac{1}{\lambda} \sum_{j=2}^{n} \binom{n}{j} \frac{(j-1)!(n-j)!}{n!} - \frac{1}{\lambda} \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} \frac{(j-1)!(n-j)!}{n!},$$

$$= \frac{1}{\lambda} \sum_{j=2}^{n} \frac{1}{j} - \frac{1}{\lambda} \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \frac{\binom{n-j+1}{j}}{j \binom{n}{j}}, \tag{2.11}$$

where $x = e^{-\lambda t}$. Furthermore,

$$\frac{\binom{n-j+1}{j}}{\binom{n}{j}} = \frac{(n-j+1)!(n-j)!}{n!(n-2j+1)!},$$

$$= \frac{(n-j)(n-j-1)\cdots(n-2j+2)}{n(n-1)\cdots(n-j+2)},$$

$$= \prod_{l=0}^{j-2} \frac{n-l-j}{n-l}. \tag{2.12}$$

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Substituting Eq. (2.12) into Eq. (2.11), we finally obtain the result in Eq. (2.9). Hence, the proof is complete. $\qquad\square$

In addition, when components have the Weibull lifetime distribution, we propose an approximate value of the MTTF for the fast calculation.

Corollary 2.2. The MTTF of a consecutive-2-out-of-n:G system when $F(t) = 1 - \exp[-(\lambda t)^m]$ $(m > 0)$ is given approximately by

$$\tilde{\mu}_{n,2} = \frac{1}{\lambda}\left(\sum_{j=2}^{n}\frac{1}{j}\right)^{1/m} - \frac{1}{m\lambda}\left[\sum_{j=2}^{\lfloor\frac{n+1}{2}\rfloor}\frac{1}{j}\prod_{l=0}^{j-2}\left(1-\frac{j}{n-l}\right)\right]^{1/m}. \qquad (2.13)$$

Proof. Nakagawa [82] has given the result of the approximate value of MTTF for a k-out-of-n system when all components have the Weibull lifetime distribution $F(t) = 1 - \exp[-(\lambda t)^m]$. Denote that $\tilde{\mu}_{n,k}$ is the such result and

$$\tilde{\mu}_{n,k} = \frac{1}{\lambda}\left(\sum_{j=k}^{n}\frac{1}{j}\right)^{1/m}. \qquad (2.14)$$

Then the result in Eq. (2.13) is straightforward from Eq. (2.14). Hence, the proof is complete. $\qquad\square$

Table 2.1 presents the values of MTTF when $F(t) = 1 - \exp(-t^m)$ $(m > 0)$. When $m = 1$, we calculate $\mu_{n,2}$ in Eq. (2.9). When $m \neq 1$, we calculate the exact values $\mu_{n,2}$ in Eq. (2.8) and the approximate values $\tilde{\mu}_{n,2}$ in Eq. (2.13). We also calculate the values of $(\mu_{n,2} - \tilde{\mu}_{n,2})$. From the results, the differences among approximate values and exact values are small enough. Therefore, for an easier calculation, it is reasonable to use the approximate values of MTTF when components follow Weibull lifetime distributions.

In addition, when the number of components is a random variable with a truncated Poisson distribution, the MTTF is

$$\begin{aligned}
\mu_{\theta,2} &= \int_0^\infty R_G(2,\theta;t)\mathrm{d}t, \\
&= \sum_{n=2}^{\infty} p_n \int_0^\infty R_G(2,n;t)\mathrm{d}t, \\
&= \frac{\sum_{n=2}^{\infty}(\theta^n/n!)\int_0^\infty\left[1-\sum_{j=0}^{\lfloor(n+1)/2\rfloor}\binom{n-j+1}{j}\bar{F}(t)^j F(t)^{n-j}\right]\mathrm{d}t}{\mathrm{e}^\theta - 1 - \theta}. \qquad (2.15)
\end{aligned}$$

When θ is large, suppose that $(\mathrm{e}^\theta - 1 - \theta)\mathrm{e}^{-\theta} = 1 - \mathrm{e}^{-\theta} - \theta\mathrm{e}^{-\theta} \approx 1$. In this

Table 2.1: MTTF of a consecutive-2-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp(-t^m)$.

n	$m = 1$	$m = 2$			$m = 3$		
	$\mu_{n,2}$	$\mu_{n,2}$	$\tilde{\mu}_{n,2}$	$\mu_{n,2} - \tilde{\mu}_{n,2}$	$\mu_{n,2}$	$\tilde{\mu}_{n,2}$	$\mu_{n,2} - \tilde{\mu}_{n,2}$
3	0.667	0.742	0.709	0.033	0.798	0.758	0.040
5	0.950	0.925	0.844	0.081	0.937	0.856	0.081
10	1.327	1.119	1.001	0.118	1.071	0.963	0.108
20	1.701	1.281	1.138	0.143	1.175	1.053	0.122
30	1.917	1.365	1.211	0.154	1.227	1.100	0.127
40	2.070	1.422	1.261	0.161	1.261	1.130	0.131
50	2.187	1.463	1.298	0.165	1.286	1.153	0.133
60	2.283	1.496	1.327	0.169	1.305	1.171	0.134
80	2.433	1.547	1.372	0.175	1.335	1.199	0.136
90	2.495	1.567	1.390	0.177	1.347	1.209	0.138
100	2.549	1.584	1.406	0.178	1.357	1.219	0.138

case, the MTTF in Eq. (2.15) is approximately given by

$$\tilde{\mu}_{\theta,2} = \mathrm{e}^{-\theta} \sum_{n=2}^{\infty} \frac{\theta^n}{n!} \int_0^{\infty} \left[1 - \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} \bar{F}(t)^j F(t)^{n-j} \right] \mathrm{d}t. \qquad (2.16)$$

In particular, when $F(t) = 1 - \exp(-\lambda t)$, the exact value of MTTF, from Eqs. (2.9) and (2.15), is

$$\mu_{\theta,2} = \frac{1}{\lambda} \frac{\sum_{n=2}^{\infty} (\theta^n/n!) \left(\sum_{j=2}^{n} \frac{1}{j} - \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \frac{1}{j} \prod_{l=0}^{j-2} (1 - \frac{j}{n-l}) \right)}{\mathrm{e}^{\theta} - 1 - \theta}, \qquad (2.17)$$

and the approximate value is given by

$$\tilde{\mu}_{\theta,2} = \frac{\mathrm{e}^{-\theta}}{\lambda} \sum_{n=2}^{\infty} \frac{\theta^n}{n!} \left(\sum_{j=2}^{n} \frac{1}{j} - \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \frac{1}{j} \prod_{l=0}^{j-2} (1 - \frac{j}{n-l}) \right). \qquad (2.18)$$

Table 2.2 presents $\mu_{\theta,2}$ in Eq. (2.17) and $\tilde{\mu}_{\theta,2}$ in Eq. (2.18) when $F(t) = 1 - \exp(-\lambda t)$. The results of the experiments indicates that when θ is large, the assumption that $1 - \mathrm{e}^{-\theta} - \theta \mathrm{e}^{-\theta} \approx 1$ would be apposite.

2.2 Reliability Analysis of a Consecutive-k-out-of-n:G System

In this section, we extend the simple consecutive-2-out-of-n:G system to consecutive-k-out-of-n:G system, where $2 \leq k \leq n$. As mentioned in Section 1.1, a consecutive-

Table 2.2: MTTF of a consecutive-2-out-of-N:G system when component lifetime distribution is $F(t) = 1 - \exp(-\lambda t)$.

θ	$\lambda = 1$		$\lambda = 2$		$\lambda = 3$	
	$\mu_{\theta,2}$	$\tilde{\mu}_{\theta,2}$	$\mu_{\theta,2}$	$\tilde{\mu}_{\theta,2}$	$\mu_{\theta,2}$	$\tilde{\mu}_{\theta,2}$
5	0.926	0.889	0.463	0.444	0.309	0.296
10	1.298	1.297	0.649	0.648	0.433	0.432
20	1.686	1.686	0.843	0.843	0.562	0.562
30	1.908	1.908	0.954	0.954	0.636	0.636
40	2.063	2.063	1.031	1.031	0.688	0.688
50	2.182	2.182	1.091	1.091	0.727	0.727
60	2.278	2.278	1.139	1.139	0.759	0.759
80	2.430	2.430	1.215	1.215	0.810	0.810
90	2.492	2.492	1.246	1.246	0.831	0.831
100	2.547	2.547	1.273	1.273	0.849	0.849

k-out-of-n:F system with n components in a line fails if and only if at least k consecutive components failed. The reliability of consecutive-k-out-of-n:F systems has been well studied by many papers [12, 13, 16, 22, 38, 43, 49–52, 62]. For the results of dynamic reliability properties, these can be referred to [28–30, 33, 53, 83, 85]

Kuo *et al.* [54] state that the consecutive-k-out-of-n:G and F systems are the mirror images of each other. However, few papers focused on the reliability formulations of consecutive-k-out-of-n:G systems [37, 54, 113, 119, 120]. In this Section, we first present the existing recursive equations for computing the reliability of consecutive-k-out-of-n:G systems. We then propose a closed form of reliability which is more concise, and give the MTTF of the consecutive-k-out-of-n:G system.

2.2.1 Existing Methods for Computing the System Reliability

Kuo *et al.* [54] focused on the consecutive-k-out-of-n:G system where components are independent and have reliabilities $a_1, \cdots, a_n$. Then the reliability of consecutive-k-out-of-n:G systems is given by

$$R_G(k, n; a_1, \cdots, a_n) = R_G(k, n-1; a_1, \cdots, a_n) +$$

$$[1 - R_G(k, n-k-1; a_1, \cdots, a_n)](1 - a_{n-k}) \prod_{i=n-k+1}^{n} a_i. \tag{2.19}$$

When the components are i.i.d. with the same component reliability a, then

$$R_G(k, n; a) = R_G(k, n-1; a) + [1 - R_G(k, n-k-1; a)](1-a)a^k. \tag{2.20}$$

Furthermore, Gera [37] proposed another formulation of recursive equation with i.i.d. components and

$$R_G(k, n; a) = a^k + (1 - a) \sum_{l=1}^{k} a^{l-1} R_G(k, n - l; a). \tag{2.21}$$

Although those recursive algorithms are computationally efficient, they have the usual disadvantage associated with a recursive algorithm of being a black box grinding out only numbers. The dependence of the reliability on the system parameters is hidden in the equations. For the Bernoulli model, reliabilities can be computed by using a combinatorial approach which is more explicit in nature. Fortunately, the closed-form for computing the number of ways of having working consecutive-k-out-of-n:F systems conditional on j failed components was obtained [43]. We propose the reliability of consecutive-k-out-of-n:G systems in closed expression with explicit sums by using the existing results.

2.2.2 System Reliability

Consider a consecutive-k-out-of-n:G system (Con/k/n:G system) which consists of n ($n \geq k$) independent and identically distributed (i.i.d.) components. It is assumed that the components and the system are either working or failed. The system reliability of a consecutive-k-out-of-n:F system (Con/k/n:F system) has been well studied in literature, and the relationship between Con/k/n:F system and Con/k/n:G system was proposed by Kuo *et al.* [54]. Then, we can obtain the system reliability of a Con/k/n:G system by using the existing closed expression of the system reliability of a Con/k/n:F system and the relationship between these two systems. Consider a consecutive-k-out-of-n:F system, which consists of n components with the failed probability $F_f(t)$ and the reliability $\bar{F}_f(t)$ at time t. Let $N_F(j, k, n)$ denote the number of ways to arrange j failed components in a line such that no k or more failed components are consecutive, that is, system is working. Then, the reliability of the consecutive-k-out-of-n:F system is expressed as

$$R_F(k, n; t) = \sum_{j=0}^{n} N_F(j, k, n) F_f(t)^j \bar{F}_f(t)^{n-j}. \tag{2.22}$$

Hwang [43] first proposed the closed formulation of $N_F(j, k, n)$, then Kuo and Zuo [59] improved the formulation as

$$N_F(j, k, n) = \sum_{i=0}^{\lfloor j/k \rfloor} (-1)^i \binom{n - j + 1}{i} \binom{n - ik}{n - j} \quad (j = 0, 1, \cdots, n), \tag{2.23}$$

where $\lfloor a \rfloor$ means the largest integer less than or equal to real value a.

Then we give the relationship between a Con/k/n:F system and a Con/k/n:G system.

Lemma 2.1 [54]. Assume that the components in consecutive-k-out-of-n systems are independent but do not necessarily have the same lifetime distributions. Denote $R_G(k, n; a_1, \cdots, a_n)$ is the reliability of a consecutive-k-out-of-n:G system, and $R_F(k, n; b_1, \cdots, b_n)$ is the reliability of a consecutive-k-out-of-n:F system. Then if $a_i = 1 - b_i$ $(i = 1, \cdots, n)$, we have the result that

$$R_G(k, n; a_1, \cdots, a_n) = 1 - R_F(k, n; b_1, \cdots, b_n).$$

Then we focus on the closed formulation of the reliability for consecutive-k-out-of-n:G systems. Denote that $N_G(j, k, n)$ is the number of combinations to arrange j $(j = k, \cdots, n)$ working components such that at least k consecutive components are working, then by using the duality relationship between consecutive-k-out-of-n:F systems and consecutive-k-out-of-n:G systems in Lemma 2.1, we have the following result.

Theorem 2.2. The reliability of a consecutive-k-out-of-n:G system with i.i.d. components is

$$R_G(k, n; t) = \sum_{j=k}^{n} N_G(j, k, n) \bar{F}(t)^j F(t)^{n-j}, \tag{2.24}$$

where

$$N_G(j, k, n) = \binom{n}{j} - \sum_{i=0}^{\lfloor j/k \rfloor} (-1)^i \binom{n-j+1}{i} \binom{n-ik}{n-j} (j = k, \cdots, n). \tag{2.25}$$

The proof is straightforward from Lemma 2.1.

2.2.3 MTTF

Using the reliability of consecutive-k-out-of-n:G systems, we can easily obtain the MTTF.

Theorem 2.3. The MTTF of a consecutive-k-out-of-n:G system with i.i.d. components is

$$\mu_n = \int_0^\infty R_G(k, n; t) \mathrm{d}t,$$
$$= \sum_{j=k}^{n} N_G(j, k, n) \int_0^\infty \bar{F}(t)^j F(t)^{n-j} \mathrm{d}t, \tag{2.26}$$

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where $N_G(j, k, n)$ is given in Eq. (2.25).

In particular, when all components have the common exponential lifetime distribution, we propose a more concise form of the MTTF.

Corollary 2.3. The MTTF of a consecutive-k-out-of-n:G system with component lifetime distribution $F(t) = 1 - \exp(-\lambda t)$ is

$$\mu_n = \frac{1}{\lambda} \sum_{j=k}^{n} \frac{N_G(j, k, n)}{j\binom{n}{j}}, \tag{2.27}$$

where $N_G(j, k, n)$ is given in Eq. (2.25).

Proof. When all components have the exponential life distribution, the MTTF of a consecutive-k-out-of-n:G system becomes

$$\mu_n = \sum_{j=k}^{n} N_G(j, k, n) \int_0^{\infty} e^{-j\lambda t}(1 - e^{-\lambda t})^{n-j} dt,$$

$$= \sum_{j=k}^{n} N_G(j, k, n) \int_0^1 \frac{1}{\lambda} x^{j-1}(1 - x)^{n-j} dx,$$

$$= \frac{1}{\lambda} \sum_{j=k}^{n} \frac{N_G(j, k, n)}{j\binom{n}{j}}. \tag{2.28}$$

□

2.3   Summary

This chapter focused on the system reliability analysis of a consecutive-k-out-of-n:G system, including the reliability and the mean time to failure (MTTF). All components were assumed to be independent and have the identical lifetime distribution. In a simple case that $k = 2$, we obtained a simple formulation of the system reliability and derived the MTTF. Furthermore, when k has a general value, we proposed a closed form of the system reliability. The MTTF of consecutive-k-out-of-n:G systems was also obtained.

Chapter 3

Number of Failed Components for Coherent Systems

A linear consecutive-k-out-of-n:F system and a linear consecutive-k-out-of-n:G system are two well known typical types of coherent systems. A formal definition of a coherent system is given by Barlow and Proschan [6]. Informally speaking, a coherent system is that every component is relevant for the system and the lifetime is non-decreasing function of components lifetimes. The notion of coherence is central in reliability analysis since any system without it would rightly be judged to be fundamentally flawed and subject to alteration.

In addition, as mentioned in Chapter 2, the system reliability analysis is the foundation in system performance evaluation. On the other hand, the number of failed components in a working or failed system is also an effective evaluation criteria. When a system fails, the number of failed components gives the information that how many spare components should be available to replace failed components. Furthermore, the number of failed/working components when the system is working at a particular time also gives useful information to understand the behavior of the system. If the number of failed components when system is working is near the maximum number of failures that causes system failure, then we could consider to take maintenance and estimate that how many spare components should be prepared. Besides, the system also has the possibility that the failure occurs before the particular time. The expected number of failed components in this situation helps us determine the number of spare components. The studies on number of failed/working components can be found in some papers. Papastavridis [86] first studied the distribution and the expected number of failed components for consecutive-k-out-of-n:F systems with independent and identically lifetime components. Eryilmaz [31] studied the distribution and expected value of the number of working components for a working consecutive-k-out-of-n:F/G system. Eryilmaz [32] also considered anthor typical type of coherent systems, which is k-out-of-n:G systems (the system works if and only if at least k components works), and discussed the expected number of working components. Furthermore,

they applied the results to the optimization problem and proposed the optimal replacement time by considering to replace failed components only. More recent servey on the number of failed/working components can be found in [34, 36, 112].

According to the literature servey, all researches on the number of failed/working components were based on a particular type of coherent system. It is necessary to summarize general formulas for the expected number of failed/working components for any coherent system. The aim of this chapter is to give the general calculation formulas for these expected values. Section 3.1 gives the calculation for the expected number of failed components when a system fails. Section 3.2 discusses the expected number of failed components at a particular time. Section 3.3 presents numerical results for two well known general coherent systems. Finally, we summarize the contributions of this chapter.

3.1 Number of Failed Components at the Time of System Failure

In this section, we give the distribution and expected value of the number of failed components for a failed system. As mentioned before, the number of failed components for a failed system helps us to determine that how many spare components should be available to replace failed components. Consider a coherent system consisting of n i.i.d. components. Denote that T_i is the lifetime of component i ($i = 1, 2, \cdots, n$), T is the lifetime of a coherent system consisting of components with lifetime $T_1, \cdots, T_n$, and $T_{[i]}$ is the ith order statistic of the n component failure times, that is, the time of the ith component failure. Then, we define the random variable X as the number of failed components at the moment of system failure and

$$\Pr\{X = i\} = \Pr\{T = T_{[i]}\} = s_i, \tag{3.1}$$

where s_i is the probability that the system failed upon the occurrence of the ith component failure. Such vector $(s_1, \cdots, s_n)$ has been called the system signature by Samaniego [93]. In this chapter, we use the system signature to discuss the distribution and expected value of the number of failed components. In general, this vector depends on both the structure function ϕ, and the joint distribution of component lifetimes. However, as pointed out in [93], the signature vector is not related to the component lifetime distribution function when the component lifetimes are independent and identical with a common continuous distribution, i.e., the signature vector only depends on the structure function. System signature has been found to be a useful tool in a variety of applications including the evaluation of the reliability characteristics of systems such as the system failure rate, and the comparison of the performance of complex systems. For an excellent review on system signature and its applications, see [11, 19, 48, 70, 94].

Knowing the signature of a system is equivalent to knowing the distribution of the number of failed components at the moment when system failure occurs. As a result, the expected number of failed components at the time of system failure for any coherent system, which once proposed by Eryilmaz [34], is

$$E[X] = \sum_{i=1}^{n} i \cdot s_i. \tag{3.2}$$

Boland [11] proposed the expression of the system signature for any coherent system by considering the number of path sets of the system with exactly j working components, where path sets are the sets of working components which lead to the system working. Let N_j denotes the number of path sets of the system with exactly j working components and exactly $(n-j)$ failed components, then the system signature can be calculated by

$$s_{n-j} = \frac{N_{j+1}}{\binom{n}{j+1}} - \frac{N_j}{\binom{n}{j}}, \tag{3.3}$$

for $j = 0, 1, \cdots, n - 1$.

Thus, using Eqs. (3.2) and (3.3), and let i denotes the number of failed components, we have the following result.

Theorem 3.1. For any coherent system, the expected number of failed components at the time of system failure is expressed as

$$E[X] = \sum_{i=1}^{n} i \cdot \left[\frac{N_{n-i+1}}{\binom{n}{i-1}} - \frac{N_{n-i}}{\binom{n}{i}} \right]. \tag{3.4}$$

3.2 Number of Failed Components at Time t

In this section, we focus on the number of failed components at a particular time t. Further, there are two cases existing, where case 1 is that system failure occurs before time t, and case 2 is that system is working at time t. Figure 3.1 shows two cases for the expected number of failed components.

(1) Case 1: System fails before time t

Assume that all components have independent and identically lifetime distribution $F(t)$ and reliability $\bar{F}(t)$. Define the random variable $X(t)$ be the number of failed components when system fails before time t. Then, we have the following result.

Theorem 3.2. For any coherent system, the expected number of failed components when system failure occurs before time t is

$$E[X(t)] = \sum_{i=1}^{n} i \cdot s_i \sum_{j=i}^{n} \binom{n}{j} F(t)^j \bar{F}(t)^{n-j}, \tag{3.5}$$
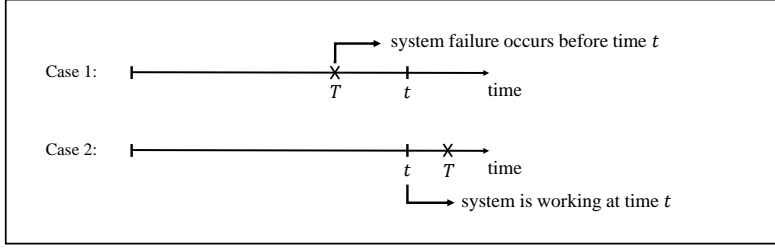
Figure 3.1: Case analysis for the expected number of failed components.

where s_i is the system signature proposed in Eq. (3.3).

Proof. Using the same notation in Section 3.1, where T is the system lifetime with n components and $T_{[i]}$ is the ith order statistic of the n component failure times. Then

$$
\begin{aligned}
E[X(t)] &= \sum_{i=1}^{n} i \cdot \Pr\{T = T_{[i]}, T \leq t\}, \\
&= \sum_{i=1}^{n} i \cdot \Pr\{T \leq t | T = T_{[i]}\} \cdot \Pr\{T = T_{[i]}\}, \\
&= \sum_{i=1}^{n} i \cdot \Pr\{T_{[i]} \leq t\} \cdot \Pr\{T = T_{[i]}\},
\end{aligned}
$$

(3.6)

where $\Pr\{T = T_{[i]}\}$ means the system signature, and $\Pr\{T_{[i]} \leq t\}$ is the distribution of the ith failed component which is expressed as

$$
\Pr\{T_{[i]} \leq t\} = \sum_{j=i}^{n} \binom{n}{j} F(t)^j \bar{F}(t)^{n-j}.
$$

Then, we can easily obtain the result in Theorem 3.2. $\qquad\square$

On the other hand, we propose anothor method to calculate such expected value by considering the number of failed components until time t. Define the event that

$$
B_i = \{T_{[i]} \leq t < T_{[i+1]}\},
$$

(3.7)

then $E[X(t)|B_i]$ is the expected number of failed components which causes system failure before time t, under the condition that there are exactly i components failed until time t. Clearly, under the condition that i components failed until time t,

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the number of failed components which causes system failure is a random variable from 1 to i. Thus, the expression of $E[X(t)|B_i]$ is

$$E[X(t)|B_i] = \sum_{j=1}^{i} j \cdot s_j. \tag{3.8}$$

Then for any coherent system, we have

$$E[X(t)] = \sum_{i=1}^{n} \Pr\{B_i\} \cdot E[X(t)|B_i],$$

$$= \sum_{i=1}^{n} \Pr\{T_{[i]} \le t < T_{[i+1]}\} \cdot E[X(t)|B_i],$$

$$= \sum_{i=1}^{n} \binom{n}{i} F(t)^i \bar{F}(t)^{n-i} \sum_{j=1}^{i} j \cdot s_j. \tag{3.9}$$

Obviously, Eqs. (3.5) and (3.9) are the same. Here, we simply give the proof.

Proof.

$$\sum_{i=1}^{n} \binom{n}{i} F(t)^i \bar{F}(t)^{n-i} \sum_{j=1}^{i} j \cdot s_j,$$

$$= \binom{n}{1} F(t) \bar{F}(t)^{n-1} \sum_{j=1}^{1} j \cdot s_j + \binom{n}{2} F(t)^2 \bar{F}(t)^{n-2} \sum_{j=1}^{2} j \cdot s_j +$$

$$\cdots + \binom{n}{n-1} F(t)^{n-1} \bar{F}(t) \sum_{j=1}^{n-1} j \cdot s_j + \binom{n}{n} F(t)^n \sum_{j=1}^{n} j \cdot s_j,$$

$$= s_1 \binom{n}{1} F(t) \bar{F}(t)^{n-1} + (s_1 + 2s_2) \binom{n}{2} F(t)^2 \bar{F}(t)^{n-2} + \cdots +$$

$$[s_1 + 2s_2 + \cdots + (n-1)s_{n-1}] \binom{n}{n-1} F(t)^{n-1} \bar{F}(t) + (s_1 + \cdots + ns_n) \binom{n}{n} F(t)^n,$$

$$= s_1 \left[\binom{n}{1} F(t) \bar{F}(t)^{n-1} + \binom{n}{2} F(t)^2 \bar{F}(t)^{n-2} + \cdots + \binom{n}{n} F(t)^n \right]$$

$$+ 2s_2 \left[\binom{n}{2} F(t)^2 \bar{F}(t)^{n-2} + \cdots + \binom{n}{n-1} F(t)^{n-1} \bar{F}(t) + \binom{n}{n} F(t)^n \right]$$

$$+ \cdots$$

$$+ (n-1)s_{n-1} \left[\binom{n}{n-1} F(t)^{n-1} \bar{F}(t) + \binom{n}{n} F(t)^n \right]$$

$$+ ns_n \binom{n}{n} F(t)^n,$$

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$$= \sum_{i=1}^{n} i \cdot s_i \sum_{j=i}^{n} \binom{n}{j} F(t)^j \bar{F}(t)^{n-j}.$$

Hence, the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

(2) Case 2: System is working at time t

Define the random variable $S(t)$ be the number of failed components at time t while system is working. Using the same notations mentioned before, where N_j denotes the number of path sets of the system with exactly j working components, $F(t)$ is the liftetime distribution of a component, and $\bar{F}(t)$ is the reliability of a component. Then, we have the following result.

Theorem 3.3. For any coherent system, the expected number of failed components when system is working at time t is

$$E[S(t)] = \sum_{i=0}^{n-1} i \cdot N_{n-i} \cdot F(t)^i \bar{F}(t)^{n-i}, \qquad (3.10)$$

where N_{n-i} is the number of path sets of the system with exactly $(n-i)$ working components.

Proof. Using the same notation in Section 3.1, where T is the system lifetime with n components and $T_{[i]}$ is the ith order statistic of the n component failure times. Then

$$E[S(t)] = \sum_{i=0}^{n-1} i \cdot \Pr\{T_{[i]} \leq t < T_{[i+1]}, T > t\},$$

$$= \sum_{i=0}^{n-1} i \cdot \Pr\{B_i, T > t\},$$

$$= \sum_{i=0}^{n-1} i \cdot \Pr\{T > t | B_i\} \cdot \Pr\{B_i\},$$

where $\Pr\{T > t | B_i\}$ is the probability that system is working under the condition where there are i failed components until time t. Then, using Eq. (3.3), we have

$$\Pr\{T > t | B_i\} = \sum_{j=i+1}^{n} s_j = \frac{N_{n-i}}{\binom{n}{i}}.$$
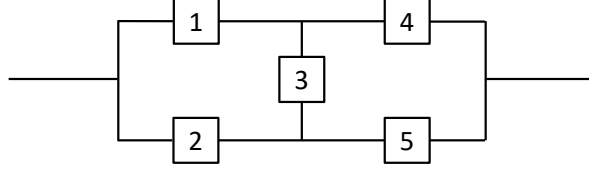
Figure 3.2: Diagram of a bridge structure system.

As a result,

$$E[S(t)] = \sum_{i=0}^{n-1} i \cdot \Pr\{T > t | B_i\} \cdot \Pr\{B_i\},$$

$$= \sum_{i=0}^{n-1} i \cdot \frac{N_{n-i}}{\binom{n}{i}} \cdot \binom{n}{i} F(t)^i \bar{F}(t)^{n-i},$$

$$= \sum_{i=0}^{n-1} i \cdot N_{n-i} \cdot F(t)^i \bar{F}(t)^{n-i}.$$

$\square$

3.3   Illustrative Examples

In this Section, we present numerical experiments for two well-known coherent systems, a bridge structure system and a consecutive-k-out-of-n:F system. These special structures have been widely used in system design.

3.3.1   Bridge Structure System

The bridge structure system is being considered as a system with five components. Figure 3.2 shows the structure of the bridge system.

Table 3.1 gives the number of path sets N_j of a bridge structure system. For example, when there are three working components, the path sets are $\{1, 4, 2\}$, $\{1, 4, 3\}$, $\{1, 4, 5\}$, $\{2, 5, 1\}$, $\{2, 5, 3\}$, $\{2, 5, 4\}$, $\{1, 3, 5\}$, $\{2, 3, 4\}$, and the number of path sets N_3 is eight. Using Eq. (3.3), we obtain the system signature of the bridge structure system in Table 3.2. Furthermore, we can easily obtain the expected number of failed components $E[X]$ in Eq. (3.2), and

$$E[X] = \sum_{i=1}^{5} i \cdot s_i = 3. \tag{3.11}$$

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Table 3.1: Path sets N_j of the bridge structure system.

N_1	N_2	N_3	N_4	N_5
0	2	8	5	1

Table 3.2: System signature of the bridge structure system.

s_1	s_2	s_3	s_4	s_5
0	1/5	3/5	1/5	0

In particular, from Eqs. (3.5) and (3.9), the expected number of failed components when system failure occurs before time t is

$$
\begin{aligned}
E[X(t)] &= \sum_{i=1}^{5} i \cdot s_i \sum_{j=i}^{5} \binom{5}{j} F(t)^j \bar{F}(t)^{5-j}, \\
&= \sum_{i=1}^{5} \left(\sum_{j=1}^{i} j \cdot s_j \right) \binom{5}{i} F(t)^i \bar{F}(t)^{5-i},
\end{aligned}
\tag{3.12}
$$

where s_i is obtained in Table 3.2.

Furthermore, when system is working at time t, from Eq. (3.10), the expected number of failed components is

$$
E[S(t)] = \sum_{i=0}^{4} i \cdot N_{5-i} \cdot F(t)^i \bar{F}(t)^{5-i},
\tag{3.13}
$$

where N_j is obtained in Table 3.1.

We then give some figures to illustrate $E[X(t)]$ and $E[S(t)]$ of the bridge structure system. Assume that all components have lifetime distribution $F(t) = 1 - \exp(-t^m)$ ($m > 0$). From Fig. 3.3, we find that the limit values of $E[X(t)]$ are the same to the value of $E[X]$ in Eq. (3.11), which are not related to the component lifetime distribution. Furthermore, when the time is large enough, the values of $E[S(t)]$ are limit to zero, which means that when the time is large, the system is almost failed.

3.3.2 Consecutive-k-out-of-n:F System

Consecutive-k-out-of-n:F systems are of special importance in reliability theory since they have been used to model and establish optimal designs of various engineering systems. Microwave stations of a telecom network and oil pipeline system are the applications for consecutive-k-out-of-n:F systems [59].

Before we illustrate the expected number of failed components for the consecutive-k-out-of-n:F system, we first give another method proposed by Yun and Endharta [112]. Such method is to enumerate all possible paths of a particular
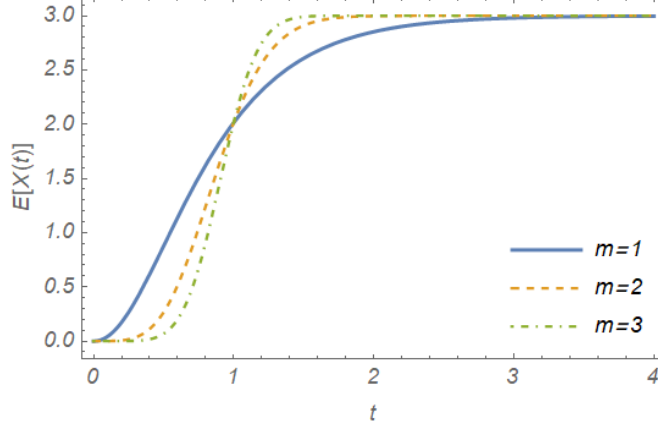
Figure 3.3: $E[X(t)]$ for a bridge structure system when component lifetime distribution is $F(t) = 1 - \exp(-t^m)$.

consecutive-k-out-of-n:F system and to calculate the corresponding expected number of failed components of each path. Denote P is the number of paths, π_j is the probability that the system failure follows path j, and W_j is the number of steps until system failure in path j. Then the expected number of failed components at the time of system failure can be estimated as

$$E[X] = \sum_{j=1}^{P} \pi_j \cdot W_j. \tag{3.14}$$

Furthermore, let α_{ji} be the sum of failure rates of working components in step i in path j. Then, as proposed in [112], the expected number of failed components when system failure occurs before time t is

$$E[X(t)] = \sum_{j=1}^{P} \pi_j \cdot W_j \cdot F_j(t), \tag{3.15}$$

where

$$F_j(t) = 1 - \sum_{i=0}^{W_j-1} A_{ji} e^{-\alpha_{ji} t},$$

and

$$A_{ji} = \prod_{m=0, m \neq i}^{W_j-1} \frac{\alpha_{jm}}{\alpha_{jm} - \alpha_{ji}}.$$

Finally, the expected number of failed components at time t when system is

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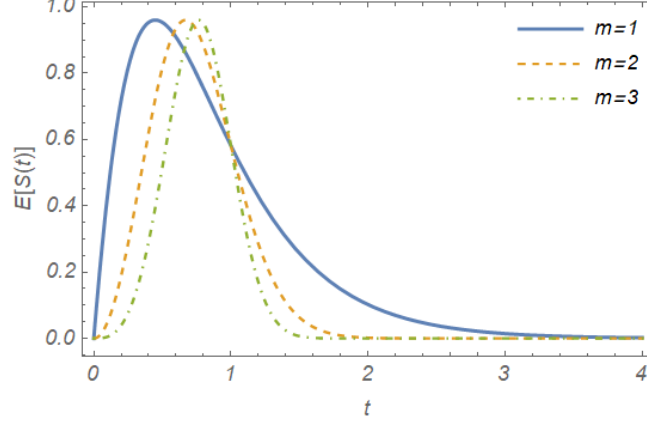
Figure 3.4: $E[S(t)]$ for a bridge structure system when component lifetime distribution is $F(t) = 1 - \exp(-t^m)$.

working is

$$E[S(t)] = \sum_{j=1}^{P} \pi_j \sum_{i=0}^{W_j-1} \sum_{m=0}^{i-1} \frac{iA_{jm}\alpha_{jm}}{\alpha_{ji} - \alpha_{jm}} (\mathrm{e}^{-\alpha_{jm}t} - \mathrm{e}^{-\alpha_{ji}t}), \tag{3.16}$$

where

$$A_{jm} = \prod_{l=0, l \neq m}^{i-1} \frac{\alpha_{jl}}{\alpha_{jl} - \alpha_{jm}}.$$

For an example, we consider a consecutive-2-out-of-4:F system with 18 paths and give details in Table 3.3. Assume that all components hold the same failure rate λ. Based on Table 3.3, we have

$$E[X] = \frac{5}{2},$$

$$E[X(t)] = \frac{5}{2} - \frac{3}{2}\mathrm{e}^{-4\lambda t} + 8\mathrm{e}^{-3\lambda t} - 9\mathrm{e}^{-2\lambda t},$$

$$E[S(t)] = 2\mathrm{e}^{-4\lambda t} - 8\mathrm{e}^{-3\lambda t} + 6\mathrm{e}^{-2\lambda t}.$$

Obviously, this method is not efficient and it should calculate each W_j, α_{ji} for the corresponding path. On the other hand, using the method proposed in this paper, we can calculate these expected values by more concise formulas, where there is no need to enumerate all possible paths.

From Eq. (3.3), the signature of a consecutive-k-out-of-n:F system is derived as

$$s_i = \frac{N_F(i-1, k, n)}{\binom{n}{i-1}} - \frac{N_F(i, k, n)}{\binom{n}{i}}, \tag{3.17}$$

Table 3.3: Paths to system failure for a consecutive-2-out-of-4:F system with component failure rate λ.

j	step i 0 state	α_{j0}	1 state	α_{j1}	2 state	α_{j2}	3 state	W_j	π_j
1	1111	4λ	1110	3λ	1100			2	1/12
2	1111	4λ	1110	3λ	1010	2λ	1000	3	1/24
3	1111	4λ	1110	3λ	1010	2λ	0010	3	1/24
4	1111	4λ	1110	3λ	0110	2λ	0010	3	1/24
5	1111	4λ	1110	3λ	0110	2λ	0101	3	1/24
6	1111	4λ	1101	3λ	1100			2	1/12
7	1111	4λ	1101	3λ	1001			2	1/12
8	1111	4λ	1101	3λ	0101	2λ	0001	3	1/24
9	1111	4λ	1101	3λ	0101	2λ	0100	3	1/24
10	1111	4λ	1011	3λ	0011			2	1/12
11	1111	4λ	1011	3λ	1001			2	1/12
12	1111	4λ	1011	3λ	1010	2λ	1000	3	1/24
13	1111	4λ	1011	3λ	1010	2λ	0010	3	1/24
14	1111	4λ	0111	3λ	0011			2	1/12
15	1111	4λ	0111	3λ	0101	2λ	0001	3	1/24
16	1111	4λ	0111	3λ	0101	2λ	0100	3	1/24
17	1111	4λ	0111	3λ	0110	2λ	0010	3	1/24
18	1111	4λ	0111	3λ	0110	2λ	0100	3	1/24

where $N_F(i, k, n)$ is the number of path sets of a consecutive-k-out-of-n:F system with exactly i failed components, and the expression is given in Eq. (2.23).

Clearly, when $i < k$, $s_i \equiv 0$. Then, the range of i is from k to n, and the expected number of failed components when system failure occurs is

$$E[X] = \sum_{i=k}^{n} i \cdot \left(\frac{N_F(i-1, k, n)}{\binom{n}{i-1}} - \frac{N_F(i, k, n)}{\binom{n}{i}} \right). \tag{3.18}$$

We then focus on the number of failed components at time t. In particular, when system failure occurs before time t, from Eqs. (3.5) and (3.9), the expected number of failed components is

$$E[X(t)] = \sum_{i=k}^{n} i \cdot \left(\frac{N_F(i-1, k, n)}{\binom{n}{i-1}} - \frac{N_F(i, k, n)}{\binom{n}{i}} \right) \sum_{j=i}^{n} \binom{n}{j} F(t)^j \bar{F}(t)^{n-j},$$

$$= \sum_{i=k}^{n} \binom{n}{i} F(t)^i \bar{F}(t)^{n-i} \sum_{j=k}^{i} j \cdot \left(\frac{N_F(j-1, k, n)}{\binom{n}{j-1}} - \frac{N_F(j, k, n)}{\binom{n}{j}} \right). \tag{3.19}$$

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Table 3.4: Expected number of failed components when system failure occurs for a consecutive-k-out-of-n:F system.

k	n	$E[X]$	k	n	$E[X]$
	5	3.9		8	6.9
3	8	4.9	5	15	10.3
	10	5.4		25	14.8
	6	5.1		12	11.6
4	12	7.7	10	20	17.5
	20	10.6		30	24.4

Furthermore, when system is working at time t, from Eq. (3.10), the expected number of failed components is

$$E[S(t)] = \sum_{i=0}^{n-1} i \cdot \left(\sum_{j=i+1}^{n} s_j \right) \binom{n}{i} F(t)^i \bar{F}(t)^{n-i},$$

$$= \sum_{i=0}^{n-1} i \cdot N_F(i,k,n) F(t)^i \bar{F}(t)^{n-i}, \tag{3.20}$$

We consider the consecutive-2-out-of-4:F system with component failure rate λ again and obtain the expected values from Eqs. (3.18), (3.19) and (3.20), and

$$E[X] = \frac{5}{2},$$

$$E[X(t)] = \frac{5}{2}(1 - \mathrm{e}^{-\lambda t})^4 + 10(1 - \mathrm{e}^{-\lambda t})^3 \mathrm{e}^{-\lambda t} + 6(1 - \mathrm{e}^{-\lambda t})^2 \mathrm{e}^{-2\lambda t},$$

$$E[S(t)] = 4(1 - \mathrm{e}^{-\lambda t})\mathrm{e}^{-3\lambda t} + 6(1 - \mathrm{e}^{-\lambda t})^2 \mathrm{e}^{-2\lambda t},$$

which are the same to the values obtained by the method from Yun and Endharta [112].

Table 3.4 gives some results of the expected values of $E[X]$ for consecutive-k-out-of-n:F systems under several values of k and n.

For the expected values of $E[X(t)]$, we give some graphs in Fig. 3.5(a) and 3.5(b). Assume that all components have lifetime distribution $F(t) = 1 - \exp(-t^m)$ where $m > 0$. From the graphs, we find that when $t \to \infty$, the limit values of $E[X(t)]$ are not related to the parameter m and are all equal to the values of $E[X]$ for the corresponding k and n in Table 3.4.

Furthermore, we illustrate $E[S(t)]$ in Eq. (3.20). From the graphs in Fig. 3.6(a) and 3.6(b), we observe that when $t \to \infty$, such expected values are limited to zero, which indicates that the system is naturally failed when time is infinite.

For another typical type of coherent system, the consecutive-k-out-of-n:G system, we will discuss the number of failed components in detail in Chapter 5.
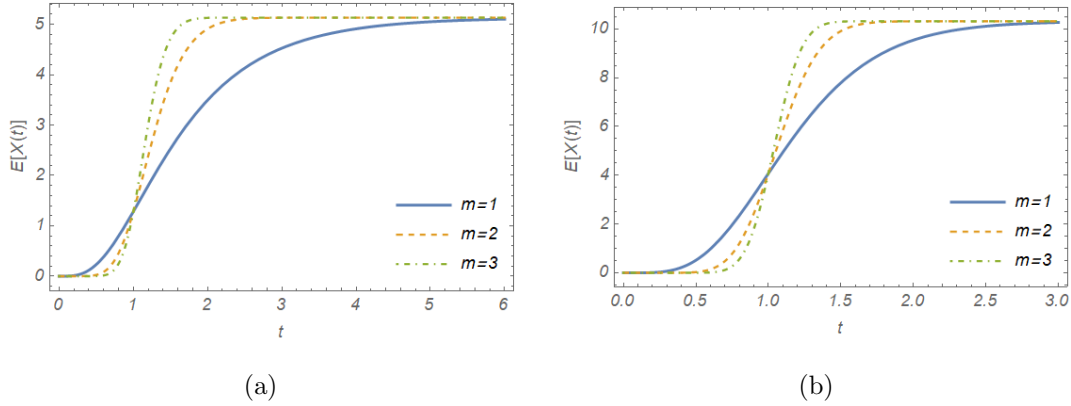
Figure 3.5: $E[X(t)]$ for a consecutive-k-out-of-n:F system when component lifetime distribution is $F(t) = 1 - \exp(-t^m)$ ((a) $k = 4, n = 6$, (b) $k = 5, n = 15$).
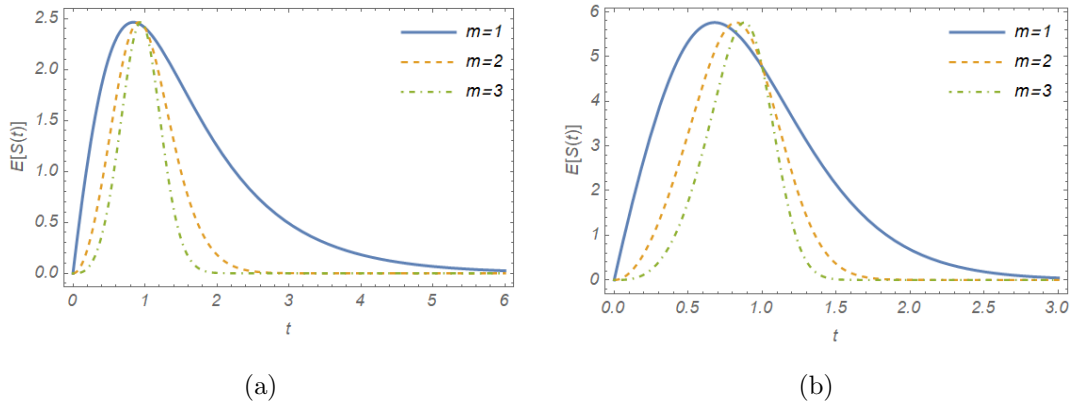
Figure 3.6: $E[S(t)]$ for a consecutive-k-out-of-n:F system when component lifetime distribution is $F(t) = 1 - \exp(-t^m)$ ((a) $k = 4, n = 6$, (b) $k = 5, n = 15$).

3.4 Summary

In this Chapter, we dealt with the general calculation formulas of the expected number of failed components for coherent systems. We first considered the expected value of number of failed components under the situation that system failure occurs. Then, we considered the expected value of number of failed components at a particular time t, whether system is failed before time t, or is working at time t. The illustrative examples were given, which include a bridge structure system and a consecutive-k-out-of-n:F system.

Chapter 4

Design and Maintenance Policies

As mentioned in Chapter 1, with the complexity of consecutive-k-out-of-n:G systems, it is necessary to increase system availability and reduce the probability of system failure, which can be operated by the optimization. The aim of this chapter is to propose the optimal design policy and the optimal maintenance policy for consecutive-k-out-of-n:G systems. In detail, at the design phase, a system designer should determine the system configurations, e.g., the number of components. If the number of components is small, then the system will fail frequently and consequently, the maintenance cost will be higher. On the other hand, too many components will cause the waste of resources. Therefore, we need to determine the optimal number of components. At the operation phase, a system is necessary to be maintained, and we consider the age replacement for the consecutive-k-out-of-n:G system. Therefore, the optimal replacement time should be determined. In this thesis, we choose the expected cost rates as objective functions. We use the results of the system reliability and MTTF which are proposed in Chapter 2 to build the model of the expected cost rates. Section 4.1 considers a simple structure where $k = 2$ and proposes the optimal results for the consecutive-2-out-of-n:G system. Section 4.2 proposes the optimal results for the consecutive-k-out-of-n:G system where $k \geq 3$. Finally, we summarize the contributions of this chapter.

4.1 Consecutive-2-out-of-n:G System

In this section, we focus on the simple case of the consecutive-k-out-of-n:G system when $k = 2$, and use the results of the system reliability and MTTF to build the models of the expected cost rates. By minimizing the expected cost rates, we derive the optimal number of components in design phase and the optimal replacement time in operation phase. As discussed in Section 2.1, we divide two situations where the number of components is a fixed value or a random value.
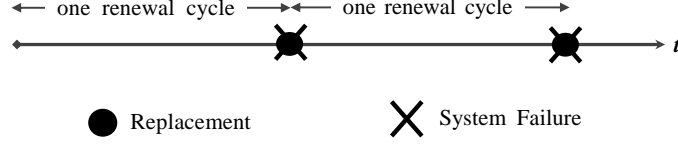
Figure 4.1: Process of the replacement after system failure.

4.1.1 Constant Number of Components

(1) Optimal number of components

We first discuss the optimal system design for the consecutive-2-out-of-n:G system. Usually, when the system failure occurs, it is necessary to replace the system as a new one in order to operate, and the replacement cost for the failed system is required. We consider a renewal cycle throughout the operation which means the interval of two operation processes [91]. Then, the expected cost rate is equal to the expected total cost during a renewal cycle divided by the expected length of a renewal cycle. Figure 4.1 shows the process of such replacement. We try to minimize the expected cost rate to obtain the optimal number of components.

To build the model of the expected cost rate, the following cost factors are considered: C_1 represents the acquisition cost for each component, and C_R represents the replacement cost for the failed system. Obviously, the expected cost for one renewal cycle is $(nC_1 + C_R)$, and the expected time of a renewal cycle is the MTTF of the system. Then the expected cost rate is

$$C_1(n; 2) = \frac{nC_1 + C_R}{\mu_{n,2}} \ (n = 2, 3, \cdots),$$ (4.1)

where $\mu_{n,2}$ is the expression of MTTF of the consecutive-2-out-of-n:G system, which is proposed in Eq. (2.8). Then we consider to find the optimal number of components.

Theorem 4.1. If a consecutive-2-out-of-n:G system is replaced after system failure, then there exists an unique optimal n^* which minimizes the expected cost rate.

Proof. From $C_1(n + 1; 2) - C_1(n; 2) \geq 0$, we have

$$\frac{\mu_{n,2}}{\mu_{n+1,2} - \mu_{n,2}} - n \geq \frac{C_R}{C_1} \ (n = 2, 3, \cdots).$$ (4.2)

Letting $G_1(n; 2)$ be the left-hand side of Eq. (4.2), and we have

$$\Delta G_1(n; 2) = G_1(n+1; 2) - G_1(n; 2),$$
$$= \frac{\mu_{n+1,2}[2\mu_{n+1,2} - (\mu_{n,2} + \mu_{n+2,2})]}{(\mu_{n+2,2} - \mu_{n+1,2})(\mu_{n+1,2} - \mu_{n,2})},$$
$$= \frac{\mu_{n+1,2}(\Delta\mu_{n,2} - \Delta\mu_{n+1,2})}{\Delta\mu_{n,2}\Delta\mu_{n+1,2}}, \tag{4.3}$$

where $\Delta\mu_{n,2} = \mu_{n+1,2} - \mu_{n,2}$. Clearly, $\Delta\mu_{n,2}$ and $\Delta\mu_{n+1,2}$ are both greater than 0. Then, to prove the monotonicity of the $G_1(n; 2)$, we consider to confirm the value of $(\Delta\mu_{n,2} - \Delta\mu_{n+1,2})$. From Eq. (2.20), we give the recursive formula of the reliability of a consecutive-2-out-of-n:G system where

$$R_G(2, n; t) = R_G(2, n-1; t) + [1 - R_G(2, n-3; t)] F(t)\bar{F}(t)^2. \tag{4.4}$$

Then, we give the calculation formula of $(\Delta\mu_{n,2} - \Delta\mu_{n+1,2})$ and

$$\Delta\mu_{n,2} - \Delta\mu_{n+1,2}$$
$$= \int_0^\infty [R_G(2, n+1; t) - R_G(2, n; t)]\mathrm{d}t - \int_0^\infty [R_G(2, n+2; t) - R_G(2, n+1; t)]\mathrm{d}t,$$
$$= \int_0^\infty [R_G(2, n-1; t) - R_G(2, n-2; t)]F(t)\bar{F}(t)^2\mathrm{d}t. \tag{4.5}$$

Apparently, the system reliability increases strictly with n for a consecutive-k-out-of-n:G system. As a result, $R_G(2, n-1; t) \geq R_G(2, n-2; t)$, and then the value of $(\Delta\mu_{n,2} - \Delta\mu_{n+1,2})$ is greater than zero. Finally, we obtain that $\Delta G_1(n; 2) \geq 0$. Thus, $G_1(n; 2)$ is proved to increase strictly with n and there exists an unique n^* which firstly satisfies Eq. (4.2), where the minimal value of the expected cost rate is $C_1(n^*; 2)$. □

Table 4.1 presents some results of the optimal number of components when $F(t) = 1 - \exp(-t^m)$ $(m > 0)$. n^* is the exact value of the optimal number of components using the value of MTTF in Eq. (2.8), and $\tilde{n}^*$ is the approximate value of the optimal number of components, using the value of MTTF in Eq. (2.13). From the table, the optimal number of components which minimizes the expect cost rate is obtained. Furthermore, when C_R/C_1 is large, the results of $\tilde{n}^*$ are almost the same to the results of n^*, that is, the approximate value of MTTF can be used in such situation.

(2) Optimal replacement time

When the system is in operation phase, it is important to make maintenance to avoid system failure and keep the system working. We focus on the age replacement and the system will be replaced at the planned time t or the time of system failure, which occurs first. Figure 4.2 shows the process of the age replacement in one

Table 4.1: Optimal number of components for a consecutive-2-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp(-t^m)$.

C_R/C_1	$m = 1$	$m = 2$		$m = 3$	
	n^*	n^*	$\tilde{n}^*$	n^*	$\tilde{n}^*$
20	12	7	7	6	5
40	19	11	10	8	8
60	25	14	14	11	10
80	30	17	17	13	12
100	36	20	20	15	15
150	48	27	26	20	19
200	60	33	33	24	24
300	81	45	45	32	32
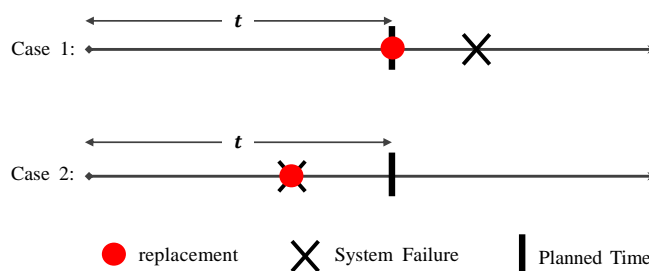
Figure 4.2: Process of age replacement in one renewal cycle.

renewal cycle. We consider the expected cost rate as the objective function and try to minimize it in order to obtain the optimal replacement time.

Assume that the replacement time is negligible and all components are replaced as new ones. Clearly, the mean time to replacement of a renewal cycle is expressed as

$$E[\min(t, T)] = \int_0^t R_G(2, n; x)\mathrm{d}x, \tag{4.6}$$

where $\lim_{t\to\infty} E[\min(t, T)] = \mu_{n,2}$ in Eq. (2.8).

Under the above maintenance policy and use the same notation C_1 and C_R, the expected cost rate is

$$C_1(t; 2, n) = \frac{nC_1 + C_R[1 - R_G(2, n; t)]}{\int_0^t R_G(2, n; x)\mathrm{d}x}. \tag{4.7}$$

It is assumed that each component has the failure rate $h(t) \equiv f(t)/\bar{F}(t)$ where $h(\infty) = \lim_{t\to\infty} h(t)$, and system has the failure rate $H(t; 2, n)$. Then we focus on the optimal replacement time which minimizes $C_1(t; 2, n)$.

Proposition 4.1. If a consecutive-2-out-of-n:G system has a increasing failure rate (IFR) and

$$2h(\infty)\mu_{n,2} > \frac{nC_1 + C_R}{C_R}, \tag{4.8}$$

then there exists a finite and unique optimal t_a^* $(0 < t_a^* < \infty)$ which minimizes the expected cost rate $C_1(t; 2, n)$, where $h(t)$ is the component failure rate and $\mu_{n,2}$ is the MTTF of the system.

Proof. Differentiating $C_1(t; 2, n)$ with respect to t and setting it equal to 0, we have

$$H(t; 2, n) \int_0^t R_G(2, n; x)\mathrm{d}x + R_G(t; 2, n) = \frac{nC_1 + C_R}{C_R}, \tag{4.9}$$

where

$$H(t; 2, n) = -\frac{\mathrm{d}R_G(t; 2, n)/\mathrm{d}t}{R_G(t; 2, n)}. \tag{4.10}$$

Letting $L_1(t; 2, n)$ be the left-hand side of Eq. (4.9) and differentiate $L_1(t; 2, n)$ with t, we have

$$\frac{\mathrm{d}L_1(t; 2, n)}{\mathrm{d}t} = \frac{\mathrm{d}H(t; 2, n)}{\mathrm{d}t} \int_0^t R_G(2, n; x)\mathrm{d}x. \tag{4.11}$$

In addition,

$H(t; 2, n) =$
$h(t)\cdot$
$$\frac{\sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor}(2n - 3j + 1)\binom{n-j}{j-1}\bar{F}(t)^j F(t)^{n-j-1} + \binom{n-\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{n+1}{2} \rfloor}(n - \lfloor \frac{n+1}{2} \rfloor)\bar{F}(t)^{\lfloor \frac{n+1}{2} \rfloor}F(t)^{n-\lfloor \frac{n+3}{2} \rfloor}}{\sum_{j=2}^{n}\binom{n}{j}\bar{F}(t)^{j-1}F(t)^{n-j} - \sum_{j=2}^{\lfloor \frac{n+1}{2} \rfloor}\binom{n-j+1}{j}\bar{F}(t)^{j-1}F(t)^{n-j}},$$
$$\tag{4.12}$$

and when $t \to \infty$, we have $\lim_{t\to\infty} \bar{F}(t) \to 0$ and $\lim_{t\to\infty} F(t) \to 1$. Therefore,

$$H(\infty; 2, n) \equiv \lim_{t\to\infty} H(t; 2, n),$$
$$= h(\infty) \lim_{t\to\infty} \frac{(2n - 3 + 1)\binom{n-1}{1-1}\bar{F}(t)}{\binom{n}{2}\bar{F}(t) - \binom{n-2+1}{2}\bar{F}(t)},$$
$$= h(\infty) \lim_{t\to\infty} \frac{2n - 2}{\binom{n}{2} - \binom{n-1}{2}},$$
$$= 2h(\infty). \tag{4.13}$$

Furthermore, it is easy to obtain that $L_1(0; 2, n) = 1$ and $L_1(\infty; 2, n) = 2h(\infty)\mu_{n,2}$. Hence, if $H(t; 2, n)$ increases strictly with t, then Eq. (4.11) is greater than 0, and $L_1(t; 2, n)$ increases strictly with respect to t from 1 to $2h(\infty)\mu_{n,2}$. Thus, according to Eq. (4.9), we can obtain the result in Proposition 4.1. $\qquad\square$

Next, we consider some specific situations to discuss the result proposed in Proposition 4.1. We first focus on the situation that the value of n satisfies $n \leq 2k$. In this section, as $k = 2$, we discuss the situation that $n = 3$ and 4. When components have exponential lifetime distribution where $F(t) = 1 - \exp(-\lambda t)$, we have

$$\frac{\mathrm{d}H(t; 2, 3)}{\mathrm{d}t} = \lambda^2 \frac{2\mathrm{e}^{-5\lambda t}}{R_G^2(2, 3; t)} > 0,$$

$$\frac{\mathrm{d}H(t; 2, 4)}{\mathrm{d}t} = \lambda^2 \frac{6\mathrm{e}^{-5\lambda t}}{R_G^2(2, 4; t)} > 0,$$

where both systems have the same limit system failure rate $H(\infty; 2, n) = 2\lambda$. Using the result of MTTF in Eq. (2.8) and Eq. (4.8) in Proposition 4.1, we have the following result.

Theorem 4.2. For a consecutive-2-out-of-n:G system and $n = 3, 4$, if all components have exponential lifetime distribution $F(t) = 1 - \exp(-\lambda t)$, then the system has IFR with limit 2λ. Furthermore,

(1) When $2\left[\sum_{j=2}^n \frac{1}{j} - \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \frac{1}{j} \prod_{l=0}^{j-2} \left(1 - \frac{j}{n-l}\right)\right] > (nC_1 + C_R)/C_R$, then there exists a finite and unique ${t_a}^*$ which minimizes the expected cost rate in Eq. (4.7), and

$$C_1(t_a^*; 2, n) = C_R \cdot H(t_a^*; 2, n). \tag{4.14}$$

(2) When $2\left[\sum_{j=2}^n \frac{1}{j} - \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \frac{1}{j} \prod_{l=0}^{j-2} \left(1 - \frac{j}{n-l}\right)\right] \leq (nC_1 + C_R)/C_R$, then system should be replaced after system failure and ${t_a}^* = \infty$.

In addition, when components have Weibull lifetime distribution and $n = 3, 4$, the optimal replacement time of the system is discussed. Before judging the monotonicity of the system failure rate under this condition, we first give the following result proposed by Navarro *et al.* [84].

Theorem 4.3 (Navarro *et al.* [84]). The lifetimes of consecutive-k-out-of-n:G systems with IFR i.i.d. component lifetimes are IFR for all $2k \geq n$.

As known well, for a Weibull distribution $F(t) = 1 - \exp[-(\lambda t)^m]$, when $m > 1$, each component has increasing failure rate (IFR), and when $0 < m < 1$, each component has decreasing failure rate (DFR). Therefore, we can easily obtain the following result.
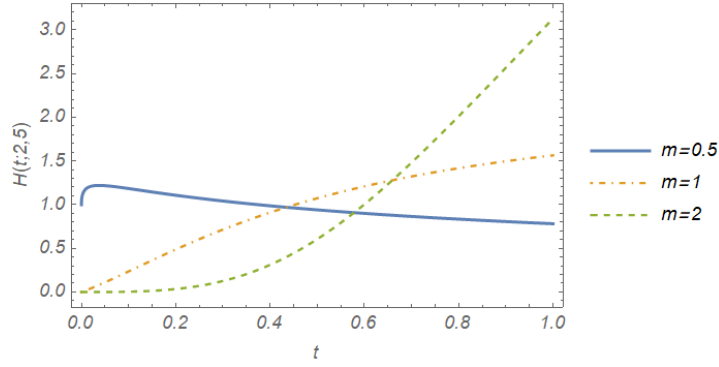
Figure 4.3: $H(t; 2, 5)$ of a consecutive-2-out-of-5:G system when component life-time distribution is $F(t) = 1 - \exp(-t^m)$.
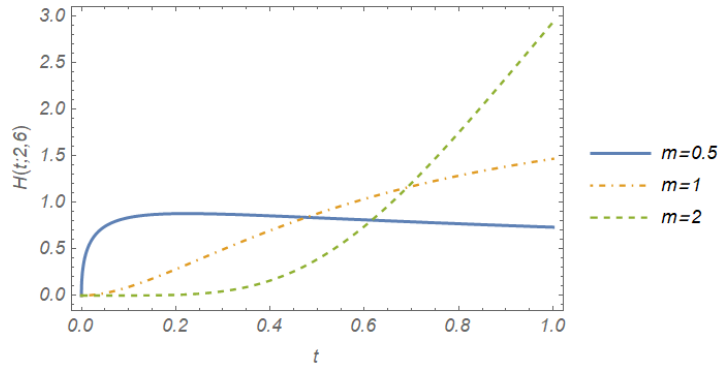
Figure 4.4: $H(t; 2, 6)$ of a consecutive-2-out-of-6:G system when component life-time distribution is $F(t) = 1 - \exp(-t^m)$.

Theorem 4.4. For a consecutive-2-out-of-n:G system and $n = 3, 4$, if all components have Weibull lifetime distribution $F(t) = 1 - \exp[-(\lambda t)^m]$ and $m > 1$, then the system has IFR and there inevitably exists the optimal $t_a{}^*$ which minimizes the expected cost rate.

Furthermore, we focus on the situation that the value of n satisfies $n > 2k$, that is, $n > 4$, and all components have the Weibull lifetime distribution $F(t) = 1 - \exp[-(\lambda t)^m]$ $(m > 0)$. According to our numerical experiments, we infer that when $m = 1$, system has IFR with limit 2λ; when $m > 1$, system has IFR with no limit. Figures 4.3 and 4.4 give some examples for the graphs of $H(t; 2, n)$ when $n = 5$ and 6, respectively.

Table 4.2 presents the results of the optimal replacement time when $F(t) = 1 - \exp(-t^m)$ $(m \geq 1)$ with $C_R/C_1 = 8$ and $C_R/C_1 = 15$. From the table, the optimal replacement time which minimizes the expect cost rate is obtained.

Table 4.2: Optimal replacement time for a consecutive-2-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp(-t^m)$.

n	$C_R/C_1 = 8$			$C_R/C_1 = 15$		
	$m = 1$	$m = 2$	$m = 3$	$m = 1$	$m = 2$	$m = 3$
5	1.254	0.665	0.689	0.646	0.555	0.617
8	1.474	0.836	0.825	0.858	0.726	0.759
10	1.703	0.918	0.884	1.000	0.806	0.820
15	2.485	1.065	0.984	1.307	0.949	0.922
20	∞	1.170	1.050	1.571	1.046	0.988
25	∞	1.253	1.099	1.815	1.120	1.036
30	∞	1.323	1.138	2.058	1.181	1.074
40	∞	1.441	1.199	2.628	1.275	1.131

Furthermore, "∞" means that the optimal replacement time is infinite and it is considered to replace system after system failure.

4.1.2 Random Number of Components

In this subsection, we consider the consecutive-2-out-of-N:G system with a random number of components N discussed in Subsection 2.1.1, and give the preventive replacement policy. When the value of N holds a truncated Poisson distribution with parameter θ, the mean time to replacement becomes

$$\int_0^t R_G(2,\theta;x)\mathrm{d}x = \sum_{n=2}^{\infty} \frac{\theta^n/n!}{\mathrm{e}^\theta - 1 - \theta} \int_0^t R_G(2,n;x)\mathrm{d}x. \tag{4.15}$$

Thus, using the same cost factors C_1 and C_R, the expected cost rate becomes

$$C_1(t;2,\theta) = \frac{E[N]C_1 + C_R[1 - R_G(2,\theta;t)]}{\int_0^t R_G(2,\theta;x)\mathrm{d}x}, \tag{4.16}$$

where $E[N]$ is given in Eq. (2.6) and $R_G(2,\theta;t)$ is given in Eq. (2.7).

In this subsection, we consider the situation that all components have exponential lifetime distribution. Then, we give the result of the optimal replacement time which minimizes $C_1(t;2,\theta)$ in Eq. (4.16).

Theorem 4.5. Assume that a consecutive-2-out-of-N:G system has a random variable N which follows a truncated Poisson distribution with parameter θ, and components have exponential lifetime distribution $F(t) = 1 - \exp(-\lambda t)$. Thus, if

$$\sum_{n=2}^{\infty} \frac{\theta^n}{n!} \left[2 \left(\sum_{j=2}^{n} \frac{1}{j} - \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \frac{1}{j} \prod_{l=0}^{j-2}(1 - \frac{j}{n-l}) \right) - 1 \right] > \frac{\theta(\mathrm{e}^\theta - 1)C_1}{C_R},$$

then there exists a finite and unique optimal $t_b{}^*$ which minimizes the expected cost rate.

Proof. As assumed in Theorem 4.5,

$$\Pr\{N = n\} = p_n = \frac{\theta^n/n!}{\sum_{k=2}^{\infty} \theta^k/k!},$$

then, the expected cost rate in Eq. (4.16) becomes

$$
\begin{aligned}
C_1(t; 2, \theta) &= \frac{\frac{\theta(e^\theta-1)}{e^\theta-1-\theta}C_1 + C_R \frac{\sum_{n=2}^{\infty}(\theta^n/n!)W(t)}{e^\theta-1-\theta}}{\frac{\sum_{n=2}^{\infty}(\theta^n/n!)\int_0^t[1-W(x)]\mathrm{d}x}{e^\theta-1-\theta}}, \\
&= \frac{\theta(e^\theta-1)C_1 + C_R \sum_{n=2}^{\infty}(\theta^n/n!)W(t)}{\sum_{n=2}^{\infty}(\theta^n/n!)\int_0^t[1-W(x)]\mathrm{d}x}, \quad (4.17)
\end{aligned}
$$

where

$$W(t) = \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} e^{-j\lambda t}(1-e^{-\lambda t})^{n-j}.$$

Differentiating $C_1(t; 2, \theta)$ in Eq. (4.17), we have

$$H(t; 2, \theta)\sum_{n=2}^{\infty} \frac{\theta^n}{n!}\int_0^t[1-W(x)]\mathrm{d}x - \sum_{n=2}^{\infty} \frac{\theta^n}{n!}W(t) = \frac{\theta(e^\theta-1)C_1}{C_R}, \quad (4.18)$$

where $H(t; 2, \theta)$ is the system failure rate of a consecutive-2-out-of-N:G system and

$$H(t; 2, \theta) = \frac{\sum_{n=2}^{\infty}(\theta^n/n!)(\mathrm{d}W(t)/\mathrm{d}t)}{\sum_{n=2}^{\infty}(\theta^n/n!)[1-W(t)]}. \quad (4.19)$$

Letting $L_1(t; 2, \theta)$ be the left-hand side of Eq. (4.18) and differentiate it with t, we have

$$\frac{\mathrm{d}L_1(t; 2, \theta)}{\mathrm{d}t} = \frac{\mathrm{d}H(t; 2, \theta)}{\mathrm{d}t}\int_0^t[1-W(x)]\mathrm{d}x. \quad (4.20)$$

As we have observed numerically and graphically, $H(t; 2, \theta)$ increases strictly in t. When $t \to \infty$, $\lim_{t\to\infty} e^{-\lambda t} \to 0$ and $\lim_{t\to\infty} 1 - e^{-\lambda t} \to 1$. Then, using the result in Eq. (4.12), we have

$$
\begin{aligned}
H(\infty; 2, \theta) &\equiv \lim_{t\to\infty} H(t; 2, \theta), \\
&= \lambda \lim_{t\to\infty} \frac{\frac{\theta^2}{2!}\binom{2}{1}e^{-\lambda t} + \sum_{n=3}^{\infty} \frac{\theta^n}{n!}\left[(2n-2)\binom{n-1}{0}e^{-\lambda t}\right]}{\frac{\theta^2}{2!}\binom{2}{2}e^{-\lambda t} + \sum_{n=3}^{\infty} \frac{\theta^n}{n!}\left[\binom{n}{2}e^{-\lambda t} - \binom{n-2+1}{2}e^{-\lambda t}\right]}, \\
&= \lambda \lim_{t\to\infty} \frac{\theta^2 e^{-\lambda t} + \sum_{n=3}^{\infty} \frac{\theta^n}{n!}(2n-2)e^{-\lambda t}}{\frac{\theta^2}{2}e^{-\lambda t} + \sum_{n=3}^{\infty} \frac{\theta^n}{n!}(n-1)e^{-\lambda t}}, \\
&= 2\lambda. \quad (4.21)
\end{aligned}
$$

Figure 4.5: $H(t; 2, \theta)$ of a consecutive-2-out-of-N:G system when the number of components follows a truncated Poisson distribution with parameter θ (component lifetime distribution is $F(t) = 1 - \exp(-t)$).

Hence, the value of Eq. (4.20) is greater than zero, that is, the left-hand side of Eq. (4.18) increases strictly with time t with limit

$$L_1(\infty; 2, \theta) \equiv \lim_{t \to \infty} L_1(t; 2, \theta),$$

$$= \sum_{n=2}^{\infty} \frac{\theta^n}{n!} \left[2 \left(\sum_{j=2}^{n} \frac{1}{j} - \sum_{j=2}^{\lfloor (n+1)/2 \rfloor} \frac{1}{j} \prod_{l=0}^{j-2} (1 - \frac{j}{n-l}) \right) - 1 \right].$$

Then, if $L_1(\infty; 2, \theta) > \theta(e^\theta - 1)C_1/C_R$, there exists an unique and finite $t_b{}^*$ ($0 < t_b{}^* < \infty$) which satisfies Eq. (4.18) with minimal expected cost rate $C_1(t_b^*; 2, \theta)$.

Figures 4.5 and 4.6 show some examples of the system failure rate $H(t; 2, \theta)$ when $\theta = 5, 10, 20$ and $\lambda = 1, 2$. $\qquad \square$

Table 4.3 presents the results of the optimal replacement time when the number of components is random and all components have the lifetime distribution $F(t) = 1 - \exp(-\lambda t)$. "$\infty$" means that the optimal replacement time is infinite and it is considered to replace system after system failure. From the table, we confirmed that the proposed replacement policy when the number of components is unknown is available.

4.2 Consecutive-k-out-of-n:G System

In this section, we focus on the two optimization problems for the consecutive-k-out-of-n:G system where $k \geq 3$ and give the corresponding expected cost rates. By minimizing the expected cost rates, we derive the optimal results according to the optimization problems. Furthermore, in operation phase, the systems with
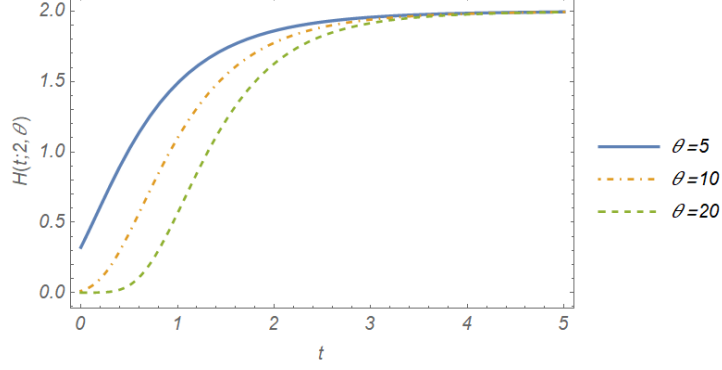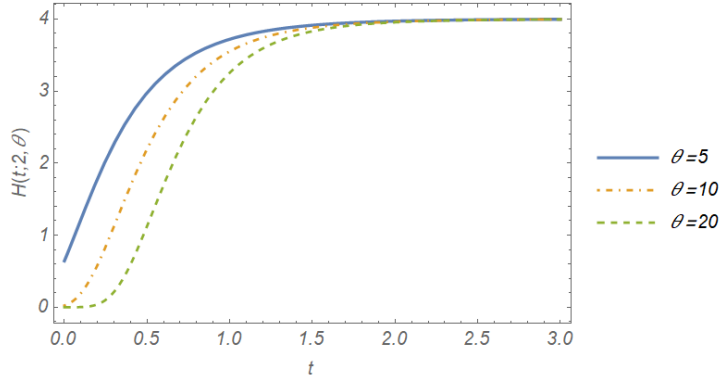
Figure 4.6: $H(t; 2, \theta)$ of a consecutive-2-out-of-N:G system when the number of components follows a truncated Poisson distribution with parameter θ (component lifetime distribution is $F(t) = 1 - \exp(-t)$).

Table 4.3: Optimal replacement time for a consecutive-2-out-of-N:G system when the number of components follows a truncated Poisson distribution with parameter θ (component lifetime distribution is $F(t) = 1 - \exp(-\lambda t)$).

θ	$C_R/C_1 = 8$			$C_R/C_1 = 15$		
	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
5	1.611	0.806	0.537	0.840	0.426	0.280
10	1.857	0.928	0.619	1.069	0.535	0.356
20	∞	∞	∞	1.603	0.802	0.534
30	∞	∞	∞	2.084	1.042	0.695
40	∞	∞	∞	2.656	1.328	0.885
50	∞	∞	∞	∞	∞	∞
60	∞	∞	∞	∞	∞	∞
80	∞	∞	∞	∞	∞	∞

different values of n have different optimal replacement time. Therefore, in subsection 4.2.3, we will analyze the optimization problem and determine the optimal number of components and replacement time simultaneously, which can give the most minimum value of the expected cost rate. All components are assumed to have independent and identically lifetime distribution (i.i.d.) with reliability $\bar{F}(t)$ and unreliability $F(t)$.

4.2.1 Optimal Number of Components

We consider the same process of replacement in Section 4.1.1. We consider the same cost factors C_1 and C_R, then the total expected cost for one renewal cycle is $(nC_1 + C_R)$, and the expected time of a renewal cycle is the MTTF of

the consecutive-k-out-of-n:G system which is obtained in Eq. (2.26). Then, the expected cost rate is

$$C_1(n;k) = \frac{nC_1 + C_R}{\mu_n},$$

$$= \frac{nC_1 + C_R}{\sum_{j=k}^{n} N_G(j,k,n) \int_0^\infty \bar{F}(t)^j F(t)^{n-j} \mathrm{d}t} \quad (n = k, k+1, \cdots), \qquad (4.22)$$

where $N_G(j,k,n)$ is obtained in Eq. (2.25).

We try to obtain the optimal number of components by minimizing the expected cost rate $C_1(n;k)$.

Theorem 4.6. If a consecutive-k-out-of-n:G system with i.i.d. components is replaced after system failure, then there exists a finite and unique optimal $n_1{}^*$ which minimizes the expected cost rate.

Proof. We consider the following inequality $C_1(n+1;k) - C_1(n;k) \geq 0$, which equivalent to

$$\frac{\mu_n}{\mu_{n+1} - \mu_n} - n \geq \frac{C_R}{C_1} \quad (n = k, k+1, \cdots). \qquad (4.23)$$

Denote the left-hand side of Eq. (4.23) as $G_1(n;k)$ and we have

$$\Delta G_1(n;k) = G_1(n+1;k) - G_1(n;k),$$

$$= \frac{\mu_{n+1}[2\mu_{n+1} - (\mu_n + \mu_{n+2})]}{(\mu_{n+2} - \mu_{n+1})(\mu_{n+1} - \mu_n)},$$

letting $\Delta \mu_n = \mu_{n+1} - \mu_n$, then we rewrite $\Delta G_1(n;k)$ as

$$\Delta G_1(n;k) = \frac{\mu_{n+1}(\Delta \mu_n - \Delta \mu_{n+1})}{\Delta \mu_n \Delta \mu_{n+1}}.$$

Clearly, $\Delta \mu_n$ and $\Delta \mu_{n+1}$ are both greater than 0. Then, to prove the monotonicity of $G_1(n;k)$, it is necessary to confirm the value of $(\Delta \mu_n - \Delta \mu_{n+1})$. We use the recursive formula of the reliability of a consecutive-k-out-of-n:G system which is given in Eq. (2.20), and

$$R_G(k,n;t) = R_G(k,n-1;t) + [1 - R_G(k,n-k-1;t)]F(t)\bar{F}(t)^k.$$

Then, we have

$$\Delta \mu_n - \Delta \mu_{n+1}$$
$$= \int_0^\infty [R_G(k,n+1;t) - R_G(k,n;t)] \, \mathrm{d}t - \int_0^\infty [R_G(k,n+2;t) - R_G(k,n+1;t)] \, \mathrm{d}t,$$
$$= \int_0^\infty [R_G(k,n-k+1;t) - R_G(k,n-k;t)]F(t)\bar{F}(t)^k \mathrm{d}t. \qquad (4.24)$$

It is well known that for the consecutive-k-out-of-n:G system, the system reliability is increasing with n. Therefore, $R_G(k, n-k+1; t) \geq R_G(k, n-k; t)$, and the value of Eq. (4.24) is greater than 0. That is, $\Delta G_1(n; k) \geq 0$ and the left-hand side of Eq. (4.23) increases strictly with n. Thus, there exists a finite and unique n_1^* which firstly satisfies Eq. (4.23). $\square$

In particular, when all components have the Weibull lifetime distribution, we obtain the following results.

Corollary 4.1. If a consecutive-k-out-of-n:G system has i.i.d. components with lifetime distribution $F(t) = 1 - \exp[-(\lambda t)^m]$, then there exists an unique optimal number of components n_1^* which minimizes the expected cost rate when system is replaced after system failure. Furthermore,

(1) $n_1^* = k$, if

$$\frac{k^{-1/m}}{k^{-1/m} - (k+1)^{-1/m}} - k \geq C_R/C_1.$$

(2) $n_1^* > k$, if

$$\frac{k^{-1/m}}{k^{-1/m} - (k+1)^{-1/m}} - k < C_R/C_1.$$

Proof. We have proved that the left-hand side of Eq. (4.23), which is denoted as $G_1(n; k)$ increases strictly with n. Then, we calculate the minimum value of $G_1(n; k)$ when components have Weibull lifetime distribution. Clearly, $R_G(k, k; t) = \bar{F}(t)^k$, and $R_G(k, k+1; t) = 2\bar{F}(t)^k F(t) + \bar{F}(t)^{k+1}$. Then,

$$G_1(k; k) = \frac{\int_0^\infty R_G(k, k, t)\mathrm{d}t}{\int_0^\infty [R_G(k, k+1, t) - R_G(k, k, t)]\mathrm{d}t} - k,$$

$$= \frac{\int_0^\infty \bar{F}(t)^k \mathrm{d}t}{\int_0^\infty \bar{F}(t)^k F(t)\mathrm{d}t} - k. \tag{4.25}$$

In particular, when $F(t) = 1 - \exp[-(\lambda t)^m]$,

$$G_1(k; k) = \frac{\int_0^\infty \mathrm{e}^{-k(\lambda t)^m}\mathrm{d}t}{\int_0^\infty \mathrm{e}^{-k(\lambda t)^m}\mathrm{d}t - \int_0^\infty \mathrm{e}^{-(k+1)(\lambda t)^m}\mathrm{d}t} - k, \tag{4.26}$$

and from the Gamma formula, we have

$$\int_0^\infty \mathrm{e}^{-k(\lambda t)^m}\mathrm{d}t = \frac{1}{\lambda}\Gamma(1 + 1/m)k^{-1/m}. \tag{4.27}$$

Then Eq. (4.26) becomes

$$G_1(k; k) = \frac{1/\lambda \cdot \Gamma(1 + 1/m)k^{-1/m}}{1/\lambda \cdot \Gamma(1 + 1/m)[k^{-1/m} - (k+1)^{-1/m}]} - k,$$

$$= \frac{k^{-1/m}}{k^{-1/m} - (k+1)^{-1/m}} - k, \tag{4.28}$$
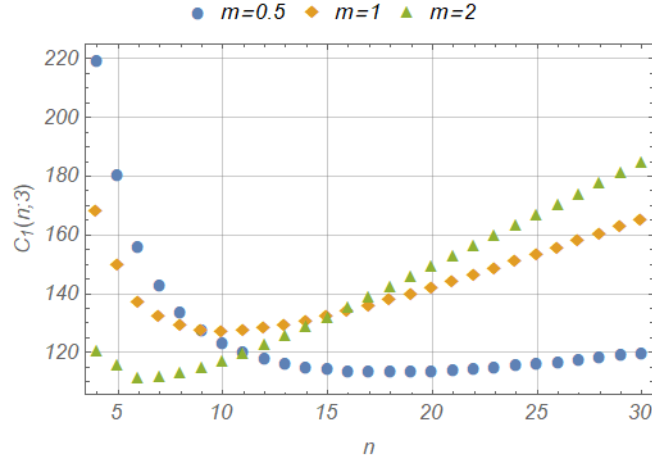
Figure 4.7: $C_1(n;3)$ for a consecutive-3-out-of-n:G system (component lifetime distribution is $F(t) = 1 - \exp(-t^m)$, $C_1 = 5, C_R = 50$).

and we can easily derive the results in Corollary 4.1. □

Figure 4.7 shows some graphs of the expected cost rate $C_1(n;k)$ for a consecutive-3-out-of-n:G system when components have Weibull lifetime distribution, and $C_1 = 5$, $C_R = 50$.

Table 4.4 presents the numerical experiment results of the optimal number of components $n_1{}^*$ and the corresponding expected cost rate $C_1(n_1{}^*;k)$ when all components have the lifetime distribution $F(t) = 1 - \exp(-t^m)$. The table investigates the effect of the system parameters. We observe that an increase of the ratio C_R/C_1 leads to an increase in $n_1{}^*$, since the system reliability should be appropriately increased by considering more components when the replacement cost for a failed system C_R is greater. On the other hand, we find out that under some cases, the optimal $n_1{}^*$ is equal to the corresponding k, that is, the optimal system design is considered to be a series system.

4.2.2 Optimal Replacement Time

To obtain the optimal replacement time during operation phase, we focus on the age replacement again, which is introduced in Section 4.1. The replacement will be done at the planned time t or the time of system failure, which occurs first. Clearly, the mean time to replacement of a renewal cycle is expressed as

$$E[\min(t,T)] = \int_0^t R_G(k,n;x)\mathrm{d}x, \tag{4.29}$$

where $\lim_{t\to\infty} E[\min(t,T)] = \mu_n$ in Eq. (2.26). We consider the cost factors, where C_1 represents the acquisition cost for each component and C_R represents the re-

Table 4.4: Optimal number of components and the corresponding expected cost rate for consecutive-k-out-of-n:G systems when all components lifetime distribution follow $F(t) = 1 - \exp(-t^m)$.

m	k	$C_1 = 5, C_R = 50$		$C_1 = 5, C_R = 100$		$C_1 = 5, C_R = 150$		$C_1 = 5, C_R = 250$	
		$n_1{}^*$	$C_1(n_1{}^*; k)$	$n_1{}^*$	$C_1(n_1{}^*; k)$	$n_1{}^*$	$C_1(n_1{}^*; k)$	$n_1{}^*$	$C_1(n_1{}^*; k)$
	3	10	127.273	14	185.163	18	236.322	25	328.596
	4	12	190.385	17	269.684	21	339.810	28	466.114
1	5	14	264.000	19	365.462	23	455.398	30	617.367
	10	22	783.881	29	1006.133	33	1204.759	42	1562.990
	15	30	1548.387	37	1907.692	43	2224.079	52	2800.199
	3	6	111.527	9	175.368	11	234.034	15	343.571
	4	8	142.804	11	218.760	13	287.841	17	416.960
2	5	10	175.749	12	262.163	15	341.023	19	488.192
	10	10	356.825	20	487.009	21	608.684	27	834.026
	15	15	546.274	30	740.052	30	888.063	34	1181.076
	3	6	101.394	7	164.432	9	223.589	11	335.918
	4	8	124.337	8	193.413	10	260.507	13	386.141
3	5	5	143.618	10	221.835	11	295.385	15	433.042
	10	10	241.264	10	361.895	20	459.477	20	643.268
	15	15	345.222	15	483.311	15	621.400	30	837.381

placement cost for a failed system. Under the condition that the system fails before t, the expected cost for one renewal cycle is $(nC_1 + C_R)$; on the other hand, when system is replaced at time t while system is working, the total cost for replacement is nC_1. As a result, the expected cost rate is

$$C_1(t; k, n) = \frac{nC_1 + C_R[1 - R_G(k, n; t)]}{\int_0^t R_G(k, n; x)\mathrm{d}x}. \tag{4.30}$$

Then, we focus on the optimal replacemnet time which minimizes Eq. (4.30) and give the following result.

Proposition 4.2. If a consecutive-k-out-of-n:G system with i.i.d. components has an increasing failure rate (IFR), then there exists an unique $t_1{}^*$ which minimizes the expected cost rate $C_1(t; k, n)$.

Proof. Differentiating $C_1(t; k, n)$ in Eq. (4.30) with respect to t and setting it equal to 0, we have

$$H(t; k, n) \int_0^t R_G(k, n; x)\mathrm{d}x + R_G(k, n; t) = \frac{nC_1 + C_R}{C_R}, \tag{4.31}$$

where $H(t; k, n)$ is the system failure rate. Further, letting $L_1(t; k, n)$ be the left-hand side of Eq. (4.31) and differentiating $L_1(t; k, n)$ with t, we have:

$$\frac{\mathrm{d}L_1(t; k, n)}{\mathrm{d}t} = \frac{\mathrm{d}H(t; k, n)}{\mathrm{d}t} \int_0^t R_G(k, n; x)\mathrm{d}x. \tag{4.32}$$

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Hence, if $H(t; k, n)$ is strictly increasing with $t \geq 0$, then $L_1(t; k, n)$ increases strictly with t and a unique replacement time t_1^* which satisfies Eq. (4.31) exists. $\square$

In particular, we consider the situation that all components have the Weibull lifetime distribution, where $F(t) = 1 - \exp[-(\lambda t)^m]$ $(m > 0)$. As mentioned in Subsection 4.1.1, when $m \geq 1$, each component has no decreasing failure rate; when $0 < m < 1$, each component has decreasing failure rate. Then, using the result in Theorem 4.3, we obtain that the system failure rate $H(t; k, n)$ increases strictly with t when $m \geq 1$ and $2k \geq n$. Furthermore, we try to judge the limit value of $H(t; k, n)$. For an easier calculation, we consider the method of calculating the reliability for any coherent system in [48]. Then, the reliability of a consecutive-k-out-of-n:G system can be expressed as

$$R_G(k, n; t) = \sum_{i=k-1}^{n-1} s_{n-i} \cdot \sum_{j=i+1}^{n} \binom{n}{j} \bar{F}(t)^j F(t)^{n-j}, \tag{4.33}$$

where s_{n-i} is the probability that the $(n-i)$th component failure causes system failure, i.e., system signature. From Eq. (4.33), the system failure rate of the consecutive-k-out-of-n:G system is expressed as

$$\begin{aligned} H(t; k, n) &= -\frac{\mathrm{d}R_G(k, n; t)/\mathrm{d}t}{R_G(k, n; t)}, \\ &= \frac{f(t)}{\bar{F}(t)} \frac{\sum_{j=k}^{n} j \cdot s_{n-j+1} \binom{n}{j} [\bar{F}(t)/F(t)]^j}{\sum_{j=k}^{n} \left(\sum_{i=k-1}^{j-1} s_{n-i} \right) \binom{n}{j} [\bar{F}(t)/F(t)]^j}. \end{aligned} \tag{4.34}$$

In addition, when $t \to \infty$, we have $\lim_{t\to\infty} \bar{F}(t) \to 0$ and $\lim_{t\to\infty} F(t) \to 1$. Then,

$$\begin{aligned} H(\infty; k, n) &\equiv \lim_{t\to\infty} H(t; k, n), \\ &= h(\infty) \lim_{t\to\infty} \frac{k \cdot s_{n-k+1} \binom{n}{k} \bar{F}(t)^k}{s_{n-k+1} \binom{n}{k} \bar{F}(t)^k}, \\ &= kh(\infty). \end{aligned} \tag{4.35}$$

As a result, we obtain the following results.

Theorem 4.7. When a consecutive-k-out-of-n:G system $(2k \geq n)$ consists of i.i.d. components with Weibull lifetime distribution $F(t) = 1 - \exp[-(\lambda t)^m]$ $(m \geq 1)$, and $H(t; k, n)$ is the system failure rate with limit $kh(\infty)$, μ_n is the MTTF of the system, then

(1) when $kh(\infty)\mu_n > (nC_1 + C_R)/C_R$, there exists an unique and finite t_1^* $(0 < t_1^* < \infty)$ which minimizes the expected cost rate and

$$C_1(t_1^*; k, n) = C_R \cdot H(t_1^*; k, n). \tag{4.36}$$

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(2) when $kh(\infty)\mu_n \leq (nC_1 + C_R)/C_R$, then the expected cost rate decreases strictly with t, and the system is suggested to be replaced when system failure occurs.

Furthermore, we discuss the optimal replacement time of the consecutive-k-out-of-n:G system when $2k < n$. In particular, when $F(t) = 1 - \exp[-(\lambda t)^m]$, Eq. (4.34) becomes

$$H(t; k, n) = m\lambda^m t^{m-1} g(t; k, n), \tag{4.37}$$

where

$$g(t; k, n) = \frac{\sum_{j=k}^{n} j \cdot s_{n-j+1} \binom{n}{j} (\exp[(\lambda t)^m] - 1)^{-j}}{\sum_{j=k}^{n} \left(\sum_{i=k-1}^{j-1} s_{n-i} \right) \binom{n}{j} (\exp[(\lambda t)^m] - 1)^{-j}}.$$

Then, we check the shape of the rational function $g(t; k, n)$ when $2k < n$ through computational experiment and find that $g(t; k, n)$ increases strictly with t for any value of m ($m > 0$). Then we infer that when $2k < n$ and $m \geq 1$, $H(t; k, n)$ increases with t, and the optimal replacement time t_1^* which minimizes the expected cost rate in Eq. (4.30) uniquely exists.

We give some examples to illustrate our results of the monotonicity of $H(t; k, n)$. In Fig. 4.8, we plot some graphs of the system failure rate for a consecutive-3-out-of-n:G system under the condition that $F(t) = 1 - \exp(-t^m)$ with $m = 0.3$. Figure 4.8(a) considers the situation that $2k \geq n$, and Fig. 4.8(b) considers the situation that $2k < n$. From the graphs, we confirmed that when components have DFR, the system failure rate $H(t; k, n)$ is also not increasing, which indicates that an unique and finite optimal replacement time does not exist. Furthermore, we illustrate the system failure rate of the consecutive-3-out-of-n:G system when $m = 2$ in Fig. 4.9. From Fig. 4.9(a) and 4.9(b), we confirmed that when components have IFR, $H(t; k, n)$ is increasing when $2k \geq n$ and $2k < n$.

4.2.3 Optimal Number of Components and Replacement Time

We have analyzed the optimal number of components in Subsection 4.2.1 and the optimal replacement time for given k and n in Subsection 4.2.2. In addition, the system with different values of n have different optimal replacement time and the corresponding minimum expected cost rate during operation stage. Therefore, we analyze this optimization problem to determine the optimal n_1^* and t_1^* simultaneously, which can give the most minimum value of the expected cost rate. However, it is not possible to check all values of n, so we consider to check a sufficient large n to find the optimal solution. We use the following steps to check all optimal solutions with Mathematica.

Step 0. $n = k - 1$.
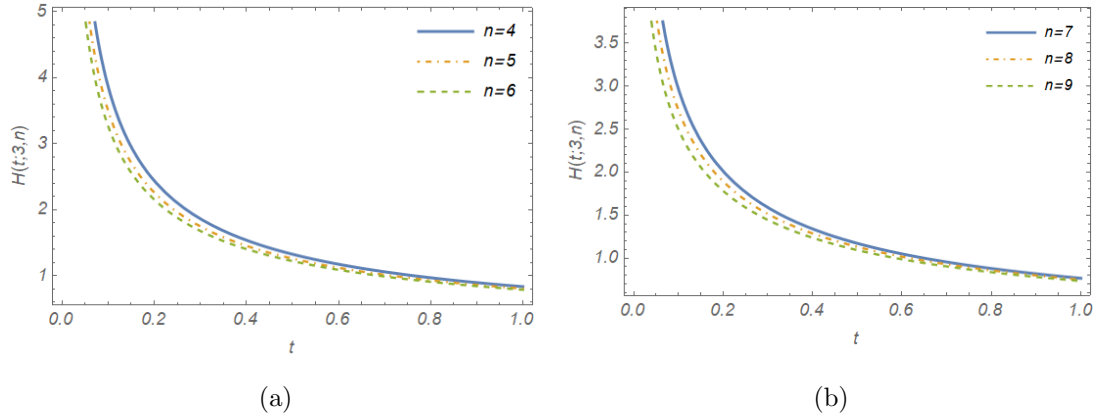
Figure 4.8: $H(t; k, n)$ for a consecutive-3-out-of-n:G system when component life-time distribution is $F(t) = 1 - \exp(-t^m)$, $m = 0.3$ ((a) $n \le 2k$, (b) $n > 2k$).
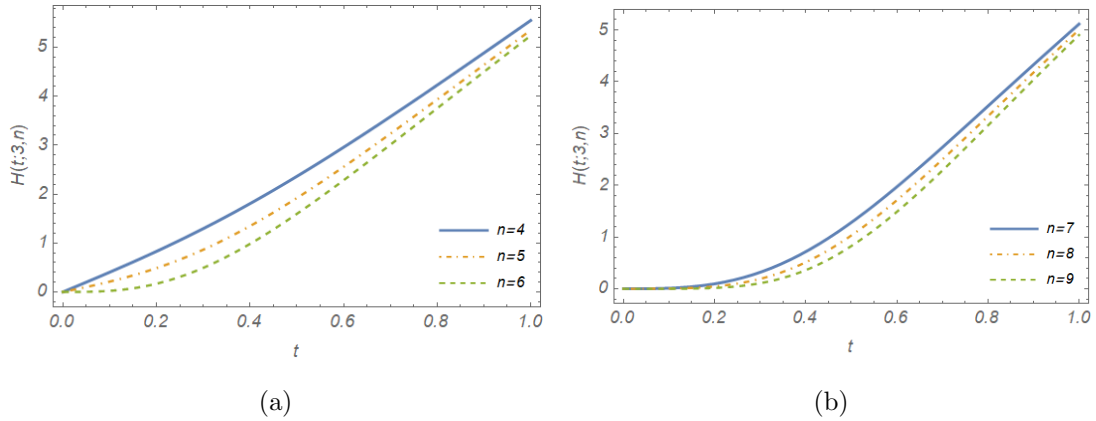
Figure 4.9: $H(t; k, n)$ for a consecutive-3-out-of-n:G system when component life-time distribution is $F(t) = 1 - \exp(-t^m)$, $m = 2$ ((a) $n \le 2k$, (b) $n > 2k$).

Step 1. $n = n + 1$.

Step 2. For the given n, the optimal replacement time and the corresponding expected cost rate are derived by minimizing Eq. (4.30).

Step 3. If $n < n_{max}$ (sufficient large number), then go to Step 1.

Step 4. Compare the expected cost rates for all n and obtain the optimal $n_1{}^*$ and $t_1{}^*$ which give the minimum value of the expected cost rate.

We give the numerical experiments for the proposed optimization problems in subsection 4.2.2 and subsection 4.2.3. The optimal $n_1{}^*$ and $t_1{}^*$ for the given k, C_1, C_R and m are obtained by checking the sufficient large number of n, which are

Table 4.5: Optimal replacement time and the corresponding expected cost rate when for a consecutive-k-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp(-t^m)$.

m	k	$C_1=5, C_R=50$			$C_1=5, C_R=100$			$C_1=5, C_R=150$			$C_1=5, C_R=250$		
		n	t_1^*	$C_1(t_1^*;k,n)$	n	t_1^*	$C_1(t_1^*;k,n)$	n	t_1^*	$C_1(t_1^*;k,n)$	n	t_1^*	$C_1(t_1^*;k,n)$
1	3	8	1.098	128.633	11	0.613	169.370	13	0.544	193.900	15	0.485	225.951
		9	1.110	126.988	12	0.649	168.213	14	0.575	193.185	16	0.511	225.325
		10	**1.149**	**126.583**	**13**	**0.684**	**167.928**	**15**	**0.605**	**193.164**	**17**	**0.536**	**225.323**
		11	1.199	126.875	14	0.719	168.259	16	0.633	193.670	18	0.560	225.810
		12	1.258	127.684	15	0.752	169.055	17	0.661	194.579	19	0.583	226.685
	4	10	1.889	192.511	14	0.568	257.932	16	0.466	299.488	18	0.392	353.676
		11	1.749	190.903	15	0.590	256.946	17	0.486	298.495	19	0.409	352.290
		12	**1.723**	**190.377**	**16**	**0.613**	**256.765**	**18**	**0.505**	**298.246**	**20**	**0.425**	**351.675**
		13	1.774	190.777	17	0.636	257.210	19	0.524	298.593	21	0.441	351.685
		14	1.874	191.715	18	0.659	258.136	20	0.543	299.422	22	0.457	352.204
	5	12	∞	265.934	16	0.542	359.443	19	0.423	420.110	22	0.350	499.951
		13	∞	264.460	17	0.557	357.995	20	0.437	419.061	23	0.361	498.821
		14	**∞**	**264.000**	**18**	**0.572**	**357.412**	**21**	**0.451**	**418.775**	**24**	**0.373**	**498.401**
		15	∞	264.423	19	0.589	357.543	22	0.464	419.102	25	0.384	498.579
		16	∞	265.635	20	0.606	358.261	23	0.478	419.942	26	0.395	499.262
	10	20	∞	785.714	27	∞	1007.140	31	0.397	1201.440	38	0.250	1479.470
		21	∞	784.337	28	∞	1006.170	32	0.400	1200.240	39	0.254	1478.630
		22	**∞**	**783.881**	**29**	**∞**	**1006.133**	**33**	**0.404**	**1199.790**	**40**	**0.258**	**1478.480**
		23	∞	784.259	30	∞	1006.970	34	0.409	1200.010	41	0.262	1478.940
		24	∞	785.400	31	∞	1008.650	35	0.414	1200.850	42	0.266	1479.940
2	3	4	0.420	95.774	4	0.304	131.275	6	0.361	116.458	7	0.358	132.133
		5	0.483	91.013	5	0.370	118.983	7	0.411	116.062	8	0.401	129.433
		6	**0.518**	**85.657**	**6**	**0.410**	**104.090**	**8**	**0.454**	**115.421**	**9**	**0.438**	**128.227**
		7	0.570	87.548	7	0.462	104.638	9	0.491	115.974	10	0.472	129.904
		8	0.611	89.736	8	0.504	105.276	10	0.525	118.344	11	0.503	132.048
	4	6	0.453	127.396	6	0.336	169.920	8	0.348	163.003	9	0.331	186.205
		7	0.485	123.569	7	0.373	158.071	9	0.383	162.995	10	0.361	183.947
		8	**0.505**	**119.700**	**8**	**0.396**	**146.062**	**10**	**0.413**	**162.770**	**11**	**0.389**	**182.335**
		9	0.541	122.091	9	0.432	146.541	11	0.440	163.211	12	0.413	182.346
		10	0.570	124.729	10	0.462	147.602	12	0.464	164.778	13	0.436	184.480
	5	8	0.464	161.051	8	0.348	210.166	8	0.295	248.114	12	0.336	241.324
		9	0.484	158.281	9	0.370	199.703	9	0.320	230.344	13	0.357	239.860
		10	**0.497**	**155.558**	**10**	**0.386**	**189.160**	**10**	**0.337**	**211.954**	**14**	**0.376**	**239.403**
		11	0.523	158.437	11	0.412	190.569	11	0.363	212.329	15	0.393	240.369
		12	0.545	161.530	12	0.435	192.082	12	0.387	212.561	16	0.410	242.959
	10	**10**	**0.345**	**344.940**	18	0.349	440.190	18	0.298	503.688	18	0.246	601.489
		11	0.377	348.035	19	0.354	435.420	19	0.303	493.194	19	0.253	579.489
		12	0.403	350.109	**20**	**0.358**	**430.777**	**20**	**0.308**	**482.896**	**20**	**0.258**	**557.605**
		13	0.423	351.596	21	0.368	434.211	21	0.318	485.125	21	0.269	558.101
		14	0.438	352.737	22	0.378	437.794	22	0.328	487.433	22	0.278	558.419

the bold fonts in Table 4.5. We also selected values of $n_1^* - 2$, $n_1^* - 1$, $n_1^* + 1$, $n_1^* + 2$ and corresponding t_1^* and $C_1(t_1^*;k,n)$ in Eq. (4.30). The symbol ∞ in Table 4.5 means that the optimal replacement time is infinity, and we suggest that the system is replaced after it failed. Furthermore, we observe that when the optimal replacement time is infinity, the optimal n_1^* which give the minimum value of the expected cost rate are the same to the results in Table 4.4 with the corresponding k and cost factors.

4.3 Summary

In this Chapter, we proposed two optimization problems of the consecutive-k-out-of-n:G system. At the system design phase, we referred the model of the expected cost rate in Nakagawa [79] and built the expected cost rate for the consecutive-k-out-of-n:G system by using the results of MTTF obtained in Chapter 2. By minimizing the expected cost rate, the optimal number of components was derived. In addition, at the operation phase, we built the expected cost rate for the preventive replacement model and obtained the results of the optimal replacement time. Finally, we also determined the optimal $n_1{}^*$ and $t_1{}^*$ simultaneously. To investigate the proposed optimal policies, we performed the numerical experiments and analyzed the results.

Chapter 5

Design and Maintenance Policies by Considering Number of Failed Components

In Chapter 4, we discussed the optimization problems for consecutive-k-out-of-n:G systems by considering the policy to replace all components. It would be reasonable because the life of the working components at the time of replacement may be damaged and will not be as new as ones at the next operation cycle. From an economical view, it is possible to replace failed components only. In particular, if the failure rates of the components are approximately constant, then the working components will be as new ones at the beginning of the next operation without any life damage. As a result, we consider that the lifetimes of components follow the exponential distribution where the failure rate is constant and only failed components will be replaced by new ones. Chapter 3 has proposed the expected number of failed components for any coherent system. Using the results in Chapter 3, we can derive the expected values of the number of failed components for the consecutive-k-out-of-n:G systems. The aim of this chapter is to give the improved optimal policies for consecutive-k-out-of-n:G systems by considering to replace failed components only at the time of maintenance. In this Chapter, lifetimes of all components are assumed to follow the same exponential distribution. Section 5.1 obtains the optimal number of components for the consecutive-k-out-of-n:G system. Section 5.2 obtains the optimal replacement time for the consecutive-k-out-of-n:G system. Section 5.3 considers the integrated optimization problem, where the optimal number of components and the optimal replacement time are determined simultaneously. Section 5.4 gives the comparison of the optimal results between the replacement of all components and the replacement of failed components only. Finally, we summarize the contributions of this chapter.

5.1 System Design

In this section, we discuss the optimal number of components at system design phase again. Using the renewal cycle which has been introduced in Chapter 4, and only failed components are considered to be replaced when system failure occurs. We first give the expected number of failed components at the time of system failure. Then, we build the model of the expected cost rate and derive the optimal number of components by minimizing this expected cost rate.

5.1.1 Expected Number of Failed Components when System Fails

Theorem 3.1 has given the expression of the expected number of failed components when system fails for any coherent system, where

$$E[X] = \sum_{i=1}^{n} i \cdot s_i,$$

$$= \sum_{i=1}^{n} i \cdot \left[\frac{N_{n-i+1}}{\binom{n}{i-1}} - \frac{N_{n-i}}{\binom{n}{i}} \right], \tag{5.1}$$

where s_i is the system signature, and N_{n-i} is the number of path sets with the $(n-i)$ working components.

We then focus on the system signature of the consecutive-k-out-of-n:G system, and it is easy to derive the expression by using the results of the path sets obtained in Chapter 2. Consider the number of path sets with j working components which is given in subsection 2.2.2, and

$$N_G(j, k, n) = \binom{n}{j} - \sum_{i=0}^{\lfloor j/k \rfloor} (-1)^i \binom{n-j+1}{i} \binom{n-ik}{n-j} \quad (j = k, \cdots, n). \tag{5.2}$$

Then according to the definition of the system signature, we have the following result.

Lemma 5.1. The system signature of a consecutive-k-out-of-n system is expressed as

$$s_{n-j} = \frac{N_G(j+1, k, n)}{\binom{n}{j+1}} - \frac{N_G(j, k, n)}{\binom{n}{j}} \quad (j = k-1, k, \cdots, n-1), \tag{5.3}$$

where j is the number of working components.

Using the result in Eq. (3.2), we give the following theorem.

Theorem 5.1. The expected number of failed components at the moment of system failure of a consecutive-k-out-of-n:G system is

$$E[X_n] = \sum_{j=k-1}^{n-1} (n-j) \cdot \left[\frac{N_G(j+1,k,n)}{\binom{n}{j+1}} - \frac{N_G(j,k,n)}{\binom{n}{j}} \right], \qquad (5.4)$$

where $N_G(j,k,n)$ is given in Eq. (2.25).

5.1.2 Optimal Number of Components

Consider a consecutive-k-out-of-n:G system which consists of n $(n \geq k)$ i.i.d. components. Assume that the components and the system are either working or failed. When system failure occurs, failed components will be replaced and the replacement time is negligible. The lifetimes of components follow the exponential distribution in which the failure rate is constant. We consider the same cost factors in Chapter 4, where C_1 represents the acquisition cost for each failed component, and C_R represents the replacement cost for a failed system. Then, the expected cost rate is given by

$$C_2(n;k) = \frac{E[X_n]C_1 + C_R}{\mu_n} \quad (k = n, n+1, \cdots), \qquad (5.5)$$

where $E[X_n]$ is obtained in Eq. (5.4) and μ_n is the MTTF of the system.

In particular, when all components lifetime distribution follow $F(t) = 1 - \exp(-\lambda t)$, we have

$$\mu_n = \frac{1}{\lambda} \sum_{j=k}^{n} \frac{N_G(j,k,n)}{j\binom{n}{j}},$$

which has been proposed in Eq. (2.27).

We try to find the optimal number of components which minimizes the expected cost rate in Eq. (5.5).

Proposition 5.1. Denote that $\Delta\mu_n = \mu_{n+1} - \mu_n$ and $\Delta E[X_n] = E[X_{n+1}] - E[X_n]$, then for a consecutive-k-out-of-n:G system, if

$$\frac{\Delta E[X_n]}{\Delta \mu_n} < \frac{\Delta E[X_{n+1}]}{\Delta \mu_{n+1}}, \qquad (5.6)$$

for any value of n $(n \geq k)$, then there exists an unique optimal $n_2{}^*$ which minimizes the expected cost rate.

Proof. Considering the inequality $C_2(n+1;k) - C_2(n;k) \geq 0$, which equivalent to

$$\frac{\mu_n E[X_{n+1}] - \mu_{n+1} E[X_n]}{\mu_{n+1} - \mu_n} \geq \frac{C_R}{C_1}, \qquad (5.7)$$
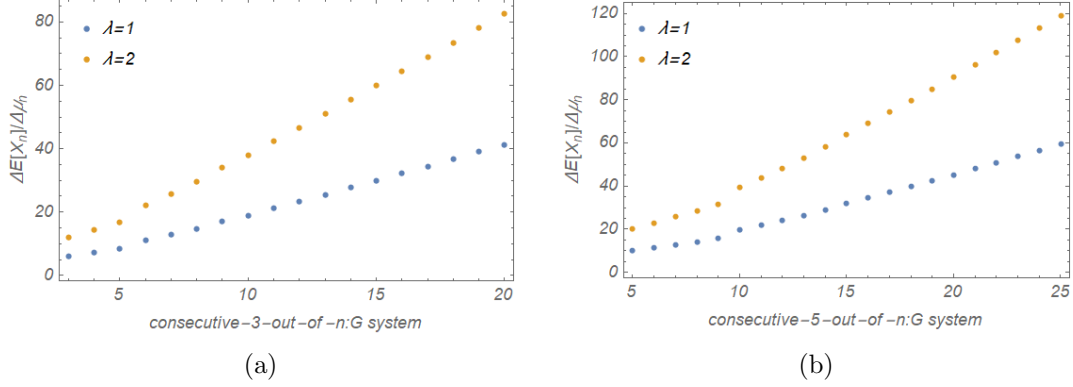
Figure 5.1: $\Delta E[X_n]/\Delta\mu_n$ for a consecutive-k-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp(-\lambda t)$.

then, denoting the left-hand side of Eq. (5.7) be $G_2(n;k)$ and we have

$$\Delta G_2(n;k) = G_2(n+1;k) - G_2(n;k),$$
$$= \frac{\mu_{n+1}(\Delta\mu_n \cdot \Delta E[X_{n+1}] - \Delta\mu_{n+1} \cdot \Delta E[X_n])}{\Delta\mu_{n+1} \cdot \Delta\mu_n}. \tag{5.8}$$

Clearly, $\Delta\mu_n$ and $\Delta E[X_n]$ are both greater than 0 for any value of n. Thus, if Eq. (5.6) is satisfied, then $\Delta G_2(n;k) > 0$ and $G_2(n;k)$ increases strictly with n from 1 ($G_2(k;k) = 1$). Overall, there inevitably exists an unique $n_2{}^*$ which firstly satisfies Eq. (5.7) can minimize the expected cost rate $C_2(n;k)$. $\qquad\square$

In particular, when all components have exponential lifetime distribution, we checked the monotonicity of $\Delta E[X_n]/\Delta\mu_n$ by computational experiments and confirmed the existence of the optimal number of components. Figures 5.1(a) and 5.1(b) give some examples of $\Delta E[X_n]/\Delta\mu_n$ under the different values of parameters. Furthermore, using the expression of the MTTF in Eq. (2.27), we have

$$G_2(n;k) = \frac{\mu_n E[X_{n+1}] - \mu_{n+1} E[X_n]}{\mu_{n+1} - \mu_n},$$
$$= \frac{\left(\sum_{j=k}^{n} \frac{N_G(j,k,n)}{j\binom{n}{j}}\right) E[X_{n+1}] - \left(\sum_{j=k}^{n+1} \frac{N_G(j,k,n+1)}{j\binom{n+1}{j}}\right) E[X_n]}{\sum_{j=k}^{n+1} \frac{N_G(j,k,n+1)}{j\binom{n+1}{j}} - \sum_{j=k}^{n} \frac{N_G(j,k,n)}{j\binom{n}{j}}},$$

which is not related to the parameter λ, in other words, the value of $n_2{}^*$ is not related to the component failure rate.

In Table 5.1, we give some results of the expected number of failed components at the time of system failure for a consecutive-k-out-of-n:G system for several values of k and n.

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Table 5.1: Expected number of failed components for a consecutive-k-out-of-n:G system.

k	n	$E[X_n]$	k	n	$E[X_n]$
	5	2.1		8	2.1
3	8	4.1	5	10	3.1
	10	5.6		15	5.7
	10	1.5		12	1.4
8	15	3.0	10	20	3.5
	25	6.7		30	6.6

Table 5.2: Optimal number of components and the corresponding expected cost rate for a consecutive-k-out-of-n:G system (component lifetime distribution is $F(t) = 1 - \exp(-\lambda t)$).

λ	k	$C_1 = 5, C_R = 10$		$C_1 = 5, C_R = 20$		$C_1 = 5, C_R = 50$		$C_1 = 5, C_R = 100$	
		$n_2{}^*$	$C_2(n_2{}^*; k)$	$n_2{}^*$	$C_2(n_2{}^*; k)$	$n_2{}^*$	$C_2(n_2{}^*; k)$	$n_2{}^*$	$C_2(n_2{}^*; k)$
	5	8	6.90	10	9.74	16	13.35	23	25.86
0.1	10	15	13.96	20	19.64	29	33.38	40	52.03
	20	29	28.09	40	39.58	56	67.12	76	104.50
	40	58	56.37	80	79.54	108	134.67	147	209.51
	5	8	34.52	10	48.70	16	82.84	23	129.32
0.5	10	15	69.79	20	98.21	29	166.91	40	260.16
	20	29	140.46	40	197.89	56	335.61	76	522.48
	40	58	281.84	80	397.71	108	673.34	147	1047.57

Next, we give numerical experiments for the optimal number of components under the condition that all components have exponential lifetime distribution. Table 5.2 presents the optimal $n_2{}^*$ and the corresponding $C_2(n_2{}^*; k)$, respectively for given λ, k, C_1 and C_R. The results are obtained numerically and the effect of the system parameters are studied. We observe that an increase of C_R leads to an increase in $n_2{}^*$. It is reasonable because when the replacement cost for a failed system C_R is higher, the possibility of system failure should be controlled and we can increase the system reliability by increasing the number of components. On the other hand, for the same k and cost factors C_1 and C_R, the optimal number of components are the same when $\lambda = 0.1$ and 0.5, which verified that $n_2{}^*$ is not related to the component failure rate.

5.2 Maintenance Policy

In this section, we consider the age replacement policy for the consecutive-k-out-of-n:G system again. In one renewal cycle, the failed components are considered to be replaced at time t or the time of system failure, which occurs first. We first discuss the expected number of failed components at time t. In this problem, we

consider the following 2 cases. Case 1 is that system fails before time t. Case 2 is that system is working at time t. Furthermore, we build the expected cost rate by considering to replace failed components only, and derive the optimal replacement time which minimizes the expected cost rate.

5.2.1 Expected Number of Failed Components at Time t

(1) Case 1: System fails before time t

Denote that T is the lifetime of the system, $X(T)$ is the number of failed components when system fails with lifetime T. Furthermore, $F(t)$ is the lifetime distribution of a component, $\bar{F}(t)$ is the reliability of a component, $T_{[n-j]}$ is the $(n-j)$th order statistic of the n component failure times, and s_{n-j} is the system signature. Then, the conditional random variable $(X(T)|T \leq t)$ represents the number of failed components at the time of system failure under the condition that system fails before time t. Therefore,

$$\Pr\{X(T) = n - j | T \leq t\} = \frac{\Pr\{T = T_{[n-j]}, T \leq t\}}{\Pr\{T \leq t\}}, \tag{5.9}$$

where

$$\begin{aligned}
\Pr\{T = T_{[n-j]}, T \leq t\} &= \Pr\{T \leq t | T = T_{[n-j]}\} \Pr\{T = T_{[n-j]}\}, \\
&= \Pr\{T_{[n-j]} \leq t\} \Pr\{T = T_{[n-j]}\}, \\
&= \left(\sum_{i=0}^{j} \binom{n}{i} \bar{F}(t)^i F(t)^{n-i}\right) \cdot s_{n-j}. \tag{5.10}
\end{aligned}$$

Then, the expected number of failed components under the condition that a consecutive-k-out-of-n:G system fails before time t is obtained as

$$\begin{aligned}
E[X(T) | T \leq t] &= \sum_{j=k-1}^{n-1} (n-j) \cdot \Pr\{X(T) = n - j | T \leq t\}, \\
&= \frac{\sum_{j=k-1}^{n-1} (n-j) \cdot s_{n-j} \sum_{i=0}^{j} \binom{n}{i} \bar{F}(t)^i F(t)^{n-i}}{\Pr\{T \leq t\}}. \tag{5.11}
\end{aligned}$$

(2) Case 2: System is working at time t

Next, we consider the number of failed components when the system is working at time t. Using the same notations $F(t)$, $\bar{F}(t)$, $T_{[n-j]}$ in Case 1, then the conditional random variable $(S(t)|T > t)$ represents the number of failed components at time t, under the condition that system is working. Therefore,

$$\Pr\{S(t) = n - j | T > t\} = \frac{\Pr\{T_{[n-j]} \leq t < T_{[n-j+1]}, T > t\}}{\Pr\{T > t\}}, \tag{5.12}$$

where

$$\Pr\{T_{[n-j]} \le t < T_{[n-j+1]}, T > t\}$$
$$= \Pr\{T > t | T_{[n-j]} \le t < T_{[n-j+1]}\} \cdot \Pr\{T_{[n-j]} \le t < T_{[n-j+1]}\},$$

and $\Pr\{T > t | T_{[n-j]} \le t < T_{[n-j+1]}\}$ is the proportion of the number of combinations that system is working with $(n-j)$ failed components which is expressed as

$$\Pr\{T > t | T_{[n-j]} \le t < T_{[n-j+1]}\} = \frac{N_G(j,k,n)}{\binom{n}{j}},$$

where $N_G(j,k,n)$ is the number of path sets of the consecutive-k-out-of-n:G system with j working components.

As a result, we have

$$\Pr\{T_{[n-j]} \le t < T_{[n-j+1]}, T > t\}$$
$$= \Pr\{T > t | T_{[n-j]} \le t < T_{[n-j+1]}\} \cdot \Pr\{T_{[n-j]} \le t < T_{[n-j+1]}\},$$
$$= \frac{N_G(j,k,n)}{\binom{n}{j}} \binom{n}{j} \bar{F}(t)^j F(t)^{n-j},$$
$$= N_G(j,k,n) \bar{F}(t)^j F(t)^{n-j}.$$

Finally, we obtain the expected value $E[S(t)|T > t]$ as

$$
\begin{aligned}
E[S(t)|T > t] &= \sum_{j=k}^{n} (n-j) \cdot \Pr\{S(t) = n-j | T > t\}, \\
&= \frac{\sum_{j=k}^{n} (n-j) \Pr\{S(t) = n-j, T > t\}}{\Pr\{T > t\}}, \\
&= \frac{\sum_{j=k}^{n} (n-j) \cdot N_G(j,k,n) \bar{F}(t)^j F(t)^{n-j}}{\Pr\{T > t\}}.
\end{aligned}
\tag{5.13}
$$

5.2.2 Optimal Replacement Time

We use the same assumptions and cost factors given in Section 4.2.2. Then, under the condition that system is working at time t, the number of failed components will be replaced and the expected replacement cost is

$$Q_1(t) = C_1 \cdot E[S(t)|T > t], \tag{5.14}$$

where $E[S(t)|T > t]$ is obtained in Eq. (5.13).

On the other hand, if the maintenance is performed at the time of system failure which is occurred before time t, then the expected replacement cost becomes

$$Q_2(t) = C_R + C_1 \cdot E[X(T)|T \le t], \tag{5.15}$$

where $E[X(T)|T \leq t]$ is obtained in Equation (5.11).

Furthermore, using the mean time to replacement of a renewal cycle proposed in Eq. (4.29), the expected cost rate is

$$\begin{aligned}
C_2(t; k, n) &= \frac{Q_1(t) \cdot \Pr\{T > t\} + Q_2(t) \cdot \Pr\{T \leq t\}}{E[\min(t, T)]}, \\
&= \frac{C_1 \cdot N(t) + C_R[1 - R_G(k, n; t)]}{\int_0^t R_G(k, n; x)\mathrm{d}x},
\end{aligned} \tag{5.16}$$

where

$$N(t) = \sum_{j=k-1}^{n-1} (n-j)s_{n-j} \sum_{i=0}^{j} \binom{n}{i} \bar{F}(t)^i F(t)^{n-i} + \sum_{j=k}^{n} (n-j)N_G(j, k, n)\bar{F}(t)^j F(t)^{n-j}.$$

It is interesting to find the optimal replacement time t_2^* by minimizing the expected cost rate in Eq. (5.16). However, it is difficult to judge the monotonicity of the cost function analytically. Differentiating the cost function in Eq. (5.16) with respect to t and setting it equal to 0, we have

$$\left[C_1 \frac{\mathrm{d}N(t)}{\mathrm{d}t} - C_R \frac{\mathrm{d}R_G(k, n; t)}{\mathrm{d}t} \right] \frac{\int_0^t R_G(k, n; x)\mathrm{d}x}{R_G(k, n; t)} - [C_1 \cdot N(t) - C_R \cdot R_G(k, n; t)] = C_R. \tag{5.17}$$

Letting the left-hand side of Eq. (5.17) be $L_2(t; k, n)$ and we have $L_2(0; k, n) = C_R$. As we have observed numerically and graphically, the monotonicity of $L_2(t; k, n)$ has the following two cases when all components have exponentical lifetime distributions. Case one is that $L_2(t; k, n)$ increases with t when $n < 3k$. In the other case, when $n \geq 3k$, the value of $L_2(t; k, n)$ is firstly less than C_R and then is greater than C_R with only one value that $L_2(t_2^*; k, n) = C_R$. Figures 5.2(a) and 5.2(b) give some graphs of $L_2(t; k, n)$ for a consecutive-3-out-of-n:G system when $C_1 = 5, C_R = 50$ and $C_1 = 5, C_R = 100$. We consider two cases of the values of n, where $n = 5$ and $n = 40$.

Therefore, we indicate that when $n < 3k$, $C_2(t; k, n)$ increases strictly with t; when $n \geq 3k$, there exists an unique t_2^* where $L_2(t_2^*; k, n) = C_R$ ($t_2^* > 0$) and the corresponding expected cost rate $C_2(t_2^*; k, n)$ is the minimal result.

Then, we give the results of the numerical experiments for the expected number of failed components and the optimal replacement time. In Figs. 5.3(a) and 5.3(b), we plot the graphs of $E[S(t)|T > t]$ for a consecutive-k-out-of-n:G system in Eq. (5.13) when $k = 5, n = 10$ and $k = 8, n = 25$, respectively. The component lifetime parameter values are chosen to be $\lambda = 1, 2$ and $m = 1, 2$. From the graphs, we find out that when t is large, the limit values of $E[S(t)|T > t]$ are the same, which are not related to the λ and m.

We then plot the graphs of $E[X(T)|T \leq t]$ for a consecutive-k-out-of-n:G system in Eq. (5.11). From Figs. 5.4(a) and 5.4(b), we observe that with different

(a)   (b)
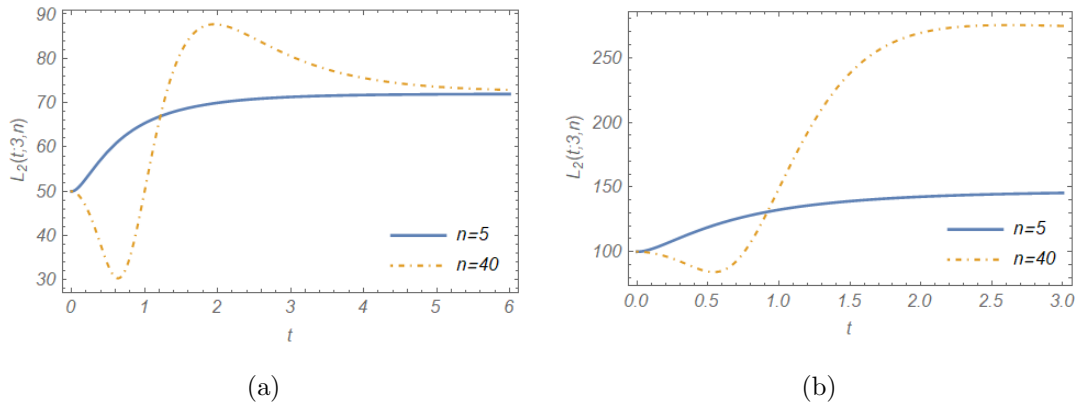
Figure 5.2: $L_2(t; k, n)$ for a consecutive-3-out-of-n:G system when component life-time distribution is $F(t) = 1 - \exp(-t)$ ((a) $C_1 = 5, C_R = 50$, (b) $C_1 = 5, C_R = 100$).

(a)   (b)

Figure 5.3: $E[S(t)|T > t]$ for a consecutive-k-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp[-(\lambda t)^m]$ ((a) $k = 5, n = 10$, (b) $k = 8, n = 25$).

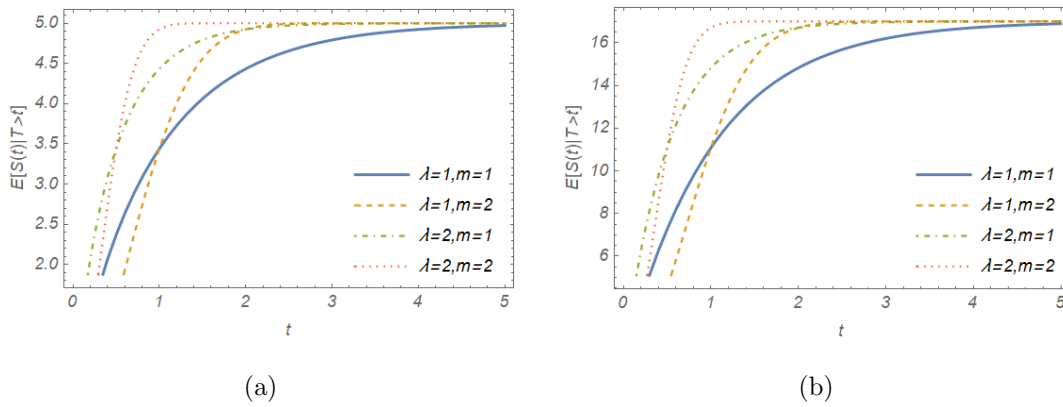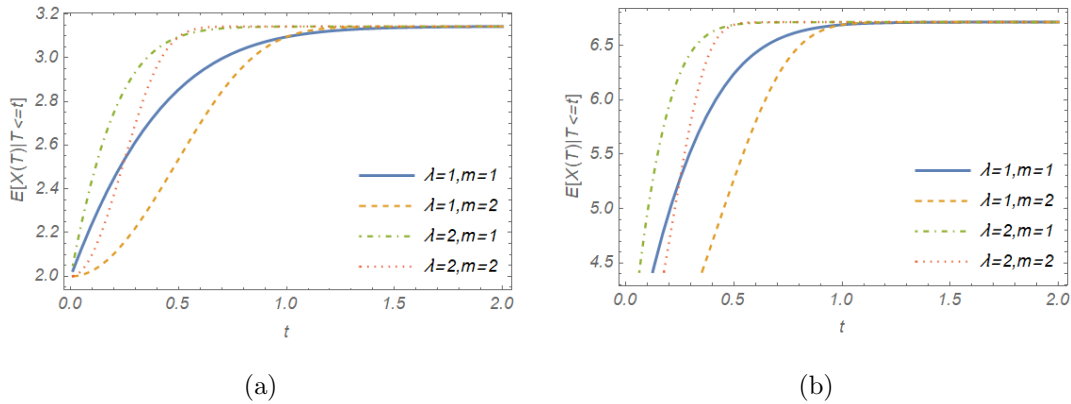Figure 5.4: $E[X(T)|T \leq t]$ for a consecutive-k-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp[-(\lambda t)^m]$ ((a) $k = 5, n = 10$, (b) $k = 8, n = 25$).

combinations of λ and m, $E[X(T)|T \leq t]$ also have the same limit values. In addition, these limit values are equal to the values of $E[X_n]$ with the corresponding k and n in Table 5.1.

In addition, we compute the optimal replacement time which minimizes the expected cost rate in Eq. (5.16). Also, all components have exponential lifetime distribution. In Table 5.3, we give the optimal replacement time t_2^* and the corresponding cost rate $C_2(t_2^*; k, n)$ for several values of λ, k, n, C_1 and C_R. From the Table, we observe that an increase of C_R leads to a decrease in t_2^*. It is reasonable because for the large value of C_R, it is necessary to avoid system failure by replacing the failed components earlier preventively. It is also obvious that if the value of component failure rate λ increases, the optimal replacement time is also implemented earlier to avoid system failure.

5.3 Optimal Number of Components and Replacement Time

We have analyzed the optimal number of components n_2^* in Section 5.1, and the optimal replacement time t_2^* for given k and n in Section 5.2. In addition, from Table 5.3, we find that with the same λ and k, different values of n have different values of minimum expected cost rate. Therefore, similar to subsection 4.2.3, we consider the optimization problem to determine the optimal n_2^* and t_2^* simultaneously, which can give the minimal value among the minimum expected cost rates with different values of n and the same values of λ and k. We consider to check a sufficient large n to find the optimal solution. We use the same steps in subsection 4.2.3 to check all optimal solutions with Mathematica.

Table 5.3: Optimal replacement time and the corresponding expected cost rate for a consecutive-k-out-of-n:G system (component lifetime distribution is $F(t) = 1 - \exp(-\lambda t)$).

λ	k	n	$C_1=5, C_R=20$		$C_1=5, C_R=50$		$C_1=5, C_R=100$		$C_1=5, C_R=200$	
			t_2^*	$C_2(t_2^*;k,n)$	t_2^*	$C_2(t_2^*;k,n)$	t_2^*	$C_2(t_2^*;k,n)$	t_2^*	$C_2(t_2^*;k,n)$
0.1	3	10	1.31	4.84	0.54	4.93	0.28	4.96	0.15	4.98
		12	2.66	5.57	1.37	5.75	0.90	5.83	0.62	5.88
		15	4.67	6.54	2.60	6.87	1.89	7.02	1.42	7.13
	4	12	0.44	5.94	0.16	5.98	0.08	5.99	0.04	5.99
		15	1.47	7.19	0.75	7.33	0.48	7.39	0.31	7.43
		20	3.27	9.05	1.91	9.36	1.40	9.51	1.07	9.62
	5	15	0.30	7.45	0.11	7.48	0.06	7.49	0.03	7.50
		20	1.36	9.60	0.75	9.76	0.51	9.84	0.35	9.89
		25	2.44	11.57	1.48	11.87	1.10	12.01	0.84	12.12
0.5	3	10	0.26	24.20	0.11	24.66	0.06	24.82	0.03	24.91
		12	0.53	27.84	0.27	28.75	0.18	29.14	0.12	29.41
		15	0.94	32.72	0.52	34.34	0.38	35.09	0.28	35.64
	4	12	0.09	29.69	0.03	29.89	0.02	29.94	0.01	29.97
		15	0.29	35.97	0.15	36.67	0.10	36.96	0.06	37.16
		20	0.65	45.23	0.38	46.81	0.28	47.56	0.21	48.10
	5	15	0.060	37.231	0.02	37.40	0.01	37.45	0.01	37.48
		20	0.272	48.004	0.15	48.81	0.10	49.18	0.07	49.42
		25	0.489	57.830	0.30	59.34	0.22	60.07	0.17	60.60

Step 0. $n = k - 1$.

Step 1. $n = n + 1$.

Step 2. For the given n, the optimal replacement time and the corresponding expected cost rate are derived by minimizing Eq. (5.16).

Step 3. If $n < n_{max}$ (sufficient large number), then go to Step 1.

Step 4. Compare the expected cost rates for all n and find the optimal n_2^* and t_2^* which give the minimum value of the expected cost rate.

We give the numerical experiments for the proposed program to give the optimal results of n_2^*, t_2^* and $C_2(t_2^*; k, n_2^*)$. The optimal n_2^* and t_2^* for the given k, C_1 and C_R are obtained by checking the sufficient large number of n, which are given in Table 5.4.

5.4 Comparison of the Optimization Problems Between Replacing All Components and Replacing Failed Components

Table 5.4: Optimal number of components and replacement time, and the corresponding expected cost rate for a consecutive-k-out-of-n:G system (component lifetime distribution is $F(t) = 1 - \exp(-\lambda t)$).

λ	k	$C_1=5,C_R=20$			$C_1=5,C_R=40$			$C_1=5,C_R=50$			$C_1=5,C_R=100$		
		$n_2{}^*$	$t_2{}^*$	$C_2(t_2{}^*;k,n_2{}^*)$	$n_2{}^*$	$t_2{}^*$	$C_2(t_2{}^*;k,n_2{}^*)$	$n_2{}^*$	$t_2{}^*$	$C_2(t_2{}^*;k,n_2{}^*)$	$n_2{}^*$	$t_2{}^*$	$C_2(t_2{}^*;k,n_2{}^*)$
	3	9	0.682	4.430	9	0.307	4.467	9	0.241	4.474	9	0.116	4.487
	4	12	0.436	5.939	12	0.201	5.971	12	0.158	5.977	12	0.077	5.989
	5	15	0.302	7.446	15	0.142	7.474	15	0.112	7.479	15	0.055	7.490
0.1	6	18	0.222	8.952	18	0.105	8.977	18	0.083	8.982	18	0.041	8.991
	7	21	0.170	10.457	21	0.081	10.479	21	0.065	10.483	21	0.032	10.492
	8	24	0.135	11.961	24	0.065	11.981	24	0.051	11.985	24	0.025	11.992
	3	9	0.136	22.148	9	0.061	22.334	9	0.048	22.368	9	0.023	22.436
	4	12	0.087	29.694	12	0.040	29.854	12	0.032	29.884	12	0.015	29.943
	5	15	0.060	37.231	15	0.028	37.370	15	0.022	37.397	15	0.011	37.449
0.5	6	18	0.044	44.760	18	0.021	44.884	18	0.017	44.908	18	0.008	44.954
	7	21	0.034	52.284	21	0.016	52.395	21	0.013	52.416	21	0.006	52.459
	8	24	0.027	59.804	24	0.013	59.904	24	0.010	59.924	24	0.005	59.962

Obviously, the performance of the optimal results by replacing failed components in this Chapter should be better than the results by replacing all components in Chapter 4. For a more direct explanation, we illustrate the graphs of the expected cost rates and compare the minimum values of the expected cost rates. We only consider the situation that all components have the exponential lifetime distribution.

5.4.1 Optimal Number of Components

We first compare the optimal number of components and the corresponding expected cost rates in Eq. (4.22) and Eq. (5.5). Some graphs for the expected cost rates are given. Assume that $k = 5$ and 10, $\lambda = 0.1$, $C_1 = 5$ and $C_R = 50$. From Figs. 5.5(a) and 5.5(b), we can easily observe that the values of $C_2(n;k)$ are always less than the values of $C_1(n;k)$. Table 5.7 shows the optimal expected cost rates and the comparison results between $C_1(n_1{}^*;k)$ and $C_2(n_2{}^*;k)$. From the values of the ratio $C_2(n_2{}^*;k)/C_1(n_1{}^*;k)$, we observe that the minimum of $C_2(n_2{}^*;k)$ outperforms the minimum of $C_1(n_1{}^*;k)$.

5.4.2 Optimal Replacement Time

We then compare the optimal replacement time and the corresponding expected cost rates in Eq. (4.30) and Eq. (5.16). We give some graphs for the comparison of the expected cost rates. Assume that $k = 4$, $n = 20$ and 50, $\lambda = 1$, $C_1 = 5$ and $C_R = 100$. From Figs. 5.6(a) and 5.6(b), no matter that $C_1(t;k,n)$ has a minimum or decreases strictly with a limit value, the minimum of $C_2(t;k,n)$ is always fully less than the minimum of $C_1(t;k,n)$ or the limit value of $C_1(t;k,n)$.

Table 5.8 shows the expected cost rates and the comparison results between $C_1(t_1{}^*;k,n)$ and $C_2(t_2{}^*;k,n)$. ∞ means that with the given parameters, the trend

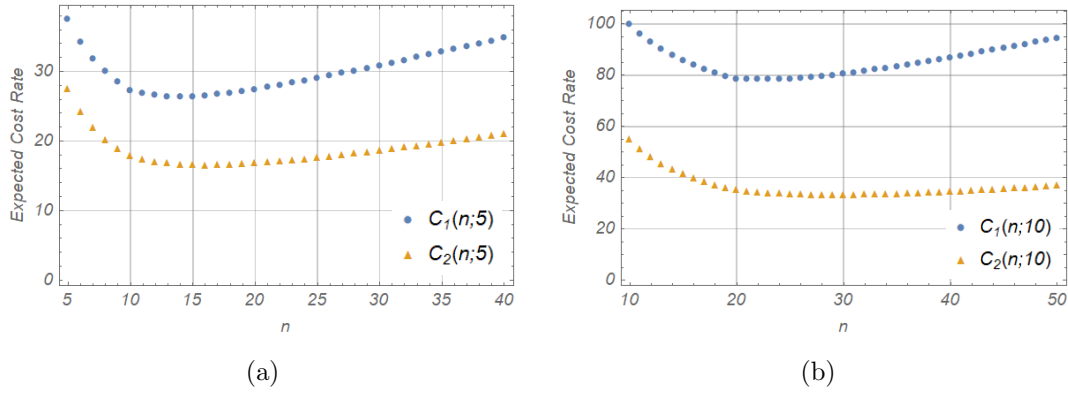Figure 5.5: $C_1(n; k)$ and $C_2(n; k)$ for a consecutive-k-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp(-\lambda t)$, $\lambda = 0.1$, $C_1 = 5$, $C_R = 50$.
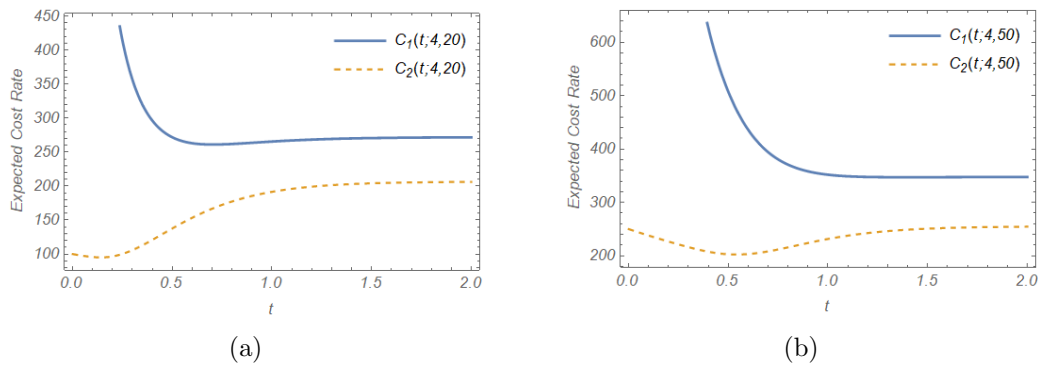

Figure 5.6: $C_1(t; k, n)$ and $C_2(t; k, n)$ for a consecutive-4-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp(-t)$, $C_1 = 5$, $C_R = 100$.

of $C_1(t; k, n)$ decreases trictly with t to a limit $C_1(\infty; k, n)$, and the optimal replacement time is the moment of system failure. From the values of the ratio $C_2(t_2^*; k, n)/C_1(t_1^*; k, n)$, we also observe that the minimum of $C_2(t_2^*; k, n)$ outperforms the minimum of $C_1(t_1^*; k, n)$.

5.4.3 Optimal Number of Components and Replacement Time

Finally, for the simultaneous optimization of the number of components and replacement time, we also give the comparison of the minimum expected cost rates in subsection 4.2.3 and Section 5.3. Table 5.9 summarizes the experimental results. n_1^*, t_1^* and $C_1(t_1^*; k, n_1^*)$ are the results of the simultaneous optimization of the number of components and replacement time by considering to replace all components. n_2^*, t_2^* and $C_2(t_2^*; k, n_2^*)$ are the results of the simultaneous optimization of the number of components and replacement time by considering to replace failed components only. Also,

$$\text{Effec} = \frac{C_1(t_1^*; k, n_1^*) - C_2(t_2^*; k, n_2^*)}{C_1(t_1^*; k, n_1^*)}.$$

From the value of the Effec., we observe that $C_2(t_2^*; k, n_2^*)$ outperforms $C_1(t_1^*; k, n_1^*)$.

5.5 Summary

In this Chapter, we gave the improved optimal policies from an economical view that only the failed components were replaced. We first obtained the expected number of failed components at the time of system failure for the consecutive-k-out-of-n:G system. Using this result, we gave the expected cost rate. By minimizing this expected value, we proposed the optimal number of components. Second, we considered the age replacement at the system operation phase. To propose the model of the expected cost rate, we first obtained the expected number of failed components at a particular time t, wherein include two cases that system is failed before time t, or system is still working at time t. Using the results of the expected number of failed components, the model of the expected cost rate was proposed and the existence of the optimal replacement time was determined by numerical experiments and graphs. Finally, we considered the optimal n_2^* and t_2^* simultaneously. The comparison of the optimal policies between the replacement of all components and the replacement of failed components only was also given, which can observe the advantages of the policies proposed in this Chapter directly.

Figure 5.7: Comparisons between $C_1(n_1^*;k)$ and $C_2(n_2^*;k)$ for a consecutive-k-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp(-\lambda t)$.

λ	k	$C_1 = 5, C_R = 50$					$C_1 = 5, C_R = 100$				
		n_2^*	$C_2(n_2^*;k)$	n_1^*	$C_1(n_1^*;k)$	$\frac{C_2(n_2^*;k)}{C_1(n_1^*;k)}$	n_2^*	$C_2(n_2^*;k)$	n_1^*	$C_1(n_1^*;k)$	$\frac{C_2(n_2^*;k)}{C_1(n_1^*;k)}$
	5	16	82.84	14	132.00	63%	23	129.32	19	182.73	71%
	10	29	166.91	22	391.94	43%	40	260.16	29	503.07	52%
0.5	20	56	335.61	40	1280.49	26%	76	522.48	45	1530.35	34%
	40	108	673.34	80	4555.56	15%	147	1047.57	80	5061.73	21%

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Figure 5.8: Comparisons between $C_1(t_1*;k,n)$ and $C_2(t_2*;k,n)$ for a consecutive-k-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp(-\lambda t)$.

λ	k	n	$C_1=5,C_R=50$					$C_1=5,C_R=100$				
			t_2*	$C_2(t_2*;k,n)$	t_1*	$C_1(t_1*;k,n)$	$\frac{C_2(t_2*;k,n)}{C_1(t_1*;k,n)}$	t_2*	$C_2(t_2*;k,n)$	t_1*	$C_1(t_1*;k,n)$	$\frac{C_2(t_2*;k,n)}{C_1(t_1*;k,n)}$
0.5	3	10	0.11	24.66	2.30	63.29	39.0%	0.06	24.82	1.15	85.87	28.9%
		12	0.27	28.75	2.52	63.84	45.0%	0.18	29.14	1.30	84.11	34.7%
		15	0.52	34.34	2.95	66.00	52.0%	0.38	35.09	1.51	84.53	41.5%
	4	12	0.03	29.89	3.45	95.19	31.4%	0.02	29.94	1.05	131.63	22.7%
		15	0.15	36.67	4.07	96.54	38.0%	0.10	36.96	1.18	128.47	28.8%
		20	0.38	46.81	∞	101.96	45.9%	0.28	47.56	1.41	130.54	36.4%
	5	15	0.02	37.40	∞	132.21	28.3%	0.01	37.45	1.06	180.97	20.7%
		20	0.15	48.81	∞	137.21	35.6%	0.10	49.18	1.21	179.13	27.5%
		25	0.30	59.34	∞	145.16	40.9%	0.22	60.07	1.39	183.77	32.7%

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Figure 5.9: Comparisons between $C_1(t_1^*; k, n_1^*)$ and $C_2(t_2^*; k, n_2^*)$ for a consecutive-k-out-of-n:G system when component lifetime distribution is $F(t) = 1 - \exp(-\lambda t)$.

		$C_1 = 5, C_R = 50$							$C_1 = 5, C_R = 100$						
λ	k	n_1^*	t_1^*	$C_1(t_1^*; k, n_1^*)$	n_2^*	t_2^*	$C_2(t_2^*; k, n_2^*)$	Effec	n_1^*	t_1^*	$C_1(t_1^*; k, n_1^*)$	n_2^*	t_2^*	$C_2(t_2^*; k, n_2^*)$	Effec
0.1	3	10	11.487	12.658	9	0.241	4.474	0.647	13	6.843	16.793	9	0.116	4.487	0.733
	4	12	17.228	19.038	12	0.158	5.977	0.686	16	6.132	25.677	12	0.077	5.989	0.767
	5	14	∞	26.400	15	0.112	7.479	0.717	18	5.724	35.741	15	0.055	7.490	0.790
	8	19	∞	54.609	24	0.051	11.985	0.781	24	7.517	71.792	24	0.025	11.992	0.833
	10	22	∞	78.388	30	0.045	14.988	0.809	29	∞	100.613	30	0.031	14.998	0.851
0.5	3	10	2.297	63.292	9	0.048	22.368	0.647	13	1.369	83.964	9	0.023	22.436	0.733
	4	12	3.446	95.188	12	0.032	29.884	0.686	16	1.226	128.382	12	0.015	29.943	0.767
	5	14	∞	132.000	15	0.022	37.397	0.717	18	1.145	178.706	15	0.011	37.449	0.790
	8	19	∞	273.046	24	0.010	59.924	0.781	24	1.503	358.959	24	0.005	59.962	0.833
	10	22	∞	391.941	30	0.009	74.941	0.809	29	∞	503.067	30	0.006	74.983	0.851

Chapter 6

Conclusions

This chapter summarizes the main contributions of this thesis and describes various future perspectives.

6.1 Summary

This thesis targeted the consecutive-k-out-of-n:G system, and we provided some system evaluation criteria, including system reliability and mean time to failure (MTTF). Also, based on the reliability properties, we discussed two types of optimization problems which include the optimal number of components in system design phase and the optimal replacement time in system operation phase. The contributions of this thesis are summarized as follows.

Chapter 1 firstly explained the background of this study. Reliability is now becoming a critical performance metric of a component or a system, and reliability engineering plays an important role to keep high reliability of a component or a system. The definition of consecutive-k-out-of-n systems which were discussed in this thesis and some examples of applications were introduced. In addition, optimization problems discussed in this thesis were introduced, which include design problems and maintenance problems. Literature reviews related to this thesis were also detailed and systematically classified. Finally, the research scope, objective and the organization of this thesis were given.

Chapter 2 focused on the system reliability evaluation, that is, to compute system reliability and MTTF. The system reliability evaluation is a fundamental step in system performance evaluation. Furthermore, the system reliability and MTTF are necessary to build the mathematical models of optimization problems. Although there are many research results on the system reliability studies of consecutive-k-out-of-n:G systems, the calculation formulas are all the recursive expressions which have the disadvantage associated with a recursive algorithm. As a result, we proposed a closed form of the system reliability which is more concise. On the other hand, we considered two cases by dividing the values of k, where case 1 is that $k = 2$, and case 2 is that k has a general value. When $k = 2$, the system

reliability was easily obtained and the expression of reliability was simple. We also considered the situation that the number of components is a random variable N which follows a truncated Poisson distribution. Finally, we obtained the reliability of the system when k is a general value. The MTTF of consecutive-k-out-of-n:G systems was also obtained.

Chapter 3 dealt with the general calculation formulas of the expected number of failed components for coherent systems. A coherent system is that every component is relevant for the system and the lifetime is non-decreasing function of components lifetimes. Obviously, the consecutive-k-out-of-n:G system is a typical type of coherent systems. The number of failed or working components in a working or failed system gives the important information. When a system is failed, the number of failed components gives the information that how many spare components should be available to replace failed components. Furthermore, the number of failed or working components at a particular time gives important information of the behavior of the system and gives ideas that how many spare components should be prepared for replacement. According to the existing literature results, researchers always focus on the number of failed or working components for a particular type of coherent systems and under one of the situations mentioned before. As a result, we proposed the general formulas for the expected number of failed components of any coherent system at a particular time, or at the time of system failure. The illustrative examples were given, which include a bridge structure system and a consecutive-k-out-of-n:F system. Yun and Endharta [112] once discussed the expected number of failed components for a consecutive-k-out-of-n:F system. They listed all possible paths of a particular system and calculated the corresponding expected number of failed components of each path. Obviously, this method is not efficient. We used the proposed method in this thesis and confirmed the same results in Yun and Endharta [112].

Chapter 4 considered two optimization problems for the consecutive-k-out-of-n:G system. At the system design phase, the system configuration was determined, e.g., the number of components. Although the system reliability increases with the increase of n for a consecutive-k-out-of-n:G system, the large number of components will cause the waste of resources. Therefore, we obtained the optimal number of components. In addition, at the operation phase, the maintenance was necessary to improve the system availability, and the optimal replacement time was determined. We referred the models of expected cost rates in Nakagawa [79] and built the models of the expected cost rates for the consecutive-k-out-of-n:G system under the assumption that all components were replaced at the time of replacement. In detail, we first discussed the optimization problems under a simple case that $k = 2$. When n is constant, we proposed the optimal number of components n^* and the optimal replacement time $t_a{}^*$, respectively. On the other hand, we considered the situation that the number of components is a random variable and the optimal replacement time $t_b{}^*$ was derived. Furthermore, when $k \geq 3$, we derived the general results about the optimal number of components and optimal

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replacement time. Finally, as different values of n have different values of optimal replacement time, we also found the optimal $n_1{}^*$ and $t_1{}^*$ simultaneously which gave the minimal value among the minimum expected cost rates with different values of n. To investigate the proposed optimal policies, we performed the numerical experiments and analyzed the results.

In Chapter 5, we gave the improved optimal policies from an economical view that only the failed components were replaced at the time of maintenance. We assumed that all components follwed the exponential lifetime distributions where the failure rate is constant, and then the working components without replacement would be as new as ones after maintenance. In Section 5.1, we discussed the optimal number of components at system design phase. Using the results of the expected number of failed components at the time of system failure for any coherent system in Section 3.1, we easily derived such expected value for the consecutive-k-out-of-n:G system. By building the model of the expected cost rate and minimizing it, the results of the optimal number of components n_2^* were obtained. In Section 5.2, we considered the optimal replacement time at system operation phase. We first obtained the expected number of failed components at a particular time t. In detail, when the lifetime of the system is less than the planned time t, using the results in Section 3.2.1, the expected number of failed components under this situation was derived. On the other hand, if the system is working at the planned time t, using the results in Section 3.2.2, the expected number of failed components under this situation was derived. Then, using the results of the expected number of failed components, the model of the expected cost rate was built and the existence of the optimal replacement time t_2^* was discussed by numerical experiments and graphs. In Section 5.3, we found the optimal $n_2{}^*$ and $t_2{}^*$ simultaneously by using the same method in Section 4.2.3. In Section 5.4, we compared the optimal policies between the replacement of all components and the replacement of failed components only. For the direct explanation, we illustrated the graphs of the expected cost rates and compared the optimal results of the number of components, replacement time, and the corresponding minimum expected cost rates.

In summary, this thesis obtained the optimal number of components at system design phase and the optimal replacement time at system operation phase of consecutive-k-out-of-n:G systems. These optimal results are considered to be able to enhance the reliability and availability of practical systems that can be expressed as consecutive-k-out-of-n:G systems. We expect that these optimization problems could provide the optimal design of practical systems and the most economical replacement policy during system operation phase. Although we considered the consecutive-k-out-of-n:G system with i.i.d. components which may be too idealistic, and two replacement models used in this thesis were simple, the optimization problems for consecutive-k-out-of-n:G systems were first studied in this thesis. Thus, this study would also be a clue for giving methods for deciding the optimization policies and enhancing the system reliability and availability.

6.2 Future Work

This section describes various interesting topics for possible future developments and research.

In this thesis, we proposed the time-based maintenance policy, where the maintenance is done at the planned time or the time of system failure which occurs first. From a more practical view, a condition-based maintenance has a more practical operation significance and is the most modern and popular maintenance technique. We will consider a condition-based maintenance for the consecutive-k-out-of-n:G system and the lifetime of the system is monitored through its operation condition. When the condition of failed components reaches a certain state, the maintenance will be considered. This method is considered to be able to reduce unnecessary maintenance actions and eliminate the risks associated with preventive maintenance actions.

Pham and Wang [89] classified the maintenance according to the degree of repair, where include the replacement, minimal repair, imperfect repair, worse repair and worst repair. According to the serveys in literature, many papers considered the maintenance policy by combining the imperfect repair or minimal repair with replacement, which could reduce the unnecessary replacement costs. Considering a consecutive-k-out-of-n:G system, we plan to develop a preventive replacement policy by considering the minimal repair or imperfect repair.

This thesis is based on the assumption that all components are independent and identical. In other words, the components are regarded as single types of components and do not affect with each other. However, in a practical situation, a system may consist of multi-type components, or the components lifetime would affect with other components lifetime. Then, we will focus on the situation that components are in a non i.i.d. cases and try to propose the maintenance policies for such complex system.

A consecutive-k-out-of-n system can be regarded as a one-dimensional system, and this system can be extended to two-or d-dimensional versions ($d \geq 3$). The introduction of the two-dimension consecutive-k-out-of-n system is given in [92] and [10]. The two-dimension consecutive-k-out-of-n system consists of mn components arranged into a rectangular pattern with m rows and n columns. The optimization problems are also needed to be discussed in detail for the two-or d-dimension consecutive-k-out-of-n systems.

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Acknowledgments