A construction of special Lagrangian submanifolds by generalized perpendicular symmetries

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Abstract

We show a method to construct a special Lagrangian submanifold L' from a given special Lagrangian submanifold L in a Calabi-Yau manifold with the use of generalized perpendicular symmetries. We use moment maps of the actions of Lie groups, which are not necessarily abelian. By our method, we construct some non-trivial examples in non-flat Calabi-Yau manifolds T^*S^n which equipped with the Stenzel metrics.

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1 Introduction

In 1982, Harvey and Lawson introduced a special class of submanifolds, namely calibrated submanifolds in their paper [4]. Calibrated submanifolds has a strong property that they realize volume minimizing submanifolds in the homological class. Particularly, in Calabi-Yau manifolds M, there are calibrations $\Re e^{\sqrt{-1}\theta}\Omega$ for the holomorphic volume form Ω which is compatible with the Calabi-Yau structure on M and $\theta \in \mathbb{R}$. Submanifolds which are calibrated by $\Re e^{\sqrt{-1}\theta}\Omega$ are called special Lagrangian submanifolds. Because special Lagrangian submanifolds play an important role for understanding mirror symmetries and the SYZ-conjecture, which asserts that for a complex 3-dimensional compact Calabi-Yau manifold M and its mirror $\tilde{M}$, there exist special Lagrangian torus fibrations $\pi : M \to B$ and $\tilde{\pi} : \tilde{M} \to B$, many mathematicians pay attention to their constructions and singularities.

Let us review the history of constructions of special Lagrangian submanifolds, regarding their ambient spaces and methods of constructions. At first $\mathbb{C}^n$ was chosen for an ambient space and in there various examples and methods of constructing special Lagrangian submanifolds were given by Joyce in a series of his papers [10]–[14]. On the other hand, Stenzel gave examples of non-flat Calabi-Yau structures on the conormal bundles over compact rank one symmetric spaces. Next special Lagrangian submanifolds are constructed in those spaces (first in T^*S^n, and recently in $\mathrm{T}^*\mathbb{C}P^n$).

One of the useful method of constructing special Lagrangian submanifolds is called the moment map technique which was introduced by Joyce in [13]. This method needs large symmetries, and by using these symmetries we can reduce PDEs for being special Lagrangian submanifolds to ODEs on the orbit spaces. Using this method, Joyce constructed special Lagrangian submanifolds in $\mathbb{C}^n(\cong \mathrm{T}^*\mathbb{R}^n)$ invariant under a subgroup of $SU(n)$. With this method special Lagrangian submanifolds were also studied in T^*S^n by Anciaux [1], Ionel and Min-Oo [9], Hashimoto and Sakai [6], Hashimoto and Mashimo [5], and in $\mathrm{T}^*\mathbb{C}P^n$ by Arai and Baba [2]. All of these examples were cohomogeneity one.

Another method was introduced by Harvey and Lawson [4] which is called the bundle technique. With the use of this method, Karigiannis and Min-Oo [15] constructed special Lagrangian submanifolds in T^*S^n, and Ionel and Ivey [8] in $\mathrm{T}^*\mathbb{C}P^n$.

Aside from these two typical methods, Joyce [13] showed a way to construct a special Lagrangian submanifold L' in $\mathbb{C}^n$ from another given special Lagrangian submanifold L by using actions of an abelian group which acts perpendicularly to L. This method has advantage that we need not deal with the PDE for L' to be a special Lagrangian submanifold (it is "already achieved" by the given special Lagrangian submanifold L), and that large symmetries are not necessarily needed.

In this paper we generalize this Joyce's result above using "perpendicular symmetries" in three points. Firstly we generalize ambient spaces to general Calabi-Yau manifolds from $\mathbb{C}^n$. Secondly we do not assume the commutativity of Lie groups. Thirdly we generalize the condition that the group acts perpendicularly to a given special Lagrangian submanifold. By this method we also construct non-trivial examples of special Lagrangian submanifolds in Calabi-Yau manifolds T^*S^n equipped with the Stenzel metrics.

The method to construct special Lagrangian submanifolds in this paper is summarized as follows: Let (M, I, ω, Ω) be a connected Calabi-Yau manifold and H a connected Lie group which acts on M preserving I. Here, we denote a complex structure, a Kähler form and a holomorphic volume form on M by I, ω and Ω respectively. Let $\mathfrak{h}$, $\mathfrak{h}^*$ and $Z(\mathfrak{h}^*)$ be the Lie algebra of H, its dual and the center of $\mathfrak{h}^*$ respectively. Assume the H-action is Hamiltonian, i.e. (M, ω, H) has a moment map $\mu : M \to \mathfrak{h}^*$. Let L be a special Lagrangian submanifold of

(M, I, ω, Ω). Suppose that for $c \in Z(\mathfrak{h}^*)$, V_c is a submanifold of M which satisfies $V_c \subset \mu^{-1}(c) \cap L$ and $\dim H + \dim V_c = \frac{1}{2} \dim M$. Assume that the actions of H are "(generalized) perpendicular actions" to L on V_c (not necessarily on whole of L). Then $H \cdot V_c$ is a special Lagrangian submanifold.

Konno [16] showed, in general Calabi-Yau manifolds, a method of constructing Lagrangian mean curvature flows by using perpendicular actions of abelian groups for given special Lagrangian submanifolds, and constructed some examples. This paper is inspired from the study by Konno.

2 Preliminaries

In this section, we review some fundamental facts about Calabi-Yau manifolds, their special Lagrangian submanifolds, group actions, and moment maps.

2.1 Special Lagrangian submanifolds

We begin with the definition of Lagrangian submanifolds in symplectic manifolds.

Let (M, ω) be a symplectic manifold. A submanifold L of (M, ω) is *isotoropic* if $\omega|_L \equiv 0$. If an isotropic submanifold L is of half-dimension of M, it is called a *Lagrangian submanifold*.

Next we see the definition of special Lagrangian submanifolds. It is a particular submanifold of a Calabi-Yau manifold which is defined as follows:

Definition 2.1. A *Calabi-Yau manifold* is a quadruple (M, I, ω, Ω) such that (M, I) is a complex manifold equipped with a Kähler form ω and a holomorphic volume form Ω which satisfy the following relation:

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \overline{\Omega}.$$

If L is an oriented Lagrangian submanifold of a Calabi-Yau manifold (M, I, ω, Ω), there exists a function $\theta : L \to \mathbb{R}/2\pi\mathbb{Z}$, which is called the *Lagrangian angle* satisfying

$$\iota^*\Omega = e^{\sqrt{-1}\theta} \mathrm{vol}_{\iota^* g}.$$

Here g is the Kähler metric, $\iota : L \to M$ is the embedding, and $\mathrm{vol}_{\iota^* g}$ is the volume form on L with respect to the induced metric $\iota^* g$. Even if L is not orientable, we can locally define the Lagrangian angle with the formula above. With the use of the Lagrangian angle θ of a Lagrangian submanifold L, the mean curvature vector $\mathcal{H}_p$ at $p \in L$ is expressed as follows:

$$\mathcal{H}_p = I_{\iota(p)}(\iota_{*p}(\nabla_{\iota^* g}\theta)_p) \in \mathrm{T}^{\perp}_{\iota(p)}\iota(L),$$

where $\nabla_{\iota^* g}\theta$ is the gradient of the function θ with respect to the induced metric $\iota^* g$.

The definition of a special Lagrangian submanifold is given by the following:

Definition 2.2. Let (M, I, ω, Ω) be a Calabi-Yau manifold. A *special Lagrangian submanifold* of (M, I, ω, Ω) is a Lagrangian submanifold such that its Lagrangian angle is constant $\theta \equiv \theta_0$. θ_0 is called the *phase* of the special Lagrangian submanifold.

From the formula of the mean curvature vector above, we can see that a special Lagrangian submanifold is a minimal submanifold. More strongly it is known that a special Lagrangian submanifold is homologically volume minimizing.

2.2 Group actions and moment maps

In this subsection we review the fundamental notions of group actions and moment maps.

Let H be a Lie group which acts on M. We denote the translation of $h \in H$ by $L_h : M \to M$. For each $p \in M$, the orbit and the isotropy subgroup at p are denoted by $H \cdot p$ and H_p respectively.

Letting $\mathfrak{h}$ denote the Lie algebra of H, any $\xi \in \mathfrak{h}$ induces a fundamental vector field $\xi^{\#}$ on M, defined as follows:

$$\xi^{\#}_p = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)p \quad (p \in M),$$

where $\exp(t\xi)$ denotes the 1-parameter subgroup of H associated to ξ.

H acts on $\mathfrak{h}^*$ by the *coadjoint action*:

$$\mathrm{Ad}^*_h : \mathfrak{h}^* \to \mathfrak{h}^*,$$

where $h \in H$, and for $c \in \mathfrak{h}^*$, $\mathrm{Ad}^*_h c$ is defined as follows:

$$\langle \mathrm{Ad}^*_h c, \xi \rangle = \langle c, \mathrm{Ad}_{h^{-1}}\xi \rangle \quad (\xi \in \mathfrak{h}).$$

Here $\langle \cdot, \cdot \rangle$ is the pairing of $\mathfrak{h}$ and $\mathfrak{h}^*$. We call

$$Z(\mathfrak{h}^*) = \{c \in \mathfrak{h}^* \mid \mathrm{Ad}^*_h c = c, h \in H\}$$

the *center* of $\mathfrak{h}^*$. If H is abelian, then it holds that $Z(\mathfrak{h}^*) = \mathfrak{h}^*$.

Definition 2.3. Let H be a Lie group acting on a symplectic manifold (M, ω). A *moment map* $\mu : M \to \mathfrak{h}^*$ is an H-equivariant map that satisfies, for any $\xi \in \mathfrak{h}$, the following:

$$-\mathfrak{i}(\xi^\#)\omega = d\langle \mu(\cdot), \xi \rangle,$$

where $\mathfrak{i}$ is the interior product.

If (M, ω, H) has a moment map, the H-action is called *Hamiltonian.* A Hamiltonian action preserves the symplectic form ω.

Proposition 2.4. *Let (M, ω) be a symplectic manifold, H a Lie group with a moment map $\mu : M \to \mathfrak{h}^*$, p a point in M. If there exists a point $q \in H \cdot p$ such that $\mu(q)$ is in $Z(\mathfrak{h}^*)$, then μ is constant on $H \cdot p$ and the H-orbit $H \cdot p$ is isotropic. Conversely, if the H-orbit $H \cdot p$ is connected and isotropic, then μ is constant on $H \cdot p$ and $\mu(p)$ is in $Z(\mathfrak{h}^*)$.*

Proof. First we assume that $\mu(q) \in Z(\mathfrak{h}^*)$ for $q \in H \cdot p$. Let r be an arbitrary point in $H \cdot p$ and $h \in H$ such that $r = hq$. Since $H \cdot p$ is homogeneous, it holds that $\mathrm{T}_r(H \cdot p) = \{\xi_r^\# \mid \xi \in \mathfrak{h}\}$. For any $\xi_1, \xi_2 \in \mathfrak{h}$, we have

$$\begin{aligned}
\omega_r((\xi_1)_r^\#, (\xi_2)_r^\#) =& \langle (d\mu)_r (\xi_1)_r^\#, \xi_2 \rangle \\
=& \left\langle \left. \frac{d}{dt} \right|_{t=0} \mu(\exp(t\xi_1)r), \xi_2 \right\rangle \\
=& \left\langle \left. \frac{d}{dt} \right|_{t=0} \mu(\exp(t\xi_1)hq), \xi_2 \right\rangle \\
=& \left\langle \left. \frac{d}{dt} \right|_{t=0} \left(\mathrm{Ad}^*_{\exp(t\xi_1)h}\right)\mu(q), \xi_2 \right\rangle \\
=& 0.
\end{aligned}$$

Hence, we see that $H \cdot p$ is isotropic. The map μ is constant on $H \cdot p$ since for any $r' = h'q \, (h' \in H)$ it holds that $\mu(r') = \mathrm{Ad}^*_{h'}\mu(q) = \mu(q)$.

Next we assume that $H \cdot p$ is connected and isotropic. For any $\xi \in \mathfrak{h}$, define $\mu_\xi : M \to \mathbb{R}$ by

$$\mu_\xi(p) := \langle \mu(p), \xi \rangle.$$

Then we have

$$d\mu_\xi = d\langle \mu(\cdot), \xi \rangle = -\omega(\xi^\#, \cdot).$$

Fix an arbitrary $q \in H \cdot p$ and let $q = hp$ for $h \in H$. For any $Y \in \mathrm{T}_q(H \cdot p)$, we have

$$Y(\mu_\xi) = (d\mu_\xi)_q(Y) = -\omega_q(\xi_q^\#, Y) = 0.$$

Hence, noting that $H \cdot p$ is connected, we see that μ_ξ is constant on $H \cdot p$ for any $\xi \in \mathfrak{h}$, i.e., $\langle \mu(\cdot), \xi \rangle$ is constant on $H \cdot p$ for any $\xi \in \mathfrak{h}$. Therefore, we see that μ is constant on $H \cdot p$, i.e., $H \cdot p \subset \mu^{-1}(\mu(p))$. Then since it holds that

$$\mathrm{Ad}_h^* \mu(p) = \mu(hp) = \mu(p) \quad (\forall h \in H),$$

we see that $\mu(p) \in Z(\mathfrak{h}^*)$. □

3 The Stenzel metrics

In this section we overview the method by Stenzel [23] for constructing Ricci-flat Kähler metrics on the cotangent bundles of compact rank one symmetric spaces, using the cohomogeneity one group actions. We also construct practically the Stenzel metrics on T^*S^n and $\mathrm{T}^*\mathbb{C}P^n$. Particularly, the Stenzel metrics g_{Stz} on T^*S^n are used later for constructing special Lagrangian submanifolds in the Calabi-Yau manifolds T^*S^n which are equipped with them.

3.1 General constructions of Ricci-flat Kähler metrics by cohomogeneity one actions

Generally, for a Kähler potential ψ on a complex manifold (M, I), its Ricci form $\mathbf{Ric}(\psi)$ is given by the following:

$$\mathbf{Ric}(\psi) = -\sqrt{-1}\partial\bar{\partial} \log \det \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j}.$$

Here, $(z_1, \cdots, z_n)$ is an arbitrary holomorphic coordinates with respect to the complex structure I and $n = \dim_{\mathbb{C}} M$. Therefore, $\mathbf{Ric}(\psi) = 0$, the condition for the Ricci-flatness is given as a fourth order partial differential equation.

Suppose that the determinant of the Hessian of ψ has the form of "a positive constant × the square of the absolute value of some holomorphic function". That is, for some $C > 0$ and some holomorphic function hol, suppose it holds that

$$\det \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} = C|\mathrm{hol}|^2. \qquad (3,1)$$

Generally, for the product $f_1 \cdots f_k$ of finite numbers of holomorphic or anti-holomorphic functions $f_i \, (i = 1, \cdots, k)$, we have $\partial\bar{\partial} \log f_1 \cdots f_k = 0$. Hence, we see that such ψ satisfies $\mathbf{Ric}(\psi) = 0$. Therefore, the second order partial differential equation (3,1) gives us one of the classes that satisfy the condition of the Ricci-flatness.

The Ricci-flat Kähler metrics which were constructed by Stenzel [23] are given as the solutions for the partial differential equations (3,1). Stenzel showed that the second order partial differential equations (3,1) can be reduced to the second order ordinary differential equations with the use of the symmetries of cohomogeneity one actions which the compact rank one symmetric spaces have. The compact rank one symmetric spaces are classified as follows:

G/K	G	K	$\dim_{\mathbb{R}} G/K$
$S^n\ (n \geq 2)$	$SO(n+1)$	$SO(n)$	n
$\mathbb{R}P^n\ (n \geq 2)$	$SO(n+1)$	$O(n)$	n
$\mathbb{C}P^n\ (n \geq 1)$	$SU(n+1)$	$S(U(1) \times U(n))$	$2n$
$\mathbb{H}P^n\ (n \geq 1)$	$Sp(n+1)$	$Sp(1) \times Sp(n)$	$4n$
CaP^2	F_4	$Spin(9)$	16

It is known that in the case of $M = \mathrm{T}^*S^2$, the Stenzel metric coincides with the hyperkähler metric on T^*S^2 discovered by Eguchi and Hanson [3]. Lee [17] explicitly described the Stenzel metrics on each cotangent bundle of compact rank one symmetric spaces except for the case of $G/K = \mathrm{Ca}P^2$.

The principle of the constructions by Stenzel is based on the following theorem.

Theorem 3.1. *Let (M, I) be a complex manifold with the complex dimension n which satisfies the following conditions:*

(i) there exists a Lie group G which acts on M with cohomogeneity one preserving I,

(ii) there exists a G-invariant, nonvanishing, holomorphic volume form Ω on M, and

(iii) there exists a G-invariant, strictly plurisubharmonic function $\rho : M \to [0, \infty)$ such that the induced function $\rho : M/G \to [0, \infty)$ on the G-orbit space M/G is injective.

Let $\Sigma = \rho(M)$. Then, the followings hold:

(1) there exist G-invariant functions ν_1 and ν_2 on M given by the followings:

$$\nu_1 = \partial\bar{\partial}\rho(\mathrm{grad}_{\mathbb{C}}\rho, \overline{\mathrm{grad}_{\mathbb{C}}\rho}),$$
$$|\mathrm{hol}|^2\nu_2 = \det \frac{\partial^2}{\partial z_i \partial \bar{z}_j}\rho.$$

Here, $\mathrm{grad}_{\mathbb{C}}\rho$ is the complex gradient of ρ which is defined by the following:

$$\partial\bar{\partial}\rho(\mathrm{grad}_{\mathbb{C}}\rho, \cdot) = \bar{\partial}\rho(\cdot).$$

The function hol *is a nonvanishing local holomorphic function on holomorphic coordinates $(z_1, \cdots, z_n)$ of M. The function ν_1 is non-negative valued. The function ν_2 is positive valued and is determined up to the product of the square of the absolute value of some local holomorphic function on M which is G-invariant. In addition, for an arbitrary real valued function f which is defined on some open set $\tilde{\Sigma} \subset \mathbb{R}$ such that $\Sigma \subset \tilde{\Sigma}$, the following holds on M:*

$$\det \frac{\partial^2(f\circ\rho)}{\partial z^i \partial \bar{z}^j} = |\mathrm{hol}|^2 \Big\{(f'\circ\rho)^n + (f'\circ\rho)^{n-1}(f''\circ\rho)(\nu_1\circ\rho)\Big\}(\nu_2\circ\rho). \quad (3,2)$$

(2) Let f be the solution of the ordinary differential equation with variable ρ

$$\Big\{(f'(\rho))^n + (f'(\rho))^{n-1}f''(\rho)\nu_1(\rho)\Big\}\nu_2(\rho) = C \quad (C > 0) \qquad (3,3)$$

which is smoothly defined on some open set $\tilde{\Sigma} \subset \mathbb{R}$ such that $\Sigma \subset \tilde{\Sigma}$. Then, the function $\psi = f \circ \rho$ is a Ricci-flat Kähler potential on M if and only if $0 < f'$ on Σ.

The existences of G-actions and a strictly plurisubharmonic function ρ which satisfy the conditions above are crucial important for this method. For the latter, Patrizio and Wong [22] studied such functions on the cotangent bundles of compact rank one symmetric spaces in detail.

We prepare the following lemma for the proof of this theorem.

Lemma 3.2. *Let (M, I) be a complex manifold, Ω a nonvanishing holomorphic volume form on M, and $\psi : M \to \mathbb{R}$ a strictly plurisubharmonic function. Then, there exists a positive valued function $F_\psi : M \to (0, \infty)$ such that*

$$(\sqrt{-1}\partial\bar{\partial}\psi)^n = (\sqrt{-1})^{n^2} F_\psi \Omega \wedge \overline{\Omega}.$$

In addition, F_ψ satisfies the following relation for some holomorphic function hol*:*

$$\det \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} = |\mathrm{hol}|^2 F_\psi. \tag{3,4}$$

Proof of Lemma 3.2. By direct calculations, we have

$$(\sqrt{-1}\partial\bar{\partial}\psi)^n = (\sqrt{-1})^{n^2} n! \det \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

On the other hand, since there exists a local holomorphic function $\widetilde{\mathrm{hol}}$ such that $\Omega = \widetilde{\mathrm{hol}}\, dz_1 \wedge \cdots \wedge dz_n$, we have

$$\Omega \wedge \overline{\Omega} = |\widetilde{\mathrm{hol}}|^2 dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

Hence, we have

$$(\sqrt{-1}\partial\bar{\partial}\psi)^n = (\sqrt{-1})^{n^2} n! |\widetilde{\mathrm{hol}}|^{-2} \det \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} \Omega \wedge \overline{\Omega}.$$

Note that the function $\widetilde{\mathrm{hol}}$ is nonvanishing because Ω is nonvanishing and that $\det \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} > 0$ because ψ is strictly plurisubharmonic. Let $F_\psi := n! |\widetilde{\mathrm{hol}}|^{-2} \det \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j}$, then F_ψ is the function which satisfies the claim of the Lemma 3.2. In fact, we have

$$0 < \det \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} = |\mathrm{hol}|^2 F_\psi$$

with $\mathrm{hol} = (n!)^{-1/2} \widetilde{\mathrm{hol}}$. □

Proof of Theorem 3.1. On an arbitrary holomorphic coordinates $(z_1, \cdots, z_n)$, it holds that

$$\mathrm{grad}_{\mathbb{C}} \rho = \rho^i \frac{\partial}{\partial z_i}.$$

Here, $\rho^i = \sum_{k=1}^n \frac{\partial \rho}{\partial \bar{z}_k} \rho^{k\bar{i}}$ and $\sum_{i=1}^n \rho^{k\bar{i}} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_l} = \delta_{kl}$ with the Kronecker delta δ_{kl}.

First, we show that there exist the functions ν_1 and ν_2 in the theorem. Since G preserves I, and ρ is G-invariant, the function $\partial\bar{\partial}\rho(\mathrm{grad}_{\mathbb{C}}\rho, \overline{\mathrm{grad}_{\mathbb{C}}\rho})$ is G-invariant on M. Since ρ is a strictly plurisubharmonic function and $\mathrm{grad}_{\mathbb{C}}\rho$ is a $(1,0)$-differential form, this function is non-negative valued. Thus we can define the G-invariant non-negative valued function $\nu_1 = \partial\bar{\partial}\rho(\mathrm{grad}_{\mathbb{C}}\rho, \overline{\mathrm{grad}_{\mathbb{C}}\rho}) : M \to [0, \infty)$.

Since the positive valued function F_ρ which is obtained by Lemma 3.2 satisfies the relation

$$(\sqrt{-1}\partial\bar{\partial}\rho)^n = (\sqrt{-1})^{n^2} F_\rho \Omega \wedge \overline{\Omega}$$

and ρ and Ω are G-invariant, the function F_ρ is also G-invariant. We define $\nu_2 := F_\rho$. Then, from the relation (3,4) in Lemma 3.2, it holds that

$$\det \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} = |\mathrm{hol}|^2 \nu_2.$$

Note that F_ρ is determined depending on Ω. Let $\tilde{\Omega}$ be another G-invariant, nonvanishing, holomorphic volume form on M. Let $\tilde{F}_\rho$ be the positive valued function which corresponds to $\tilde{\Omega}$ in Lemma 3.2. Then, there exists some holomorphic function $\widetilde{\mathrm{hol}}$ such that $\tilde{\Omega} = \widetilde{\mathrm{hol}}\Omega$. Since Ω and $\tilde{\Omega}$ are G-invariant and nonvanishing, $\widetilde{\mathrm{hol}}$ is also G-invariant and nonvanishing. We have

$$\begin{aligned}(\sqrt{-1})^{n^2} F_\rho \Omega \wedge \overline{\Omega} = (\sqrt{-1}\partial\bar{\partial}\rho)^n =& (\sqrt{-1})^{n^2} \tilde{F}_\rho \tilde{\Omega} \wedge \overline{\tilde{\Omega}} \\ =& (\sqrt{-1})^{n^2} \tilde{F}_\rho \widetilde{\mathrm{hol}}\Omega \wedge \overline{\widetilde{\mathrm{hol}}\Omega} \\ =& (\sqrt{-1})^{n^2} |\widetilde{\mathrm{hol}}|^2 \tilde{F}_\rho \Omega \wedge \overline{\Omega}.\end{aligned}$$

Thus we see that

$$F_\rho = |\widetilde{\mathrm{hol}}|^2 \tilde{F}_\rho$$

and $|\widetilde{\mathrm{hol}}|^2$ is G-invariant positive valued function. Hence we have verified that ν_2 in the theorem exists.

Since ρ is injective on the G-orbit space, ν_1 and ν_2 can be seen as a ρ-variable functions: $\nu_i = \nu_i(\rho)$ for $i = 1, 2$. Let $f : \tilde{\Sigma} \to \mathbb{R}$ be a smooth function which is defined on some open set $\tilde{\Sigma} \subset \mathbb{R}$ such that $\Sigma \subset \tilde{\Sigma}$. Then, by direct calculations, we have

$$\begin{aligned}&\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \det(f \circ \rho) \\ =& \left\{ (f' \circ \rho)^n + (f' \circ \rho)^{n-1} (f'' \circ \rho) \partial\bar{\partial}\rho(\mathrm{grad}_{\mathbb{C}}\rho, \overline{\mathrm{grad}_{\mathbb{C}}\rho}) \right\} \det \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} \\ =& |\mathrm{hol}|^2 \left\{ (f' \circ \rho)^n + (f' \circ \rho)^{n-1} (f'' \circ \rho)(\nu_1 \circ \rho) \right\} (\nu_2 \circ \rho)\end{aligned}$$

on M. Thus we have shown (1) of the theorem.

Next we show the claim of (2). Since the two functions

$$\rho : M \to \Sigma,$$
$$f : \tilde{\Sigma} \to \mathbb{R}$$

are smooth, $\psi = f \circ \rho : M \to \mathbb{R}$ is also smooth. Hence, it is sufficient for verifying the claim of (2) to confirm the followings:

(I) The function ψ is strictly plurisubharmonic on M,

(II) $\mathbf{Ric}(\psi) \equiv \mathbf{0}$ on M.

First we show the condition (I). The condition (I) means that $\sqrt{-1}\partial\bar{\partial}\psi(\cdot, I\cdot)$ is positive definite on T_pM for each $p \in M$, and it is equivalent to that $\partial\bar{\partial}\psi(\cdot, \bar{\cdot})$ is positive definite on $\mathrm{T}_p^{(1,0)}M$ for each $p \in M$. Here, $\mathrm{T}_p^{(1,0)}M$ is the space of all $(1,0)$-vectors in $\mathrm{T}_pM^{\mathbb{C}}$.

Define the complex $(n-1)$-dimensional vector space $\mathrm{ann}(\partial\rho)_p$ for each $p \in M$ by the following:

$$\mathrm{ann}(\partial\rho)_p = \left\{ v \in \mathrm{T}_p^{(1,0)}M \mid \partial\rho(v) = 0 \right\}.$$

We take a basis $(Z_1, \cdots, Z_n)$ in $\mathrm{T}_p^{(1,0)}M$ such that

- $(Z_1, \cdots, Z_n)$ is orthonormal with respect to the Hermitian inner product $(\partial\bar{\partial}\rho)_p$, and
- $(Z_1, \cdots, Z_{n-1})$ is a basis in $\mathrm{ann}(\partial\rho)_p$.

Here, we note that $\partial\bar{\partial}\rho$ is positive definite since ρ is strictly plurisubharmonic. Since it holds that

$$\partial\bar{\partial}(f \circ \rho) = f''\partial\rho \wedge \bar{\partial}\rho + f'\partial\bar{\partial}\rho,$$

we have, for $j, k \in \{1, \cdots, n-1\}$,

$$\partial\bar{\partial}\psi(Z_j, \bar{Z}_k) = f'\partial\bar{\partial}\rho(Z_j, \bar{Z}_k) = f'\delta_{jk},$$
$$\partial\bar{\partial}\psi(Z_j, \bar{Z}_n) = f'\partial\bar{\partial}\rho(Z_j, \bar{Z}_n) = 0.$$

Hence, we have the expression of the quadratic form $\partial\bar{\partial}\psi(\cdot, \bar{\cdot})$ with respect to $(Z_1, \cdots, Z_n)$ as follows:

$$\partial\bar{\partial}\psi(\cdot, \bar{\cdot}) = \left[\begin{array}{ccc|c} f' & & & \\ & \ddots & & 0 \\ & & f' & \\ \hline & 0 & & \alpha_n \end{array}\right] \quad (\alpha_n \in \mathbb{R}).$$

If $f' > 0$ on Σ, we see that $\alpha_n > 0$ since it holds that

$$(f')^{n-1}\alpha_n = \det \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} = C|\mathrm{hol}|^2 > 0$$

by the ordinary differential equation (3,3) in the theorem. Hence, we see that each of n eigenvalues of $\partial\bar{\partial}\psi(\cdot,\bar{\cdot})$ is positive, i.e., $\partial\bar{\partial}\psi(\cdot,\bar{\cdot})$ is positive definite if and only if $f' > 0$ on Σ. Thus we have shown the condition (I).

The condition (II) clearly holds since

$$\mathbf{Ric}(\psi) = -\sqrt{-1}\partial\bar{\partial} \log \det \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j}$$

and

$$\det \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} = C|\mathrm{hol}|^2$$

due to the ordinary differential equation (3,3) and the relation (3,2) in the theorem. □

Remark 3.3. When $\rho(M) \subset [1, \infty)$ (and it is always possible to consider $\tilde{\rho} := \rho + c$ with some $c > 0$ instead of ρ), by the variable change $u = \cosh^{-1} \rho$, the ordinary differential equation (3,3) with variable ρ

$$\left\{ \left(f'(\rho)\right)^n + \left(f'(\rho)\right)^{n-1} f''(\rho)\nu_1(\rho) \right\} \nu_2(\rho) = C$$

is deformed into the equation with variable u

$$\frac{\frac{d}{du}(\dot{F}(u))^n}{\cosh u \sinh^{n-1} u}(\nu_2 \circ \cosh)(u) = nC.$$

Here, $\dot{F}$ denotes the differentiation of F by u. In particular, if $f' > 0$ and $u > 0$, we have $\dot{F} > 0$ by $\dot{F} = f' \sinh u$. From the ordinary differential equation with respect to F, we also have

$$\frac{\left(\dot{F}(u)\right)^{n-1} \ddot{F}(u)}{\cosh u \sinh^{n-1} u}(\nu_2 \circ \cosh)(u) = C.$$

Hence, $\ddot{F} > 0$ if $f' > 0$ and $u > 0$. This is used when we construct special Lagrangian submanifolds later.

The following lemma gives us a way to calculate $f'(\rho)$.

Lemma 3.4. *Let ν_1, ν_2 be the functions in Theorem 3.1 (1). Let $\lambda : \Sigma_1 \to \mathbb{R}$ be the solution defined on some open set $\Sigma_1 \subset \mathbb{R}$ of the following ordinary differential equation with variable ρ:*

$$\{\lambda(\rho)\nu_1(\rho)\nu_2(\rho)\}' = n\lambda(\rho)\nu_2(\rho).$$

Define the function Λ by

$$\Lambda(\rho) = \frac{nC\displaystyle\int_c^\rho \lambda(s)ds}{\lambda(\rho)\nu_1(\rho)\nu_2(\rho)},$$

where $C > 0$ is the constant in the ordinary differential equation (3,3) in the Theorem 3.1 and $c \in \Sigma_1$. Then if Λ is defined on an open set $\Sigma_2 \subset \mathbb{R}$, it holds that

$$(f'(\rho))^n = \Lambda(\rho)$$

on $\Sigma_1 \cap \Sigma_2$.

Proof of Lemma 3.4. We have

$$\begin{aligned} & n\lambda\{\nu_2(f')^n + \nu_1\nu_2(f')^{n-1}f''\} = n\lambda\nu_2(f')^n + n\lambda\nu_1\nu_2(f')^{n-1}f'' \\ =& \left\{\frac{d}{d\rho}(\lambda\nu_1\nu_2)\right\}(f')^n + n\lambda\nu_1\nu_2(f')^{n-1}f'' = \frac{d}{d\rho}\{(\lambda\nu_1\nu_2)(f')^n\} \end{aligned}$$

on Σ_1. Hence, the ordinary differential equation (3,3) in Theorem 3.1 is deformed into the following:

$$nC\lambda = \{(\lambda\nu_1\nu_2)(f')^n\}'.$$

Integrating the both sides with the initial condition zero, we have

$$nC\int_c^\rho \lambda ds = \lambda\nu_1\nu_2(f')^n.$$

Hence, we have

$$(f'(\rho))^n = \frac{nC\displaystyle\int_c^\rho \lambda ds}{\lambda\nu_1\nu_2}$$

on $\Sigma_1 \cap \Sigma_2$. $\square$

3.2 The Stenzel metrics on the cotangent bundles of compact rank one symmetric spaces

Stenzel [23] showed that the Theorem 3.1 can be applied to the compact rank one symmetric spaces T^*G/K as follows.

By Helgarson [7], the following holds:

Lemma 3.5. *A compact Riemannian manifold X is a rank one symmetric space if and only if its linear isotropy subgroup at a point $p \in X$ acts on the unit sphere of $\mathrm{T}^*_p X$ transitively.*

This indicates that G acts on the cotangent bundle of a compact rank one symmetric space G/K with cohomogeneity one. Hence, T^*G/K satisfies the condition (i) of Theorem 3.1.

Generally, it is known that for a compact connected Lie group G, there exists a unique complex connected Lie group $G_{\mathbb{C}}$ such that its Lie algebra $\mathfrak{g}_{\mathbb{C}}$ coincides the complexification of the real Lie algebra $\mathfrak{g}$ of G and that G is a maximal compact subgroup of $G_{\mathbb{C}}$. Similarly, we can consider $K_{\mathbb{C}}$ for any closed subgroup K of G. Then, $K_{\mathbb{C}}$ is isomorphic to some complex closed subgroup of $G_{\mathbb{C}}$. Hence, we can consider $G_{\mathbb{C}}/K_{\mathbb{C}}$. By Matsushima [18], Morimoto and Nagano [19], and Nagano [21], if G is a semisimple Lie group additionally, then the following holds:

Lemma 3.6. *Let G be a compact connected semisimple Lie group, K a closed subgroup of G. Then, the complex manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$ is G-equivariantly diffeomorphic to T^*G/K. In addition, $G_{\mathbb{C}}/K_{\mathbb{C}}$ is a Stein manifold, that is, for a sufficiently large number $N \in \mathbb{N}$, the complex manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$ is embedded into $\mathbb{C}^N$ as a complex manifold.*

By this lemma, T^*G/K has the canonical complex structure derived from its corresponding Stein structure $(M = G_{\mathbb{C}}/K_{\mathbb{C}}, I)$.

By Stenzel [23], the following holds:

Lemma 3.7. *Let G be a compact semisimple Lie group, K a connected closed subgroup, $M := G_{\mathbb{C}}/K_{\mathbb{C}}$. Then, there exists a $G_{\mathbb{C}}$-invariant nonvanishing holomorphic volume form Ω on M.*

When $G/K = \mathbb{R}P^n$, since $K = O(n)$ is disconnected, this lemma does not hold. However, it does not matter because the Stenzel metrics on $\mathrm{T}^*\mathbb{R}P^n$ are constructed from the ones on T^*S^n. By this lemma, we see that the condition (ii) of Theorem 3.1 holds in T^*G/K.

According to Theorem 2.1 in [22] the following holds:

Lemma 3.8. *Let (M, I) be a complex n-dimensional Stein manifold which corresponds to the cotangent bundle of a compact rank one symmetric space T^*G/K, B the submanifold in M which corresponds to the zero section of T^*G/K. Then there exists a real analytic, strictly plurisubharmonic exhaustion $\rho : M \to [1, \infty)$ which has the following properties:*

(1) $\rho(p) = 1 \Leftrightarrow p \in B$,

(2) the variable change $u = \cosh^{-1} \rho$ satisfies the homogeneous Monge-Ampère equation $(\partial\bar{\partial}u)^n = 0$ on $M \backslash B$.

Since Patrizio and Wong [22] explicitly exhibited ρ for each G/K, we can consider whether this ρ is G-invariant and injective on the G-orbit space. We show that these conditions hold in each cases of T^*S^n and $\mathrm{T}^*\mathbb{C}P^n$.

3.3 The Stenzel metrics on T^*S^n and $\mathrm{T}^*\mathbb{C}P^n$

We identify the tangent bundle and the cotangent bundle of the n-sphere S^n by the canonical Riemannian metric of S^n, and describe it by

$$\mathrm{T}^*S^n = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1, x \centerdot \xi = 0\},$$

where " $\centerdot$ " is the canonical real inner product on the Euclidean space $\mathbb{R}^{n+1}$ and $\|x\| = \sqrt{x \centerdot x}$ for each $x \in \mathbb{R}^{n+1}$. We occasionally denote ${}^t(x_1, \cdots, x_{n+1})$, ${}^t(\xi_1, \cdots, \xi_{n+1})$ by x, ξ respectively. $SO(n+1)$ acts on T^*S^n by $h \cdot (x, \xi) = (hx, h\xi)$ for $h \in SO(n+1)$ with cohomogeneity one. The principal orbit at a point (x, ξ) equals a sphere bundle with a radius of $\|\xi\|$.

Let Q^n be a complex quadric hypersurface in $\mathbb{C}^{n+1}$ as follows:

$$Q^n = \left\{ z = {}^t(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} z_i^2 = 1 \right\}.$$

Szöke [24] gave an $SO(n+1)$-equivariant diffeomorphism $\Phi : \mathrm{T}^*S^n \to Q^n$ defined by:

$$\Phi(x, \xi) = \cosh(\|\xi\|)x + \sqrt{-1}\frac{\sinh(\|\xi\|)}{\|\xi\|}\xi.$$

We can induce a complex structure to Q^n from $\mathbb{C}^{n+1}$.

By [22], the following holds:

Lemma 3.9. *In the Stein manifold (Q^n, I), corresponding to T^*S^n, the function $\rho : Q^n \to [1, \infty)$ defined by*

$$\rho(z_1, \cdots, z_{n+1}) = \sum_{i=1}^{n+1} |z_i|^2$$

is a strictly plurisubharmonic function which satisfies the conditions of Lemma 3.8.

Note that $\Sigma = \rho(M) = [1, \infty)$.

By Lemma 3.5, we can directly verify that this ρ is G-invariant and injective on the G-orbit space. Therefore, (Q^n, I) satisfies the conditions (i), (ii), and (iii) of Theorem 3.1.

We can construct a Ricci-flat Kähler metric by applying Theorem 3.1 to (Q^n, I). Define

$$U_{n+1} = \{(z_1, \cdots, z_{n+1}) \in Q^n \mid z_{n+1} \neq 0\},$$

and take the following holomorphic coordinates

$$U_{n+1} \ni (z_1, \cdots, z_{n+1}) \mapsto (\frac{z_1}{z_{n+1}}, \cdots, \frac{z_n}{z_{n+1}}) =: (w_1, \cdots, w_n) \in \mathbb{C}^n.$$

Then, by direct calculations, we have

$$\det \frac{\partial^2 \rho}{\partial w_i \partial \bar{w}_j} = \frac{1}{|z_{n+1}|^2} \rho.$$

Hence, we define $\nu_2(\rho) = \rho$.

Next we consider about $\partial\bar{\partial}\rho(\mathrm{grad}_{\mathbb{C}}\rho, \overline{\mathrm{grad}_{\mathbb{C}}\rho})$. For this, the following lemma by Patrizio and Wong [22] is useful:

Lemma 3.10. *Let (M, I) be a complex manifold, τ a real valued function with one variable, ρ the real valued function on M. Then, $\tau \circ \rho$ satisfies the homogeneous Monge-Ampère equation if and only if the following equation holds:*

$$\partial\bar{\partial}\rho(\mathrm{grad}_{\mathbb{C}}\rho, \overline{\mathrm{grad}_{\mathbb{C}}\rho}) = -\frac{\tau'(\rho)}{\tau''(\rho)}.$$

By this lemma and Lemma 3.8, we have

$$\partial\bar{\partial}\rho(\mathrm{grad}_{\mathbb{C}}\rho, \overline{\mathrm{grad}_{\mathbb{C}}\rho}) = \frac{\rho^2 - 1}{\rho}$$

on $\Sigma^\circ = (1, \infty)$. Since $\partial\bar{\partial}\rho(\mathrm{grad}_{\mathbb{C}}\rho, \overline{\mathrm{grad}_{\mathbb{C}}\rho})$ is a smooth G-invariant function on M, it is differentiable with the variable ρ at $\rho = 1$ from right. Hence, it is smoothly extendable to an open set $\tilde{\Sigma}$ such that $[1, \infty) \subset \tilde{\Sigma}$. In particular, we have $\partial\bar{\partial}\rho(\mathrm{grad}_{\mathbb{C}}, \overline{\mathrm{grad}_{\mathbb{C}}\rho}) = (\rho^2 - 1)/\rho$ on $\Sigma = [1, \infty)$.

Applying ν_1, ν_2 which we obtained above to Theorem 3.1, we have

$$\det \frac{\partial^2}{\partial w_i \partial \bar{w}_j}(f \circ \rho) = \frac{1}{|\, z_{n+1} \,|^2} \Big\{ \rho(f' \circ \rho)^n + (f' \circ \rho)^{n-1}(f'' \circ \rho)(\rho^2 - 1) \Big\}.$$

The ordinary differential equation in Lemma 3.4 is given by the following:

$$\frac{d}{d\rho}\{\lambda(\rho^2 - 1)\} = n\lambda\rho.$$

Solving this equation with the initial condition zero, we have

$$\lambda(\rho) = (\rho^2 - 1)^{\frac{n-2}{2}}.$$

Integrating the both sides of Lemma 3.4 with the initial condition zero, we have

$$f'(\rho) = (nC)^{\frac{1}{n}} \left\{ \frac{\int_1^\rho (s^2 - 1)^{\frac{n-2}{2}} ds}{(\rho^2 - 1)^{\frac{n}{2}}} \right\}^{\frac{1}{n}}.$$

Let $\mathcal{F}_1 := \int_1^\rho (s^2 - 1)^{(n-2)/2} ds$ and $\mathcal{F}_2 := (\rho^2 - 1)^{n/2}$. Then we have

$$\frac{d}{d\rho}\mathcal{F}_1 = (\rho^2 - 1)^{\frac{n-2}{2}}, \quad \frac{d}{d\rho}\mathcal{F}_2 = n\rho(\rho^2 - 1)^{\frac{n-2}{2}}.$$

Hence, by the l'Hôpital's rule, we have $f'(\rho) \to C^{1/n} > 0$ $(\rho \to 1)$. Thus we see that $f'(\rho) > 0$ on $[1, \infty)$.

Consequently, we have the following result:

Proposition 3.11. *In (Q^n, I), corresponding to $\mathrm{T}^* S^n$, for the solution f of the following ordinary differential equation, $\psi = f \circ \rho$ is a Ricci-flat Kähler potential on $G \cdot \Sigma \cong Q^n$:*

$$\rho(f'(\rho))^n + (f'(\rho))^{n-1} f''(\rho)(\rho^2 - 1) = C > 0. \tag{3,5}$$

Next we construct the Stenzel metrics on $\mathrm{T}^*\mathbb{C}P^n$.

Firstly, based on [22], we show a Stein manifold which corresponds to $\mathrm{T}^*\mathbb{C}P^n$.

$\mathbb{C}P^n$ is embedded into $\mathbb{C}P^n \times \mathbb{C}P^n$ as follows: For $z \in \mathbb{C}^{n+1}$, define $[z] := \{\alpha z \mid \alpha \in \mathbb{C}\}$. Then, the embedding is

$$[z] \mapsto ([z], [\bar{z}]).$$

The image of $\mathbb{C}P^n$ by this embedding is the following fixed point set to the involution $([z], [w]) \mapsto ([\bar{w}], [\bar{z}])$ in $\mathbb{C}P^n \times \mathbb{C}P^n$:

$$\{([z], [w]) \in \mathbb{C}P^n \times \mathbb{C}P^n \mid [z] = [\bar{w}]\}.$$

$\mathbb{C}P^n \times \mathbb{C}P^n$ is embedded into $\mathbb{C}P^N (N = (n+1)^2 - 1)$ by the map $\mathcal{S}$ called the *Segre embedding* as follows: We denote $([z], [w]) \in \mathbb{C}P^n \times \mathbb{C}P^n$ by the homogeneous coordinates $((z_0 : \cdots : z_n), (w_0 : \cdots : w_n))$. Then $\mathcal{S}$ is defined by

$$\mathcal{S}([z], [w]) = (\zeta_{\alpha\beta}).$$

Here, $\zeta_{\alpha\beta} = z_\alpha w_\beta (0 \leq \alpha, \beta \leq n)$ and $(\zeta_{\alpha\beta})$ are the homogeneous coordinates in $\mathbb{C}P^N$. Then, it holds that

$$\mathcal{S}(\mathbb{C}P^n \times \mathbb{C}P^n) = \{\zeta \in \mathbb{C}P^N \mid \zeta_{ij}\zeta_{kl} - \zeta_{il}\zeta_{kj} = 0,\ i, j, k, l = 0, 1, \cdots, n\}.$$

Define the hyperplane $\mathbb{C}P^{N-1}_\infty$ in $\mathbb{C}P^N$ by the following:

$$\mathbb{C}P^{N-1}_\infty = \left\{ \zeta \in \mathbb{C}P^N \;\middle|\; \sum_{\alpha=0}^{n} \zeta_{\alpha\alpha} = 0 \right\}.$$

$\mathbb{C}P^N - \mathbb{C}P^{N-1}_\infty$ is isomorphic to $\mathbb{C}^N$ as a complex manifold. By [22], the following holds:

Lemma 3.12. *$\mathcal{S}(\mathbb{C}P^n \times \mathbb{C}P^n) - \mathbb{C}P^{N-1}_\infty$ is a Stein manifold which corresponds to $\mathrm{T}^*\mathbb{C}P^n$.*

Let $M^{\mathrm{II}}_{2n} := \mathcal{S}(\mathbb{C}P^n \times \mathbb{C}P^n) - \mathbb{C}P^{N-1}_\infty$. Then, we have the following:

Lemma 3.13. *Define the function $\mathcal{N} : M^{\mathrm{II}}_{2n} \to [1, \infty)$ by the following:*

$$\mathcal{N}(\zeta) = \frac{\displaystyle\sum_{0 \leq \alpha, \beta \leq n} |\zeta_{\alpha\beta}|^2}{\left| \displaystyle\sum_{0 \leq \alpha \leq n} \zeta_{\alpha\alpha} \right|^2}.$$

Then, the function

$$\rho = 2\mathcal{N} - 1 : M_{2n}^{\mathrm{II}} \to [1, \infty)$$

is a strictly plurisubharmonic function which satisfies the conditions of Lemma 3.8.

We show that ρ is $G = SU(n+1)$-invariant and injective on the G-orbit space. First, we define an action of $SL(n+1,\mathbb{C})$ on $\mathbb{C}P^n \times \mathbb{C}P^n - \mathcal{S}^{-1}(\mathbb{C}P^{N-1}_\infty)$ by

$$g \cdot ([z],[w]) = ([gz], [{}^t g^{-1} w]),$$

where $g \in SL(n+1,\mathbb{C})$, and $([z],[w]) \in \mathbb{C}P^n \times \mathbb{C}P^n - S^{-1}(\mathbb{C}P^{N-1}_\infty)$.

On the other hand, $\mathcal{N}$ is described on $\mathbb{C}P^n \times \mathbb{C}P^n - \mathcal{S}^{-1}(\mathbb{C}P^{N-1}_\infty)$ by the following:

$$\mathcal{N}([z],[w]) = \frac{\left(\sum_{0\le\alpha\le n} |z_\alpha|^2\right)\left(\sum_{0\le\alpha\le n} |w_\alpha|^2\right)}{\left|\sum_{0\le\alpha\le n} z_\alpha w_\alpha\right|}.$$

By this expression, we directly verify that $\mathcal{N}$ is invariant with respect to the actions of $SU(n+1)$. Hence, $\rho = 2\mathcal{N} - 1$ is also $SU(n+1)$-invariant.

Noting that the isotropy subgroup of $SU(n+1)$ at the point $[e_1] \in \mathbb{C}P^n$ is $S(U(1)\times U(n))$, we can directly verify that the following set $\mathcal{O}$ is an orbit space with respect to the actions of $SU(n+1)$:

$$\mathcal{O} = \left\{ ([e_1], [\cos\theta e_1 + \sin\theta e_2]) \in \mathbb{C}P^n \times \mathbb{C}P^n - \mathcal{S}^{-1}(\mathbb{C}P^{N-1}_\infty) \;\middle|\; 0 \le \theta < \frac{\pi}{2} \right\}.$$

Here, $e_1 = {}^t(1,0,\cdots,0), e_2 = {}^t(0,1,0,\cdots,0) \in \mathbb{C}^{n+1}$. Then, since

$$\mathcal{N}([e_1], [\cos\theta e_1 + \sin\theta e_2]) = 1/\cos^2\theta,$$

we see that $\mathcal{N}$ is injective on $\mathcal{O}$, and so is ρ.

By direct calculations, we have

$$\det \frac{\partial^2}{\partial z_i \partial \bar{z}_j}\rho = \rho(\rho+1)^{n-1}|\mathrm{hol}|^2.$$

By Lemma 3.10, we also have

$$\partial\bar{\partial}\rho(\mathrm{grad}_{\mathbb{C}}\rho, \overline{\mathrm{grad}_{\mathbb{C}}\rho}) = \frac{\rho^2 - 1}{\rho}.$$

Hence, we have $\nu_1(\rho) = \rho(\rho+1)^{n-1}$ and $\nu_2(\rho) = (\rho^2-1)/\rho$ on $\Sigma = [1, \infty)$ similarly as in the case of T^*S^n.

Then the ordinary differential equation of Lemma 3.4 is given by the following:

$$\lambda\rho(\rho+1)^{n-1} = \frac{d}{d\rho}\left\{\frac{1}{2n}\lambda(\rho-1)(\rho+1)^n\right\}.$$

Solving this equation with the initial condition zero, we have $\lambda = (\rho-1)^{n-1}$. Then, it holds that

$$\left\{\frac{1}{2n}\lambda(\rho-1)(\rho+1)^n(f')^{2n}\right\}' = \frac{C}{n}\{(\rho-1)^n\}'.$$

Integrating the both sides with the initial condition zero, we have

$$\frac{1}{2}(\rho-1)^n(\rho+1)^n(f')^{2n} = C(\rho-1)^n.$$

Hence, we have

$$f' = \frac{2^{\frac{1}{2n}}}{\sqrt{\rho+1}} > 0$$

on Σ. Consequently, we obtain the following result by Theorem 3.1:

Proposition 3.14. *In M^{II}_{2n}, corresponding to $\mathrm{T}^*\mathbb{C}P^n$, for the solution f of the following ordinary differential equation, $\psi = f \circ \rho$ is a Ricci-flat Kähler potential on $G \cdot \Sigma \cong M^{2n}_{\mathrm{II}}$:*

$$\rho(\rho+1)^{n-1}(f'(\rho))^{2n} + (\rho-1)(\rho+1)^n(f'(\rho))^{2n-1}f''(\rho) = C > 0.$$

4 Transformations of holomorphic volume forms

In this section, we retain the notation as in Section 2. We show a formula (Proposition 4.2) corresponding to transformations of holomorphic volume forms $L_h^*\Omega$. We use this formula to calculate the Lagrangian angle of a Lagrangian immersion which we finally construct in Theorem 5.5.

Let (M, I) be a complex manifold and Ω a holomorphic volume form on M. Let H be a Lie group which acts on M preserving I. Then the map

$$(L_h)^* : A^k(M)^{\mathbb{C}} \to A^k(M)^{\mathbb{C}}$$

defined by

$$\omega \mapsto L_h^* \omega$$

preserves types of complex differential k-forms ($k \in \mathbb{N}$), where $A^k(M)^{\mathbb{C}}$ is the complex vector space which consists of all complex k-forms on M. Hence $L_h^*\Omega$ is an $(n, 0)$-form. Therefore there exists a holomorphic function f_h that satisfies $L_h^*\Omega = f_h\Omega$.

Next we introduce a Calabi-Yau structure into M, and assume that H-actions preserve its Kähler structure. Then we can see that the holomorphic function f_h satisfies $|f_h| \equiv 1$ as follows.

Proposition 4.1. *Let (M, I, ω, Ω) be a $2n$-dimensional Calabi-Yau manifold and H a Lie group which acts on M preserving I and ω. Then f_h satisfies that $|f_h|$ equals a constant 1 on M.*

Proof. The quadruple $(M, (L_{h*})^{-1} \circ I \circ (L_{h*}), L_h^*\omega, L_h^*\Omega)$ is also a Calabi-Yau manifold for any $h \in H$, since H preserves I and ω. Therefore, we have

$$\begin{aligned}
&(-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \overline{\Omega} = \frac{\omega^n}{n!} = \frac{(L_h^*\omega)^n}{n!} \\
=&(-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n L_h^*\Omega \wedge \overline{L_h^*\Omega} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n |f_h|^2 \Omega \wedge \overline{\Omega}.
\end{aligned}$$

Comparing the both sides, we obtain $|f_h| \equiv 1$. □

By Proposition 4.1 we know the following: Because a holomorphic function which has a constant norm on a connected space has to be constant, f_h is a $U(1)$-valued constant function on a connected Calabi-Yau manifold. Therefore we can define a map

$$c : H \to U(1)$$

by

$$c(h) = c_h := f_h.$$

The map c is a homomorphism between Lie groups. In fact for $h_1, h_2 \in H$, we have

$$c_{h_2} c_{h_1} \Omega = L_{h_2}^*(L_{h_1}^*\Omega) = L_{h_1 h_2}^*\Omega = c_{h_1 h_2}\Omega.$$

Therefore $c_{h_2 h_1} = c_{h_1} c_{h_2} = c_{h_2} c_{h_1}$, and c is a homomorphism.

Using this fact, the next Proposition 4.2 expresses transformations of a holomorphic volume form in a connected Calabi-Yau manifold in terms of a Lie algebra. We assume H to be connected so that we express any $h \in H$ as $h = \exp \eta_1 \cdots \exp \eta_l$ by $\eta_1, \cdots, \eta_l \in \mathfrak{h}$. For h, such $\eta_1, \cdots, \eta_l \in \mathfrak{h}$ are not unique, however the following holds for any of them.

Proposition 4.2. *Let (M, I, ω, Ω) be a connected Calabi-Yau manifold and H a connected Lie group which acts on M preserving I and ω. Then there exists $a_H \in \mathfrak{h}^*$ such that for any $h \in H$, it holds that*

$$L_h^* \Omega = e^{\sqrt{-1}\langle a_H, \eta_1 + \cdots + \eta_l \rangle} \Omega,$$

where

$$\eta_1, \cdots, \eta_l \in \mathfrak{h} \text{ such that } h = \exp \eta_1 \cdots \exp \eta_l.$$

Proof. Because $c : H \to U(1)$ defined above is a homomorphism, the following commutative relation holds between c and $(dc)_e : \mathfrak{h} \cong \mathrm{T}_e H \to \mathfrak{u}(1)$:

$$c \circ \exp \xi = e^{(dc)_e \xi}.$$

In fact, since c makes a one-parameter subgroup $\exp(t\xi)$ of H into a one-parameter subgroup $c(\exp(t\xi))$ of $U(1)$, there exists $\sqrt{-1}\alpha \in \mathfrak{u}(1)$ ($\alpha \in \mathbb{R}$) such that $c(\exp(t\xi)) = \exp_{U(1)}(t(\sqrt{-1}\alpha)) = e^{\sqrt{-1}t\alpha}$. By differentiating the both sides, we obtain

$$\sqrt{-1}\alpha = \left.\frac{d}{dt}\right|_{t=0} e^{\sqrt{-1}t\alpha} = \left.\frac{d}{dt}\right|_{t=0} c(\exp(t\xi)) = (dc)_e \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) = (dc)_e \xi.$$

Thus we see $\sqrt{-1}\alpha = (dc)_e \xi$ and $(c \circ \exp)(t\xi) = e^{t(dc)_e \xi}$. When $t = 1$, we obtain $c \circ \exp(\xi) = e^{(dc)_e \xi}$.

Because H is connected, for each $h \in H$, there exist finite $\eta_1, \cdots, \eta_l \in \mathfrak{h}$ such that $h = \exp \eta_1 \cdots \exp \eta_l$. Then, we have

$$\begin{aligned} c_h &= c(\exp \eta_1 \cdots \exp \eta_l) = c(\exp \eta_1) \cdots c(\exp \eta_l) = e^{(dc)_e \eta_1} \cdots e^{(dc)_e \eta_l} \\ &= e^{\sqrt{-1}\langle -\sqrt{-1}(dc)_e, \eta_1 \rangle} \cdots e^{\sqrt{-1}\langle -\sqrt{-1}(dc)_e, \eta_l \rangle} = e^{\sqrt{-1}\langle -\sqrt{-1}(dc)_e, \eta_1 + \cdots + \eta_l \rangle}. \end{aligned}$$

Therefore noting $\mathfrak{u}(1) = \{\sqrt{-1}\varphi \in \mathbb{C} \mid \varphi \in \mathbb{R}\}$ and letting $a_H := -\sqrt{-1}(dc)_e$, we can define a linear map $a_H : \mathfrak{h} \to \mathbb{R}$, i.e., $a_H \in \mathfrak{h}^*$ and the claim of the proposition holds. □

By Proposition 4.2, transformations of a holomorphic volume form are expressed in terms of a Lie algebra. This enables us to explicitly show the Lagrangian angle of a Lagrangian immersion $(H/K) \times V \to M$ which we construct in the next section in terms of the Lie algebra $\mathfrak{h}$ at each $(hK, p) \in (H/K) \times V$. Here K is a closed Lie subgroup of H and V is a submanifold in M.

Corollary 4.3. *Let $(M, I, \omega, \Omega, H)$ be same as Proposition 4.2. Then $a_H = 0$ if and only if the H-action preserves Ω, namely it preserves the Calabi-Yau structure.*

5 Special Lagrangian construction

In this section we show a construction of special Lagrangian submanifolds by "(generalized) perpendicular symmetries", using the formula (Proposition 4.2) which we proved in the previous section. We construct an isotropic immersion, especially a Lagrangian immersion in Proposition 5.3. We give a formula that express the Lagrangian angle of this Lagrangian immersion in Theorem 5.5. We finally construct a special Lagrangian immersion in Corollary 5.7 by considering a condition to have constant Lagrangian angle.

5.1 Immersions

First with the use of group actions, we construct an immersion which is fundamental for our constructions. This immersion has a form $H \cdot V$ for a submanifold V in M. When H is abelian, it might be natural to assume that the action is free. Otherwise, we may need to consider singular orbits. To control them, we add a condition that the isotropy subgroup H_p at each point $p \in V$ is a constant K. Lemma 5.2 is one of important properties that these immersions have.

Proposition 5.1. *Let M be a manifold and H a Lie group which acts on M. Let $\mathfrak{h}$ be the Lie algebra of H and V a submanifold of M. Assume the followings:*

(Imm-H) $\xi_p^\# \notin \mathrm{T}_pV \backslash \{0\}$ *for any $p \in V$ and any $\xi \in \mathfrak{h}$, and*

(Imm-istp) *the isotropy subgroup at p is a constant K for any $p \in V$.*

Define a map $\phi : (H/K) \times V \to M$ by $\phi(hK, p) = hp$. Then ϕ is an immersion.

Lemma 5.2. *Assume the conditions of Proposition 5.1. For any $(hK, p) \in (H/K) \times V$, any $\xi \in \mathfrak{h}$, and any $v \in \mathrm{T}_pV$, the following holds:*

$$\phi_{*(hK,p)} \left(\left. \frac{d}{dt} \right|_{t=0} h \exp(t\xi)K, v \right) = (L_h)_{*p}(\xi_p^\# + v).$$

Proof of Lemma 5.2. By (Imm-istp), the map ϕ is well-defined.

Fix an arbitrary point $(hK, p) \in (H/K) \times V$. First we show the following:

$$\mathrm{T}_{hK}(H/K) = \left\{ \left. \frac{d}{dt} \right|_{t=0} h \exp(t\xi) K \;\middle|\; \xi \in \mathfrak{h} \right\}.$$

For $g \in H$, define

$$\tau_g : (H/K) \to (H/K)$$

by

$$\tau_g(hK) = ghK.$$

The map τ_g is an element of Diff(H/K), here Diff(H/K) is the space of all diffeomorphisms on H/K. We have $\mathrm{T}_K(H/K) = \{ \frac{d}{dt}\big|_{t=0} \exp(t\xi)K \mid \xi \in \mathfrak{h}\}$. We also have

$$(\tau_h)_{*K} \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi)K = \left. \frac{d}{dt} \right|_{t=0} h \exp(t\xi)K.$$

The linear map $(\tau_h)_{*K} : \mathrm{T}_K(H/K) \to T_{hK}(H/K)$ is an isomorphism. Therefore the claim above holds.

Let $\gamma(t)$ be a curve in V that satisfies $\gamma(0) = p$ and $\gamma'(0) = v$. Then we have

$$(\phi_*)_{(hK,p)} \left(\left. \frac{d}{dt} \right|_{t=0} h \exp(t\xi)K, 0 \right) = \left. \frac{d}{dt} \right|_{t=0} h \exp(t\xi)p = (L_h)_{*p} \xi_p^{\#},$$

$$(\phi_*)_{(hK,p)}(0, v) = \left. \frac{d}{dt} \right|_{t=0} h\gamma(t) = (L_h)_{*p} v.$$

Thus Lemma 5.2 has been proved. □

Proof of Proposition 5.1. To prove Proposition 5.1, it is sufficient to show that if for any $\xi \in \mathfrak{h}$ and any $v \in \mathrm{T}_pV$,

$$\phi_{*(hK,p)} \left(\left. \frac{d}{dt} \right|_{t=0} h \exp(t\xi)K, v \right) = (L_h)_{*p}(\xi_p^{\#} + v) = 0,$$

then

$$\left. \frac{d}{dt} \right|_{t=0} h \exp(t\xi)K = 0, \quad v = 0.$$

Since $(L_h)_{*p}$ is an isomorphism, if $(L_h)_{*p}(\xi_p^{\#} + v) = 0$, then $\xi_p^{\#} + v = 0$. By (Imm-H), a pair $(\xi_p^{\#}, v)$ is linearly independent. Hence we have $\xi_p^{\#} = 0$ and $v = 0$ from $\xi_p^{\#} + v = 0$. If we define a map

$$j : (H/K) \to (H \cdot p)$$

by

$$j(hK) \mapsto hp,$$

the map j is a diffeomorphism. With the isomorphism $j_{*hK} : \mathrm{T}_{hK}(H/K) \to \mathrm{T}_{hp}(H \cdot p)$, we have

$$\left.\frac{d}{dt}\right|_{t=0} h\exp(t\xi)K \mapsto \left.\frac{d}{dt}\right|_{t=0} h\exp(t\xi)p = L_{h*p}\xi_p^{\#}.$$

Thus we see that $\left.\frac{d}{dt}\right|_{t=0} h\exp(t\xi)K = 0$ if and only if $\xi_p^{\#} = 0$. □

5.2 Isotropic immersions

Next we introduce a symplectic structure to a manifold M, and show a condition for the immersions of Proposition 5.1 to be isotropic in Proposition 5.3.

Proposition 5.3. *Let (M, ω) be a $2n$-dimensional symplectic manifold and H a Lie group which acts on M and has a moment map μ. Let $\mathfrak{h}$ be the Lie algebra of H and c an element of $\mathfrak{h}^*$. Let V_c be a submanifold of M that satisfies $V_c \subset \mu^{-1}(c)$.*

Assume (Imm-H), (Imm-istp), and the followings:

- *(Istp-V_c)* V_c *is isotropic, and*
- *(Istp-cnt)* c *is an element of* $Z(\mathfrak{h}^*)$*, the center of* $\mathfrak{h}^*$*.*

Define a map $\phi_c : (H/K) \times V_c \to M$ by $\phi(hK, p) = hp$. Then ϕ_c is an isotropic immersion.

In addition, if the following condition holds, ϕ_c is a Lagrangian immersion:

- *(Lag-dim)* $\dim H/K + \dim V_c = n$.

Lemma 5.4. *Assume the settings of Proposition 5.3 except the conditions (Imm-H), (Imm-istp), (Istp-V_c), (Istp-cnt), and (Lag-dim). Then $\omega_p(\xi_p^{\#}, v) = 0$ for any $p \in V_c$, any $v \in \mathrm{T}_pV_c$, and any $\xi \in \mathfrak{h}$.*

Proof of Lemma 5.4. Noting $(d\mu)_p v = 0$, we have

$$\omega_p(\xi_p^{\#}, v) = -d(\langle \mu(\cdot), \xi\rangle)_p v = -\langle (d\mu)_p v, \xi\rangle = 0.$$

Thus we have shown Lemma 5.4. □

Proof of Proposition 5.3. Since the map ϕ_c is an immersion by Proposition 5.1, it is sufficient for proving Proposition 5.3 to show that $\phi_c^*\omega$ is a constant 0 on $(H/K) \times V_c$. For two arbitrary elements $(\frac{d}{dt}\big|_{t=0} h\exp(t\xi_i)K, v_i) \in \mathrm{T}_{hK}(H/K) \times \mathrm{T}_pV_c (i = 1, 2)$, we have

$$
\begin{aligned}
&(\phi_c^*\omega)_{(hK,p)} \left((\frac{d}{dt}\bigg|_{t=0} h\exp(t\xi_1)K, v_1), (\frac{d}{dt}\bigg|_{t=0} h\exp(t\xi_2)K, v_2) \right) \\
=&\omega_{hp} \left((\phi_c)_{*(hK,p)}(\frac{d}{dt}\bigg|_{t=0} h\exp(t\xi_1)K, v_1), (\phi_c)_{*(hK,p)}(\frac{d}{dt}\bigg|_{t=0} h\exp(t\xi_2)K, v_2) \right) \\
=&\omega_{hp} \left((L_h)_{*p}\{(\xi_1)_p^\# + v_1\}, (L_h)_{*p}\{(\xi_2)_p^\# + v_2\} \right) \\
=&\omega_p((\xi_1)_p^\# + v_1, (\xi_2)_p^\# + v_2) \\
=&\omega_p((\xi_1)_p^\#, (\xi_2)_p^\#) + \omega_p((\xi_1)_p^\#, v_2) + \omega_p(v_1, (\xi_2)_p^\#) + \omega_p(v_1, v_2).
\end{aligned}
$$

The first term is equal to zero by (Istp-cnt) and Proposition 2.4, the second and third terms are zero by Lemma 5.4, and the forth term is zero by (Istp-V_c). Thus we see that ϕ is an isotropic immersion. In addition, if (Lag-dim) holds, this immersion is Lagrangian by the definition of Lagrangian submanifolds. □

5.3 Lagrangian angle and special Lagrangian construction

We constructed a Lagrangian immersion in Proposition 5.3. We show a condition for this immersion to be a special Lagrangian immersion by using the Lagrangian angle. In Theorem 5.5, with the use of a formula for transformations of holomorphic volume forms (Proposition 4.2), we give explicitly the Lagrangian angle of a Lagrangian immersion of Proposition 5.3.

Lemma 5.8 is used for calculations of the Lagrangian angle.

Theorem 5.5. *Let $(M, g, I, \omega, \Omega)$ be a connected $2n$-dimensional Calabi-Yau manifold and H a connected Lie group which acts on M preserving I and has a moment map μ. Let $\mathfrak{h}$ be the Lie algebra of H and L a Lagrangian submanifold of M that has a local Lagrangian angle θ. Let c be an element of $\mathfrak{h}^*$ and V_c a submanifold of M that satisfies $V_c \subset \mu^{-1}(c) \cap L$. Assume (Imm-istp), (Istp-cnt), (Lag-dim), and the following (LagAng-H):*

(LagAng-H) For any $p \in V_c$ and any $\xi \in \mathfrak{h}$, the following (i) and (ii) hold:

(i) $\xi_p^\# \in \mathrm{T}_p^\perp L \oplus \mathrm{T}_pV_c$,

(ii) $\xi_p^{\#} \notin \mathrm{T}_p V_c \backslash \{0\}$.

Define $\phi_c : (H/K) \times V_c \to M$ *as in Proposition 5.3. Then locally the following holds:*

$$(\phi_c^* \Omega)_{(hK,p)} = \pm e^{\sqrt{-1}\theta_c} \mathrm{vol}_{\phi_c^* g},$$

where $\theta_c : (H/K) \times V_c \to M$ *is defined by*

$$\theta_c(hK, p) = \langle a_H, \eta_1 + \cdots + \eta_l \rangle + \theta(p) - \frac{\pi}{2} \dim(H/K),$$

and

$$\eta_1, \cdots, \eta_l \in \mathfrak{h} \text{ such that } h = \exp \eta_1 \cdots \exp \eta_l.$$

Remark 5.6. In Theorem 5.5, we do not assume the conditions (Imm-H) and (Istp-V_c) in Proposition 5.3 to make ϕ_c a Lagrangian immersion. However under the conditions of Theorem 5.5, they hold. In fact, (Imm-H) holds by (LagAng-H). Since L is a Lagrangian submanifold and $V_c \subset L$, (Istp-V_c) holds.

From Theorem 5.5 we immediately obtain the following corollary. Constructions of special Lagrangian submanifolds are directly based on this corollary.

Corollary 5.7. *Assume the conditions of Theorem 5.5. In addition, if* θ *is constant on* V_c *(e.g.* L*: a special Lagrangian submanifold) and the* H*-actions preserve the Calabi-Yau structure on* M*, i.e.,* $a_H = 0$*, then* ϕ_c *is a special Lagrangian immersion.*

We prepare the next Lemma 5.8 for the proof of Theorem 5.5.

Lemma 5.8. *Under the conditions of Theorem 5.5, for any* $p \in V_c$ *there exist*

$$\xi_1, \cdots, \xi_m \in \mathfrak{h}, \quad v_1, \cdots, v_{n-m}, w_1, \cdots, w_m \in \mathrm{T}_p V_c$$

that satisfy the followings:

(1) *For any* $h \in H$*,*

$$\left((\frac{d}{dt}\Big|_{t=0} h \exp(t\xi_1) K, w_1), \cdots, (\frac{d}{dt}\Big|_{t=0} h \exp(t\xi_m) K, w_m), (0, v_1), \cdots, (0, v_{n-m}) \right)$$

is an orthonormal basis in $\mathrm{T}_{(hK,p)}((H/K) \times V_c)$ *with respect to* $\phi_c^* g$*,*

(2) $(\xi_j)_p^{\#} + w_j \in \mathrm{T}_p^{\perp} L$ *for* $j = 1, \cdots, m$*, and*

(3) $\left(I_p\{(\xi_1)^\#_p + w_1\}, \cdots, I_p\{(\xi_m)^\#_p + w_m\}, v_1, \cdots, v_{n-m}\right)$ *is an orthonormal basis in* T_pL *with respect to* $\iota^* g$.

Here $m = \dim(H/K)$, *and* $\iota : L \to M$ *is the embedding.*

Proof of Lemma 5.8. First we show Lemma 5.8 (1) and (2). By Lemma 5.2, we have

$$\left(\phi_{c*(hK,p)}(\frac{d}{dt}\bigg|_{t=0} h\exp(t\xi_j)K, w_j), \phi_{c*(hK,p)}(0, v_i)\right) = (L_{h*p}\{(\xi_j)^\#_p + w_j\}, L_{h*p}v_i).$$

Since L_{h*p} is isometric, it is enough for showing (1) and (2) to verify that there exist ξ_j, v_i, and w_j ($i = 1, \cdots, n-m,\ j = 1, \cdots, m$) such that $((\xi_j)^\#_p + w_j, v_i)$ is an orthonormal system of T_pM and $(\xi_j)^\#_p + w_j \in \mathrm{T}^\perp_p L$.

Noting (Lag-dim), let (v_i) ($i = 1, \cdots, n-m$) be an orthonormal basis of T_pV_c with respect to the metric on V induced from g. By (LagAng-H), it holds that $\mathrm{T}_p(H \cdot p) \cap \mathrm{T}_pL = \{0\}$. Hence, noting (Lag-dim) again, we can take $\eta_j \in \mathfrak{h}$ ($j = 1, \cdots, m$) such that $((\eta_j)^\#_p)$ is a basis of $\mathrm{T}_p(H \cdot p)$ and $((\eta_1)^\#_p, \cdots, (\eta_m)^\#_p, v_1, \cdots, v_{n-m})$ is linearly independent in T_pM.

By (LagAng-H), there exist $u_j \in \mathrm{T}^\perp_p L \backslash \{0\}$ and $z_j \in \mathrm{T}_pV_c$ ($j = 1, \cdots, m$) that decompose $(\eta_j)^\#_p$ into direct summations as follows:

$$(\eta_j)^\#_p = u_j + z_j \quad (j = 1, \cdots, m).$$

(u_j) is linearly independent. In fact, if u_1 is expressed by $u_1 = b_2u_2 + \cdots + b_mu_m$ for $b_j \in \mathbb{R}$ such that ${}^t(b_2, \cdots, b_m) \neq \mathbf{0}$, we have

$$\begin{aligned} &(\eta_1)^\#_p - z_1 = b_2((\eta_2)^\#_p - z_2) + \cdots + b_m((\eta_m)^\#_p - z_m) \\ \Leftrightarrow\ &(\eta_1)^\#_p - \{b_2(\eta_2)^\#_p + \cdots + b_m(\eta_m)^\#_p\} = z_1 - (b_2z_2 + \cdots + b_mz_m). \end{aligned}$$

Because the left-hand side belongs to $\mathrm{T}_p(H \cdot p)$, there exists $\eta \in \mathfrak{h}$ such that $\eta^\#_p$ equals the left-hand side. If $\eta^\#_p \neq 0$, then $\eta^\#_p \in \mathrm{T}_pV_c \backslash \{0\}$ because the right-hand side belongs to T_pV_c. This is contrary to (LagAng-H). On the other hand, if $\eta^\#_p = 0$, this is contrary to that $(\eta_j)^\#_p$ is linearly independent because the left-hand side equals 0. For $j = 2, \cdots, m$, we can verify the same assertion. Thus (u_j) is linearly independent.

Therefore, noting $u_j \in \mathrm{T}^\perp_p L$, there exists $A \in GL(m, \mathbb{R})$ such that $(\tilde{u}_1 \cdots, \tilde{u}_m) = (u_1, \cdots, u_m)A$ is an orthonormal system in $\mathrm{T}^\perp_p L$. Because $\mathrm{T}^\perp_p L \perp \mathrm{T}_pV_c$, $(v_i, \tilde{u}_j)$ is an orthonormal system in T_pM.

Thus, if we define $\xi_j \in \mathfrak{h}$ and $w_j \in \mathrm{T}_p V_c$ by

$$\begin{aligned}((\xi_1)^\#_p, \cdots, (\xi_m)^\#_p) =&((\eta_1)^\#_p, \cdots, (\eta_m)^\#_p)A,\\ (w_1, \cdots, w_m) =&(-z_1, \cdots, -z_m)A,\end{aligned}$$

then $\tilde{u}_j = (\xi_j)^\#_p + w_j$ and Lemma 5.8 (1) and (2) hold.

Next we show Lemma 5.8 (3). By (1) and (2), it is enough for showing the claim of (3) to verify $I_p\{(\xi_j)^\#_p + w_j\} \in \mathrm{T}^\perp_p V_c$ ($j = 1, \cdots, m$).

$I_p(\xi_j)^\#_p \in \mathrm{T}^\perp_p V_c$ because $0 = \omega_p((\xi_j)^\#_p, v_i) = g_p(I_p(\xi_j)^\#_p, v_i)$ by Lemma 5.4. On the other hand, $I_p w_j \in \mathrm{T}^\perp_p V_c$ because V_c is isotropic and $0 = \omega_p(w_j, v_i) = g_p(I_p w_j, v_i)$. Thus Lemma 5.8 (3) has been verified. □

Proof of Theorem 5.5. Let $\mathcal{X}^{(0,1)}(M)$ be the set of complex vector fields of type $(0,1)$ on M. For any $\eta \in \mathfrak{h}$, it holds that $\eta^\# + \sqrt{-1}I\eta^\# \in \mathcal{X}^{(0,1)}(M)$. Since Ω is a complex differential form of type $(n, 0)$ on M, we have

$$\mathfrak{i}(\eta^\#)\Omega = -\sqrt{-1}\mathfrak{i}(I\eta^\#)\Omega.$$

Take (ξ_j, v_i, w_j) in Lemma 5.8 for $i = 1, \cdots, n-m$ and $j = 1, \cdots, m$. Then, noting $(L_h^*\Omega) = e^{\sqrt{-1}\langle a_H, \eta_1 + \cdots + \eta_l\rangle}\Omega$, we have

$$\begin{aligned}&(\phi_c^*\Omega)_{(hK,p)}\left(\cdots, (\frac{d}{dt}\Big|_{t=0} h\exp(t\xi_j)K, w_j), \cdots; \cdots, (0, v_i), \cdots\right)\\ =&\Omega_{hp}(\cdots, (L_h)_{*p}\{(\xi_j)^\#_p + w_j\}, \cdots; \cdots, (L_h)_{*p}v_i, \cdots)\\ =&(L_h^*\Omega)_p(\cdots, (\xi_j)^\#_p + w_j, \cdots; \cdots, v_i, \cdots)\\ =&(-\sqrt{-1})^m(L_h^*\Omega)_p(\cdots, I_p\{(\xi_j)^\#_p + w_j\}, \cdots; \cdots, v_i, \cdots)\\ =&(-\sqrt{-1})^m e^{\sqrt{-1}\langle a_H, \eta_1 + \cdots + \eta_l\rangle}\Omega_p(\cdots, I_p\{(\xi_j)^\#_p + w_j\}, \cdots; \cdots, v_i, \cdots)\\ =&(-\sqrt{-1})^m e^{\sqrt{-1}\langle a_H, \eta_1 + \cdots + \eta_l\rangle}(\iota^*\Omega)_p(\cdots, I_p\{(\xi_j)^\#_p + w_j\}, \cdots; \cdots, v_i, \cdots)\\ =&(-\sqrt{-1})^m e^{\sqrt{-1}\langle a_H, \eta_1 + \cdots + \eta_l\rangle}e^{\sqrt{-1}\theta}(\mathrm{vol}_{\iota^*g})_p(\cdots, I_p\{(\xi_j)^\#_p + w_j\}, \cdots; \cdots, v_i, \cdots)\\ =&\pm e^{\sqrt{-1}(\langle a_H, \eta_1 + \cdots + \eta_l\rangle + \theta - \frac{\pi}{2}m)}.\end{aligned}$$

By Lemma 5.8 (1), we have

$$\mathrm{vol}_{\phi_c^*g}\left(\cdots, (\frac{d}{dt}\Big|_{t=0} h\exp(t\xi_j)K, w_j), \cdots; \cdots, (0, v_i), \cdots\right) = \pm 1.$$

Thus Theorem 5.5 has been proved. □

Joyce pointed out in [13] that the commutativity of a Lie group is a necessary condition for the group action to be perpendicular to whole of L. However, to construct a special Lagrangian submanifold, we need for a group action to be perpendicular to L on V_c. Similarly the condition that L is a special Lagrangian submanifold, that is, the condition that the Lagrangian angle is constant on L is reduced to on V_c. The perpendicular condition is also weakened as above. This situation, roughly speaking, indicates that a special Lagrangian submanifold may be constructed by Corollary 5.7, if $H \cdot V_c$ (not necessarily each fundamental vector $\xi_p^\#$ at $p \in V_c$) is perpendicular to L for some $c \in Z(\mathfrak{h}^*)$.

As a special case of the condition (LagAng-H) if we assume that each fundamental vector $\xi_p^\#$ is perpendicular to L, we obtain the next corollary. In this case we need not assume (Istp-cnt).

Corollary 5.9. *Let $(M, g, I, \omega, \Omega)$ be a connected $2n$-dimensional Calabi-Yau manifold and H a connected Lie group which acts on M preserving I and has a moment map μ. Let $\mathfrak{h}$ be the Lie algebra of H and L a Lagrangian submanifold of M with a local Lagrangian angle θ. Let c be an element of $\mathfrak{h}^*$ and V_c a submanifold of M such that $V_c \subset \mu^{-1}(c) \cap L$. Assume (Imm-istp), (Lag-dim), and (LagAng-H)$'$ as follows:*

(LagAng-H)$'$ $\quad \xi_p^\# \perp \mathrm{T}_p L$ *for any* $p \in V_c$, *and any* $\xi \in \mathfrak{h}$.

Define $\phi_c : (H/K) \times V_c \to M$ as in Proposition 5.3. Then locally the following holds:

$$(\phi_c^* \Omega)_{(hK,p)} = \pm e^{\sqrt{-1}\theta_c} \mathrm{vol}_{\phi_c^* g},$$

where $\theta_c : (H/K) \times V_c \to M$ is defined by

$$\theta_c(hK, p) = \langle a_H, \eta_1 + \cdots + \eta_l \rangle + \theta(p) - \frac{\pi}{2} \dim(H/K),$$

and

$$\eta_1, \cdots, \eta_l \in \mathfrak{h} \text{ such that } h = \exp \eta_1 \cdots \exp \eta_l.$$

Proof. It is sufficient to verify that under the conditions of Corollary 5.9, (Imm-H), (Istp-V_c), and (Istp-cnt) hold.

(Imm-H) holds by (LagAng-H)$'$. (Istp-V_c) holds as in Remark 5.6. Finally to show (Istp-cnt), we fix an arbitrary point $hp \in H \cdot p$ ($h \in H$). We have

$$\mathrm{T}_{hp}(H \cdot p) = \left\{ \left. \frac{d}{dt} \right|_{t=0} h \exp(t\xi) p \;\middle|\; \xi \in \mathfrak{h} \right\} = \{ (L_h)_{*p} \xi_p^\# \mid \xi \in \mathfrak{h} \}.$$

Noting $I_p\xi_p^\# \in \mathrm{T}_pL$ and $\eta_p^\# \in \mathrm{T}_p^\perp L$ because of the assumption that L is Lagrangian and (LagAng-H)$'$, we have

$$\omega_{hp}((L_h)_{*p}\xi_p^\#, (L_h)_{*p}\eta_p^\#) = (L_h^*\omega)_p(\xi_p^\#, \eta_p^\#) = g_p(I_p\xi_p^\#, \eta_p^\#) = 0.$$

Therefore $H \cdot p$ is isotropic. This is equivalent to $\mu(p) \in Z(\mathfrak{h}^*)$. □

Corollary 5.10. *Assume the conditions of Corollary 5.9. In addition, if θ is constant on V_c (e.g. L: a special Lagrangian submanifold) and the H-actions preserve the Calabi-Yau structure on M, i.e., $a_H = 0$, then ϕ_c is a special Lagrangian immersion.*

6 Examples in T^*S^n

In this section, with the use of the results above, we construct non-trivial examples of special Lagrangian submanifolds in non-flat Calabi-Yau manifolds T^*S^n which equipped with the Stenzel metrics. In Subsection 6.1, we review some notions about the Stenzel metrics on T^*S^n, and make sure some facts that is used to construct our examples. In Subsection 6.2, we construct two examples by using the actions of an abelian group. One of them is based on Corollary 5.5 of generalized perpendicular conditions. In Subsection 6.3, we construct an example based on Corollary 5.10 by using the actions of a non-abelian group.

Through this section, we use some notations. We denote $\mathbf{e}_i$ the column k-vector whose i-th element equals one and the any other element equals to zero in k-dimensional real or complex Euclidean space for some $k \in \mathbb{N}$. We define

$$\xi_{ij} := E_{ji} - E_{ij} \in M(k, \mathbb{R}),$$

where E_{ij} denotes the $k \times k$-matrix whose (i, j)-component is 1 and all the others are 0 for some $k \in \mathbb{N}$.

6.1 Stenzel metric on T^*S^n

In [23], Stenzel constructed complete Ricci-flat Kähler metrics on the cotangent bundles of compact rank one symmetric spaces. This gives us examples of non-flat Calabi-Yau manifolds. In this paper, we denote this Calabi-Yau structure by $(\mathrm{T}^*S^n, I, \omega_{\mathrm{Stz}}, \Omega_{\mathrm{Stz}})$. We construct our examples of special Lagrangian submanifolds in $(\mathrm{T}^*S^n, I, \omega_{\mathrm{Stz}}, \Omega_{\mathrm{Stz}})$.

As seen in Subsection 3.3, we identify the tangent bundle and the cotangent bundle of the n-sphere S^n:

$$\mathrm{T}^*S^n = \{(x,\xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1, x \centerdot \xi = 0\}.$$

$SO(n+1)$ acts on T^*S^n by $h \cdot (x,\xi) = (hx, h\xi)$ for $h \in SO(n+1)$ with cohomogeneity one. The principal orbit at a point (x,ξ) equals to a sphere bundle with a radius of $\|\xi\|$.

A complex quadric hypersurface Q^n is defined by the following:

$$Q^n = \left\{ z = {}^t(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} z_i^2 = 1 \right\}.$$

Q^n is $SO(n+1)$-equivariantly diffeomorphic to T^*S^n with the Szöke's map $\Phi : \mathrm{T}^*S^n \to Q^n$ in [24] defined by

$$\Phi(x,\xi) = \cosh(\|\xi\|)x + \sqrt{-1}\frac{\sinh(\|\xi\|)}{\|\xi\|}\xi.$$

We can induce a complex structure to Q^n from $\mathbb{C}^{n+1}$. Stenzel constructed Ricci-flat Kähler metrics with respect to these complex structures. We denoted this by I above. Therefore when we use the complex structure for studying the perpendicular conditions later, we do the calculations not in T^*S^n but in Q^n. The Kähler form ω_{Stz} that Stenzel constructed is given as follows:

$$\omega_{\mathrm{Stz}} = \sqrt{-1}\partial\bar{\partial} f(\rho) = \sqrt{-1}\sum_{i,j=1}^{n+1} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} f(\rho) dz_i \wedge d\bar{z}_j,$$

here $\rho = \|z\|^2 = \sum_{i=1}^{n+1} z_i \bar{z}_i$ is the strictly plurisubharmonic function defined in Lemma 3.9 and f is a smooth real function satisfies the ordinary differential equation (3,5) in Proposition 3.11:

$$\rho(f'(\rho))^n + (f'(\rho))^{n-1} f''(\rho)(\rho^2 - 1) = C > 0.$$

Through this section, we retain the notation as in Remark 3.3. By Remark 3.3, the functions f and F with $f = F \circ \cosh^{-1}$ have properties that $\dot{F} > 0, \ddot{F} > 0$ on $\mathrm{T}^*S^n \backslash S^n$ and $f' > 0$ on T^*S^n.

The actions of $SO(n+1)$ preserve the Calabi-Yau structure of $(\mathrm{T}^*S^n, I, \omega_{\mathrm{Stz}}, \Omega_{\mathrm{Stz}})$. Hence, for $a_H = a_{SO(n+1)} \in \mathfrak{h}^* = \mathfrak{so}(n+1)^*$ of Proposition 4.2 determined by $(\mathrm{T}^*S^n, I, \omega_{\mathrm{Stz}}, \Omega_{\mathrm{Stz}}, SO(n+1))$, we have $a_H = 0$ by Corollary 4.3.

A moment map $\mu : Q^n \to \mathfrak{so}(n+1)^*$ with respect to $(\mathrm{T}^*S^n, \omega_{\mathrm{Stz}}, SO(n+1))$ is given in [1] as follows:

$$(\mu(z))(X) = f'(\rho) Iz \,\bullet\, Xz, \quad (z \in Q^n, X \in \mathfrak{so}(n+1)), \tag{6,1}$$

here " $\bullet$ " denotes the canonical real inner product on $\mathbb{C}^{n+1}$.

Finally, we give a basic fact for preparing an original special Lagrangian submanifold to construct a new one: Karigiannis and Min-Oo showed in [15] that a conormal bundle $\mathrm{T}^{*\perp}N$ in T^*S^n for a submanifold N in S^n is a special Lagrangian submanifold if and only if N is an austere submanifold of S^n. Especially, a totally geodesic submanifold of S^n is an austere submanifold.

6.2 The case of $H = U(1), L_1 = \mathrm{T}^{*\perp}S^2, L_2 = \mathrm{T}^{*\perp}S^1 \subset \mathrm{T}^*S^5$

Let M be the cotangent bundle of 5-sphere T^*S^5, L_1 the conormal bundle of a totally geodesic submanifold S^2 of S^5, and L_2 the conormal bundle of a totally geodesic submanifold S^1 of S^5 as follows:

$$L_1(\cong \mathrm{T}^{*\perp}S^2) = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \end{bmatrix} \right) \middle| \; \|x\| = 1, \xi_j \in \mathbb{R} \; (j = 2, 4, 6) \right\},$$

$$L_2(\cong \mathrm{T}^{*\perp}S^1) = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} \right) \middle| \; \|x\| = 1, \xi_j \in \mathbb{R} \; (j = 2, 4, 5, 6) \right\}.$$

Because these S^2 and S^1 are totally geodesic submanifolds of S^5, they are austere submanifolds. Hence their conormal bundles $\mathrm{T}^{*\perp}S^2$ and $\mathrm{T}^{*\perp}S^1$ are special Lagrangian submanifolds of T^*S^5. We use the polar coordinates $x_1 = \cos\varphi_1 \cos\varphi_2, x_3 = \cos\varphi_1 \sin\varphi_2, x_5 = \sin\varphi_1$ for L_1 and $x_1 = \cos\varphi, x_3 = \sin\varphi$ for L_2.

Let H be the $U(1)$-action of the Hopf-fibration $S^5 \to \mathbb{C}P^2$, that is, the diagonal $U(1) \cong SO(2)$-action represented as follows:

$$H(\cong SO(2)) = \left\{ \left[\begin{array}{c|c|c} h & & \\ \hline & h & \\ \hline & & h \end{array} \right] \in GL(6,\mathbb{R}) \;\middle|\; h \in SO(2) \right\}.$$

The Lie algebra $\mathfrak{h}$ is given as follows:

$$\mathfrak{h}(\cong \mathfrak{so}(2)) = \operatorname{span}\{\eta\},$$

here $\eta = \xi_{12} + \xi_{34} + \xi_{56}$ and ξ_{ij} is as mentioned at the beginning of this section. Note that the isotropy subgroup of this $SO(2)$-action is trivial at any point $p \in L_1$ and L_2. Hence the condition (Imm-istp) holds for any point $p \in L_1$ and L_2.

We obtain an explicit expression of the moment map (6,1) by direct calculations.

Lemma 6.1. *Define μ_η by $\mu_\eta(z) = \langle \mu(z), \eta \rangle$ for $z \in \Phi(L_j)$ $(j = 1, 2)$. Then $\mu_\eta(z)$ equals*

$$\begin{cases} -\mathcal{K}(\|\xi\|)(\cos\varphi_1 \cos\varphi_2 \xi_2 + \cos\varphi_1 \sin\varphi_2 \xi_4 + \sin\varphi_1 \xi_6) & \text{on } \Phi(L_1)\setminus\{\|\xi\| = 0\}, \\ -\mathcal{K}(\|\xi\|)(\cos\varphi \xi_2 + \sin\varphi \xi_4) & \text{on } \Phi(L_2)\setminus\{\|\xi\| = 0\}. \end{cases}$$

Here,

$$\mathcal{K}(\|\xi\|) = \frac{f'(\cosh(2\|\xi\|))\sinh(2\|\xi\|)}{\|\xi\|},$$

and f is the solution of the ordinary differential equation (3,5) in Proposition 3.11.

Under these preparations, we obtain the following:

Proposition 6.2. *Let $(M, I, \omega_{\mathrm{Stz}}, \Omega_{\mathrm{Stz}}, L_j, H)$ be as above. Let $V_c^{(j)} := L_j \cap \mu^{-1}(c)$ for each $c \in \mathfrak{h}^*$ and $j = 1, 2$.*

(1) $H \cdot V_c^{(1)}$ is a special Lagrangian submanifold for any $c \in \mathfrak{h}^*$ such that $V_c^{(1)} \neq \emptyset$.

(2) $H \cdot V_c^{(2)}$ is a special Lagrangian submanifold for any $c \in \mathfrak{h}^*$ such that $V_c^{(2)} \neq \emptyset$.

Proof. Note that $Z(\mathfrak{h}^*) = \mathfrak{h}^*$ because H is abelian. Thus (Istp-cnt) automatically holds. The condition (Imm-istp) also holds as mentioned above. Therefore we see that (Istp-cnt) and (Imm-istp) hold in the any case of (1) and (2).

(1) First we show the proposition above for $j = 1$. This proof is based on Corollary 5.10. We will show in order (1-I) the perpendicular condition: the H-action satisfies (LagAng-H)$'$ on L_1, and (1-II) the submanifold condition: $V_c^{(1)} \neq \emptyset$ is a submanifold of M and (Lag-dim) holds for $(V_c^{(1)}, H, K)$.

(1-I) First we assume $\|\xi\| \neq 0$. By direct calculations, for $z \in \Phi(L_1)$, the fundamental vector $\eta_z^\#$ and $I_z\eta_z^\#$ are given as follows:

$$\eta_z^\# = \cosh(\|\xi\|) \begin{bmatrix} 0 \\ \cos\varphi_1\cos\varphi_2 \\ 0 \\ \cos\varphi_1\sin\varphi_2 \\ 0 \\ \sin\varphi_1 \end{bmatrix} + \sqrt{-1}\frac{\sinh(\|\xi\|)}{\|\xi\|} \begin{bmatrix} -\xi_2 \\ 0 \\ -\xi_4 \\ 0 \\ -\xi_6 \\ 0 \end{bmatrix},$$

$$I_z\eta_z^\# = \frac{\sinh(\|\xi\|)}{\|\xi\|} \begin{bmatrix} \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \\ 0 \end{bmatrix} + \sqrt{-1}\cosh(\|\xi\|) \begin{bmatrix} 0 \\ \cos\varphi_1\cos\varphi_2 \\ 0 \\ \cos\varphi_1\sin\varphi_2 \\ 0 \\ \sin\varphi_1 \end{bmatrix}.$$

On the other hand, using the coordinates above, we have a basis of $\mathrm{T}_z\Phi(L_1)$ as follows:

$$\begin{aligned}
\frac{\partial}{\partial\varphi_1} &= \cosh(\|\xi\|)(-\sin\varphi_1\cos\varphi_2\mathbf{e}_1 - \sin\varphi_1\sin\varphi_2\mathbf{e}_3 + \cos\varphi_1\mathbf{e}_5), \\
\frac{\partial}{\partial\varphi_2} &= \cosh(\|\xi\|)(-\cos\varphi_1\sin\varphi_2\mathbf{e}_1 + \cos\varphi_1\cos\varphi_2\mathbf{e}_3), \\
\frac{\partial}{\partial\xi_j} &= \frac{\sinh(\|\xi\|)}{\|\xi\|}\xi_j x + \sqrt{-1}\left\{\frac{\xi_j}{\|\xi\|^2}\mathcal{F}\xi + \frac{\sinh(\|\xi\|)}{\|\xi\|}\mathbf{e}_j\right\} \quad (j = 2, 4, 6),
\end{aligned}$$

where

$$\mathcal{F} = \mathcal{F}(\|\xi\|) = \cosh(\|\xi\|) - \frac{\sinh(\|\xi\|)}{\|\xi\|}.$$

Since L_1 is a Lagrangian submanifold of a Kähler manifold M, it is sufficient for verifying $\eta_z^\# \in \mathrm{T}_z^\perp\Phi(L_1)$ to show $I_z\eta_z^\# \in \mathrm{T}_z\Phi(L_1)$. For generating the imaginary part of $I_z\eta_z^\#$ by $\left(\frac{\partial}{\partial\xi_j}\right)$ $(j = 2, 4, 6)$, the following is necessary: there exists

$(a_2, a_4, a_6) \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$ which satisfies

$$A \begin{bmatrix} a_2 \\ a_4 \\ a_6 \end{bmatrix} = \begin{bmatrix} \cosh(\|\xi\|)\cos\varphi_1\cos\varphi_2 \\ \cosh(\|\xi\|)\cos\varphi_1\sin\varphi_2 \\ \cosh(\|\xi\|)\sin\varphi_1 \end{bmatrix}, \tag{6,2}$$

where

$$A = \begin{bmatrix} \frac{\xi_2^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|} & \frac{\xi_2\xi_4}{\|\xi\|^2}\mathcal{F} & \frac{\xi_2\xi_6}{\|\xi\|^2}\mathcal{F} \\ \frac{\xi_2\xi_4}{\|\xi\|^2}\mathcal{F} & \frac{\xi_4^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|} & \frac{\xi_4\xi_6}{\|\xi\|^2}\mathcal{F} \\ \frac{\xi_2\xi_6}{\|\xi\|^2}\mathcal{F} & \frac{\xi_4\xi_6}{\|\xi\|^2}\mathcal{F} & \frac{\xi_6^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|} \end{bmatrix}.$$

We verify that $\mathrm{rank}A = 3$ if $\|\xi\| \neq 0$ as follows. By elementary transformations of matrices, we have

$$A \to \frac{1}{\mathcal{F}_1\mathcal{F}_2^2} \begin{bmatrix} \mathcal{F}_1 & * & * \\ 0 & \frac{\sinh(\|\xi\|)}{\|\xi\|}\mathcal{F}_2 & * \\ 0 & 0 & \frac{\sinh(\|\xi\|)}{\|\xi\|}\mathcal{F}_3 \end{bmatrix},$$

where

$$\begin{aligned} \mathcal{F}_1 &= \frac{\xi_2^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|}, \\ \mathcal{F}_2 &= \frac{\xi_2^2 + \xi_4^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|}, \\ \mathcal{F}_3 &= \left\{ \frac{\xi_2^2 + \xi_4^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|} \right\} \left\{ \frac{\xi_2^2 + \xi_6^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|} \right\} - \frac{\xi_4^2\xi_6^2}{\|\xi\|^4}\mathcal{F}^2. \end{aligned}$$

With the use of series expansions

$$\mathcal{F}(x) = \cosh x - \frac{\sinh x}{x} = \sum_{n=1}^{\infty} \left\{ \frac{1}{(2n)!} - \frac{1}{(2n+1)!} \right\} x^{2n},$$

we see that $\mathcal{F} > 0$ if $\|\xi\| \neq 0$. This indicates that $F_1, F_2 > 0$ if $\|\xi\| \neq 0$. By direct calculations, we also obtain

$$\mathcal{F}_3 = \frac{\mathcal{F}^2}{\|\xi\|^4}\xi_2^2(\xi_2^2 + \xi_4^2 + \xi_5^2) + \frac{\mathcal{F}\sinh(\|\xi\|)}{\|\xi\|^3}(2\xi_2^2 + \xi_4^2 + \xi_5^2) + \frac{\sinh^2(\|\xi\|)}{\|\xi\|^2},$$

and thus we see that $F_3 > 0$ if $\|\xi\| \neq 0$. Hence, we see that $\mathrm{rank} A = 3$ if $\|\xi\| \neq 0$.

Therefore, (6,2) has a non-trivial solution for each $\|\xi\| \neq 0$. For this solution (a_2, a_4, a_6), we can verify that there exists $(b_1, b_2) \in \mathbb{R}^2$ which satisfies

$$b_1 \frac{\partial}{\partial \varphi_1} + b_2 \frac{\partial}{\partial \varphi_2} + a_2 \frac{\partial}{\partial \xi_2} + a_4 \frac{\partial}{\partial \xi_4} + a_6 \frac{\partial}{\partial \xi_6} = I_z \eta_z^{\#} = \frac{\sinh(\|\xi\|)}{\|\xi\|} \begin{bmatrix} \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \\ 0 \end{bmatrix}.$$

In fact, the left-hand side is also written as follows:

$$b_1 \cosh(\|\xi\|)(p^{\perp})_1 + b_2 \cosh(\|\xi\|)(p^{\perp})_2 + \frac{\sinh(\|\xi\|)}{\|\xi\|}(a_2\xi_2 + a_4\xi_4 + a_6\xi_6)p,$$

where

$$p = \begin{bmatrix} \cos\varphi_1 \cos\varphi_2 \\ 0 \\ \cos\varphi_1 \sin\varphi_2 \\ 0 \\ \sin\varphi_1 \\ 0 \end{bmatrix}, (p^{\perp})_1 = \begin{bmatrix} -\sin\varphi_1 \cos\varphi_2 \\ 0 \\ -\sin\varphi_1 \sin\varphi_2 \\ 0 \\ \cos\varphi_1 \\ 0 \end{bmatrix}, (p^{\perp})_2 = \begin{bmatrix} -\cos\varphi_1 \sin\varphi_2 \\ 0 \\ \cos\varphi_1 \cos\varphi_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $(p, (p^{\perp})_1, (p^{\perp})_2)$ is an orthonormal basis of $\mathbb{R}^3$ with respect to the canonical real inner product, the condition to verify is reduced to the following condition:

$$\begin{bmatrix} \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \\ 0 \end{bmatrix} \cdot p = \begin{bmatrix} \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ 0 \\ a_4 \\ 0 \\ a_6 \\ 0 \end{bmatrix}.$$

By using the relation (6,2), we see that the left-hand side equals

$$\frac{1}{\cosh(\|\xi\|)} \begin{bmatrix} \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \\ 0 \end{bmatrix} \cdot A \begin{bmatrix} a_2 \\ 0 \\ a_4 \\ 0 \\ a_6 \\ 0 \end{bmatrix} = \frac{1}{\cosh(\|\xi\|)} A \begin{bmatrix} \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ 0 \\ a_4 \\ 0 \\ a_6 \\ 0 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ 0 \\ a_4 \\ 0 \\ a_6 \\ 0 \end{bmatrix}.$$

Thus we see that $I_z\eta_z^{\#} \in \mathrm{T}_z\Phi(L_1)$. Hence, (LagAng-$H$)$'$ holds if $\|\xi\| \neq 0$.

When $\|\xi\| = 0$, by taking a limit $\|\xi\| \to 0$, we have

$$\frac{\partial}{\partial \xi_j} \to \sqrt{-1}\mathbf{e}_j \quad (j = 2, 4, 6).$$

Thus we can also verify that $I_z\eta_z^{\#} \in \mathrm{T}_z\Phi(L_1)$ when $\|\xi\| = 0$.

(1-II) Note that $\mu(L_1 \cap \{\|\xi\| = 0\}) = 0$. When $\|\xi\| \neq 0$, we use the following fact: There exists a neighborhood U_p around $p \in V_c^{(1)}$ in L_1 such that $V_c^{(1)} \cap U_p$ is a submanifold of L_1 (therefore of M), if $(\nabla\mu_\eta)_p \neq \mathbf{0} \in \mathrm{T}_pL_1$, and then $\dim V_c^{(1)} = \dim L_1 - 1 = 5 - 1 = 4$. Here ∇ is the gradient with respect to the induced metric $\iota^* g_{\mathrm{Stz}}$ by the inclusion map $\iota : L_1 \to M$. By direct calculations, we obtain

$$\nabla\eta_\mu = \begin{bmatrix} \mathcal{K}(\sin\varphi_1\cos\varphi_2\xi_2 + \sin\varphi_1\sin\varphi_2\xi_4 - \cos\varphi_1\xi_6) \\ \mathcal{K}(\cos\varphi_1\sin\varphi_2\xi_2 - \cos\varphi_1\cos\varphi_2\xi_4) \\ -\mathcal{G}\xi_2(\cos\varphi_1\cos\varphi_2\xi_2 + \cos\varphi_1\sin\varphi_2\xi_4 + \sin\varphi_1\xi_6) - \mathcal{K}\cos\varphi_1\cos\varphi_2 \\ -\mathcal{G}\xi_4(\cos\varphi_1\cos\varphi_2\xi_2 + \cos\varphi_1\sin\varphi_2\xi_4 + \sin\varphi_1\xi_6) - \mathcal{K}\cos\varphi_1\sin\varphi_2 \\ -\mathcal{G}\xi_6(\cos\varphi_1\cos\varphi_2\xi_2 + \cos\varphi_1\sin\varphi_2\xi_4 + \sin\varphi_1\xi_6) - \mathcal{K}\sin\varphi_1 \end{bmatrix},$$

with respect to $(\frac{\partial}{\partial\varphi_1}, \frac{\partial}{\partial\varphi_2}, \frac{\partial}{\partial\xi_2}, \frac{\partial}{\partial\xi_4}, \frac{\partial}{\partial\xi_6})$. Here $\mathcal{K} = \mathcal{K}(\|\xi\|)$ is as defined above and

$$\mathcal{G} = \mathcal{G}(\|\xi\|) = \\ \frac{1}{\|\xi\|}\left\{2f''(\cosh(2\|\xi\|))\sinh^2(2\|\xi\|) + f'(\cosh(2\|\xi\|))\left(2\cosh(2\|\xi\|) - \frac{\sinh(2\|\xi\|)}{\|\xi\|}\right)\right\}.$$

By the assumption, ${}^t(\xi_2, \xi_4, \xi_6) \neq \mathbf{0}$. If $\sin\varphi_1\cos\varphi_2\xi_2 + \sin\varphi_1\sin\varphi_2\xi_4 - \cos\varphi_1\xi_6 \neq 0$ or $\cos\varphi_1\sin\varphi_2\xi_2 - \cos\varphi_1\cos\varphi_2\xi_4 \neq 0$, the first or second component of $\nabla\mu_\eta$ cannot be zero because $K > 0$ if $\|\xi\| \neq 0$. Hence, $\xi \centerdot (p^\perp)_1 = \xi \centerdot (p^\perp)_2 = 0$ is necessary for $\nabla\mu_\eta = \mathbf{0}$, that is, $\xi = \alpha p$ for some $\alpha \in \mathbb{R}\backslash\{0\}$ is necessary. Clearly, $\mathcal{G} \neq 0$ is also necessary. Then, noting $\alpha^2 = \|\xi\|^2$, we have

$$\nabla\mu_\eta = \begin{bmatrix} 0 \\ 0 \\ -\cos\varphi_1\cos\varphi_2(\mathcal{G}\|\xi\|^2 + \mathcal{K}) \\ -\cos\varphi_1\sin\varphi_2(\mathcal{G}\|\xi\|^2 + \mathcal{K}) \\ -\sin\varphi_1(\mathcal{G}\|\xi\|^2 + \mathcal{K}) \end{bmatrix}.$$

By direct calculations, we also have

$$\mathcal{G}\|\xi\|^2 + \mathcal{K} = \ddot{F}(2\|\xi\|).$$

By the properties of Stenzel's Kähler potential that $\ddot{F} > 0$ if $\|\xi\| \neq 0$ (see Remark 3.3), we thus see that $\nabla\mu_\eta \neq \mathbf{0}$ on $L_1 \backslash \{\|\xi\| = 0\}$. When $\|\xi\| = 0$, it is sufficient to verify $V_0^{(1)}$ is a submanifold of M. By the expression of the moment map in Lemma 6.1, we have

$$V_0^{(1)} = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \end{bmatrix} \right) \middle| \ \|x\| = 1, \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} \cdot \begin{bmatrix} \xi_2 \\ \xi_4 \\ \xi_6 \end{bmatrix} = 0 \right\}.$$

This is diffeomorphic to $\mathrm{T}S^2$. Therefore $V_c^{(1)} \neq \emptyset$ is a submanifold of M for any $c \in \mathfrak{h}^*$, and (Lag-dim) holds for $(V_c^{(1)}, H, K)$. Thus we have proven (1) of the proposition.

(2) This proof is based on Corollary 5.7. (2-I) the generalized perpendicular condition: the H-action satisfies (LagAng-H) on L_2. To show it, first we assume $\|\xi\| \neq 0$. By direct calculations, we have

$$\eta_z^{\#} = \cosh(\|\xi\|) \begin{bmatrix} 0 \\ \cos\varphi \\ 0 \\ \sin\varphi \\ 0 \\ 0 \end{bmatrix} + \sqrt{-1}\frac{\sinh(\|\xi\|)}{\|\xi\|} \begin{bmatrix} -\xi_2 \\ 0 \\ -\xi_4 \\ 0 \\ -\xi_6 \\ \xi_5 \end{bmatrix}.$$

We set the following strategy. First we decompose $I_z\eta_z^{\#}$ as follows:

$$I_z\eta_z^{\#} = I_z(\eta_z^{\#})_1 + I_z(\eta_z^{\#})_2,$$

where

$$I_z(\eta_z^{\#})_1 = \frac{\sinh(\|\xi\|)}{\|\xi\|} \begin{bmatrix} \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \sqrt{-1}\cosh(\|\xi\|) \begin{bmatrix} 0 \\ \cos\varphi \\ 0 \\ \sin\varphi \\ 0 \\ 0 \end{bmatrix}, \ I_z(\eta_z^{\#})_2 = \frac{\sinh(\|\xi\|)}{\|\xi\|} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \xi_6 \\ -\xi_5 \end{bmatrix}.$$

Then assume that $I_z(\eta_z^\#)_1 \in \mathrm{T}_z\Phi(L_2)$. Since $I_z(\eta_z^\#)_2$ clearly has no $\mathrm{T}_z\Phi(L_2)$-components, we see that the decomposition above is a direct decomposition with respect to $\mathrm{T}_zQ^n = \mathrm{T}_z\Phi(L_2) \oplus \mathrm{T}_z^\perp\Phi(L_2)$. Since

$$(\eta_z^\#)_2 = \sqrt{-1}\frac{\sinh(\|\xi\|)}{\|\xi\|}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\xi_6 \\ \xi_5 \end{bmatrix}$$

and μ depend neither on fifth nor sixth component of the imaginary part of $\mathbb{C}^{n+1} \cong \mathrm{T}_z\mathbb{C}^{n+1} \supset \mathrm{T}_zQ^n$, we have

$$\langle (d\mu_\eta)_z, (\eta_z^\#)_2 \rangle = 0,$$

namely $(\eta_z^\#)_2 \in \mathrm{T}_z\mu^{-1}(\mu(z))$. Hence we have that $(\eta_z^\#)_2 \in \mathrm{T}_z\mu^{-1}(\mu(z)) \cap \mathrm{T}_z\Phi(L_2) = \mathrm{T}_z\Phi(V_{\mu(z)}^{(2)})$. Noting that $(\eta_z^\#)_1 \neq \mathbf{0}$ for any $z \in \Phi(L_2)$, we thus see that (LagAng-H) holds if $I_z(\eta_z^\#)_1 \in \mathrm{T}_z\Phi(L_2)$. We can actually verify $I_z(\eta_z^\#)_1 \in \mathrm{T}_z\Phi(L_2)$ as follows. For generating the imaginary part of $I_z(\eta_z^\#)_1$ by $\left(\frac{\partial}{\partial \xi_j}\right)$ $(j = 2, 4, 5, 6)$, the following is necessary: there exists $(a_2, a_4, a_5, a_6) \in \mathbb{R}^4 \backslash \{\mathbf{0}\}$ which satisfies

$$B\begin{bmatrix} a_2 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} \cosh(\|\xi\|)\cos\varphi \\ \cosh(\|\xi\|)\sin\varphi \\ 0 \\ 0 \end{bmatrix}, \tag{6,3}$$

where

$$B = \begin{bmatrix} \frac{\xi_2^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|} & \frac{\xi_2\xi_4}{\|\xi\|^2}\mathcal{F} & \frac{\xi_2\xi_5}{\|\xi\|^2}\mathcal{F} & \frac{\xi_2\xi_6}{\|\xi\|^2}\mathcal{F} \\ \frac{\xi_2\xi_4}{\|\xi\|^2}\mathcal{F} & \frac{\xi_4^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|} & \frac{\xi_4\xi_5}{\|\xi\|^2}\mathcal{F} & \frac{\xi_4\xi_6}{\|\xi\|^2}\mathcal{F} \\ \frac{\xi_2\xi_5}{\|\xi\|^2}\mathcal{F} & \frac{\xi_4\xi_5}{\|\xi\|^2}\mathcal{F} & \frac{\xi_5^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|} & \frac{\xi_5\xi_6}{\|\xi\|^2}\mathcal{F} \\ \frac{\xi_2\xi_6}{\|\xi\|^2}\mathcal{F} & \frac{\xi_4\xi_6}{\|\xi\|^2}\mathcal{F} & \frac{\xi_5\xi_6}{\|\xi\|^2}\mathcal{F} & \frac{\xi_6^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|} \end{bmatrix}.$$

We can verify that $\mathrm{rank}B = 4$ if $\|\xi\| \neq 0$ as follows. By elementary transformations of matrices, we have

$$B \to \frac{1}{(\mathcal{F}_1)^3(\mathcal{F}_2)^2\tilde{\mathcal{F}}_3}\begin{bmatrix} \mathcal{F}_1 & * & * & * \\ 0 & \frac{\sinh(\|\xi\|)}{\|\xi\|}\mathcal{F}_2 & * & * \\ 0 & 0 & \frac{\sinh(\|\xi\|)}{\|\xi\|}\tilde{\mathcal{F}}_3 & * \\ 0 & 0 & 0 & \mathcal{F}_4 \end{bmatrix},$$

where

$$\tilde{\mathcal{F}}_3 = \left\{\frac{\xi_2^2+\xi_4^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|}\right\}\left\{\frac{\xi_2^2+\xi_5^2}{\|\xi\|^2}\mathcal{F} + \frac{\sinh(\|\xi\|)}{\|\xi\|}\right\} - \frac{\xi_4^2\xi_5^2}{\|\xi\|^4}\mathcal{F}^2,$$

$$\mathcal{F}_4 = \mathcal{F}_3\tilde{\mathcal{F}}_3 - \left\{\frac{\xi_5\xi_6}{\|\xi\|^2}\mathcal{F}_1\mathcal{F}\right\}^2,$$

$\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are same as the proof of (1). The functions $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ satisfy > 0 if $\|\xi\| > 0$ as we verified them there. We have

$$\begin{aligned}
&\mathcal{F}_4 = \\
&\left\{\frac{\xi_2^4+\xi_2^2\xi_4^4+\xi_2^2\xi_6^2}{\|\xi\|^4}\mathcal{F}^2 + \frac{\sinh(\|\xi\|)}{\|\xi\|^3}(2\xi_2^2+\xi_4^2+\xi_6^2)\mathcal{F} + \frac{\sinh^2(\|\xi\|)}{\|\xi\|^2}\right\} \\
&\times\left\{\frac{\xi_2^4+\xi_2^2\xi_4^4+\xi_2^2\xi_5^2}{\|\xi\|^4}\mathcal{F}^2 + \frac{\sinh(\|\xi\|)}{\|\xi\|^3}(2\xi_2^2+\xi_4^2+\xi_5^2)\mathcal{F} + \frac{\sinh^2(\|\xi\|)}{\|\xi\|^2}\right\} \\
&-\left\{\frac{\xi_2^4\xi_5^2\xi_6^2}{\|\xi\|^8}\mathcal{F}^4 + 2\frac{\sinh(\|\xi\|)}{\|\xi\|^7}\xi_2^2\xi_5^2\xi_6^2\mathcal{F}^3 + \frac{\sinh^2(\|\xi\|)}{\|\xi\|^6}\xi_5^2\xi_6^2\mathcal{F}^2\right\}
\end{aligned}$$

For the sake of ease, we denote this formula above by

$$\mathcal{F}_4 = \{l_1+l_2+l_3\}\times\{r_1+r_2+r_3\} - \{t_1+t_2+t_3\}.$$

We can see that $l_i \times r_j, t_i \geq 0$ if $\|\xi\| \neq 0$ for $i,j = 1,2,3$ and that $l_3 \times r_3 = \frac{\sinh^4(\|\xi\|)}{\|\xi\|^4} > 0$ if $\|\xi\| \neq 0$. Since $l_1 \times r_1 \geq t_1$, $l_1 \times r_2 + l_2 \times r_1 \geq t_2$ and $l_2 \times r_2 \geq t_3$, we thus see that $\mathcal{F}_4 > 0$ if $\|\xi\| \neq 0$. Hence, $\mathrm{rank} B = 4$ and the equation (6,3) has a non-trivial solution if $\|\xi\| \neq 0$. We can verify in the same way of the proof of (1) that for this solution (a_2, a_4, a_5, a_6), there exists $b \in \mathbb{R}\backslash\{0\}$ which satisfies

$$b\frac{\partial}{\partial\varphi} + a_2\frac{\partial}{\partial\xi_2} + a_4\frac{\partial}{\partial\xi_4} + a_5\frac{\partial}{\partial\xi_5} + a_6\frac{\partial}{\partial\xi_6} = I_z(\eta_z^\#)_1.$$

Therefore, (LagAng-H) holds at any point $p \in L_2\backslash\{\|\xi\| = 0\}$. When $\|\xi\| = 0$, we can also compute $I_z\eta_z^\# \in \mathrm{T}_z\Phi(L_2)$ by taking a limit $\|\xi\| \to 0$. This indicates that the stronger condition (LagAng-H)$'$ holds at any point $p \in L_2 \cap \{\|\xi\| = 0\}$. Thus we see that (2-I) holds.

(2-II) the submanifold condition: $V_c^{(2)} \neq \emptyset$ is a submanifold of M and (Lag-dim) holds for $(V_c^{(2)}, H, K)$. We first prove this for $c \neq 0$ in the same way of the proof of (1) which show that $\nabla \mu_\eta \neq \mathbf{0}$ on $L_2 \backslash \{\|\xi\| = 0\}$. We note that $\|\xi\| \neq 0$ if $\mu(z) \neq 0$. By direct calculations, we have

$$\nabla \eta_\mu = \begin{bmatrix} \mathcal{K}(\sin\varphi\xi_2 - \cos\varphi\xi_4) \\ -\xi_2\mathcal{G}(\cos\varphi\xi_2 + \sin\varphi\xi_4) - \mathcal{K}\cos\varphi \\ -\xi_4\mathcal{G}(\cos\varphi\xi_2 + \sin\varphi\xi_4) - \mathcal{K}\sin\varphi \\ -\xi_5\mathcal{G}(\cos\varphi\xi_2 + \sin\varphi\xi_4) \\ -\xi_6\mathcal{G}(\cos\varphi\xi_2 + \sin\varphi\xi_4) \end{bmatrix},$$

with respect to $(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\xi_2}, \frac{\partial}{\partial\xi_4}, \frac{\partial}{\partial\xi_5}, \frac{\partial}{\partial\xi_6})$. Here $\mathcal{K} = \mathcal{K}(\|\xi\|)$ and $\mathcal{G} = \mathcal{G}(\|\xi\|)$ are as defined in the proof of (1). We can see that for $\nabla \mu_\eta = \mathbf{0}$, it is necessary (i) $\xi_2 = \xi_4 = 0$, (ii)${}^t(\xi_2, \xi_4) \bullet {}^t(-\sin\varphi, \cos\varphi) = 0$, that is, $\xi = \alpha \cdot {}^t(\cos\varphi, \sin\varphi)$ for some $\alpha \in \mathbb{R}\backslash\{0\}$, (iii) $\mathcal{G} \neq 0$, and (iv) $\xi_5 = \xi_6 = 0$. Then, noting $\alpha^2 = \|\xi\|^2$, we have

$$\nabla \mu_\eta = \begin{bmatrix} 0 \\ -\cos\varphi(\mathcal{G}\|\xi\|^2 + \mathcal{K}) \\ -\sin\varphi(\mathcal{G}\|\xi\|^2 + \mathcal{K}) \\ 0 \\ 0 \end{bmatrix}.$$

Since $\mathcal{G}\|\xi\|^2 + \mathcal{K} = \ddot{F}\,(2\|\xi\|)$ and $\ddot{F} > 0$ if $\|\xi\| \neq 0$, we thus see that $\nabla \mu_\eta \neq \mathbf{0}$ on $L_2 \backslash \{\|\xi\| = 0\}$. For $c = 0$, $V_0^{(2)}$ is expressed as follows:

$$V_0^{(2)} = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} \right) \middle| \; \|x\| = 1, \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \bullet \begin{bmatrix} \xi_2 \\ \xi_4 \end{bmatrix} = 0 \right\}.$$

Ignoring fifth and sixth components, this is diffeomorphic to $TS^1 \cong S^1 \times \mathbb{R}$. Fifth and sixth components constitute a plane unrelated to the base manifold. Thus we see $V_0^{(2)} \cong S^1 \times \mathbb{R}^3$ and $(V_0^{(2)}, H, K)$ satisfies (Lag-dim). Thus we see that (2-II) holds. $\square$

6.3 The case of $H = SO(2)\times SO(2)\times SO(3)$, $L = \mathrm{T}^{*\perp}S^2 \subset \mathrm{T}^*S^6$

Let M be the cotangent bundle of 6-sphere T^*S^6 and L the conormal bundle of a totally geodesic submanifold S^2 of S^6 as follows:

$$L(\cong \mathrm{T}^{*\perp}S^2) = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \\ \xi_7 \end{bmatrix} \right) \;\middle|\; \|x\| = 1,\ \xi_j \in \mathbb{R}\ (j = 2,4,6,7) \right\}.$$

Define H as follows:

$$\begin{aligned} &H(\cong SO(2)\times SO(2)\times SO(3)) \\ &= \left\{ \left[\begin{array}{c|c|c} h_1 & 0 & 0 \\ \hline 0 & h_2 & 0 \\ \hline 0 & 0 & h_3 \end{array}\right] \in GL(7,\mathbb{R}) \;\middle|\; h_1, h_2 \in SO(2),\ h_3 \in SO(3) \right\}. \end{aligned}$$

Note that H is non-abelian. The Lie algebra $\mathfrak{h}$ of H is given as follows:

$$\mathfrak{h}(\cong \mathfrak{so}(2)\oplus\mathfrak{so}(2)\oplus\mathfrak{so}(3)) = \mathrm{span}\{\xi_{12}, \xi_{34}, \xi_{56}, \xi_{57}, \xi_{67}\},$$

here ξ_{ij} is as mentioned at the beginning of this section.

We obtain an explicit expression of the moment map of (6,1) by direct calculations. Define μ_{ij} for the basis $(\xi_{12}, \xi_{34}, \xi_{56}, \xi_{57}, \xi_{67})$ of $\mathfrak{h}$ and $z \in \Phi(L)$ by $\mu_{ij}(z) = \langle \mu(z), \xi_{ij}\rangle$.

Lemma 6.3. *For (M, I, ω, H, μ) above, and $z \in \Phi(L)\backslash\{\|\xi\| = 0\}$, we have*

$$\begin{aligned} \mu_{12}(z) &= -\mathcal{K}(\|\xi\|)\cos\varphi_1\cos\varphi_2\xi_2, \\ \mu_{34}(z) &= -\mathcal{K}(\|\xi\|)\cos\varphi_1\sin\varphi_2\xi_4, \\ \mu_{56}(z) &= -\mathcal{K}(\|\xi\|)\sin\varphi_1\xi_6, \\ \mu_{57}(z) &= -\mathcal{K}(\|\xi\|)\sin\varphi_1\xi_7, \\ \mu_{67}(z) &\equiv 0. \end{aligned}$$

Here, we use the polar coordinates $x_1 = \cos\varphi_1\cos\varphi_2, x_3 = \cos\varphi_1\sin\varphi_2, x_5 = \sin\varphi_1$, and

$$\mathcal{K}(\|\xi\|) = \frac{f'(\cosh(2\|\xi\|))\sinh(2\|\xi\|)}{\|\xi\|}.$$

We obtain the following result:

Proposition 6.4. *Let $(M, I, \omega_{\mathrm{Stz}}, \Omega_{\mathrm{Stz}}, L, H)$ be as above. Define a rank two subbundle $\hat{L}$ of L as follows:*

$$\hat{L} = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \;\middle|\; \|x\| = 1,\ \xi_j \in \mathbb{R}\ (j = 2, 4) \right\}.$$

Let $\hat{L}^{\mathrm{pr}}$ be the set of all points $p \in \hat{L}$ such that the isotropy subgroup H_p satisfies $H_p \subset H_q$ for all $q \in \hat{L}$. For $(c_1, c_2) \in \mathbb{R}^2$, define $V_{(c_1,c_2)}$ and $\hat{V}_{(c_1,c_2)}$ by

$$V_{(c_1,c_2)} = \hat{L}^{\mathrm{pr}} \cap \{p \in M \mid \mu_{12}(p) = c_1, \mu_{34}(p) = c_2, \mu_{ij}(p) = 0\},$$
$$\hat{V}_{(c_1,c_2)} = \hat{L} \cap \{p \in M \mid \mu_{12}(p) = c_1, \mu_{34}(p) = c_2, \mu_{ij}(p) = 0\},$$

where $(i, j) = (5, 6), (5, 7), (6, 7)$. Then for any $(c_1, c_2) \neq (0, 0) \in \mathbb{R}^2$ such that $V_{(c_1,c_2)} \neq \emptyset$, $H \cdot V_{(c_1,c_2)}$ is a special Lagrangian submanifold of M, and $H \cdot \hat{V}_{(0,0)}$ is a union of five connected special Lagrangian submanifolds of M.

Proof. The proof for $V_{(c_1,c_2)}$ is based on Corollary 5.10, and one for $\hat{V}_{(0,0)}$ on direct calculations. As we saw in the proof of Corollary 5.9, $V_{(c_1,c_2)}$ has to be included in the inverse image of the center $Z(\mathfrak{h}^*)$ of $\mathfrak{h}^*$ with the moment map μ. Hence, noting the $\mathfrak{so}(3)$-part of $\mathfrak{h} \cong \mathfrak{so}(2) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(3)$, we see that we can apply our construction for the part such that $\mu_{ij}(p) = 0$ $((i, j) = (5, 6), (5, 7), (6, 7))$ in L. This indicates that $\xi_6 = \xi_7 = 0$ is necessary. That is, the place in where we have to check the conditions of Corollary 5.9 is $\hat{L} \subset L$.

By the definition of $V_{(c_1,c_2)}$, at any point $p \in V_{(c_1,c_2)}$, the isotropy subgroup H_p is the following one-parameter subgroup K generated by ξ_{67}:

$$K(\cong SO(2)) = \left\{ \left[\begin{array}{c|c} E_5 & \\ \hline & h \end{array} \right] \;\middle|\; h \in SO(2) \right\},$$

here E_5 is the unit 5×5-matrix. Thus we see that (Imm-istp) holds.

First we prove the proposition for $V_{(c_1,c_2)}$ for $(c_1, c_2) \neq (0, 0)$. Since $\mu(\hat{L} \cap \{\|\xi\| = 0\}) = \mathbf{0}$, we can assume $\|\xi\| \neq 0$ in this case. As same in Proposition 6.2, conditions we have to check are the followings: (I) the perpendicular condition: the H-action satisfies (LagAng-H)$'$ on $\hat{L}\backslash\{\|\xi\| = 0\}$, and (II) the submanifold condition: $V_{(c_1,c_2)} \neq \emptyset$ is a submanifold of M and (Lag-dim) holds for $(V_{(c_1,c_2)}, H, K)$. We can verify these in the same way as Proposition 6.2.

Finally we study $\hat{V}_{(0,0)}$ generally rather than $V_{(0,0)}$, including non-principal points. By Lemma 6.3, We obtain that

$$\hat{V}_{(0,0)} = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \middle| \begin{array}{c} \|x\| = 1, \\ \xi_2, \xi_4 \in \mathbb{R}, \\ x_1\xi_2 = x_3\xi_4 = 0 \end{array} \right\}.$$

$\hat{V}_{(0,0)}$ is not a smooth manifold. However it is a union, which is not disjoint, of the following five connected manifolds:

$$\hat{V}_{(0,0)} = \hat{V}^{S^2}_{(0,0)} \cup \hat{V}^{S^1\times\mathbb{R}}_{(0,0),(1)} \cup \hat{V}^{S^1\times\mathbb{R}}_{(0,0),(3)} \cup \hat{V}^{\mathbb{R}^2}_{(0,0),(1)} \cup \hat{V}^{\mathbb{R}^2}_{(0,0),(-1)},$$

here

$$\hat{V}^{S^2}_{(0,0)} = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ 0 \\ 0 \end{bmatrix}, \mathbf{0} \right) \middle| \|x\| = 1 \right\}, \quad \hat{V}^{S^1\times\mathbb{R}}_{(0,0),(1)} = \left\{ \left(\begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \middle| \begin{array}{c} \|x\| = 1, \\ \xi_2 \in \mathbb{R} \end{array} \right\},$$

$$\hat{V}^{S^1\times\mathbb{R}}_{(0,0),(3)} = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \\ x_5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \xi_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \middle| \begin{array}{c} \|x\| = 1, \\ \xi_4 \in \mathbb{R} \end{array} \right\}, \quad \hat{V}^{\mathbb{R}^2}_{(0,0),(\epsilon)} = \left\{ \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \epsilon \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \middle| \xi_2, \xi_4 \in \mathbb{R} \right\},$$

and $\epsilon = \pm 1$. We can see that each set $\hat{V}^{W}_{(0,0)}$ is a 2-dimensional connected submanifold of M diffeomorphic to W. Each $\hat{V}^{W}_{(0,0)}$ has non-principal orbits with respect to the action of H on $H \cdot \hat{V}^W_{(0,0)}$. Hence it does not satisfy (Imm-istp). However we can directly verify that each $H \cdot \hat{V}^W_{(0,0)}$ for $\hat{V}^{S^2}_{(0,0)}$, $\hat{V}^{S^1\times\mathbb{R}}_{(0,0),(j)}(j=1,3)$, and $\hat{V}^{\mathbb{R}^2}_{(0,0),(\epsilon)}(\epsilon=\pm 1)$ is a special Lagrangian submanifold of M diffeomorphic to S^6, $\mathrm{T}^{*\perp}S^4$, and $\mathrm{T}^{*\perp}S^2$ respectively. $\square$

We chose $SO(2)\times SO(2)\times SO(3)$ as a Lie group H for the special Lagrangian submanifold $L \subset \mathrm{T}^*S^6$ of Proposition 6.4 rather than $SO(2)\times SO(5)$ because of two reasons. First, since the center of the Lie algebra $\mathfrak{h} \cong \mathfrak{so}(2)\oplus\mathfrak{so}(2)\oplus\mathfrak{so}(3)$ has two dimensions, we could obtain two-parameters of special Lagrangian submanifolds $H \cdot V_{(c_1,c_2)}$. Second, for $p, q \in \mathbb{N}$ such that $p+q=n+1$, special Lagrangian submanifolds which are $SO(p)\times SO(q)$-invariant in $(\mathrm{T}^*S^n, I, g_{\mathrm{Stz}}, \omega_{\mathrm{Stz}})$ have been already obtained by Hashimoto and Sakai in [6], and they showed that such special Lagrangian submanifolds are cohomogeneity one with respect to $SO(p)\times SO(q)$. In the case of Proposition 6.4, we can verify that $SO(2)\times SO(2)\times SO(3)$ acts on $H \cdot V_{(c_1,c_2)}$ with cohomogeneity two.

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