Extra Dimensions,
Modified Gravity and Inflation

Hiroshi Nakada

Department of Physics,
Graduate School of Science and Engineering,
Tokyo Metropolitan University

DISSERTATION
submitted for the degree of
Doctor of Science

2019

Abstract

An extra dimension was introduced for the first time by Kaluza and Klein in 1926, in order to unify gravitational field and electromagnetic field. More extra dimensions are often required by modern unified theories such as string theory. Another way of dealing with extra dimensions is the brane-world whose extra dimensions can only be checked by gravity. In the thesis, the ideas of extra dimensions and brane-world are applied for a description of cosmological inflation in the early Universe in the context of the $f(R)$ gravity theory. In order to cause the inflationary expansion of the Universe, a scalar field called inflaton is necessary. The $f(R)$ gravity is the modified gravitational theory in which the standard (Einstein-Hilbert) action is replaced by a function f of Ricci scalar R. The special case, known as Starobinsky model of $R+R^2$ gravity, is the very successful inflationary model in four dimensions, in line with all current observations of the Cosmic Microwave Background (CMB) radiation. We unified the Randall-Sundrum brane-world with the Starobinsky model in five dimensions and found that Starobinsky modified gravity does not destroy the Randall-Sundrum solution. Next, we considered the function $f(R) = R + \gamma R^n - 2\Lambda$ in higher (D) spacetime dimensions with the cosmological constant Λ, and described spontaneous compactification down to four spacetime dimensions, as the inflationary model. We found that it requires $D = n/2$, whereas the extra gauge $(p-1)$-form is needed for consistent (spontaneous) Freund-Rubin-type compactification. The cases of $D = 8$ and $D = 12$ were studied in detail, and the predictions for the CMB tensor-to-scalar ratio were calculated.

Contents

Introduction

The appearance of Einstein's general relativity has greatly changed thinking of physicists in two ways. One is to describe the laws of physics in terms of geometry. Einstein formulated gravitational theory by using Riemannian geometry with gravity as spacetime geometry. This made a great influence on physicists at that time, it became a trigger to consider the unified theory, to geometrically and physically describe the physical laws unifiedly, that is, as the unified theory. The other is to make the spacetime to be the dynamical object. As a result, the Universe where we live is not static, but it can be thought as a dynamic object. In fact, Friedmann derived his equation that represents the expansion and contraction of the Universe, and then discovered that the Universe continues to expand. It was later confirmed by observation of Hubble. This is the beginning of the modern cosmology about the evolution of the Universe.

Currently, it is known that superstring theory and M-theory are promising as the candidates for unified theory, but the theory first proposed as a unified theory was a five-dimensional theory by Kaluza [1]. Kaluza considered the gravitational theory in five-dimensional spacetime with one extra dimension and showed that four-dimensional gravitational field and electromagnetic field appear from the five-dimensional metric. However, Kaluza assumed that all fields do not depend on the extra dimension, and did not explain why the extra dimension is invisible. Klein showed that the charge of the electromagnetic field is quantized by adding Kaluza's theory to the proposal that the extra dimension is small and closed to a circle [2]. Also, this assumption explains that extra dimension can not be seen. It is called Kaluza-Klein compactification treating the extra dimension that is closed and can not be seen because it is small. After that, as the unified theory was actively studied by the development of the gauge theory and the Standard Model of elementary particles, research to unify the higher-dimensional gravity and general gauge theory was generalized.

Kaluza-Klein compactification cannot describe why extra dimension is small. From this reason, research on spontaneous compactification and its stabilization was conducted. Specific models of the spontaneous compactification use of the gauge fields [3–5], the higher powers of the scalar curvature [6], compactification by quantum fluctuations of the scalar field [7], etc. Research of compactfications like these, is still actively conducted to derive a four-dimensional effective theory from String theory and M-theory.

In the Kaluza-Klein compactification, the size of extra dimension and its stabilization become a problem, but in the brane-world picture, the extra dimensions cannot be seen because particles of Standard Model other than gravity are localized in three-dimensional brane, so such problems may not occur. The brane-world model is the alternative to Kaluza-Klein compactification handling the extra dimension, such as ADD model [8, 9], Randall-Sundrum model [10, 11], and DGP model [12]. The two models proposed by Randall-Sundrum got a lot of attention. The first one is the two brane model, to solve the hierarchy problem (although the problem of stabilizing the distance between two branes remains). The second model, the one brane model, gained a lot of attention as a model allowing infinitely large extra dimensions.

Meanwhile, cosmology has also been constantly developed along with the development of the observation technologies. The current Standard Model of cosmology is called the ΛCDM model, which is a theory that consistently explains from 10^{-22}s to the present since the Universe was born. The cosmological constant Λ represents the dark energy and is necessary to explain the current expansion of the Universe, CDM stands for cold dark matter, and is required for the formation of the large scale structure of the Universe. Although ΛCDM is supported from observations, one does not know the identity of dark energy, dark matter, but also one has fundamental problems of Big Bang cosmology. It is the flatness problem and the horizon problem. The inflation theory was proposed independently by Sato [13] and Guth [14] to make these fundamental solutions. This is the theory that assumes an exponential expansion in the early Universe, and is considered to be the origin of fluctuations for making the anisotropy of Cosmic Microwave Background(CMB) and the large scale structure. This inflation is thought to be caused by a scalar field called inflaton, and the origins of this inflaton and its potential are still actively studied at present.

Modified gravity is proposed to solve these cosmological problems, and there are various models according to the problem. In order to explain inflation we need to introduce a new matter called inflaton. Also, the dark matter is unknown matter, and the dark energy has properties different from ordinary matter. On the other hand, modified gravity modifies Einstein's theory of gravity, so that inflation and dark energy can be incorporated into theory as geometrical properties of spacetime.

The $f(R)$ theory is a theory replacing Einstein-Hilbert Lagrangian with a function $f(R)$ of scalar curvature [15, 16]. This theory has long been studied extensively as a model of dark energy and inflation. As a special case, the Starobinsky model is known as an inflation model that satisfies current observations well [17]. This model was suggested by Starobinsky when inflation was not yet proposed. He showed that adding the R^2 term as quantum effect of gravity to Einstein-Hilbert action leads to a de Sitter solution.

We studied the modified gravity in higher-dimensional spacetime. In [18], we applied the Randall-Sundrum two-brane model to the five-dimensional Starobinsky model and investigated its stability. As a result of concrete calculation, it was found that the scalar filed contributing from R^2 term is stabilized to minimum by its potential, and does not destroy the Randall-Sundrum model.

In [19], we consider the Lagrangian $R+\gamma R^n-2\Lambda$ and the condition to derive the Starobinsky type inflaton potential which has a plateau. We assume that the spontaneous compactification from the D-dimensional spacetime to the four-dimensional spacetime occurred by some mechanism before the inflation.

S. P. Otero, F. G. Pedro, and C. Wieck developed a model to realize spontaneous compactification and stabilization of the volume of extra dimensions by adding $(p-1)$-form gauge field to our model. [20] According to that model, spontaneous compactification and modulus stabilization are realized at $p=n$. In [21], we analyze the case of $D=8$ and consider the possibility of its embedding in $D=8$ supergravity as the natural origin of the $(p-1)$-form gauge field.

This thesis is organized as follows. Chapter 1 represents the review of the Kaluza-Klein compactification. In Chapter 2, we review of Randall-Sundrum brane models and our first work [18]. In Chapter 3, we review the standard cosmology, modified gravity and our results of the inflation from modified gravity in higher-dimensional spacetime.

The notation in this thesis is as follows. We use the natural units $\hbar=c=1$ and the D-dimensional spacetime signature $(-,+,\cdots,+)$. We denote spacetime vector indices in D-dimensions by capital latin letters $A,B,\ldots=0,1,\ldots,D-1$, and spacetime vector indices in four-dimensions by lower case greek letters $\alpha,\beta,\ldots=0,1,2,3$ and spacetime vector indices in extra dimensions by lower latin letters $a,b,\ldots$. The X means the D-dimensional coordinates, x means the four-dimensional coordinates and y means the extra coordinates.

Chapter 1

Kaluza-Klein Theory

It was Kaluza who introduced the extra dimension for the first time in 1921 [1]. He introduced the fifth dimension to unify the gravitational field and the electromagnetic field in five-dimensional spacetime. Specifically, as a component of a five-dimensional metric, the four-dimensional gravitational field and the electromagnetic field are considered, and under the hypothesis of the cylinder condition that those fields do not depend on the fourth spatial direction called a extra dimension, he succeeded in deriving the four-dimensional Einstein-Maxwell equations. (The equation of scalar field corresponding to dilaton was also derived, but at the time such scalar was treated as a constant.) But, he didn't explain why we can't see the fifth dimension.

In 1926, Klein suggested the quantum interpretation of the Kaluza's theory by assuming the fifth dimension to be a circle and very small [2]. He succeeded to explain why we can't see the fifth dimension, and added to validity of Klein's cylinder condition. Also, he showed such a small extra space leads to a quantization of charge of electromagnetic filed. In that case, the charge is quantized depending on the radius of extra space.

Such spacetime is topologically represented as $M^4 \times S^1$, which can be interpreted as a circle sticking to each point of the four-dimensional spacetime. For example, the surface of a straw is a two-dimensional surface, but when viewed from far enough it looks like a one-dimensional line. In this way, by assuming that the size of the extra dimension is sufficiently small, we can not see the extra dimension. Such a way of treating the extra dimension is called Kaluza-Klein theory. At present, the size of the Kaluza-Klein circle should be much less than $10^{-15}cm$, i.e. much less than a quark size.

Since we can unify $U(1)$ gauge theory with five-dimensional gravity, it is natural to think about generalizing it and unify the general gauge theory. For that purpose, we had to think about the higher-dimensional spacetime with more than five dimensions. This generalized theory is developed in [22–24]. In [25], Witten gave the realistic model of Kaluza-Klein theory which can describe $SU(3) \times SU(2) \times U(1)$ gauge theory.

After that, with the development of the unified theory such as superstring theory or M-theory and supergravity as its effective theory, the necessity of extra dimensions was uplifted to be debated again. The four-dimensional Kaluza-Klein compactification has been extensively studied.

In this Chapter, we describe the Kaluza-Klein theory and its generalizations, and the spontaneous compactification of extra dimensions.

1.1 Dimensional reduction

First, we describe how we can see the five-dimensional massless scalar field in four-dimensional spacetime.

The fifth coordinate y is taken on the circle whose radius is r. So, it is required y to be periodic: $y \sim y + 2\pi r$. Assuming the four-dimensional spacetime to be Minkowski spacetime M^4, this five-dimensional spacetime is written as $M^4 \times S^1$.

The five-dimensional massless scalar field $\phi(x, y)$ obeys five-dimensional Klein-Gordon equation:

$$\partial_A \partial^A \phi(x, y) = 0 \tag{1.1.1}$$

Since y is periodic, we can perform the expansion of $\phi(x, y)$:

$$\phi(x, y) = \sum_{n=-\infty}^{\infty} \phi_n(x) e^{-i\frac{n}{r}y} \tag{1.1.2}$$

Substituting this to (1.1.1), we obtain

$$\left(\partial_\mu \partial^\mu - \frac{n^2}{r^2} \right) \phi_n(x) = 0 \tag{1.1.3}$$

This equation looks like the scalar Klein-Gordon equation with mass $m^2 = n^2/r^2$ in four-dimensional spacetime. So, the massless scalar field in five dimensional spacetime with compact extra dimension behaves as the massive scalar field in four-dimensional spacetime. In the equation (1.1.3), if we choose $n = 0$, this equation reads:

$$\partial_\mu \partial^\mu \phi_0(x) = 0 \tag{1.1.4}$$

Then, $\phi_0(x)$ is a massless scalar field. Such the $n = 0$ mode is called the zero mode, and the $n \neq 0$ modes are called the KK modes. However, KK mode particles have not been detect in the particle experiments so far until now.

If the radius of extra spacetime is very small, the mass $\sim r^{-2}$ (at $n \neq 0$) becomes very large. This means it becomes more difficult to detect the KK mode as the extra spacetime becomes smaller. So, if we take the size of extra dimension to Planck length l_P, and we can ignore KK modes.

Since KK modes were not detected in experiments, the compactification radius r can be restricted from above. Their masses would thus have to be greater, $n/r >$

1TeV, which implies a strong constraint on r:

$$r \leq 10^{-21}\text{cm}. \tag{1.1.5}$$

It is difficult to detect so small extra dimension directly.

This procedure used to derive the effective theory from higher-dimensional fundamental theory is called dimensional reduction. In general, lower spin fields appear from original higher-dimensional fields. For example, the four-dimensional vector field and scalar field come from the five-dimensional vector field. We describe below four-dimensional metric, vector and scalar field from the five-dimensional metric.

1.2 Kaluza-Klein theory as a unified theory

1.2.1 Unification of gravitational field and $U(1)$ gauge field

Let us consider the gravitational field in five-dimensional spacetime, which unifies the four-dimensional gravitational field and electromagnetic field. We assume the five-dimensional spacetime is $M^4 \times S^1$. Also, as we mentioned above, the extra coordinate y is to be a circle whose radius is r.

The five-dimensional Einstein-Hilbert action is

$$S_5 = \frac{1}{2\kappa_5^2} \int d^5X \sqrt{-g_5} R^{(5)} \tag{1.2.1}$$

where, subscript "5" means the five-dimensional quantities. From this action, the vacuum Einstein equation is derived:

$$R^{(5)}_{AB} = 0 \tag{1.2.2}$$

The action (1.2.1) is invariant under the five-dimensional general coordinate transformation:

$$X^A \to X'^A = X^A - \xi^A(X). \tag{1.2.3}$$

The five-dimensional metric transforms under this transformation as

$$\delta g_{AB} = \partial_C g_{AB} \xi^C + g_{CB} \partial_A \xi^C + g_{AC} \partial_B \xi^C \tag{1.2.4}$$

where $\xi^A(X)$ is arbitrary vector field parameter.

For the above five-dimensional action, considering the decomposition of the metric and taking the integration over the extra dimension leads to the four-dimensional effective theory. Before that, in order the four-dimensional theory, to be obtained from the extra dimension, several physical requirements are imposed on the theory.

The first requirement is that all fields appearing in four-dimensions do not depend on extra dimensional coordinate (cylinder condition). This may be considered as taking out the zero mode only by the dimensional reduction explained in the previous section. This means that the extra dimension is sufficiently small, and KK modes of the effective theory at the energy scales lower than its large masses, cannot be observed in elementary particle experiments.

The second requirement is that the theory is invariant against translating along the extra coordinate.This means the parameters of general coordinate transformation $\xi^A(X)$ satisfy the following requirements:

$$\xi^\mu = \xi^\mu(x^\mu), \quad \xi^y = ay + \epsilon(x^\mu) \tag{1.2.5}$$

where a is an arbitrary constant and $\epsilon(x)$ is a function of four-dimensional coordinates.

Next, let us form the metric decomposition, which was suggested by Kaluza to satisfy the above requirements:

$$g_{AB} = e^{\phi/\sqrt{3}} \begin{bmatrix} g_{\mu\nu} + e^{-\sqrt{3}\phi} A_\mu A_\nu & e^{-\sqrt{3}\phi} A_\mu \\ e^{-\sqrt{3}\phi} A_\nu & e^{-\sqrt{3}\phi} \end{bmatrix} \tag{1.2.6}$$

where $g_{\mu\nu}(x)$ is four-dimensional metric, $A_\mu(x)$ is vector field, $\phi(x)$ is scalar field. Although it is not obvious that $A_\mu(x)$ is an Abelian gauge field, by imposing the above physical requirement, it can be shown that five-dimensional general coordinate transformation acts as the gauge transformation of $A_\mu(x)$. From equation (1.2.5), calculating and rearranging the transformations for each component of the five-dimensional metric, we can obtain the transformation laws for each field:

$$\delta g_{\mu\nu} = \partial_\rho g_{\mu\nu} \xi^\rho + g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho \tag{1.2.7}$$

$$\delta A_\mu = \partial_\mu \xi^\rho A_\rho + \xi^\rho \partial_\rho A_\mu + \partial_\mu \epsilon(x) \tag{1.2.8}$$

It can be seen that the five-dimensional general coordinate transformation acts as the U(1) gauge transformation for A_μ. Therefore, A_μ is regarded as a U(1) gauge field.

By substituting the above metric to the five-dimensional Einstein-Hilbert action and performing y-integral, it is possible to obtain the four-dimensional effective action

$$S_4 = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g_4} \left(R^{(4)} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} e^{-\sqrt{3}\phi} F_{\mu\nu} F^{\mu\nu} \right) \tag{1.2.9}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of Maxwell field, and

$$\kappa_4^2 = \kappa_5^2 / 2\pi R \tag{1.2.10}$$

is the four-dimensional gravitational constant. Therefore, the five-dimensional Einstein-Hilbert action is found to be a sum of Einstein-Hilbert action and Maxwell action coupled to the scalar ϕ and the scalar field action in four-dimensional spacetime.

If we set the scalar field to be zero, the action becomes the Einstein-Maxwell action. However, it is not allowed since there is interaction between those lower dimensional fields. To consider this, let us focus on field equations. The field equations are obtained by varying the action with respect to each field:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{1.2.11}$$

$$\nabla^{\mu}\left(e^{-\sqrt{3}\phi}F_{\mu\nu}\right) = 0 \tag{1.2.12}$$

$$\nabla_{\mu}\nabla^{\mu}\phi = -\frac{1}{2}\sqrt{3}e^{-\sqrt{3}\phi}F_{\mu\nu}F^{\mu\nu} \tag{1.2.13}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the Maxwell field and the scalar field:

$$\kappa^2 T_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\partial_\rho\phi\partial^\rho g_{\mu\nu}\right) + \frac{1}{2}e^{-\sqrt{3}\phi}\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}\right) \tag{1.2.14}$$

The field equation (1.2.13) means it is not allowed to set the scalar field to be zero because there is the field strength of the Maxwell field as the source field on the right-hand-side of the equation (1.2.13).

1.2.2 Charge quantization

Klein showed the electric charge of electromagnetism is quantized by introducing the fifth dimension [2]. Let us consider the scalar field in five-dimensional spacetime again. The action of the scalar field is given by

$$S = -\int d^5X\sqrt{-g_5}\frac{1}{2}\partial_A\phi(x,y)\partial^A\phi(x,y). \tag{1.2.15}$$

We can perform the Fourier expansion since the fifth dimension is periodic:

$$\phi(x,y) = \sum_{n=-\infty}^{\infty}\phi_n(x)e^{-i\frac{n}{R}y} \tag{1.2.16}$$

We consider the general coordinate transformation about fifth dimension

$$y \to y' = y + \xi\epsilon(x) \tag{1.2.17}$$

The scalar field transforms under this transformation as

$$\phi(x,y) = \sum_{n=-\infty}^{\infty}\phi_n(x)e^{-i\frac{n}{R}y}e^{-i\frac{n}{R}\xi\epsilon(x)}. \tag{1.2.18}$$

It is necessary to introduce a gauge field A_μ, in order to make the action invariant under that transformation. We introduce the gauge field via the $U(1)$ gauge covariant derivative:

$$S = \int d^4x\sqrt{-g_4}\left[\left(\partial_\mu - i\frac{nkA_\mu}{R}\right)\phi_n\left(\partial^\mu - i\frac{nkA^\mu}{R}\right)\phi_n - \frac{n^2}{R^2}\phi_n\right] \tag{1.2.19}$$

where k a the normalization constant for A_μ we set $k = 1$ in equation (1.2.6). This implies the charge quantization as

$$q_n = -\frac{nk}{R} \tag{1.2.20}$$

in units of k/R. Since the charged particle with charge q_n has the mass $m_n = n/R$, we assume all charged particles ($n \neq 0$) have the masses on the Planck scale.

1.2.3 Scalar-tensor gravity

Let us consider the decomposition of five-dimensional metric as follows:

$$g_{AB} = \begin{bmatrix} g_{\mu\nu} & 0 \\ 0 & \Phi \end{bmatrix} \tag{1.2.21}$$

Substituting this metric to five-dimensional Einstein-Hilbert action, the effective four-dimensional action is obtained as follows:

$$S_4 = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g_4}\, \Phi^{\frac{1}{2}} R^{(4)}. \tag{1.2.22}$$

To separate the Ricci scalar and Φ, we consider the next transformation called Weyl transformation:

$$g_{\mu\nu} \rightarrow \Phi^{-\frac{1}{3}} g_{\mu\nu} \tag{1.2.23}$$

$$\Phi \rightarrow \Phi^{\frac{2}{3}} \tag{1.2.24}$$

Such parametrization in which action is in the form of Einstein-Hilbert action is called Einstein frame. Furthermore, to obtain canonical kinetic term of the scalar field, we rescale the Φ:

$$\phi = \frac{1}{\sqrt{3}\kappa} \ln\left(\frac{\Phi}{\Phi_0}\right) \tag{1.2.25}$$

where Φ_0 is the vacuum expectation value of Φ. Thus, we obtain the following four-dimensional effective action:

$$S_4 = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \tilde{R} - \int d^4x \sqrt{-\tilde{g}} \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \tag{1.2.26}$$

This action is equivalent to Einstein-Hilbert action with the scalar field.

1.3 Generalization to non-Abelian gauge theories

In five-dimensional Kaluza-Klein theory, we can unify four-dimensional gravitational field and electromagnetic field in five-dimensional gravitational field. However, it does not include a general gauge field, i.e. a non-Abelian gauge field. That is because the general coordinate transformation in the S^1 compactified spacetime corresponded to the $U(1)$ gauge transformation. In other words, we have to assume that the non-abelian gauge symmetry appears by taking the compact spacetime of higher dimension.

The transformation that does not change the metric is called the isometry transformation, and this transformation exists when the spacetime has the symmetry. A consideration of the isometry transformations on a compact n-dimensional manifold shows that the Lie algebra made by the generators of the isometry transformations corresponds to the symmetry of the gauge field appearing in the effective action in four-dimensional spacetime . By doing so, we consider the generalized Kaluza-Klein theory which can unify the non-Abelian gauge field by assuming spacetime of $(4+n)$-dimension and a compactification of n-dimensions on a manifold with certain symmetry. In this section, we assume a compact manifold K^n of extra dimensions. This means we don't consider why extra dimensions are compactified. Why extra dimensions are compact is solved by considering a spontaneous compaction. We will describe spontaneous compaction in the next sections.

1.3.1 Isometry transformations

We described above the Kaluza-Klein requirement which is the transformation along the extra coordinate that doesn't change the theory. This means

$$\delta g_{mn} = 0. \tag{1.3.1}$$

Such transformation which doesn't change the metric is called the isometry transformation. Even if the spacetime dimension becomes higher, one imposes the same requirement, and one considers the isometry transformations for the metric.

We start from the general coordinate transformations.

$$X^A \to X'^A = X^A - K_a^A(X)\epsilon^a(X) \tag{1.3.2}$$

If the metric is invariant under this transformation, there exist a vector satisfying the equation:

$$\nabla_A K_B + \nabla_B K_A = 0 \tag{1.3.3}$$

This equation is called Killing equation. And vectors K^A which satisfy this equation are called Killing vectors. The Killing equation means that a change of the coordinates along the Killing vector does not change the metric. In other words, the spacetime has a symmetry along the Killing vector. The Killing vector is also defined by Lie derivative of metric:

$$\mathcal{L}_K g = 0 \tag{1.3.4}$$

Since the Lie bracket of two Killing vectors also becomes the Killing vector, the Killing vectors form Lie algebra:

$$K_b^m \partial_m K_c^n - K_c^m \partial_m K_b^n = -C_{abc} K_a^n \tag{1.3.5}$$

where C_{abc} are the structure constants.

If we write T_a as the generator of the corresponding infinitesimal isometry transformation, the generators satisfy the relation.

$$[T_a, T_b] = iC_{abc}T_c \tag{1.3.6}$$

The non-Abelian gauge fields appearing in the four-dimensional effective theory are determined by the algebra formed by the Killing vectors.

For example, when the extra coordinates are compactified on S^2, this spacetime has symmetry of isometry group $SO(3)$. At this time, the gauge field of $SO(3)$ group appears in four-dimensional effective theory. In general, symmetry of $SO(n+1)$ appears when compactified on S^n.

A compact manifold K^n is arbitrary otherwise. Then, we describe more special Einstein manifolds as an example. Einstein manifolds are defined by the real curvature in the form:

$$R_{AB} = kg_{AB} \tag{1.3.7}$$

where k is a constant. We can consider the energy-momentum tensor, which causes compactification of extra dimensions, by substituting this curvature to Einstein equations. Before we find the matter which causes compactification, we describe the Riemann tensor of the n-sphere S^n as:

$$R_{ABCD} = (g_{AC}g_{BD} - g_{AD}g_{BC})r^{-2} \tag{1.3.8}$$

where r is radius of the sphere. Here, we assume $D = (4+n)$-dimensional spacetime which is compactified to $M^4 \times S^n$. M^4 is four-dimensional Minkowski spacetime. From equation (1.3.8), we have

$$R_{\alpha\beta\gamma\delta} = 0 \tag{1.3.9}$$

and

$$R_{abcd} = (g_{ac}g_{bd} - g_{ad}g_{bc})r^{-2} \tag{1.3.10}$$

Therefore, the Ricci tensor and the scalar curvature on the n-sphere are

$$R_{ab} = (n-1)g_{ab}r^{-2}, \tag{1.3.11}$$

$$R = n(n-1)r^{-2}. \tag{1.3.12}$$

In the next section, we assume that the n-dimensional part of the $(4+n)$-dimensional spacetime is compactified on the manifold K^n.

1.3.2 $(4+n)$-dimensional Kaluza-Klein theory

We assume the $(4+n)$-dimensional spacetime which has the topology $M^4 \times K^n$. The manifold K^n is n-dimensional spacetime which is compactified with some internal symmetry. We start from the higher-dimensional metric:

$$g_{AB} = \begin{bmatrix} g_{\mu\nu} + g_{mn}K^m_a(y)A^a_\mu(x)K^n_b(y)A^b_\nu(x) & K^m_a(y)A^a_\mu(x) \\ K^n_a(y)A^a_\nu(x) & g_{mn}(y) \end{bmatrix} \tag{1.3.13}$$

where $g_{\mu\nu}$ is four-dimensional metric, $g_{mn}(y)$ is the metric on the K^n, A_μ is the gauge field, and $K_a^n(y)$ is Killing vector on the K^n.

To consider the isometry transformation on the K^n, we introduce the $(4+n)$-dimensional general coordinate transformations:

$$\delta X^A = -\xi^A(x,y) \tag{1.3.14}$$

where

$$\xi^\mu(x,y) = 0, \quad \xi^m(x,y) = K_a^m(y)\epsilon^a(x) \tag{1.3.15}$$

Under this transformations, we investigate how the (μn) component of five-dimensional metric transform and, obtain the transformation law of the gauge field A_μ:

$$\delta A_\mu^a(x) = \partial_\mu \epsilon(x)^a - f_{bc}^a A_\mu(x)^b \epsilon^c(x) \tag{1.3.16}$$

This is the transformation law of the non-Abelian gauge field. So, A_μ is regarded as the non-Abelian gauge field indeed.

We obtain the effective action by substituting the above metric to the Einstein-Hilbert action in D-dimensional spacetime,

$$S_D = \frac{1}{2\kappa_D^2}\int d^DX\sqrt{-g_D}R^{(D)} \tag{1.3.17}$$

Then, we have

$$S_D = \int d^ny\sqrt{-g_n}\frac{1}{2\kappa_D^2}\int d^4x\sqrt{-g_4}R^{(4)} \tag{1.3.18}$$

$$-\int d^ny\sqrt{-g_n}K_a^m(y)K_b^n(y)g_{mn}(y)\frac{1}{8\kappa_D^2}\int d^4x\sqrt{-g}F_{\mu\nu}{}^{a}F^{\mu\nu b} \tag{1.3.19}$$

where,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc}A_\mu^b A_\nu^c \tag{1.3.20}$$

Here, we assume the volume of compact manifold K^n is

$$V_n = \frac{1}{\kappa_D^2}\int d^ny\sqrt{-g_y} \equiv \frac{1}{\kappa_4^2} \tag{1.3.21}$$

and the norm of the Killing vector is

$$K_a^m(y)K_b^n(y)g_{mn}(y) = 2\kappa_4^2\delta_{ab} \tag{1.3.22}$$

Substituting the five-dimensional metric to the action (1.3.18) with these assumptions, we obtain the four-dimensional effective theory:

$$S_4 = \frac{1}{2\kappa_4^2}\int d^4xR^{(4)} - \frac{1}{4}\int d^4xF_{\mu\nu}^aF^{a\mu\nu} \tag{1.3.23}$$

This is the action of Einstein gravity coupled to the non-Abelian gauge field in four-dimensional spacetime.

1.3.3 SU(3)×SU(2)×U(1) gauge theory

In the standard model of elementary particles, the gauge fields of strong and electroweak interactions belong to the gauge group of $SU(3)\times SU(2)\times U(1)$. A compact manifold K^n that reproduces the symmetries $SU(3)\times SU(2)\times U(1)$ was considered by Witten [25]. Witten showed that the compact manifold K^n must have dimension of more than seven. The reason is that the smallest dimension of a manifold with isometry group G is given by coset space G/H, where H is the largest submanifold of G with linearly realized symmetry, when G is given by

$$SU(3)\times SU(2)\times U(1), \tag{1.3.24}$$

it has the dimension $8+3+1=12$. This is determined from the number of generators. The H is given by

$$H = SU(2)\times U'(1)\times U''(1) \tag{1.3.25}$$

and has the dimension $3+1+1=5$. So, the minimal dimension of G/H is $12-5=7$, and hence, $D=4+7=11$.

1.4 Compactification mechanism

We showed the gauge field can be unified with the gravitational field by using compact extra dimensions.However, the spacetime $M^4\times K^n$ is not always the vacuum solution of $(4+n)$-dimensional Einstein equations. It was showed the five-dimensional pure Kaluza-Klien theory has the $M^4\times S^1$ spacetime as a solution, but generally, the $(4+n)$-dimensional $(n>1)$ Kaluza-Klein theory doesn't have the n dimensional compact solution [26]. According to [27], after computing quantum effects in the five-dimensional Kaluza-Klein theory, the fifth dimension becomes small and compact.

Therefore, in general, for the extra dimensions to be compact, it is necessary to introduce matter which induces the energy-momentum tensor or a certain modification of the Einstein gravity in the higher dimensions.

We start from the $(4+n)$-dimensional Einstein equation with matter:

$$R^{(D)}_{AB} - \frac{1}{2}g_{AB}R^{(D)} - \frac{1}{2}g_{AB}\Lambda^{(D)} = \kappa^2 T_{AB} \tag{1.4.1}$$

Here, in order to construct the spacetime $M^4\times K^n$, where M^4 is four-dimensional Einstein spacetime, with n-dimensional compactified spacetime K^n, the components of the metric do not have to be the Minkowski metric $\eta_{\alpha\beta}$. It means that $(\alpha\beta)$ components of the Einstein equation satisfy

$$R_{\alpha\beta} = 0. \tag{1.4.2}$$

The $(\alpha\beta)$-components of the energy-momentum tensor from the Lorentz invariance are

$$T_{\alpha\beta} = \frac{\alpha}{\kappa^2}\eta_{\alpha\beta} \tag{1.4.3}$$

where the α is a constant. Next, if we assume extra dimensions are compactified on the Einstein manifold (see the definition in the Subsection 1.3.1), the (ab) components of Einstein equation satisfy

$$R_{ab} = kg_{ab}, \quad k > 0 \tag{1.4.4}$$

where k is a constant. Therefore, the (ab) components of the energy-momentum tensor are

$$T_{ab} = \frac{\alpha'}{\kappa^2}g_{ab}. \tag{1.4.5}$$

Combining with (1.4.3), we can obtain

$$R_{ab} = (\alpha' - \alpha). \tag{1.4.6}$$

This relation means that the condition the spacetime K^n is compactified reads

$$\alpha' - \alpha > 0. \tag{1.4.7}$$

1.4.1 Freund-Rubin compactification

Freund and Rubin suggested spontaneous compactification by p-form gauge fields [3]. In particular, eleven-dimensional supergravity is unique and include the four-form gauge field strength in its action. Freund and Rubin assumed that the fields $F = dA$ can cause compactification of extra dimensions.

We start from the D-dimensional action:

$$S_D = \int d^D X\sqrt{-g_D}\left[\frac{1}{2\kappa_D^2}R^{(D)} - \frac{1}{48}F_{ABCD}F^{ABCD}\right] \tag{1.4.8}$$

From this action we can obtain field equations of gravitational field and gauge field:

$$R^{(D)}_{AB} - \frac{1}{2}g_{AB}R^{(D)} = \kappa_D^2 T_{AB} \tag{1.4.9}$$

$$\frac{1}{\sqrt{-g_D}}\partial_A\left(\sqrt{-g_D}F^{ABCD}\right) = 0 \tag{1.4.10}$$

where T_{AB} is the energy-momentum tensor:

$$T_{AB} = -\frac{1}{6}\left(F_{CDEA}F^{CDE}{}_{B} - \frac{1}{8}F_{CDEF}F^{CDEF}g_{AB}\right) \tag{1.4.11}$$

The equation (1.4.10) has a solution

$$F^{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{-g_4}}\epsilon^{\alpha\beta\gamma\delta}F \tag{1.4.12}$$

Here F is constant, $\epsilon^{\alpha\beta\gamma\delta}$ is Levi-Civita symbol.For this solution, the $(\alpha\beta)$-components and the (ab)-components of the energy-momentum tensor become

$$T_{\alpha\beta} = \frac{F^2}{2} sgn(g_4) g_{\alpha\beta} \tag{1.4.13}$$

$$T_{ab} = -\frac{F^2}{2} sgn(g_4) g_{ab} \tag{1.4.14}$$

Remembering the requirements of compactification we described above, we obtain

$$\frac{\alpha}{\kappa^2} = \frac{F^2}{2} \tag{1.4.15}$$

Using this relation, the scalar curvatures of each manifold M^4 and M^n are

$$R_4 = -\frac{F^2}{\kappa^2}\frac{4(n-1)}{n+2}, \tag{1.4.16}$$

$$R_n = \frac{F^2}{\kappa^2}\frac{3n}{n+2}. \tag{1.4.17}$$

This means the manifold M^n is compactified and its curvature is positive. Also, the manifold M^4 becomes not Minkowski spacetime but AdS spacetime. If we want the manifold M^4 to be Minkowski spacetime, we need to add the cosmological constant as

$$\Lambda = \kappa_4^2(n-1)F^2 \tag{1.4.18}$$

in the action (1.4.8).

1.4.2 Compactification due to the gauge fields

Introducing gauge fields for spontaneous compactification seems to be not inline with the purpose of Kaluza-Klein theory of unifying the gauge fields in higher-dimensional gravity. However, this has several advantages. Generally, the action of gauge field is

$$S_F = -\frac{1}{4}\int d^{4+n}X\sqrt{-g_{4+n}}(F_a)_{AB}(F_a)^{AB} \tag{1.4.19}$$

and the energy momentum tensor is

$$T_{AB} = -\left((F_a)_{AC}(F_a)_B{}^C - \frac{1}{4}(F_a)_{CD}F_a{}^{CD}g_{AB}\right) \tag{1.4.20}$$

As we saw in Freund-Rubin's compactification, the structure of the compact space is determined by how we choose the solution of the gauge field. For example, compactification from six dimensions on S^2 is done by considering the monopole solution [4]. Also, the S^4 compactification is done by considering the instanton solution [5]. Ref [28] classifies the seven-dimensional compact spaces for deriving the four-dimensional effective theory through spontaneous compactification from $D = 11$ supergravity.

1.4.3 Stabilizing the moduli

Compactification by the gauge field gives the curvature to the compact space, but we do not guarantee that size is stable. It is known that we can stabilize the size of extra dimensions by introducing a potential of dilaton field, as was demonstrated by S. Carroll, J. Geddes, M. Hoffman, R. Wald in [29], where the condition of stabilization on homogeneous compact spaces was discussed.

Here, as a concrete example, we show that six-dimensional spacetime is compactified to four-dimensions with the two-form field strength, the size of extra dimensions is stabilized by the dilaton potential, and the non-compact spacetime becomes that of Minkowski.

Also, in order to become a four-dimensional Minkowski spacetime, we show that we need to include cosmological constant in the six-dimensional action. This method is also used in our research.

We start from the six-dimensional action:

$$S = \frac{1}{2\kappa^2}\int d^6x\sqrt{-g_6}\,[R_6 - \Lambda_6] - \int d^6x\sqrt{-g_6}\frac{1}{4}F_{AB}F^{AB} \tag{1.4.21}$$

where Λ_6 is a six-dimensional a cosmological constant, F_{AB} is the two-form field strength, $F = dA$.

To realize $M^4 \times S^2$ compactification, we use the ansatz for the metric:

$$ds^2 = g_{AB}(X)dX^A dX^B = g_{\alpha\beta}(x)dx^\alpha dx^\beta + e^{2\chi(x)}g_{ab}(y)dy^a dy^b \tag{1.4.22}$$

where $\chi(x)$ is dilaton depending on four-dimensional spacetime.

Next, we consider the integration over extra coordinates. For this metric, we can decompose the determinant and the scalar curvature as

$$\sqrt{-g_6} = \sqrt{-g_4}e^{2\chi}\sqrt{g_y}, \tag{1.4.23}$$

and

$$R_6 = R_4 + e^{-2\chi}R_y - 4e^{-\chi}g^{\alpha\beta}\nabla_\alpha\nabla_\beta e^\chi - 2e^{-2\chi}g^{\alpha\beta}\nabla_\alpha e^\chi\nabla_\beta e^\chi. \tag{1.4.24}$$

The gravitational part of the action becomes

$$S_g = \frac{1}{2\kappa^2}\int d^4x d^2y\sqrt{-g_4}\sqrt{g_y}e^{2\chi}\left[R_4 + e^{-2\chi}R_y - 2e^{-2\chi}g^{\alpha\beta}\partial_\alpha e^\chi\partial_\beta e^\chi - 2\Lambda\right] \tag{1.4.25}$$

Here we define the volume of compact space $\mathcal{V}$ and the four-dimensional gravitational constant κ_4 as follows:

$$\mathcal{V} = \int d^2y\sqrt{g_y}, \tag{1.4.26}$$

$$\frac{1}{2\kappa_4^2} \equiv \frac{\mathcal{V}}{2\kappa_6^2}. \tag{1.4.27}$$

So, we obtain the four-dimensional gravitational action with the scalar field χ:

$$S_g = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g_4}\left[e^{2\chi}R_4 + R_y - 2e^{2\chi}g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi - 2e^{2\chi}\Lambda\right] \tag{1.4.28}$$

However, this is not the minimally coupled action. We have to change the coordinate system to reproduce the Einstein-like gravitational theory. It can be realized by applying the Weyl transformation.

The Weyl transformation is the transformation of the metric as follows:

$$g_{\alpha\beta} = \Omega^{-2}g\tilde{g}_{\alpha\beta}, \quad \sqrt{-g_4} = \Omega^{-4}\sqrt{-\tilde{g}_4} \tag{1.4.29}$$

Under this transformation, the scalar curvature transforms:

$$R_4 = \Omega^2[\tilde{R}_4 + 6\tilde{\Box}f - 6\tilde{g}^{\alpha\beta}f_\alpha f_\beta] \tag{1.4.30}$$

where

$$f = \ln\Omega, \quad f_\alpha = \frac{\partial_\alpha\Omega}{\Omega} \tag{1.4.31}$$

Substituting (1.4.29) and (1.4.30) to the action (1.4.28), we obtain

$$S_g = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-\tilde{g}_4}[e^{2\chi}\Omega^{-2}(\tilde{R}_4 - 6\tilde{g}^{\alpha\beta}f_\alpha f_\beta - 2\Omega^{-2}e^{-2\chi}\tilde{g}^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi \tag{1.4.32}$$

$$+\,\Omega^{-4}R_y - 2\Omega^{-4}e^{2\chi}\Lambda]. \tag{1.4.33}$$

We choose Ω to obtain the action in the Einstein frame, as follows:

$$\Omega = e^\chi \tag{1.4.34}$$

Then

$$f = \chi, \quad f_\alpha = \partial_\alpha\chi \tag{1.4.35}$$

From these results, we obtain

$$S_g = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{\tilde{g}_4}[R_4 + e^{-4\chi}R_y - 8\tilde{g}^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi - 2e^{-2\chi}\Lambda]. \tag{1.4.36}$$

To make the action of scalar field canonical, we rescale χ,

$$\frac{2\sqrt{2}}{\kappa_4}\chi = \phi \tag{1.4.37}$$

Finally, the action of the gravitational part becomes

$$S_g = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{\tilde{g}_4}R_4 + \int d^4x\sqrt{-\tilde{g}_4}\left(-\frac{1}{2}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V'(\phi)\right) \tag{1.4.38}$$

where

$$V(\phi) = \frac{1}{2\kappa_4^2}\left(2e^{-\frac{\kappa_4}{\sqrt{2}}\phi}\Lambda - ke^{-\frac{2\kappa_4}{\sqrt{2}}\phi}\right), \quad k \equiv R_y \tag{1.4.39}$$

This action includes the canonical scalar field with its potential.

Next, let us consider the action of the two-form filed. We apply the dimensional reduction for that action, as we did for the action of gravitational field,

$$S_F = -\frac{1}{4}\int d^6X\sqrt{-g_6}F_{AB}F^{AB} = -\frac{1}{4}\int d^6X\sqrt{-g_6}g^{AC}g^{BD}F_{AB}F^{CD} \tag{1.4.40}$$

$$= -\frac{1}{4}\int d^4x\sqrt{-g_4}d^2y\sqrt{g_y}e^{-2\chi}g^{ac}g^{bd}F_{ab}F_{cd} \tag{1.4.41}$$

$$= -\frac{1}{4}\int d^4x\sqrt{-g_4}e^{-2\chi}F^2 \tag{1.4.42}$$

where we have defined the constant F as follows:

$$\int d^2y\sqrt{g_y}g^{ac}g^{bd}F_{ab}F_{cd} = F^2 \tag{1.4.43}$$

Next, we apply the Weyl transformation:

$$S_F = -\frac{1}{4}\int d^4x\sqrt{-\tilde{g}_4}\Omega^{-4}e^{-2\chi}F^2 \tag{1.4.44}$$

$$= -\frac{1}{4}\int d^4x\sqrt{-\tilde{g}_4}e^{-6\chi}F^2 \tag{1.4.45}$$

$$= -\frac{1}{4}\int d^4x\sqrt{-\tilde{g}_4}e^{-\frac{3\kappa_4}{\sqrt{2}}\phi}F^2 \tag{1.4.46}$$

Adding this to the potential, we obtain the total potential:

$$V(\phi) = \frac{1}{2\kappa_4^2}\left(2e^{-\frac{\kappa_4}{\sqrt{2}}\phi}\Lambda - ke^{-\frac{2\kappa_4}{\sqrt{2}}\phi} + \frac{\kappa_4^2}{2}e^{-\frac{3\kappa_4}{\sqrt{2}}\phi}F^2\right) \tag{1.4.47}$$

We show this potential in Figure 1.1. From this graph, it is obvious this potential has a minimum and it stabilizes the size of the extra space. Also, the higher dimensional cosmological constant $\Lambda \neq 0$ is required for the minimum of $V(\phi)$ to be zero, because the four-dimensional spacetime is an Anti-de Sitter spacetime when $\Lambda = 0$.

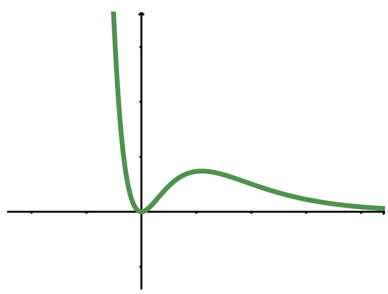


Figure 1.1: A sketch of the scalar potential

Chapter 2

Brane-world models

The Kaluza-Klein theory described in the previous chapter has the concept that extra dimensions can't be seen because they are small and compact. As the alternative to this compactification, there is another approach called brane world as a concept of extra dimensions. This is the idea that we can't see the spatial extra dimensions because interactions of the standard model of elementary particles, other than gravity are localized in the three dimensional membrane.This development began with Rubakov showed that there exists a solution for fermions localized on the domain wall [30].

The first model of brane-world was proposed by Arkani-Hamed, Dimopoulos and Dvali [8, 9]. This model is called the ADD model and was proposed to solve the hierarchy problem in particle physics. In this model, it is assumed that the extra dimension is closed like S^1, but the particles of the standard model propagate in four dimensions, only the gravity can propagate in the extra dimension. This time, the strength of gravity depends on the size of the extra dimension. Since the size of the extra dimension does not have to be small, it is called the large extra dimension model.

After that, Randall and Sundrum proposed a new brane model to solve the problem of hierarchy in a different way, which assumed an extra dimension as an orbifold S^1/Z^2, with two branes of opposite tensions (RSI) [10]. Furthermore, by removing one brane from RSI, they succeeded in introducing an infinitely large extra dimension(RSII) [11]. This proposal has received much attention.

We considered the RSI model in modified gravity and analyzed its stability. As a result, we found that coupling of the RSI model to Starobinsky gravity does not destroy stability of the usual RSI model.

In this Chapter, we consider the brane-world handling of extra dimensions are the alternative to KK theory with focus on the two brane models proposed by Randall and Sundrum (RSI). After that, we consider the RSI model and its stability in the modified gravity which is our research result in [18].

2.1 Domain wall solution

Rubakov showed that there is a solution confined to the domain wall for the five-dimensional fermion from the kink solution of the scalar field in the five-dimensional spacetime [30]. This picture is the key of the assumption in brane-world that the standard model particles are confined on the brane.

Let us start from five-dimensional scalar field with potential:

$$S = \int d^4x dy \sqrt{-g_5} \left(-\frac{1}{2} \partial_A \phi \partial^A \phi - V(\phi) \right) \tag{2.1.1}$$

where

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{2.1.2}$$

and λ is the coupling constant. This scalar filed has the kink solution:

$$\phi_v = v \tanh \left(\sqrt{\frac{\lambda v^2}{2}} y \right) \tag{2.1.3}$$

This solution has asymptotics

$$\phi_v(y \to \infty) = +v \tag{2.1.4}$$

$$\phi_v(y \to -\infty) = -v \tag{2.1.5}$$

It describes a domain wall separating two classical vacua at $y = 0$, thus introducing five-dimensional fermion to this model. The five-dimensional action of the fermion interacting with the scalar field ϕ is

$$S = \int d^5X \sqrt{-g} \left(\mathrm{i}\bar{\Psi}\Gamma^A \partial_A \Psi - g\phi\bar{\Psi}\Psi \right) \tag{2.1.6}$$

where g is Yukawa coupling constant. In the domain wall background, the Dirac equation is

$$\mathrm{i}\Gamma^A \partial_A \Psi - \phi_v \Psi = 0 \tag{2.1.7}$$

For this fermion, the mass in four dimension is given by

$$\gamma^\mu \partial_y \Psi = -m\Psi \tag{2.1.8}$$

and there is the zero mode solution Ψ_0 with $m = 0$:

$$\gamma^5 \partial_y \Psi_0 = g\phi_v(y)\Psi_0 \tag{2.1.9}$$

The zero mode is left-handed (or right-handed) in the four dimensions. So, we can write this zero mode as

$$\Psi_0 = e^{\int_0^y dy' g\phi_v(y')} \psi_{L\ (R)}(p) \tag{2.1.10}$$

where ψ is the solution of the Weyl equation. This means zero mode is localized near $y = 0$ and it decays at large y.

2.2 ADD model

The first phenomenological model of brane world, whose size of extra dimension is large, was suggested by N.Arkani-Hamed, S.Dimopoulos and G.Dvali in 1998 [8,9]. The motivation of this model is to solve the hierarchy problem in the particle physics. This approach (called ADD model) considers the brane whose tension is neglected after embedding into the higher dimensions with flat and compact extra dimensions (Figure 2.1). The large extra dimension is allowed because of confinement of the matter fields on the brane, and only gravity can propagate in the extra dimensions. Here, the dimension of spacetime is D-dimensional ($D > 4$), and the fundamental mass scale in D-dimensional spacetime is denoted by M, in order to distinguish it from the Plank mass M_{pl} in the four-dimensional spacetime. The gravitational action in the D-dimensional spacetime is:

$$S = \frac{M^{D-2}}{2} \int d^D X \sqrt{-g_D} \left(R^{(D)} \right) \quad , \tag{2.2.1}$$

where

$$M^{D-2} = M^{n+2} \tag{2.2.2}$$

is the D-dimensional fundamental mass scale, $n = D - 4$ is the number of extra dimensions. The subscript (D) denotes D-dimensional geometric quantities. In this model , the long distance four-dimensional gravity is mediated by the graviton zero mode, whose wave function is homogeneous over extra dimensions. Hence, the four-dimensional effective action describing long distance gravity is obtained from equation (2.2.1) by taking the metric to be independent of extra dimensions and integrating over extra coordinates:

$$S_{eff} = \frac{M^{D-2} V_n}{2} \int d^4 x \sqrt{-g_4} R^{(4)} \quad , \tag{2.2.3}$$

where $V_D \sim r^n$ is the volume of extra dimensions whose radius is r. So, the four-dimensional Planck mass is determined by the volume of extra dimensions:

$$M_{Pl} = M(Mr)^{\frac{n}{2}}. \tag{2.2.4}$$

If the size of extra dimensions is large compared to the fundamental length M^{-1}, the Planck mass is much larger than the fundamental gravity scale M. Then, the hierarchy between M_{pl} and M_{EW} is entirely due to the large size of extra dimensions. Assuming that $M \sim 1TeV$, one calculates from (2.2.4) the value of r:

$$r \sim M^{-1} \left(\frac{M_{pl}}{M} \right)^{\frac{2}{n}} \sim 10^{\frac{32}{n}} \cdot 10^{-17} \text{cm}. \tag{2.2.5}$$

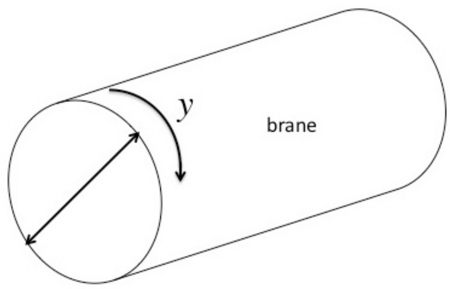
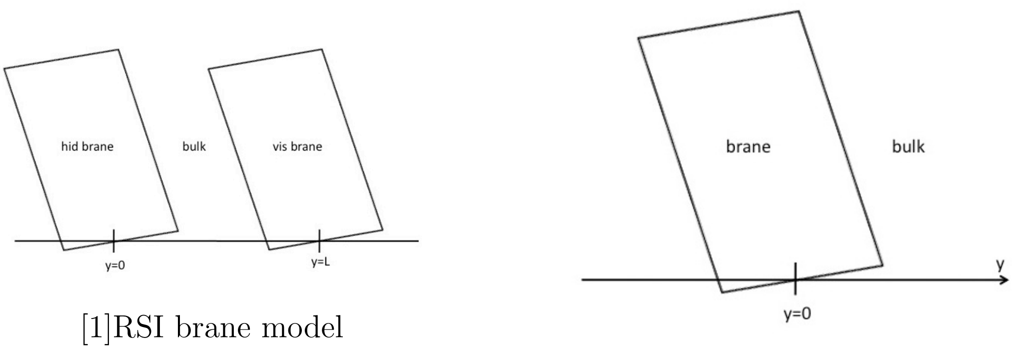


Figure 2.1: ADD brane model

2.3 Randall-Sundrum I model

In 1999, L.Randall and R.Sundrum suggested two brane-world models (Figure 2.2). The first one called RSI is two brane model to solve the hierarchy problem [10]. The second model called RSII is the model removing to the infinity one brane from the RSI [11]. We describe the geometry of RS models, and their basic idea.



[1]RSI brane model

[2]RSII brane model

Figure 2.2: RS brane models

2.3.1 Warped geometry

The RSI assumes the existence of one extra dimension compactified on a circle whose upper and lower halves are identified, called S^1/Z_2 orbifold (Figure 2.3). The orbifold condition is inspired from Horava-Witten model [31]. This contruction entails two fixed points, $y = 0$ and $y = \pi r \equiv L$. Randall and Sundrum assumed two branes which have opposite tensions and they are located at those points.

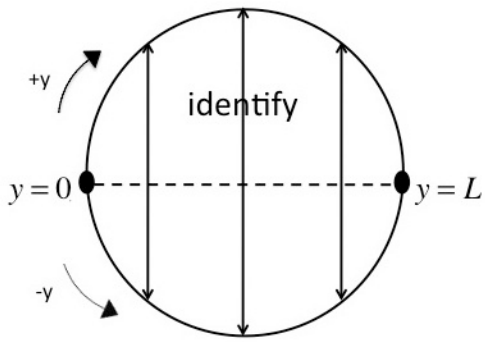


Figure 2.3: S^1/Z_2 orbifold

There is the five-dimensional cosmological constant Λ_5 in the action. As it will be seen later, Λ_5 takes a negative value required in order to make flat four-dimensional spacetime on the brane.

The action of this set-up is

$$S = S_{gravity} + S_{hid} + S_{vis} \tag{2.3.1}$$

$$S_{gravity} = \int d^4x \int_{-L}^{L} dy\sqrt{-g_5}(2M_5^3 R^{(5)} - \Lambda_5) \tag{2.3.2}$$

$$S_{hid} = \tau_{hid} \int d^4x dy \sqrt{-g_{hid}}\sqrt{g_y}\delta(y) \tag{2.3.3}$$

$$S_{vis} = \tau_{vis} \int d^4x dy \sqrt{-g_{vis}}\sqrt{g_y}\delta(y-L). \tag{2.3.4}$$

where M_5 is the fundamental mass scale in the five-dimensional spacetime, τ is the absolute tension of branes, the subscript "5" means five-dimensional values, "vis" and "hid" mean values on the visible brane and hidden brane respectively. The five-dimensional Einstein equations are

$$G_{AB} = R_{AB} - \frac{1}{2}g_{AB}R = \frac{1}{2M^3}(\Lambda_5 + \tau_{vis}g_{\alpha\beta}\delta^{\alpha}(y-L)_A\delta^{\beta}(y-L)_B$$
$$+\tau_{hid}g_{\alpha\beta}\delta^{\alpha}_A(y)\delta^{\beta}_B(y)). \tag{2.3.5}$$

Since the solution of these equations should fit the real world, we require the metric should preserve the Poincare invariance. This leads to the following Ansatz:

$$ds^2 = e^{-2A(y)}\eta_{\alpha\beta}dx^{\alpha}dx^{\beta} + dy^2. \tag{2.3.6}$$

The prefactor $e^{-2A(y)}$, called the "warp factor", is written as the exponential for convenience. Its dependence on the extra coordinate y causes this metric to be non-factorizable, which means that, unlike the metrics appearing in the usual Kaluza-Klein scenarios, it cannot be expressed as a direct product of the four-dimensional Minkowski spacetime and a manifold of extra dimensions.

To determine the $A(y)$, we must calculate (2.3.5) with this Ansatz. The corresponding (non-vanishing) Christoffel symbols, Ricci tensor and Ricci scalar are

$$\Gamma^5_{\alpha\beta} = A' e^{-2A} \eta_{\alpha\beta} \ , \tag{2.3.7}$$
$$R_{\alpha\beta} = \left(A'' - 4A'^2\right) e^{-2A} \eta_{\alpha\beta} \ , \tag{2.3.8}$$
$$R_{55} = 4A'' - 4A'^2 \ , \tag{2.3.9}$$
$$R = 8A'' - 20A'^2 \ , \tag{2.3.10}$$

respectively, where the primes denote the derivatives with respect to y.

The $(5,5)$ component of Einstein equation gives

$$G_{55} = 6A'^2 = \frac{-\Lambda_5}{2M^3} \equiv k^2. \tag{2.3.11}$$

where the prime means differentiation respect to y. In order for A to have a real solution, Λ_5 must be negative, and it means that the space between the branes is set to be Anti-de Sitter space. Anti-de Sitter spacetime is defined as the homogenous spacetime with a negative cosmological constant. Since M and Λ_5 are constants, we denote the RHS of (2.3.11) as k^2. Also, (2.3.11) yields l^{-2} as the curvature radius of Anti-de Sitter space. So, l means the scale of extra dimension. Integrating over y and considering the orbifold symmetry, we get

$$A(y) = k|y|. \tag{2.3.12}$$

We get the background metric in the Randall-Sundrum model as

$$ds^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \tag{2.3.13}$$

with $-L \leq y \leq L$.

Next, we look at the $\mu\nu$ components of Einstein equations. The Einstein tensor is

$$G_{\mu\nu} = (6A'^2 - 3A'')g_{\mu\nu}. \tag{2.3.14}$$

Also, from (3.8) we have

$$A' = sgn(y)k. \tag{2.3.15}$$

The term $sgn(y)$ may be written as a combination of Heaviside function as

$$sgn(y) = \Theta(y) - \Theta(-y). \tag{2.3.16}$$

The heaviside function is defined as

$$\Theta(y) = \begin{cases} 1 & (y > 0) \\ 0 & (y < 0) \end{cases} \tag{2.3.17}$$

Its derivative is the delta function. Let us consider the branes located at $y = 0$ and $y = L$:

$$A'' = 2k(\delta(y) - \delta(y - L)) \tag{2.3.18}$$

Plugging those results into (3.10) gives

$$G_{\mu\nu} = 6k^2 g_{\mu\nu} - 6k(\delta(y) - \delta(y - L))g_{\mu\nu}. \tag{2.3.19}$$

Comparing this to the energy-momentum tensor, we get the relations:

$$\frac{-\Lambda_5}{2M^3} = 6k^2 \tag{2.3.20}$$

$$\tau_{hid} = -\tau_{vis} = 12kM^3. \tag{2.3.21}$$

So, the absolute values of the tension on each brane coincide, but their signatures are opposite. Moreover, we need the fine-tuning between Λ_5 and τ, in order to reconstruct the four-dimensional Minkowski spacetime.

2.3.2 Exponential hierarchy

Given the metric in the RSI model, we look at how hierarchy problem is solved. In the Standard Model (SM) of particle physics, the mass of W and Z bosons is generated by Higgs mechanism. The mass of the weak interaction intermediate boson determines the range of weak interaction, and its value is determined by the vacuum expectation value (VEV) of Higgs field. So, if the VEV of Higgs field is suppressed on the visible brane, the hierarchy is generated.

The RS I model requires our world to have a negative tension brane. In other words, the SM fields are confined on the visible brane whose tension is negative. The action of Higgs field on the visible brane is

$$S_{Higgs} = \int d^4x \sqrt{-g_{vis}}[g^{\alpha\beta}_{vis} D_\alpha H^\dagger D_\beta H - \lambda(H^\dagger H - v^2)^2], \tag{2.3.22}$$

where H denotes the Higgs field and v denotes the VEV of Higgs field. Using $g_{\alpha\beta} = e^{-2kL}\eta_{\alpha\beta}$, the action becomes

$$S = \int d^4x e^{-4kL}[e^{2kL}\eta^{\alpha\beta} D_\alpha H^\dagger D_\beta H - \lambda(H^\dagger H - v^2)^2], \tag{2.3.23}$$

Redefining the Higgs field as $H = e^{kL}\tilde{H}$, the action becomes

$$S_{Higgs} = \int d^4x [\eta^{\alpha\beta} D_\alpha \tilde{H}^\dagger D_\beta \tilde{H} - \lambda(\tilde{H}^\dagger \tilde{H} - (e^{-kL}v^2))^2]. \tag{2.3.24}$$

So, the Higgs VEV is exponentially suppressed on the "visible" brane as

$$v_{eff} = e^{-kL}v. \tag{2.3.25}$$

If the value of the bare Higgs mass is of the order of the Planck scale, the physical Higgs mass could be warped down to the weak scale. Since $M_{EW} \simeq 10^{-16} M_{pl}$, the appropriate value for the size of the extra dimension is given by

$$kL \simeq ln10^{16} \simeq 35. \tag{2.3.26}$$

It's necessary to know whether the strength of gravity on the brane is affected by this mechanism. To check it, we need to get the four dimensional gravitational action from the five dimensional action,

$$S = M_5^3 \frac{1 - e^{-2kL}}{k} \int d^4x \sqrt{-g_0}^{(4)} R(h_{\alpha\beta}) \tag{2.3.27}$$

where

$$M_{pl}^2 = M^3 \frac{1 - e^{-2kL}}{k}. \tag{2.3.28}$$

We see that it weakly depends on L. So, if kL becomes large, M_{pl} is not suppressed.

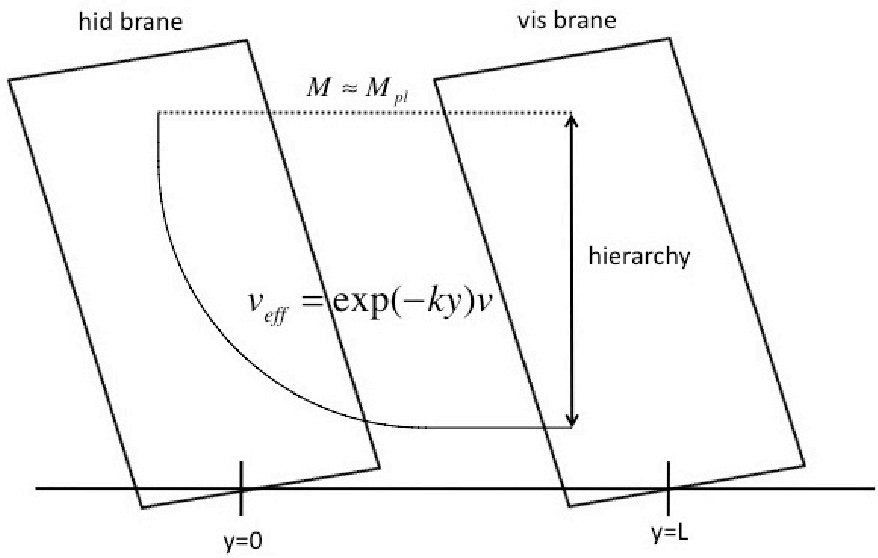


Figure 2.4: Generating the exponential hierarchy.

2.4 Randall-Sundrum II brane model

The RSII model is a single brane model, which is obtained by removing the negative tension brane to $y \to \infty$ in the RSI model [11]. This means we are living in the positive tension brane in the RSII, and this model dose not solve the hierarchy problem. On the other hand, this model succeeds in introducing infinitely large extra dimensions.

In general, introducing the infinitely large extra dimensions breaks Newton's law. In the n dimensional spacetime, Newton's law becomes:

$$F = G_N \frac{mM}{r^{n-2}} \tag{2.4.1}$$

So, introducing the infinitely large extra dimensions appears to be forbidden. But RSII reconstructs the four-dimensional effective gravity by confinement of the graviton zero mode on the brane.

It can be verified by considering the metric of RSI and calculating the potential for the graviton zero-mode and the KK mode. Although detailed calculation is not carried out here, the zero-mode is localized in the positive tension brane and the KK mode can be propagated to extra dimensions. Calculating the potential of graviton's zero-mode and the KK mode, one can reconstruct the four-dimensional gravity if the correction of the graviton KK mode is small enough.

Here, we consider the contribution of KK gravitons exchange into gravitational potential on the brane . Each KK graviton produces the potential of Yukawa type, so the total contribution is

$$\begin{aligned} V_{KK}(r) &= -{}^{(5)}G m_1 m_2 \int_0^\infty dm [h^m_{\mu\nu}(0)]^2 \frac{e^{-mr}}{r} && (2.4.2) \\ &\approx -\text{const} \frac{G_N m_1 m_2}{r} \frac{1}{r^2 k^2} && (2.4.3) \end{aligned}$$

Since, the graviton zero mode affects the conventional Newton potential, the total gravitational potential is

$$V(r) = G_N \frac{m_1 m_2}{r} (1 + \frac{\alpha}{r^2 k^2}) \tag{2.4.4}$$

where α is a constant.

This means, if r is large, the correction term can be neglected. Also, the correction term must be neglected in the $r \sim O(mm)$ scale, because Newton's law is confirmed down to the scale of $O(mm)$ from experiments. This fact restricts the fundamental mass scale. To neglect the correction term, we require

$$k^{-1} \lesssim 0.1\text{mm}. \tag{2.4.5}$$

In RSII model, the Planck mass is determined from (2.4.5) by removing the brane, $y \to \infty$:

$$M_{pl}^2 = \frac{M^3}{k} \tag{2.4.6}$$

So, using this relation, we get the restriction of the fundamental mass scale:

$$M \gtrsim 10^{10}\text{GeV} \tag{2.4.7}$$

2.5 Randall-Sundrum Brane-World in Modified Gravity

The success of brane-world poses the question of its stability against possible modifications of Einstein gravity. Such modifications are inevitable because the Einstein gravity is known to be non-renormalizable, either in four or higher spacetime dimensions. In this section we positively answer this question by modifying gravity in five spacetime dimensions via adding the simplest higher-order term proportional to the scalar curvature squared, in the context of the RSI brane-world model.

2.5.1 Setup

Our gravitational action in five dimensions reads

$$S_{\rm gr.} = \frac{1}{2\kappa_5^2}\int d^5X\sqrt{-g_5}(R+\alpha R^2-\Lambda) \quad , \tag{2.5.1}$$

where we have introduced the gravitational coupling constant κ_5, the spacetime scalar curvature R, the modified gravity coupling constant α, and a cosmological constant Λ.

As in the RSI model, we introduce a 'visible' brane (where SM fields are confined) located at $y=0$ with tension $\tau_{\rm RS}$, and a 'hidden' brane located at $y=L$ and having the negative tension $-\tau_{\rm RS}$, where the coordinate y parametrizes extra dimension with the orbifold identification $-y\sim y$. Accordingly, our full action is given by

$$S = S_{\rm gr.}+S_{\rm vis.}+S_{\rm hid.} \quad , \tag{2.5.2}$$

where

$$\begin{aligned} S_{\rm hid.} &= -\tau_{\rm RS}\int d^4xdy\sqrt{-g}\,\delta(y-L) \quad , & (2.5.3)\\ S_{vis.} &= \tau_{\rm RS}\int d^4xdy\sqrt{-g}\,\delta(y) \quad . & (2.5.4)\end{aligned}$$

2.5.2 Duality transformation to scalar-tensor gravity

Let us replace $R+\alpha R^2$ by $(1+2\alpha B)R-\alpha B^2$ in the gravitational action (2.5.1), by using a new scalar field B in five dimensions, as

$$S_{\rm gr.} = \frac{1}{2\kappa_5^2}\int d^5X\sqrt{-g_5}[(1+2\alpha B)R-\alpha B^2-\Lambda] \ . \tag{2.5.5}$$

The equation of motion of the field B is algebraic and reads $B=R$. Hence, after substituting it back to the action (2.5.5) we get the equivalent action (2.5.1).

The action (2.5.5) can be brought to Einstein frame by a Weyl transformation in five-dimensions.

$$g_{AB} = \Omega^{-2}\tilde{g}_{AB}, \quad \sqrt{-g} = \Omega^{-5}\sqrt{-\tilde{g}} \ , \tag{2.5.6}$$

with a suitably chosen factor Ω. By using the induced relation

$$R = \Omega^2[\tilde{R} + 8\tilde{\Box} f - 12\tilde{g}^{AB} f_{,A} f_{,B}] \ , \tag{2.5.7}$$

where the tildes refer to the transformed quantities, and the definitions

$$f = \ln \Omega \qquad \text{and} \qquad f_{,A} = \frac{\partial_A \Omega}{\Omega} \ , \tag{2.5.8}$$

we find the gravitational action as follows:

$$S_{\text{gr.}} = \frac{1}{2\kappa_5^2} \int d^5X \sqrt{-\tilde{g}} \Omega^{-5}[(1+2\alpha\phi)\Omega^2(\tilde{R}+8\tilde{\Box} f-12\tilde{g}^{AB} f_{,A} f_{,B})-\alpha B^2-\Lambda] \ . \tag{2.5.9}$$

Hence, to get the transformed gravitational action in Einstein frame, we set

$$\Omega^3 = e^{3f} = 1 + 2\alpha B \ . \tag{2.5.10}$$

It yields

$$f = \frac{1}{3} \ln(1 + 2\alpha B) \tag{2.5.11}$$

and

$$S_{\text{gr.}} = \frac{1}{2\kappa_5^2} \int d^5X \sqrt{-\tilde{g}} [\tilde{R} - 12\tilde{g}^{AB} \partial_A f \partial_B f - e^{-5f}(\alpha B^2 + \Lambda)] \ . \tag{2.5.12}$$

A canonically normalized scalar kinetic term is obtained after rescaling

$$\phi = 2\sqrt{3} f/\kappa \ , \tag{2.5.13}$$

so that we find

$$B = \frac{\exp\left(\frac{\sqrt{3}\kappa\phi}{2}\right) - 1}{2\alpha} \tag{2.5.14}$$

and the scalar potential

$$V(\phi) = \frac{1}{8\alpha\kappa^2} \left[\exp\left(\frac{\kappa\phi}{2\sqrt{3}}\right) - 2\exp\left(-\frac{\kappa\phi}{\sqrt{3}}\right) + (1 + 4\alpha\Lambda)\exp\left(-\frac{5\kappa\phi}{2\sqrt{3}}\right)\right] . \tag{2.5.15}$$

As a result, the gravitational action takes the form

$$S_{\text{gr.}} = \frac{1}{2\kappa^2} \int d^5x \sqrt{-\tilde{g}} \tilde{R} + \int d^5x \sqrt{-\tilde{g}} \left[-\frac{1}{2}\tilde{g}^{AB} \partial_A \phi \partial_B \phi - V(\phi)\right] \ . \tag{2.5.16}$$

Our main result of this Subsection is the scalar potential (2.5.15) induced by the modified $(R + R^2)$ gravity in five spacetime dimensions.

2.5.3 Scalar dynamics in RSI brane-world

As is demonstrated in the previous Subsection, the net effect of adding the R^2 term to the gravitational action amounts to the presence of the extra dynamical scalar field ϕ minimally coupled to gravity and having the scalar potential (2.5.15) in five dimensions.

Note that the five-dimensional cosmological constant Λ enters both (2.5.15) and (2.5.16) via the factor

$$\beta = 1 + 4\alpha\Lambda \ , \tag{2.5.17}$$

while the scalar potential is bounded from below provided that

$$\beta > 0 \ . \tag{2.5.18}$$

In addition, demanding the five-dimensional cosmological constant to be negative, as is needed in the RSI model, we get

$$\beta \leq 1 \ . \tag{2.5.19}$$

The profile of the scalar potential at $\beta = 1/2$ is given in Figure 2.5.

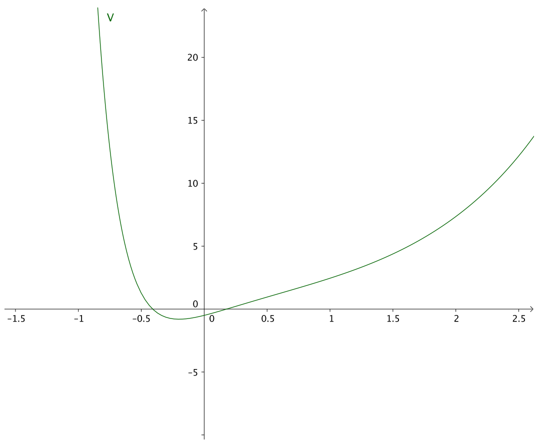


Figure 2.5: The profile of the scalar potential $8\alpha\kappa^2 V(\phi) = \tilde{V}(\tilde{\phi})$ for $\tilde{\phi} = \frac{\kappa}{2\sqrt{3}}\phi$ and $\beta = 0.5$. The value of the scalar potential at its minimum is given by $\tilde{V}(\tilde{\phi_0}) \approx -0.81$. There are two solutions to $V = 0$: one at $\tilde{\phi}_{\text{left}} \approx -0.4$ and another at $\tilde{\phi}_{\text{right}} \approx 0.17$, on the left and on the right of the AdS minimum, respectively.

The minimum of the scalar potential (2.5.15) is achieved at

$$\tilde{\phi_0} = \frac{\kappa}{2\sqrt{3}}\phi_0 = \frac{1}{3} ln \left(-2 + \sqrt{4 + 5\beta} \right) \tag{2.5.20}$$

so that, in particular, $\tilde{\phi_0} = \phi_0 = 0$ for $\beta = 1$.

The value of the scalar potential at its minimum for any $0 < \beta \leq 1$ is given by

$$8\alpha\kappa^2 V(\phi_0) = \tilde{V}(\tilde{\phi}_0) = \frac{-6\left(\sqrt{4+5\beta}-2-\beta\right)}{\left(-2+\sqrt{4+5\beta}\right)^{5/3}} \quad . \tag{2.5.21}$$

The $\tilde{V}_0(\tilde{\phi}_0)$ is an increasing negative function of β for $0 < \beta < 1$, and it vanishes at $\beta = 1$.

Since $\beta = 1$ implies $\Lambda = 0$, we assume that $0 < \beta < 1$ in what follows. Then the value of the scalar potential at its minimum, defined by (2.5.21), is always negative, which corresponds to an AdS vacuum.

2.5.4 Modified RSI model

The Einstein equations for the action (2.5.16) in five dimensions read

$$\tilde{G}_{AB} = \tilde{R}_{AB} - \frac{1}{2}\tilde{g}_{AB}\tilde{R} = \kappa^2 T_{AB} \ , \tag{2.5.22}$$

where we have introduced the total energy-momentum tensor

$$T_{AB} = T^{\phi}_{AB} + T^{\text{vis.}}_{AB} + T^{\text{hid.}}_{AB} \tag{2.5.23}$$

as a sum of three contributions,

$$\begin{aligned} T^{\phi}_{AB} &= \partial_A\phi\partial_B\phi + \tilde{g}_{AB}\left(-\frac{1}{2}\tilde{g}^{MN}\partial_M\phi\partial_N\phi - V(\phi)\right) \ , && (2.5.24)\\ T^{\text{hid.}}_{AB} &= e^{-\frac{5\kappa}{2\sqrt{3}}\phi}\tilde{g}_{AB}\tau_{\text{RS}}\delta(y-L) \ , && (2.5.25)\\ T^{\text{vis.}}_{AB} &= -e^{-\frac{5\kappa}{2\sqrt{3}}\phi}\tilde{g}_{AB}\tau_{\text{RS}}\delta(y) \ . && (2.5.26)\end{aligned}$$

The RS Ansatz for the five-dimensional spacetime metric with the four-dimensional Poincaré symmetry is given by

$$ds^2 = e^{-2A}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \ , \tag{2.5.27}$$

where the warp factor $A(y)$ only depends upon the coordinate y of the hidden (fifth) dimension,and $\eta_{\mu\nu}$ is Minkowski metric in four other dimensions.

From equations (2.3.8)∼(2.3.10), the Einstein equations take the form

$$\begin{aligned} G_{55} &= 6(A')^2 = \kappa^2 T_{55} = \kappa^2\left[\frac{1}{2}(\phi')^2 - V(\phi)\right] \ , && (2.5.28)\\ G_{\mu\nu} &= -3\left(A'' + 2A'^2\right)e^{-2A}\eta_{\mu\nu} = \kappa^2 T_{\mu\nu} \ , && (2.5.29)\\ T_{\mu\nu} &= \tilde{g}_{\mu\nu}\left[-\frac{1}{2}\tilde{g}^{55}(\phi')^2 - V(\phi)\right] && (2.5.30)\\ &+ e^{-\frac{5\kappa}{2\sqrt{3}}\phi}e^{-2A}\eta_{\mu\nu}\tau_{\text{RS}}\delta(y-L) && (2.5.31)\\ &- e^{-\frac{5\kappa}{2\sqrt{3}}\phi}e^{-2A}\eta_{\mu\nu}\tau_{\text{RS}}\delta(y) \ , && (2.5.32)\end{aligned}$$

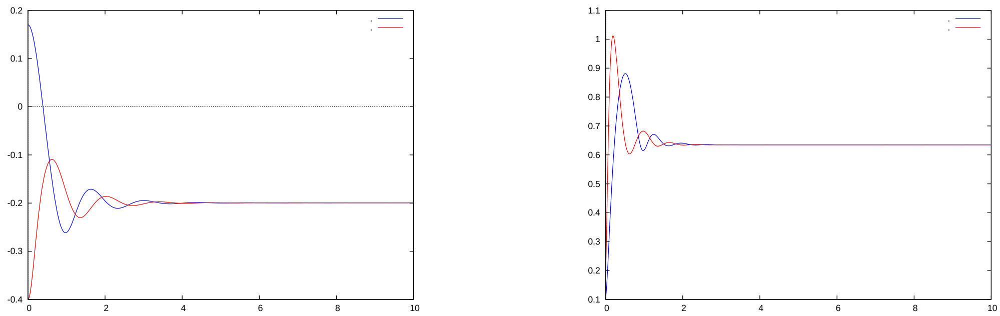
where we have assumed that the scalar field ϕ only depends upon y, because it is also required by the four-dimensional Poincaré invariance (on the visible brane).

The equation of motion of the scalar field $\phi(y)$ coupled to gravity reads

$$\phi'' + 4A'\phi' + \frac{V(\phi)}{d\phi} = 0 \ . \tag{2.5.33}$$

As is already seen in Figure 2.5, the scalar potential is very (exponentially) steep, whereas the right-hand-side of equation (2.5.28) must be positive, so that a slow roll is impossible. Therefore, the scalar field is quickly at its minimum (AdS vacuum) ϕ_0 in five dimensions.

We numerically verified the scalar stabilization with the potential (2.3.18) by using Runge-Kutta method and the MAXIMA software in application to a system of two coupled ordinary differential equations (2.5.28) and (2.5.33). When choosing the parameters as $\beta = 1/2$ (as in Figure 2.5) and $\alpha = 1/96$ (it fixes the scale of V in equation (2.5.33), the initial conditions $\phi' = V = A' = 0$ are consistent with (2.5.28). The corresponding numerical solutions to the functions $\tilde{\phi}(y)$ and $A'(y)$ are given in Figure 2.6. We conclude that the functions $\tilde{\phi}(y)$ and $A'(y)$ quickly approach constant values indeed, independently upon their initial conditions.



(a) (b)

Figure 2.6: (a) the functions $\tilde{\phi}(y)$, and (b) the functions $A'(y)$, with the "left" (red) and "right" (blue) initial conditions, respectively.

Once the scalar field is at its minimum (AdS vacuum) ϕ_0 in five dimensions, the standard RSI scenario applies, being described by the solution

$$A(y) = k|y| \ , \qquad -L \le y \le L \ , \quad k > 0 \ . \tag{2.5.34}$$

In particular, the size L of the 5th dimension is related to the Planck mass $M_{\rm Pl}$ in four dimensions as

$$\kappa^2 M_{\rm Pl}^2 = \frac{1 - e^{-2kL}}{2k} \ . \tag{2.5.35}$$

All that is consistent with the other Einstein equations provided that

$$k^2 = -\kappa^2 V(\phi_0) \equiv -\kappa^2 V_{\text{min}} \quad \text{and} \quad \tau_{\text{RS}} = 12k \quad . \tag{2.5.36}$$

Then the negative cosmological constant in five dimensions is compensated on the visible brane by its tension, so that we get a Minkowski vacuum there.

It is worth mentioning that the effective value of τ_{RS} gets modified in our approach against the RSI one by the factor of $e^{-\frac{5\kappa}{2\sqrt{3}}\phi_0}$, because of equations (2.5.25) and (2.5.26).

As in [10], the hierarchy between the electro-weak scale and the Planck scale M_{Pl} in four dimensions (on the visible brane) is achieved via the presence of the exponential factor e^{-kL} induced by the extra dimension in the vacuum expectation value of Higgs field, so that we must require $kL \approx \ln 10^{16} \approx 35$. Then the exponential term in equation (2.5.36) becomes very small and can be ignored. The Newtonian limit of RSI model leads to the similar exponentially small corrections to Newton law of gravity, which are not in conflict with observations.

2.5.5 Summary of Section 2.5

Given the phenomenological viability of Randall-Sundrum brane world, as the established and reasonable alternative to Kaluza-Klein compactification, it makes sense to analyze stability of the RS brane-world against possible modifications of gravity, as well as against quantum gravity corrections. In our investigation, we did a small step in this direction by proving stability of the RSI model against the simplest modification of the higher-dimensional gravity described by adding the scalar curvature squared term in five dimensions.

An impact of the R^2-modified gravity on the RSI model can be simply described in terms of a single dynamical scalar field with the particular scalar potential (2.5.15). It is clear from our construction that this scalar has the gravitational origin as spin-0 part of five-dimensional spacetime metric. We found that the value of the RSI parameter k is determined by dynamics of that scalar in the fifth dimension.

Chapter 3

Cosmological inflation from higher dimensions

The purpose of studying cosmology is to explain how the Universe began, and predict what will happen in the future. Currently, from the observation of CMB etc., the evolution process from 10^{-22} seconds to the present time since the Universe began has been coherently explained from theory and observations. However, in order to explain the evolution up to the present Universe, it is also known that it is necessary to introduce things that we do not understand yet (dark energy and dark matter).

Regarding the beginning of the Universe, Big Bang cosmology explains that the Universe started from a high temperature and high density state, and continued to expand and reached the present state, but does not explain why expansion started. As explained below, the inflation theory was introduced to explain some problems of Big Bang cosmology in the beginning of the expansion of the Universe.

In order to compose the galaxies and get the large scale structure of the Universe, invisible gravitational source called dark matter is necessary. Since its observation by electromagnetic waves is not possible, dark matter is expected to be new kind of matter which does not have electromagnetic interaction.

As regards the future of the Universe, the latest observations found that the Universe is in the accelerating expansion. From the Friedmann equation, since the expansion rate should decelerate with the standard matter, the dynamics of the Universe has the properties that cannot be explained with the known matter. This phenomenon is called dark energy. Even more surprisingly, it is known that dark energy occupies about 70% of the energy of the Universe.

The current standard cosmology is called ΛCDM model, where Λ is the cosmological constant representing dark energy, and CDM means Cold Dark Matter. From the current observations, both dark energy and dark matter are considered to be necessary to reproduce the present Universe.

The Friedmann-Einstein cosmology has the fundamental problems of flatness and horizon. In order to solve them, the inflation, which is said to have caused an exponential expansion in the early Universe, was proposed. Although inflation was caused by a scalar field called inflaton, the studies to explain inflaton naturally were actively performed because it is unknown how this scalar field appears.

In chapter 1 we introduced Kaluza-Klein's higher-dimensional gravitational theory and its problems, chapter 2 focused on Randall-Sudrum model, introducing how extra dimensions are used instead of Kaluza-Klein theory and examined five-dimensional corrections. Our research is based on applying Randall-Sundrum ansatz to modified gravity theory.

In this chapter, we first introduce the standard cosmology and then describe the inflationary theory which solves the problems mentioned above.

3.1 Standard Cosmology

In Einstein's general relativity, spacetime is dynamical. As a result, the Universe becomes dynamical. In research about expansion and contraction, both theory and observations have made remarkable progress so far. In this Section we describe the standard model of cosmology that is constructed from the current observations and theory. In addition, we describe some problems of standard cosmology, describe inflation proposed to solve them, and describe the modified gravity as the inflationary model.

3.1.1 FLRW metric and Friedmann equation

In 1921, Friedmann assumed a homogeneous isotropic spacetime and derived his equation which describes expansion and contraction of the Universe. The metric, which represents a homogeneous isotropic spacetime is called Friedmann-Lemaître-Robertson-Walker (FLRW) metric, and is written as follows:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right] \tag{3.1.1}$$

where $a(t)$ is the scale factor, K is the topologically constant:

$$K = \begin{cases} +1, & \text{closed Universe} \\ 0, & \text{flat Universe} \\ -1, & \text{open Universe} \end{cases} \tag{3.1.2}$$

and

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{3.1.3}$$

The spatial part of FLRW metric (in square brackets) describes the maximally symmetric three-dimensional space.

The energy-momentum tensor is restricted to the form of perfect fluid:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{3.1.4}$$

where ρ is the energy density, p is the pressure, and u_μ is the four-velocity of matter.

The Friedmann equation can be derived from Einstein equations by substituting the FLRW metric (3.1.1) and the energy-momentum tensor (3.1.4) to Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}. \tag{3.1.5}$$

One obtains from the $(0,0)$ component and (i,j) components of the Einstein equations that

$$H^2 = \frac{\kappa^2}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \tag{3.1.6}$$

$$\dot{H} = -\frac{\kappa^2}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{3.1.7}$$

The dot represents the time differentiation, and $H = \dot{a}/a$ is called the Hubble function which represents the expansion rate of the Universe. Also, from the conservation law of the energy-momentum $\nabla^\mu T_{\mu\nu} = 0$, we obtain the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0 \ . \tag{3.1.8}$$

The energy density related to the evolution of the Universe changes as the scale factor evolves. This is determined by the equation of state, and is expressed as follows:

$$\rho = wp \ , \ \ w = \text{const.} \tag{3.1.9}$$

By the substituting this equation to (3.1.8), we obtain

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \tag{3.1.10}$$

and this means

$$\rho \propto a^{-3(1+w)} \tag{3.1.11}$$

The w depends on the kind of energies: for example, the energy density of a non-relativistic matter decreases in the inverse proportion to the volume as $\rho_M \propto a^{-3}$. In summary, the equations of state classified as follows:

$$w = \begin{cases} 0 \ , \ \text{matter} \\ \frac{1}{3} \ , \ \text{radiation} \\ -1 \ , \ \text{dark energy (cosmological constant)} \end{cases} \tag{3.1.12}$$

The ρ in the Friedmann equation (3.1.6) includes all of the matter, it can be represented as $\rho = \rho_m + \rho_r + \rho_\Lambda$.

3.1.2 Cosmological parameters and current observations

The parameters that characterize the Friedmann equation are called cosmological parameters, and they are determined by current cosmological observations. The subscript 0 is attached to the amount at the current time, and the current Hubble parameter H_0 is called the Hubble constant. Considering the Friedmann equation at the current time, assuming $\Lambda = 0$, the energy density, when the curvature of space becomes zero, is

$$\rho_c = \frac{3H_0^2}{\kappa^2} \tag{3.1.13}$$

and is called the critical density. Due to this critical density, the current energy density, the cosmological constant, and spatial curvature are conventionally divided by ρ_c, in order to get the dimensionless cosmological parameters are defined as follows:

$$\Omega_m = \frac{\rho_m}{\rho_c}, \quad \Omega_r = \frac{\rho_r}{\rho_c}, \quad \Omega_\Lambda = \frac{\Lambda}{\rho_c}, \quad \Omega_K = -\frac{K}{(aH_0)^2} \tag{3.1.14}$$

From these, Friedmann equation becomes

$$1 - \Omega = \Omega_K \tag{3.1.15}$$

where $\Omega = \Omega_m + \Omega_r + \Omega_\Lambda$. These Ω's correspond to the parts of each type of energy that exist in the Universe. In addition, in terms of the cosmological parameters, the Friedmann equation reads

$$\dot{a}^2 = H_0^2 \left[\frac{\Omega_{0,M}}{a} + \frac{\Omega_{0,r}}{a^2} + \Omega_{0,K} + \Omega_{0,\Lambda} a^2 \right] \tag{3.1.16}$$

Therefore, the expansion of the Universe is determined by the cosmological Ω parameters.

The values of cosmological parameters are determined by current observations of type Ia supernova, the Cosmic Microwave Background (CMB) and baryonic acoustic oscillations with the results [32],

$$H_0 = 67.36 \pm 0.54 [km\ s^{-1} Mpc^{-1}]\ , \quad \Omega_{0,m} = 0.0484\ , \quad \Omega_{0,CDM} = 0.258\ ,$$
$$\Omega_{0,\Lambda} = 0.6847 \pm 0.0073\ , \quad \Omega_{0,r} = 0.0016 \tag{3.1.17}$$

3.2 Inflation

Sato [13] and Guth [14] independently proposed the inflation paradigm for solving the problem of Big Bang cosmology. Prior to that, Starobinsky explained that the Universe could accelerate and expand due to the quantum gravity effects. The last model is called the Starobinsky model, and even now it is well matched with the observation. Details of this model will be discussed below.

3.2.1 Problems of standard cosmology

There were two major problems in the standard Big Bang cosmology such as the flatness problem and the horizon problem. And inflation was introduced to solve them. We discuss the mechanism of Inflation in the next section, in order to show how these problems are solved.

1. Flatness problem
   As is shown in the previous section, it is known that the curvature of the present Universe is almost zero. Let us consider the early Universe and Friedmann equation (3.1.14). If Ω is a bit larger than 1, the Universe collapses immediately, and if it is even smaller than 1 it expands at once. In other words, in the early Universe, Ω has to take a value very close to 1. This means the curvature of the Universe also has to take a value very close to zero in the early Universe. But there is no good reason why the curvature of the Universe is almost zero without inflation.

   An inflationary solution is very effective in erasing memory of what happened before it, so its initial conditions become irrelevant. The attractor drives Ω to 1 during inflation — it gives a solution to the flatness problem.

2. Horizon problem
   When considering the FLRW metric, we assumed that the Universe is homogeneous and isotropic. This means that there is no special place in the Universe. It is confirmed by CMB observations that the temperature distribution is isotropic with accuracy of 10^{-5}. However, there is a contradiction because the regions exceeding the range of causality (Hubble horizon) have the same information.

   Inflation $\ddot{a} > 0$ implies

   $$\frac{d}{dt}\left(\frac{H^{-1}}{a}\right) = \frac{-\ddot{a}}{\dot{a}^2} < 0. \tag{3.2.1}$$

   Hence, during inflation, the observed (causally connected) part of the Universe was inside the Hubble horizon H^{-1}/a, so there was enough time for everything to be homogenized.

3.2.2 Slow-roll inflation

There are several types of inflation models that solve the above problems, but we introduce only slow-roll inflation as the best solution.

Inflation is thought to be caused by a scalar field called inflaton. The energy density and pressure of the scalar field are

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.2.2}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{3.2.3}$$

where $V(\phi)$ is the potential of inflaton. Substituting this to Friedmann equation yields

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]. \tag{3.2.4}$$

Also, from the equation (3.1.8), the dynamics of the scalar field is governed by

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{3.2.5}$$

In order to solve the problems we mentioned above, the inflation must continue long enough. For that purpose, the potential energy should be more dominant than the kinetic energy, and should be enough time for this. This statement is called the slow-roll condition and is represented as $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$, $\ddot{\phi} \ll 3H\dot{\phi}$. Under these conditions, equations (3.2.4) and (3.2.5) can be approximated to the form:

$$H^2 \simeq \frac{\kappa^2}{3}V(\phi) \tag{3.2.6}$$

$$3H\dot{\phi} \simeq -V(\phi) \tag{3.2.7}$$

The slow-roll condition can be represented them as the restrictions for inflaton potential as follows:

$$\epsilon \equiv \frac{1}{2\kappa^2}\left(\frac{V'}{V}\right)^2 \ll 1, \quad \eta \equiv \frac{1}{\kappa^2}\left|\frac{V''}{V}\right| \ll 1 \tag{3.2.8}$$

So, the potential of slow-roll inflation must satisfy these conditions.

The duration of inflation is measured by the e-folding number as the amount representing how big the scale factor has become, and it is defined as follows:

$$N_e \equiv \ln\frac{a_f}{a_i} = \int_{t_i}^{t_f} H dt = -\kappa^2 \int_{\phi_i}^{\phi_f} \frac{V(\phi)}{V'(\phi)} d\phi \tag{3.2.9}$$

From this, in order for N_e to be sufficiently large, the potential should have a part with a small inclination (plateau).

There are also the tensor-to-scalar ratio r and the power spectral index n_s from the observations discriminating the models of inflation and related to tensor and scalar fluctuations respectively, during inflation. According to Planck satellite observations [33], the tensor-to-scalar ratio and power spectral index are restricted to

$$r < 0.064\ , \quad n_s \approx 0.9649 \pm 0.0042. \tag{3.2.10}$$

Any viable inflationary model must satisfy these restrictions. The Starobinsky model is well known as the inflationary model that well meets these restrictions.

Since inflation requires a scalar field, the origin of this scalar field is widely discussed. Modified gravity does lead to such new inflaton field. So that, the inflation in modified gravity has gravitational origin due to geometry of spacetime.

3.3 Modified gravity

In the Einstein equations, the left-hand side represents the geometry of spacetime and the right-hand side represents the distribution of matter. The modified gravity is a theory which has modified Einstien-Hilbert action, and it corresponds to modifying the left side of the Einstein equation. The advantage to think like this is that we can incorporate the effects of inflation and dark energy into the theory as a geometric property of spacetime. Here, we introduce the f(R) theory which is often used for construction of inflation and dark energy models.

3.3.1 $f(R)$ gravity

The $f(R)$ theory is a theory in which the scalar curvature of Einstein-Hilbert action is replaced with arbitrary function of scalar curvature, $f(R)$. The model is decided by what kind of function $f(R)$ to use. For example, as will be specifically described below,

$$f(R) = R + \frac{1}{6M^2}R^2 \tag{3.3.1}$$

is the inflation model called Starobinsky model.

The action of $f(R)$ gravity is

$$S = \frac{1}{2\kappa^2}\int d^DX\sqrt{-g_D}f(R) + \mathcal{L}_m \tag{3.3.2}$$

where $\mathcal{L}_m$ is Lagrangian of matter fields. The energy momentum tensor of the matter is defined as

$$T_{AB} = \frac{-2}{\sqrt{-g_D}}\frac{\delta\mathcal{L}_m}{\delta g^{AB}} \tag{3.3.3}$$

The field equation of $f(R)$ gravity can be obtained by varying the action (3.3.2):

$$f'(R)R_{AB} - \frac{1}{2}f(R)g_{AB} - [\nabla_A\nabla_B - g_{AB}\Box]f'(R) = \kappa^2 T_{AB} \tag{3.3.4}$$

where $f'(R) = df/dR$, $\Box \equiv \nabla_A\nabla^A$, and using the relations:

$$\delta f(R) = \frac{\partial f(R)}{\partial R}\delta R \tag{3.3.5}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{AB}\delta g^{AB} \tag{3.3.6}$$

$$\delta R = +R_{AB}\delta g^{AB} + g_{AB}\nabla^C\nabla_C\delta g^{AB} - \nabla_A\nabla_B\delta g^{AB} \tag{3.3.7}$$

3.3.2 From $f(R)$ gravity to scalar-tensor gravity

The $f(R)$ gravity can be made to have the same form as Brans-Dicke theory by introducing an auxiliary field. Consider the following action that introduces a new field:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}[f(\chi) + f'(\chi)(R-\chi)] + S_M(g_{\mu\nu}, \psi) \tag{3.3.8}$$

Here, when we vary this action with respect to χ, we obtain

$$f''(\chi)(R-\chi) = 0 \tag{3.3.9}$$

From this equation, if $f''(\chi) \neq 0$, we get $\chi = R$. Substituting this to the action is equivalent to action (3.3.2), so it can be said that these two actions are classically equivalent.

We redefine $f'(\chi) = \phi$, so that the action becomes

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}[\phi R - V(\phi)] + S_M(g_{\mu\nu}, \psi) \tag{3.3.10}$$

where $V(\phi)$ is

$$V(\phi) = \chi(\phi)\phi - f(\chi(\phi)) \tag{3.3.11}$$

The Brans-Dicke theory is defined by the action

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega_{BD}}{2\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu - V(\phi)\right] + S_M \tag{3.3.12}$$

Here ω_{BD} is called Brans-Dicke parameter, and, hence, $f(R)$ theory is equivalent to the $\omega_{BD} = 0$ Brans-Dicke action.

Next, we consider the Weyl transformation

$$g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu}, \quad \sqrt{-g} = \Omega^{-4}\sqrt{-\tilde{g}} \tag{3.3.13}$$

Under this transformation, Ricci scalar transforms [34] as

$$R = \Omega^2(\tilde{R} + 6\Box\omega - 6\tilde{g}^{\mu\nu}\omega_\mu\omega_\nu) \tag{3.3.14}$$

where

$$\omega = \ln\Omega, \quad \omega_\mu = \frac{\partial_\mu\Omega}{\Omega} \tag{3.3.15}$$

Substituting these equations to the action (3.3.10), we obtain

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,(f'(\chi)R - V(\chi)) \tag{3.3.16}$$

$$= \frac{1}{2\kappa^2}\int d^4x\sqrt{-\tilde{g}}\Omega^{-4}\left[f'(\chi)\Omega^2(\tilde{R} + 6\Box f - 6\tilde{g}^{\mu\nu}f_\mu f_\nu) - V(\chi)\right] \tag{3.3.17}$$

Then, if we choose

$$\Omega^2 = f'(\chi) \tag{3.3.18}$$

we obtain the action in the Einstein frame:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega - f'(\chi)^{-2}V(\chi)\right] \tag{3.3.19}$$

Moreover, to make the scalar field canonical, we rescale the scalar field as

$$\phi = \sqrt{\frac{6}{\kappa^2}}\omega \tag{3.3.20}$$

So, we obtain the gravitational action with the canonical scalar field in the Einstein frame,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\tilde{g}}\tilde{R} + \int d^4x\sqrt{-\tilde{g}}\left(-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right), \tag{3.3.21}$$

where the potential $V(\phi)$ is

$$V(\phi) = \frac{f'R - f}{2\kappa^2 f'^2}. \tag{3.3.22}$$

Therefore, though the $f(R)$ theory is a theory which modified Einstein's theory of gravity, by performing the field redefinitions it becomes the theory of the standard (Einstein) gravity theory with the physical scalar field.

3.3.3 Starobinsky model

Starobinsky considered the quantum effect of gravity and added a correction term to Einstein-Hilbert action. He showed that de Sitter spacetime will be realized. At that time there was no inflation theory, but this model still remains strong and survives as a model consistent with the observations [33].

Here, although it is different from the method of Starobinsky at the time, we show that this model causes inflation by the method used in $f(R)$ theory, and derive the observables. Starobinsky model is the $f(R)$ theory with the function chosen as

$$f(R) = R + \frac{1}{6M^2}R^2 \tag{3.3.23}$$

We start from the action

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R + \frac{1}{6M^2}R^2\right), \tag{3.3.24}$$

and perform the following field substitution:

$$R + \frac{1}{6M^2}R^2 \to (1+\chi)R - \frac{3}{2}M^2\chi^2. \tag{3.3.25}$$

Then, the action becomes

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[(1+\chi)R - \frac{3}{2}M^2\chi^2\right] \tag{3.3.26}$$

If we take the variation with respect to χ, we obtain

$$\chi = \frac{1}{3M^2}R\ . \tag{3.3.27}$$

The equivalence of the actions (3.3.24) and (3.3.26) can be confirmed by substituting (3.3.27) to (3.3.26). Next, performing the Weyl transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, we obtain

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\tilde{g}}\Omega^2\left[(1+\chi)(\tilde{R} + 6\Box f - 6\tilde{g}^{\mu\nu}\omega_\mu\omega_\nu) - \frac{3}{2}M^2\chi^2\right] \tag{3.3.28}$$

To make the action to the Einstein frame, we choose

$$\Omega^{-2} = (1+\chi) \tag{3.3.29}$$

and after rescaling the scalar field, we find the canonical form

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\tilde{g}}\tilde{R} + \int d^4x\sqrt{-\tilde{g}}\left(-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) \tag{3.3.30}$$

where

$$\phi = \sqrt{\frac{6}{\kappa^2}}\chi \tag{3.3.31}$$

and

$$V(\phi) = \frac{3}{4}\frac{M^2}{\kappa^2}\left(1 - e^{-\sqrt{\frac{2}{3}}\kappa\phi}\right)^2 \tag{3.3.32}$$

We show the shape potential in the Figure 3.1. This potential is going to satisfy the slow-roll condition because it has a plateau, and the slow-roll parameters are less than one.

Generally, the inflaton potential causing slow-roll inflation can be approximated to the following form during inflation:

$$V(\phi) \approx V_0 - V_1 e^{-\alpha\phi}\ , \tag{3.3.33}$$

This means, the slow-roll inflation is characterized by three parameters: V_0, V_1 and α. The V_1 is obviously unimportant because it can be easily changed to any desired value by a shift of the field ϕ in equation (3.3.33). The V_0 determines the scale of inflation. And the value of α determines the key observational parameter r related to primordial gravity waves and known as the tensor-to-scalar ratio,

$$r = \frac{8}{\alpha^2 N_e^2} \tag{3.3.34}$$

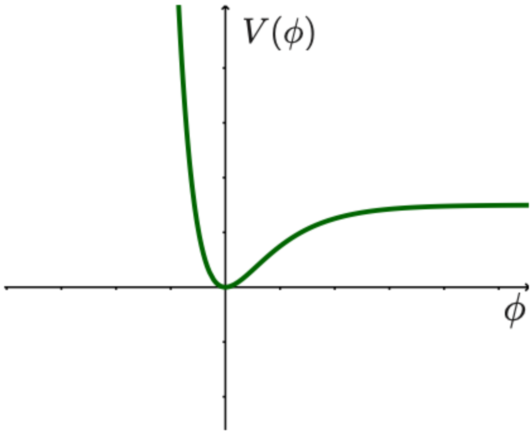


Figure 3.1: The most important feature of Starobinsky inflaton potential is having a plateau. This plateau is crucial for the slow-roll condition and can cause inflation for a sufficient time during the inflaton rolls down the plateau. Also, the minimum value of potential is Minkowski vacuum.

As regards the other CMB spectral tilts (inflationary observables), the scalar spectral index n_s and its running $dn_s/d\ln k$, their values derived from the potential (3.3.32) are the same as those in the Starobinsky case, namely

$$n_s \approx 1 - \frac{2}{N_e} \qquad \text{and} \qquad \frac{dn_s}{d\ln k} \approx -\frac{(1-n_s)^2}{2} \approx -\frac{2}{N_e^2} \quad . \tag{3.3.35}$$

The Planck data [33] sets the upper bound on r (with 95% of CL) as

$$r < 0.064, \quad n_S = 0.9649 \pm 0.0042, \quad \frac{dn_s}{d\ln k} = -0.005 \pm 0.013. \tag{3.3.36}$$

It can be confirmed the Starobinsky model satisfies the observed values by substituting $\alpha_S = \sqrt{\frac{2}{3}}$ and the best fit $N_e = 55$ to (3.3.34), which leads to

$$r \sim 0.004, \quad n_s \sim 0.96 \tag{3.3.37}$$

3.4 Inflation from $(R+\gamma R^n - 2\Lambda)$ Gravity in Higher Dimensions

The higher-dimensional $(R + \gamma R^n - 2\Lambda)$ gravity models and their *spontaneous* compactification to four dimensions were systematically studied in Refs. [35–38] that ruled out their phenomenological applicability to dark energy (because of negative cosmological constants) and early Universe inflation (because of low values of the scalar index n_s and the e-foldings number N_e). First, we relax the requirements of Refs. [35–37] by dropping the condition of spontaneous compactification, i.e. we do not impose the equations of motion in D dimensions, and ignore all moduli related to the compact dimensions. The compactification details should be addressed in a more fundamental framework (like supergravity or superstrings) with more fields representing extra degrees of freedom and more couplings involved. In this section, we simply assume that it is possible *before* inflation. The four-dimensional inflationary models based on a higher-dimensional $(R + \gamma R^n - 2\Lambda)$ gravity were previously considered in Ref. [38], though only for $D < 8$ where they were found to be not viable because of low values of n_s and N_e.

We study the $(R + \gamma R^n - 2\Lambda)$ gravity models in *higher* $(D > 4)$ spacetime dimensions with the cosmological constant Λ, in an effort to derive some scalar potentials of the type (3.3.32) or (3.3.33), leading to the very *specific* values of α that have their origin in a higher-dimensional modified gravity. We stress that we do *not* mean a cosmological inflation in higher dimensions. We assume that our Universe was born multi-dimensional, and then four spacetime dimensions became infinite, while the others curled up by some unknown mechanism *before inflation*. We exploit the fact that the Weyl transform, as part of the duality transformation between Jordan and Einstein frames, depends upon D [39,40]. We apply the duality transformation to $f(R)$ gravity in D dimensions, get the scalaron (inflaton) scalar potential, and after that dimensionally reduce it (by integrating over flat compact dimensions) to four (infinite) spacetime dimensions. The cosmological inflation is assumed to be taking place after compactification (and after moduli stabilization, if any).

3.4.1 Our setup

We denote spacetime vector indices in D dimensions by capital latin letters $A, B, \ldots = 0, 1, \ldots, D-1$, and spacetime vector indices in four dimensions by lower case greek letters $\alpha, \beta, \ldots = 0, 1, 2, 3$. In this Section we proceed along the lines of Ref. [18], though in D dimensions and with arbitrary n.

Our starting point is the following gravitational action in a D-dimensional curved spacetime:

$$S_{\text{grav.}} = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g_D}(R + \gamma R^n - 2\Lambda) \quad , \tag{3.4.1}$$

where $\kappa > 0$ is the gravitational coupling constant of (mass) dimension $\frac{1}{2}(-D+2)$, $\gamma > 0$ is the new (modified gravity) coupling constant of (mass) dimension $(-2n+2)$, and Λ is the cosmological constant of (mass) dimension 2, in D dimensions. Unlike Refs. [35–38], after the Legendre-Weyl transform of the action (3.4.1) to the (dual) scalar-tensor gravity in D dimensions (see below), we demand the scalar potential to have an infinite plateau of a positive height (for large field values).

After a substitution

$$R + \gamma R^n \longrightarrow (1+B)R - \left(\frac{1}{\gamma n}\right)^{\frac{1}{n-1}} \left(\frac{n-1}{n}\right) B^{\frac{n}{n-1}} \quad , \tag{3.4.2}$$

where we have introduced the new scalar field B, the action (3.4.1) takes the form

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g_D} \left[(1+B)R - \left(\frac{1}{\gamma n}\right)^{\frac{1}{n-1}} \left(\frac{n-1}{n}\right) B^{\frac{n}{n-1}} - 2\Lambda \right] \quad . \tag{3.4.3}$$

The field B enters the action (3.4.3) algebraically, while its "equation of motion" reads $B = \gamma n R^{n-1}$. After substituting the latter back into the action (3.4.3) we get the original action (3.4.1). Hence, the actions (3.4.1) and (3.4.3) are classically equivalent.

Next, we apply a Weyl transformation with the space-time-dependent parameter $\Omega(x)$,

$$g_{AB} = \Omega^{-2}\tilde{g}_{AB}, \quad \sqrt{-g} = \Omega^{-D}\sqrt{-\tilde{g}} \quad , \tag{3.4.4}$$

where we have introduced the new spacetime metric $\tilde{g}_{AB}$ in D dimensions. As a result of this transformation, the corresponding scalar curvatures are related by

$$R = \Omega^2[\tilde{R} + 2(D-1)\tilde{\Box} f - (D-1)(D-2)\tilde{g}^{AB} f_A f_B] \quad , \tag{3.4.5}$$

where

$$f = \ln\Omega \ , \qquad f_A = \frac{\partial_A \Omega}{\Omega} \ , \tag{3.4.6}$$

and the covariant wave operator $\tilde{\Box} = \tilde{D}^A \tilde{D}_A$ in D dimensions.

The Weyl-transformed (and equivalent via the field-redefinition (3.4.4)) action S is given by

$$\begin{aligned} S =& \frac{1}{2\kappa^2} \int d^D x \sqrt{-\tilde{g}_D}\Omega^{-D}[(1+B)\Omega^2(\tilde{R} + 2(D-1)\tilde{\Box} f \\ & -(D-1)(D-2)\tilde{g}^{AB} f_A f_B) - \left(\frac{1}{\gamma n}\right)^{\frac{1}{n-1}} \frac{n-1}{n} B^{\frac{n}{n-1}} - 2\Lambda] \ . \end{aligned} \tag{3.4.7}$$

Hence, in order to get the corresponding action in Einstein frame, we should choose the local parameter Ω as

$$\Omega^{D-2} = e^{(D-2)f} = 1 + B \quad . \tag{3.4.8}$$

We thus find

$$f = \frac{1}{D-2}\ln(1+B) \tag{3.4.9}$$

and

$$S = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-\tilde{g}_D}\left[\tilde{R} - (D-1)(D-2)\tilde{g}^{AB}\partial_A f\partial_B f \right.$$
$$\left. - e^{-Df}\left(\frac{1}{\gamma n}\right)^{\frac{1}{n-1}}\frac{n-1}{n}B^{\frac{n}{n-1}} - 2e^{-Df}\Lambda\right] \quad . \tag{3.4.10}$$

As is clear from equation (3.4.10), we should also rescale the scalar field f, in order to get the canonically normalized scalar kinetic terms, as

$$\phi = \sqrt{\frac{(D-1)(D-2)}{\kappa^2}}\, f \quad . \tag{3.4.11}$$

As a result, in terms of the canonical scalar ϕ, we find

$$B = e^{(D-2)\kappa\phi/\sqrt{(D-1)(D-2)}} - 1 \quad , \tag{3.4.12}$$

the scalar potential

$$2\kappa^2 V(\phi) = \left(\frac{1}{\gamma n}\right)^{\frac{1}{n-1}}\left(\frac{n-1}{n}\right)\left[e^{(D-2)\kappa\phi/\sqrt{(D-1)(D-2)}} - 1\right]^{\frac{n}{n-1}} \times$$
$$\times\, e^{-D\kappa\phi/\sqrt{(D-1)(D-2)}} + 2\Lambda e^{-D\kappa\phi/\sqrt{(D-1)(D-2)}} \quad , \tag{3.4.13}$$

and the standard scalar-tensor gravity action in Einstein frame in D dimensions,

$$S = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-\tilde{g}_D}\tilde{R} + \int d^Dx\sqrt{-\tilde{g}_D}\left[-\frac{1}{2}\tilde{g}^{AB}\partial_A\phi\partial_B\phi - V(\phi)\right] \quad . \tag{3.4.14}$$

We assume that the D-dimensional action (3.4.14) is then "compactified" to four infinite spacetime dimensions before inflation. Applying dimensional reduction (i.e. taking all fields to be independent upon compact flat $(D-4)$ dimensions), we have

$$\int d^Dx = V_{D-4}\int d^4x \ , \quad \phi = \phi_4/\sqrt{V_{D-4}} \ , \quad \kappa = \kappa_4\sqrt{V_{D-4}} \ , \quad V = V_4/V_{D-4} \ , \tag{3.4.15}$$

so that $\kappa\phi = \kappa_4\phi_4$ and $\kappa^2 V = \kappa_4^2 V_4$, where we have introduced the volume V_{D-4} of compact dimensions, with the subscripts "4" referring to four spacetime dimensions.

It gives rise to the standard four-dimensional action (in Einstein frame, and with a canonical scalar ϕ_4)

$$S_{\rm inf.}[\tilde{g}_4,\phi_4] = \frac{1}{2}\int d^4x\sqrt{-\tilde{g}_4}\tilde{R}_4 + \int d^4x\sqrt{-\tilde{g}_4}\left[-\frac{1}{2}\tilde{g}_4^{\mu\nu}\partial_\mu\phi_4\partial_\nu\phi_4 - V_4(\phi_4)\right] \tag{3.4.16}$$

that we are going to consider as our inflationary model in four spacetime dimensions. In what follows we stay in four spacetime dimensions. However, the higher dimension D and the power n enter the four-dimensional scalar potential $V_4(\phi_4)$ as the parameters, according to equations (3.4.13) and (3.4.15).

3.4.2 The scalar potential

To study our scalar potential in four spacetime dimensions, we rescale the relevant quantities by introducing the notation

$$\lambda = \left(\frac{n}{n-1}\right)\left(\frac{1}{\gamma n}\right)^{-\frac{1}{n-1}} 2\Lambda \ , \tag{3.4.17}$$

$$\tilde{\phi} = \frac{\phi_4}{\sqrt{(D-1)(D-2)}} \ , \tag{3.4.18}$$

and

$$\tilde{V}(\tilde{\phi}) = \frac{2V_4(\phi_4)}{\left(\frac{1}{\gamma n}\right)^{\frac{1}{n-1}}\left(\frac{n-1}{n}\right)} \ . \tag{3.4.19}$$

Then the scalar potential in D dimensions takes the simple form

$$\tilde{V}(\tilde{\phi}) = \left[e^{(D-2)\tilde{\phi}} - 1\right]^{\frac{n}{n-1}} e^{-D\tilde{\phi}} + \lambda e^{-D\tilde{\phi}} \ , \tag{3.4.20}$$

where we have also used equations (3.4.13) and (3.4.15). Demanding this scalar potential to have *a plateau* of a positive height for $\tilde{\phi} \to \infty$, like that in equation (3.3.32), we get the condition

$$\left[e^{(D-2)\tilde{\phi}}\right]^{\frac{n}{n-1}} e^{-D\tilde{\phi}} = 1 \tag{3.4.21}$$

that implies (*cf.* Refs. [37, 39])

$$n = \frac{D}{2} \ . \tag{3.4.22}$$

Substituting it back to equation (3.4.20) yields the potential

$$\tilde{V}(\tilde{\phi}) = \left[1 - e^{-(D-2)\tilde{\phi}}\right]^{\frac{D}{D-2}} + \lambda e^{-D\tilde{\phi}} \ . \tag{3.4.23}$$

Let us write down the power $D/(D-2) = p/q$ in terms of mutually prime positive integers p and q. Should q be even, it leads to the obstruction $\tilde{\phi} \geq 0$ of the real scalar field $\tilde{\phi}$ because its scalar potential becomes imaginary for $\tilde{\phi} < 0$. For example, it happens when $D = 6$ and $D = 10$. Avoiding such situation puts a severe restriction on the allowed values of D in our approach. Similarly, since n is also the power of R in equation (3.4.1), and R can take negative values, we conclude that n must be integer and, hence, D must be *even.* The allowed dimensions are thus must be multiples of four, with the lowest values beyond four being $D = 8$ and $D = 12$. These two cases are studied in more detail in the next section.

Requiring the scalar potential to be bounded from below is needed for stability. It the limit $\tilde{\phi} \to -\infty$, the leading term in the scalar potential (3.4.23) is given by

$$\lim_{\tilde{\phi}\to-\infty} \tilde{V}(\tilde{\phi}) \approx \left[(-1)^{\frac{D}{D-2}} + \lambda\right] e^{-D\tilde{\phi}} \ , \tag{3.4.24}$$

so that we have to restrict the parameter λ as

$$\lambda \geq -(-1)^{\frac{D}{D-2}} \quad . \tag{3.4.25}$$

If we require the existence of a minimum of the scalar potential, describing the classical vacuum after inflation, we need the existence of a real (finite) solution to $\frac{d\tilde{V}}{d\tilde{\phi}} = 0$. We find it at

$$\tilde{\phi}_0 = \frac{1}{D-2} \ln\left(1 + \lambda^{\frac{D-2}{2}}\right) \tag{3.4.26}$$

with

$$\tilde{V}(\tilde{\phi}_0) = \lambda \left(1 + \lambda^{\frac{D-2}{2}}\right)^{\frac{-2}{D-2}} \quad . \tag{3.4.27}$$

It gives rise to a bit stronger condition,

$$\lambda > (-1)^{\frac{2}{D-2}} \ , \tag{3.4.28}$$

and amounts to $\lambda > -1$ in the allowed dimensions.

A stronger condition arises by demanding the scalar potential minimum to correspond either a Minkowski or a de Sitter vacuum. According to equation (3.4.27), $\tilde{V}(\tilde{\phi}_0) \geq 0$ implies

$$\lambda \geq 0 \ . \tag{3.4.29}$$

Finally, demanding the second derivative of the scalar potential at its minimum to be finite and positive or, equivalently, requiring a finite positive scalaron mass, restricts λ by

$$\lambda > 0 \ , \tag{3.4.30}$$

and implies *positive* cosmological constants in both D and four dimensions, because of equations (3.4.17) and (3.4.27). Under the conditions above, with $n = \frac{D}{2}$, we find the four-dimensional scalar potential as

$$V_4(\phi_4) = \left(\frac{2}{\gamma D}\right)^{\frac{2}{D-2}} \left(\frac{D-2}{2D}\right) \left(1 - e^{-\sqrt{\frac{D-2}{D-1}}\phi_4}\right)^{\frac{D}{D-2}} + \Lambda e^{-\sqrt{\frac{D^2}{(D-1)(D-2)}}\phi_4} \ . \tag{3.4.31}$$

Taylor expansion of the potential around its minimum at $\phi_4^{(0)}$,

$$V_4(\phi_4) = V_4(\phi_4^{(0)}) + \frac{1}{2}\frac{dV_4^2(\phi_4^{(0)})}{d\phi_4^2}(\phi_4 - \phi_4^{(0)})^2 + \dots \ , \tag{3.4.32}$$

yields the cosmological constant δ in four dimensions as

$$V_4(\phi_4^{(0)}) = \Lambda \left[1 + \frac{\gamma D}{2}\left(\frac{2D}{D-2}\Lambda\right)^{\frac{D-2}{2}}\right]^{\frac{-2}{D-2}} \equiv \delta, \tag{3.4.33}$$

and the inflaton mass M as

$$\frac{dV_4^2(\phi_4^{(0)})}{d\phi_4^2} = \frac{2D}{(D-1)(D-2)}\Lambda\left[1+\frac{\gamma D}{2}\left(\frac{2D}{D-2}\Lambda\right)^{\frac{D-2}{2}}\right]^{-\frac{2}{D-2}} \times \tag{3.4.34}$$
$$\times\left[\frac{\gamma D}{2}\left(\frac{2D}{D-2}\Lambda\right)^{\frac{D-2}{2}}\right]^{-1} \equiv M^2 \quad .$$

Equations (3.4.33) and (3.4.34) can be considered as a system of two equations on the two parameters Λ and γ of our model, because the observational values of δ and M are known as $\delta = \mathcal{O}(10^{-120})$ and $M \approx 3\times 10^{-6}$, respectively. We find

$$\Lambda^{\frac{D-2}{2}} = \frac{2D}{(D-1)(D-2)}\delta^{\frac{D}{2}}M^{-2} + \delta^{\frac{D-2}{2}} \tag{3.4.35}$$

and

$$\gamma = \frac{4}{(D-1)(D-2)}\left(\frac{2D}{D-2}\right)^{-\frac{D-2}{2}}\delta M^{-2}\left[\frac{2D}{(D-1)(D-2)}\delta^{\frac{D}{2}}M^{-2} + \delta^{\frac{D-2}{2}}\right]^{-1} \quad . \tag{3.4.36}$$

Because of the tiny value of the cosmological constant δ, the solutions can be greatly simplified to

$$\Lambda = \delta \quad , \tag{3.4.37}$$

as expected, and

$$\gamma = \frac{4}{(D-1)(D-2)}\left(\frac{D-2}{2D}\right)^{\frac{D-2}{2}} M^{-2}\delta^{\frac{4-D}{2}} \quad . \tag{3.4.38}$$

As a check, when $D=4$ and $\Lambda=0$, we recover the Starobinsky model of Subsec. 3.3.3, with $\gamma_4 = 1/(6M^2) \approx \frac{1}{54}10^{12}$. Otherwise, for any D, we find

$$\gamma \approx \frac{4}{9(D-1)(D-2)}\left(\frac{D-2}{2D}\right)^{\frac{D-2}{2}} 10^{12+60(D-4)} \quad . \tag{3.4.39}$$

The parameter γ in $D>4$ generically diverges when $\Lambda \to +0$ (unless $M^4\delta^{D-4} = const. > 0$), which also implies $\delta \to +0$ and $M \to +\infty$ because the 2nd derivative (3.4.34) of the scalar potential (3.4.31) becomes infinite at the minimum in this limit. It is remarkable that a (finite) positive cosmological constant ensures M to be finite too.

3.4.3 Examples: $D = 8$ and $D = 12$

In this Subsection, we specify our results to the two particular cases, having the special dimensions $D=8$ and $D=12$, respectively, and with $\lambda \geq 0$.

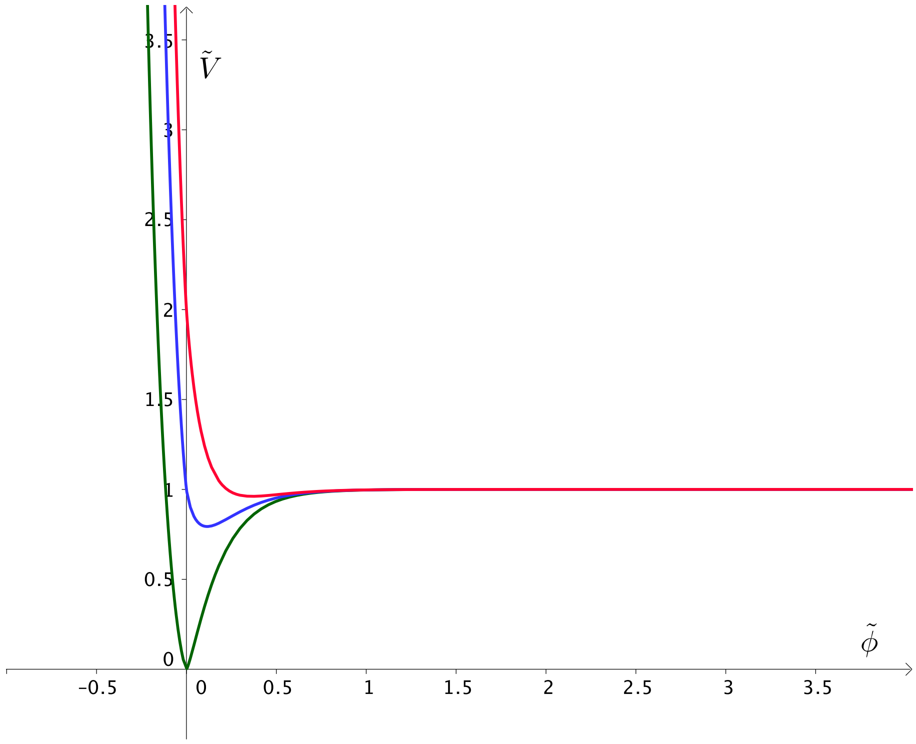


Figure 3.2: The profile of the four-dimensional scalar potential obtained from $D = 8$ dimensions. The green line describes the case of $\lambda = 0$, the blue line is of $\lambda = 1$, and the red line is of $\lambda = 2$, respectively.

- As regards $D = 8$ and $n = 4$, the scalar potential (3.4.23) reads

$$\tilde{V}(\tilde{\phi}) = \left(1 - e^{-6\tilde{\phi}}\right)^{\frac{4}{3}} + \lambda e^{-8\tilde{\phi}} \quad . \tag{3.4.40}$$

It has the absolute minimum at

$$\tilde{\phi}_0 = \frac{1}{6} \ln\left(1 + \lambda^3\right) \quad , \tag{3.4.41}$$

where it has a value (the cosmological constant)

$$\tilde{V}(\lambda) = \lambda(1 + \lambda^3)^{-\frac{1}{3}} \quad . \tag{3.4.42}$$

A profile of the four-dimensional inflaton scalar potential, originating from $D = 8$, is given in Figure 3.2.

According to equation (3.4.39), the parameter γ in $D = 8$ has the value $\gamma_8 \approx \frac{1}{2^8 \cdot 7} 10^{252}$. Since its (mass) dimension is $2 - D = -6$, the relevant scale in $D = 8$ is given by

$$\gamma_8^{-1/6} \approx 3.485 \cdot 10^{-42} M_{\rm Pl} \quad . \tag{3.4.43}$$

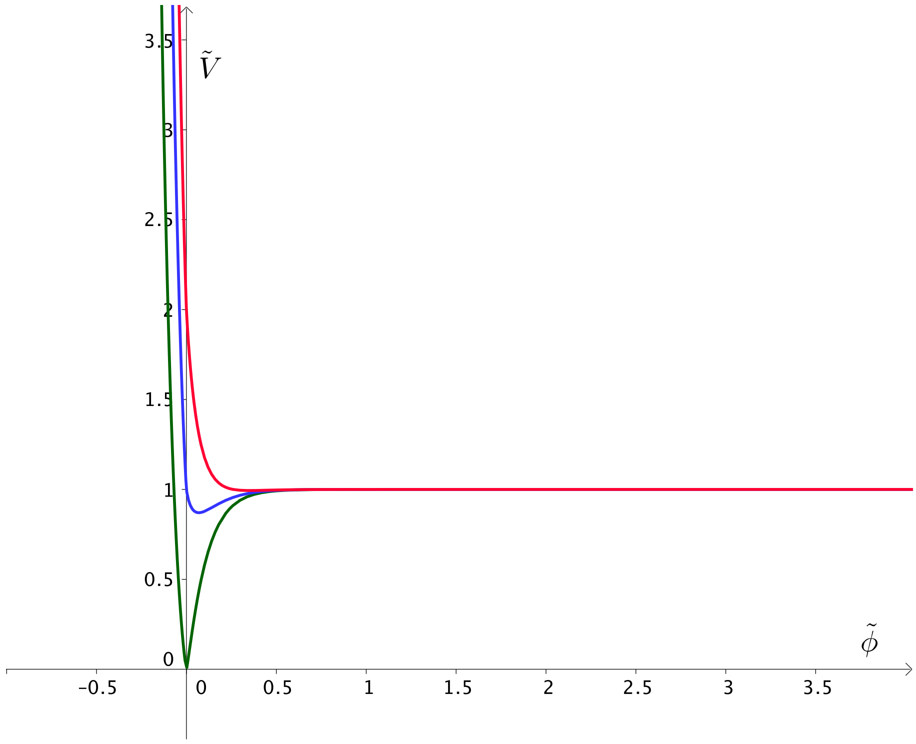


Figure 3.3: The profile of the scalar potential potential obtained from $D = 12$ dimensions. The green line describes the case of $\lambda = 0$, the blue line is of $\lambda = 1$, and the red line is of $\lambda = 2$, respectively.

- As regards $D = 12$ and $n = 6$, the scalar potential (3.4.23) reads

$$\tilde{V}(\tilde{\phi}) = \left(1 - e^{-10\tilde{\phi}}\right)^{\frac{6}{5}} + \lambda e^{-12\tilde{\phi}} \quad . \tag{3.4.44}$$

It has the absolute minimum at

$$\tilde{\phi}_0 = \frac{1}{10} \ln\left(1 + \lambda^5\right) \quad , \tag{3.4.45}$$

where it has a value (the cosmological constant)

$$\tilde{V}(\lambda) = \lambda(1 + \lambda^5)^{-\frac{1}{5}} \quad . \tag{3.4.46}$$

A profile of the four-dimensional inflaton scalar potential, originating from $D = 12$, is given in Figure 3.3.

According to equation (3.4.39), the parameter γ in $D = 12$ has the value $\gamma_{12} \approx \frac{5^4}{2^9 \cdot 3^7 \cdot 11} 10^{492}$. Since its (mass) dimension is $2 - D = -10$, the relevant scale in $D = 12$ is given by

$$\gamma_{12}^{-1/10} \approx 1.696 \cdot 10^{-49} M_{\rm Pl} \quad . \tag{3.4.47}$$

3.4.4 Summary of Section 3.4

In this section we derived the inflaton scalar potential from higher ($D > 4$) dimensions, in the context of the D-dimensional ($R + \gamma R^n - 2\Lambda$) gravity, by using the Starobinsky model of chaotic large-field inflation in $D = 4$ as a prototype. We assumed that a compactification of the extra dimensions took place *before* inflation. We found that this requires a positive cosmological constant and $n = D/2$. We calculated the corresponding scalar potential and the values of its parameters for any D, and specified our results to the two special cases, $D = 8$ and $D = 12$. Our models predict the viable values of n_s and r for N_e around $55 \div 5$.

Our scalar potentials in their slow-roll part fall into the class of the inflationary plateau-type potentials describing chaotic large-field inflation and having the form (3.5.70) with

$$\alpha = \sqrt{\frac{D-2}{D-1}} \quad , \tag{3.4.48}$$

because of equation (3.4.31). In particular, we have $\alpha_4 = \alpha_s = \sqrt{2/3}$, $\alpha_8 = \sqrt{6/7}$ and $\alpha_{12} = \sqrt{10/11}$.

According to equation (3.5.72), the value of the α-parameter dictates the observable value of the tensor-to-scalar ratio r as

$$r = \frac{8(D-1)}{(D-2)N_e^2} \quad . \tag{3.4.49}$$

In particular, we find $r_4 = \frac{12}{N_e^2}$, $r_8 = \frac{28}{3N_e^2}$, and $r_{12} = \frac{44}{5N_e^2}$. All those values are in agreement with current observations, and give the sharp (though very close) predictions for future measurements of r.

The microscopic parameters of our higher-dimensional model were tuned to get the *same* inflaton mass as that of the original Starobinsky model, in four dimensions. It implies that inflaton oscillations after inflation, near the minimum of the potential, will be the *same* too. When a conventional matter action is added to our modified gravity model, the Weyl rescaling of metric results in the *universal* couplings (via the GR covariant derivatives) of inflaton ϕ to all matter fields with powers of $\exp(-\alpha\kappa\phi)$. Since the values of α we found from higher dimensions in equation (3.4.48) are only slightly higher of the (Starobinsky) value at $D = 4$, while all those couplings are suppressed by Planck mass, we expect that the D-dependence in any reheating process will be (physically) negligible.

The predictions of this inflationary model in $D = 8$ and $D = 12$ for the CMB observables are in agreement with the current observations. However, since dynamics of the extra dimensions is ignored, their size is not under control, so that stabilization of extra dimensions is still needed — this is known as the moduli stabilization problem in the literature.

3.5 Inflation from higher dimensions via spontaneous compactification

In our previous Section, we proposed a derivation of the viable inflaton scalar potential from the higher (D) dimensional ($R + \gamma R^n - 2\Lambda$) gravity, by giving up the condition of spontaneous compactificaton of extra dimensions and ignoring the moduli, i.e. just assuming that the compactification happened before inflation and it can be made spontaneous by adding some more fields. As a result, the inflaton scalar potential in four spacetime dimensions turns out to be dependent upon the parameters (γ, Λ, D, n), while the viable inflationary phenomenology requires

$$n = D/2 \ , \tag{3.5.1}$$

with the dimension D being a multiple of four. The condition (3.5.1) arises by demanding the existence of a *plateau* with a positive height for the inflationary scalar potential, as is apparently favoured by the Planck mission observations [32, 33, 41], and is the case in the famous Starobinsky inflationary model (in Subsection 3.3.3), though is in contrast to Refs. [35–38] where the scalar potential was demanded to vanish before the onset of inflation. Our results were significantly enhanced in Ref. [20] where a spontaneous compactification and stabilization of the volume of extra dimensions was achieved by adding a single $(p-1)$-form gauge field having a non-vanishing flux in compact dimensions and obeying the condition

$$p = n \ . \tag{3.5.2}$$

In this section we extend our analysis in the first relevant higher dimension $D = 8$, and consider an embedding of the $D = 8$ modified gravity model into a (modified) $D = 8$ supergravity.

3.5.1 The $D = 8$ model and its $D = 4$ compactification

It is the demand of having a plateau for the scalar potential in higher D dimensions that results in the condition (3.5.1) [19, 39]. But it is still insufficient for moduli stabilization that requires at least one p-form field obeying the condition (3.5.2) [20].

Therefore, our minimal model in $D = 8$ is defined by the action

$$S = \frac{M_8^6}{2} \int d^8X \sqrt{-g_8} \left[R_8 + \gamma_8 R_8^4 - 2\Lambda_8 - g^{A_1B_1} g^{A_2B_2} g^{A_3B_3} g^{A_4B_4} F_{A_1A_2A_3A_4} F_{B_1B_2B_3B_4} \right] . \tag{3.5.3}$$

It depends upon two fields, a metric g_{AB} and a 3-form gauge potential A_{ABC}, whose field strength 4-form is $F = dA$, and has three parameters: the gravitational mass scale M_8, the (modified gravity) coupling constant $\gamma_8 > 0$ and the cosmological constant $\Lambda_8 > 0$, all in $D = 8$ dimensions.

Applying the Legendre-Weyl transform to the action (3.5.3) in $D = 8$ results in the dual (classically equivalent) action

$$S_{\rm dual} = \frac{M_8^6}{2}\int d^8X\sqrt{-\tilde{g}_8}\left[\tilde{R}_8 - 42\tilde{g}^{AB}\partial_A f\partial_B f - M_8^2\tilde{V}(f)\right. \tag{3.5.4}$$

$$\left. -\tilde{g}^{A_1B_1}\tilde{g}^{A_2B_2}\tilde{g}^{A_3B_3}\tilde{g}^{A_4B_4}F_{A_1A_2A_3A_4}F_{B_1B_2B_3B_4}\right] \quad , \tag{3.5.5}$$

depending upon three fields, the Weyl transformed (new) metric $\tilde{g}_{AB}$, the 4-form $F = dA$, and the real scalaron $f(X)$ having the scalar potential

$$\tilde{V}(f) = a^{-2}(1-e^{-6f})^{\frac{4}{3}} + 2e^{-8f}\tilde{\Lambda}_8 \quad , \tag{3.5.6}$$

in terms of the (dimensionless) coupling constants

$$\gamma_8 \equiv M_8^{-6}\tilde{\gamma}_8 \quad , \quad \Lambda_8 \equiv M_8^2\tilde{\Lambda}_8 \quad , \quad \frac{3}{4}\left(\frac{1}{4\tilde{\gamma}_8}\right)^{\frac{1}{3}} \equiv a^{-2} \quad . \tag{3.5.7}$$

Under the Weyl transform (3.4.4), the Ω-factors are *cancelled* against each other, so that the action for the 4-form gauge fields in (3.5.3) remains unchanged,

$$S_8[\tilde{g}_{AB}, F_4] = -\frac{M_8^6}{2}\int d^8X\sqrt{-\tilde{g}_8}\,\tilde{g}^{A_1B_1}\cdots\tilde{g}^{A_4B_4}F_{A_1\ldots A_4}F_{B_1\ldots B_4} \quad . \tag{3.5.8}$$

The dual action (3.5.4) has the standard form of Einstein's gravity coupled to the matter fields (f, A) and having the scalar potential (3.5.6) in $D = 8$. This scalar potential has a plateau of the positive height a^{-2} for large positive values of f.

Let us consider a compactification of the $D = 8$ theory (3.5.4) on a 4-sphere S^4 with the warp factor χ, down to *four* spacetime dimensions, i.e. in a curved spacetime with the local structure

$$\mathcal{M}_8 = \mathcal{M}_4 \times e^{2\chi}S^4 \quad . \tag{3.5.9}$$

The 8-dimensional coordinates (X^A) can then be decomposed into the 4-dimensional spacetime coordinates (x^α) with $\alpha = 0, 1, 2, 3$, and the coordinates (y^a) of four compact dimensions of S^4, with $a, b, \ldots = 1, 2, 3, 4$. The compactification ansatz reads

$$ds_8^2 = \tilde{g}_{AB}dX^AdX^B = g_{\alpha\beta}dx^\alpha dx^\beta + e^{2\chi}g_{ab}dy^ady^b \quad , \tag{3.5.10}$$

where $g_{\alpha\beta} = g_{\alpha\beta}(x)$, $g_{ab} = g_{ab}(y)$ and $\chi = \chi(x)$, with the normalization

$$\int d^4y\sqrt{g_y} = M_8^{-4} \quad . \tag{3.5.11}$$

Taking into account the S^4 Euler number equal to 2, yields

$$\int d^4y\sqrt{g_y}R_y = 2M_8^{-2} \quad , \tag{3.5.12}$$

where R_y is the scalar curvature of the sphere S^4. The decomposition (3.5.10) also implies

$$\sqrt{-\tilde{g}_8} = e^{4\chi}\sqrt{-g_4}\sqrt{g_y} \tag{3.5.13}$$

and

$$\tilde{R}_8 = R + e^{-2\chi}R_y - 8e^{-\chi}\tilde{\Box}e^{\chi} - 12e^{-2\chi}g^{\alpha\beta}\partial_\alpha e^{\chi}\partial_\beta e^{\chi} \quad , \tag{3.5.14}$$

where we have introduced the Ricci scalar R and the generally covariant wave operator $\tilde{\Box} = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ in four spacetime dimensions.

The volume $\mathcal{V}$ of four (compact) extra dimensions is given by

$$\mathcal{V} = \int d^4y\sqrt{\det(e^{2\chi}g_{ab})} = e^{4\chi}M_8^4 \quad , \tag{3.5.15}$$

so that the warp factor χ can be identified with the volume modulus of the sphere S^4.

A substitution of equations (3.5.10), (3.5.13) and (3.5.14) into the action (3.4.14), and an integration over the compact dimensions by using equations (3.5.11) and (3.5.12), lead to the action

$$S_4[g_{\alpha\beta}, f, \chi] = \frac{M_8^2 e^{4\chi_0}}{2}\int d^4x\sqrt{-g}\left(\frac{e^{\chi}}{e^{\chi_0}}\right)^4 \left[R + 2M_8^2e^{-2\chi} \right.$$
$$\left. +12g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi - 42g^{\alpha\beta}\partial_\alpha f\partial_\beta f - M_8^2\tilde{V}(f)\right] \quad , \tag{3.5.16}$$

where we have introduced the vacuum expectation value $\langle\chi\rangle_0 = \chi_0 = const.$

The action (3.5.16) is still in a Jordan frame, so that the wrong sign of the kinetic term of the field χ is not necessarily a problem. The Weyl transformation with the parameter $h(x)$ to the Einstein frame is given by

$$g_{\alpha\beta} = e^{-2h}\hat{g}_{\alpha\beta}, \quad h = 2(\chi - \chi_0) \quad . \tag{3.5.17}$$

It implies

$$g^{\alpha\beta} = e^{2h}\hat{g}^{\alpha\beta}, \quad \sqrt{-g} = e^{-4h}\sqrt{-\hat{g}} \ , \tag{3.5.18}$$

and

$$R = e^{2h}\left[\hat{R} + 6\hat{g}^{\alpha\beta}\nabla_\alpha\nabla_\beta h - 6\hat{g}^{\alpha\beta}\partial_\alpha h\partial_\beta h\right] \quad . \tag{3.5.19}$$

Accordingly, the action (3.5.16) gets transformed to

$$S_4[\hat{g}_{\alpha\beta}, f, \chi] = \frac{M_8^2 e^{4\chi_0}}{2}\int d^4x\sqrt{-\hat{g}_4}\left\{\hat{R} - 12\hat{g}^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi \right.$$
$$\left. -42\hat{g}^{\alpha\beta}\partial_\alpha f\partial_\beta f - \left(\frac{e^{\chi}}{e^{\chi_0}}\right)^{-4} M_8^2\left[\tilde{V}(f) - 2e^{-2\chi}\right]\right\} \ , \tag{3.5.20}$$

with the *physical* signs in front of all the kinetic terms. This also fixes the four-dimensional (reduced) Planck mass as

$$M_{\rm Pl}^2 \equiv \kappa^{-2} = M_8^2 e^{4\chi_0} \ . \tag{3.5.21}$$

Therefore, we have

$$S_4[\hat{g}_{\alpha\beta}, f, \chi] = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-\hat{g}} \Big[\hat{R} - 12\hat{g}^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi \\ - 42\hat{g}^{\alpha\beta}\partial_\alpha f \partial_\beta f - e^{-4\chi} M_{\rm Pl}^2 \left(\tilde{V}(f) - 2e^{-2\chi} \right) \Big] \quad . \tag{3.5.22}$$

Similarly, applying the compactification ansatz (3.5.10) to the 4-form action (3.5.8) in 8 dimensions yields

$$S_{8,F}[\tilde{g}_{AB}, F] = -\frac{M_8^6}{2} \int d^4x \sqrt{-g} \int d^4y \sqrt{g_y}\, e^{-4\chi} g^{a_1 b_1} \cdots g^{a_4 b_4} F_{a_1 \ldots a_4} F_{b_1 \ldots b_4} \quad . \tag{3.5.23}$$

After defining the (dimensionless) *flux* parameter F^2 as

$$\int d^4y \sqrt{g_y}\, g^{a_1 b_1} \cdots g^{a_4 b_4} F_{a_1 \ldots a_4} F_{b_1 \ldots b_4} = M_8^{-2} F^2 = const. \quad , \tag{3.5.24}$$

and using the Weyl transformation (3.5.17), the action (3.5.23) reduces to

$$\begin{aligned} S_{4,F}[\hat{g}_{AB}, \chi] &= -\frac{M_8^2 e^{4\chi_0}}{2} \int d^4x \sqrt{-g} \left(\frac{e^{\chi}}{e^{\chi_0}} \right)^4 e^{-8\chi} M_8^2 F^2 \\ &= -\frac{M_8^2 e^{4\chi_0}}{2} \int d^4x \sqrt{-\hat{g}}\, e^{-4h} \left(\frac{e^{\chi}}{e^{\chi_0}} \right)^4 e^{-8\chi} M^2 F^2 \\ &= -\frac{M_{\rm Pl}^4}{2} \int d^4x \sqrt{-\hat{g}}\, e^{-12\chi} F^2 \quad . \end{aligned} \tag{3.5.25}$$

The total action in 4 dimensions is given by a sum of equations. (3.5.22) and (3.5.25),

$$S_4[\hat{g}_{AB}, \chi, f] = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g} \Big[\hat{R} - 12\hat{g}^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi \\ -42\hat{g}^{\alpha\beta}\partial_\alpha f \partial_\beta f - M_{\rm Pl}^2 \left(e^{-4\chi}\tilde{V}(f) - 2e^{-6\chi} - e^{-12\chi} F^2 \right) \Big] \quad . \tag{3.5.26}$$

The canonical scalar fields $\hat{\chi}$ and $\hat{f}$ are thus given by

$$\hat{\chi} = 2\sqrt{3} M_{\rm Pl}\chi \quad \text{and} \quad \hat{f} = \sqrt{42} M_{\rm Pl} f \quad , \tag{3.5.27}$$

and the two-scalar potential in four dimensions reads

$$M_{\rm Pl}^{-4} V(\chi, f) = \left[a^{-2}(1 - e^{-6f})^{\frac{4}{3}} + 2\tilde{\Lambda}_8 e^{-8f} \right] e^{-4\chi} - 2e^{-6\chi} + F^2 e^{-12\chi} \quad . \tag{3.5.28}$$

This compactification results in the following $D = 4$ action :

$$S_4[\hat{g}_{AB}, \chi, f] = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g} \Big[\hat{R} - 12\hat{g}^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi \\ -42\hat{g}^{\alpha\beta}\partial_\alpha f \partial_\beta f - M_{\rm Pl}^2 \left(e^{-4\chi}\tilde{V}(f) - 2e^{-6\chi} - e^{-12\chi} F^2 \right) \Big] \quad , \tag{3.5.29}$$

of three fields: a metric $\hat{g}_{\alpha\beta}(x)$, the scalaron $f(x)$ and the S^4 (volume) modulus $\chi(x)$, with the scalar potential depending upon the parameters $(a, \tilde{\Lambda}_8)$ *and* the 4-form gauge field strength *flux* F defined by the integration

$$\int d^4y \sqrt{g_y}\, g^{a_1 b_1} \cdots g^{a_4 b_4} F_{a_1 \ldots a_4} F_{b_1 \ldots b_4} = M_8^{-2} F^2 \tag{3.5.30}$$

over the S^4. The full two-scalar potential in $D = 4$ thus reads

$$M_{\rm Pl}^{-4} V(\chi, f) = \left[a^{-2} (1 - e^{-6f})^{\frac{4}{3}} + 2\tilde{\Lambda}_8 e^{-8f} \right] e^{-4\chi} - 2e^{-6\chi} + F^2 e^{-12\chi} \quad . \tag{3.5.31}$$

We have restored the reduced Planck scale $M_{\rm Pl}$ in equations. (3.5.29) and (3.5.31) for reader's convenience.

3.5.2 The study of scalar potential

In this section we investigate the scalar potential (3.5.28) in four dimensions. It depends upon two fields, the inflaton f and the modulus χ, and has three parameters $(a^{-2}, F^2, \tilde{\Lambda}_8)$ originating from eight dimensions.

The potential (3.5.28) has a Minkowski vacuum (f_0, χ_0) defined by the equations

$$\left.\frac{\partial V}{\partial f}\right|_{f=f_0} = \left.\frac{\partial V}{\partial \chi}\right|_{\chi=\chi_0} = V|_{f=f_0,\ \chi=\chi_0} = 0 \quad . \tag{3.5.32}$$

The solution to these three equations is given by

$$e^{6f_0} = 1 + (2\tilde{\Lambda}_8 a^2)^3 \quad \text{and} \quad e^{6\chi_0} = 2F^2 \ , \tag{3.5.33}$$

together with a condition of the parameters,

$$\frac{2}{3}\tilde{\Lambda}_8 = \left(\frac{1}{16F^2 - 256\tilde{\gamma}_8} \right)^{1/3} \quad , \tag{3.5.34}$$

where we have used the third relation (3.5.7) between $\tilde{\gamma}_8$ and a.

The second derivatives of the scalar potential (3.5.28) at the critical point (3.5.33) determine the masses of the canonically normalized scalars (3.5.27) as

$$m_{\hat{f}_0}^2 = \left.\frac{\partial^2 V}{\partial f^2}\right|_{f=f_0} \frac{1}{42 M_{\rm Pl}^2} = \frac{M_{\rm Pl}^2}{56 F^2} \left(\frac{F^2}{\tilde{\gamma}_8} - 16 \right) \quad , \tag{3.5.35}$$

and

$$m_{\hat{\chi}_0}^2 = \left.\frac{\partial^2 V}{\partial \chi^2}\right|_{\chi=\chi_0} \frac{1}{12 M_{\rm Pl}^2} = \frac{M_{\rm Pl}^2}{F^2} \quad , \tag{3.5.36}$$

where we have used (3.5.34) also. Equations (3.5.34) and (3.5.35) imply the same condition

$$\frac{F^2}{\tilde{\gamma}_8} > 16 \tag{3.5.37}$$

for both the existence of a Minkowski vacuum and its stability, respectively.

At the onset of inflation ($f = +\infty$), the scalar potential of the modulus χ is given by

$$M_{\rm Pl}^{-4} V(\chi) = a^{-2} e^{-4\chi} - 2e^{-6\chi} + F^2 e^{-12\chi} \tag{3.5.38}$$

that only depends upon two (free) parameters (a^{-2}, F^2).

The critical points of the potential (3.5.38) are determined by the condition

$$a^{-2} - 3e^{2\chi_c} + 3F^2 e^{-8\chi_c} = 0 \tag{3.5.39}$$

that has the form of the *depressed quartic* equation

$$z^4 + qz + r = 0 \tag{3.5.40}$$

in terms of

$$z = e^{-2\chi_c} \quad , \qquad q = \frac{-1}{F^2} < 0 \ , \qquad r = \frac{1}{3a^2F^2} > 0 \ . \tag{3.5.41}$$

The *quartic* discriminant is given by

$$\frac{\Delta_4}{27 \cdot 256} = (r/3)^3 - (q/4)^4 \quad , \tag{3.5.42}$$

while writing down an explicit solution to (3.5.40) depends upon the sign of Δ_4.

By using the auxiliary (Ferrari's) resolvent cubic equation

$$m^3 - rm - q^2/8 = 0 \quad , \tag{3.5.43}$$

we can factorize the left-hand-side of the quartic equation (3.5.40) as

$$\left(z^2 + m + \sqrt{2m}z - \frac{q}{2\sqrt{2m}} \right) \left(z^2 + m - \sqrt{2m}z + \frac{q}{2\sqrt{2m}} \right) = 0 \ . \tag{3.5.44}$$

Because each term in the first factor is positive in our case, we get a *quadratic* equation from the vanishing second factor whose two roots are given by

$$z_{1,2} = \sqrt{\frac{m}{2}} \left[1 \pm \sqrt{-\frac{q}{m} - \sqrt{2m}} \right] \ . \tag{3.5.45}$$

These two roots precisely correspond to the existence of a local (meta-stable) minimum and a local maximum of the potential (3.5.38), with $-\infty < \chi_{\rm min.} < \chi_{\rm max.} < +\infty$.

The *cubic* discriminant $\Delta_3 = 4r^3 - 27(q^2/8)^2$ of the *depressed* cubic equation (3.5.43) is simply related to Δ_4 as

$$\frac{\Delta_3}{4 \cdot 27} = (r/3)^3 - (q/4)^4 = \frac{\Delta_4}{27 \cdot 256} \quad . \tag{3.5.46}$$

When $\Delta_{3,4} \geq 0$, three real solutions to the cubic equation (3.5.43) are given by the Vieté formula

$$m_k = 2\sqrt{r/3}\cos\theta_k \quad , \qquad k = 0, 1, 2 \ , \tag{3.5.47}$$

where

$$\theta_k = \frac{1}{3}\arccos\left(\frac{3q^2}{16r}\sqrt{3/r}\right) - \frac{2\pi k}{3} \ , \tag{3.5.48}$$

while we should choose the highest (positive) root. The condition $\Delta_{3,4} \geq 0$ implies

$$\frac{F^2}{\tilde{\gamma}_8} \geq 27 \quad . \tag{3.5.49}$$

When $\Delta_{3,4} \leq 0$ or, equivalently, $F^2/\tilde{\gamma}_8 \leq 27$, the angle (3.5.48) does not exist. Instead, we should use the Vieté's substitution in Ferrari's equation with

$$m = w + \frac{r}{3w} \ , \qquad r > 0 \quad , \tag{3.5.50}$$

that yields a *quadratic* equation for w^3,

$$(w^3)^2 - \frac{q^2}{8}w^3 + \frac{r^3}{27} = 0 \quad , \tag{3.5.51}$$

whose roots are

$$w^3_{1,2} = (q/4)^2\left[1 \pm \sqrt{1 - \frac{(r/3)^3}{(q/4)^4}}\right] \quad . \tag{3.5.52}$$

Going back to the critical condition (3.5.39) in the form

$$F^2 = e^{6\chi_c}\left[1 - \frac{1}{3}a^{-2}e^{2\chi_c}\right] \quad , \tag{3.5.53}$$

and inserting it into the potential (3.5.38) yields the *height* of the inflationary potential $V_{\rm plateau}$ at the onset of inflation,

$$M_{\rm Pl}^{-4}V_{\rm plateau} = e^{-6\chi_c}\left[\frac{2}{3}a^{-2}e^{2\chi_c} - 1\right] \quad . \tag{3.5.54}$$

Demanding its positivity, $V_{\rm plateau} > 0$, gives us the restriction (3.5.37) again.

The second derivative of the potential (3.5.38) at the critical point (3.5.39) is given by

$$\left.\frac{\partial^2 V}{\partial\chi^2}\right|_{\chi=\chi_c} = 8e^{-6\chi_c}\left(9 - 4a^{-2}e^{2\chi_c}\right) \quad . \tag{3.5.55}$$

Its positivity (stability) implies

$$\frac{F^2}{\tilde{\gamma}_8} < 54 \quad . \tag{3.5.56}$$

Taken together with (3.5.37) and (3.5.49), this implies that the values of the ratio $F^2/\tilde{\gamma}_8$ have to be restricted as follows:

$$\begin{aligned} 16 < \frac{F^2}{\tilde{\gamma}_8} \leq 27 \quad , \qquad \Delta_{3,4} \leq 0 \quad , \\ 27 \leq \frac{F^2}{\tilde{\gamma}_8} < 54 \quad , \qquad \Delta_{3,4} \geq 0 \quad . \end{aligned} \tag{3.5.57}$$

Because of $1 < F^2/(16\tilde{\gamma}_8) \equiv 1 + \delta < (\frac{3}{2})^3$, it is instructive to investigate the case of $0 < \delta \ll 1$ describing the strong stabilization of the modulus χ. In this case, (3.5.33) and (3.5.53) give rise to

$$0 < \chi_c - \chi_0 \approx \frac{1}{12}\delta \ll 1 \quad , \tag{3.5.58}$$

leading to a *single-field inflation* driven by inflaton (scalaron) f indeed.

The physical *hierarchy* of scales reads

$$m_{\hat{f}_0} < m_{\hat{\chi}_0} \ll M_{\rm KK} \ll M_{\rm Pl} \quad . \tag{3.5.59}$$

The KK scale in our case is given by $M_{\rm KK} \approx e^{-\chi_0} M_{\rm Pl}$, where the presence of the warp factor is dictated by the compactification ansatz (3.5.10).

The condition $M_{\rm KK} \ll M_{\rm Pl}$ implies

$$2F^2 \gg 1 \tag{3.5.60}$$

because of (3.5.33). The condition $m_{\hat{\chi}_0} \ll M_{\rm KK}$ implies

$$F^2 \gg \sqrt{2} \tag{3.5.61}$$

that is slightly stronger than (3.5.60). Both conditions can be easily satisfied by taking $F^2 \gg 1$.

The remaining condition $m_{\hat{f}_0} < m_{\hat{\chi}_0}$ implies $F^2/\tilde{\gamma}_8 < 72$ that is already satisfied under the conditions (3.5.57). However, it is not possible to get $m_{\hat{f}_0} \ll m_{\hat{\chi}_0}$ here. It has a stable Minkowski vacuum and a plateau with a positive height provided that

$$1 < F^2/(16\tilde{\gamma}_8) = 1 + \delta < (\frac{3}{2})^3 \quad , \tag{3.5.62}$$

where the inequality on the right hand side is also needed to ensure a positive mass squared of the modulus χ at the onset of inflation — see equation (3.5.56).

For generic values of δ in equation (3.5.62) one gets a two-field inflation. However, the modulus χ is strongly stabilized when $\delta \ll 1$ that implies only a small shift of the minimum of χ during inflation, from χ_c to χ_0, as

$$0 < \chi_c - \chi_0 \approx \frac{1}{12}\delta \ll 1 \quad , \tag{3.5.63}$$

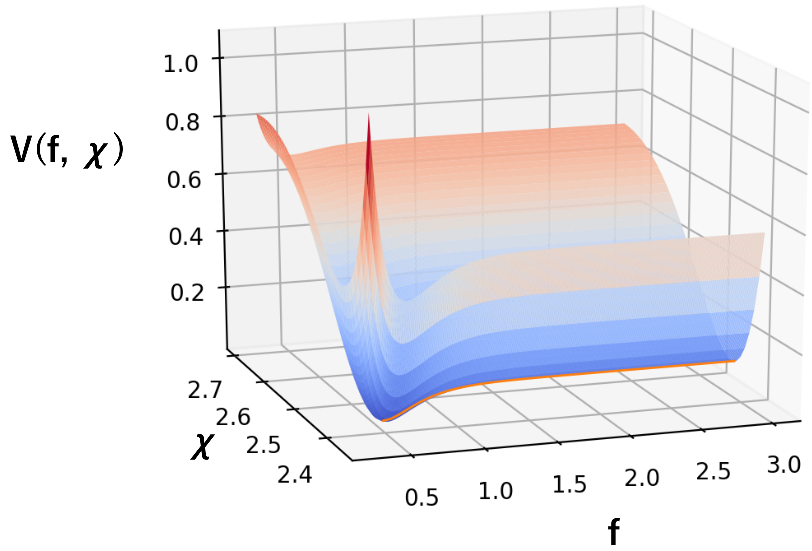


Figure 3.4: The profile of the scalar potential (3.5.31) for the numerical input $F^2 = 10^6$, $\tilde{\gamma}_8 = 6 \cdot 10^4$ and $\tilde{\Lambda}_8 \approx 0.0174$. The bottom line shows the inflationary trajectory.

and results in a *single-field inflation* driven by the inflaton (scalaron) f in $D = 4$.

A profile of the scalar potential in $D = 4$ is given in Figure 3.4. It should be mentioned that the cosmological constant in $D = 8$ is given by equation (3.5.34) that implies

$$\tilde{\Lambda}_8 = \frac{\delta^{-1/3}}{2a^2} \quad , \tag{3.5.64}$$

where we have used equation (3.5.7) also. In particular, it means that δ cannot vanish.

3.5.3 Towards supergravity embedding of our model

In this Subsection we explore a possibility of embedding our 8-dimensional model (3.5.3) into a $D = 8$ supergravity. First, supergravity may be the natural origin of the p-form field because higher-dimensional supergravities usually include such fields. Second, the supergravity extensions of modified gravity certainly exist in $D = 4$ [42, 43], and they should also exist in higher dimensions $D \leq 11$.

Unfortunately, to the best of our knowledge, no fully supersymmetric extension of any $(R + R^4)$ gravity in higher $(8 \leq D \leq 11)$ dimensions was ever derived, so that our investigation in this Subsection cannot be fully consistent and compelling, unlike that in the previous Subsections. Moreover, any standard (two-derivative) supergravity theory does not allow a positive cosmological constant in its action (it would break supersymmetry), so that the origin of the cosmological constant in

$D = 8$ can only be either due to a spontaneous supersymmetry breaking or/and some nonperturbative effects. So, this Section ends up with a conjecture.

A good starting point of this investigation is the maximally supersymmetric $D = 11$ supergravity, because of its uniqueness. It can be modified by the quartic scalar curvature term and then compactified down to $D = 8$ on a compact manifold (3-sphere S^3) — see Appendix A for details. Moreover, the $SO(3)$ non-abelian isometries of the S^3 can be gauged, thus producing the non-abelian gauge fields and a scalar potential in $D = 8$. Taken together, it leads to the bosonic part of the (modified and gauged) $D = 8$ supergravity action, having the form (A.14),

$$S_8 = \int d^8x \frac{e}{2\kappa^2}[R + \tilde{\gamma}e^{2\kappa\phi}R^4 - \kappa^2 e^{2\kappa\phi}F^{\alpha}_{\mu\nu}F^{\mu\nu}_{\alpha} - 2\kappa^2\partial_\mu\phi\partial^\mu\phi - V(T) - P_{\mu ij}P^{\mu ij}$$
$$-\frac{1}{2}\kappa^2 e^{-4\kappa\phi}\partial_\mu B\partial^\mu B - \frac{\tilde{\kappa}^2}{12}e^{2\kappa\phi}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} - \frac{\kappa^3}{432}e^{-1}\varepsilon^{\mu_1\ldots\mu_8}G_{\mu_1\ldots\mu_4}G_{\mu_5\ldots\mu_8}B] + \ldots\ , \tag{3.5.65}$$

in terms of the following $D = 8$ fields: a metric $g_{\mu\nu}$, dilaton ϕ, the $SO(3)$ gauge field strength $F^{\alpha}_{\mu\nu}$, the vector fields $P_{\mu ij}$, the 4-form gauge field strength $G_{\mu\nu\rho\sigma}$ and $(5 + 1)$ scalars (T, B) whose scalar potential is

$$V(T) = \frac{g^2}{4\kappa^2}e^{-2\kappa\phi}(T_{ij}T^{ij} - \frac{1}{2}T^2)\ . \tag{3.5.66}$$

The supergravity (3.5.65) has the required *quartic* scalar curvature term and the gauge 3-form kinetic term given by the gauge field strength *4-form* squared, while the abelian vector fields $P_{\mu ij}$ are merely the spectators here. Hence, (3.5.65) could be the supersymmetric extension of our action (3.5.3) provided that (i) the dilaton ϕ is stabilized, and (ii) a positive cosmological constant is generated. One usually assumes in the literature that the dilaton potential is generated by quantum non-perturbative corrections beyond the supergravity level. And the cosmological constant may be generated by the non-perturbative vacuum expectation value

$$\left\langle \kappa^2 e^{2\kappa\phi}F^{\alpha}_{\mu\nu}F^{\mu\nu}_{\alpha}\right\rangle = 2\Lambda_8\ . \tag{3.5.67}$$

Unfortunately, we do not have means to compute the dilaton vacuum expectation value and the gluon condensate (3.5.67) in $D = 8$.

3.5.4 Inflation

Once the modulus χ is strongly stabilized, the inflaton potential (3.5.31) takes the form ($M_{\rm Pl} = 1$)

$$e^{4\chi_0}a^2V(f) = \left(1 - e^{-6f}\right)^{\frac{4}{3}} + \lambda e^{-8f} - \lambda(1 + \lambda^3)^{-\frac{1}{3}}\ . \tag{3.5.68}$$

with $\lambda = 2a^2\tilde{\Lambda}_8 = \delta^{-1/3}$. This potential has the absolute minimum at

$$f_0 = \frac{1}{6}\ln\left(1 + \lambda^3\right)\ , \tag{3.5.69}$$

where it vanishes in the Minkowski vacuum. A profile of the scalar potential (3.5.68) is given in Figure 3.5.

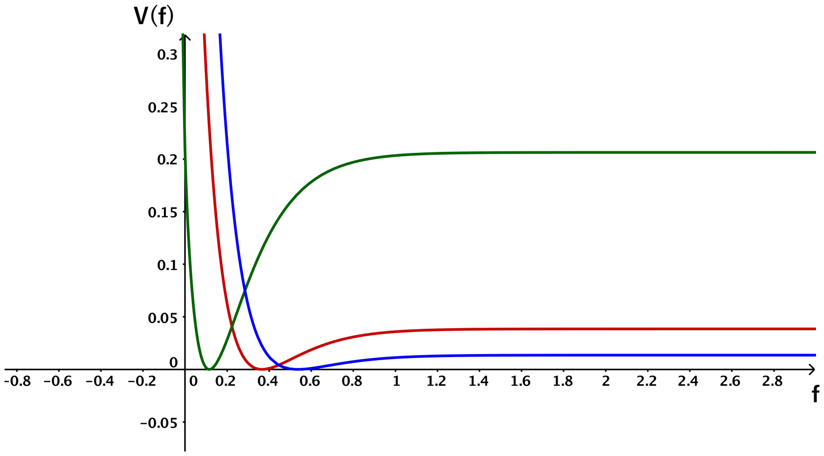


Figure 3.5: The profile of the scalar potential (3.5.68) for $\lambda = 1$ (green), $\lambda = 2$ (red) and $\lambda = 2.88$ (blue).

During inflationary slow roll along the plateau, the scalar potential (3.5.68) can be approximated as

$$V(\phi) = V_0 - V_1 e^{-\alpha\phi} \quad , \tag{3.5.70}$$

with

$$\alpha = \sqrt{\frac{6}{7}} \quad . \tag{3.5.71}$$

This value of α determines the key observational parameter r related to primordial gravity waves and known as the tensor-to-scalar ratio,

$$r = \frac{8}{\alpha^2 N_e^2} = \frac{28}{3N_e^2} \quad . \tag{3.5.72}$$

The Planck data [33] sets the upper bound on r (with 95% of CL) as $r < 0.064$. Our result (3.5.72) is clearly below this bound.

As regards the other CMB spectral tilts (the inflationary observables), the scalar spectral index n_s and its running $dn_s/d\ln k$, their values derived from the potential (3.5.70) are

$$n_s \approx 1 - \frac{2}{N_e} \qquad \text{and} \qquad \frac{dn_s}{d\ln k} \approx -\frac{(1-n_s)^2}{2} \approx -\frac{2}{N_e^2} \quad , \tag{3.5.73}$$

i.e. they are the same as in the Starobinsky model (in Subsection 3.3.3).

The microscopic parameters of our model can be easily tuned to get the *same* inflaton mass M, so that our effective inflationary model obtained from the higher ($D = 8$) dimensions is almost indistinguishable from the original Starobinsky model having $\alpha_s = \sqrt{2/3}$.

When a conventional matter action is added to our gravity model, Weyl rescaling of the metric result in the *universal* couplings (via the GR covariant derivatives) of inflaton f to all matter fields with powers of $\exp(-\alpha\kappa_4 f)$. The value (3.5.71) of α derived from $D = 8$ is only slightly different from the Starobinsky value $\alpha_s = \sqrt{2/3}$, while all the matter couplings to the scalaron are suppressed by the Planck mass. Therefore, the impact of higher dimensions on the inflationary observables and reheating is very small in our approach.

3.5.5 Summary of Section 3.5

We used the Starobinsky inflationary model of the $(R + R^2)$ gravity in four dimensions as the prototype for deriving the new inflationary models from higher dimensions. Among the advantages of this approach are (i) its geometrical nature, as only gravitational interactions are used, (ii) consistency with the current astronomical observations of CMB, (iii) the clear physical nature of inflaton (scalaron) as the spin-0 part of metric. In this section we focused on $D = 8$ dimensions only. In our scenario, the Universe was born multi-dimensional, and then four spacetime dimensions became infinite, while the others curled up by some unknown mechanism before inflation.

In higher-dimensions, it turned out to be necessary to include a cosmological constant and a gauge (form) field, with the strong conditions on the higher dimension, the power n of the scalar curvature and the rank of the form, see equations (3.5.1) and (3.5.2). The moduli stabilization and the scale hierarchy are also possible to achieve, while both are non-trivial in the present context. It may also be possible to embed our $D = 8$ modified gravity model into the modified $D = 8$ supergravity and, ultimately, into the modified $D = 11$ supergravity.

As regards the observational predictions of our model, it leads to the certain sharp value (3.5.72) of the CMB tensor-to-scalar ratio that is, however, only slightly different from that of the original Starobinsky model.

Conclusion

In this thesis, our aim was to construct an inflationary model from a higher dimensional theory. To that end, it was necessary to (i) describe extra dimensions in Kaluza-Klein field theory and Randall-Sundrum brane-world, (ii) derive the spontaneous compactification of extra dimensions and stabilize the extra dimensions, (iii) describe slow-roll inflation in modified gravity.

In Chapter 2, we reviewed Randall-Sundrum brane-world and investigated the Randall-Sundrum brane-world model modified by Starobinsky gravity.

The impact of the R^2-modified gravity on the RSI model can be simply described in terms of a single dynamical scalar field with the particular scalar potential (2.5.15). It is clear from our construction that this scalar has the gravitational origin as spin-0 part of five-dimensional spacetime metric. We found that the value of the RSI parameter k is determined by dynamics of that scalar in the fifth dimension.

It makes sense to analyze stability of the RS brane-world against possible modifications of gravity, as well as against quantum gravity corrections. We did a small step in this direction by proving stability of the RSI model against the simplest modification of the higher-dimensional gravity described by adding the scalar curvature squared term in five dimensions. However, this model cannot include the slow-roll inflation on the brane.

In Chapter 3, we reviewed the standard cosmology and modified gravity as a model of inflation, and investigated the Starobinsky-type inflationary model from the D-dimensional ($D > 4$) modified gravity ($R + \gamma R^n - 2\Lambda$). There are two steps in our construction of the consistent inflationary model.

First, in Section 3.4, we derived the inflaton scalar potential from higher ($D > 4$) dimensions, in the context of the D-dimensional ($R + \gamma R^n - 2\Lambda$) gravity, by using the Starobinsky model of chaotic large-field inflation in $D = 4$ as a prototype. We assumed that a compactification of the extra dimensions took place *before* inflation. We found that this requires a positive cosmological constant and $n = D/2$. We calculated the corresponding scalar potential and the values of its parameters for any D, and specified our results to the two special cases, $D = 8$ and $D = 12$.

Our scalar potentials in their slow-roll part fall into the class of the inflationary plateau-type potentials describing chaotic large-field inflation and having the form (3.5.70) with

$$\alpha = \sqrt{\frac{D-2}{D-1}} \ , \tag{3.5.74}$$

because of equation (3.4.31). In particular, we have $\alpha_4 = \alpha_s = \sqrt{2/3}$, $\alpha_8 = \sqrt{6/7}$ and $\alpha_{12} = \sqrt{10/11}$.

According to equation (3.4.49), we find $r_4 = \frac{12}{N_e^2}$, $r_8 = \frac{28}{3N_e^2}$, and $r_{12} = \frac{44}{5N_e^2}$. All those values are in agreement with current observations, and give the sharp (though very close) predictions for future measurements of r. However, since dynamics of the extra dimensions was ignored, their size was not under control, so that stabilization of extra dimensions was still needed.

Second, in Section 3.5, we investigated the D=8, $(R + \gamma R^n - 2\Lambda)$ gravity and its D=4 spontaneous compactification with modulus stabilization. In our scenario, the Universe was born multi-dimensional, and then four spacetime dimensions became infinite, while the others curled up by some unknown mechanism before inflation. The inflation happened after the compactification and the moduli stabilization.

In higher-dimensions, it turned out to be necessary to include a cosmological constant and a gauge (form) field, with the strong conditions on the higher dimension, the power n of the scalar curvature and the rank of the form as $p = n = D/2$. The moduli stabilization and the scale hierarchy are also possible to achieve, while both are non-trivial in the present context. It may also be possible to embed our $D = 8$ modified gravity model into the modified $D = 8$ supergravity and, ultimately, into the modified $D = 11$ supergravity.

As regards the observational predictions of our model, it leads to the certain value (3.4.49) of the CMB tensor-to-scalar ratio that is, however, only slightly different from that of the original Starobinsky model.

Our results may be used for studying inflation and moduli stabilization in more general frameworks, such as unification of fields and forces, KK theories of gravity, supergravity and superstrings, and braneworld.

Acknowledgements

I would like to thank my academic advisor, Prof. S. Ketov, for his carefully thought feedback and valuable comments. I am also indebted to Dr. T. Yumibayashi, Dr. Y. Wakimoto, Dr. Y. Aldabergenov and Dr. Y. Sakai, whose comments made enormous contribution to my work.

I was supported by a scholarship of Tokyo Metropolitan University for graduate students during my time as a doctor course student.

I would also like to express my sincere gratitude to my family for their moral support and warm encouragements.

Appendix A : $D = 8$ gauged supergravity

The $D = 8$ supergravity (with 16 supercharges) received relatively little attention in the literature versus the supergravities in $D = 10$ and $D = 11$. For our purposes, we need a $D = 8$ supergravity modified by the quartic scalar curvature term and having a scalar potential. In this Appendix we recall the $SU(2)$ gauged $N = 2$ supergravity in $D = 8$, which was derived by Salam and Sezgin [44] by using the Scherk-Schwarz-type dimensional reduction [45] of the 11-dimensional supergravity [46].

The 11-dimensional supergravity [46] is *unique*, so that it is the good point to start with. Its standard action is well known, while its existence can be related to the existence of the 11-dimensional supermultiplet containing the 11-dimensional spacetime scalar curvature $\mathscr{R}$ among its field components. Therefore, there is little doubt that the $(\mathscr{R} + \mathscr{R}^4)$ *supergravity* action in $D = 11$ also exist, though (to the best of our knowledge) it was never constructed in the literature. So, assuming its existence, we write down the relevant part of its bosonic terms as

$$S_{11} = \int d^{11}X \frac{E}{2\tilde{\kappa}^2} \left(\mathscr{R} + \tilde{\gamma}\mathscr{R}^4 - \frac{\tilde{\kappa}^2}{12} G_{ABCD}G^{ABCD} + \frac{8\tilde{\kappa}^3}{144^2} \varepsilon^{A_1 \ldots A_{11}} G_{A_1\ldots} G_{A_5\ldots} V_{\ldots A_{11}} \right) , \tag{A.1}$$

where we have simply added the quartic curvature term (with the coupling constant $\tilde{\gamma}$) to the standard bosonic action of the 11-dimensional supergravity. Of course, adding the $\mathscr{R}^4$ term also requires adding its supersymmetric completion that is going to result in more bosonic terms in the action. However, because all extra terms are going to be the higher-derivative couplings of the bosonic 3-form field, also non-minimally coupled to gravity, we assume that these extra couplings are irrelevant for the *scalar* sector of the theory (see below). [1]

As regards our notation, we denote $E \equiv \det E_M{}^A$ in terms of an *elfbein* $E_M{}^A$ in $D = 11$. Here we denote the 11-dimensional Lorentz indices by early capital latin letters as $A, B, C, \ldots$, while the middle capital latin letters $M, N, P, \ldots$ are used for the 11-dimensional Einstein (curved) indices. The $\tilde{\kappa}$ is the gravitational constant in

[1] It is worth mentioning that our approach is apparently *different* from M-theory, because we treat the $\mathscr{R}^4$ term nonperturbatively, so that its presence leads to the new physical degrees of freedom in $D = 11$, which are absent in the standard $D = 11$ supergravity, similarly to the $(R + R^4)$ gravity in lower dimensions.

$D = 11$. The scalar curvature $\mathscr{R}$ is defined in terms of the spin connection

$$\omega_{ABC} \equiv E_A^M \omega_{MBC} = \frac{1}{2}\eta_{CE}(E_A^M E_B^N - E_B^M E_A^N)\partial_M E_N^E - \frac{1}{2}\eta_{AE}(E_B^M E_C^N - E_C^M E_B^N)\partial_M E_N^E$$
$$+\frac{1}{2}\eta_{BE}(E_C^M E_A^N - E_A^M E_C^N)\partial_M E_N^E \quad \text{(A.2)}$$

as

$$\mathscr{R} = \omega_{ABC}\omega^{CAB} + \omega_A\omega^A - 2E^{-1}\partial_M(EE^M{}_A\omega^A) \ , \quad \text{(A.3)}$$

where $\omega_A \equiv \eta^{BC}\omega_{BCA}$ and η_{AB} is Minkowski metric in $D = 11$. The 4-form field strength G_{ABCD} is defined in terms of the 3-form gauge potential V_{ABC} as

$$G_{ABCD} = 4\partial_{[A}V_{BCD]} + 12\omega_{[AB}{}^E V_{CD]E} \ . \quad \text{(A.4)}$$

To dimensionally reduce the modified $D = 11$ supergravity to eight dimensions on a sphere S^3, we use the ansatz [44]

$$E_M{}^A = \begin{pmatrix} e^{-\tilde{\kappa}\phi/3}e_\mu^a & 0 \\ 2\tilde{\kappa}e^{2\tilde{\kappa}\phi/3}A_\mu^\alpha L_\alpha^i & e^{2\tilde{\kappa}\phi/3}L_\alpha^i \end{pmatrix} \ , \quad E^M{}_A = \begin{pmatrix} e^{\tilde{\kappa}\phi/3}e_a^\mu & -2\tilde{\kappa}e^{\tilde{\kappa}\phi/3}e_a^\mu A_\mu^\alpha \\ 0 & e^{-2\tilde{\kappa}\phi/3}L_i^\alpha \end{pmatrix} \ , \quad \text{(A.5)}$$

where we have introduced the 8-dimensional Lorentz indices $a, b, c, \ldots$ and the 8-dimensional Einstein indices $\mu, \nu, \rho, \ldots$, as well as the 3-dimensional (compact) Lorentz and Einstein indices, $i, j, k, \ldots$ and $\alpha, \beta, \gamma, \ldots$, respectively. The dilaton ϕ represents the volume modulus of the 3-sphere, the e_μ^a is an 8-dimensional *achtbein*, the L_α^i is the unimodular matrix ($\det L_\alpha^i = 1$) having 5 scalars, and the A_μ^α is a set of 8-dimensional vectors.

The Scherk-Schwarz dimensional reduction is used to gauge symmetries of a compact manifold in the reduced theory by allowing the fields to depend on the compact coordinates [45]. Let us denote the non-compact coordinates by $\{x\}$, and the compact coordinates by $\{y\}$, and then factorize the y-dependence as

$$e_\mu^a(x,y) = e_\mu^a(x) \ , \quad A_\mu^\alpha(x,y) = U^{-1\alpha}{}_\beta(y)A_\mu^\beta(x) \ , \quad L_\alpha^i(x,y) = U_\alpha{}^\beta(y)L_\beta^i(x) \ , \quad \text{(A.6)}$$

where $U_\alpha{}^\beta(y)$ are elements of the gauge group $SU(2)$ in our case. The $SU(2)$ structure constants

$$f_{\alpha\beta}^\gamma \equiv U_\alpha^{-1\alpha'}U_\beta^{-1\beta'}(\partial_{\beta'}U_{\alpha'}{}^\gamma - \partial_{\alpha'}U_{\beta'}{}^\gamma) = -\frac{g}{2\tilde{\kappa}}\varepsilon_{\alpha\beta\delta}g^{\delta\beta} \quad \text{(A.7)}$$

are *y-independent*, where we have introduced the $SU(2)$ gauge coupling constant g and the 3-dimensional Levi-Civita tensor $\varepsilon_{\alpha\beta\gamma}$.

Substituting the ansatz (A.5) into (A.2) reduces the spin connection components as [44]

$$\begin{aligned}
\omega_{abc} &= e^{\tilde{\kappa}\phi/3}(\tilde{\omega}_{abc} - \frac{1}{3}\tilde{\kappa}\eta_{ab}\partial_c\phi + \frac{1}{3}\tilde{\kappa}\eta_{ac}\partial_b\phi) \ , \\
\omega_{abi} &= \tilde{\kappa}e^{4\tilde{\kappa}\phi/3}F_{abi} \ , \\
\omega_{aij} &= e^{\tilde{\kappa}\phi/3}Q_{aij} \ , \\
\omega_{iab} &= -\tilde{\kappa}e^{4\tilde{\kappa}\phi/3}F_{abi} \ , \\
\omega_{ija} &= e^{\tilde{\kappa}\phi/3}(P_{aij} + \frac{2}{3}\tilde{\kappa}\delta_{ij}\partial_a\phi) \ , \\
\omega_{ijk} &= -\frac{g}{4\tilde{\kappa}}e^{-2\tilde{\kappa}\phi/3}(\varepsilon_{jkl}T_i^{\ l} + \varepsilon_{kli}T_j^{\ l} - \varepsilon_{lij}T_k^{\ l}) \ ,
\end{aligned} \tag{A.8}$$

where we have used the notation

$$\begin{aligned}
F^{\alpha}_{\mu\nu} &= \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + g\varepsilon_{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu \ , \\
P_{\mu ij} + Q_{\mu ij} &= L^\alpha_i(\delta^\beta_\alpha\partial_\mu - g\varepsilon_\alpha{}^\beta{}_\gamma A^\gamma_\mu)L_{\beta j} \ , \\
T^{ij} &= L^i_\alpha L^j_\beta\delta^{\alpha\beta} \ ,
\end{aligned} \tag{A.9}$$

with $P_{\mu ij}$ representing the symmetric part of the r.h.s. of (A.9), and $Q_{\mu ij}$ representing the antisymmetric part. The fields L^i_α are subject to the relations [44]

$$L^i_\alpha L^j_\beta\delta_{ij} = g_{\alpha\beta}, \quad L^i_\alpha L^j_\beta g^{\alpha\beta} = \delta^{ij} \ , \tag{A.10}$$

where $g_{\alpha\beta}$ is the metric of the compact manifold (S^3).

As regards V_{ABC} and G_{ABCD}, their relevant components are

$$\begin{aligned}
\varepsilon_{\alpha\beta\gamma}B &\equiv e^{2\tilde{\kappa}\phi}L^i_\alpha L^j_\beta L^k_\gamma V_{ijk} \ , \\
\varepsilon_{\alpha\beta\gamma}\partial_\mu B &\equiv e^{5\tilde{\kappa}\phi/3}e^a_\mu L^i_\alpha L^j_\beta L^k_\gamma G_{aijk} \ ,
\end{aligned} \tag{A.11}$$

where B is another *scalar* field.

Equations (A.5), (A.8), and (A.11) allow us to rewrite the 11-dimensional action (A.1) as

$$\begin{aligned}
S_{11} = \int d^8x d^3y \ U(y)\frac{e}{2\tilde{\kappa}^2}[&R + \tilde{\gamma}e^{2\tilde{\kappa}\phi}R^4 - \tilde{\kappa}^2 e^{2\tilde{\kappa}\phi}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha \\
&- 2\tilde{\kappa}^2\partial_\mu\phi\partial^\mu\phi - \frac{g^2}{4\tilde{\kappa}^2}e^{-2\tilde{\kappa}\phi}(T_{ij}T^{ij} - \frac{1}{2}T^2) \\
&- P_{\mu ij}P^{\mu ij} - \frac{1}{2}\tilde{\kappa}^2 e^{-4\tilde{\kappa}\phi}\partial_\mu B\partial^\mu B - \frac{\tilde{\kappa}^2}{12}e^{2\tilde{\kappa}\phi}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} \\
&- \frac{\tilde{\kappa}^3}{432}e^{-1}\varepsilon^{\mu_1\ldots\mu_8}G_{\mu_1\ldots\mu_4}G_{\mu_5\ldots\mu_8}B] + \ldots \ ,
\end{aligned} \tag{A.12}$$

where $U(y) \equiv \det U_\alpha{}^\beta(y)$, $T \equiv T_i{}^i$, R is the 8-dimensional scalar curvature and the dots stand for irrelevant terms. Since the only y-dependent function is $U(y)$, one can perform y-integration with

$$\int d^3y U(y) = V_0 \ , \qquad \text{(A.13)}$$

defining the invariant volume V_0 of the compact manifold (S^3). With the gravitational coupling $\kappa = \tilde{\kappa}/\sqrt{V_0}$ in $D = 8$, rescaling dilaton as $\phi \to \phi/\sqrt{V_0}$ (and similarly for the other fields A^α_μ, B and $V_{\mu\nu\rho}$) and rescaling the gauge coupling as $g \to g\sqrt{V_0}$ leads to the action

$$S_8 = \int d^8x \frac{e}{2\kappa^2}[R + \tilde{\gamma}e^{2\kappa\phi}R^4 - \kappa^2 e^{2\kappa\phi}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha - 2\kappa^2\partial_\mu\phi\partial^\mu\phi - V(T) - P_{\mu ij}P^{\mu ij}$$
$$-\frac{1}{2}\kappa^2 e^{-4\kappa\phi}\partial_\mu B\partial^\mu B - \frac{\tilde{\kappa}^2}{12}e^{2\kappa\phi}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} - \frac{\kappa^3}{432}e^{-1}\varepsilon^{\mu_1\ldots\mu_8}G_{\mu_1\ldots\mu_4}G_{\mu_5\ldots\mu_8}B] + \ldots \ , \qquad \text{(A.14)}$$

whose scalar poitential is given by [44]

$$V(T) = \frac{g^2}{4\kappa^2}e^{-2\kappa\phi}(T_{ij}T^{ij} - \frac{1}{2}T^2) \ . \qquad \text{(A.15)}$$

The vacuum expectation value of the dilaton ϕ is supposed to be determined by non-perturbative effects in superstring theory, whereas the vacuum expectation value of the scalar potential $V(T)$ vanishes. Therefore, the only possibility of generating a positive cosmological constant in the action (A.14) is via a formation of the non-perturbative $SU(2)$ gluon condensate in $D = 8$ dimensions, see Subsection (3.5.3).

Bibliography

[1] Theodor Kaluza. On the Unification Problem in Physics. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1921:966–972, 1921.

[2] Oskar Klein. Quantum Theory and Five-Dimensional Theory of Relativity. *Z. Phys.*, 37:895–906, 1926.

[3] Peter G. O. Freund and Mark A. Rubin. Dynamics of Dimensional Reduction. *Phys. Lett.*, B97:233–235, 1980.

[4] S. Randjbar-Daemi, Abdus Salam, and J. Strathdee. Spontaneous compactification in six-dimensional Einstein-Maxwell theory. *Nuclear Physics B*, 214(3):491 – 512, 1983.

[5] S. Randjbar-Daemi, Abdus Salam, and J. Strathdee. Instanton induced compactification and fermion chirality. *Physics Letters B*, 132(1):56 – 60, 1983.

[6] C. Wetterich. Spontaneous compactification in higher dimensional gravity. *Physics Letters B*, 113(5):377 – 381, 1982.

[7] Philip Candelas and Steven Weinberg. Calculation of Gauge Couplings and Compact Circumferences from Selfconsistent Dimensional Reduction. *Nucl. Phys.*, B237:397, 1984.

[8] Nima Arkani-Hamed, Savas Dimopoulos, and G. R. Dvali. The Hierarchy problem and new dimensions at a millimeter. *Phys. Lett.*, B429:263–272, 1998.

[9] Ignatios Antoniadis, Nima Arkani-Hamed, Savas Dimopoulos, and G. R. Dvali. New dimensions at a millimeter to a Fermi and superstrings at a TeV. *Phys. Lett.*, B436:257–263, 1998.

[10] Lisa Randall and Raman Sundrum. A Large mass hierarchy from a small extra dimension. *Phys. Rev. Lett.*, 83:3370–3373, 1999.

[11] Lisa Randall and Raman Sundrum. An Alternative to compactification. *Phys. Rev. Lett.*, 83:4690–4693, 1999.

[12] G. R. Dvali, Gregory Gabadadze, and Massimo Porrati. 4-D gravity on a brane in 5-D Minkowski space. *Phys. Lett.*, B485:208–214, 2000.

[13] Katsuhiko Sato. First-order phase transition of a vacuum and the expansion of the Universe. *Monthly Notices of the Royal Astronomical Society*, 195(3):467–479, 1981.

[14] Alan H. Guth. The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems. *Phys. Rev.*, D23:347–356, 1981.

[15] Thomas P. Sotiriou and Valerio Faraoni. f(R) Theories Of Gravity. *Rev. Mod. Phys.*, 82:451–497, 2010.

[16] Antonio De Felice and Shinji Tsujikawa. f(R) theories. *Living Rev. Rel.*, 13:3, 2010.

[17] Alexei A. Starobinsky. A New Type of Isotropic Cosmological Models Without Singularity. *Phys. Lett.*, B91:99–102, 1980.

[18] Hiroshi Nakada and Sergei V. Ketov. Randall-Sundrum braneworld in modified gravity. *Phys. Rev.*, D94(10):103503, 2016.

[19] Sergei V. Ketov and Hiroshi Nakada. Inflation from $(R + \gamma R^n - 2\Lambda)$ Gravity in Higher Dimensions. *Phys. Rev.*, D95(10):103507, 2017.

[20] Santiago Pajón Otero, Francisco G. Pedro, and Clemens Wieck. $R + \alpha R^n$ Inflation in higher-dimensional Space-times. *JHEP*, 05:058, 2017.

[21] Hiroshi Nakada and Sergei V. Ketov. Inflation from higher dimensions. *Phys. Rev.*, D96(12):123530, 2017.

[22] Y. M. Cho. Higher - Dimensional Unifications of Gravitation and Gauge Theories. *J. Math. Phys.*, 16:2029, 1975.

[23] Y. M. Cho and Peter G. O. Freund. Nonabelian Gauge Fields in Nambu-Goldstone Fields. *Phys. Rev.*, D12:1711, 1975.

[24] Y. M. Cho and Pong Soo Jang. Unified Geometry of Internal Space with Space-Time. *Phys. Rev.*, D12:3789, 1975.

[25] Edward Witten. Search for a Realistic Kaluza-Klein Theory. *Nucl. Phys.*, B186:412, 1981.

[26] Thomas Appelquist and Alan Chodos. The Quantum Dynamics of Kaluza-Klein Theories. *Phys. Rev.*, D28:772, 1983.

[27] Thomas Appelquist and Alan Chodos. Quantum Effects in Kaluza-Klein Theories. *Phys. Rev. Lett.*, 50:141, 1983.

[28] M. J. Duff, B. E. W. Nilsson, and C. N. Pope. Kaluza-Klein Supergravity. *Phys. Rept.*, 130:1–142, 1986.

[29] Sean M. Carroll, James Geddes, Mark B. Hoffman, and Robert M. Wald. Classical stabilization of homogeneous extra dimensions. *Phys. Rev.*, D66:024036, 2002.

[30] V. A. Rubakov and M. E. Shaposhnikov. Do We Live Inside a Domain Wall? *Phys. Lett.*, 125B:136–138, 1983.

[31] Petr Horava and Edward Witten. Eleven-dimensional supergravity on a manifold with boundary. *Nucl. Phys.*, B475:94–114, 1996.

[32] N. Aghanim et al. Planck 2018 results. VI. Cosmological parameters. 2018.

[33] Y. Akrami et al. Planck 2018 results. X. Constraints on inflation. 2018.

[34] Y. Fujii and K. Maeda. *The scalar-tensor theory of gravitation.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.

[35] U. Gunther, P. Moniz, and A. Zhuk. Asymptotical AdS from nonlinear gravitational models with stabilized extra dimensions. *Phys. Rev.*, D66:044014, 2002.

[36] U. Gunther, P. Moniz, and A. Zhuk. Nonlinear multidimensional cosmological models with form fields: Stabilization of extra dimensions and the cosmological constant problem. *Phys. Rev.*, D68:044010, 2003.

[37] Uwe Gunther, Alexander Zhuk, Valdir B. Bezerra, and Carlos Romero. AdS and stabilized extra dimensions in multidimensional gravitational models with nonlinear scalar curvature terms R^{-1} and R^4. *Class. Quant. Grav.*, 22:3135–3167, 2005.

[38] Tamerlan Saidov and Alexander Zhuk. On the problem of inflation in nonlinear multidimensional cosmological models. *Phys. Rev.*, D79:024025, 2009.

[39] John D. Barrow and S. Cotsakis. Inflation and the Conformal Structure of Higher Order Gravity Theories. *Phys. Lett.*, B214:515–518, 1988.

[40] Kei-ichi Maeda. Towards the Einstein-Hilbert Action via Conformal Transformation. *Phys. Rev.*, D39:3159, 1989.

[41] P. A. R. Ade et al. Improved Constraints on Cosmology and Foregrounds from BICEP2 and Keck Array Cosmic Microwave Background Data with Inclusion of 95 GHz Band. *Phys. Rev. Lett.*, 116:031302, 2016.

[42] Sergei V. Ketov. Supergravity and Early Universe: the Meeting Point of Cosmology and High-Energy Physics. *Int. J. Mod. Phys.*, A28:1330021, 2013.

[43] Sergei V. Ketov and Takahiro Terada. Old-minimal supergravity models of inflation. *JHEP*, 12:040, 2013.

[44] Abdus Salam and E. Sezgin. d = 8 SUPERGRAVITY. *Nucl. Phys.*, B258:284, 1985.

[45] Joel Scherk and John H. Schwarz. How to Get Masses from Extra Dimensions. *Nucl. Phys.*, B153:61–88, 1979.

[46] E. Cremmer, B. Julia, and Joel Scherk. Supergravity Theory in Eleven-Dimensions. *Phys. Lett.*, B76:409–412, 1978.