Moduli of diffeomorphisms with homoclinic tangencies

ホモクリニック接触を持つ微分同相写像のモジュライ（英文）

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Contents

Introduction

This thesis concerns the topological conjugacy problem for diffeomorphisms on a closed manifold M. A diffeomorphism f on M is called structurally stable if any diffeomorphism g close to f is topologically conjugate to f. The structural stability for diffeomorphisms are well studied by many authors. In particular, R. Mañé (1987) and others proved that, in the C^1 category, f is structurally stable if and only if f is an Axiom A diffeomorphism with the strong transversality condition. On the other hand, if f has a basic set which has a homoclinic tangency, then it is never structurally stable. So, if f has a homoclinic tangency, then any neighborhood of f in the space of diffeomorphisms contains both diffeomorphisms g which are topologically conjugate and non-conjugate to f. Thus, we need topological conjugacy invariants to decide whether a given g is topologically conjugate to f or not.

A *modulus* $m(f)$ for a diffeomorphism f is a topological conjugacy invariant for f, that is, $m(f) = m(g)$ holds for any $g : M \to M$ which is contained in a certain class of diffeomorphisms on M and topologically conjugate to f. The aim of this thesis is to present new moduli for diffeomorphisms of dimensions two and three.

This thesis is organized as follows.

In Chapter 1, we present definitions, notions and concepts needed in this thesis. Besides, we introduce several preceding results on moduli.

In Chapter 2, we study moduli for 2-dimensional diffeomorphisms with cubic homoclinic tangencies (two-sided tangencies of the lowest order) under certain open conditions. Ordinary arguments used in previous studies of conjugacy invariants associated with one-sided tangencies do not work in the two-sided case. We present a new method which is applicable to the two-sided case.

In Chapter 3, we investigate moduli of a 3-dimensional diffeomorphism f with a sadldle focus p and a homoclinic quadratic tangency q. It is shown there that, for most of such diffeomorphisms, all the eigenvalues of $Df(p)$ are moduli and the restriction of a conjugacy homeomorphism to a local unstable manifold is a uniquely determined linear conformal map.

Chapter 1

Basic definitions and concepts

In this chapter, we present some of definitions, notions and concepts needed in this thesis. Refer to [De, Ro1, Ro2] and so on for other standard results on dynamical systems.

1.1 Hyperbolic fixed points of diffeomorphisms

Let M be a C^r $(1 \leq r \leq \infty)$ manifold and $\mathrm{Diff}^r(M)$ the space of C^r diffeomorphisms on M with C^r topology. Suppose that f is an element of $\mathrm{Diff}^r(M)$. For a point $x \in M$, the *orbit* $\mathcal{O}(x)$ of x for f is defined as $\mathcal{O}(x) = \{f^n(x)\,;\, n \in \mathbb{Z}\}$, where f^0 is the identity map on M, f^n is the composition of f with itself n times if $n > 0$ and f^n is the composition of f^{-1} with itself $-n$ times if $n < 0$. A point $p \in M$ is called a *periodic point* for f if $p = f^n(p)$ holds for some positive integer n. The minimum of such an n is called the *period* of p. A point $p \in M$ is called a *fixed point* for f if $p = f(p)$ holds, that is, a fixed point is a periodic point with period one.

Suppose that p is a fixed point for f. Then the derivative $Df(p)$ of f at p is a linear map on the tangent space $T_p(M)$ at p. By an identification of $T_p(M)$ with $\mathbb{R}^m$ as vector spaces, one can regard the linear map on $T_p(M)$ with that on $\mathbb{R}^m$, where m is the dimension of M.

Definition 1.1. A fixed point p for f is called *hyperbolic* if the absolute value $|\lambda|$ of any eigenvalue λ of $Df(p)$ is different from one. The hyperbolic fixed point p is called a *sink* if the absolute value $|\lambda|$ of any eigenvalue λ of $Df(p)$ is less than one. The hyperbolic fixed point p is called a *source* if the absolute value $|\lambda|$ of any eigenvalue λ of $Df(p)$ is greater than one. A hyperbolic fixed point which is neither a sink nor a source is said to be a *saddle*.

Figures 1.1 and 1.2 illustrate hyperbolic fixed points in the case of $\dim M = 2$ and $\dim M = 3$, respectively, where all the eigenvalues of $Df(p)$ are real.

We also consider the case that some of eigenvalues are non-real. If $Df(p)$ have non-real eigenvalues $re^{\pm\sqrt{-1}\theta}$, then f acts on a neighborhood of p as the combination of a rotation and an expansion or contraction. In the case of $\dim M = 3$, we have several phase portraits of f near p. The hyperbolic fixed point p is a sink if $Df(p)$ has a real eigenvalue $0 < \lambda < 1$

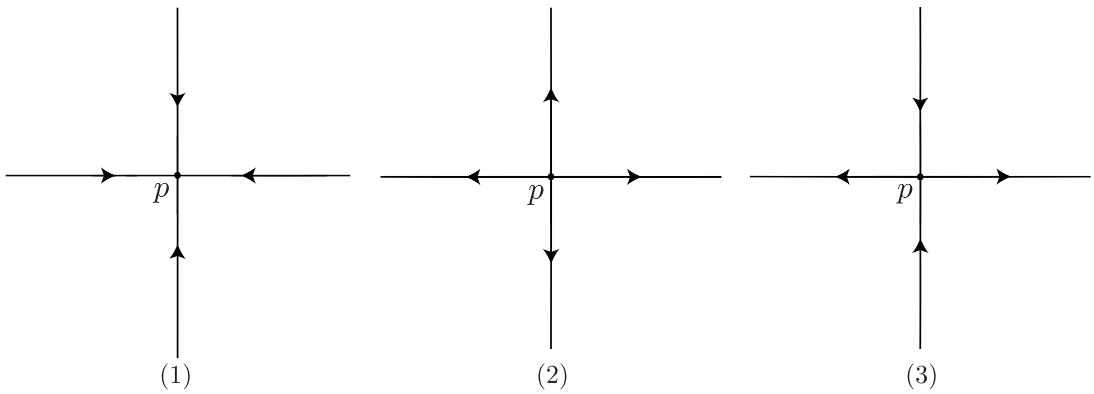

Figure 1.1: The case of $\dim M = 2$. p is a sink in (1), a source in (2) and a saddle in (3).

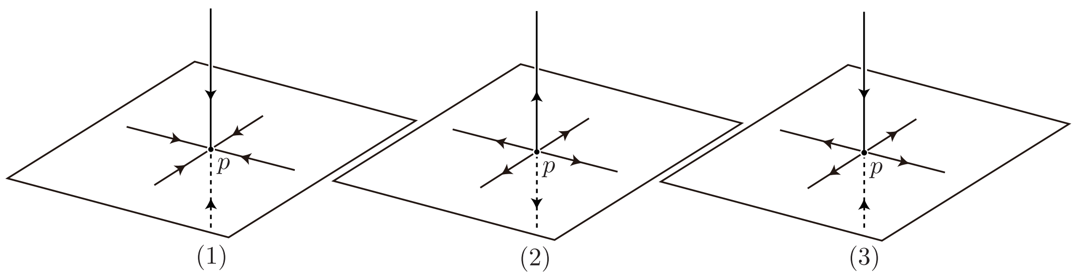

Figure 1.2: The case of $\dim M = 3$. p is a sink in (1), a source in (2) and a saddle in (3).

and non-real eigenvalues $re^{\pm\sqrt{-1}\theta}$ with $r < 1$. The hyperbolic fixed point p is a source if $Df(p)$ has a real eigenvalue $\lambda > 1$ and non-real eigenvalues $re^{\pm\sqrt{-1}\theta}$ with $r > 1$. If the hyperbolic fixed point p is neither a sink nor source, then it is called a *saddle focus*. See Figure 1.3. In Section 3, we study moduli of 3-dimensional diffeomorphisms having saddle foci with a real eigenvalue $0 < \lambda < 1$ and non-real eigenvalues $re^{\pm\sqrt{-1}\theta}$ with $r > 1$. See Figure 1.3 (3).

The following linearization theorem is called the Hartman-Grobman theorem. See the Chapter 5 in [Ro1] for the proof.

Theorem 1.2 (Hartman-Grobman Theorem)**.** *Let $f : M \to M$ be a C^r diffeomorphism with a hyperbolic fixed point p. Then, there exist neighborhoods U, V of p with $U \cup f(U) \subset V$ and a homeomorphism $h : V \to T_p(M)$ with $h(p) = \mathbf{0}$ and such that the following diagram is commutative.*

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & f(U) \\ {\scriptstyle h|_U}\downarrow & & \downarrow{\scriptstyle h|_{f(U)}} \\ T_p(M) & \xrightarrow[Df(p)]{} & T_p(M) \end{array}$$

By Theorem 1.2, we can call the linear map $Df(p)$ a *linearized map* or *linearization* of f at p. Moreover, by Taylor's theorem, we know that the linear map $Df(p)$ approximates

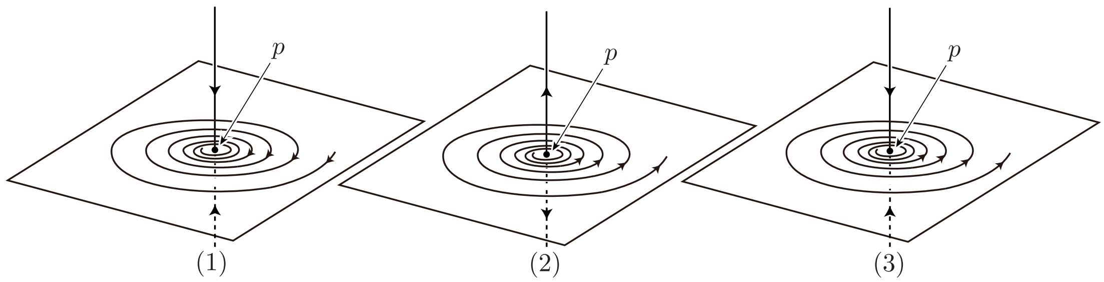

Figure 1.3: The case of $\dim M = 3$. p is a sink in (1), a source in (2) and a saddle focus in (3).

f near p.

1.2 Heteroclinic and homoclinic tangencies

Let f be a C^r diffeomorphism on M and $p \in M$ a fixed point for f. The *stable* and *unstable manifolds* $W^s(p)$ and $W^u(p)$ of p are defined as

$$W^s(p) = \{x \in M\,;\, f^n(x) \to p \ (n \to \infty)\},$$
$$W^u(p) = \{x \in M\,;\, f^{-n}(x) \to p \ (n \to \infty)\}.$$

Moreover, we define the *local stable* and *local unstable manifolds* $W^s_{\rm loc}(p)$ and $W^u_{\rm loc}(p)$ of p as

$$W^s_{\rm loc}(p) = \{x \in U(p)\,;\, f^n(x) \in U(p) \ \text{for any } n \in \mathbb{N}, \lim_{n\to\infty} f^n(x) = p\},$$
$$W^u_{\rm loc}(p) = \{x \in U(p)\,;\, f^{-n}(x) \in U(p), \ \text{for any } n \in \mathbb{N}, \lim_{n\to\infty} f^{-n}(x) = p\},$$

where $U(p)$ is a sufficiently small neighborhood of p in M.

The following theorem is called the Stable Manifold Theorem. This theorem shows that the local stable manifold $W^s_{\rm loc}(p)$ and local unstable manifold $W^u_{\rm loc}(p)$ are C^r submanifolds of M. See the Chapter 5 in [Ro1] for the proof.

Theorem 1.3 (Stable Manifold Theorem)**.** *Let $f : M \to M$ be a diffeomorphism and let $p \in M$ be a saddle fixed point for f. Then the local stable manifold $W^s_{\rm loc}(p)$ of p is a C^r submanifold of M tangent to the subspace of $T_p(M)$ spanned by the eigenvectors with contracting eigenvalues. Similarly, the local unstable manifold $W^u_{\rm loc}(p)$ of p is a C^r submanifold of M tangent to the subspace of $T_p(M)$ spanned by the eigenvectors with expanding eigenvalues.*

We say that the dimension of $W^s_{\rm loc}(p)$ is the *stable index* of p and denote it by $\mathrm{ind}^s(p)$. Then $\mathrm{ind}^u(p) = \dim M - \mathrm{ind}^s(p)$ is called the *unstable index* of p. For the definitions of stable and unstable manifolds,

$$W^s(p) = \bigcup_{n\geq 1} f^{-n}(W^s_{\rm loc}(p)), \quad W^u(p) = \bigcup_{n\geq 1} f^{n}(W^u_{\rm loc}(p)).$$

This implies that $W^s(p)$ and $W^u(p)$ are the images of injective C^r immersions from $\mathbb{R}^s$ and $\mathbb{R}^u$ to M, respectively, where $s = \mathrm{ind}^s(p)$ and $u = \mathrm{ind}^u(p)$.

Let p_1 and p_2 are two distinct saddle type fixed points of a diffeomorphism f on M. A point $q \in M$ is called a *heteroclinic point* associated with p_1 and p_2 if $q \in W^s(p_1) \cap W^u(p_2)$, i.e., $\lim_{n\to\infty} f^n(q) = p_1$, $\lim_{n\to\infty} f^{-n}(q) = p_2$. We say that the point q is a *transverse heteroclinic point* if $W^s(p_1)$ and $W^u(p_2)$ intersect transversely at q, namely, $T_q(M) = T_q(W^s(p_1)) \oplus T_q(W^u(p_2))$ holds. When q is a non-transverse intersection point, q is called a *heteroclinic tangency* associated with p_1 and p_2. See Figure 1.4.

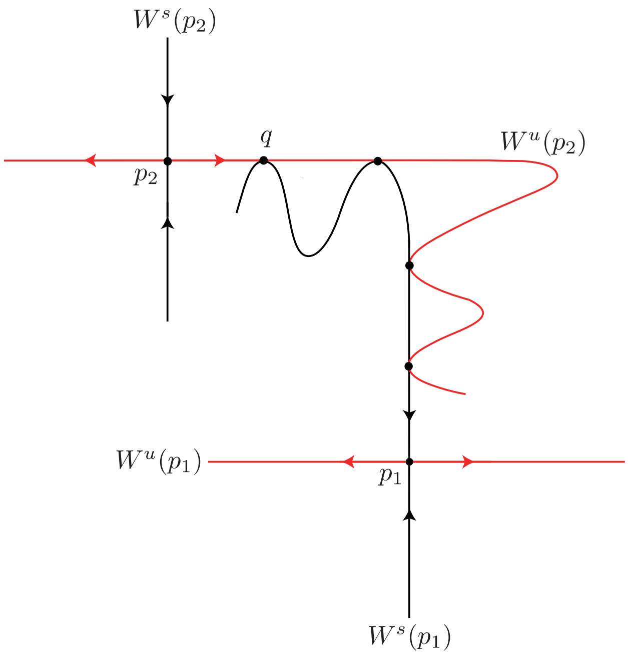

Figure 1.4: q is one of heteroclinic tangencies associated with p_1 and p_2.

Let p is a saddle fixed point of a diffeomorphism f on M. A point $q \in M$ is called a *homoclinic point* associated with p if $q \in W^s(p) \cap W^u(p) \setminus \{p\}$, i.e., $q \neq p$ and $\lim_{n\to\infty} f^n(q) = p$ and $\lim_{n\to\infty} f^{-n}(q) = p$. We say that the point q is called a *transverse homoclinic point* if $W^s(p)$ and $W^u(p)$ intersect transversely at q. When q is a non-transverse intersection point, q is called a *homoclinic tangency* associated with p. See Figure 1.5.

Let f be a C^r $(n \leq r \leq \infty)$ diffeomorphism with a heteroclinic or homoclinic tangency q. We fix a Riemannian metric on M and define the order of tangency as follows. The tangency is of *order n* if the limit

$$\lim_{\substack{w \in W^s_{\mathrm{loc}}(p),\\ w \to q}} \frac{d(w, W^u(p))}{[d(w,q)]^n}$$

exists and is not zero, where d is the distance on M induced from this metric. If $n = 2$ (resp. $n = 3$), then the tangency q is called *quadratic* (resp. cubic). If n is even, then the tangency

q is said to be *one-sided*. If n is odd, then the tangency q is *two-sided*. See Figures 1.5 and 1.6. Homoclinic tangencies have been studied by Newhouse, Palis and Takens and so on since the seventies. For example, see [dM, dMP, dMvS, KS1, KS2, NPT, Ni, Pa, Po, PT].

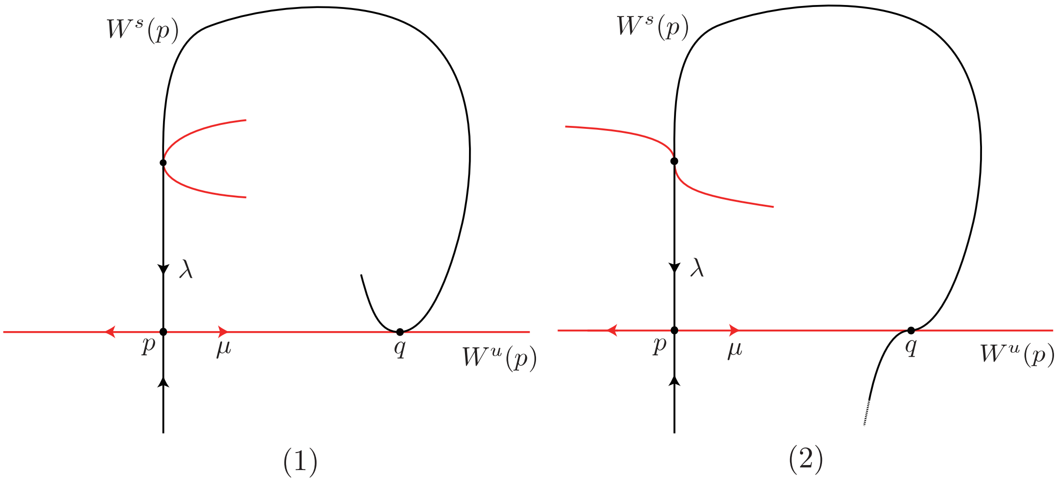

Figure 1.5: The case of $\mathrm{ind}^s(p) = \mathrm{ind}^u(p) = 1$. q is a homoclinic quadratic tangency in (1) and a homoclinic cubic tangency in (2).

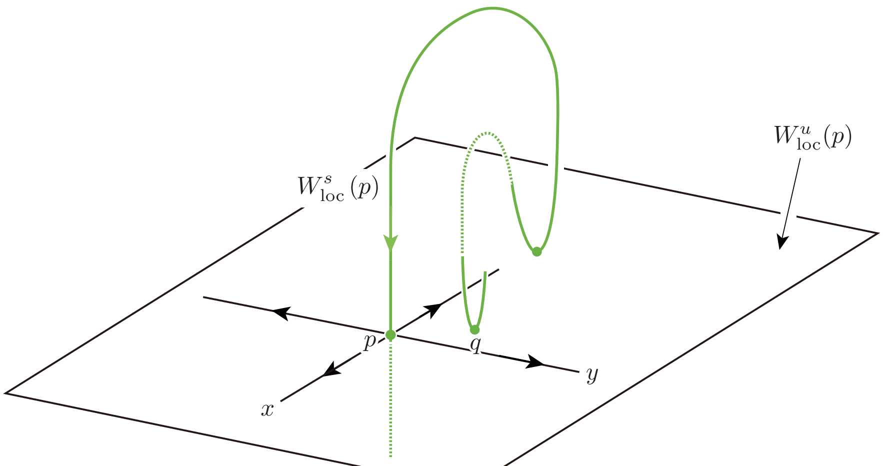

Figure 1.6: The case of $\mathrm{ind}^s(p) = 1$ and $\mathrm{ind}^u(p) = 2$. p is a saddle point and q is a homoclinic quadratic tangency associated with p.

Now, we define hyperbolic invariant sets for a diffeomotphism f. A subset S of M is said to be *positively invariant* if $f(x) \in S$ for all $x \in S$, i.e., $f(S) \subset S$. On the other hand, a subset S of M is said to be *negatively invariant* if $f^{-1}(S) \subset S$. Such an S is said to be an *invariant set* of f if $f(S) = S$. Notice that any periodic orbit and the orbit of a heteroclinic or a homoclinic point are typical examples of invariant sets for f. We denote by $\|\cdot\|_x$ the norm on the tangent space $T_x(M)$ at $x \in M$ induced from the Riemannian metric on M. A closed invariant set Λ for f is said to be *hyperbolic* if it satisfies the following conditions.

(1) At each point $x \in \Lambda$, the tangent space to M splits as the direct sum of subspaces $\mathbb{E}^u_x$ and $\mathbb{E}^s_x$, i.e., $T_x(M) = \mathbb{E}^u_x \oplus \mathbb{E}^s_x$.

(2) The splitting is invariant under the action of the derivative map, i.e., $Df_x(\mathbb{E}^u_x) = \mathbb{E}^u_{f(x)}$ and $Df_x(\mathbb{E}^s_x) = \mathbb{E}^s_{f(x)}$.

(3) There exist $0 < \lambda < 1$ and $C > 0$ independent of x such that, for all $n \geq 0$,

$$\|Df^n_x(\boldsymbol{v}^s)\|_{f^n(x)} \leq C\lambda^n \|\boldsymbol{v}^s\|_x \ \text{ for } \boldsymbol{v}^s \in \mathbb{E}^s_x,$$
$$\|Df^{-n}_x(\boldsymbol{v}^u)\|_{f^{-n}(x)} \leq C\lambda^n \|\boldsymbol{v}^u\|_x \ \text{ for } \boldsymbol{v}^u \in \mathbb{E}^u_x$$

hold.

Notice that the closure of the orbit of a transverse heteroclinic or homoclinic point is a simple example of a hyperbolic invariant set for f. For a *Morse-Smale diffeomorphism*, the set $\mathrm{Per}(f)$ of all periodic points is a finite hyperbolic invariant set. For an *Anosov diffeomorphism*, e.g. the toral Anosov automorphisms, the ambient manifold M itself is a hyperbolic invariant set. We have many curious examples of hyperbolic invariant sets other than them, e.g. horseshoes, the Plykin attractor, the solenoid, or some invariant sets of Hénon-like maps. For example, see [De, Ro2].

As in the case of hyperbolic fixed points, we can define the stable and unstable manifolds for a hyperbolic invariant set as follows. Let Λ be a hyperbolic invariant set for f. The *stable* and *unstable manifolds* $W^s(x)$ and $W^u(x)$ of $x \in \Lambda$ are defined as

$$W^s(x) = \Big\{ y \in M \,;\, \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0 \Big\},$$
$$W^u(x) = \Big\{ y \in M \,;\, \lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0 \Big\}.$$

The unions

$$W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x), \quad W^u(\Lambda) = \bigcup_{x\in\Lambda} W^u(x)$$

are called the *stable* and *unstable manifolds* for Λ, respectively. For $\varepsilon > 0$, we identify the neighborhoods of each point $x \in \Lambda$ in M with $U_\varepsilon(x) = \mathbb{E}^s_x(\varepsilon) \times \mathbb{E}^u_x(\varepsilon)$, where $\mathbb{E}^s_x(\varepsilon) = \{\boldsymbol{v} \in \mathbb{E}^s_x \,;\, \|\boldsymbol{v}\|_x < \varepsilon\}$ and $\mathbb{E}^u_x(\varepsilon) = \{\boldsymbol{v} \in \mathbb{E}^u_x \,;\, \|\boldsymbol{v}\|_x < \varepsilon\}$. We define the *local stable* and *local unstable manifolds* $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ of $x \in \Lambda$ *of size* ε as

$$W^s_\varepsilon(x) = \Big\{ y \in U_\varepsilon(x) \,;\, f^j(y) \in U_\varepsilon(f^j(x)) \ \text{ for } j \geq 0 \Big\},$$
$$W^u_\varepsilon(x) = \Big\{ y \in U_\varepsilon(x) \,;\, f^{-j}(y) \in U_\varepsilon(f^{-j}(x)) \ \text{ for } j \geq 0 \Big\}.$$

Now, we extend Stable Manifold Theorem to the case of hyperbolic invariant sets. See the Chapter 8 in [Ro1] for the proof.

Theorem 1.4 (Stable Manifold Theorem for hyperbolic invariant sets)**.** *Let f be a C^r $(1 \le r \le \infty)$ diffeomorphism on M and let Λ be a compact hyperbolic invariant set for f. Then there is an $\varepsilon > 0$ such that, for each $x \in \Lambda$, there are two C^r embedded disks $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ which are tangent to $\mathbb{E}^s_x$ and $\mathbb{E}^u_x$, respectively, and satisfy the following conditions.*

- *$W^s_\varepsilon(x)$ is represented by the graph of a C^r function $\sigma^s_x : \mathbb{E}^s_x(\varepsilon) \to \mathbb{E}^u_x(\varepsilon)$ with $\sigma^s_x(\mathbf{0}_x) = \mathbf{0}_x$ and $D\sigma^s_x(\mathbf{0}) = \mathbf{0}$:*

$$W^s_\varepsilon(x) = \Big\{(\sigma^s_x(\boldsymbol{v}), \boldsymbol{v})\,;\, \boldsymbol{v} \in \mathbb{E}^s_x(\varepsilon)\Big\}.$$

  Besides, the function σ^s_x and its derivatives vary continuously on x. Similarly, there is a C^r function $\sigma^u_x : \mathbb{E}^u_x(\varepsilon) \to \mathbb{E}^s_x(\varepsilon)$ with $\sigma^u_x(\mathbf{0}_x) = \mathbf{0}_x$ and $D\sigma^u_x(\mathbf{0}) = \mathbf{0}$:

$$W^u_\varepsilon(x) = \Big\{(\boldsymbol{u}, \sigma^u_x(\boldsymbol{u}))\,;\, \boldsymbol{u} \in \mathbb{E}^u_x(\varepsilon)\Big\}.$$

  The function σ^u_x and its derivatives also vary continuously on x.

- *There exist $0 < \lambda < 1$ and $C \ge 1$ such that*

$$\begin{aligned} W^s_\varepsilon(x) &\subset \Big\{y \in U_\varepsilon(x)\,;\, d(f^j(x), f^j(y)) \le C\lambda^j d(x,y) \ \text{ for } \ j \ge 0\Big\},\\ W^u_\varepsilon(x) &\subset \Big\{y \in U_\varepsilon(x)\,;\, d(f^{-j}(x), f^{-j}(y)) \le C\lambda^j d(x,y) \ \text{ for } \ j \ge 0\Big\}. \end{aligned}$$

By Theorem 1.4, we have

$$W^s(x) = \bigcup_{n \ge 0} f^{-n}(W^s_\varepsilon(f^n(x))), \quad W^u(x) = \bigcup_{n \ge 0} f^{n}(W^s_\varepsilon(f^{-n}(x))).$$

Notice that $W^s(x)$ and $W^u(x)$ are just the images of injective C^r immersions from $\mathbb{R}^s$ and $\mathbb{R}^u$ to M but not necessarily the images of embeddings, where $s = \dim \mathbb{E}^s_x$ and $u = \dim \mathbb{E}^u_x$. Horseshoes or toral Anosov automorphisms are typical examples of such diffeomorphisms. See the Chapter 8 in [Ro1].

1.3 Topological conjugacy and structural stability

For two diffeomorphisms f and g, if the orbits for f one-to-one correspond to those for g with the same behavior, then we regard that f and g have essentially the same dynamical systems. For example, mutually conjugate linear maps satisfy the property. For classifying such diffeomorphisms, we introduce the notion of topological conjugacy.

Definition 1.5. We say that two diffeomorphisms f and g on a $C^r(1 \le r \le \infty)$ manifold M are *topologically conjugate* to each other if there exists a homeomorphism $h : M \to M$ with $h \circ f = g \circ h$. This homeomorphism h is called a *topological conjugacy* between f and g.

Let p be a periodic point for f with period n and set $p' = h(p)$. Then p' satisfies $g^n(p') = g^n(h(p)) = h(f^n(p)) = h(p) = p'$. Thus, the point p' is also a periodic point for g with the same period n.

A subset D of M is called a *fundamental domain* of f if any non-periodic orbit of f intersects D exactly in one point. Fundamental domains are often used to construct topological conjugacies between diffeomorphisms. For example, let f be a linear map on $\mathbb{R}^2$ with real contracting eigenvalues and g another linear map on $\mathbb{R}^2$ with non-real contacting eigenvalues. Take a unit circle C on $\mathbb{R}^2$, then one can have a pair of annuli A_f and A_g in $\mathbb{R}^m$ bounded by $C \cup f(C)$ and $C \cup g(C)$, respectively. See Figure 1.7. Then $A'_f = A_f \setminus f(C)$ and $A'_g = A_g \setminus g(C)$ are fundamental domains for f and g, respectively. There exists a homeomorphism $\tilde{h} : A_f \to A_g$ with

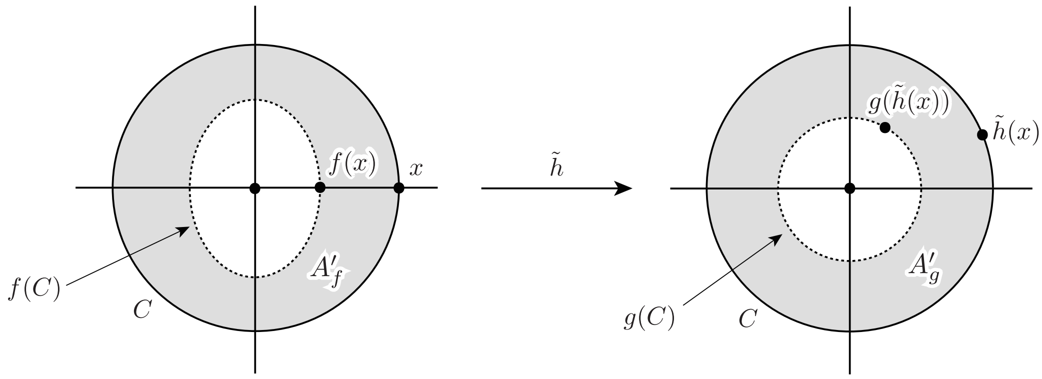

Figure 1.7: Fundamental domains A'_f of f and A'_g of g.

$$\tilde{h}(f(x)) = g(\tilde{h}(x)) \tag{1.1}$$

for any $x \in C$. Extend $\tilde{h}$ to the map $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$h(x) = g^{-n(x)}(\tilde{h}(f^{n(x)}(x)))$$

for $x \in \mathbb{R}^2 \setminus \{0\}$ and $h(0) = 0$ for $0 \in \mathbb{R}^2$, where $n(x)$ is a uniquely determined integer with $f^{n(x)}(x) \in A'_f$. By (1.1), h is a well defined homeomorphism on $\mathbb{R}^2$, which is a topological conjugacy between f and g.

For a given diffeomorphism f, we are interested in the topological conjugacy class constructed by diffeomorphisms close to f. Thus we introduce the notion of the structural stability for diffeomorphisms. A diffeomorphism $f \in \mathrm{Diff}^r(M)$ is called *structurally stable* if there exists a neighborhood $\mathcal{N} \subset \mathrm{Diff}^r(M)$ of f such that, for any $g \in \mathcal{N}$, f and g are topologically conjugate.

Remark 1.6. In this definition of structural stability, the condition that h is a homeomorphism is crucial. We suppose that h is a diffeomorphism. Then h is called a C^r $(1 \leq r \leq \infty)$ conjugacy between f and g. If f has a fixed point p, then, by the chain rule of composition maps, $Dh(p)Df(p) = Dg(h(p))Dh(p)$ holds. This shows that $Df(p)$ and $Dg(h(p))$

are similar matrices via the matrix $Dh(p)$. Thus they have the same eigenvalues. On the other hand, for any $f \in \mathrm{Diff}^r(M)$ and any fixed point p of f, there exists $g \in \mathrm{Diff}^r(M)$ arbitrarily C^r close to f such that the eigenvalues of $Dg(p')$ are different from those of $Df(p)$, where p' is the fixed point of g corresponding to p. Namely, any neighborhood of $f \in \mathrm{Diff}^r(M)$ contains an element which is not C^r conjugate to f. Thus, any diffeomorphism with a fixed point is not structurally stable with respect to C^r conjugacy.

1.4 C^r convergence of unstable manifolds

Let p be a hyperbolic fixed point of a diffeomorphism f on M and $U(p)$ a sufficiently small neighborhood of p in M. Take a disk D embedded in M of dimension $\mathrm{ind}^u(p)$ which intersects transversely the local stable manifold $W^s_{\mathrm{loc}}(p)$ at a single point z_0. For any $n \in \mathbb{N}$, let D_n be the component of $f^n(D) \cap U(p)$ containing $f^n(z_0)$. Then, D_n uniformly C^r converges to $W^u_{\mathrm{loc}}(p)$ as $n \to \infty$. Figure 1.8 illustrates the cases of $\mathrm{ind}^s(p) = \mathrm{ind}^u(p) = 1$ and $\mathrm{ind}^s(p) = 1$, $\mathrm{ind}^u(p) = 2$. More precisely, we have the following theorem called Inclination Lemma. See the Chapter 5 in [Ro1] for the proof.

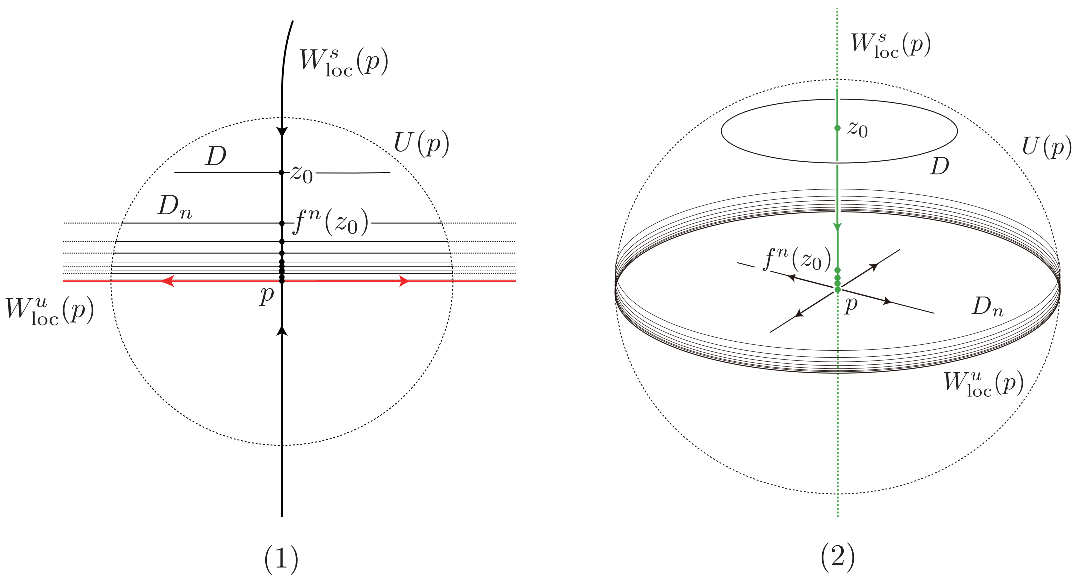

Figure 1.8: (1) The case of $\mathrm{ind}^s(p) = \mathrm{ind}^u(p) = 1$. (2) The case of $\mathrm{ind}^s(p) = 1$ and $\mathrm{ind}^u(p) = 2$.

Theorem 1.7 (Inclination Lemma). *Let f be a C^r $(1 \le r \le \infty)$ diffeomorphism on M and $p \in M$ a saddle fixed point. Assume that M has a coordinate neighborhood of p such that $W^s_{\mathrm{loc}}(p) \subset \mathbb{R}^s \times \{0\}$ and $W^u_{\mathrm{loc}}(p) \subset \{0\} \times \mathbb{R}^u$, where $s = \mathrm{ind}^s(p)$ and $u = \mathrm{ind}^u(p)$, if necessary by changing the coordinates suitably. Then, for any C^r submanifold D with $\dim(D) = u$ intersecting $W^s_{\mathrm{loc}}(p)$ transversely at $z_0 = (x_0, 0) \in W^s_{\mathrm{loc}}(p) \times \{0\}$, the component D_n of $f^n(D) \cap U(p)$ containing $f^n(z_0)$ uniformly C^r converges to $W^u_{\mathrm{loc}}(p)$ as $n \to \infty$.*

We consider the case that a diffeomorphism f has a homoclinic tangency r associated with a saddle fixed point p. First, suppose that $\dim M = 2$ and r is either a quadratic or cubic homoclinic tangency. It is not hard to show that $W^u(p)$ and $W^s(p)$ have a transverse intersection point z in a neighborhood of r under suitable open conditions of f. For example, see [GS1, GS2] if r is a quadratic tangency and Lemma 1.2 in [KS1] if r is a cubic tangency. Figure 1.9 illustrates the situations. Take an arc D^u in $W^u(p)$ such that the interior of D^u contains z. Then there exists an integer N such that $f^N(z) \in U(p)$. Let D^u_0 be the connected component of $f^N(D^u) \cap U(p)$ containing $f^N(z)$. Let D^u_n be the component of $f^{N+n}(D^u) \cap U(p)$ containing $f^{N+n}(z)$. By Inclination Lemma (Theorem 1.7), D^u_n C^r converges to $W^u_{\mathrm{loc}}(p)$ as $n \to \infty$.

Next, we consider the case that $\dim M = 3$ and $\mathrm{ind}^s(p) = 1$, $\mathrm{ind}^u(p) = 2$. By [Ni], under certain open conditions of f, there exists a transverse intersection point z of $W^u(p)$ and $W^s(p)$ near r. As in the case of $\dim M = 2$, there exists a disk $\widetilde{D}^u$ in $W^u(p)$ such that the interior of $\widetilde{D}^u$ contains z. Again by Inclination Lemma, we can take the disk $\widetilde{D}^u_n$ C^r converging to $W^u_{\mathrm{loc}}(p)$ as $n \to \infty$. The sequences $\{D^u_n\}$ and $\{\widetilde{D}^u_n\}$ are crucial in arguments of Chapters 2 and 3, respectively.

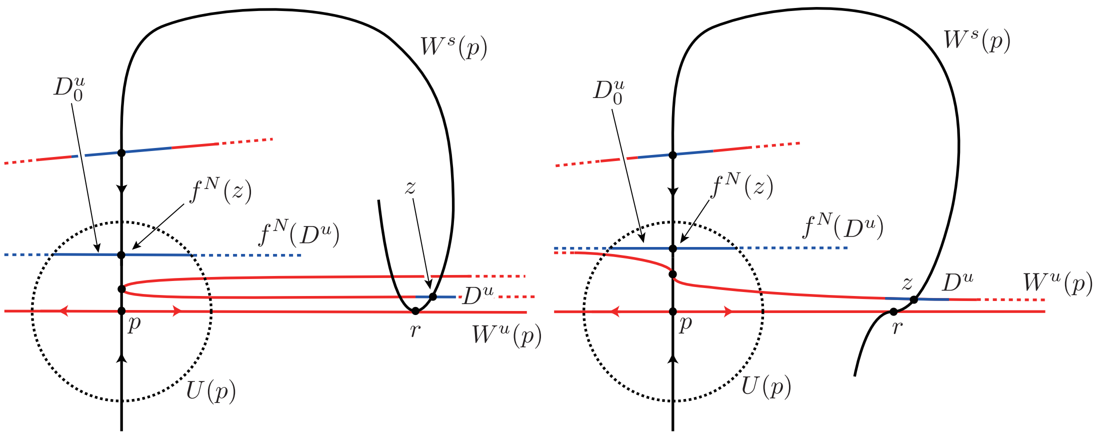

Figure 1.9: (1) r is a homoclinic quadratic tangency. (2) r is a homoclinic cubic tangency.

1.5 Motivation and preceding results

Structurally stable diffeomorphisms have no heteroclinic or homoclinic tangencies. On the other hand, diffeomorphisms with heteroclinic or homoclinic tangencies are typical examples of structurally unstable diffeomorphisms. For such a diffeomorphism f, we need topological conjugacy invariants to dicide whether a given diffeomorphism g is topological conjugate to f or not. See Figure 1.10. Such topological conjugacy invariants are called modulus.

Definition 1.8. For a subspace $\mathcal{N}$ of the diffeomorphism space $\mathrm{Diff}^r(M)$ with $r \geq 1$, we

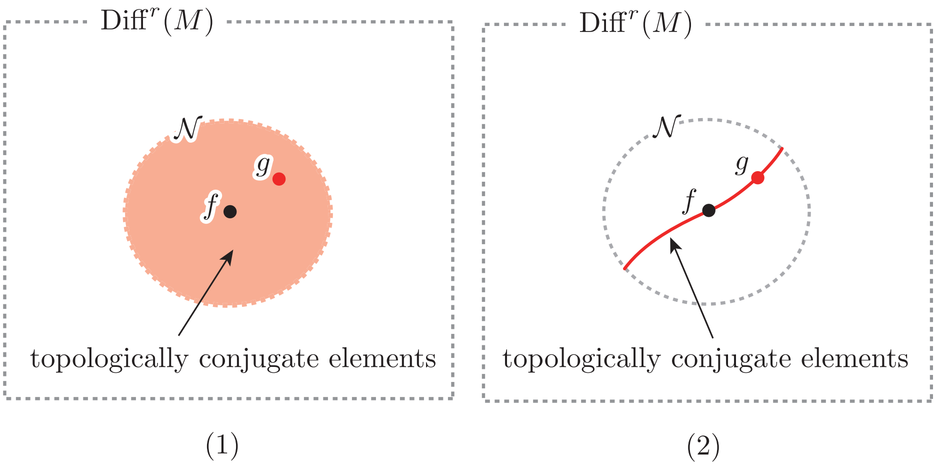

Figure 1.10: (1) The case that f is a structurally stable diffeomorphism. (2) The case that f is structurally unstable diffeomorphism.

say that a value $m(f)$ determined by $f \in \mathrm{Diff}^r(M)$ is a *modulus* in $\mathcal{N}$ if $m(g) = m(f)$ holds for any $g \in \mathcal{N}$ topologically conjugate to f.

The topological classification of structurally unstable diffeomorphisms on a manifold M is an important subject in the study of dynamical systems. Palis [Pa] suggested that moduli play important roles in such a classification. The research of dynamical systems with moduli have been originated by Palis, de Melo and Takens. Subsequently, Posthumus, van Strien and others have studied enthusiastically this subject. See [dM, dMP, dMvS, GPvS, NPT, Pa, PT, Ta]. Our study in this thesis is based on results of Palis [Pa], de Melo [dM] and Posthumus [Po].

We will finish this section by introducing their results. First, we consider the case of $\dim M = 2$. Suppose that f_i $(i = 0, 1)$ are elements of $\mathrm{Diff}^2(M)$ with two saddle fixed points p_i, q_i such that $W^u(p_i)$ and $W^s(q_i)$ have a quadratic heteroclinic tangency r_i and there exists a homeomorphism $h : M \to M$ with $f_1 = h \circ f_0 \circ h^{-1}$, $h(p_0) = p_1$, $h(q_0) = q_1$ and $h(r_0) = r_1$. See Figure 1.11 (1). Then, under some moderate conditions, Palis [Pa] proved that $\dfrac{\log|\lambda_0|}{\log|\mu_0|} = \dfrac{\log|\lambda_1|}{\log|\mu_1|}$, where λ_i is the contracting eigenvalue of $Df(p_i)$ and μ_i is the expanding eigenvalue of $Df(q_i)$. This means that $m(f_i) = \dfrac{\log|\lambda_i|}{\log|\mu_i|}$ is one of moduli.

Following his result, de Melo [dM] studied the moduli of the stability of two-dimensional diffeomorphisms f, that is, a minimal set of moduli which parametrizes the topological conjugacy classes of f in $\mathrm{Diff}^r(M)$. He detected moduli of stability for some classes of two-dimensional diffeomorphisms. In [dM], he also showed that the restrictions of the conjugacy homeomorphism h on each $W^s(p_0) \setminus \{p_0\}$ and $W^u(q_0) \setminus \{q_0\}$ are local diffeomorphisms if $\dfrac{\log|\lambda_0|}{\log|\mu_0|}$ is irrational.

Subsequently, Posthumus [Po] proved that the homoclinic version of Palis and de Melo's results. In fact, he proved that, if f_i $(i = 0, 1)$ has a saddle fixed point p_i with a homoclinic

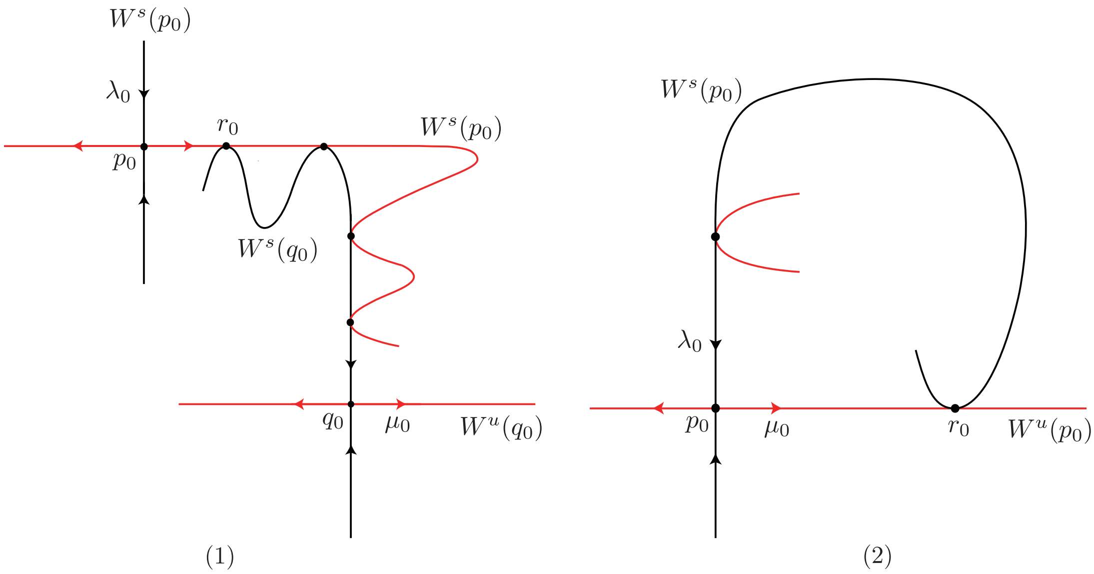

Figure 1.11: (1) The situation in Palis' case. (2) The situation in Posthumus' case.

quadratic tangency r_i, then $\dfrac{\log|\lambda_0|}{\log|\mu_0|} = \dfrac{\log|\lambda_1|}{\log|\mu_1|}$ holds, where λ_i and μ_i are the contracting and expanding eigenvalues of $Df(p_i)$, respectively. See Figure 1.11 (2). Moreover, if $\dfrac{\log|\lambda_0|}{\log|\mu_0|}$ is irrational, then the eigenvalues are also moduli, that is, $\lambda_0 = \lambda_1$ and $\mu_0 = \mu_1$.

For 2-dimensional diffeomorphisms, various results related to moduli concerning eigenvalues are obtained by some authors; see for example [dMP, dMvS, GPvS, PT]. However, in all of these results, the assumption that the tangency is quadratic or one-sided is crucial. In fact, some of their arguments do not work in the case that q is a two-sided tangency, see Remark 2.9 for the reason.

Chapter 2

Moduli of surface diffeomorphisms with cubic tangencies

In this chapter, we study conjugacy invariants for 2-dimensional diffeomorphisms with cubic homoclinic tangencies (two-sided tangencies of the lowest order) under certain open conditions. Some of arguments used in previous works of conjugacy invariants associated with one-sided tangencies do not work in the two-sided case. We present a new method which is applicable to the two-sided case.

2.1 Moduli of surface diffeomorphisms with cubic tangencies

The following is the main result in this chapter.

Theorem 2.1. *Suppose that M is a closed surface with Riemannian metric. Let f_i $(i = 0, 1)$ be elements of $\mathrm{Diff}^3(M)$ each of which has a saddle fixed point p_i and a homoclinic cubic tangency q_i associated with p_i and satisfies the following conditions.*

(A1) *For $i = 0, 1$, there exists a neighborhood $U(p_i)$ of p_i in M such that $f|_{U(p_i)}$ is linear.*

(A2) *f_0 is topologically conjugate to f_1 by a homeomorphism $h : M \to M$ with $h(p_0) = p_1$ and $h(q_0) = q_1$.*

(A3) *Each f_i $(i = 0, 1)$ satisfies the small expanding condition and one of the adaptable conditions with respect to (p_i, q_i) in Section 2.8.*

Then (M1) *and* (M2) *hold, where λ_i, μ_i are the eigenvalues of $Df_0(p_i)$ with $0 < |\lambda_i| < 1 < |\mu_i|$.*

(M1) $\dfrac{\log|\lambda_0|}{\log|\mu_0|} = \dfrac{\log|\lambda_1|}{\log|\mu_1|}$.

(M2) *Moreover, if $\dfrac{\log|\lambda_0|}{\log|\mu_0|}$ is irrational, then $\mu_0 = \mu_1$ and $\lambda_0 = \lambda_1$.*

Here we say that f_0 satisfies the *small expanding condition* at p_0 if $|\mu_0| = 1 + \varepsilon$ with $0 < \varepsilon < \varepsilon_0$ for the constant ε_0 given in Lemma 2.6. Note that this condition depends on local expressions of f_0 such as (2.2) near p_0 and (2.5) near $f_0^{m_0}(q_0)$. In Section 2.2, we present a codimension two submanifold $\mathcal{C}$ of $\mathrm{Diff}^3(M)$ such that any element of $\mathcal{C}$ sufficiently close to f_0 also satisfies (A3). In the case that f is of class C^∞, we know from Sternberg [St] and Takens [Ta] that (A1) is an open dense condition in $\mathrm{Diff}^\infty(M)$.

Though we only consider the case of cubic tangencies, we believe that our method still works in the case of two-sided tangencies of higher order. So we propose the following question.

Question 2.2. Is it possible to generalize our theorem to the case where diffeomorphisms have two-sided homoclinic tangencies of higher order?

We will finish the introduction by outlining the proof of the main theorem. Let f_0 be a diffeomorphism satisfying the conditions of Theorem 2.1. We may assume that q_0 and $r_0 = \varphi(q_0)$ are contained in $W^u_{\mathrm{loc}}(p_0)$ and $W^s_{\mathrm{loc}}(p_0)$ respectively, where $\varphi = f_0^{m_0}$ for some positive integer m_0. For the proof of Theorem 2.1, we need to find out a useful connection between the eigenvalues μ_i and λ_i for $i = 0, 1$. By applying Inclination Lemma (Lemma 1.7), we have a sequence $\{\alpha^u_n\}$ of arcs in $W^u(p_0)$ which meet $W^s_{\mathrm{loc}}(p_0)$ transversely at single points $z_0\lambda_0^n$ and C^3 converge to a sub-arc of $W^u_{\mathrm{loc}}(p_0)$. See Figure 2.5. Then $\varphi(\alpha^u_n)$ contains an S-shaped arc $\gamma'_{0,n}$ framed by the rectangle S_n as illustrated in Figure 2.1. We note that

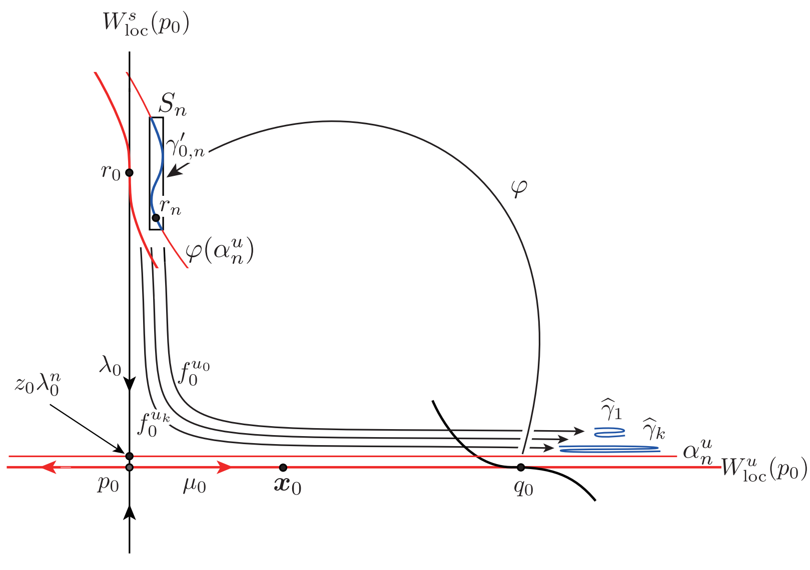

Figure 2.1: $\gamma'_{0,n}$ is an S-shaped arc. $\widehat{\gamma}_1$ and $\widehat{\gamma}_k$ are compressed S-shaped arcs near q_0 induced from $\gamma'_{0,n}$.

such arcs $\gamma'_{0,n}$ are subtle and vanish eventually as $n \to \infty$. See Figures 2.6 and 2.13. Since h is not supposed to be smooth, one can not expect that h sends $\gamma'_{0,n}$ to an S-shaped curve in $W^u(p_1)$. However Intersection Lemma (Lemma 2.7) shows that it actually holds, which is a key lemma in our argument. For the proof, we send $\gamma'_{0,n}$ to a curve $\widehat{\gamma}_1$ in a small neighborhood of q_0 by $f_0^{u_0}$ for some $u_0 \in \mathbb{N}$ and pull it back near r_0 by φ. Repeating this process many times, one can amplify $\widehat{\gamma}_1$ and finally have a compressed S-shaped curve $\widehat{\gamma}_k$ near q the diameter of which is substantial so that it can be distinguished by h. From this fact, we know that $h(\widehat{\gamma}_k)$ intersects a compressed S-shaped curve $\widehat{\gamma}_k^*$ in $W^u(p_1)$. It follows that there exists a sequence $\{r_n\}$ with $r_n \in \gamma'_{0,n}$ as illustrated in Figure 2.1 such that $\bar{r}_n = h(r_n)$ is contained in the corresponding S-shaped curve $\bar{\gamma}'_{0,n}$ in $W^u(p_1)$. We note that the images of r_n, $\bar{r}_n$ by the orthogonal projections to the first coordinates are represented as $az_0\lambda_0^n + o(\lambda_0^n)$, $\bar{a}\bar{z}_0\lambda_1^n + o(\lambda_1^n)$ respectively for some non-zero constants a, $\bar{a}$. One can take subsequences $\{n(k)\}$, $\{m(k)\}$ of $\mathbb{N}$ such that $f_0^{m(k)}(r_{n(k)})$ converges to a point $x_0 \in W^u_{\mathrm{loc}}(p_0)$. Then $f_1^{m(k)}(\bar{r}_{n(k)})$ also converges to $h(x_0) \in W^u_{\mathrm{loc}}(p_1)$. By using this fact, we will show that $\lim_{k\to\infty} \dfrac{m(k)}{n(k)} = -\dfrac{\log\lambda_0}{\log\mu_0}$ and $\lim_{k\to\infty} \dfrac{m(k)}{n(k)} = -\dfrac{\log\lambda_1}{\log\mu_1}$. This proves the assertion (M1). The assertion (M2) is proved by (M1) together with standard arguments in [dM, Po].

2.2 Preliminaries

Let $\{a_n\}$, $\{b_n\}$ be sequences with non-zero entries. Then $a_n \approx b_n$ means that $\dfrac{a_n}{b_n} \to 1$ as $n \to \infty$, and $a_n \sim b_n$ means that there exist constants C and C' independent of n with $0 < C' < 1 < C$ and satisfying $C' \le \dfrac{a_n}{b_n} \le C$ for any n. Suppose next that $\{a_n\}$, $\{b_n\}$ are sequences with non-negative entries. If there exists a constant $C' > 0$ independent of n and satisfying $a_n \le C'b_n$ for any n, then we denote the property by $a_n \precsim b_n$.

Throughout the remainder of this chapter, we suppose that M is a closed connected surface and $f : M \to M$ is a C^3 diffeomorphism with a saddle fixed point p. Let μ, λ be the eigenvalues of $Df(p)$ with

$$0 < |\lambda| < 1 < |\mu|. \tag{2.1}$$

Suppose moreover that f is C^3 linearizable in a neighborhood $U(p)$ of p in M. Then there exists a C^3 coordinate (x, y) on $U(p)$ satisfying the following condition:

$$f(x, y) = (\mu x, \lambda y) \tag{2.2}$$

for any $(x, y) \in U(p)$. In particular, this implies that $p = (0, 0)$, $W^u_{\mathrm{loc}}(p) := \{(x, y) \in U(p);\ y = 0\} \subset W^u(p)$ and $W^s_{\mathrm{loc}}(p) := \{(x, y) \in U(p);\ x = 0\} \subset W^s(p)$.

Let $\mathcal{C}$ be the subspace of $\mathrm{Diff}^3(M)$ consisting of elements $f \in \mathrm{Diff}^3(M)$ satisfying the following conditions (C1)–(C3).

(C1) f has a saddle periodic point p.

(C2) There exists a homoclinic cubic tangency q associated with p.

(C3) f satisfies the adaptable conditions in the sense of Section 2.8 with respect to p, q.

Note that $\mathcal{C}$ is a codimension two submanifold of $\mathrm{Diff}^3(M)$.

Let q be a cubic tangency of $W^u(p)$ and $W^s(p)$. We assume that q is contained in $W^u_{\mathrm{loc}}(p) \subset U(p)$ if necessary replacing q by $f^{-n}(q)$ with sufficiently large $n \in \mathbb{N}$. For the point q, there exists $m_0 \in \mathbb{N}$ such that $r := f^{m_0}(q) \in W^s_{\mathrm{loc}}(p) \subset U(p)$. Then one can rearrange the linearizing coordinate on $U(p)$ so that $q = (1, 0)$, $r = (0, 1)$. Moreover, we may suppose that

$$U(p) = [-2, 2] \times [-2, 2],\ W^u_{\mathrm{loc}}(p) = [-2, 2] \times \{0\},\ W^s_{\mathrm{loc}}(p) = \{0\} \times [-2, 2].$$

Let $U(q)$, $U(r)$ be sufficiently small neighborhoods of q, r in $U(p)$ respectively. Then the component $L^s(q)$ of $W^s(p) \cap U(q)$ containing q is represented as

$$L^s(q) = \{(x + 1, y) \in U(q);\ y = v(x)\},$$

where v is a C^3 function satisfying

$$v(0) = v'(0) = v''(0) = 0 \quad \text{and} \quad v'''(0) \neq 0. \tag{2.3}$$

Similarly, the component $L^u(r)$ of $W^u(p) \cap U(r)$ containing r is represented as

$$L^u(r) = \{(x, y + 1) \in U(r);\ x = w(y)\},$$

where w is a C^3 function satisfying

$$w(0) = w'(0) = w''(0) = 0 \quad \text{and} \quad w'''(0) \neq 0, \tag{2.4}$$

see Figure 2.2

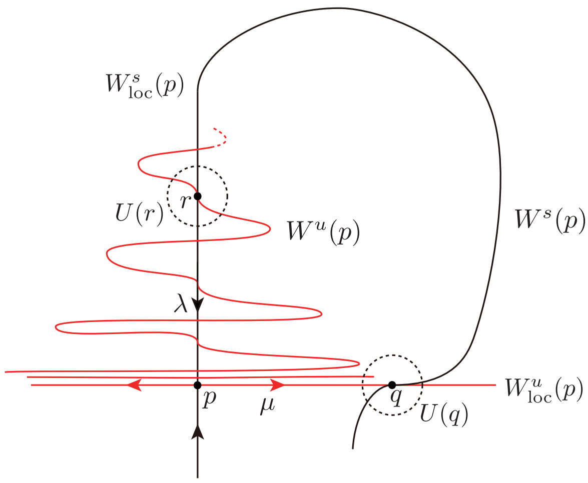

Figure 2.2: q and r are homoclinic cubic tangencies associated with p.

Recall that $q = (1, 0)$, $r = (0, 1)$ are cubic tangencies between $W^s(p)$ and $W^u(p)$ and $f^{m_0}(q) = r$ for some $m_0 \in \mathbb{N}$. We set $f^{m_0} = \varphi$ for short. By (2.3) and (2.4), φ is represented in $U(q)$ as follows for some constants a, b, c, d, e.

$$\varphi(x + 1, y) = (ay + bxy + cx^3 + H_1(x + 1, y), 1 + dx + ey + H_2(x + 1, y)), \tag{2.5}$$

where H_1, H_2 are C^3 functions satisfying the following conditions.

$$\begin{aligned} H_1(1,0) &= \partial_x H_1(1,0) = \partial_y H_1(1,0) = \partial_{xx} H_1(1,0) = \partial_{xy} H_1(1,0) \\ &= \partial_{xxx} H_1(1,0) = 0, \\ H_2(1,0) &= \partial_x H_2(1,0) = \partial_y H_2(1,0) = 0. \end{aligned} \tag{2.6}$$

Since φ is a diffeomorphism,

$$a, d \neq 0.$$

The fact that q is a cubic tangency implies

$$c \neq 0.$$

Here we put the following extra open condition.

$$b \neq 0. \tag{2.7}$$

By (2.5) and (2.6), the Jacobian matrix of φ at $(x+1, y)$ is given as follows.

$$\begin{aligned} D\varphi(x+1,y) &= \begin{bmatrix} by + 3cx^2 + \partial_x H_1(x+1,y) & a + bx + \partial_y H_1(x+1,y) \\ d + \partial_x H_2(x+1,y) & e + \partial_y H_2(x+1,y) \end{bmatrix} \\ &= \begin{bmatrix} by + 3cx^2 + o(x^2) + o(y) + O(xy) & a + bx + o(x) + O(y) \\ d + O(x) + O(y) & e + O(x) + O(y) \end{bmatrix}. \end{aligned} \tag{2.8}$$

Here we only consider the case satisfying the following condition, which belongs to Case II_{++} in Section 2.8.

$$0 < \lambda < 1, \mu > 1, a > 0, b < 0, c > 0, d < 0. \tag{2.9}$$

See Figure 2.3 for the situation of $W^u_{\mathrm{loc}}(p)$ and $W^s_{\mathrm{loc}}(p)$ in the case of (2.9). Note that (2.9) implies the extra condition (2.7).

One can set $\mu = 1 + \varepsilon$ for some $\varepsilon > 0$. We only consider the case that ε is sufficiently small.

Consider the rectangle $R_\varepsilon = [1+\varepsilon, (1+\varepsilon)^3] \times [0, \varepsilon^3]$ in $U(q)$. By (2.5),

$$\begin{aligned} \varphi(1+\varepsilon, 0) &= (c\varepsilon^3 + o(\varepsilon^3), 1 + d\varepsilon + o(\varepsilon)), \\ \varphi(1+\varepsilon, \varepsilon^3) &= ((a+c)\varepsilon^3 + o(\varepsilon^3), 1 + d\varepsilon + o(\varepsilon)), \\ \varphi((1+\varepsilon)^3, 0) &= (27c\varepsilon^3 + o(\varepsilon^3), 1 + 3d\varepsilon + o(\varepsilon)), \\ \varphi((1+\varepsilon)^3, \varepsilon^3) &= ((a+27c)\varepsilon^3 + o(\varepsilon^3), 1 + 3d\varepsilon + o(\varepsilon)). \end{aligned} \tag{2.10}$$

Let $\mathrm{pr}_x : U(p) \to W^u_{\mathrm{loc}}(p)$ and $\mathrm{pr}_y : U(p) \to W^s_{\mathrm{loc}}(p)$ be the orthogonal projections with respect to the linearizing coordinate on $U(p)$. Then there exist constants τ_0, τ_1 with $0 < \tau_0 < \tau_1$ independent of ε and satisfying

$$\mathrm{pr}_x(\varphi(R_\varepsilon)) \subset [\tau_0\varepsilon^3, \tau_1\varepsilon^3]. \tag{2.11}$$

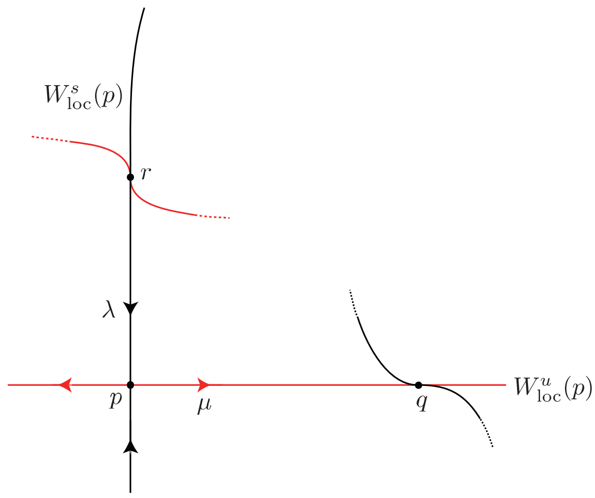

Figure 2.3: The case of II_{++}.

Since $d < 0$ by (2.9), it follows from (2.10) that

$$\mathrm{pr}_y(\varphi(R_\varepsilon)) \subset [1 + 3.5d\varepsilon, 1 + 0.5d\varepsilon] \subset [1 + 4d\varepsilon, 1]. \tag{2.12}$$

For any $\boldsymbol{x} \in R_\varepsilon$, let $u_0 = u_0(\boldsymbol{x})$ be a uniquely determined positive integer such that $f^i(\varphi(\boldsymbol{x})) \in U(p)$ for $i = 1, \ldots, u_0$ and $\mathrm{pr}_x(f^{u_0}(\varphi(\boldsymbol{x}))) \subset ((1+\varepsilon)^2, (1+\varepsilon)^3]$. Since $\mathrm{pr}_x(f^{u_0}(\varphi(\boldsymbol{x}))) = \mu^{u_0}\mathrm{pr}_x(\varphi(\boldsymbol{x}))$,

$$1 < (1+\varepsilon)^2 < \mu^{u_0}\mathrm{pr}_x(\varphi(\boldsymbol{x})) < \tau_1\mu^{u_0}\varepsilon^3.$$

Since $\mathrm{pr}_y(\varphi(\boldsymbol{x})) < 1$ by (2.12), it follows that

$$\mathrm{pr}_y(f^{u_0}(\varphi(\boldsymbol{x}))) = \lambda^{u_0}\mathrm{pr}_y(\varphi(\boldsymbol{x})) < \lambda^{u_0}.$$

Consider the following conditions for $\varepsilon > 0$:

$$\tau_1 < \varepsilon^{-1} \quad \text{and} \quad (1+\varepsilon)^{\frac{3}{2}} = \mu^{\frac{3}{2}} < \lambda^{-1}. \tag{2.13}$$

If these conditions are satisfied, then the following inequalities

$$1 < 1 + \varepsilon < \mu^{u_0}\mathrm{pr}_x(\varphi(\boldsymbol{x})) < \tau_1\mu^{u_0}\varepsilon^3 < \mu^{u_0}\varepsilon^2 \tag{2.14}$$

hold. This implies that

$$\mathrm{pr}_y(f^{u_0}(\varphi(\boldsymbol{x}))) = \lambda^{u_0}\mathrm{pr}_y(\varphi(\boldsymbol{x})) < \lambda^{u_0} < \mu^{-\frac{3}{2}u_0} < \varepsilon^3.$$

Thus the positive integer $u_0(\boldsymbol{x})$ satisfies

$$f^{u_0(\boldsymbol{x})}(\varphi(\boldsymbol{x})) \in R_\varepsilon \tag{2.15}$$

for all $\boldsymbol{x} \in R_\varepsilon$. See Figure 2.4.

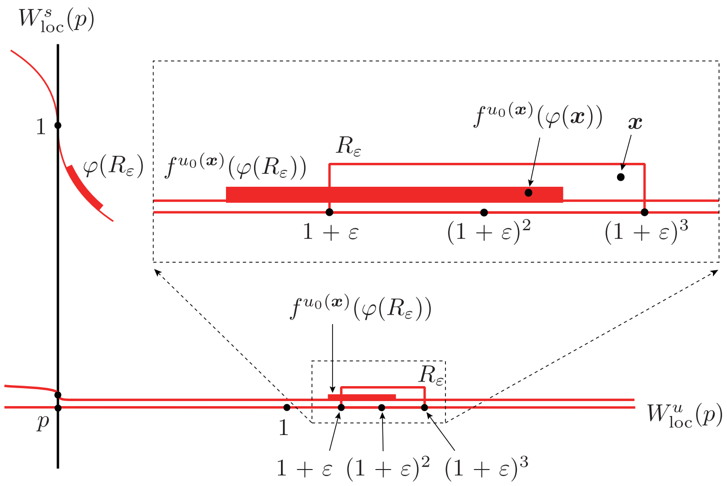

Figure 2.4: The rectangles R_ε and $f^{u_0(\boldsymbol{x})}(\varphi(R_\varepsilon))$ for $\boldsymbol{x} \in R_\varepsilon$.

2.3 Sequence of Rectangles

Let $f : M \to M$ be a C^3 diffeomorphism given in Section 2.2. In particular, f satisfies the linearizing condition (2.2) on $U(p)$. As is seen in Subsection 1.4, $W^u(p)$ and $W^s(p)$ have a transverse intersection point other than p. Let δ^u be a segment in $W^u_{\rm loc}(p)$ with $\mathrm{Int}\delta^u \supset \{p, q\}$. Then, by Inclination Lemma (Theorem 1.7), there exists a sequence $\{\alpha^u_n\}_{n=0}^\infty$ of arcs in $W^u(p)$ C^3 converging to δ^u and satisfying the following conditions:

- α^u_0 meets $W^s_{\rm loc}(p)$ transversely in a single point $\boldsymbol{z}_0 = (0, z_0)$.
- Each α^u_n contains $f^n(\boldsymbol{z}_0) = (0, z_0\lambda^n)$, and the intersection $\tilde{\alpha}^u_n = \alpha^u_n \cap U(q)$ is an arc meeting $L^s(q)$ transversely in a single point c_n for any sufficiently large $n > 0$.

See Figure 2.5. Note that α^u_0 is represented by the graph of a C^3-function $y_0 : \delta^u \to \mathbb{R}_+$,

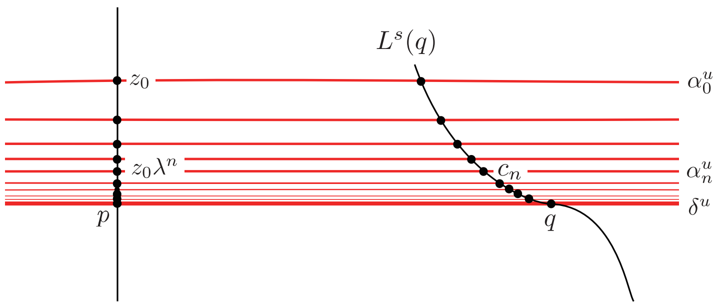

Figure 2.5: A sequence $\{\alpha^u_n\}_{n=0}^\infty$ C^3-converging to δ^u.

that is, $\alpha^u_0 = \{(x, y_0(x)) ; x \in \delta^u\}$. Then each α^u_n is represented by the graph of the function $y_n : \delta^u \to \mathbb{R}_+$ with

$$y_n(x) = \lambda^n y_0(\mu^{-n}x) \quad \text{for} \quad x \in \delta^u. \tag{2.16}$$

We parametrise $\tilde{\alpha}_n^u$ in $[(1+\varepsilon)^{-3}, (1+\varepsilon)^3]$ by $\alpha_n(t) = (t+1, \tilde{y}_n(t))$ with $(1+\varepsilon)^{-3} - 1 \le t \le (1+\varepsilon)^3 - 1$, where $\tilde{y}_n(t) = y_n(t+1)$. By (2.5) and (2.8),

$$\varphi(\alpha_n(t)) = (a\tilde{y}_n(t) + bt\tilde{y}_n(t) + ct^3 + \text{h.o.t.}, 1 + dt + e\tilde{y}_n(t) + \text{h.o.t.}), \tag{2.17}$$

$$\begin{aligned} D\varphi(\alpha_n(t))(\alpha_n'(t)) = (a\tilde{y}_n'(t) + b\tilde{y}_n(t) + bt\tilde{y}_n'(t) + 3ct^2 + \text{h.o.t.}, \\ d + e\tilde{y}_n'(t) + \text{h.o.t.}), \end{aligned} \tag{2.18}$$

where the primes represent the derivative on t and 'h.o.t.' denotes the sum of the higher order terms on t. By (2.16),

$$|\tilde{y}_n'(t)| = |y_n'(t+1)| = \lambda^n \mu^{-n} |y_0'(\mu^{-n}(t+1))|.$$

Suppose that σ is the maximum of $|y_0'(x)|$ on δ^u. Then

$$|\tilde{y}_n'(t)| = |y_n'(t+1)| = \lambda^n \mu^{-n} |y_0'(\mu^{-n}(t+1))| \le \lambda^n \mu^{-n} \sigma$$

for any $n \in \mathbb{N}$. This implies that

$$|\tilde{y}_n'(t)| \precsim \lambda^n \mu^{-n}. \tag{2.19}$$

Suppose that $d\varphi_{\alpha_n(t)}(\alpha_n'(t))$ is vertical at $t = t_n$. Then $\lim_{n\to\infty} t_n = 0$ and, by (2.18),

$$b\tilde{y}_n(t_n) + (a + bt_n)\tilde{y}_n'(t_n) \approx -3ct_n^2.$$

Since $\tilde{y}_n(t) \approx \lambda^n z_0$ and $|\tilde{y}_n'(t)| \precsim \lambda^n \mu^{-n}$, this condition is equivalent to

$$3ct_n^2 \approx -b\tilde{y}_n(t_n) \approx -b\lambda^n z_0. \tag{2.20}$$

It follows that, for all sufficiently large n, $d\varphi_{\alpha_n(t)}(\alpha_n'(t))$ is vertical at two points $t_{n,\pm}$ with

$$t_{n,\pm} \approx \pm\sqrt{\frac{-bz_0}{3c}}\lambda^{\frac{n}{2}}. \tag{2.21}$$

Let $\tilde{t}_{n,\pm}$ be the elements of $[(1+\varepsilon)^{-3} - 1, (1+\varepsilon)^3 - 1]$ with $\tilde{t}_{n,-} < t_{n,-}$, $t_{n,+} < \tilde{t}_{n,+}$ such that $\varphi(\alpha_n(\tilde{t}_{n,\pm}))$ is the intersection point of $\varphi(\alpha_n(t))$ and the vertical line $L_{n,\pm}$ tangent to $\varphi(\alpha_n(t))$ at $\varphi(\alpha_n(t_{n,\mp}))$. Let S_n be the smallest orthogonal rectangle in $U(r)$ containing the four points $\varphi(\alpha_n(\tilde{t}_{n,-}))$, $\varphi(\alpha_n(t_{n,-}))$, $\varphi(\alpha_n(t_{n,+}))$, $\varphi(\alpha_n(\tilde{t}_{n,+}))$. See Figure 2.6.

Now we will estimate the size of S_n. Let D_n be the distance between S_n and $W^s_{\text{loc}}(p)$. Then

$$\begin{aligned} D_n &\approx a\tilde{y}_n(t_{n,+}) + bt_{n,+}\tilde{y}_n(t_{n,+}) + ct_{n,+}^3 \\ &\approx az_0\lambda^n + bz_0\sqrt{\frac{-bz_0}{3c}}\lambda^{\frac{3}{2}n} - \frac{bz_0}{3}\sqrt{\frac{-bz_0}{3c}}\lambda^{\frac{3}{2}n} \sim \lambda^n. \end{aligned} \tag{2.22}$$

By (2.5), the width $W_{0,n}$ of S_n is represented as

$$\begin{aligned} W_{0,n} &\approx (a\tilde{y}_n(t_{n,-}) + bt_{n,-}\tilde{y}_n(t_{n,-}) + ct_{n,-}^3) - (a\tilde{y}_n(t_{n,+}) + bt_{n,+}\tilde{y}_n(t_{n,+}) + ct_{n,+}^3) \\ &= a(\tilde{y}_n(t_{n,-}) - \tilde{y}_n(t_{n,+})) + b(t_{n,-}\tilde{y}_n(t_{n,-}) - t_{n,+}\tilde{y}_n(t_{n,+})) + c(t_{n,-}^3 - t_{n,+}^3). \end{aligned}$$

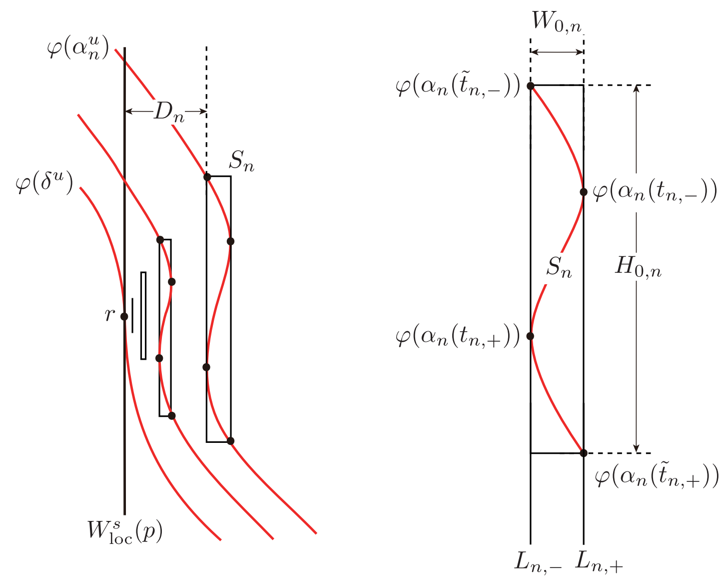

Figure 2.6: The smallest orthogonal rectangle S_n.

It follows from Mean Value Theorem together with (2.19) that

$$|\tilde{y}_n(t_{n,-}) - \tilde{y}_n(t_{n,+})| \precsim \lambda^n \mu^{-n} |t_{n,-} - t_{n,+}| \sim \lambda^{\frac{3}{2}n} \mu^{-n}.$$

Moreover, by (2.21), we have

$$\begin{aligned} c(t_{n,-}^3 - t_{n,+}^3) &\approx c\left(t_{n,-}\left(\frac{-b\tilde{y}_n(t_{n,-})}{3c}\right) - t_{n,+}\left(\frac{-b\tilde{y}_n(t_{n,+})}{3c}\right)\right) \\ &= -\frac{b}{3}(t_{n,-}\tilde{y}_n(t_{n,-}) - t_{n,+}\tilde{y}_n(t_{n,+})). \end{aligned}$$

Since

$$\begin{aligned} t_{n,-}\tilde{y}_n(t_{n,-}) - t_{n,+}\tilde{y}_n(t_{n,+}) &= (t_{n,-} - t_{n,+})\tilde{y}_n(t_{n,-}) + t_{n,+}(\tilde{y}_n(t_{n,-}) - y_n(t_{n,+})) \\ &\approx -\sqrt{\frac{-bz_0}{3c}}\lambda^{\frac{n}{2}} \cdot z_0\lambda^n + O\left(\lambda^{\frac{n}{2}} \cdot \lambda^{\frac{3}{2}n}\mu^{-n}\right) \sim -\lambda^{\frac{3}{2}n}, \end{aligned}$$

we have

$$W_{0,n} \approx O(\lambda^{\frac{3}{2}n}\mu^{-n}) + \frac{2b}{3}(t_{n,-}\tilde{y}_n(t_{n,-}) - t_{n,+}\tilde{y}_n(t_{n,+})) \sim \lambda^{\frac{3}{2}n}. \tag{2.23}$$

Next we estimate the height $H_{0,n}$ of S_n. For that, we estimate $W_{0,n}$ again by using $\tilde{t}_{n,+}$ and $t_{n,+}$ instead of $t_{n,-}$ and $t_{n,+}$. Since $\tilde{t}_{n,+} > t_{n,+}$, one can set $\tilde{t}_{n,+} = t_{n,+} + \rho_n\lambda^{\frac{n}{2}}$ for some $\rho_n > 0$.

$$\begin{aligned} W_{0,n} &\approx a(\tilde{y}_n(\tilde{t}_{n,+}) - \tilde{y}_n(t_{n,+})) + b(\tilde{t}_{n,+}\tilde{y}_n(\tilde{t}_{n,+}) - t_{n,+}\tilde{y}_n(t_{n,+})) + c(\tilde{t}_{n,+}^3 - \tilde{t}_{n,-}^3) \\ &= (a + b\tilde{t}_{n,+})(\tilde{y}_n(\tilde{t}_{n,+}) - \tilde{y}_n(t_{n,+})) + b(\tilde{t}_{n,+} - t_{n,+})\tilde{y}_n(t_{n,+}) + c(\tilde{t}_{n,+}^3 - t_{n,-}^3). \end{aligned}$$

Again by Mean Value Theorem together with (2.19),

$$|\tilde{y}_n(\tilde{t}_{n,+}) - \tilde{y}_n(t_{n,+})| \precsim \lambda^n\mu^{-n} \cdot \rho_n\lambda^{\frac{n}{2}} = \rho_n\lambda^{\frac{3}{2}n}\mu^{-n}.$$

Moreover, we have

$$(\tilde{t}_{n,+} - t_{n,+})\tilde{y}_n(t_{n,+}) \sim \rho_n\lambda^{\frac{n}{2}} \cdot \lambda^n = \rho_n\lambda^{\frac{3}{2}n}$$

and

$$\begin{aligned}\tilde{t}^3_{n,+} - t^3_{n,+} &= 3\rho_n^2\lambda^n t_{n,+} + 3\rho_n\lambda^{\frac{n}{2}}t^2_{n,+} + \rho_n^3\lambda^{\frac{3}{2}n} \\ &\approx \left(3\rho_n\sqrt{\frac{-bz_0}{3c}} - \frac{3bz_0}{c} + \rho_n^2\right)\rho_n\lambda^{\frac{3}{2}n}.\end{aligned}$$

This shows that

$$W_{0,n} \sim \left(a\mu^{-n} + bz_0 + 3\rho_n\sqrt{\frac{-bz_0}{3c}} - \frac{3bz_0}{c} + \rho_n^2\right)\rho_n\lambda^{\frac{3}{2}n}.$$

Since $W_{0,n} \sim \lambda^{\frac{3}{2}n}$, it follows that $\rho_n \sim 1$ and hence $\tilde{t}_{n,+} \sim \lambda^{\frac{n}{2}}$. Similarly $-\tilde{t}_{n,-} \sim \lambda^{\frac{n}{2}}$. This implies that

$$|\tilde{t}_{n,\pm}| \sim \lambda^{\frac{n}{2}}. \tag{2.24}$$

Therefore we have

$$\begin{aligned}H_{0,n} &= (1 + d\tilde{t}_{n,-} + e\tilde{y}_n(\tilde{t}_{n,-})) - (1 + d\tilde{t}_{n,+} + e\tilde{y}_n(\tilde{t}_{n,+})) \\ &= d(\tilde{t}_{n,-} - \tilde{t}_{n,+}) + e(\tilde{y}_n(\tilde{t}_{n,-}) - \tilde{y}_n(\tilde{t}_{n,+})) \sim \lambda^{\frac{n}{2}} + O(\lambda^{\frac{3}{2}n}\mu^{-n}) \sim \lambda^{\frac{n}{2}}.\end{aligned} \tag{2.25}$$

In particular, $\{S_n\}$ is a sequence of rectangles converging to the cubic tangency r.

2.4 Slope Lemma

Let $\boldsymbol{v} = \begin{bmatrix} u \\ v \end{bmatrix} \in T_{\boldsymbol{x}}(M)$ be a tangent vector at $\boldsymbol{x} \in U(p)$ with $u \neq 0$. Then we say that $|vu^{-1}|$ is the (absolute) *slope* of $\boldsymbol{v}$ and denote it by Slope($\boldsymbol{v}$).

Consider any tangent vector $\boldsymbol{v}_0 = \begin{bmatrix} 1 \\ \delta \end{bmatrix} \in T_{\boldsymbol{x}}(M)$ at $\boldsymbol{x} = (x+1, y) \in R_\varepsilon$ with $|\delta| \leq \varepsilon^{\frac{5}{2}}$. We set $\boldsymbol{v}_0' = D\varphi(x+1, y)(\boldsymbol{v}_0)$ and $\boldsymbol{v}_1 = Df^{u_0}(\varphi(x+1, y))(\boldsymbol{v}_0')$. By (2.8),

$$\mathrm{Slope}(\boldsymbol{v}_0') \approx \frac{|d + e\delta|}{|3cx^2 + a\delta|}.$$

Since $\varepsilon \leq x$ and $|\delta| \leq \varepsilon^{\frac{5}{2}}$,

$$\begin{aligned}\mathrm{Slope}(\boldsymbol{v}_0') &\approx \frac{|d + e\delta|}{|3cx^2 + a\delta|} \leq \frac{|d| + |e\delta|}{|3cx^2| - |a\delta|} \leq \frac{|d| + |e\varepsilon^{\frac{5}{2}}|}{|3c\varepsilon^2| - |a\varepsilon^{\frac{5}{2}}|} \\ &= \frac{|d| + |e\varepsilon^{\frac{5}{2}}|}{|3c| - |a\varepsilon^{\frac{1}{2}}|}\varepsilon^{-2} = \frac{|d| + |e\varepsilon^{\frac{5}{2}}|}{|3c| - |a\varepsilon^{\frac{1}{2}}|}\varepsilon^{\frac{1}{2}} \cdot \varepsilon^{-\frac{5}{2}}.\end{aligned}$$

By taking $\varepsilon_1 > 0$ sufficiently small, for any $0 < \varepsilon \leq \varepsilon_1$, we have

$$\mathrm{Slope}(\boldsymbol{v}_0') \leq 2\frac{|d| + |e\varepsilon^{\frac{5}{2}}|}{|3c| - |a\varepsilon^{\frac{1}{2}}|}\varepsilon^{\frac{1}{2}} \cdot \varepsilon^{-\frac{5}{2}} \leq \frac{|3d|}{|2c|}\varepsilon^{\frac{1}{2}} \cdot \varepsilon^{-\frac{5}{2}} \leq 1 \cdot \varepsilon^{-\frac{5}{2}}.$$

Then,by (2.13) and (2.14), we have

$$\mathrm{Slope}(\boldsymbol{v}_1) = \mathrm{Slope}(\boldsymbol{v}_0')\lambda^{u_0}\mu^{-u_0} \leq \varepsilon^{-\frac{5}{2}}\lambda^{u_0}\mu^{-u_0} \leq \varepsilon^{-\frac{5}{2}}\mu^{-\frac{5}{2}u_0} \leq \varepsilon^{-\frac{5}{2}}\varepsilon^{5} = \varepsilon^{\frac{5}{2}}.$$

Thus we get the following lemma. See Figure 2.7.

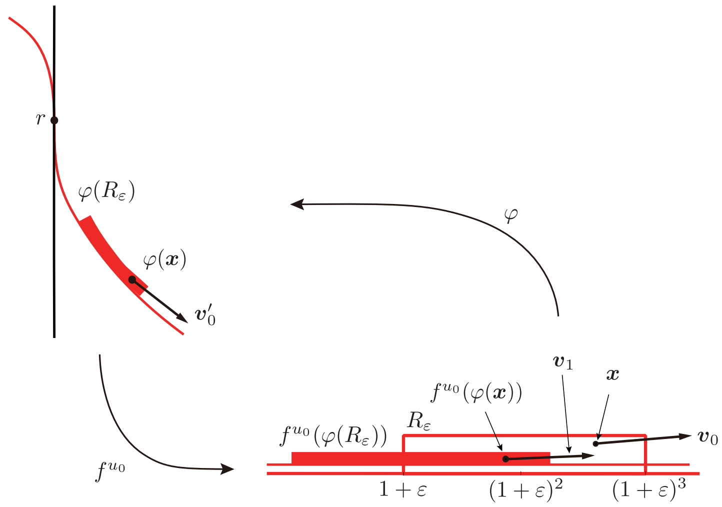

Figure 2.7: The tangent vectors $\boldsymbol{v}_0$, $\boldsymbol{v}_0'$ and $\boldsymbol{v}_1$.

Lemma 2.3 (Slope Lemma I)**.** *Suppose that f satisfies the conditions* (2.13)*. Then there exists a constant $\varepsilon_1 > 0$ such that, if $\varepsilon \in (0, \varepsilon_1]$, then*

$$\mathrm{Slope}(\boldsymbol{v}_0') \leq \varepsilon^{-\frac{5}{2}} \quad \textit{and} \quad \mathrm{Slope}(\boldsymbol{v}_1) \leq \varepsilon^{\frac{5}{2}} \tag{2.26}$$

for any tangent vector $\boldsymbol{v}_0 \in T_{\boldsymbol{x}}(M)$ at $\boldsymbol{x} = (x+1, y) \in R_\varepsilon$ with $\mathrm{Slope}(\boldsymbol{v}_0) \leq \varepsilon^{\frac{5}{2}}$.

Fix a sufficiently small $s > 0$ and set $\mathrm{pr}_x(S_n) = [s_n^-, s_n^+]$ for $n \in \mathbb{N}$. If n is sufficiently large, then $[s_n^-, s_n^+] \subset (0, s]$. Let $\beta_n^u(s)$ be the component of $\varphi(\alpha_n^u) \cap \mathrm{pr}_x^{-1}((0, s])$ containing $\varphi(\alpha_n([\tilde{t}_{n,-}, \tilde{t}_{n,+}]))$. For any $\boldsymbol{x} \in \beta_n^u(s)$, let $j_n(\boldsymbol{x})$ be a positive integer such that $f^j(\boldsymbol{x}) \in U(p)$ for $j = 1, \ldots, j_n(\boldsymbol{x})$ and $\mathrm{pr}_x(f^{j_n(\boldsymbol{x})}(\boldsymbol{x})) \in [1+\varepsilon, (1+\varepsilon)^3]$. For any $\varepsilon > 0$, one can take s so that $\mathrm{pr}_x(f^{j_n(\boldsymbol{x})}(\boldsymbol{x})) \in R_\varepsilon$ for any $\boldsymbol{x} \in \beta_n^u(s)$. Let $\boldsymbol{v}(\boldsymbol{x})$ be a unit vector tangent to $\beta_n^u(s)$ at $\boldsymbol{x}$.

The following result is applied to f_1 in the proof of Theorem 2.1.

Lemma 2.4 (Slope Lemma II)**.** *Let ε_1 be the constant given in Lemma 2.3. For any $\varepsilon \in (0, \varepsilon_1]$, there exist $s > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\mathrm{Slope}(Df^{j_n(\boldsymbol{x})}(\boldsymbol{x})(\boldsymbol{v}(\boldsymbol{x}))) < \varepsilon^{\frac{5}{2}}$$

if $n \geq n_0$ and $\boldsymbol{x} \in \beta_n^u(s) \setminus S_n$.

Proof. We only consider the case where $\boldsymbol{x}$ is an element of $\beta_n^u(s) \setminus S_n$ with $\mathrm{pr}_x(\boldsymbol{x}) \geq s_n^+$. Then $t \geq \tilde{t}_{n,+}$ holds if $\varphi(\alpha_n(t)) = \boldsymbol{x}$. The proof in the case of $\mathrm{pr}_x(\boldsymbol{x}) \leq s_n^-$ is done quite similarly. Since $\rho_n \sim 1$ and $\tilde{t}_{n,+} = t_{n,+} + \rho_n \lambda^{\frac{n}{2}}$, $t - t_{n,+} \geq \tilde{t}_{n,+} - t_{n,+} \sim \lambda^{\frac{n}{2}}$. This implies that

$$t^2 - t_{n,+}^2 \sim t^2 \gtrsim \lambda^n. \tag{2.27}$$

In fact, if $t - t_{n,+} \geq \frac{t}{2}$, then $t^2 - t_{n,+}^2 = (t - t_{n,+})(t + t_{n,+}) > \frac{t^2}{2}$ and hence (2.27) holds. On the other hand, if $t - t_{n,+} \leq \frac{t}{2}$, then $t \leq 2t_{n,+}$ and so $t \sim \lambda^{\frac{n}{2}}$. It follows that $t + t_{n,+} \sim \lambda^{\frac{n}{2}}$ and $t - t_{n,+} \sim \lambda^{\frac{n}{2}}$. Then $t^2 - t_{n,+}^2 \sim \lambda^n \sim t^2$. Thus (2.27) holds.

We set $\xi_n(t) = \mathrm{pr}_x(\boldsymbol{x}) = \mathrm{pr}_x(\varphi(\alpha_n(t)))$. By (2.17),

$$\begin{aligned} \xi_n(t) &= a\tilde{y}_n(t) + bt\tilde{y}_n(t) + ct^3 + \text{h.o.t.}, \\ \xi_n'(t) &= a\tilde{y}_n'(t) + b\tilde{y}_n(t) + bt\tilde{y}_n'(t) + 3ct^2 + \text{h.o.t.}. \end{aligned} \tag{2.28}$$

From the definition of $j_n(\boldsymbol{x})$,

$$\mu^{j_n(\boldsymbol{x})}\xi_n(t) = \mu^{j_n(\boldsymbol{x})}\mathrm{pr}_x(\boldsymbol{x}) = \mathrm{pr}_x(f^{j_n(\boldsymbol{x})}(\boldsymbol{x})) \in [1+\varepsilon, (1+\varepsilon)^3].$$

This implies that $\mu^{j_n(\boldsymbol{x})}\xi_n(t) \sim 1$. We note that $\xi_n'(t_{n,+}) = 0$. By Mean Value Theorem, $\tilde{y}_n(t) - \tilde{y}_n(t_{n,+}) = \tilde{y}_n'(c)(t - t_{n,+})$ for some $t_{n,+} < c < t$. From this fact together with (2.16), (2.19), (2.27) and (2.28), we know that

$$\xi_n'(t) = \xi_n'(t) - \xi_n'(t_{n,+}) \sim t^2 - t_{n,+}^2 \sim t^2.$$

By (2.18), $\mathrm{Slope}(\boldsymbol{v}(\boldsymbol{x})) \sim t^{-2}$. Hence we have

$$\mathrm{Slope}(Df^{j_n(\boldsymbol{x})}(\boldsymbol{x})(\boldsymbol{v}(\boldsymbol{x}))) = \mathrm{Slope}(\boldsymbol{v}(\boldsymbol{x})) \cdot \frac{\lambda^{j_n(\boldsymbol{x})}}{\mu^{j_n(\boldsymbol{x})}} \sim t^{-2}\lambda^{j_n(\boldsymbol{x})}\xi_n(t). \tag{2.29}$$

Now we need to consider the following two cases.
Case 1. $ct^3 \leq a\tilde{y}_n(t)$. By (2.28), $\xi_n(t) \sim \lambda^n$. Since $t^{-2} \lesssim \lambda^{-n}$ by $t \gtrsim \lambda^{\frac{n}{2}}$, it follows from (2.29) that

$$\mathrm{Slope}(Df^{j_n(\boldsymbol{x})}(\boldsymbol{x})(\boldsymbol{v}(\boldsymbol{x}))) \lesssim \lambda^{-n}\lambda^{j_n(\boldsymbol{x})}\lambda^n = \lambda^{j_n(\boldsymbol{x})}.$$

Case 2. $ct^3 \geq a\tilde{y}_n(t)$. Again by (2.28), we have $\xi_n(t) \sim t^3$. Then, by (2.29),

$$\mathrm{Slope}(Df^{j_n(\boldsymbol{x})}(\boldsymbol{x})(\boldsymbol{v}(\boldsymbol{x}))) \sim t^{-2}\lambda^{j_n(\boldsymbol{x})}t^3 = t\lambda^{j_n(\boldsymbol{x})} \lesssim \lambda^{j_n(\boldsymbol{x})}.$$

Let $n_0(s)$ be the minimum positive integer with $s_{n_0(s)}^+ < s$. Since $n_0(s)$ goes to infinity as $s \to +0$, one can take $s = s(\varepsilon) > 0$ such that our desired inequality holds for any $\boldsymbol{x} \in \beta_n^u(s) \setminus S_n$. □

2.5 Sequence of rectangle-like boxes

Now we will define a sequence $\{B_{k,n}\}_{k=1}^{\infty}$ of rectangle-like boxes and estimate the sizes of them.

Recall that $\mathrm{pr}_x(S_n) = [s_n^-, s_n^+]$. Let i_n be the positive integer with $(1+\varepsilon)^2 < \mu^{i_n} s_n^+ \leq (1+\varepsilon)^3$. By (2.15), $f^{i_n}(S_n)$ is contained in R_ε for any sufficiently large n. We set $f^{i_n}(S_n) = B_{1,n} = B_1$ for short. Since $s_n^+ \sim \lambda^n$ by (2.22) and (2.23), we have

$$\mu^{i_n}\lambda^n \sim 1. \tag{2.30}$$

We denote the width and height of B_1 and the distance between B_1 and $W^u_{\mathrm{loc}}(p)$ by $W_{1,n} = W_1$, $H_{1,n} = H_1$ and $L_{1,n} = L_1$ respectively. It follows from (2.22), (2.23) and (2.25) that

$$W_{1,n} \sim \lambda^{\frac{3}{2}n}\mu^{i_n} \sim \lambda^{\frac{n}{2}}, \quad H_{1,n} \sim \lambda^{\frac{n}{2}+i_n}, \quad L_{1,n} \sim \lambda^{i_n}. \tag{2.31}$$

Note that, for any sufficiently large n, $H_1 \ll L_1 \ll W_1$. Consider a closed interval δ_1 in $W^u_{\mathrm{loc}}(p)$ which is a small neighborhood of $\mathrm{pr}_x(B_1)$.

Let $v_i^{(1)}$, $e_i^{(1)}$ $(i = 0, 1, 2, 3)$ be the vertices and edges of B_1 as illustrated in Figure 2.8 (a). We consider the image $\varphi(B_1)$. By Lemma 2.3, for $i = 0, 2$,

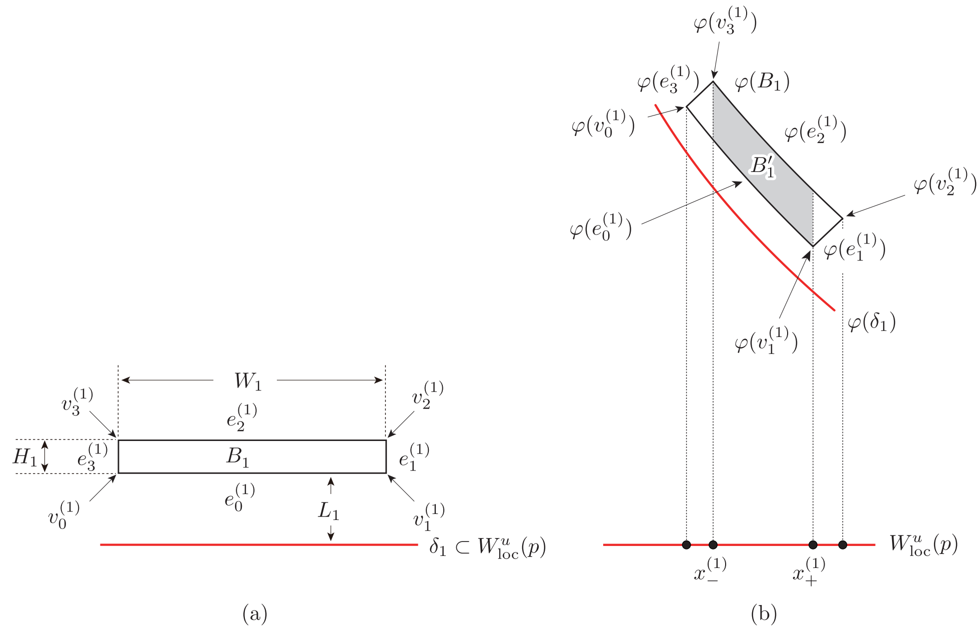

Figure 2.8: The rectangle B_1 and the parallelogram-like box B_1'.

$$\mathrm{diam}(\mathrm{pr}_x(\varphi(e_i^{(1)}))) \gtrsim \varepsilon^{\frac{5}{2}} W_1 \sim \varepsilon^{\frac{5}{2}}\lambda^{\frac{n}{2}}.$$

On the other hand, for $i = 1, 3$,

$$\mathrm{diam}(\mathrm{pr}_x(\varphi(e_i^{(1)}))) \lesssim H_1 \sim \lambda^{\frac{n}{2}+i_n}.$$

Since $\lambda^{i_n}\varepsilon^{-\frac{5}{2}}$ can be supposed to be arbitrarily small for all sufficiently large n,

$$
\begin{aligned}
x_+^{(1)} - x_-^{(1)} &\gtrsim \varepsilon^{\frac{5}{2}} W_1 - O(\lambda^{\frac{n}{2}+i_n}) \sim \varepsilon^{\frac{5}{2}}\lambda^{\frac{n}{2}} - O(\lambda^{\frac{n}{2}+i_n}) \\
&= \varepsilon^{\frac{5}{2}}\lambda^{\frac{n}{2}}\left(1 - \frac{O(\lambda^{i_n})}{\varepsilon^{\frac{5}{2}}}\right) \sim \varepsilon^{\frac{5}{2}}\lambda^{\frac{n}{2}}.
\end{aligned} \tag{2.32}
$$

where $x_+^{(1)} = \mathrm{pr}_x(\varphi(v_1^{(1)}))$ and $x_-^{(1)} = \mathrm{pr}_x(\varphi(v_3^{(1)}))$, see Figure 2.8 (b). Let B_1' be the intersection $\mathrm{pr}_x^{-1}\big([x_-^{(1)}, x_+^{(1)}]\big) \cap \varphi(B_1)$. Any compact region in $U(p)$ like B_1' is called a *parallelogram-like box*.

Let u_1 be the positive integer with $(1+\varepsilon)^2 < \mu^{u_1}x_+^{(1)} \leq (1+\varepsilon)^3$. By (2.15), $f^{u_1}(B_1')$ is contained in R_ε for any sufficiently large n. We denote $f^{u_1}(B_1')$ by B_2. We call that any compact region in $U(p)$ like B_2 is a *rectangle-like box*.

Let B be either a parallelogram-like or rectangle-like box. The *horizontal width* of B is the diameter of the interval $\mathrm{pr}_x(B)$. The *vertical height* of B is the maximum of the lengths of $\eta(x_0)$ with $x_0 \in \mathrm{pr}_x(B)$, where $\eta(x_0)$ is the intersection of B and the vertical line $x = x_0$. See Figure 2.9 in the case of $B = B_1'$. Suppose that B is a rectangle-like

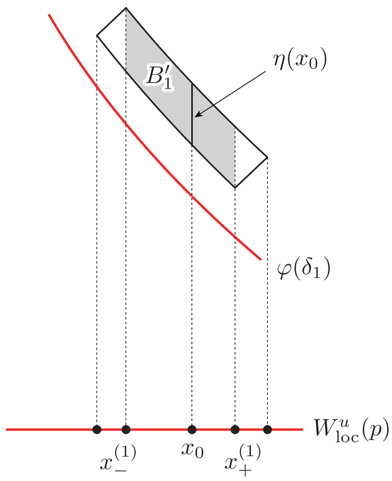

Figure 2.9: A vertical segment $\eta(x_0)$ connecting the opposite pair of edges of the parallelogram-like box B_1'.

box and δ is an almost horizontal arc in $U(q)$ with $B \cap \delta = \emptyset$ and $\mathrm{pr}_x(B) \subset \mathrm{pr}_x(\delta)$. Then the *vertical distance* between B and δ is the maximum of $\sigma(x_1)$ with $x_1 \in \mathrm{pr}_x(B)$, where $\sigma(x_1)$ is the length of the shortest segment in the vertical line $x = x_1$ connecting B with δ.

Let δ_2 be a sub-arc of $f^{u_1}(\varphi(\delta_1)) \subset W^u(p)$ such that $\mathrm{pr}_x(\delta_2)$ is a small neighborhood of $\mathrm{pr}_x(B_2)$ in $W^u_{\mathrm{loc}}(p)$. See Figure 2.10. We denote the horizontal width and vertical height of B_2 and the vertical distance between B_2 and δ_2 by W_2, H_2 and L_2 respectively. By (2.14), (2.31) and (2.32),

$$
W_2 = (x_+^{(1)} - x_-^{(1)})\mu^{u_1} \gtrsim \varepsilon^{\frac{5}{2}}\lambda^{\frac{n}{2}}\mu^{u_1} \geq \varepsilon^{-\frac{1}{2}}\tau_1^{-1}\lambda^{\frac{n}{2}} \sim \varepsilon^{-\frac{1}{2}}W_1. \tag{2.33}
$$

For any x_0 with $x_-^{(1)} \le x_0 \le x_+^{(1)}$, $\eta(x_0)$ is a vertical segment connecting $\varphi(e_0^{(1)})$ with $\varphi(e_2^{(1)})$. By this fact together with (2.26), one can show the vertical height H_1' of B_1' satisfies $H_1' \precsim H_1\varepsilon^{-\frac{5}{2}}$. It follows from (2.13) and (2.14) that

$$H_2 = \lambda^{u_1} H_1' \precsim \mu^{-\frac{3}{2}u_1}\varepsilon^{-\frac{5}{2}} H_1 < (\varepsilon^2)^{\frac{3}{2}}\varepsilon^{-\frac{5}{2}} H_1 = \varepsilon^{\frac{1}{2}} H_1. \tag{2.34}$$

Let L_2 be the vertical distance between B_2 and δ_2. By using an argument similar to that for the estimation (2.34), we have

$$L_2 \precsim \varepsilon^{\frac{1}{2}} L_1. \tag{2.35}$$

The following lemma is obtained immediately from (2.33), (2.34) and (2.35).

Lemma 2.5. *Let $\varepsilon_1 > 0$ be the constant given in Lemma 2.3. Then there exists a constant $\varepsilon_0 \in (0, \varepsilon_1]$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, the inequalities*

$$W_2 \ge 10W_1, \quad H_2 \le 10^{-1}H_1 \quad \textit{and} \quad L_2 \le 10^{-1}L_1$$

hold.

If $\mu = 1+\varepsilon$ for an $\varepsilon \in (0, \varepsilon_0)$, then we say that f satisfies the *small expanding conditions* at p.

We repeat the process as above. Let B_2' be the subset of $\varphi(B_2)$ cobounded by the vertical lines $x = x_-^{(2)}$ and $x = x_+^{(2)}$ passing through two of the four vertices of $\varphi(B_2)$ and satisfying $[x_-^{(2)}, x_+^{(2)}] \subset \mathrm{Int}(\mathrm{pr}_x(\varphi(B_2)))$. Let u_2 be the positive integer with $(1+\varepsilon)^2 < \mu^{u_2} x_+^{(2)} \le (1+\varepsilon)^3$ and $f^{u_2}(B_2') \subset R_\varepsilon$ for sufficient large $n \in \mathbb{N}$. Set $B_3 = f^{u_2}(B_2')$. Let δ_3 be a sub-arc of $f^{u_2}(\varphi(\delta_2))$ such that $\mathrm{pr}_x(\delta_3)$ is a small neighborhood of $\mathrm{pr}_x(B_3)$ in $W^u_{\mathrm{loc}}(p)$. We denote the horizontal width and vertical height of B_3 and the vertical distance between B_3 and δ_3 by W_3, H_3 and L_3 respectively.

The objects B_k', u_k, B_{k+1}, δ_k, W_{k+1}, H_{k+1}, L_{k+1} $(k = 3, 4, 5, \dots)$ are defined inductively if

$$B_j \subset R_\varepsilon \tag{2.36}$$

for $j = 1, 2, \dots, k$.

The top and bottom sides of the rectangle B_1 are horizontal and $\gamma_1 = B_1 \cap W^u(p)$ consists of three proper arcs in B_1. By Slope Lemma I (Lemma 2.3), for $k = 2, 3, \cdots$, the top and bottom sides of the rectangle-like box B_k are almost horizontal and $\gamma_k = B_k \cap W^u(p)$ consists of three proper arcs in B_k. See Figure 2.12. Thus we have the following lemma.

Lemma 2.6. *Let $\varepsilon_0 > 0$ be the constant given in Lemma 2.5. For any $\varepsilon \in (0, \varepsilon_0]$, there exists the maximum integer $k_0 = k_0(\varepsilon, n)$ satisfying* (2.36). *Moreover,*

$$W_{k+1} \ge 10W_k, \quad H_{k+1} \le 10^{-1}H_k \quad \textit{and} \quad L_{k+1} \le 10^{-1}L_k \tag{2.37}$$

hold for any $k = 1, 2, \dots, k_0$.

See Figure 2.10 for the situation of Lemma 2.6. We note that, since $W_1 = W_{1,n} \sim \lambda^{\frac{n}{2}}$ by (2.31), $\lim_{n\to\infty} k_0(\varepsilon, n) = \infty$ for a fixed ε with $0 < \varepsilon \le \varepsilon_0$.

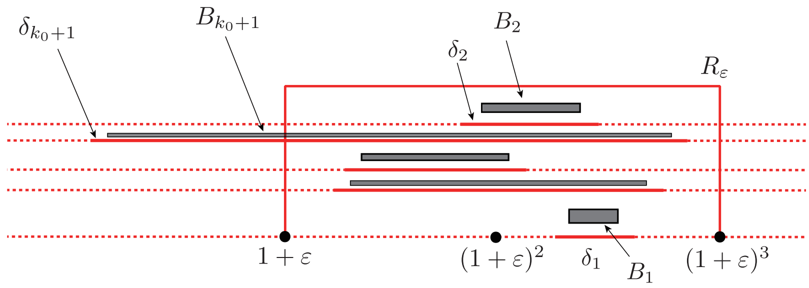

Figure 2.10: The pairs of the rectangle-like box B_k and the sub-arc δ_k of $W^u(p)$ for $k = 1, 2, \ldots, k_0 + 1$.

2.6 Intersection Lemma

Recall that $\mathcal{C}$ is the codimension two submanifold of $\mathrm{Diff}^3(M)$ defined in Section 2.2. Let f_0, f_1 be elements of $\mathcal{C}$ satisfying the conditions (A1)–(A3) in Theorem 2.1. In particular, $\varepsilon > 0$ is taken so that Slope Lemmas I and II (Lemmas 2.3 and 2.4) hold. Moreover, we suppose that f_0, f_1 satisfy the condition (2.9), which is one of the adaptable cases given in Section 2.8.

From now on, we set $f_0 = f$ and use the notations in Sections 2.2–2.5. Here the subscription '0' is omitted from the notations. For example, $\lambda_0 = \lambda$, $\mu_0 = \mu$, $p_0 = p$, $q_0 = q$ and so on. We also set $f_1 = \overline{f}$ and represent the notations for $\overline{f}$ by adding bars to the corresponding notations for f, e.g. $\overline{\lambda}$, $\overline{\mu}$, $\overline{p}$, $\overline{q}$, $\overline{m}_0$, $\overline{S}_n$, $\overline{B}_k$, $\overline{W}_k$ and so on.

Let $h : M \to M$ be a homeomorphism with $\overline{f} = h \circ f \circ h^{-1}$. Here we note that $h(r)$ is not necessarily equal to $\overline{r}$. In fact, $h(r) = \overline{r}$ if and only if $m_0 = \overline{m}_0$ or equivalently $\overline{\varphi} = h \circ \varphi \circ h^{-1}$. We may assume that $m_0 \leq \overline{m}_0$ if necessary replacing f and $\overline{f}$. Then $h(f^{\overline{m}_0 - m_0}(r)) = \overline{r}$. Since the constants appeared in (2.5) depend on the coordinate on $U(p)$, one can not replace the coordinates on $U(p)$ or $U(\overline{p})$ so as to satisfy $h(r) = \overline{r}$.

For any C^1 arc α in $U(p)$, the union of the end points of α is denoted by $\partial\alpha$. When any vector tangent to α is not vertical, the maximum $\mathrm{Slope}(\alpha)$ of $\mathrm{Slope}(\boldsymbol{v}(\boldsymbol{x}))$ for vectors $\boldsymbol{v}(\boldsymbol{x})$ tangent to α at $\boldsymbol{x} \in \alpha$ is well defined.

If $s > 0$ is small enough, then $\gamma'_{0,n} = \beta^u_n(s) \cap S_n$ is equal to $\alpha^u_n \cap S_n$ for any sufficiently large $n \in \mathbb{N}$.

The following is a key lemma for the proof of Theorem 2.1.

Lemma 2.7 (Intersection Lemma). *Let $\gamma'_{0,n} = \beta^u_n(s) \cap S_n$ and $\overline{\gamma}'_{0,n} = \overline{\beta}^u_n(\overline{s}) \cap \overline{S}_n$. Then there exists an $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$,*

$$h(f^{\overline{m}_0 - m_0}(\gamma'_{0,n})) \cap \overline{\gamma}'_{0,n} \neq \emptyset. \tag{2.38}$$

Proof. We suppose that, for any $n_0 \in \mathbb{N}$, there would exist $n > n_0$ such that

$$\gamma'^{*}_{0,n} \cap \overline{\gamma}'_{0,n} = \emptyset,$$

where $\gamma'^{*}_{0,n} = h \circ f^{\overline{m}_0 - m_0}(\gamma'_{0,n})$, and introduce a contradiction.

Recall that $i_n \in \mathbb{N}$ satisfies $(1+\varepsilon)^2 < f^{i_n}(s_n^+) \le (1+\varepsilon)^3$ and $f^{i_n}(S_n) \subset R_\varepsilon$. For short, we set

$$\gamma_1 = \gamma_{1,n} := f^{i_n}(\gamma'_{0,n}) \quad \text{and} \quad \gamma_1^* = \gamma_{1,n}^* := h(\gamma_1).$$

Then $\gamma_1^* = \overline{f}^{\,i_n-(\overline{m}_0-m_0)}(\gamma_{0,n}^*)$. Since $h(q) = \overline{q}$ and $\overline{f} = h\circ f\circ h^{-1}$, we have $h(1) = 1,\ h(1+\varepsilon) = 1+\overline{\varepsilon},\ h\big((1+\varepsilon)^2\big) = (1+\overline{\varepsilon})^2,\ h\big((1+\varepsilon)^3\big) = (1+\overline{\varepsilon})^3$ and $1+\overline{\varepsilon} < \mathrm{pr}_x(\gamma_1^*) \le (1+\overline{\varepsilon})^3$. Strictly, γ_1^* may slightly exceed $R_{\overline{\varepsilon}}$. Then we may rearrange our argument so that Lemmas 2.3 and 2.4 for $\overline{f}$ still hold if γ_1^* is contained in a sufficiently small neighborhood of $R_{\overline{\varepsilon}}$. Then, by applying Lemma 2.4 to $\overline{f}$, one can show that γ_1^* is a sub-arc almost parallel to $\overline{\delta}_1 \subset W^u_{\mathrm{loc}}(\overline{p})$ and $\mathrm{Slope}(\gamma_1^*) < \overline{\varepsilon}^{\frac{5}{2}}$ for any sufficiently large n.

The intersection $\gamma_1' = \varphi(\gamma_1) \cap B_1'$ consists of mutually disjoint three arcs connecting the vertical sides of B_1'. See Figure 2.11. We set $\gamma_2 = f^{u_1}(\gamma_1')$ and $\gamma_2^* = h(\gamma_2)$. Note that

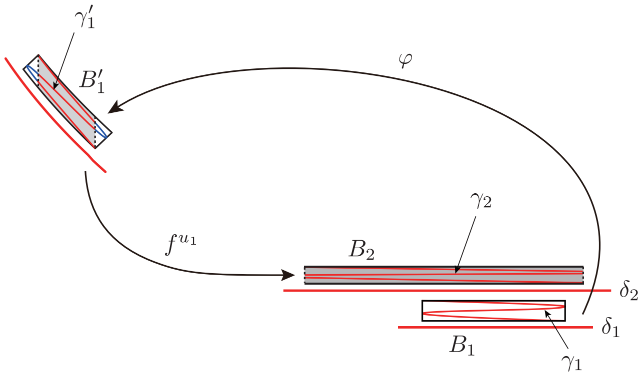

Figure 2.11: γ_1' is a disjoint union of arcs connecting the vertical sides of B_1' and γ_2 is a disjoint union of three proper arcs connecting the vartical sides of B_2.

γ_2 is a disjoint union of three proper arcs in B_2 connecting the vertical sides of B_2. Let $\widehat{\gamma}_2^*$ be the smallest arc in $W^u(\overline{p})$ containing γ_2^*. By applying Lemma 2.4 to $\overline{f}$, we have $\mathrm{Slope}(\widehat{\gamma}_2^*) < \overline{\varepsilon}^{\frac{5}{2}}$. In particular, $\widehat{\gamma}_2^*$ is almost parallel to $\overline{\delta}_2$. Repeating the same argument, one can have sequences $\{\gamma_k\}$ satisfying the following conditions.

- Each γ_k is a disjoint union of three proper arcs in B_k connecting the vertical sides of B_k.
- For each $\gamma_k^* = h(\gamma_k)$, the smallest arc $\widehat{\gamma}_k^*$ in $W^u(\overline{p})$ containing γ_k^* is almost parallel to $\overline{\delta}_k$.

See Figure 2.12.

Take $\boldsymbol{x} \in \gamma_k$ arbitrarily and set $\boldsymbol{x}^* = h(\boldsymbol{x}) \in \gamma_k^*$. Since h is uniformly continuous on R_ε, for any $\overline{l} > 0$, there exists $l > 0$ independent of $\boldsymbol{x}$ such that $h \circ f^{\overline{m}_0-m_0}(N_l(\boldsymbol{x})) \subset N_{\overline{l}}(\boldsymbol{x}^*)$, where $N_l(\boldsymbol{x})$ is the l-neighborhood of $\boldsymbol{x}$ and $N_{\overline{l}}(\boldsymbol{x}^*)$ is the $\overline{l}$-neighborhood of $\boldsymbol{x}^*$ in M. If n is sufficiently large, then $N_l(\boldsymbol{x})$ must intersect the three arcs of γ_k. However, $N_{\overline{l}}(\boldsymbol{x}^*)$ intersects only one arc of γ_k^*. This gives a contradiction. Thus (2.38) holds for all sufficiently large n. □

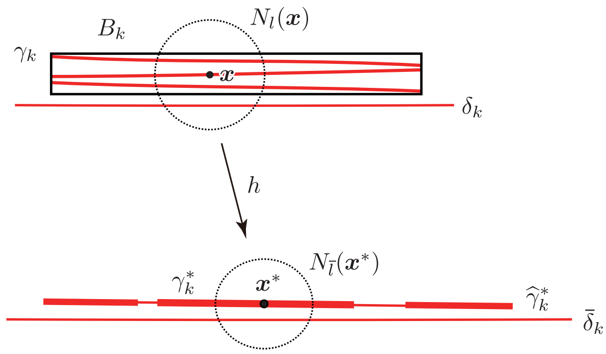

Figure 2.12: $N_l(\boldsymbol{x})$ intersects the three arcs of γ_k, but $N_{\bar{l}}(\boldsymbol{x}^*)$ intersects only one arc of γ_k^*.

2.7 Proof of Theorem 2.1

Now we are ready to prove Theorem 2.1. The proof is done by using our Intersection Lemma (Lemma 2.7) together with arguments in [dM, Pa, Po] and so on. We only consider the case where both f and $\overline{f}$ satisfy the condition (2.9), which belongs to Case II$_{++}$ in Section 2.8, and the small expanding condition at p and $\overline{p}$ respectively. The proof of any other adaptable case is done similarly.

Proof of (M1) of Theorem 2.1. By Intersection Lemma (Lemma 2.7), one can take $\overline{r}_n \in \overline{\gamma}'_{0,n} \cap h \circ f^{\overline{m}_0 - m_0}(\gamma'_{0,n})$. Since $\overline{r}_n$ converges to $\overline{r}$ as $n \to \infty$, $r_n = (h \circ f^{\overline{m}_0 - m_0})^{-1}(\overline{r}_n) \in \gamma'_{0,n}$ converges to r as $n \to \infty$. See Figure 2.13.

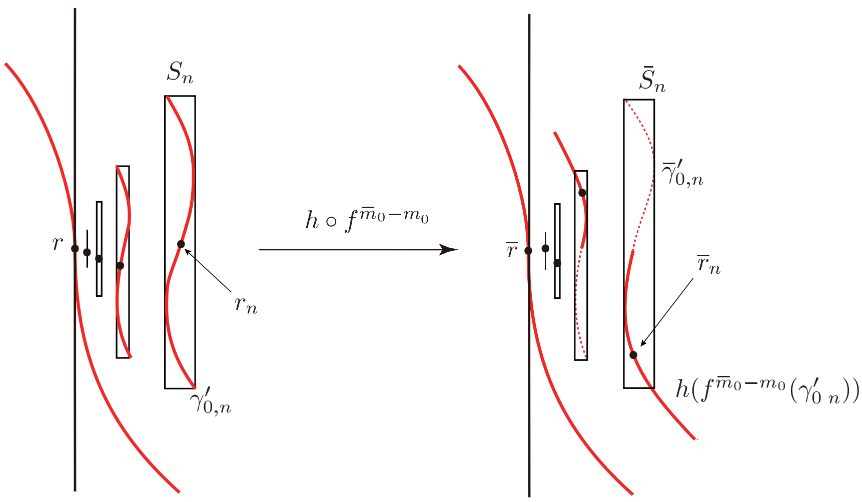

Figure 2.13: For $\overline{r}_n \in \overline{\gamma}'_{0,n} \cap h \circ f^{\overline{m}_0 - m_0}(\gamma'_{0,n})$, $r_n = (h \circ f^{\overline{m}_0 - m_0})^{-1}(\overline{r}_n) \in \gamma'_{0,n}$ converges to r as $n \to \infty$.

Let $W^u_{\mathrm{loc},+}(p)$ be the component of $W^u_{\mathrm{loc}}(p) \setminus \{p\}$ containing q. Take a fundamental domain D for f in $W^u_{\mathrm{loc},+}(p)$. Then there exist subsequences $\{r_{n(k)}\} \subset \{r_n\}$, $\{m(k)\}$ of $\mathbb{N}$ and $\boldsymbol{x}_0 \in D$ satisfying the following conditions.

- $r_{n(k)}$ converges to r as $k \to \infty$.
- $\boldsymbol{x}_{n(k)} := f^{m(k)}(r_{n(k)})$ converges to $\boldsymbol{x}_0 = (x_0, 0)$ as $k \to \infty$.
- $q_{n(k)} := \varphi^{-1}(r_{n(k)})$ converges to q as $k \to \infty$.

Then

$$\begin{aligned} x_0 &= \lim_{k\to\infty} \mathrm{pr}_x(\boldsymbol{x}_{n(k)}) = \lim_{k\to\infty} \mathrm{pr}_x(r_{n(k)})\mu^{m(k)} \\ &= \lim_{k\to\infty} az_0 \left(\lambda^{n(k)} + O\big(\lambda^{\frac{3}{2}n(k)}\big)\right) \mu^{m(k)} \\ &= \lim_{k\to\infty} az_0\lambda^{n(k)}\mu^{m(k)}. \end{aligned}$$

It follows that $\lim_{k\to\infty} \lambda^{n(k)}\mu^{m(k)} = \dfrac{x_0}{az_0}$. Then there exist constants C_0 and C_1 with $0 < C_0 < C_1$ and such that

$$C_0 < \lambda^{n(k)}\mu^{m(k)} < C_1$$

for any k. Taking the logarithms of this inequalities, we have

$$\frac{\log C_0}{n(k)\log\mu} < \frac{\log\lambda}{\log\mu} + \frac{m(k)}{n(k)} < \frac{\log C_1}{n(k)\log\mu}.$$

This shows that $\displaystyle\lim_{k\to\infty} \frac{m(k)}{n(k)} = -\frac{\log\lambda}{\log\mu}$. By applying a similar argument to $\overline{f}$, one can prove

$$\lim_{k\to\infty} \bar{\lambda}^{n(k)}\bar{\mu}^{m(k)-(\overline{m}_0-m_0)} = \frac{h(x_0)}{\bar{a}\bar{z}_0}$$

and hence $\displaystyle\lim_{k\to\infty} \frac{m(k)}{n(k)} = \lim_{k\to\infty} \frac{m(k)-(\overline{m}_0-m_0)}{n(k)} = -\frac{\log\bar{\lambda}}{\log\bar{\mu}}$. Consequently, $\dfrac{\log\lambda}{\log\mu} = \dfrac{\log\bar{\lambda}}{\log\bar{\mu}}$ holds. □

Lemma 2.8. *If* $\dfrac{\log\lambda}{\log\mu}$ *is irrational, then the restriction* $h|_{W^u_+(p)}$ *is locally* C^1 *diffeomorphic, where* $W^u_+(p)$ *is the component of* $W^u(p) \setminus \{p\}$ *containing* q*.*

Proof. Let s_n be the real number with $\mathrm{pr}_x(r_n) = \mu^{-s_n}$. Since $\mathrm{pr}_x(r_n) \approx az_0\left(\lambda^n + O\big(\lambda^{\frac{3}{2}n}\big)\right)$ by (2.17) and (2.21), we have

$$1 = \mathrm{pr}_x(r_n)\mu^{s_n} \approx az_0\left(\lambda^n + O\big(\lambda^{\frac{3}{2}n}\big)\right)\mu^{s_n} \approx az_0\lambda^n\mu^{s_n}.$$

Thus $c_n = az_0\lambda^n\mu^{s_n}$ satisfies $\lim_{n\to\infty} c_n = 1$. Moreover,

$$s_n = \frac{\log c_n}{\log\mu} - \frac{\log(az_0)}{\log\mu} - n\frac{\log\lambda}{\log\mu}. \tag{2.39}$$

Since $-\dfrac{\log\lambda}{\log\mu}$ is irrational, the set

$$\left\{-\frac{\log(az_0)}{\log\mu} - n\frac{\log\lambda}{\log\mu} \mod 1\,;\; n=1,2,\dots\right\}$$

is dense in the interval $[0,1]$. Since $\lim_{n\to\infty}\log c_n = 0$, the set $S=\{s_n \mod 1\,;\, n = 1,2,\dots\}$ is also dense in $[0,1]$.

Take a point x_0 of $[\mu^{-1},1]$ arbitrarily, and let $\sigma\in[0,1]$ be the real number with $\mu^{-\sigma} = x_0$. Since $[\mu^{-1},1]$ is a fundamental domain for f in $W^u_{\mathrm{loc},+}(p)$, it follows from the density of S that there exist subsequences $\{n(k)\}$, $\{m(k)\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty}(s_{n(k)}-m(k))=\sigma$. Then

$$\begin{aligned}x_0 = \mu^{-\sigma} &= \lim_{k\to\infty}\mu^{-s_{n(k)}+m(k)} = \lim_{k\to\infty}\mathrm{pr}_x(r_{n(k)})\mu^{m(k)}\\ &= \lim_{k\to\infty} az_0\left(\lambda^{n(k)} + O(\lambda^{\frac{3}{2}n(k)})\right)\mu^{m(k)} = \lim_{k\to\infty} az_0\lambda^{n(k)}\mu^{m(k)}.\end{aligned}$$

Thus we have $\lim\limits_{k\to\infty}\lambda^{n(k)}\mu^{m(k)} = \dfrac{x_0}{az_0}$.

Since $\overline{f}$ is conjugate to f via h, $\mathrm{pr}_x(\overline{r}_{n(k)})\bar{\mu}^{m(k)-(\overline{m}_0-m_0)}$ converges to $h(x_0)$. As above, we have

$$\lim_{k\to\infty}\bar{\lambda}^{n(k)}\bar{\mu}^{m(k)-(\overline{m}_0-m_0)} = \frac{h(x_0)}{\bar{a}\bar{z}_0}.$$

If we set $\tau = \dfrac{\log\bar{\mu}}{\log\mu} = \dfrac{\log\bar{\lambda}}{\log\lambda}$, then $\bar{\mu}=\mu^\tau$ and $\bar{\lambda}=\lambda^\tau$. It follows that

$$\frac{x_0^\tau}{a^\tau z_0^\tau} = \frac{h(x_0)}{\bar{a}\bar{z}_0}\bar{\mu}^{\overline{m}_0-m_0}.$$

Thus $h|_{W^u_{\mathrm{loc},+}(p)}$ is a C^1 diffeomorphism represented as

$$h(x) = \frac{\bar{a}\bar{z}_0}{a^\tau z_0^\tau\,\bar{\mu}^{\overline{m}_0-m_0}}x^\tau,$$

where $W^u_{\mathrm{loc},+}(p)$ is the component of $W^u_{\mathrm{loc}}(p)\setminus\{p\}$ containing q. Since $W^u_+(p)=\bigcup_{n=0}^\infty f^n(W^u_{\mathrm{loc},+}(p))$ and both f and $\overline{f}$ are C^3 diffeomorphisms, $h|_{W^u_+(p)}$ is locally C^1 diffeomorphic. This completes the proof. □

Proof of (M2) of Theorem 2.1. Take a sequence $\{q_j\}$ on $W^u_{\mathrm{loc},+}(p)$ converging to q and set $t_j=\varphi(q_j)$. See Figure 2.14. Let t_j' be the image of t_j by the horizontal projection to $W^s_{\mathrm{loc}}(p)$. Obviously, both t_j and t_j' converge to r as $k\to\infty$. There exist subsequences $\{t_{j(k)}\}$ of $\{t_j\}$, $\{l(k)\}$ of $\mathbb{N}$ and a point x_1 of $W^u_{\mathrm{loc}}(p)$ with $\lim\limits_{k\to\infty} f^{l(k)}(t_{j(k)}) = x_1$. Then the following approximations

$$x_1 \sim \mathrm{pr}_x(t_{j(k)})\mu^{l(k)} \sim [d(t'_{j(k)},r)]^3\mu^{l(k)} \sim [d(q_{j(k)},q)]^3\mu^{l(k)}$$

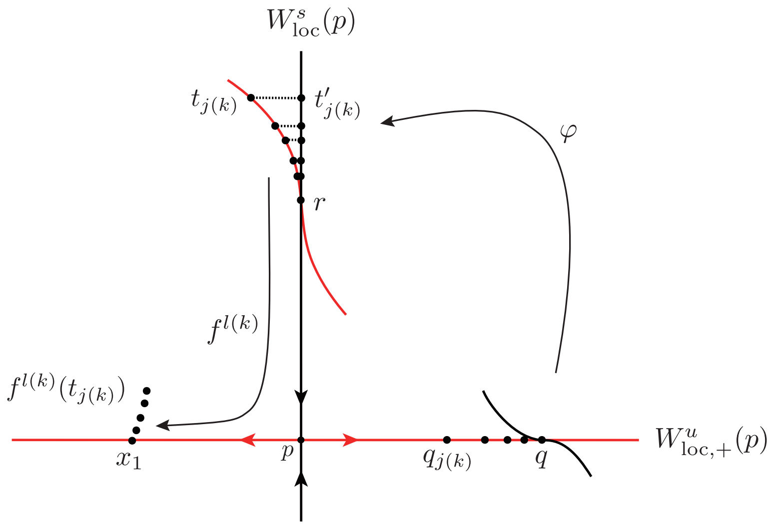

Figure 2.14: The case of II_{++}.

hold. It follows that $\mu^{-l(k)} \sim [d(q_{j(k)}, q)]^3$. Similarly, we have $\bar{\mu}^{-l(k)+(\overline{m}_0-m_0)} \sim [d(\bar{q}_{j(k)}, \bar{q})]^3$, where $\bar{q}_{j(k)} = h(q_{j(k)})$. Since $h|_{W^u_+(p)}$ is locally C^1-diffeomorphic by Lemma 2.8,

$$d(\bar{q}_{j(k)}, \bar{q}) \sim d(q_{j(k)}, q).$$

Thus

$$\left(\frac{\bar{\mu}}{\mu}\right)^{-l(k)} \sim \left(\frac{d(\bar{q}_{j(k)}, \bar{q})}{d(q_{j(k)}, q)}\right)^3 \bar{\mu}^{-(\overline{m}_0-m_0)} \sim 1.$$

This implies that $\mu = \bar{\mu}$. By (M1), we also have $\lambda = \bar{\lambda}$. This completes the proof of the part (M2). □

Remark 2.9. Some arguments used in the case that the tangency between $W^s(p)$ and $W^u(p)$ is one-sided (for example [dM, Pa, Po]) can not be applicable to the two-sided case. Here we explain the reason.

Suppose that a homoclinic tangency q_0 is one-sided, say a quadratic tangency. Take an arc γ in $U(q_0)$ meeting $W^u_{\rm loc}(p_0)$ orthogonally at q_0. Let $\{w_i\}$ be a sequence in γ converging to q_0 from above. Then

$$d(w_i, W^s(p_0)) \approx d(w_i, W^u_{\rm loc}(p_0)) \tag{2.40}$$

holds. On the other hand, their images by the conjugacy homeomorphism h satisfy

$$d(h(w_i), W^s(p_1)) \leq d(h(w_i), W^u_{\rm loc}(p_1)). \tag{2.41}$$

See Figure 2.15 (a). By using (2.40) and (2.41), one can show that $\dfrac{\log\lambda_1}{\log\mu_1} \leq \dfrac{\log\lambda_0}{\log\mu_0}$. By

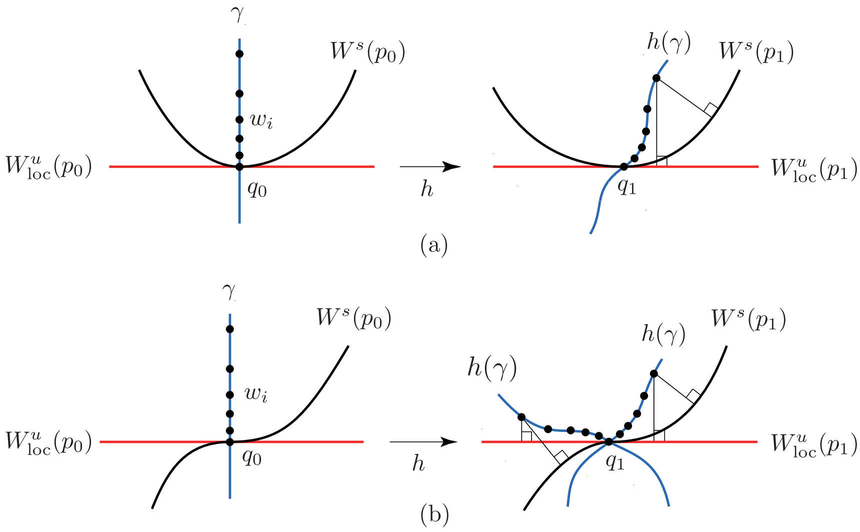

Figure 2.15: (a) The case of quadratic tangencies. (b) The case of cubic tangencies.

applying the same argument to h^{-1}, we also have $\dfrac{\log\lambda_1}{\log\mu_1} \geq \dfrac{\log\lambda_0}{\log\mu_0}$, and hence $\dfrac{\log\lambda_1}{\log\mu_1} = \dfrac{\log\lambda_0}{\log\mu_0}$.

Now we consider the case of two-sided tangencies, say cubic tangencies, and $\{w_i\}$ is a sequence as above. Then the approximation (2.40) still holds. However, the inequality (2.41) would not hold as is suggested in Figure 2.15 (b). So it might be difficult to get the inequality $\dfrac{\log\lambda_1}{\log\mu_1} \leq \dfrac{\log\lambda_0}{\log\mu_0}$ only by arguments in [dM, Pa, Po]. Thus we need another idea in the study of moduli associated with two-sided homoclinic tangencies.

2.8 Adaptable conditions

In this section, we will present conditions on the signs of a, bc, λ and μ under which any arguments presented throughout the previous sections are valid.

Recall that we have set

$$U(p) = [-2, 2] \times [-2, 2],\ W^u_{\rm loc}(p) = [-2, 2] \times \{0\},\ W^s_{\rm loc}(p) = \{0\} \times [-2, 2].$$

The union $W^u_{\rm loc}(p) \cup W^s_{\rm loc}(p)$ divides $U(p)$ to four components. The closures of these components containing $(1, 1)$, $(-1, 1)$, $(-1, -1)$ and $(1, -1)$ are called the first, second, third and fourth quadrants of $U(p)$ and denoted by Q_1, Q_2, Q_3 and Q_4, respectively. In our argument it is required that $\varphi(R_\varepsilon)$ or some substitution is in Q_1. If R_ε lies in Q_2, then we may use

$$R^-_\varepsilon = [(1+\varepsilon)^{-3}, (1+\varepsilon)^{-1}] \times [0, \varepsilon^3]$$

instead of R_ε. Then $\varphi(R^-_\varepsilon)$ is in Q_1. See Figure 2.16. Thus one can arrange the placement of $\varphi(R_\varepsilon)$ suitably under any conditions on the signs of a, bc, λ and μ.

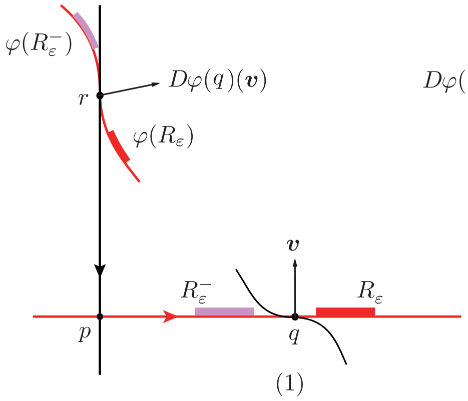

Figure 2.16: (1) $\varphi(R_\varepsilon)$ is in Q_1. (2) $\varphi(R_\varepsilon^-)$ is in Q_1.

Definition 2.10 (Adaptable condition)**.** f satisfies the *adaptable condition* with respect to (p, q) if, for all sufficiently large positive integers n (or positive even or odd integers), there exists a rectangle S_n defined as in Section 2.3 and either S_n or its image $f(S_n)$ lies in Q_1.

As was seen in Section 2.3, S_n exists if and only if there exists t_n satisfying the condition

$$3ct_n^2 \approx -b\lambda^n z_0 \tag{2.20'}$$

which corresponds to (2.20). Here z_0 is the positive constant as illustrated in Figure 2.5.

Now we will see that the existence of S_n and the placements of S_n and $f(S_n)$ are strictly determined by the signs of a, bc, λ and μ, which are classified to the sixteen cases as in Table 2.1

First we suppose that $\lambda > 0$. Then there exists t_n satisfying (2.20') if and only if $bc < 0$. Moreover, if $a > 0$, then S_n is in Q_1, which belongs to Case II_+. See Figure 2.17 (1). If $a < 0$, then S_n is in Q_2. Hence $f(S_n)$ is Q_1 if $\mu < 0$, which is in Case IV_{+-}. See Figure 2.17 (2).

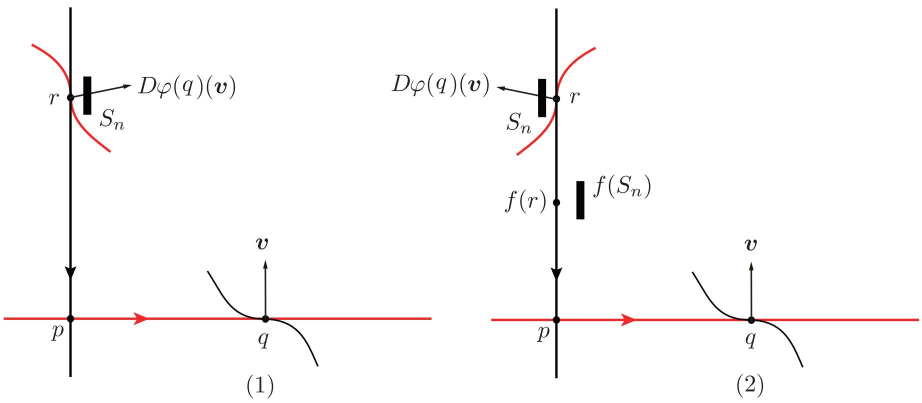

Figure 2.17: (1) The case of II_+ or II_-. (2) The case of IV_{+-} or IV_{--}.

Case			a	bc	λ	μ
I	I_+	I_{++}	$+$	$+$	$+$	$+$
		I_{+-}				$-$
	I_-	I_{-+}			$-$	$+$
		$\mathbf{I}_{--}$				$-$
II	II_+	$\mathbf{II}_{++}$	$+$	$-$	$+$	$+$
		$\mathbf{II}_{+-}$				$-$
	II_-	$\mathbf{II}_{-+}$			$-$	$+$
		$\mathbf{II}_{--}$				$-$
III	III_+	III_{++}	$-$	$+$	$+$	$+$
		III_{+-}				$-$
	III_-	$\mathbf{III}_{-+}$			$-$	$+$
		$\mathbf{III}_{--}$				$-$
IV	IV_+	IV_{++}	$-$	$-$	$+$	$+$
		$\mathbf{IV}_{+-}$				$-$
	IV_-	IV_{-+}			$-$	$+$
		$\mathbf{IV}_{--}$				$-$

Table 2.1: The shaded cells are the cases in which f satisfies the adaptable conditions.

Next we suppose that $\lambda < 0$. Then there exists t_n satisfying (2.20') if and only if either (i) $bc < 0$ and n is even or (ii) $bc > 0$ and n is odd. In the case (i), S_n is in Q_1 if $a > 0$, which belongs to Case II_-. See Figure 2.17 (1). If $a < 0$ and $\mu < 0$, then $f(S_n)$ is in Q_1, which belongs to Case IV_{--}. See Figure 2.17 (2). On the other hand, in the case (ii), S_n is in Q_1 if $a < 0$, which belongs to Case III_-. See Figure 2.18 (1). If $a > 0$ and $\mu < 0$, then $f(S_n)$ is in Q_1, which belongs to Case I_{--}. See Figure 2.18 (2).

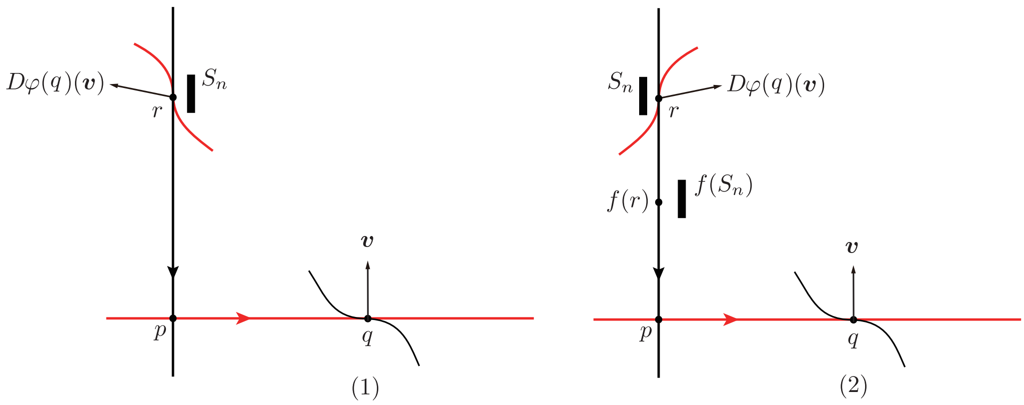

Figure 2.18: (1) The case of III_-. (2) The case of I_{--}.

Thus we have the following proposition.

Proposition 2.11. *If one of Cases* I_{--}, II, III_-, IV_{+-} and IV_{--} holds, then f satisfies the adaptable condition with respect to (p, q).

It follows from the proposition that f satisfies the adaptable condition in nine of the sixteen cases in Table 2.1.

Chapter 3

Moduli of 3-dimensional diffeomorphisms with saddle foci

In this chapter, we investigate moduli of a 3-dimensional diffeomorphism f with a sadldle focus p and a homoclinic quadratic tangency q associated with p. We show that, for most of such diffeomorphisms, all the eigenvalues of $Df(p)$ are moduli and the restriction of a conjugacy homeomorphism to a local unstable manifold is a uniquely determined linear conformal map.

3.1 Moduli of 3-dimensional diffeomorphisms with saddle foci

First, we prove the following theorem.

Theorem 3.1. *Let M be a 3-manifold and f_j $(j = 0, 1)$ elements of $\mathrm{Diff}^r(M)$ for some $r \geq 3$ which have hyperbolic fixed points p_j and homoclinic quadratic tangencies q_j positively associated with p_j and satisfy the following conditions.*

- *For $j = 0, 1$, there exists a neighborhood $U(p_j)$ of p_j in M such that $f_j|_{U(p_j)}$ is linear and $Df_j(p_j)$ has non-real eigenvalues $r_j e^{\pm\sqrt{-1}\theta_j}$ and a real eigenvalue λ_j with $r_j > 1$, $\theta_j \neq 0 \mod \pi$ and $0 < \lambda_j < 1$.*
- *f_0 is topologically conjugate to f_1 by a homeomorphism $h : M \to M$ with $h(p_0) = p_1$ and $h(q_0) = q_1$.*

Then the following (D1) *and* (D2) *hold.*

(D1) $\dfrac{\log \lambda_0}{\log r_0} = \dfrac{\log \lambda_1}{\log r_1}$.

(D2) *Either $\theta_0 = \theta_1$ or $\theta_0 = -\theta_1 \mod 2\pi$.*

Here we say that a homoclinic quadratic tangency q_0 is *positively associated* with p_0 if both $f_0^n(q_0)$ and $f_0^{-n}(\alpha)$ lie in the same component of $U(p_0) \setminus W^u_{\mathrm{loc}}(p_0)$ for a sufficiently large $n \in \mathbb{N}$ and any small curve α in $W^s(p_0)$ containing q_0. Theorem 3.1 holds also in the case when $\theta_0 = 0 \mod \pi$ or $-1 < \lambda_j < 0$ except for some rare case, see Remark 3.4 for details.

Remark 3.2. Assertion (D1) of Theorem 3.1 is implied in the case (D) of Theorem 1.1 in [NPT, Chapter III]. Assertion (D2) is also proved by Dufraine [Du2] under weaker assumptions. The author used non-spiral curves in $W^u_{\mathrm{loc}}(p)$ emanating from p. On the other hand, we employ unstable bent disks defined in Section 3.2 which are originally introduced by Nishizawa [Ni]. By using such disks, we construct a convergent sequence of mutually parallel straight segments in $W^u_{\mathrm{loc}}(p)$ which are mapped to straight segments in $W^u_{\mathrm{loc}}(h(p))$ by h, see Figure 3.9. An *advantage* of our proof is that these sequences are applicable to prove our main theorem, Theorem 3.3 below.

Results corresponding to Theorem 3.1 for 3-dimensional flows with Shilnikov cycles are obtained by Togawa [To], Carvalho-Rodrigues [CR] and for those with connections of saddle-foci by Bonatti-Dufraine [BD], Dufraine [Du1], Rodrigues [Rod] and so on. See the Section 2 in [Rod] for details. Moreover Carvalho-Rodrigues [CR] present results on moduli of 3-dimensional flows with Bykov cycles.

The following theorem is the main theorem in this chapter.

Theorem 3.3. *Under the assumptions in Theorem 3.1, suppose moreover that $\theta_0/2\pi$ is irrational. Then the following conditions hold.*

(E1) $\lambda_0 = \lambda_1$ *and* $r_0 = r_1$.

(E2) *The restriction* $h|_{W^u_{\mathrm{loc}}(p_0)} : W^u_{\mathrm{loc}}(p_0) \to W^u_{\mathrm{loc}}(p_1)$ *is a uniquely determined linear conformal map.*

In contrast to Posthumus' results for 2-dimensional diffeomorphisms, the eigenvalues λ_0 and r_0 are proved to be moduli without the assumption that $\dfrac{\log \lambda_0}{\log r_0}$ is irrational.

The restriction $h|_{W^u_{\mathrm{loc}}(p_0)}$ is said to be a *linear conformal map* if $h|_{W^u_{\mathrm{loc}}(p_0)}$ is represented as $h|_{W^u_{\mathrm{loc}}(p_0)}(z) = \rho e^{\sqrt{-1}\omega} z$ ($z \in W^u_{\mathrm{loc}}(p_0)$) for some $\rho \in \mathbb{R} \setminus \{0\}$ and $\omega \in \mathbb{R}$ under the natural identification of $W^u_{\mathrm{loc}}(p_0)$, $W^u_{\mathrm{loc}}(p_1)$ with neighborhoods of the origin in $\mathbb{C}$ via their linearizing coordinates.

For any $r_j > 1$ and $\theta_j \in \mathbb{R}$ ($j = 0, 1$), let $\varphi_j : \mathbb{C} \to \mathbb{C}$ be the map defined by $\varphi_j(z) = r_j e^{\sqrt{-1}\theta_j} z$. Then there are many choices of conjugacy homeomorphisms on $\mathbb{C}$ for φ_0 and φ_1. For example, we take two-sided Jordan curves Γ_j in $\mathbb{C}$ with $\varphi_j(\Gamma_j) \cap \Gamma_j = \emptyset$ and bounding disks in $\mathbb{C}$ containing the origin arbitrarily. Then there exists a conjugacy homeomorphism $h : \mathbb{C} \to \mathbb{C}$ for φ_0 and φ_1 with $h(\Gamma_0) = \Gamma_1$. On the other hand, Theorem 3.3 (E2) implies that we have severe constraints in the choice of conjugacy homeomorphisms for 3-dimensional diffeomorphisms as above. Intuitively, it says that only a homeomorphism h with $h|_{W^u_{\mathrm{loc}}}(p)$ linear and conformal can be a candidate for a conjugacy between f_0 and f_1. As an application of the linearity and conformality of $h|_{W^u_{\mathrm{loc}}}(p)$, we will present a new modulus for f_0 other than θ_0, λ_0, r_0, see Corollary 3.9 in Section 3.5.

3.2 Front curves and folding curves

For $j = 0, 1$, let f_j be a diffeomorphism and q_j a quadratic tangency associated with a hyperbolic fixed point p_j satisfying the conditions of Theorem 3.1. We will define in this section front curves in $W^u(p_j)$ and folding curves in $W^u_{\mathrm{loc}}(p_j)$ and show in the next section that these curves converge to straight segments which are preserved by any conjugacy homeomorphism between f_0 and f_1.

We set $f_0 = f$, $p_0 = p$, $q_0 = q$, $r_0 = r$, $\theta_0 = \theta$ and $\lambda_0 = \lambda$ for short. Similarly, let $f_1 = f'$, $p_1 = p'$, $q_1 = q'$, $r_1 = r'$, $\theta_1 = \theta'$ and $\lambda_1 = \lambda'$. Suppose that $(z, t) = (x, y, t)$ with $z = x + \sqrt{-1}y$ is a coordinate around p with respect to which f is linear. For a small $a > 0$, let $D_a(p)$ be the disk $\{z \in \mathbb{C}\,; |z| \le a\}$. We may assume that q is contained in the interior of $D_a(p) \times \{0\} \subset W^u_{\mathrm{loc}}(p)$ and $\widehat{q} = f^N(q)$ is in the interior of the upper half $W^{s+}_{\mathrm{loc}}(p) = \{0\} \times [0, a]$ of $W^s_{\mathrm{loc}}(p)$ for some $N \in \mathbb{N}$. See Figure 3.1. Let $U_a(p)$ be

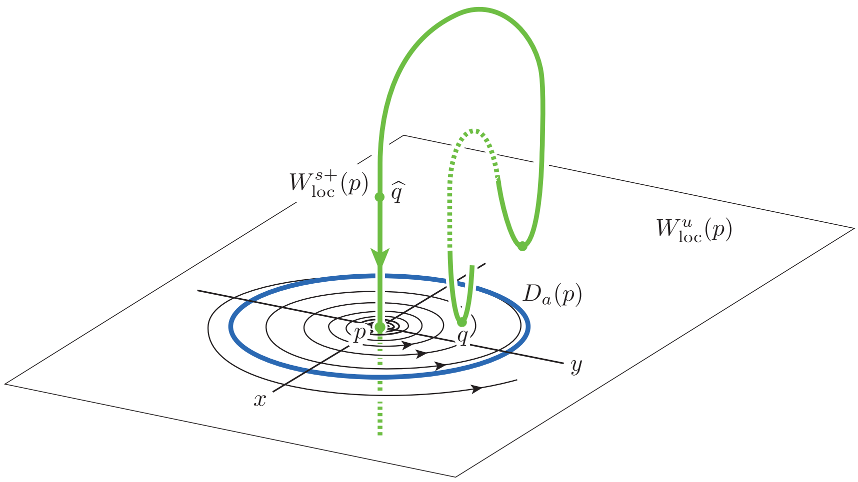

Figure 3.1: A saddle-focus p and a homoclinic quadratic tangency q in $D_a(p)$.

the circular column in the coordinate neighborhood defined by $U_a(p) = D_a(p) \times [0, a]$ and $V_{\widehat{q}}$ a small neighborhood of $\widehat{q}$ in $U_a(p)$. Suppose that $U_a(p)$ has the Euclidean metric induced from the linearizing coordinate on $U_a(p)$. By choosing the coordinate suitably and replacing θ by $-\theta$ if necessary, we may assume that the restriction $f|_{D_a(p)}$ is represented as $re^{\sqrt{-1}\theta}z$ for $z \in \mathbb{C}$ with $|z| < a$. Similarly, one can suppose that $f'|_{D_{a'}(p')}$ is represented as $r'e^{\sqrt{-1}\theta'}z$ for some $a' > 0$. The orthogonal projection $\mathrm{pr} : U_a(p) \to D_a(p)$ is defined by $\mathrm{pr}(x, y, t) = (x, y)$.

In this section, we construct an unstable bent disk $\widetilde{H}_0$ in $W^u(p) \cap U_a(p)$, the front curve $\widetilde{\gamma}_0$ in $\widetilde{H}_0$ and the folding curves γ_0 in $U_a(p)$. We also define the sequence of unstable bent disks $\widetilde{H}_m$ in $W^u(p) \cap U_a(p)$ converging to $\widetilde{H}_0$, which will be used in the next section to construct the sequence of front curves converging to $\widetilde{\gamma}_0$.

3.2.1 Construction of unstable bent disks, front curves and folding curves

We set $\widehat{q} = (0, t_0)$. Let $\widetilde{H}$ be the component of $W^u(p) \cap V_{\widehat{q}}$ containing $\widehat{q}$. One can retake the linearizing coordinate on $\mathbb{C}$ if necessary so that the line in $V_{\widehat{q}}$ passing through $\widehat{q}$ and parallel to the x-axis in $U_a(p)$ meets $\widetilde{H}$ transversely. Then $\widetilde{H}$ is represented as the graph of a C^r function $x = \varphi(y, t)$ with

$$\varphi(0, t_0) = 0, \quad \frac{\partial \varphi}{\partial t}(0, t_0) = 0 \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial t^2}(0, t_0) \neq 0. \tag{3.1}$$

By the implicit function theorem, there exists a C^{r-1} function $t = \eta(y)$ defined in a small neighborhood V of 0 in the y-axis and satisfying $\eta(0) = t_0$ and $\partial\varphi(y, \eta(y))/\partial t = 0$. Then the curve $\widetilde{\gamma}$ in $V_{\widehat{q}}$ parametrized by $\big(\varphi(y, \eta(y)), y, \eta(y)\big)$ divides $\widetilde{H}$ into two components and $\gamma = \mathrm{pr}(\widetilde{\gamma})$ is a C^{r-1} curve embedded in $D_a(p)$. Let $\widetilde{H}^+$ (resp. $\widetilde{H}^-$) be the closure of the upper (resp. lower) component of $\widetilde{H} \setminus \widetilde{\gamma}$. For a sufficiently large $n_0 \in \mathbb{N}$, the component $\widetilde{H}_0$ of $f^{n_0}(\widetilde{H}) \cap U_a(p)$ containing $q_0 = f^{n_0}(\widehat{q})$ is an *unstable bent disk* in $U_a(p)$ such that $\partial \widetilde{H}_0$ is a simple closed C^r curve in $\partial_{\mathrm{side}} U_a(p)$, where

$$\partial_{\mathrm{side}} U_a(p) = \{(x, t) \in \mathbb{C} \times \mathbb{R}\,;\, |z| = a, 0 \leq t < a\} \subset \partial U_a(p).$$

See Figure 3.2. We set $\widetilde{\gamma}_0 = f^{n_0}(\widetilde{\gamma}) \cap \widetilde{H}_0$, $\widetilde{H}_0^+ = f^{n_0}(\widetilde{H}^+) \cap \widetilde{H}_0$, $\widetilde{H}_0^- = f^{n_0}(\widetilde{H}^-) \cap \widetilde{H}_0$, $H_0 = \mathrm{pr}(\widetilde{H}_0^+) = \mathrm{pr}(\widetilde{H}_0^-)$ and $\gamma_0 = \mathrm{pr}(\widetilde{\gamma}_0)$. Then $\widetilde{\gamma}_0$ is called the *front curve* of $\widetilde{H}_0$ and γ_0 is the *folding curve* of H_0.

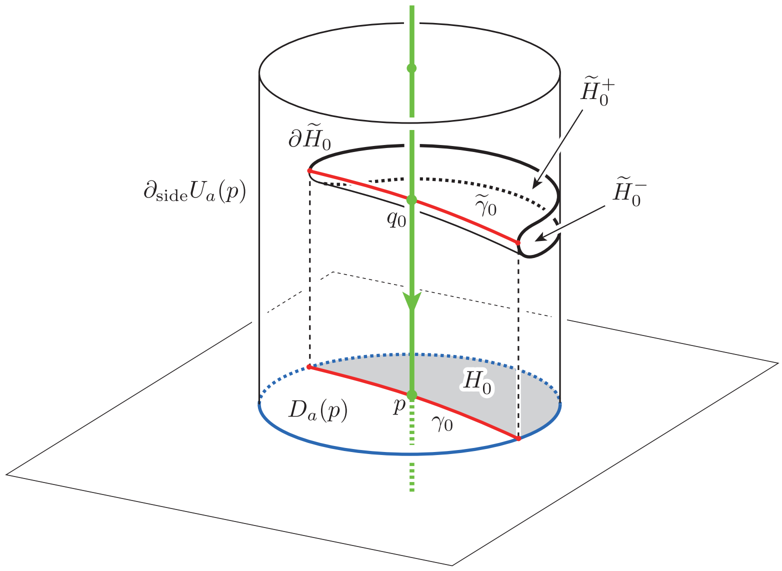

Figure 3.2: The front curve $\widetilde{\gamma}_0$ divides $\widetilde{H}_0$ into the two sheets $\widetilde{H}_0^+$ and $\widetilde{H}_0^-$. The folding curve γ_0 of H_0 is the orthogonal image of $\widetilde{\gamma}_0$.

We note that Nishizawa [Ni] has studied unstable bent disks similar to $\widetilde{H}_0$ as above in a different situation. In fact, he considered a 3-dimensional diffeomorphism g which has a saddle fixed point s such that all the eigenvalues of $Dg(s)$ are real and has a homoclinic quadratic tangency associated with s. Here we consider the component $\widetilde{H}_{0;u}^-$ of $f^u(\widetilde{H}_0^-) \cap U_a(p)$ containing $f^u(q_0)$ for $u \in \mathbb{N}$. Since the homoclinic tangency q is positively associated with p, one can show that there exists $\widetilde{H}_{0;u}^-$ which meets $W^s(p)$ transversely at a point $\widehat{z}$ near q by using an argument similar to that in [Ni, Lemma 4.4]. See Figure 3.3. To show the claim, the assumption of $\theta_0 \neq 0 \mod \pi$ in Theorem 3.1 is crucial. In fact,

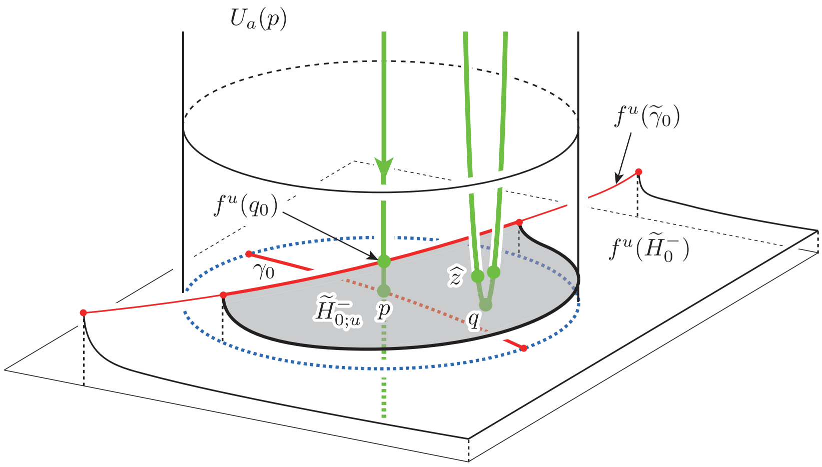

Figure 3.3: The half disk $\widetilde{H}_{0;u}^-$ meets $W^s(p)$ transversely at two points near q, one of which is $\widehat{z}$.

the condition implies that the following property:

(P) There exists an arbitrarily large u such that the interior of $H_{0;u} = \mathrm{pr}(\widetilde{H}_{0;u}^-)$ in $D_a(p)$ contains q.

Remark 3.4. (1) We here suppose $\theta = 0 \mod \pi$. Even in this case, if f has the property (P), then the component of $W^s(p)$ containing q and $W^u(p)$ have a homoclinic transverse intersection point. Then Theorems 3.1 and 3.3 will be proved quite similarly. Since $\theta = 0 \mod \pi$, all $f^u(\gamma_0)$ are tangent to a unique straight segment γ_∞ in $D_a(p)$ at p. Thus the property (P) is satisfied if γ_∞ does not pass through q.

(2) Even in the case of $-1 < \lambda < 0$, one can show that f has the property (P) similarly by using f^2 instead of f if $2\theta \neq 0 \mod \pi$. Moreover, since either q or $f(q)$ is a homoclinic tangency positively associated with p, Theorems 3.1 and 3.3 hold without the assumption that q is positively associated with p.

3.2.2 Construction of convergent sequence of unstable bent disks

Take $v \in \mathbb{N}$ such that $\widehat{z}_0 = f^v(\widehat{z})$ is a point $(0, \widehat{t})$ contained in $U_a(p)$, where $\widehat{z}$ is the transverse intersection point of $\widetilde{H}_{0;u}^-$ and $W^s(p)$ given in the previous subsection. Let D be a small disk in $W^u(p) \cap U_a(p)$ whose interior contains $\widehat{z}_0$. The *absolute slope* $\sigma(\boldsymbol{v})$ of a vector $\boldsymbol{v} = (v_1, v_2, v_3)$ in $U_a(p)$ with $(v_1, v_2) \neq (0, 0)$ is given as

$$\sigma(\boldsymbol{v}) = \frac{|v_3|}{\sqrt{v_1^2 + v_2^2}}.$$

The *maximum absolute slope* $\sigma(D)$ of D is defined by

$$\sigma(D) = \max\{\sigma(\boldsymbol{v})\,;\ \text{unit vectors } \boldsymbol{v} \text{ in } U_a(p) \text{ tangent to } D\}.$$

Fix $m_0 \in \mathbb{N}$ such that, for any $m \in \mathbb{N} \cup \{0\}$, the component D_m of $f^{m_0+m}(D) \cap U(p)$ containing $f^{m_0+m}(\widehat{z}_0)$ is a properly embedded disk in $U_a(p)$ with $\partial D_m \subset \partial_{\mathrm{side}} U_a(p)$. Note that D_m intersects $W^s_{\mathrm{loc}}(p)$ transversely at $(0, \lambda^m t_0)$, where $t_0 = \lambda^{m_0}\widehat{t}$. See Figure 3.4.

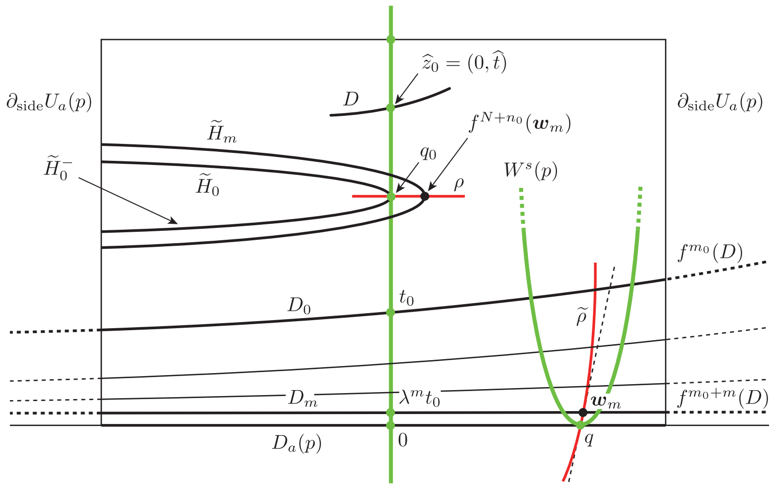

Figure 3.4: Trip from $\widetilde{H}_0^-$ to $\widetilde{H}_m$: $f^{u+v}(\widetilde{H}_0^-) \supset D$, $f^{m_0}(D) \supset D_0$, $f^m(D_0) \supset D_m$ and $f^{N+n_0}(D_m) \supset \widetilde{H}_m$, where N, n_0 are the positive integers with $f^N(q) = \widetilde{q}$ and $f^{n_0}(\widetilde{q}) = q_0$. The dotted line passing through q represents a straight segment tangent to $\widetilde{\rho}$ at q.

The maximum absolute slope of D_m satisfies

$$\sigma(D_m) \leq \sigma_0 \lambda^m r^{-m}, \tag{3.2}$$

where $\sigma_0 = \sigma(D)\lambda^{m_0} r^{-m_0}$. Consider a short straight segment ρ in $U_a(p)$ meeting $\widetilde{H}_0$ orthogonally at q_0. Then $\widetilde{\rho} = f^{-(N+n_0)}(\rho)$ is a C^r curve meeting $D_a(p)$ transversely at q, where N, n_0 are the positive integers given as above. One can choose $m_0 \in \mathbb{N}$ so that, for any $m \in \mathbb{N} \cup \{0\}$, $\widetilde{\rho}$ meets D_m transversely at a single point $\boldsymbol{w}_m = (z_m, s_m)$. Then (3.2) implies that $|t_0\lambda^m - s_m| \leq \widetilde{a}\sigma_0\lambda^m r^{-m}$, where $\widetilde{a} = \sup_{m \geq 0}\{|z_m|\} < \infty$. It follows that $s_m = t_0\lambda^m + O(\lambda^m r^{-m})$. Since $\widetilde{\rho}$ has a tangency of order at least two with a straight segment at q,

$$\mathrm{dist}(\boldsymbol{w}_m, q) = \widetilde{t}_0\lambda^m + O(\lambda^m r^{-m}) + O(\lambda^{2m}) = \widetilde{t}_0\lambda^m + o(\lambda^m) \tag{3.3}$$

for some constant $\widetilde{t}_0 > 0$. By the inclination lemma, D_m uniformly C^r converges to $D_a(p)$. A short curve in $W^s(p)$ containing q as an interior point meets D_m transversely in two points for all sufficiently large m. Let $\widetilde{H}_m$ be the component of $f^{N+n_0}(D_m) \cap U_a(p)$ containing $f^{N+n_0}(\boldsymbol{w}_m)$. Then $\widetilde{H}_m$ C^r converges to $\widetilde{H}_0$ as $m \to \infty$. By (3.1), there exist C^r functions $\varphi_m(y,t)$ C^r converging to φ and representing $\widetilde{H}_m$ as the graph of $x = \varphi_m(y,t)$. Then the front curve $\widetilde{\gamma}_m$ in $\widetilde{H}_m$ is defined as the front curve $\widetilde{\gamma}_0$ in $\widetilde{H}_0$. Since $\partial\varphi_m(y,t)/\partial t$ C^{r-1} converges to $\partial\varphi(y,t)/\partial t$, $\widetilde{\gamma}_m$ also C^{r-1} converges to $\widetilde{\gamma}_0$. Note that $\widetilde{\gamma}_m$ divides $\widetilde{H}_m$ into the upper surface $\widetilde{H}_m^+$ and the lower surface $\widetilde{H}_m^-$ with $\widetilde{\gamma}_m = \widetilde{H}_m^+ \cap \widetilde{H}_m^-$ and $H_m = \mathrm{pr}(\widetilde{H}_m) = \mathrm{pr}(\widetilde{H}_m^+) = \mathrm{pr}(\widetilde{H}_m^-)$. The image $\gamma_m = \mathrm{pr}(\widetilde{\gamma}_m)$ is called the folding curve of H_m.

3.3 Limit straight segments

A curve γ in $D_a(p)$ is called a *straight segment* if γ is a segment with respect to the Euclidean metric on $D_a(p)$. In this section, we will construct a proper straight segment $\gamma_0^\natural$ in $D_a(p)$ with $p \notin \gamma_0^\natural$ which is mapped to a straight segment in $U_{a'}(p')$ by h.

3.3.1 Sequences of folding curves converging to straight segments

Let α be an oriented C^{r-1} curve in $D_a(p)$ of bounded length. Since $r-1 \geq 2$, there exists the maximum absolute curvature $\kappa(\alpha)$ of α. If α passes near the center 0 of $D_a(p)$ and satisfies $\kappa(\alpha) < 1/a$, then α has a unique point $z(\alpha)$ with $\mathrm{dist}(0, z(\alpha)) = \mathrm{dist}(0, \alpha)$. In fact, if α had two points z_i $(i = 1,2)$ with $\mathrm{dist}(0, z_i) = \mathrm{dist}(0, \alpha)$, then for a point z_3 in α with the maximum $\mathrm{dist}(0, z_3)$ between z_1 and z_2, the curvature of α at z_3 is not less than $1/\mathrm{dist}(0, z_3) \geq 1/a$, a contradiction. We denote by $\vartheta(\alpha) \mod 2\pi$ the angle between $\widehat{\alpha}$ and the positive direction of the x-axis at 0, where $\widehat{\alpha}$ is the oriented curve in $D_a(p)$ obtained from α by the parallel translation taking $z(\alpha)$ to 0.

By (3.3), there exists a constant $\widetilde{d}_0 > 0$ such that

$$\mathrm{dist}(\widetilde{\gamma}_m, \text{the } t\text{-axis}) = \widetilde{d}_0(\widetilde{t}_0\lambda^m + o(\lambda^m)) + o(\lambda^m) = \widetilde{d}_0\widetilde{t}_0\lambda^m + o(\lambda^m). \tag{3.4}$$

Since γ_m C^{r-1} converges to γ_0, $\kappa(\gamma_m)$ also converges to $\kappa(\gamma_0)$ as $m \to \infty$. This shows that

$$\sup_m\{\kappa(\gamma_m)\} = \kappa_0 < \infty. \tag{3.5}$$

It follows that, for all sufficiently large m, there exists a unique point c_m of γ_m with

$$\mathrm{dist}(c_m, 0) = \mathrm{dist}(\gamma_m, 0) = \mathrm{dist}(\widetilde{c}_m, \text{the } t\text{-axis}) = \mathrm{dist}(\widetilde{\gamma}_m, \text{the } t\text{-axis}),$$

where $\widetilde{c}_m$ is the point of $\widetilde{\gamma}_m$ with $\mathrm{pr}(\widetilde{c}_m) = c_m$.

Fix w with $0 < w < a/2$ arbitrarily. For any $n \in \mathbb{N}$, let $m(n)$ be the minimum positive integer such that $f^n(c_m)$ is contained in $D_w(p)$ for any $m \geq m(n)$. Then $\lim_{n\to\infty} m(n) = \infty$ holds. For any $m \geq m(n)$, the component $\widetilde{H}_{m,n}$ of $f^n(\widetilde{H}_m) \cap U_a(p)$ containing $\widetilde{c}_{m,n} = f^n(\widetilde{c}_m)$ is a proper disk in $U_a(p)$ with $\partial\widetilde{H}_{m,n} \subset \partial_{\mathrm{side}}U_a(p)$. Then $\widetilde{\gamma}_{m,n} = f^n(\widetilde{\gamma}_m) \cap \widetilde{H}_{m,n}$ is the front curve of $\widetilde{H}_{m,n}$ and $\gamma_{m,n} = \mathrm{pr}(\widetilde{\gamma}_{m,n})$ is the folding curve of $H_{m,n} = \mathrm{pr}(\widetilde{H}_{m,n})$. Then $c_{m,n} = \mathrm{pr}(\widetilde{c}_{m,n})$ is a unique point of $\gamma_{m,n}$ closest to 0. Here we orient $\widetilde{\gamma}_m = \widetilde{\gamma}_{m,0}$ so that $\widetilde{\gamma}_{m,0}$ C^{r-1} converges as oriented curves to $\widetilde{\gamma}_0$ as $m \to \infty$. Suppose that $\gamma_{m,n}$ has the orientation induced from that on $\widetilde{\gamma}_{m,0}$ via $\mathrm{pr} \circ f^n$. In particular, it follows that

$$\lim_{m\to\infty} \vartheta(\gamma_{m,0}) = \vartheta(\gamma_0). \tag{3.6}$$

We set $d_{m,n} = \mathrm{dist}(c_{m,n}, 0)$. By (3.4),

$$d_{m,n} = r^n(\widetilde{d}_0\widetilde{t}_0\lambda^m + o(\lambda^m)). \tag{3.7}$$

There exist subsequences $\{m_j\}$, $\{n_j\}$ of $\mathbb{N}$ and $w\lambda/2 \le w_0 \le w$ such that

$$\lim_{j\to\infty} \widetilde{d}_0\widetilde{t}_0\lambda^{m_j}r^{n_j} = w_0. \tag{3.8}$$

If necessary taking subsequences of $\{m_j\}$ and $\{n_j\}$ simultaneously, we may also assume that $\vartheta(\gamma_{m_j,n_j})$ has a limit $\theta^\natural$. Since $f(z) = re^{\sqrt{-1}\theta}z$ on $D_a(p)$, by (3.5) we have

$$\kappa(\gamma_{m_j,n_j}) \le r^{-n_j}\kappa(\gamma_{m_j,0}) \le r^{-n_j}\kappa_0 \to 0 \quad \text{as} \quad j \to \infty.$$

Thus the following lemma is obtained immediately.

Lemma 3.5. *The sequence γ_{m_j,n_j} uniformly converges as oriented curves to an oriented straight segment $\gamma_0^\natural$ in $D_a(p)$ with $\vartheta(\gamma_0^\natural) = \theta^\natural$ and $\mathrm{dist}(\gamma_0^\natural, 0) = w_0$.*

We say that $\gamma_0^\natural$ is the *limit straight segment* of γ_{m_j,n_j}.

3.3.2 Limit straight segments preserved by the conjugacy

Let $U_{a'}(p')$, $U_{b'}(p')$ be the circular columns defined as $U_a(p)$ for some $0 < a' < b'$ which are contained in a coordinate neighborhood around p' with respect to which f' is linear. One can retake $a > 0$ and choose such a', b' so that $U_{a'}(p') \subset h(U_a(p)) \subset U_{b'}(p')$.

Let $\widetilde{H}'_{m,n}$ be the component of $h(\widetilde{H}_{m,n}) \cap U_{a'}(p')$ defined as $\widetilde{H}_{m,n}$ and $\mathrm{pr}(\widetilde{H}'_{m,n}) = H'_{m,n}$. One can define the front and folding curves $\widetilde{\gamma}'_{m,n}$, $\gamma'_{m,n}$ in $\widetilde{H}'_{m,n}$ and $H'_{m,n}$ as $\widetilde{\gamma}_{m,n}$, $\gamma_{m,n}$ in $\widetilde{H}_{m,n}$ and $H_{m,n}$ respectively. See Figure 3.5.

Since h is only supposed to be a homeomorphism, $h(\widetilde{\gamma}_{m,n}) \cap U_{a'}(p')$ would not be equal to $\widetilde{\gamma}'_{m,n}$. We will show that this equality holds in the limit. For the sequences $\{m_j\}$, $\{n_j\}$ given in Section 3.3, we set $\widetilde{H}_{m_j,n_j} = \widetilde{H}_{(j)}$, $H_{m_j,n_j} = H_{(j)}$, $\widetilde{H}'_{m_j,n_j} = \widetilde{H}'_{(j)}$ and $H'_{m_j,n_j} = H'_{(j)}$ for simplicity. Similarly, suppose that $\widehat{H}'_{(j)}$ is the component of $W^u(p') \cap U_{b'}(p')$ containing $\widetilde{H}'_{(j)}$ and $\widehat{\gamma}'_{m_j,n_1}$ is the front curve of $\widehat{H}'_{(j)}$. The distance between $\boldsymbol{x}$, $\boldsymbol{y}$ in $U_a(p)$ is denoted by $d(\boldsymbol{x}, \boldsymbol{y})$ and that between $\boldsymbol{x}'$, $\boldsymbol{y}'$ in $U_{a'}(p')$ by $d'(\boldsymbol{x}', \boldsymbol{y}')$.

The *path metric* on $\widetilde{H}_{(j)}$ is denoted by $d_{\widetilde{H}_{(j)}}$. That is, for any $\boldsymbol{x}$, $\boldsymbol{y} \in \widetilde{H}_{(j)}$, $d_{\widetilde{H}_{(j)}}(\boldsymbol{x}, \boldsymbol{y})$ is the length of a shortest piecewise smooth curve in $\widetilde{H}_{(j)}$ connecting $\boldsymbol{x}$ with $\boldsymbol{y}$. The path metrics $d_{\widetilde{H}'_{(j)}}$ on $\widetilde{H}'_{(j)}$ and $d_{\widehat{H}'_{(j)}}$ on $\widehat{H}'_{(j)}$ are defined similarly.

Lemma 3.6. (i) *For any $\varepsilon > 0$, there exists a constant $\eta(\varepsilon) > 0$ independent of $j \in \mathbb{N}$ and satisfying the following conditions.*

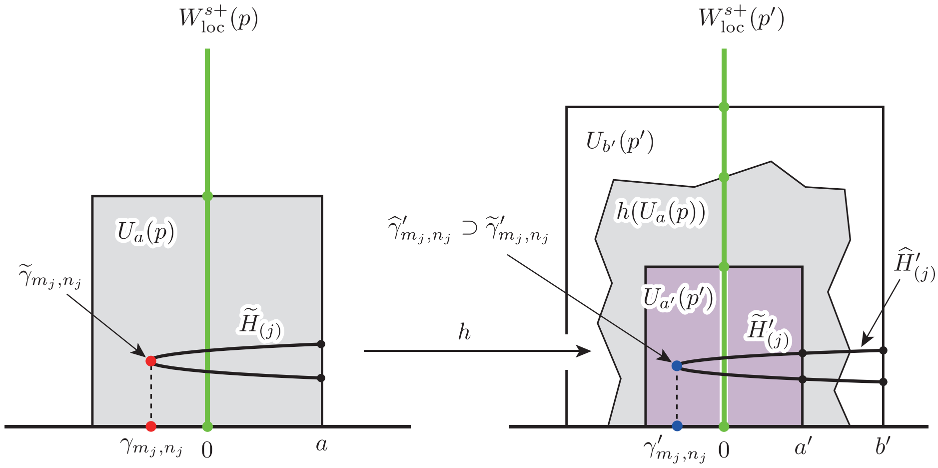

Figure 3.5: The image $h(\widetilde{H}_{(j)})$ is contained in $\widehat{H}'_{(j)}$, but $h(\widetilde{H}^{\pm}_{(j)})$ is not necessarily contained in $\widehat{H}'^{\pm}_{(j)}$.

- $\lim_{\varepsilon\to 0}\eta(\varepsilon)=0$.
- *Let* $\boldsymbol{x}$, $\boldsymbol{y}$ *be any points of* $\widetilde{H}_{(j)}$ *both of which are contained in one of* $\widetilde{H}^+_{(j)}$ *and* $\widetilde{H}^-_{(j)}$. *If* $d(\boldsymbol{x},\boldsymbol{y})<\eta(\varepsilon)$, *then* $d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{y})<\varepsilon$.

(ii) *For any* $\varepsilon>0$, *there exists a constant* $\delta(\varepsilon)>0$ *independent of* $j\in\mathbb{N}$ *and satisfying the following conditions.*

- $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0$.
- *Let* $\boldsymbol{x}$, $\boldsymbol{y}$ *be any points of* $\widetilde{H}_{(j)}$ *both of which are contained in one of* $\widetilde{H}^+_{(j)}$ *and* $\widetilde{H}^-_{(j)}$. *If* $d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{y})<\delta(\varepsilon)$ *and* $\boldsymbol{x}'=h(\boldsymbol{x})$ *and* $\boldsymbol{y}'=h(\boldsymbol{y})$ *are contained in* $\widetilde{H}'_{(j)}$, *then* $d_{\widetilde{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{y}')<\varepsilon$.

One can take these constants $\eta(\varepsilon)$, $\delta(\varepsilon)$ so that they work also for $d_{\widetilde{H}'_{(j)}}$ and $d_{\widehat{H}'_{(j)}}$.

Proof. (i) The assertion is proved immediately from the fact that $\widetilde{H}^{\pm}_{(j)}$ uniformly converges to a disk $H^\natural$ in $D_a(p)$ such that $d(\boldsymbol{x},\boldsymbol{y})=d_{H^\natural}(\boldsymbol{x},\boldsymbol{y})$ for any $\boldsymbol{x},\boldsymbol{y}\in H^\natural$.
(ii) Suppose that $\boldsymbol{x},\boldsymbol{y}\in\widetilde{H}^+_{(j)}$. First we consider the case that both $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are contained in one of $\widetilde{H}'^+_{(j)}$ and $\widetilde{H}'^-_{(j)}$, say $\widetilde{H}'^+_{(j)}$. If $d_{\widetilde{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{y}')\geq\varepsilon$, then it follows from the assertion (i) that $d'(\boldsymbol{x}',\boldsymbol{y}')\geq\eta(\varepsilon)$. Since h is uniformly continuous on $U_a(p)$, there exists a constant $\delta_1(\varepsilon)>0$ with $\lim_{\varepsilon\to 0}\delta_1(\varepsilon)=0$ and $d(\boldsymbol{x},\boldsymbol{y})\geq\delta_1(\varepsilon)$. Hence, in particular, $d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{y})\geq\delta_1(\varepsilon)$. Thus $d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{y})<\delta_1(\varepsilon)$ implies $d_{\widetilde{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{y}')<\varepsilon$.

Next we suppose that $\boldsymbol{x}'\in\widetilde{H}'^+_{(j)}$ and $\boldsymbol{y}'\in\widetilde{H}'^-_{(j)}$. Consider a shortest curve α in $\widetilde{H}_{(j)}$ connecting $\boldsymbol{x}$ and $\boldsymbol{y}$. Since $\alpha'=h(\alpha)$ is contained in $\widehat{H}'_{(j)}$, α' intersects $\widehat{\gamma}'_{m_j,n_j}$ non-trivially.

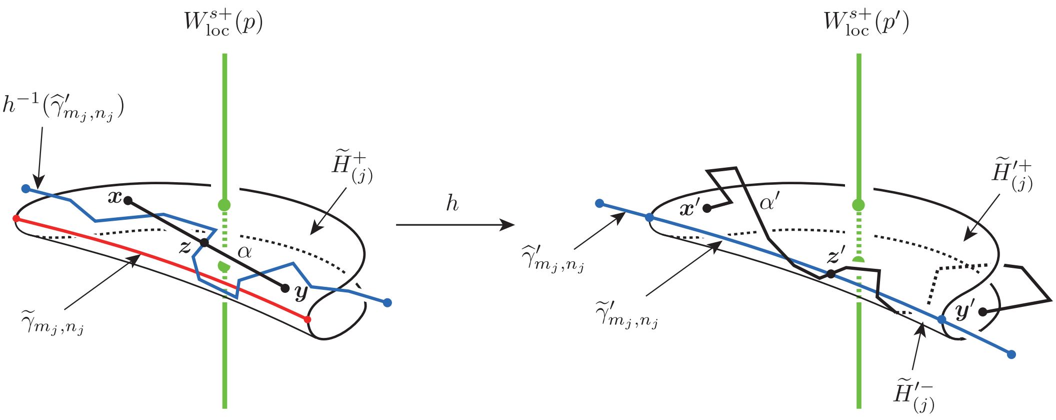

Figure 3.6: The case of $\boldsymbol{x}, \boldsymbol{y} \in \widetilde{H}^{+}_{(j)}$, $\boldsymbol{x}' \in \widetilde{H}'^{+}_{(j)}$ and $\boldsymbol{y}' \in \widetilde{H}'^{-}_{(j)}$.

Let $\boldsymbol{z}$ be one of the intersection points of α with $h^{-1}(\widetilde{\gamma}'_{m_j,n_j})$. See Figure 3.6. Suppose that $d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{y}) < \delta_1(\varepsilon/2)$. Since $d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{y}) = d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{z}) + d_{\widetilde{H}_{(j)}}(\boldsymbol{z},\boldsymbol{y})$,

$$d_{\widetilde{H}_{(j)}}(\boldsymbol{x},\boldsymbol{z}) < \delta_1(\varepsilon/2) \quad \text{and} \quad d_{\widetilde{H}_{(j)}}(\boldsymbol{z},\boldsymbol{y}) < \delta_1(\varepsilon/2).$$

Since $\boldsymbol{x}', \boldsymbol{z}' \in \widehat{H}'^{+}_{(j)}$ and $\boldsymbol{z}', \boldsymbol{y}' \in \widehat{H}'^{-}_{(j)}$, by the result in the previous case we have $d_{\widehat{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{z}') < \varepsilon/2$ and $d_{\widehat{H}'_{(j)}}(\boldsymbol{z}',\boldsymbol{y}') < \varepsilon/2$, and hence

$$d_{\widetilde{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{y}') = d_{\widehat{H}'_{(j)}}(\boldsymbol{x}',\boldsymbol{y}') < \varepsilon.$$

Thus $\delta(\varepsilon) := \delta_1(\varepsilon/2)$ satisfies the conditions of (ii). $\square$

The following result is a key of this section.

Lemma 3.7. *For any $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that, for any $j \geq j_0$,*

$$h(\widetilde{\gamma}_{m_j,n_j}) \cap \widetilde{H}'_{(j)} \subset \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)}),$$

where $\mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)})$ is the ε-neighborhood of $\widetilde{\gamma}'_{m_j,n_j}$ in $\widetilde{H}'_{(j)}$.

Figure 3.7 illustrates the situation of Lemma 3.7.

Proof. For $\sigma = \pm$, we will show that $h^{-1}(\widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\gamma'_{m_j,n_j}, \widetilde{H}'_{(j)})) \subset \widetilde{H}^{\sigma}_{(j)}$ for all sufficiently large j. Since $h^{-1}|_{U_{a'}(p')}$ is uniformly continuous, there exists $\nu(\varepsilon) > 0$ such that, for any $\boldsymbol{x}', \boldsymbol{y}' \in U_{a'}(p')$ with $d'(\boldsymbol{x}',\boldsymbol{y}') < \nu(\varepsilon)$, the inequality $d(\boldsymbol{x},\boldsymbol{y}) < \eta(\delta(\varepsilon))$ holds, where $\boldsymbol{x} = h^{-1}(\boldsymbol{x}')$, $\boldsymbol{y} = h^{-1}(\boldsymbol{y}')$. Since both $\widetilde{H}'^{+}_{(j)}$ and $\widetilde{H}'^{-}_{(j)}$ uniformly converge to the same half disk $H'^{\natural}$ in $D_{a'}(p')$, there exists $j_0 \in \mathbb{N}$ such that, for any $j \geq j_0$ and any $\boldsymbol{x}' \in \widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{(j)}, \widetilde{H}'_{(j)})$, $d'(\boldsymbol{x}',\boldsymbol{y}')$ is less than $\nu(\varepsilon)$, where $\boldsymbol{y}'$ is the element of $\widetilde{H}'^{-\sigma}_{(j)}$ with $\mathrm{pr}(\boldsymbol{x}') = \mathrm{pr}(\boldsymbol{y}')$. Then we have $d(\boldsymbol{x},\boldsymbol{y}) < \eta(\delta(\varepsilon))$. If both $\boldsymbol{x}$ and $\boldsymbol{y}$ were contained in one of $\widetilde{H}^{\sigma}_{(j)}$

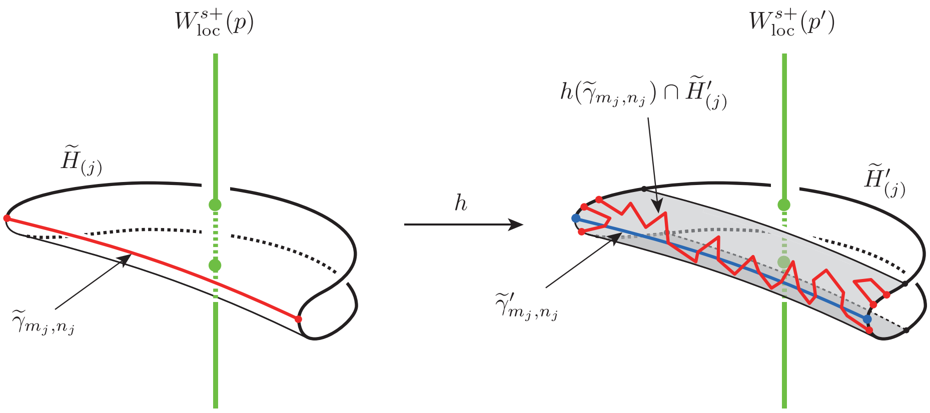

Figure 3.7: The shaded region represents $\mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)})$.

and $\widetilde{H}^{-\sigma}_{(j)}$, then by Lemma 3.6 (i) $d_{\widetilde{H}_{(j)}}(\boldsymbol{x}, \boldsymbol{y}) < \delta(\varepsilon)$. Then, by Lemma 3.6 (ii), $d_{\widetilde{H}'_{(j)}}(\boldsymbol{x}', \boldsymbol{y}')$ would be less than ε. This contradicts that $\boldsymbol{x}' \in \widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)})$ and $\boldsymbol{y}' \in \widetilde{H}'^{-\sigma}_{(j)}$. See Figure 3.8. Thus, if $\boldsymbol{y}$ is contained in $\widetilde{H}^{\sigma}_{(j)}$, then $\boldsymbol{x}$ is not in $\widetilde{H}^{\sigma}_{(j)}$. In particular, $\boldsymbol{x}$ is

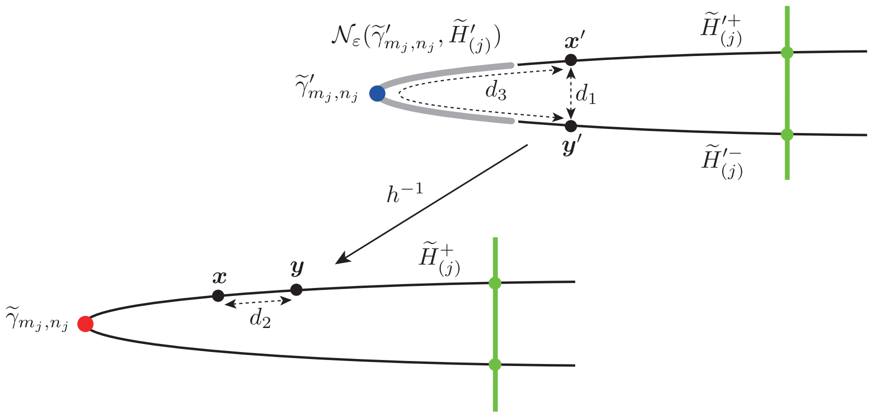

Figure 3.8: The situation which does not actually occur. $d_1 := \text{dist}(\boldsymbol{x}', \boldsymbol{y}') < \nu(\varepsilon)$, $d_2 := \text{dist}_{\widetilde{H}_{(j)}}(\boldsymbol{x}, \boldsymbol{y}) < \delta(\varepsilon)$ and $d_3 := \text{dist}_{\widetilde{H}'_{(j)}}(\boldsymbol{x}', \boldsymbol{y}') < \varepsilon$.

not contained in $\widetilde{\gamma}_{m_j,n_j} = \widetilde{H}^{+}_{(j)} \cap \widetilde{H}^{-}_{(j)}$, and so $\widetilde{\gamma}_{m_j,n_j} \cap h^{-1}(\widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m,n}, \widetilde{H}'_{(j)})) = \emptyset$. Since $h^{-1}(\widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m,n}, \widetilde{H}'_{(j)}))$ is connected, it follows that $h^{-1}(\widetilde{H}'^{\sigma}_{(j)} \setminus \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m,n}, \widetilde{H}'_{(j)})) \subset \widetilde{H}^{\sigma}_{(j)}$ for $\sigma = \pm$, and hence $h^{-1}(\mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)})) \supset \widetilde{\gamma}_{m_j,n_j} \cap h^{-1}(\widetilde{H}'_{(j)})$. This completes the proof. $\square$

From the proof of Lemma 3.7, we know that there exists a simple curve in $h(\widetilde{\gamma}_{m_j,n_j}) \cap \widetilde{H}'_{(j)}$ connecting the two components of $\partial \widetilde{H}'_{(j)} \cap \partial \mathcal{N}_\varepsilon(\widetilde{\gamma}'_{m_j,n_j}, \widetilde{H}'_{(j)})$. The following corollary says that the images of certain straight segments in $D_a(p)$ by the homeomorphism h are naturally straight segments in $D_{a'}(p')$.

Corollary 3.8. *For the limit straight segment $\gamma_0^\natural$ of γ_{m_j,n_j}, $h(\gamma_0^\natural)\cap D_{a'}(p')$ is the limit straight segment of γ'_{m_j,n_j}, i.e. $h(\gamma_0^\natural)\cap D_{a'}(p') = \gamma_0'^\natural$.*

Proof. Since $\gamma_0^\natural$ is the limit straight segment of $\widetilde{\gamma}_{m_j,n_j}$ and h is uniformity continuous, $h(\gamma_0^\natural)\cap D_{a'}(p')$ is the limit of $h(\widetilde{\gamma}_{m_j,n_j})\cap \widetilde{H}'_{(j)}$. It follows from Lemma 3.7 that $h(\gamma_0^\natural)\cap D_{a'}(p')$ is also the limit of $\mathrm{pr}(\widetilde{\gamma}'_{m_j,n_j}) = \gamma'_{m_j,n_j}$, that is, $h(\gamma_0^\natural)\cap D_{a'}(p')$ is equal to the limit straight segment of γ'_{m_j,n_j}. □

For any straight segment l in $D_a(p)$ such that $h(l)$ is also a straight segment in $D_{b'}(p')$, we denote $h(l)\cap D_{a'}(p')$ simply by $h(l)$. In particular, Corollary 3.8 implies that $h(\gamma_0^\natural) = \gamma_0'^\natural$.

3.4 Proof of Theorem 3.1

Suppose that $\mathrm{St}_a(p)$ is the set of oriented proper straight segments in $D_a(p)$ passing through 0, that is, each element of $\mathrm{St}_a(p)$ is an oriented diameter of the disk $D_a(p)$. For any $l \in \mathrm{St}_a(p)$ and $n \in \mathbb{N}$, the component of $f^n(l)\cap U_a(p)$ containing 0 is also an element of $\mathrm{St}_a(p)$. We denote the element simply by $f^n(l)$.

Since $f^n|_{D_a(p)}$ preserves angles on $D_a(p)$, by (3.6), for any $k, n \in \mathbb{N}$,

$$\vartheta(\gamma_{m,n}) - \vartheta(\gamma_{m+k,n}) = \vartheta(\gamma_{m,0}) - \vartheta(\gamma_{m+k,0}) \to \vartheta(\gamma_0) - \vartheta(\gamma_0) = 0$$

as $m\to\infty$. Moreover it follows from (3.7) that $\lim_{j\to\infty} d_{m_j+k,n_j} = w_0\lambda^k$. By these facts together with Lemma 3.5, one can show that γ_{m_j+k,n_j} uniformly converges as $m\to\infty$ to a straight segment $\gamma_k^\natural$ in $U_a(p)$ with

$$\vartheta(\gamma_k^\natural) = \theta^\natural \quad\text{and}\quad d(0,\gamma_k^\natural) = w_0\lambda^k. \tag{3.9}$$

Thus we have obtained the parallel family $\{\gamma_k^\natural\}$ of oriented straight segments in $D_a(p)$. See Figure 3.9. By Corollary 3.8, $\{\gamma_k'^\natural\}$ with $\gamma_k'^\natural = h(\gamma_k^\natural)$ is also a parallel family of oriented straight segments in $D_{a'}(p')$. Since $\gamma_k'^\natural$ is the limit of γ'_{m_j+k,n_j} as $j\to\infty$, we have the equations

$$\vartheta(\gamma_k'^\natural) = \theta'^\natural \quad\text{and}\quad d(0,\gamma_k'^\natural) = w_0'\lambda'^k. \tag{3.10}$$

corresponding to (3.9) for some $\theta'^\natural$ and $w_0' > 0$. Let $\gamma_\infty^\natural \in \mathrm{St}_a(p)$ (resp. $\gamma_\infty'^\natural \in \mathrm{St}_{a'}(p')$) be the limit of $\gamma_k^\natural$ (resp. $\gamma_k'^\natural$).

Proof of Theorem 3.1. By Lemma 3.5 and (3.7), $w_0 = \lim_{j\to\infty} \widetilde{d}_0\widetilde{t}_0\lambda^{m_j}r^{n_j}$. This implies that

$$\lim_{j\to\infty}\left(\frac{m_j}{n_j}\log\lambda + \log r\right) = \lim_{j\to\infty}\frac{1}{n_j}\log\frac{w_0}{\widetilde{d}_0\widetilde{t}_0} = 0$$

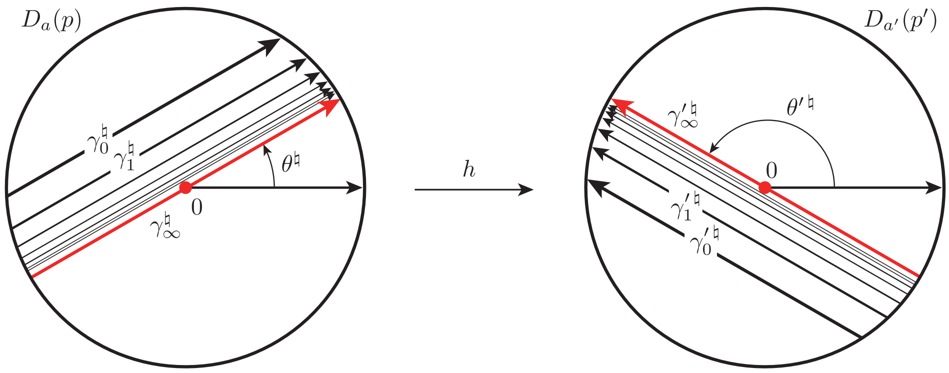

Figure 3.9: The images of the parallel straight segments $\gamma_k^\natural$ in $D_a(p)$ by h.

and hence $\lim_{j\to\infty} \dfrac{m_j}{n_j} = -\dfrac{\log r}{\log \lambda}$. Applying the same argument to $\gamma'^\natural_{m_j,n_j}$, we also have $\lim_{j\to\infty} \dfrac{m_j}{n_j} = -\dfrac{\log r'}{\log \lambda'}$. This shows the part (D1) of Theorem 3.1.

Now we will prove the part (D2). For any $n \in \mathbb{N} \cup \{0\}$, we set $f^n(\gamma_\infty^\natural) = \gamma_{\infty,n}^\natural$ and $f'^n(\gamma'^\natural_\infty) = \gamma'^\natural_{\infty,n}$. By Corollary 3.8,

$$h(\gamma_{\infty,n}^\natural) = h(f^n(\gamma_\infty^\natural)) = f'^n(h(\gamma_\infty^\natural)) = f'^n(\gamma'^\natural_\infty) = \gamma'^\natural_{\infty,n}. \tag{3.11}$$

We identify $\mathrm{St}_a(p)$ with the unit circle $S^1 = \{z \in \mathbb{C}\,;\, |z| = 1\}$ by corresponding $l \in \mathrm{St}_a(p)$ to $e^{\sqrt{-1}\vartheta(l)}$. Then the action of f on $\mathrm{St}_a(p)$ is equal to the θ-rotation R_θ on S^1 defined by $R_\theta(z) = e^{\sqrt{-1}\theta}z$.

If $\theta/2\pi = v/u$ for coprime positive integers u, v with $0 \le v < u$. Since $h(\gamma_\infty^\natural) = \gamma'^\natural_\infty$, we have $f'^k(\gamma'^\natural_\infty) \neq \gamma'^\natural_\infty$ for $k = 1, \dots, u-1$ and $f'^u(\gamma'^\natural_\infty) = \gamma'^\natural_\infty$. This implies that $\theta'/2\pi = v'/u$ for some $v' \in \mathbb{N}$ with $0 \le v' < u$. Since $h|_{D_a(p)} : D_a(p) \to D_{a'}(p')$ is a homeomorphism with the correspondence $h(R_\theta^k(\gamma_\infty^\natural)) = R_{\theta'}^k(\gamma'^\natural_\infty)$ $(k = 0, 1, \dots, u-1)$, there exists an orientation-preserving homeomorphism $\eta_0 : S^1 \to S^1$ with $\eta_0(e^{\sqrt{-1}(\theta^\natural + k\theta)}) = e^{\sqrt{-1}(\theta'^\natural + k\theta')}$ for $k = 0, 1, \dots, u-1$. We set $\Gamma = \{e^{\sqrt{-1}(\theta^\natural + k\theta)};\, k = 0, 1, \dots, u-1\}$ and $\Gamma' = \{e^{\sqrt{-1}(\theta'^\natural + k\theta')};\, k = 0, 1, \dots, u-1\}$. Then $[e^{\sqrt{-1}\theta^\natural}, e^{\sqrt{-1}(\theta^\natural+\theta)}) \cap \Gamma$ consists of v points, where $[a, b)$ denotes the positively oriented half-open interval in S^1 for $a, b \in S^1$ with $a \neq b$. Since moreover $\eta_0\big([e^{\sqrt{-1}\theta^\natural}, e^{\sqrt{-1}(\theta^\natural+\theta)}) \cap \Gamma\big) = [e^{\sqrt{-1}\theta'^\natural}, e^{\sqrt{-1}(\theta'^\natural+\theta')}) \cap \Gamma'$ consists of v' points, it follows that $v = v'$, and hence $\theta = \theta'$.

Next we suppose that $\theta/2\pi$ is irrational. Then, for any $l \in \mathrm{St}_a(p)$, there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that the sequence $\gamma_{\infty,n_k}^\natural$ uniformly converges to l as $k \to \infty$. By (3.11), $\gamma'^\natural_{\infty,n_k}$ uniformly converges to $l' = h(l)$. Since $\gamma'^\natural_{\infty,n_k} \in \mathrm{St}_{a'}(p')$, l' is also an element of $\mathrm{St}_{a'}(p')$. Thus we have a homeomorphism $\eta : S^1 \to S^1$ with respect to which R_θ and $R_{\theta'}$ are conjugate. Since the rotation number is invariant under topological conjugations, $\theta/2\pi = \theta'/2\pi \mod 1$ holds. This completes the proof of the part (D2). $\square$

3.5 Proof of Theorem 3.3

In this section, we will prove Theorem 3.3. Suppose that f, f' are elements of $\mathrm{Diff}^r(M)$ satisfying the conditions of Theorems 3.1 and $\theta/2\pi$ is irrational.

Since $\theta = \theta' \mod 2\pi$, for any $k, j \in \mathbb{N}$,

$$\vartheta(\gamma^{\natural}_{\infty,k}) - \vartheta(\gamma^{\natural}_{\infty,j}) = \vartheta(\gamma'^{\natural}_{\infty,k}) - \vartheta(\gamma'^{\natural}_{\infty,j}) = (k-j)\theta \mod 2\pi. \tag{3.12}$$

Let l_j $(j = 1, 2)$ be any elements of $\mathrm{St}_a(p)$. As in the proof of Theorem 3.1, there exist subsequences $\{n_k\}$, $\{n_j\}$ of $\mathbb{N}$ such that the sequencers $\{\gamma^{\natural}_{\infty,n_k}\}$, $\{\gamma^{\natural}_{\infty,n_j}\}$ uniformly converge to l_1 and l_2 respectively. Then, $\{\gamma'^{\natural}_{\infty,n_k}\}$, $\{\gamma'^{\natural}_{\infty,n_j}\}$ also uniformly converge to the elements $l'_1 = h(l_1)$ and $l'_2 = h(l_2)$ of $\mathrm{St}_{a'}(p')$ respectively. Then, by (3.12),

$$\vartheta(l_2) - \vartheta(l_1) = \vartheta(l'_2) - \vartheta(l'_1) \mod 2\pi. \tag{3.13}$$

For the proof of Theorem 3.3, we need another family of straight segments in $D_a(p)$. Fix an integer a_0 with

$$a_0 > \max\left\{\frac{\log(2r)}{\log(\lambda^{-1})}, \frac{\log(2r')}{\log(\lambda'^{-1})}\right\}.$$

For any $k \geq 0$, we consider the straight segment $\xi^{\natural}_k = f^k(\gamma^{\natural}_{a_0 k}) \cap D_a(p)$. By (3.9),

$$\vartheta(\xi^{\natural}_k) - \vartheta(\xi^{\natural}_0) = k\theta \mod 2\pi \quad \text{and} \quad d(0, \xi^{\natural}_k) = w_0 \lambda^{a_0 k} r^k < 2^{-k} w_0. \tag{3.14}$$

Similarly, by (3.10), $\xi'^{\natural}_k = h(\xi^{\natural}_k)$ is a straight segment in $D_{a'}(p')$ with

$$\vartheta(\xi'^{\natural}_k) - \vartheta(\xi'^{\natural}_0) = k\theta \mod 2\pi \quad \text{and} \quad d(0, \xi'^{\natural}_k) = w'_0 \lambda'^{a_0 k} r'^k < 2^{-k} w'_0. \tag{3.15}$$

Proof of Theorem 3.3. Let α be the element of $\mathrm{St}_a(p)$ with $\vartheta(\xi^{\natural}_0) - \vartheta(\alpha) = \pi/2$ and $\alpha' = h(\alpha) \in \mathrm{St}_{a'}(p')$. We will show that $\theta_{\alpha'} := \vartheta(\xi'^{\natural}_0) - \vartheta(\alpha')$ is also equal to $\pi/2 \mod 2\pi$. See Figure 3.10. In fact, since $\theta/2\pi$ is irrational, by (3.14) there exists a subsequence

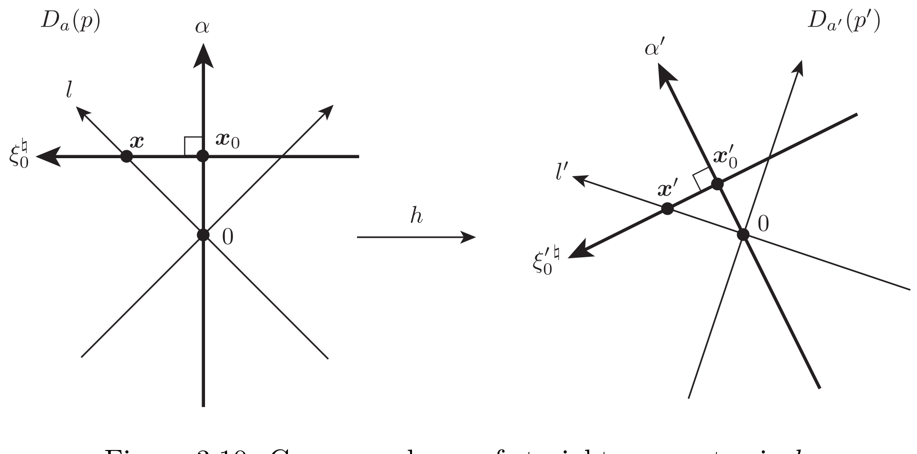

Figure 3.10: Correspondence of straight segments via h.

$\xi^\natural_{k_j}$ uniformly converges to α. Since $h|_{D_a(p)}$ is uniformly continuous, $\xi'^\natural_{k_j}$ also uniformly converges to α'. On the other hand, since $\vartheta(\xi^\natural_{k_j}) - \vartheta(\alpha) = k_j\theta + \pi/2 \mod 2\pi$ and $\vartheta(\xi'^\natural_{k_j}) - \vartheta(\alpha') = k_j\theta + \theta_{\alpha'} \mod 2\pi$,

$$\theta_{\alpha'} - \frac{\pi}{2} = \big(\vartheta(\xi'^\natural_{k_j}) - \vartheta(\alpha')\big) - \big(\vartheta(\xi^\natural_{k_j}) - \vartheta(\alpha)\big) \to 0 \mod 2\pi$$

as $j \to \infty$. Thus we have $\theta_{\alpha'} = \pi/2 \mod 2\pi$.

We denote by $z(\boldsymbol{x}) \in \mathbb{C}$ the entry of $\boldsymbol{x} \in D_a(p)$ with respect to the linearizing coordinate on $D_a(p)$. Similarly, the entry of $\boldsymbol{x}' \in D_{a'}(p')$ is denoted by $z'(\boldsymbol{x}')$. Let $\boldsymbol{x}_0$ be the intersection point of α and $\xi^\natural_0$, and let $\boldsymbol{x}'_0 = h(\boldsymbol{x}_0)$. One can set $z(\boldsymbol{x}_0) = \rho_0 e^{\sqrt{-1}\omega_0}$ and $z'(\boldsymbol{x}'_0) = \rho'_0 e^{\sqrt{-1}\omega'_0}$ for some $\rho_0 > 0$, $\rho'_0 > 0$ and $\omega_0, \omega'_0 \in \mathbb{R}$. We define the new linearizing coordinate on $D_{a'}(p')$ by using the linear conformal map such that, for any $\boldsymbol{x}' \in D_{a'}(p')$, $z'^{\,\mathrm{new}}(\boldsymbol{x}') = \rho_0\rho_0'^{-1} e^{\sqrt{-1}(\omega_0 - \omega'_0)} z'(\boldsymbol{x}')$. Then $z(\boldsymbol{x}_0) = z'^{\,\mathrm{new}}(\boldsymbol{x}'_0)$ holds.

For any $\boldsymbol{x} \in \xi^\natural_0$, there exists $l \in \mathrm{St}_a(p)$ with $\{\boldsymbol{x}\} = \xi^\natural_0 \cap l$. Then $\boldsymbol{x}' = h(\boldsymbol{x})$ is the intersection of $\xi'^\natural_0$ and $l' = h(l)$. By (3.13), $\vartheta(l) - \vartheta(\alpha) = \vartheta(l') - \vartheta(\alpha') \mod 2\pi$ and hence $z(\boldsymbol{x}) = z'^{\,\mathrm{new}}(\boldsymbol{x}')$. We say the property that h is *identical* on $\xi^\natural_0$. Since $\theta/2\pi$ is irrational, there exists $k_* \in \mathbb{N}$ satisfying

$$\frac{\pi}{3} \le \vartheta(\xi^\natural_{k_*}) - \vartheta(\xi^\natural_0) \le \frac{\pi}{2} \mod 2\pi.$$

Then $\xi^\natural_{k_*}$ meets $\xi^\natural_0$ at a single point $\boldsymbol{x}_{k_*}$ in $D_a(p)$. For $\alpha_{k_*} = f^{k_*}(\alpha)$ and $\alpha'_{k_*} = h(\alpha_{k_*})$, we have $\vartheta(\xi^\natural_{k_*}) - \vartheta(\alpha_{k_*}) = \vartheta(\xi'^\natural_{k_*}) - \vartheta(\alpha'_{k_*}) = \pi/2$. Since h is identical at $\boldsymbol{x}_{k_*}$, h is proved to be identical on $\xi^\natural_{k_*}$ by an argument as above. Then one can show inductively that, for any $n \in \mathbb{N}$, h is identical on $\xi^\natural_{nk_*}$. See Figure 3.11. By (3.14), $\lim_{n\to\infty} d(0, \xi^\natural_{nk_*}) = 0$. Since

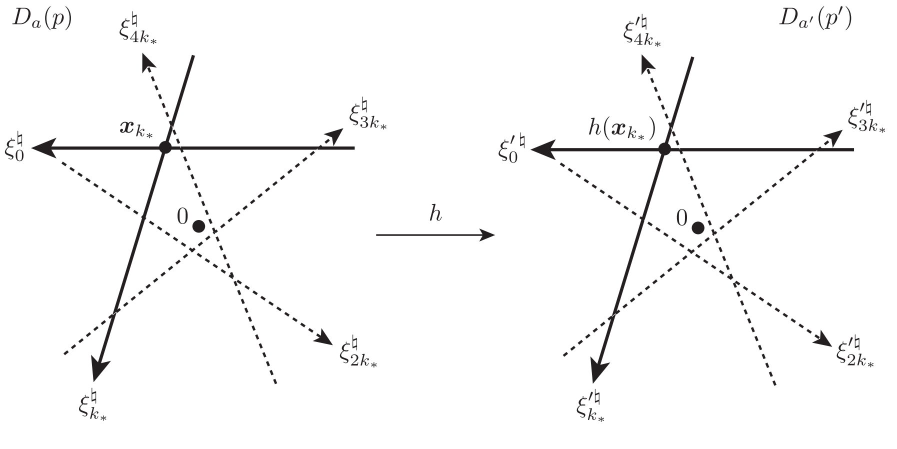

Figure 3.11: Correspondence via h with respect to the new coordinate on $D_{a'}(p')$.

moreover $k_*\theta/2\pi$ is irrational, $\overline{\bigcup_{n=1}^{\infty} \xi^\natural_{nk_*}}$ is equal to $D_a(p)$. This shows that h is identical on $D_a(p)$. In particular, this implies that $h|_{D_a(p)}$ is a linear conformal map with respect to the original coordinates. We write $z(q) = \rho_1 e^{\sqrt{-1}\omega_1}$ and $z'(q') = \rho'_1 e^{\sqrt{-1}\omega'_1}$. It follows from the assumption of $h(q) = q'$ in our theorems that $h(z) = \rho_1' \rho_1^{-1} e^{\sqrt{-1}(\omega_1' - \omega_1)} z$ for any $z \in \mathbb{C}$ with $|z| \le a$. In particular, this implies that $h|_{W^u_{\mathrm{loc}}(p)}$ is a linear conformal map. Let $\widetilde{h}$ be any other conjugacy homeomorphism between f and f' satisfying the conditions in Theorems 3.1 and 3.3. In particular, $\widetilde{h}(p) = p'$ and $\widetilde{h}(q) = q'$ hold. Since $z(q) = \rho_1 e^{\sqrt{-1}\omega_1}$ and $z'(q') = \rho_1' e^{\sqrt{-1}\omega_1'}$, one can show as above that $\widetilde{h}(z) = \rho_1' \rho_1^{-1} e^{\sqrt{-1}(\omega_1' - \omega_1)} z$ for any $z \in \mathbb{C}$ with $|z| \le a$ and hence $\widetilde{h}|_{D_a(p)} = h|_{D_a(p)}$. This shows the assertion (E2) of Theorem 3.3 and $r = r'$. Then, by the assertion (D1) of Theorem 3.1, we also have $\lambda = \lambda'$. This completes the proof. □

Let $\widehat{z}$ be the homoclinic transverse point of $W^u(p)$ and $W^s(p)$ given in Subsection 3.2.1. Fix a sufficiently large $n \in \mathbb{N}$ with $s = f^{-n}(\widehat{z}) \in D_p(a)$. Then $s' = h(s)$ is contained in $D_{b'}(p')$. The following corollary shows that $z(s)/z(q)$ is a modulus for f. Recall that $z(\boldsymbol{x}) \in \mathbb{C}$ is the entry of $\boldsymbol{x}$ with respect to the complex linearizing coordinate on $D_a(a)$. The complex number $z'(\boldsymbol{x}')$ is defined similarly for $\boldsymbol{x}' \in D_{a'}(p')$.

Corollary 3.9. *Let f, f' be elements of $\mathrm{Diff}^r(M)$ satisfying the conditions of Theorems 3.1 and 3.3, and let h be a conjugacy homeomorphism between f and f' with $h(p) = p'$ and $h(q) = q'$. If $h|_{W^u_{\mathrm{loc}}(p)}$ is orientation-preserving, then $z(s)/z(q) = z'(s')/z'(q')$. Otherwise, $z(s)/z(q) = \overline{z'(s')/z'(q')}$.*

Proof. Here we only consider the case that h is orientation-preserving. Since $h|_{D_a(p)}$ is a linear conformal map, the triangle with vertices $0, z(q), z(s)$ is similar to that with vertices $0, z'(q'), z'(s')$ with respect to Euclidean geometry. This shows $z(s)/z(q) = z'(s')/z'(q')$. □

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List of papers by Shinobu Hashimoto

[1] S. Hashimoto, Moduli of surface diffeomorphisms with cubic tangencies, to appear in *Tokyo J. Math.*

[2] S. Hashimoto, S. Kiriki, and T. Soma, Moduli of 3-dimensional diffeomorphisms with saddle foci, *Discrete Cont. Dynam. Sys.* **38** (2018), No. 10, 5021–5037.