TWISTED ALEXANDER POLYNOMIALS OF $(-2,3,2n+1)$-PRETZEL KNOTS

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Abstract. We calculate the twisted Alexander polynomials of $(-2,3,2n+1)$-pretzel knots associated to their holonomy representations.

1. Introduction

The notion of twisted Alexander polynomials was introduced by Wada [W] and Lin [L] independently in 1990s. The definition of Lin is for knots in S^3 and the definition of Wada is for finitely presented groups. The twisted Alexander polynomial is a generalization of the Alexander polynomial and is defined for the pair of a group and its representations. By Kitano and Morifuji [KM], it is known that Wada's twisted Alexander polynomials of the knot groups for any nonabelian representations into $SL_2(\mathbb{F})$ over a field $\mathbb{F}$ are polynomials. In this paper, by using the following definition due to Wada, we compute the twisted Alexander polynomials of $(-2,3,2n+1)$-pretzel knots K_n depicted in Figure 1 associated to their holonomy representations $\rho_m : G(K_n) \to SL_2(\mathbb{C})$ given in following section.

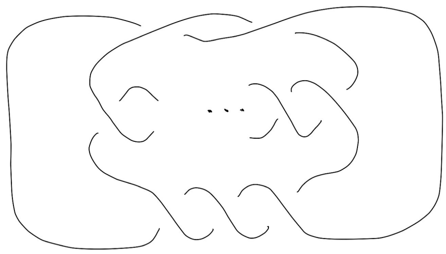

Figure 1. $(-2,3,2n+1)$-pretzel knot

Definition 1.1. Let $G(K) = \pi_1(S^3 \setminus K)$ be the knot group of a knot K presented by

$$G(K) = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_{n-1} \rangle.$$

Let Γ denote the free group generated by $x_1, \cdots, x_n$ and $\phi : \mathbb{Z}\Gamma \to \mathbb{Z}G(K)$ the natural ring homomorphism. Let $\rho : G(K) \to GL_d(\mathbb{C})$ be a d-dimensional linear representation of $G(K)$ and $\Phi : \mathbb{Z}\Gamma \to M_d(\mathbb{C}[t,t^{-1}])$ the ring homomorphism defind by

$$\Phi = (\tilde{\rho} \otimes \tilde{\alpha}) \circ \phi,$$

where $\tilde{\alpha} : \mathbb{Z}G(K) \to \mathbb{Z}\langle t, t^{-1} \rangle$ and $\tilde{\rho}$ are respective ring homomorphisms induced by the abelianization $\alpha : G(K) \to \langle t \rangle$ and ρ. We put

$$A_{i,j} = \Phi\left(\frac{\partial r_i}{\partial x_j}\right),$$

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where $\dfrac{\partial}{\partial x_j}$ denotes the Fox derivative (or free derivative) with respect to x_j, that is, a map $\mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$ satisfying the conditions

$$\frac{\partial}{\partial x_j}x_i = \delta_{ij},\ \frac{\partial}{\partial x_j}gg' = \frac{\partial}{\partial x_j}g + \frac{\partial}{\partial x_j}g',$$

where δ_{ij} denotes the Kronecker symbol and $g, g' \in \Gamma$. Then, the twisted Alexander polynomial of K is defined by

$$\Delta_{K,\rho} = \frac{\det A_{\rho,k}}{\det \Phi(x_k - 1)},$$

where $A_{\rho,k}$ is the $2(n-1) \times 2(n-1)$ matrix obtained from $A_\rho = (A_{i,j})$ by removing the k-th column, i.e.

$$A_{\rho,k} = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k-1} & A_{1,k+1} & \cdots & A_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n-1,1} & \cdots & A_{n-1,k-1} & A_{n-1,k+1} & \cdots & A_{n-1,n} \end{pmatrix}.$$

If K is hyperbolic, i.e. the complement $S^3 \setminus K$ admits a complete hyperbolic metric of finite volume, the most important representation is its holonomy representation into $SL_2(\mathbb{C})$ which is a lift of the representation into the group of orientation-preserving isometries of the hyperbolic 3-space $\mathbb{H}^3$. In fact, the twisted Alexander polynomials of some hyperbolic knots associated to their holonomy representations are computed by Dunfield, Friedl and Jackson [DFJ]. Recently, the twisted Alexander polynomials of some infinite families of knots, twist knots and genus one two-bridge knots associated to their holonomy representations, are computed by Morifuji [Mo1] and Tran [T1] and genus one two-bridge knots associated to the adjoint representations of their holonomy representations is also computed by Tran [T2].

$(-2, 3, 2n+1)$-pretzel knot is an infinite family of knots which contains the Fintushel-Stern knot i.e. $(-2, 3, 7)$-pretzel knot. It plays an important role in studying of exceptional surgeries of knots [Ma]. The A-polynomials of $(-2, 3, 2n+1)$-pretzel knot are computed by Tamura-Yokota [TY] and Garoufalidis-Mattman [GM].

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2. Holonomy representations

In this section, we give a presentation of knot group $G(K_n)$ and its holonomy representation $\rho_m : G(K_n) \to SL_2(\mathbb{C})$, where m represents the eigenvalue of the meridian of K_n.

Let L be the link depicted in Figure 2 and $E = S^3 \setminus L$. Then, the Wirtinger presentation (see [CF]) of $\pi_1(E)$ is given by

$$\langle a, b, x \mid \{axba(xb)^{-1}\}^{-1}x = xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb,\ [x, axba(xb)^{-1}] = 1\rangle,$$

where a, b and x is Wirtinger generators assigned to the corresponding pass depicted in Figure 2. Note that $E_n := S^3 \setminus K_n$ is obtained from L by $(-\frac{1}{n})$-surgery along the trivial component, that is, removing the tubular neighborhood of the trivial component and re-gluing the solid torus again. Therefore, by the van Kampen theorem, we have

$$\pi_1(E_n) = \langle a, b, x \mid \{axba(xb)^{-1}\}^{-1}x = xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb,\ x = \{axba(xb)^{-1}\}^n\rangle.$$

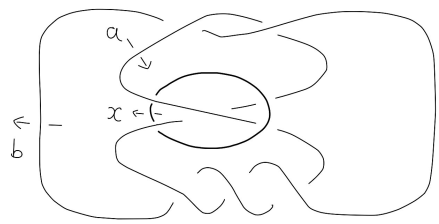

Figure 2. Link L

Proposition 2.1. *For a non-zero complex number m, there exists a representation $\rho_m : \pi_1(E_n) \to SL_2(\mathbb{C})$ such that*

$$\rho_m(a) = \begin{pmatrix} m & -\dfrac{(m^2 - s)(s^{2n+1} + 1)}{m(s+1)} \\ 0 & m^{-1} \end{pmatrix}, \quad \rho_m(b) = \frac{1}{s\alpha} \begin{pmatrix} \beta & -\dfrac{(s\alpha - m\beta)(ms\alpha - \beta)}{m\beta} \\ \beta & \dfrac{m(ms\alpha - \beta) + s\alpha}{m} \end{pmatrix},$$

and

$$\rho_m(x) = \begin{pmatrix} s^n & 0 \\ \dfrac{s^n - s^{-n}}{s^{2n+1} + 1} & s^{-n} \end{pmatrix},$$

where s is a solution to

$$\begin{aligned}
(1) \qquad 0 = {} & m^8(s-1)(s+1)^2(s^{2n} - s^2)s^{2n+2} \\
& -m^6\{s^{6n+3} + (2s^6 + s^5 - 4s^4 + s^3 + s^2 - s - 1)s^{4n+1} \\
& \qquad -(s^6 + s^5 - s^4 - s^3 + 4s^2 - s - 2)s^{2n+2} + s^6\} \\
& +m^4\{(s^2+1)s^{6n+2} + (s^6 + 2s^5 - 3s^4 - 2s^3 + 6s^2 - 4s - 2)s^{4n+3} \\
& \qquad -(2s^6 + 4s^5 - 6s^4 + 2s^3 + 3s^2 - 2s - 1)s^{2n} + (s^2 + 1)s^5\} \\
& -m^2\{s^{6n+3} + (2s^6 + s^5 - 4s^4 + s^3 + s^2 - s - 1)s^{4n+1} \\
& \qquad -(s^6 + s^5 - s^4 - s^3 + 4s^2 - s - 2)s^{2n+2} + s^6\} \\
& +(s-1)(s+1)^2(s^{2n} - s^2)s^{2n+2}
\end{aligned}$$

and α, β are given by

$$\begin{aligned}
\alpha = {} & (s^2-1)s^{2n}\{-m^6(s-1)s^2(s^{2n+1}+1) + m^4(s^{2n+2}(s^4 - 2s^2 + 3s - 1) + s^4 - 3s^3 + 2s^2 - 1) \\
& -m^2 s(s^{2n}(2s^3 - s^2 + 1) - s(s^3 - s + 2)) + s^2(s^{2n} - s^2)\}, \\
\beta = {} & m^7 s^{2n+2}(s^2-1)(s^3+1) \\
& -m^5 s^3\{s^{4n}(s^3 - s^2 + 1) + s^{2n-2}(s-1)(s^3 + s + 1)(s^3 + s^2 + 1) - (s^3 - s + 1)\} \\
& +m^3 s^2(s^3+1)(s^{2n} - 1)(s^{2n} + s^2) - ms^3(s^{2n} - s^2)(s^{2n} + s).
\end{aligned}$$

In what follows, for simplicity, we denote the right hand side of (1) by r_0.

Proof. For simplicity, put $A = \rho_m(a)$, $B = \rho_m(b)$, $X = \rho_m(x)$. By the aid of Mathematica, we have

$$AXBA(XB)^{-1} = \begin{pmatrix} s & 0 \\ \dfrac{s^2 - 1}{s(s^{2n+1}+1)} & \dfrac{1}{s} \end{pmatrix} + r_1 \begin{pmatrix} \dfrac{1}{m^3 s(s^{2n+1}+1)\alpha^2} & -\dfrac{1}{m^3 s(s+1)\alpha^2} \\ \dfrac{s+1}{m^3 s^2(s^{2n+1}+1)^2\alpha^2} & -\dfrac{1}{m^3 s^2(s^{2n+1}+1)\alpha^2} \end{pmatrix},$$

where

$$\begin{aligned}
r_1 = {} & -\alpha^2 ms(m^2 s^{2n+2} - m^2 - s^{2n+1} + s) + \alpha\beta(m^2-1)(m^2+1)s^{2n+1}(s+1) \\
& +\beta^2 ms^{2n}(m^2 s^{2n+1} - m^2 s - s^{2n+2} + 1) \equiv 0 \mod r_0.
\end{aligned}$$

Therefore, by (1), we have $X = \{AXBA(XB)^{-1}\}^n$, that is, $\rho_m(x) = \rho_m\left(\{axba(xb)^{-1}\}^n\right)$.

On the other hand, we can observe

$$AXB\{AXBA(XB)^{-1}\} \equiv XBX^{-1}\{AXBA(XB)^{-1}\}XB \mod r_0$$

and so $AXB\{AXBA(XB)^{-1}\} = XBX^{-1}\{AXBA(XB)^{-1}\}XB$ by (1). Further more, we obtain

$$\begin{aligned}
XB\{AXBA(XB)^{-1}\}^{-1}(AXB)^{-1}XB &= XB(AXB\{AXBA(XB)^{-1}\})^{-1}XB \\
&= XB(XBX^{-1}\{AXBA(XB)^{-1}\}XB)^{-1}XB \\
&= \{AXBA(XB)^{-1}\}^{-1}X
\end{aligned}$$

that is, $\rho_m\left(\{axba(xb)^{-1}\}^{-1}x\right) = \rho_m\left(xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb\right)$. This completes the proof. ∎

Remark 2.2. Since the representation ρ_m comes from the holonomy representation obtained from the ideal triangulation of E given in [TY], the holonomy representation ρ_m of $G(K_n)$ is given by the solution to (1) which maximizes the hyperbolic volume of $S^3 \setminus K_n$.

3. Calculation of the twisted Alexander polynomial

The following is the main result of this paper.

Theorem 3.1. *The twisted Alexander polynomial of K_n associated to ρ_m is given by*

$$\Delta_{K_n,\rho_m}(t) = 1 + \sum_{i=0}^{2n-1} \lambda_i (t^{i+3} + t^{4n-i+3}) + t^{4n+6},$$

where

$$\lambda_i = \begin{cases} \dfrac{(1+m^2)(Hs^{i/2+1}\beta - s(s^{i/2+1} - s^{-(i/2+1)})(\eta_1+\eta_2))}{Hm\beta} & \text{if } 0 \le i \le 2n-2 \text{ and } i \text{ is even,} \\ \dfrac{s^{(i-1)/2} - s^{-(i-1)/2}}{s - s^{-1}} & \text{if } 0 \le i \le 2n-2 \text{ and } i \text{ is odd,} \\ \dfrac{s^{n-1} - s^{-(n-1)}}{s - s^{-1}} - \dfrac{(s^2-1)\eta_1}{Hs^n\beta} & \text{if } i = 2n-1 \end{cases}$$

and we put

$$\begin{aligned} H &= 1 - m^2 s + m^2 s^{2n+1} - s^{2n+2}, \\ \eta_1 &= m\alpha - ms^{2n+1}\alpha + s^{2n}\beta + m^2 s^{2n}\beta, \\ \eta_2 &= -ms\alpha + ms^{2n+1}\alpha - s^{2n}\beta - s^{2n+1}\beta. \end{aligned}$$

To prove Theorem 3.1, it suffices to show

Proposition 3.2. *For simplicity, we put $S = s^n$ and $T = t^n$. The twisted Alexander polynomial $\Delta_{K_n,\rho_m}(t)$ is given by*

$$\begin{aligned} &\frac{S-T^2}{s-t^2}\frac{s}{S}\left(\frac{ms - mST^2 + (1+m^2)(1-s^2)StT^2}{m(1-s^2)t^2} + \frac{(1+m^2)(1-sSt^2T^2)(\eta_1+\eta_2)}{Hmt^3\beta}\right) \\ &+ \frac{1-ST^2}{1-st^2}\frac{s}{S}\left(\frac{(1+m^2)(1-s^2)S - mSt + mstT^2}{m(1-s^2)t^3} - \frac{(1+m^2)(sS - t^2T^2)(\eta_1+\eta_2)}{Hmt^3\beta}\right) \\ &+ \frac{1}{t^6} + T^4 + \frac{(1-s^2)(1+t^2)T^2\eta_1}{HSt^4\beta}. \end{aligned}$$

By multiplying t^6 and rearranging with respect to t , we obtain the formula of Theorem 3.1, when we use

$$\frac{S-T^2}{s-t^2} = \frac{S}{s}\sum_{i=0}^{n-1}\left(\frac{t^2}{s}\right)^i, \quad \frac{ST^2-1}{st^2-1} = \sum_{i=0}^{n-1}(st^2)^i.$$

4. Proof of Proposition 3.2

Recall that

$$\begin{aligned} \pi_1(E_n) &= \langle a,b,x \mid \{axba(xb)^{-1}\}^{-1}x = xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb,\ x = \{axba(xb)^{-1}\}^n\rangle \\ &= \langle a,c \mid (acac^{-1})^{n-1} = c(acac^{-1})^{-1}(ac)^{-1}c\rangle. \end{aligned}$$

Then the twisted Alexander polynomial of K_n is given by

$$\Delta_{K_n,\rho_m}(t) \quad = \quad \frac{\det\Phi\left(\dfrac{\partial}{\partial a}(acac^{-1})^{n-1} - \dfrac{\partial}{\partial a}c(acac^{-1})^{-1}(ac)^{-1}c\right)}{\det\Phi(c-1)},$$

where

$$\Phi\left(\frac{\partial}{\partial a}(acac^{-1})^{n-1} - \frac{\partial}{\partial a}c(acac^{-1})^{-1}(ac)^{-1}c\right)$$

(2)
$$= \sum_{i=1}^{n-1} t^{2(i-1)}\rho_m\left(\left\{axba(xb)^{-1}\right\}^{i-1}\right)\left\{\rho_m(1) + t^{2(n+1)}\rho_m(axb)\right\} + t^{4n+1}\rho_m(xbxba^{-1})$$
$$+t^{2n-1}\rho_m\left(xb\left\{axba(xb)^{-1}\right\}^{-1}\right) + t^{-3}\rho_m\left(xb\{axba(xb)^{-1}\}(axb)^{-1}\right).$$

For simplicity, we put

$$\gamma_1 = s\alpha - m\beta\ ,\ \gamma_2 = ms\alpha - \beta\ ,\ \gamma_3 = m^2 s(sS^2+1)\alpha.$$

By the aid of Mathematica, the first term of the right hand side of (2) is given by

$$\sum_{i=1}^{n-1} t^{2(i-1)}(AXBA(XB)^{-1})^{i-1}(E + t^{2(n+1)}AXB)$$
$$= \begin{pmatrix} \dfrac{(ST^2-st^2)(St^2\beta T^2+m\alpha)}{mst^2(st^2-1)\alpha} & -\dfrac{T^2(ST^2-st^2)(\gamma_1\eta_2+(m\alpha-\beta)\gamma_3)}{m^2 s(s+1)S\,(st^2-1)\,\alpha\beta} \\ \dfrac{mC_1\alpha - St^2T^2C_2\beta}{msS(sS^2+1)t^2(s-t^2)(st^2-1)\alpha} & \dfrac{C_3t^4T^4+C_4t^2T^4+C_5t^6T^2+C_6t^4T^2+C_7}{(s+1)S^2t^2(s-t^2)(st^2-1)\gamma_3\beta} \end{pmatrix},$$

where

$$\begin{aligned}
C_1 &= -t^4 s(s^2-1)S - T^2\{t^2(S^2-s^4) - s(S^2-s^2)\},\\
C_2 &= -t^2(t^2-1)s(s+1)S + T^2\{t^2(S^2+s^3) + s(S^2-s)\},\\
C_3 &= (s^3+S^2)\gamma_1\eta_2 - \{s^3(ms\alpha+\beta) - S^2(m\alpha-\beta)\}\gamma_3,\\
C_4 &= -s(s+S^2)\gamma_1\eta_2 + s\{s(ms\alpha+\beta) - S^2(m\alpha-\beta)\}\gamma_3,\\
C_5 &= -s(s+1)S\{\gamma_1\eta_2 + (\eta_1+\eta_2-(1+m^2S^2-sS^2)\beta)\gamma_3\},\\
C_6 &= s(s+1)S\{s\alpha\eta_2 - m(s+1)S^2\beta\gamma_2\},\\
C_7 &= s(s+1)S(st^2-1)(St^2-sT^2)\beta\gamma_3.
\end{aligned}$$

Similarly, the second term of the right hand side of (2) is given by

$$XBXBA^{-1} = \begin{pmatrix} \dfrac{S^2D_1}{\gamma_3\alpha} & \dfrac{msD_1D_2 - (sS^2+1)(sS^2D_1+m\gamma_3\alpha)\beta^2}{(s+1)\gamma_3\alpha\beta^2} \\ \dfrac{(s+1)D_2}{(sS^2+1)\gamma_3\alpha} & \dfrac{msS^2D_1D_2 + s(sS^2+1)(m^2s\alpha^2 - S^2\beta^2)D_2}{S^2(sS^2+1)\gamma_3\alpha\beta^2} - m \end{pmatrix},$$

where

$$\begin{aligned}
D_1 &= -(s+1)\alpha\gamma_2 + m(\eta_1+\gamma_2+mS^2\gamma_1)\beta,\\
D_2 &= -\alpha\eta_2 + mS^2(\eta_1+mS^2\gamma_1+\gamma_2)\beta,
\end{aligned}$$

the third term of the right hand side of (2) is given by

$$XB\left\{AXBA(XB)^{-1}\right\}^{-1} = \begin{pmatrix} \dfrac{SE_1}{ms\,(sS^2+1)\,\alpha\beta} & -\dfrac{S\gamma_1\gamma_2}{m\alpha\beta} \\ \dfrac{(s+1)E_2}{msS\,(sS^2+1)^2\,\alpha\beta} & \dfrac{E_3}{mS\,(sS^2+1)\,\alpha\beta} \end{pmatrix},$$

where

$$\begin{aligned}
E_1 &= (s^2-1)\alpha\gamma_2 + m(\eta_1+mS^2\gamma_1 - s\gamma_2)\beta,\\
E_2 &= (s-1)\alpha\eta_2 + mS^2(\eta_1+mS^2\gamma_1 - s\gamma_2)\beta,\\
E_3 &= -s\alpha\eta_2 + m(s+1)S^2\beta\gamma_2,
\end{aligned}$$

and the fourth term of the right hand side of (2) is given by

$$XB(AXBAXBA(XB)^{-1})^{-1} = \begin{pmatrix} \dfrac{mF_3}{\gamma_3^2\beta^2} & \dfrac{F_4}{m(s+1)\gamma_3\alpha\beta^2} \\ \dfrac{m(s^2-1)F_1F_2}{S^2(sS^2+1)\gamma_3^2\beta^2} & \dfrac{mF_5}{S^2\gamma_3^2\beta^2} \end{pmatrix},$$

where

$$\begin{aligned}
F_1 &= m(s+1)S^2(\eta_1+mS^2\gamma_1)\beta-\eta_2\alpha,\\
F_2 &= m(s+1)S^2(sS^2+1)\beta^2-sF_1,\\
F_3 &= -\{m\beta(\eta_1+mS^2\gamma_1)+s\gamma_1\gamma_2-\gamma_2\alpha\}F_2+ms(s+1)S^2(sS^2+1)\gamma_1\gamma_2\beta^2,\\
F_4 &= (s^2-1)\{m(\eta_1+mS^2\gamma_1)\beta-\gamma_2\alpha\}F_2\\
&\quad +\gamma_3\{m\gamma_2\alpha-(m^2\eta_1+s^2\eta_2+m^3S^2\gamma_1-s^2(S^2-1)\gamma_2)\beta-ms\gamma_1\gamma_2\}\alpha,\\
F_5 &= (s-1)(sF_1-m\gamma_3\alpha)F_2-m^2S^2(sS^2+1)\gamma_3\alpha\beta^2.
\end{aligned}$$

Therefore, the determinant of the right hand side of (2) is written as

$$\frac{\sum_{i,j}U_{i,j}t^iT^j}{m^3S^2t^6(s-t^2)(st^2-1)\beta^2\iota},$$

where

$$\begin{aligned}
U_{0,0} &= U_{4,0}=U_{6,0}=U_{2,4}=U_{10,4}=U_{6,8}=U_{8,8}=U_{12,8}=-m^3sS^2\beta^2\iota,\\
U_{2,0} &= U_{10,8}=m^3(s^2+1)S^2\beta^2\iota,\\
HU_{3,0} &\equiv HU_{9,8}\equiv -m^2(m^2+1)sS^2\beta(Hs\beta-(s^2-1)(\eta_1+\eta_2))\iota \mod r_0,\\
U_{5,0} &\equiv U_{7,8}\equiv m^2(m^2+1)sS^2\beta^2\iota \mod r_0,\\
HU_{1,2} &\equiv HU_{11,6}\equiv m^2(m^2+1)(s-1)sS\beta\eta_2\iota \mod r_0,\\
HU_{2,2} &= HU_{6,2}=HU_{8,2}=HU_{4,6}\equiv HU_{6,6}=HU_{10,6}\equiv m^3(s^2-1)sS\beta\eta_1\iota \mod r_1,\\
HU_{3,2} &\equiv HU_{9,6}\equiv m^2(m^2+1)(s-1)S\beta\{HsS^2\beta-s(sS^2+1)\eta_1-(s^2S^2+s^2+1)\eta_2\}\iota \mod r_0,\\
H^2U_{4,2} &\equiv H^2U_{8,6}\\
&\equiv m(s-1)sS\{H^2m^3\alpha\beta+H(m^2+1)(m^2s+s+1)\beta\eta_2-(m^2+1)^2(s^2-1)\eta_2(\eta_1+\eta_2)\}\iota\\
&\qquad \mod r_0,\\
HU_{5,2} &\equiv HU_{7,6}\equiv -m^2(m^2+1)(s-1)sS\beta\eta_2\iota \mod r_0,\\
HU_{7,2} &\equiv HU_{5,6}\equiv m^2(m^2+1)(s-1)sS\beta(HS^2\beta-(sS^2+1)\eta_1-(sS^2-1)\eta_2)\iota \mod r_1,\\
H^2U_{3,4} &\equiv H^2U_{9,4}\equiv -m^2(m^2+1)(s-1)^2s(s+1)\eta_1\eta_2\iota \mod r_0,\\
H^2U_{4,4} &= H^2U_{8,4}\\
&\equiv m\{H^2m^2(s^2-s+1)S^2\beta^2+(m^2+1)^2(s-1)^2s\eta_2(-HS^2\beta+(sS^2+1)\eta_1+sS^2\eta_2)\}\iota\\
&\qquad \mod r_1,\\
H^2U_{5,4} &\equiv H^2U_{7,4}\\
&\equiv -(m^2+1)(s-1)s\{(s-1)\eta_2(m^3H\alpha+(m^2+1)\eta_2)+m^2S^2H\beta(H\beta-(s+1)(\eta_1+\eta_2))\}\iota\\
&\qquad \mod r_0,\\
H^2U_{6,4} &\equiv -2ms(HmS\beta-(m^2+1)(s-1)\eta_2)(HmS\beta+(m^2+1)(s-1)\eta_2)\iota \mod r_0,
\end{aligned}$$

where we put $\iota=m^2s^2(s+1)S(sS^2+1)^3\alpha^3\beta$, and the other $U_{i,j}$'s are 0.

On the other hand, by the aid of Mathematica,

$$\begin{aligned}
\det\Phi(c-1) &= \det\left(t^{2n+1}\rho_m(xb)-\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)\\
&= \frac{mSH\beta+mSHt^2T^4\beta-(m^2+1)(s-1)tT^2\eta_2}{mSH\beta}-\frac{(S^2-1)tT^2}{mS(sS^2+1)H\alpha\beta}r_1\\
&= \frac{mSH\beta+mSHt^2T^4\beta-(m^2+1)(s-1)tT^2\eta_2}{mSH\beta}.
\end{aligned}$$

Consequently, we have

$$\Delta_{K_n,\rho_m}(t)=\frac{\sum_{i,j}V_{i,j}t^iT^j}{Hm^2St^6(s-t^2)(st^2-1)\beta},\tag{3}$$

where

$$\begin{aligned}
V_{0,0} &= V_{4,0} = V_{6,0} = V_{4,4} = V_{6,4} = V_{10,4} = -Hm^2 sS\beta,\\
V_{2,0} &= V_{8,4} = Hm^2(s^2+1)S\beta,\\
V_{3,0} &= V_{7,4} = m(m^2+1)sS\{(s^2-1)(\eta_1+\eta_2) - Hs\beta\},\\
V_{5,0} &= V_{5,4} = Hm(m^2+1)sS\beta,\\
V_{2,2} &= V_{8,2} = m^2 s(s^2-1)\eta_1,\\
V_{3,2} &= V_{7,2} = m(m^2+1)(s-1)s\{(s+1)\eta_1+\eta_2\}\\
V_{4,2} &= V_{6,2} = (s-1)s\{(m^2+1)\eta_2 + Hm^3\alpha\},\\
V_{5,2} &= -2m(m^2+1)(s-1)s\eta_2,
\end{aligned}$$

and the other $V_{i,j}$'s are 0. By the aid of Mathematica, the difference between the right hand side of (3) and the formula in Proposition 3.2 is equal to

$$\frac{s\zeta_1 + t\zeta_2 - 2t^2\zeta_1 + t^3\zeta_2 + st^4\zeta_1}{Hm^2St^3(s+1)(s-t^2)(st^2-1)\beta}T^2,$$

where

$$\begin{aligned}
\zeta_1 &= m(m^2+1)s(s+1)(HS^2\beta - s(S^2-1)\eta_1 - (sS^2-1)\eta_2),\\
\zeta_2 &= Hm^2 s(m\alpha - ms^2\alpha + s\beta + S^2\beta) - (s^2-1)(m^2\eta_1 + m^2s^3\eta_1 + s\eta_2 + m^2 s\eta_2).
\end{aligned}$$

Note that $\zeta_1 = 0$ by the definition of H, η_1 and η_2 and that

$$\zeta_2 = m\{(m^2(s^2-s+1)-s)(s^3S^2+1) - Hs(s-1)\}r_0 = 0.$$

This completes the proof of Proposition 3.2.

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