Tropical polynomials being the minimum finishing time of project networks, genera of tropicalizations of curves, and tropical ideals

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Takaaki Ito

Department of Mathematics and Information Sciences
Tokyo Metropolitan University

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1 Introduction

Tropical geometry is developing with several branches of mathematics, geometry, algebra, and applied mathematics.

Most studies are based on geometric motivations. For example, the computation of Gromov-Witten invariants is well taken up. Mikhalkin showed that the Gromov-Witten invariants of projective plane can be computed by counting tropical curves in [17]. This result made tropical geometry known as a useful combinatorial tool for problems in algebraic geometry. The second example is the theory of tropicalizations. The term"tropicalization" is a general term for the processes of associating a tropical variety to an algebraic variety, or for the resulting tropical variety itself. Gathmann introduces several ways of tropicalization in [5]. In most papers which treats tropicalizations, the coefficient field K is required to be algebraically closed. The tropicalizations of varieties over an arbitrary field are studied by Gubler in [7]. The third example is tropical intersection theory. Based on Mikhalkin's ideas in [18], which is called stable intersections, Allermann and Rau established the foundation in [1]. Katz shows in [10] that the theory in [1] is equivalent to the fan displacement rule of [3]. Jensen and Yu give another definition of stable intersections in [11], which is preferable for computations to Allermann and Rau's definition and fan displacement rule. In [16], Meyer extended the stable intersections in $\mathbb{R}^n$ to tropical toric varieties.

Recently, an algebraic foundation for tropical geometry is tried to develop. Giansiracusa and Giansiracusa define tropical schemes in [4], which are congruences on the semiring of tropical polynomials. Maclagan and Rincón found a relationship among tropical schemes, ideals in the semiring of tropical polynomials, and valuated matroids in [13]. Based on this relationship, the authors of [13] defined tropical ideals in [14], which generalize the tropicalizations of classical ideals. Viro suggests another approach in [20], which uses hyperfields.

Tropical geometry has applications to the other fields of mathematics. For instance, Kobayashi and Odagiri illustrated the adjacency of paths in project networks by using tropical varieties in [12]. Speyer and Strumfels showed that the space of phylogenetic trees coincides with the tropical Grassmannian of 2-planes.

This thesis consists of three studies, which are based on geometric, algebraic, and applied mathematical interest, respectively. The author believe that this work helps the development of comprehensive study in tropical geometry.

1.1 Preliminary

We now recall the basic definitions and facts in tropical geometry. For details of this section, see [15].

In this thesis, a *valuation* on a field K means a non-archimedean additive valuation on K, namely, a map $v : K \to \mathbb{R} \cup \{\infty\}$ such that

- $v(a) = \infty$ if and only if $a = 0$,
- $v(ab) = v(a) + v(b)$,
- $v(a + b) \geq \min\{v(a), v(b)\}$.

The *trivial valuation* is the following map:

$$a \mapsto \begin{cases} 0 & \text{if} \quad a \neq 0 \\ \infty & \text{if} \quad a = 0. \end{cases}$$

Let K be an algebraically closed field with a nontrivial valuation v. Consider the map

$$\mathrm{Val} : \mathbb{G}_m^n \to \mathbb{R}^n, (a_1, \ldots, a_n) \mapsto (-v(a_1), \ldots, -v(a_n)).$$

Here we use $-v(a_i)$ but not $v(a_i)$, because the author adopts the max-plus convention. Let X be a subvariety of a torus $\mathbb{G}_m^n$. We define the *tropicalization* $\mathrm{trop}(X)$ of X as the closure of $\mathrm{Val}(X) \subset \mathbb{R}^n$ via the Euclidean topology. $\mathrm{trop}(X)$ is the support of a polyhedral complex in $\mathbb{R}^n$.

There is another definition of tropicalizations, which uses tropical polynomials. The *tropical semifield* is the semifield $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where the addition is $a \oplus b := \max\{a, b\}$ and the multiplication is $a \odot b := a + b$. In this paper, we usually omit the symbol $\odot$. A *tropical polynomial* of $x_1, \ldots, x_n$ is a formal sum of the form

$$a_1 \mathbf{x}^{\mathbf{u}_1} \oplus a_2 \mathbf{x}^{\mathbf{u}_2} \oplus \cdots \oplus a_m \mathbf{x}^{\mathbf{u}_m}$$

for some $a_i \in \mathbb{T}$ and $\mathbf{u}_i \in \mathbb{Z}_{\geq 0}^n$, where $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \cdots x_n^{u_n}$ if $\mathbf{u} = (u_1, \ldots, u_n)$. The set $\mathbb{T}[x_1, \ldots, x_n]$ of all tropical polynomials of

$x_1, \ldots, x_n$ forms a semiring via the natural addition and multiplication. We call $\mathbb{T}[x_1, \ldots, x_n]$ the *tropical polynomial semiring*. Also we define the tropical Laurent polynomial semiring $\mathbb{T}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ as usual sense.

Each nonzero tropical Laurent polynomial defines a piecewise linear map from $\mathbb{R}^n$ to $\mathbb{R}$. For a nonzero tropical Laurent polynomial $f \in \mathbb{T}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we define the *tropical variety* $V(f)$ as

$$V(f) = \{\mathbf{w} \in \mathbb{R}^n \mid f \text{ is not differentiable at } \mathbf{w} \in \mathbb{R}^n\}.$$

For a Laurent polynomial $f = \sum_{\mathbf{u}} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we define its *tropicalization* as $\mathrm{trop}_v(f) := \bigoplus_{\mathbf{u}} (-v(a_{\mathbf{u}})) \odot \mathbf{x}^{\mathbf{u}}$, which is a tropical Laurent polynomial. Then the tropical variety $V(\mathrm{trop}(f))$ coincides with the tropicalization $\mathrm{trop}(V(f))$ (Kapranov's theorem [2]). For a subvariety X of a torus $\mathbb{G}_m^n$ defined by the ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we have the following equality:

$$\mathrm{trop}(X) = \bigcap_{f \in I} V(\mathrm{trop}(f)).$$

1.2 Outline of thesis

In Section 2 we study a characterization for tropical polynomials being the minimum finishing time of project networks. A *project network* consists of some activities, where each activity can be started after all the preceding activities have finished. We may regard the set of activities as an ordered set. By taking the Hasse diagram, a project network is represented as a directed acyclic graph. Each activity is endowed with a non-negative real number t_i, called *time cost*. We may consider that the time cost of an activity represents the time to complete the activity. The *minimum finishing time* of a project network is the minimum time taken for finishing all the activities in that network. The minimum finishing time is represented by a tropical polynomial of $t_1, \ldots, t_n$.

A tropical polynomial is called a *realizable polynomial* or an *R-polynomial* if it can be realized as the minimum finishing time of a project network. An R-polynomial satisfies the following three conditions (Proposition 2 in [12]): (1) the degree on each variable is exactly one, (2) the coefficient of each term is a unity and (3) no term is divisible by any other terms ('indivisibility'). A tropical polynomial satisfying those conditions is called *prerealizable polynomial* or

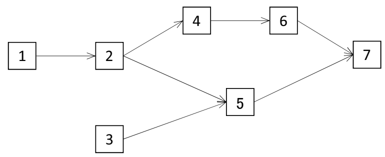

Figure 1: project network

P-polynomial. A P-polynomial is not always an R-polynomial. A simplest example of a P-polynomial which is not an R-polynomial is $t_1t_2 \oplus t_2t_3 \oplus t_3t_1$ ([12]).

A characterization of R-polynomials using poset is known (Proposition 2.1.3), but it is not effective for judging whether a given P-polynomial is an R-polynomial. In this paper, we introduce another characterization by graph theory. We do this by two steps. We first show that every R-polynomial satisfies a term extendability condition, which we will define later, and prove the following theorem.

Theorem 1.2.1. *There is a one-to-one correspondence between the set of P-polynomials $f(t) = f(t_1, \ldots, t_n)$ having term extendability and the set of simple graphs with the vertex set $[n]$. Via this correspondence, two simple graphs are isomorphic if and only if the corresponding P-polynomials coincide up to a permutation of variables.*

Secondly, by this theorem, we will give a characterization for R-polynomials in the context of graph theory. The following is our main theorem.

Theorem 1.2.2. *Let $f(t)$ be a P-polynomial of degree d with the term extendability. Then $f(t)$ is an R-polynomial if and only if there is a vertex coloring of the term graph $TG(f)$ with the color set $\{1, \ldots, d\}$ such that every increasing path of three vertices is a clique of $TG(f)$.*

By this theorem, we can give some examples of judging whether a given polynomial is an R-polynomial.

As for P-polynomials, we have a correspondence between the set of P-polynomials and the set of abstract complexes.

In Section 3, we discuss genera of the tropicalizations of curves over an algebraic function fields of one variable. The genus of the tropicalization of a subvariety of $\mathbb{G}_m^n$ is not always equal to that of original subvariety (see Example 3.1.6). In this section, we consider the tropicalizations of curves over an algebraic function field over $\mathbb{C}$ of one variable. By varying a valuation on the coefficient field, we give a tropicalization which keeps the genus.

The main theorem of this section is the following.

Theorem 1.2.3. *Let E be an elliptic curve over an algebraic function field K of one variable on $\mathbb{C}$. Suppose that the j–invariant of E is not in $\mathbb{C}$. Then there exist*

- *a finite extension L of K,*
- *an elliptic curve $C \subset \mathbb{P}^2$ over L birationally equivalent to the scalar extension $E \times_{\mathrm{Spec} K} \mathrm{Spec} L$,*
- *a valuation v on L*

such that the tropicalization of $C \cap T$ via v has genus one, where T is a big torus of $\mathbb{P}^2$.

In Section 4 we study tropical ideals in tropical polynomial function semirings. In [14], Maclagan and Rincón defined the *tropical variety* $V(I)$ associated to an ideal $I \subset \mathbb{T}[x_1, \ldots, x_n]$ as

$$V(I) = \bigcap_{f \in I} V(f).$$

A tropical variety is expected to be the support of a finite polyhedral complex. However, a counterexample is given in [14]. To avoid this problem, in [14], the authors define *tropical ideals* as follows.

Definition 1.2.4. An ideal I in $\mathbb{T}[x_1, \ldots, x_n]$ is a *tropical ideal* if, for any $f, g \in I$ and any monomial $\mathbf{x}^{\mathbf{u}}$ for which $[f]_{\mathbf{u}} = [g]_{\mathbf{u}} \neq -\infty$, there exists $h \in I$ such that $[h]_{\mathbf{u}} = -\infty$ and $[h]_{\mathbf{v}} \leq [f]_{\mathbf{v}} \oplus [g]_{\mathbf{v}}$ for all $\mathbf{v}$, with the equality holding whenever $[f]_{\mathbf{v}} \neq [g]_{\mathbf{v}}$.

Here we use the notation $[f]_{\mathbf{u}}$ to denote the coefficient of the monomial $\mathbf{x}^{\mathbf{u}}$ in f.

They show that for any tropical ideal I, the set $V(I)$ is the support of a finite polyhedral complex. Moreover, they proved that tropical ideals satisfy the ascending chain condition and also that tropical Nullstellensatz holds, which are not true for arbitrary ideals.

In this section, we consider a "function version" of tropical ideals. We define tropical ideals in the tropical polynomial function semirings $\mathbb{T}[x_1, \ldots, x_n]/\sim$, where the relation $\sim$ is defined as $f \sim g$ if and only if $f(\mathbf{a}) = g(\mathbf{a})$ for any $\mathbf{a} \in \mathbb{R}^n$. The definition of our tropical ideals is analogous to [14]. One of the advantages of considering in $\mathbb{T}[x_1, \ldots, x_n]/\sim$ instead of $\mathbb{T}[x_1, \ldots, x_n]$ is that we can always factorize a tropical polynomial function of one variable into a product of tropical polynomial functions of degree one (see Lemma 4.2.1 or [6]).

As a first step of the study of our tropical ideals, we focus on the case of one variable. The followings are our main theorems.

Theorem 1.2.5. *For any tropical polynomial function $\varphi \in \mathbb{T}[x]/\sim$, the set $\varphi \odot (\mathbb{T}[x]/\sim) := \{\varphi \odot \psi \mid \psi \in \mathbb{T}[x]/\sim\}$ is a tropical ideal in $\mathbb{T}[x]/\sim$.*

Theorem 1.2.6. *Every tropical ideal in $\mathbb{T}[x]/\sim$ is of the form $\varphi \odot (\mathbb{T}[x]/\sim)$ for some $\varphi \in \mathbb{T}[x]/\sim$.*

These theorems say that $\mathbb{T}[x]/\sim$ is like a PID. As a consequence of the theorems, it follows that our tropical ideals are closed under the intersection, and that we can add, multiply and generate tropical ideals. In fact, these properties do not hold for Maclagan and Rincón's tropical ideals.

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2 A characterization for tropical polynomials being the minimum finishing time of project networks

2.1 Project networks

In this section, we recall the relation between project networks and tropical algebra. For detail of this section, see [12].

Formally, a project network is an acyclic directed graph with no short-cuts, where a graph is said *with no short-cuts* if the following condition holds: if there are two distinct paths from activity a to activity b, then both paths consist of more than one arrow.

Proposition 2.1.1 ([12, Proposition 1]). *Let X be a finite set. There is a one-to-one correspondence between the set of partial orders on X and the set of simple directed graphs with vertex set X without cycles or short-cuts.*

The correspondence in Proposition 2.1.1 is given as follows. For a given partial order of X, we take its Hasse diagram as the corresponding graph. For a given project network with the vertex set X, we define the corresponding partial order on X so that, for each arrow, its head is greater than its tail.

Each activity is endowed with a non-negative real number t_i, called *time cost*. We may consider that the time cost of an activity represents the time to complete the activity. The *minimum finishing time* of a project network is the minimum time taken for finishing all the activities in that network. Then the minimum finishing time is a function of $t_1, \ldots, t_n$, which has following properties.

Proposition 2.1.2 ([12, Proposition 2]). *The minimum finishing time $f(t) = f(t_1, \ldots, t_n)$ can be written as a tropical polynomial of $t_1, \ldots, t_n$ satisfying the following three conditions:*

(1) *the degree on each variable is exactly one,*

(2) *the coefficient of each term is a unity,*

(3) *no term is divisible by any other terms. ('indivisibility')*

A tropical polynomial is called a *realizable polynomial* or an *R-polynomial* if it can be obtained as the minimum finishing time of a project network. Also, a tropical polynomial is called a *prerealizable* *polynomial* or a *P-polynomial* if it satisfies the condition $(1)-(3)$. A P-polynomial is not always an R-polynomial.

For a set of variables $\{t_i\}_{i\in\Lambda}$ and a subset $I \subset \Lambda$, we denote the monomial $\prod_{i\in I} t_i$ by t_I. We know the following characterization of R-polynomials.

Proposition 2.1.3 ([12, Proposition 3])**.** *Let $f(t) = \bigoplus_{I\in\mathcal{I}} t_I$ be a tropical polynomial in n variables. Then $f(t)$ is an R-polynomial if and only if there exist a poset structure on the index set $[n]$ such that*

$$I \text{ is a maximal totally ordered subset } \Leftrightarrow t_I \text{ is a term of } f(t).$$

If we want to check whether a given P-polynomial is an R-polynomial by this characterization, for example, we may list up the all poset structure on $[n]$. However, the calculation amount is not realistic. Thus we introduce another approach in the later section.

2.2 Term extendability

In this section, we introduce our key condition, called *term extendability*, which holds for every R-polynomial. For a given P-polynomial, checking for term extendability is easier than checking whether the polynomial is an R-polynomial. Many P-polynomials are excluded from the candidates for R-polynomials by restricting via this condition. Furthermore, in the next section, we will construct a correspondence between P-polynomials with term extendability and simple graphs. The correspondence is important for our new characterization. Unfortunately, there is a P-polynomial that has term extendability but is not an R-polynomial. We will see some examples of such polynomials in this section.

In the later of this section, we will estimate the number of terms of R-polynomials by using term extendability condition. In addition, we will show that if the number of terms is smaller than 5, then the term extendability condition is sufficient for a P-polynomial to be an R-polynomial.

First, we give a definition and a proposition. Let $f(t)$ be a P-polynomial of $t_1,\ldots,t_n$. For $i,j \in [n]$, we say that i and j are *comparable* in $f(t)$ if $f(t)$ has a term which is divisible by t_it_j. Note that if $f(t)$ is an R-polynomial, then i and j are comparable if and only if i and j are comparable in the usual sense in the poset of the corresponding project network.

Proposition 2.2.1. *Let $f(t) = f(t_1, \dots, t_n)$ be an R-polynomial and $I \subset [n]$ be a subset. Suppose that any two elements of I are comparable. Then $f(t)$ has a term which is divisible by t_I.*

Proof. Let N be the corresponding project network to $f(t)$. Since any two distinct elements of I are comparable, the set I forms a totally ordered vertex set of N. Then there is a maximal totally ordered vertex set J of N containing I. Therefore t_J is a term of $f(t)$, which is divisible by t_I. $\square$

Now we define the term extendability. Let $f(t) = f(t_1, \dots, t_n)$ be a P-polynomial. Then $f(t)$ has term extendability if, for any subset $I \subset [n]$ such that any two distinct elements of I are comparable in $f(t)$, there is a term of $f(t)$ divisible by t_I.

Proposition 2.2.1 means that every R-polynomial has term extendability.

Example 2.2.2. Let $h(t)$ be a P-polynomial. Suppose that we can write $h(t) = (t_1t_2 \oplus t_2t_3 \oplus t_3t_1)f(t) \oplus g(t)$ for some P-polynomials $f(t)$ and $g(t)$. Then $h(t)$ does not have term extendability. Indeed, suppose that $h(t)$ has term extendability. Let t_I be a term of $f(t)$. (If $f(t)$ is constant, let $I = \emptyset$). Consider the set $I' := I \cup \{1, 2, 3\}$. Any two elements of I' are comparable, so $h(t)$ has a term divisible by $t_{I'}$. Since $h(t)$ also has a term $t_It_1t_2$, this contradicts the indivisibility for $h(t)$. We conclude that $h(t)$ is not an R-polynomial.

There is an example that $h(t)f(t) \oplus g(t)$ has term extendability although $h(t)$ does not have.

Example 2.2.3. The polynomial $h(t) = t_1t_2t_4 \oplus t_1t_3t_5 \oplus t_2t_3t_6$ does not satisfy term extendability for $I = \{1, 2, 3\}$, while the polynomial $h(t) \oplus t_1t_2t_3 = t_1t_2t_4 \oplus t_1t_3t_5 \oplus t_2t_3t_6 \oplus t_1t_2t_3$ satisfies term extendability.

This polynomial is in fact not an R-polynomial, but $h(t) \oplus t_1t_2t_3 \oplus t_2t_4t_6 = t_1t_2t_4 \oplus t_1t_3t_5 \oplus t_2t_3t_6 \oplus t_1t_2t_3 \oplus t_2t_4t_6$ is an R-polynomial. We will show that in Example 2.3.12.

Next we estimate the number of terms of R-polynomials.

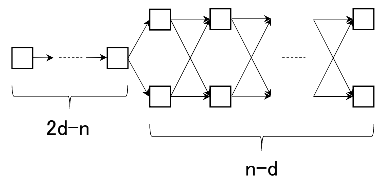

Figure 2

Proposition 2.2.4. *Let $f(t)$ be a P-polynomial having term extendability. If t_I, t_J and t_K are distinct three terms of $f(t)$, then $I \cup J \neq I \cup K$.*

Proof. Suppose that $I \cup J = I \cup K$. We use term extendability for the set $J \cup K$. To do this we show that every two distinct elements of $J \cup K$ are comparable.

Let $i, j \in J \cup K$. If $i, j \in J$ or $i, j \in K$, then i and j are obviously comparable. If $i \in J \setminus K$ and $j \in K \setminus J$, we have $i \in I \cup J = I \cup K$. Then $i \in I$. Similarly, $j \in I$. Hence i and j are comparable.

By the term extendability, $f(t)$ has a term divisible by $t_{J \cup K}$. Since $J \subsetneq J \cup K$, this contradicts the indivisibility for $f(t)$. $\square$

Corollary 2.2.5. *Let $f(t) = f(t_1, ..., t_n)$ be a P-polynomial having term extendability. Let d be the degree of $f(t)$. Then $f(t)$ has at most $\sum_{i=0}^{\min\{d,n-d\}} \binom{n-d}{i}$ terms.*

Proof. Let t_{I_0} be a term of $f(t)$ of degree d. Consider the map $t_I \mapsto I_0 \cup I$ from the set of terms of $f(t)$ to the set $\{J \subset [n] | I_0 \subset J \text{ and } \#J \leq 2d\}$. By Proposition 2.2.4, this map is injective. Then the number of terms of $f(t)$ is at most $\#\{J \subset [n] | I_0 \subset J \text{ and } \#J \leq 2d\} = \sum_{i=0}^{\min\{d,n-d\}} \binom{n-d}{i}$. $\square$

Note that this estimate is best bound if $\min\{d, n-d\} = n-d$, i.e. $2d \geq n$. Indeed, in that case, the number of terms of $f(t)$ is at most $\sum_{i=0}^{n-d} \binom{n-d}{i} = 2^{n-d}$. It can be attained by the minimum finishing time of the project network in Figure 2.

Proposition 2.2.6. *Let $f(t) = f(t_1, ..., t_n)$ be a P-polynomial. If $\deg(f(t)) \geq n-2$, then $f(t)$ is an R-polynomial if and only if $f(t)$ has term extendability.*

Proof. If $\deg(f(t)) = n$, we have $f(t) = t_1 \cdots t_n$ and so $f(t)$ is an R-polynomial.

If $\deg(f(t)) = n-1$, then $f(t)$ is a binomial by Corollary 2.2.5. Note that $f(t)$ is not a monomial because every variable appears at least once. Let $f(t) = t_I \oplus t_J$. Then $f(t)$ is the minimum finishing time of the project network in Figure 3, so $f(t)$ is an R-polynomial.

If $\deg(f(t)) = n-2$, we may assume that $f(t)$ has a term $t_{[n-2]}$. By the indivisibility, the term other than $t_{[n-2]}$ is divisible by t_{n-1} or t_n. Moreover, there is at most one term of the form $t_I t_{n-1}$ ($I \subset [n-2]$). Indeed, if $t_I t_{n-1}$ and $t_J t_{n-1}$ ($I, J \subset [n-2]$) are the terms of $f(t)$, we have $[n-2] \cup (I \cup \{n-1\}) = [n-2] \cup (J \cup \{n-1\})$, which contradicts Proposition 2.2.4. It is similar for the term of the form $t_I t_n$ and $t_I t_{n-1} t_n$ ($I \subset [n-2]$). Thus there are following 4 cases:

If $f(t)$ is of the form $t_{[n-2]} \oplus t_I t_{n-1} t_n$ ($I \subset [n-2]$), then $f(t)$ is a binomial. Therefore we can show that $f(t)$ is an R-polynomial by the same argument with the case $\deg(f(t)) = n-1$.

If $f(t)$ is of the form $t_{[n-2]} \oplus t_I t_{n-1} \oplus t_J t_n$ ($I, J \subset [n-2]$), then $f(t)$ is the minimum finishing time of the project network in Figure 4. Therefore $f(t)$ is an R-polynomial.

Suppose $f(t)$ is of the form $t_{[n-2]} \oplus t_I t_{n-1} \oplus t_J t_{n-1} t_n$ ($I, J \subset [n-2]$). By the term extendability, there must be a term of $f(t)$ which is divisible by $t_{I \cup J} t_{n-1}$. If the term is $t_J t_{n-1} t_n$, we have $I \subset J$, which contradicts the indivisibility. Thus the term is $t_I t_{n-1}$, so we have $I \supset J$. Then $f(t)$ is the minimum finishing time of the project network in Figure 5. Hence $f(t)$ is an R-polynomial.

Finally, suppose that $f(t)$ is of the form $t_{[n-2]} \oplus t_I t_{n-1} \oplus t_J t_n \oplus t_K t_{n-1} t_n$ ($I, J, K \subset [n-2]$). In the same way with the above case, we have $I \supset K$ and $J \supset K$, and hence $I \cap J \supset K$. If $I \cap J \neq K$, there is $i \in (I \cap J) \setminus K$. By the term extendability, there is a term of $f(t)$ which is divisible by $t_i t_{n-1} t_n$. However, any terms of $f(t)$ are not divisible by $t_i t_{n-1} t_n$. Thus we have $I \cap J = K$. Then $f(t)$ is the minimum finishing time of the project network in Figure 6. Hence $f(t)$ is an R-polynomial. $\square$

Corollary 2.2.7. *For $n \leq 4$, $f(t)$ is an R-polynomial if and only if $f(t)$ has term extendability.*

Remark 2.2.8. For $n = 5$, the polynomial $t_1 t_2 \oplus t_2 t_3 \oplus t_3 t_4 \oplus t_4 t_5 \oplus t_5 t_1$ is a counterexample. We will show that this polynomial is not an R-polynomial in Example 2.3.11.

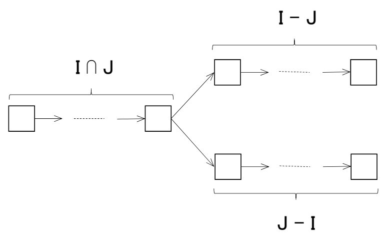

Figure 3

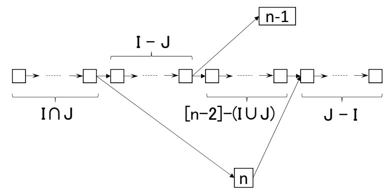

Figure 4

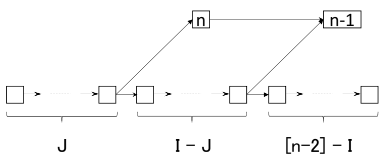

Figure 5

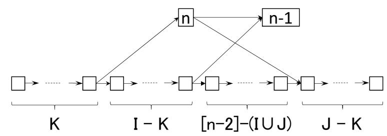

Figure 6

We remark that we may associate an abstract complex with a P-polynomial as follows. Let $f(t_1, \ldots, t_n) = \bigoplus_{I \in \mathcal{I}} t_I$ be a P-polynomial. Then the set

$$\{J \subset [n] \mid J \text{ is a subset of some } I \in \mathcal{I}\}$$

forms an abstract complex. Conversely, for a given abstract complex with the vertex set $[n]$, the tropical polynomial $\bigoplus_{I:\text{maximal face}} t_I$ is a P-polynomial. Then the following proposition is clear.

Proposition 2.2.9. *Let $\mathcal{P}_n$ be the set of P-polynomials with the variables $t_1, \ldots, t_n$ and $\mathcal{A}_n$ be the set of abstract complexes with the vertex set $[n]$. Then the above constructions give bijections between $\mathcal{P}_n$ and $\mathcal{A}_n$, which are inverse of each other. Moreover, a P-polynomial has term extendability if and only if the corresponding complex is flag complex, i.e. for any $I \subset [n]$, if $\{i, j\}$ is a simplex for all $i, j \in I$, then I is a simplex.*

2.3 Term graphs

In this section, we show Theorem 2.3.7, the main theorem of this paper. The theorem gives us a characterization for R-polynomials. As a preparation, we show the following theorem.

Theorem 2.3.1. *There is a one-to-one correspondence between the set of P-polynomials $f(t) = f(t_1, \ldots, t_n)$ having term extendability* *and the set of simple graphs with the vertex set $[n]$. Via this correspondence, two simple graphs are isomorphic if and only if the corresponding P-polynomials coincide up to a permutation of variables.*

Remark 2.3.2. This theorem follows from Proposition 2.2.9 and a well-known fact that there is a bijection between the set of flag complexes and the set of 'clique complexes'(see [19]). Here, we directly construct a bijection in Theorem 2.3.1.

First of all, let us associate a simple graph to a given P-polynomial.

Definition 2.3.3. Let $f(t) = f(t_1, \dots, t_n)$ be a P-polynomial. We define the *term graph* of $f(t)$ as the simple graph $TG(f) = (V, E)$, where $V = [n]$ is the vertex set and E is the edge set which consists of the pairs that are comparable in $f(t)$.

It is clear by definition that if t_I is a term of $f(t)$, then I forms a clique in $TG(f)$.

By the following lemma, a P-polynomial $f(t)$ can be reconstructed by the term graph $TG(f)$ if $f(t)$ has term extendability.

Lemma 2.3.4. *Let $f(t) = f(t_1, \dots, t_n)$ be a P-polynomial having term extendability. Then for any subset $I \subset [n]$, the monomial t_I is a term of $f(t)$ if and only if the set I is a maximal clique of $TG(f)$.*

Proof. We show this by the induction for $\#I$. Let d be the maximum size of the cliques of $TG(f)$. If $\#I > d$, then I is not a clique of $TG(f)$, and then t_I is not a term of $f(t)$. Thus we may assume $\#I \leq d$.

We consider the case $\#I = d$ at first. If t_I is a term of $f(t)$, then I is a clique of $TG(f)$, and the maximality follows from the definition of d. Conversely, if I is a maximal clique of $TG(f)$, then any two distinct elements of I are comparable. Therefore, by the term extendability, $f(t)$ has a term $t_{I'}$ which is divisible by t_I. Then I' is a clique of $TG(f)$ containing I. By the maximality of I, we have $I' = I$. Hence t_I is a term of $f(t)$.

Next we assume that $\#I < d$ and the statement holds for any $J \subset [n]$ such that $\#J > \#I$. If t_I is a term of $f(t)$, then I is a clique of $TG(f)$. If I is not a maximal clique, then there is a maximal clique I' containing I properly. By the assumption of induction, $t_{I'}$ is a term of $f(t)$, which contradicts the indivisibility of $f(t)$. Hence

I is maximal. Conversely, if I is a maximal clique of $TG(f)$, we can show that t_I is a term of $f(t)$ by the proof similar to the above case. $\square$

This lemma means that the map $f(t) \mapsto TG(f)$ between the two sets in Theorem 2.3.1 is injective.

For showing the surjectivity, we construct the inverse map. For a given simple graph G with the vertex set $[n]$, we associate the following tropical polynomial f_G;

$$f_G(t) = \bigoplus_{I:\text{maximal clique of } G} t_I$$

.

Lemma 2.3.5. *The polynomial $f_G(t)$ has term extendability.*

Proof. Let $I \subset [n]$ be a subset such that any two distinct elements of I are comparable in $f_G(t)$. Let i and j be distinct elements of I. Since i and j are comparable, then $f_G(t)$ has a term which is divisible by $t_i t_j$. Thus the original graph G has a clique including i and j, and then i and j are adjacent in G. Hence any two distinct elements of I are adjacent in G, i.e. I forms a clique of G. Let I' be a maximal clique including I. Then the term $t_{I'}$ of $f_G(t)$ is divisible by t_I. $\square$

Proof of Theorem 2.3.1. By Lemma 2.3.4 and Lemma 2.3.5, we obtain a one-to-one correspondence between the set of P-polynomials $f(t) = f(t_1, \ldots, t_n)$ having term extendability and the set of simple graphs with the vertex set $[n]$. The remaining part is clear. $\square$

Finally we describe a characterization for R-polynomials. To do this, we use vertex colorings of term graphs.

Definition 2.3.6. Let G be a simple graph. Assume that there is a vertex coloring of G with the color set $\{1, \ldots, d\}$. Then the sequence of vertices $v_1, \ldots, v_m$ of G is an *increasing path* if v_i and v_{i+1} are adjacent for $i = 1, \ldots, m-1$ and the colors of them are increasing.

Theorem 2.3.7. *Let $f(t)$ be a P-polynomial of degree d having term extendability. Then $f(t)$ is an R-polynomial if and only if there is a vertex coloring of the term graph $TG(f)$ with the color set $\{1, \ldots, d\}$ such that every increasing path of three vertices is a clique of $TG(f)$.*

Remark 2.3.8. The condition that every increasing path *of three vertices* is a clique of $TG(f)$ is equivalent to the condition that every increasing path is a clique of $TG(f)$. Indeed, assume that every increasing path of 3 vertices is a clique and let $v_1, \ldots, v_m$ is an increasing path. Then, for $k \leq m-2$, the sequence v_k, v_{k+1}, v_{k+2} forms an increasing path. Thus $\{v_k, v_{k+1}, v_{k+2}\}$ is a clique, and then v_k and v_{k+2} are adjacent. Hence the sequence v_k, v_{k+2}, v_{k+3} forms an increasing path for $k \leq m-3$. By repeating this argument, every pair of two distinct vertices in $v_1, \ldots, v_m$ are adjacent. It means that $\{v_1, \ldots, v_m\}$ is a clique.

The *length* of a path of a project network is the number of arrows in the path.

Lemma 2.3.9. *Let N be a project network with the vertex set $[n]$. Let d be the maximum length of paths of N. We define the subsets $V_0, \ldots, V_d \subset [n]$ as follows:*

$$V_0 := \{v \in [n] \mid v \textit{ is minimal in } [n]\},$$

$$V_k := \left\{ v \in [n] \mid v \textit{ is minimal in } [n] \setminus \bigcup_{l=0}^{k-1} V_l \right\} \ (k = 1, \ldots, d).$$

Then $V_0, \ldots, V_d$ satisfy the followings:

(1) *The set $[n]$ is the disjoint union of $V_0, \ldots, V_d$.*

(2) *Each V_k is non-empty.*

(3) *For each path of N and each $k = 0, \ldots, d$, the intersection of the path and V_k is empty or singleton.*

Proof. (1) Suppose that the set $[n] \setminus \cup_{k=0}^{d} V_k$ is not empty and let i be a minimal vertex of $[n] \setminus \cup_{k=0}^{d} V_k$. We claim that there is a vertex $v_d \in V_d$ such that $v_d < i$.

Indeed, let m be the number

$$\max\{k \mid 0 \leq k \leq d, \text{ there is a vertex } j \in V_k \text{ such that } j < i\}.$$

By the minimality of i, i is a minimal vertex of $[n] \setminus \cup_{k=0}^{m} V_k$. Thus, if $m < d$, then $i \in V_{m+1}$. It contradicts the definition of i. Hence $m = d$, and there is a vertex $v_d \in V_d$ such that $v_d < i$.

By the same proof, there are vertices $v_{d-1}, \ldots, v_0$ of N such that $v_k \in V_k$ $(k = 0, \ldots, d-1)$ and $v_0 < \cdots < v_d$. Then there is a path through $v_0, \ldots, v_d, i$, which contradicts the definition of d.

(2) We denote $[n] \setminus \bigcup_{l=0}^{k-1} V_l$ by W_k. Let $(v_0, \dots, v_d)$ be a maximal path of N and $v_i \in V_{k_i}$. We claim that $k_i < k_{i+1}$. Otherwise, we have $v_i \in W_{k_i} \subset W_{k_{i+1}}$. Hence $v_i, v_{i+1} \in W_{k_{i+1}}$ and $v_i < v_{i+1}$, but v_{i+1} is a minimal vertex of $W_{k_{i+1}}$ because $v_{i+1} \in V_{k_{i+1}}$. It is a contradiction. Thus we have $0 \leq k_0 < \cdots < k_d \leq d$, which means that $k_i = i$. Hence $v_k \in V_k \neq \emptyset$.

(3) is clear. □

Proof of Theorem 2.3.7. If $f(t)$ is an R-polynomial, let N be the corresponding project network with the vertex set $[n]$. The maximum length of paths of N is d. Take $V_1, \dots, V_d$ as Lemma 2.3.9 for N. Note that the vertex sets of $TG(f)$ and N are same, namely, are $[n]$. For each $k = 1, \dots, d$, color the vertices in V_k with k.

Let v_1, v_2, v_3 be an increasing path of $TG(f)$. For $k = 1, 2$, v_k and v_{k+1} are adjacent in $TG(f)$, so t_{v_k} and $t_{v_{k+1}}$ are comparable in $f(t)$. Hence v_k and v_{k+1} are comparable in N. Since the color of v_{k+1} is greater than that of v_k, we have $v_k < v_{k+1}$. Therefore v_1, v_2, v_3 is totally ordered in N. Then $f(t)$ has a term divisible by $t_{v_1}t_{v_2}t_{v_3}$, which means that the set $\{v_1, v_2, v_3\}$ is a clique of $TG(f)$

Conversely, if there is a vertex coloring of the term graph $TG(f)$ by d colors $1, \dots, d$ such that every increasing path is a clique of $TG(f)$, we may define the partial order of $[n]$ by the following way: For $i, j \in [n]$, i and j are comparable if and only if i and j are adjacent in $TG(f)$. The order of them is induced by the order of their colors.

Using this order, we can define the project network N on $[n]$. Let $g(t)$ be the minimum finishing time of N. We claim that $g(t) = f(t)$. Let $I \subset [n]$ be a subset. Then

t_I is a term of $g(t)$

$\Leftrightarrow$ I is the vertex set of a maximal path of N

$\Leftrightarrow$ I is the vertex set of a maximal increasing path of $TG(f)$

$\Leftrightarrow$ I forms a maximal clique of $TG(f)$

$\Leftrightarrow$ t_I is a term of $f(t)$.

Hence $g(t) = f(t)$. □

Corollary 2.3.10. *Let $f(t) = f(t_1, ..., t_n)$ be a homogeneous P-polynomial of degree 2. Then $f(t)$ is R-polynomial if and only if the term graph $TG(f)$ is a bipartite graph.*

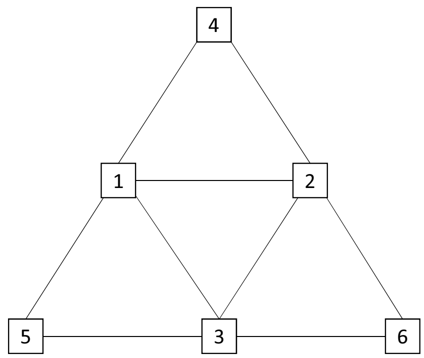

Figure 7

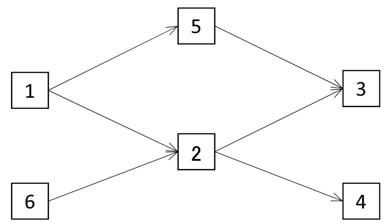

Figure 8

Example 2.3.11. The polynomial $f(t) = t_1t_2 \oplus t_2t_3 \oplus t_3t_4 \oplus t_4t_5 \oplus t_5t_1$ is not an R-polynomial. Indeed, the term graph $TG(f)$ is just a pentagon, which is not a bipartite graph.

Example 2.3.12. $g(t) := t_1t_2t_4 \oplus t_1t_3t_5 \oplus t_2t_3t_6 \oplus t_1t_2t_3$ is not an R-polynomial, but $g(t) \oplus t_2t_4t_6$ is an R-polynomial.

Indeed, the term graph $TG(g)$ is the graph in Figure 7. If $g(t)$ is an R-polynomial, there is a vertex coloring with the colors c_1, c_2 and c_3 ($c_1 < c_2 < c_3$). By symmetry, we may assume that the colors of the vertex $1, 2, 3$ are c_1, c_2, c_3 respectively. Since the vertex set $\{1, 2, 6\}$ is not a clique in $TG(g)$, the sequence $(1, 2, 6)$ is not an increasing path. Then the color of 6 is less than c_2, and hence the color of 6 is c_1. Similarly the color of 4 is c_3. Therefore the sequence $(6, 2, 4)$ is an increasing path, but the set $6, 2, 4$ is not a clique. This is a contradiction.

On the other hand, $g(t) \oplus t_2t_4t_6$ is the minimum finishing time of the project network in Figure 8. Hence $f(t)$ is an R-polynomial.

3 Genera of the tropicalizations of curves over an algebraic function fields of one variable

In general, the tropicalization of a curve in the torus $\mathbb{G}_m^n$ does not have same genus to the original curve (see Example 3.1.6). The purpose of this section is to make a tropicalization which keeps the genus. The choice of the coefficient field is important. Typical examples of the coefficient fields are the field of Puiseux series and the algebraic closure of $\mathbb{Q}_p$. These examples are local, namely, essentially they have just one valuation. In this section, we consider the case that the coefficient field has multiple valuations. The advantage of this setting is the following: Let X be a variety over a field K with multiple valuations. Since the tropicalization depends on the valuation, we obtain the family $\{\mathrm{trop}_v(X)\}_v$ of the tropicalizations of X. Thus, even if the genus changes via the tropicalization with respect to a certain valuation v, there may be another valuation w such that the tropicalization $\mathrm{trop}_w(X)$ has same genus to X.

We take here an algebraic function field over $\mathbb{C}$ of one variable as the coefficient field, and ask whether there is a valuation v such that the tropicalization $\mathrm{trop}_v(X)$ has the genus same to X. Theorem 3.2.1, the main theorem of this section, gives an answer for elliptic curves.

3.1 Preliminary

3.1.1 Tropical geometry

In this section, we recall the basic results in tropical geometry which are used in the proof of Theorem 1.2.3. See Section 1.1 too. First we define the tropicalization of a hypersurface in a torus over an arbitrary coefficient field.

Definition 3.1.1. Let X be a hypersurface in the torus $\mathbb{G}_m^n$ over a field K. Fix a valuation v on K. Let $f = \sum_{\mathbf{u}} a_{\mathbf{u}}\mathbf{x}^{\mathbf{u}} \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the defining Laurent polynomial of X. Then we define the *tropicalization* $\mathrm{trop}_v(X) \subset \mathbb{R}^n$ of X with respect to the valuation v as

$$\mathrm{trop}_v(X) = \left\{ \mathbf{w} \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \text{the maximum of } \max_{\mathbf{u}}(-v(a_{\mathbf{u}}) + \mathbf{u}\cdot\mathbf{w}) \\ \text{is attained at least twice} \end{array} \right\},$$

where $\cdot$ is the standard inner product. In this section, a *tropical variety* means a set of the form $\operatorname{trop}_v(X)$ for some X and v.

Remark 3.1.2. This definition of tropicalization is equivalent to [7], which follows from [7, Proposition 5.6, Remark 5.7, Example 5.8].

Unlike the case that K is algebraically closed, the tropicalization $\operatorname{trop}_v(X)$ of a variety X does not always coincide with the closure of $\{(v(a_1), \ldots, v(a_n)) \in \mathbb{R}^n \mid (a_1, \ldots, a_n) \in X\}$ because $v(K^\times)$ may not be dense in $\mathbb{R}$.

Proposition 3.1.3. *Fix a field K and a valuation v on K. Let $X \subset \mathbb{G}_m^n$ be the hypersurface defined by a Laurent polynomial $f \in K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$. Let L/K be a field extension and w be a valuation on L which is an extension of v. Then $\operatorname{trop}_w(X \times_{\operatorname{Spec}K} \operatorname{Spec}L) = \operatorname{trop}_v(X)$ holds.*

Proof. Follows from the definition of tropicalizations. $\square$

Fix a field K and a valuation v on K. Let $f = \sum_{\mathbf{u}} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ be a Laurent polynomial. The *support* of f is the set $\operatorname{Supp}(f) = \{\mathbf{u} \in \mathbb{Z}^n \mid a_{\mathbf{u}} \neq 0\}$. The *Newton polytope* $\operatorname{Newt}(f)$ of f is the polytope $\operatorname{conv}(\operatorname{Supp}(f))$. We define the *regular subdivision* of $\operatorname{Newt}(f)$ with respect to f as follows. Consider the polytope

$$P := \operatorname{conv}\{(\mathbf{u}, t) \in \mathbb{R}^{n+1} \mid \mathbf{u} \in \operatorname{Supp}(f),\, t \leq -v(a_{\mathbf{u}})\}.$$

A face of P is called an *upper face* if it has an outer normal vector whose last coordinate is positive. The collection of the projections of upper faces to $\operatorname{Newt}(f)$ forms a polyhedral complex. This complex is called the *regular subdivision* of $\operatorname{Newt}(f)$ with respect to f. We often omit "with respect to f" if f is clear from the context. Note that the any vertex of the regular subdivision of $\operatorname{Newt}(f)$ is a point in $\operatorname{Supp}(f)$.

Proposition 3.1.4 ([15, Proposition 3.1.6])**.** *Let $f \in K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ be a Laurent polynomial. Then the tropical variety $\operatorname{trop}_v(V(f))$ is the support of $(n-1)$-skeleton of the polyhedral complex dual to the regular subdivision of Newt(f).*

Note that in [15], the coefficient field is always algebraically closed. However, Proposition 3.1.4 holds for an arbitrary coefficient field by Proposition 3.1.3.

A *tropical curve* is a tropical variety of dimension 1. The *genus* $g(Y)$ of a tropical curve Y is defined as its first Betti number.

Let $0 \neq f \in K[x^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial. By Proposition 3.1.4, the genus $g(\mathrm{trop}_v(V(f)))$ is equal to the number of the interior points of $\mathrm{Newt}(f)$ which are the vertices of the regular subdivision of $\mathrm{Newt}(f)$.

Proposition 3.1.5. *Let $C \subset \mathbb{P}^2$ be a smooth projective plane curve of genus g. Let T be a big torus of $\mathbb{P}^2$. Then $g(\mathrm{trop}_v(C \cap T)) \leq g$.*

Proof. Let d be the degree of C. By degree–genus formula, we have $g = (d-1)(d-2)/2$. On the other hand, $C \cap T$ is defined by a polynomial $f \in K[x, y]$ of degree d. Thus $\mathrm{Supp}(f)$ is included in the triangle with the vertices $(0,0), (d,0)$ and $(0,d)$. That triangle has $(d-1)(d-2)/2$ interior points. Hence we have $g(\mathrm{trop}_v(C \cap T)) \leq (d-1)(d-2)/2$. □

In general, the equality $g(\mathrm{trop}_v(C \cap T)) = g$ does not hold.

Example 3.1.6. Suppose that $\mathrm{ch}\, K \neq 2, 3$. Let $C \subset \mathbb{P}^2$ be an elliptic curve over K defined by the Weierstrass equation $y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$. Then $C \cap T$ is defined by the equation $y^2 = 4x^3 - g_2 x - g_3$. The vertices of the regular subdivision of $\mathrm{Newt}(f)$ are in $\mathrm{Supp}(f) = \{(0,2), (3,0), (1,0), (0,0)\}$. Since any point in $\mathrm{Supp}(f)$ is not an interior point of $\mathrm{Newt}(f)$, we have $g(\mathrm{trop}_v(C \cap T)) = 0 \neq 1$.

3.1.2 Algebraic function field of one variable

An *algebraic function field* of one variable over a field k is a finitely generated field extension K/k with the transcendental degree 1 over k.

Proposition 3.1.7. *Let k be an algebraically closed field. Then the category of algebraic function field of one variable over k is equivalent to the category of nonsingular projective curves over k.*

Proof. For example, see [8, Chapter 1, Corollary 6.12]. □

Let C be a nonsingular projective curve over k. Then the corresponding field K_C is the function field of C. For a closed point $p \in C$, we define the valuation v_p on K as $v_p(f) = \mathrm{ord}_f(p)$.

The following lemma is used in the proof of 1.2.3.

Lemma 3.1.8. *Let K be an algebraic function field of one variable over k. Then, for any $f \in K \setminus k$, there exists a valuation v on K such that $v(f) < 0$.*

Proof. Let C be the nonsingular projective curve over k corresponding to K. We regard f as a k-valued rational function on C. The assumption $f \notin k$ leads to $\mathrm{div}(f) \neq 0$. Since $\deg(\mathrm{div}(f)) = 0$, there is a closed point $p \in C$ such that the coefficient of p in $\mathrm{div}(f)$ is negative. The valuation v_p corresponding to p satisfies the desired condition. $\square$

3.1.3 The Hessian form of an elliptic curve

Let K be an algebraically closed field with characteristic 0. Then the following is well known.

Proposition 3.1.9. *Every elliptic curve E over K is birationally equivalent to the curve C in $\mathbb{P}^2$ defined by the equation*

$$X^3 + Y^3 + Z^3 - DXYZ = 0$$

for some $D \in K$. The curve C is called the Hessian form *of E.*

3.2 Proof of the main theorem

In this section, we show the main theorem.

Theorem 3.2.1. *Let E be an elliptic curve over an algebraic function field K of one variable on $\mathbb{C}$. Suppose that the j–invariant of E is not in $\mathbb{C}$. Then there exist*

- *a finite extension L of K,*
- *an elliptic curve $C \subset \mathbb{P}^2$ over L birationally equivalent to the scalar extension $E \times_{\mathrm{Spec}K} \mathrm{Spec}L$,*
- *a valuation v on L*

such that the tropicalization of $C \cap T$ via v has genus 1, where T is a big torus of $\mathbb{P}^2$.

Proof. Consider the scalar extension $E' := E \times_{\mathrm{Spec}K} \mathrm{Spec}\overline{K}$, where $\overline{K}$ is an algebraic closure of K. Let $f := X^3+Y^3+Z^3-DXYZ = 0$ be the defining equation of the Hessian form of E'. Since $D \in \overline{K}$, there exists a finite extension L of K such that $D \in L$. Let $C \subset \mathbb{P}^2_L$ be the elliptic curve over L defined by the equation $f = 0$. Then C is birationally equivalent to $E \times_{\mathrm{Spec} K} \mathrm{Spec} L$. By the assumption that the j-invariant of E is not in $\mathbb{C}$, we have $D \notin \mathbb{C}$. Then, by Lemma 3.1.8, there is a valuation v on L such that $v(D) < 0$ and $v(a) = 0$ for any $a \in \mathbb{C}$.

Now we consider the tropicalization of $C \cap T$ via the valuation v. $C \cap T$ is defined by the polynomial $g := x^3 + y^3 + 1 - 3Dxy \in L[x^{\pm 1}, y^{\pm 1}]$. Since $v(1) = 0$ and $v(D) < 0$, the regular subdivision of $\mathrm{Newt}(g)$ consists of the points $(3,0), (0,3), (0,0), (1,1)$ and the any edges connecting two of that four points (see Figure 9). Then $\mathrm{Newt}(g)$ has just one interior point which is a vertex of the regular subdivision. Hence we have $g(\mathrm{trop}_v(C \cap T)) = 1$. $\square$

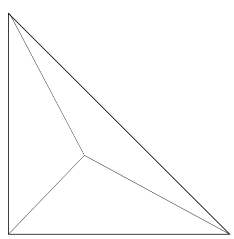

Figure 9. The regular subdivision of $\mathrm{Newt}(g)$

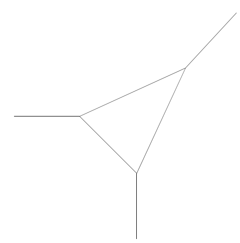

Figure 10. $\mathrm{trop}_v(C \cap T)$

4 Tropical ideals in tropical polynomial function semirings

In this section, we define tropical ideals in the tropical polynomial function semirings $\mathbb{T}[x_1, \ldots, x_n]/\sim$, where the relation $\sim$ is defined as $f \sim g$ if and only if $f(\mathbf{a}) = g(\mathbf{a})$ for any $\mathbf{a} \in \mathbb{R}^n$. The definition of our tropical ideals is given in Definition 4.1.3, which is analogous to [14]. Then we show that, in the one variable case, every principal ideal is a tropical ideal (Theorem 4.3.1), and that every tropical ideal is principal (Theorem 4.3.2). These theorems say that $\mathbb{T}[x]/\sim$ is like a PID. As a consequence of the theorems, it follows that our tropical ideals are closed under the intersection, and that we can add, multiply and generate tropical ideals. In fact, these properties do not hold for Maclagan and Rincón's tropical ideals.

This section is organized as follows. In Section 4.2, we introduce the maximum representation of a tropical polynomial function and define our tropical ideals. After we develop the theory of one variable tropical polynomial functions in Section 4.3, we prove our main theorems and their corollaries in Section 4.4. Section 4.5 is independent of the other sections. In the section, we introduce some unexpected examples of tropical ideals in Maclagan and Rincón's sense, for instance, a pair of tropical ideals whose intersection is not a tropical ideal.

4.1 Tropical polynomial function semirings

In this section, we introduce the maximum representation of tropical polynomial function and give our definition of tropical ideals.

We use the following notations. The tropical addition and multiplication on $\mathbb{T}$ or $\mathbb{T}[x_1, \ldots, x_n]$ is denoted by $\oplus$ and $\odot$, respectively. We use the notations $+, -$ and $\cdot$ for the standard addition, subtraction and multiplication, respectively. When the symbol of multiplication is omitted, it means the tropical multiplication $\odot$. For a tropical polynomial $f \in \mathbb{T}[x_1, \ldots, x_n]$ and a monomial $\mathbf{x}^{\mathbf{u}}$, we denote the coefficient of $\mathbf{x}^{\mathbf{u}}$ in f by $[f]_{\mathbf{u}}$. When the coefficient of a term of a tropical polynomial is omitted, it means that the coefficient is 0, the multiplicative unit of $\mathbb{T}$.

Now we define tropical polynomial functions. We denote by $\mathbb{T}[x_1, \ldots, x_n]/\sim$ the quotient of $\mathbb{T}[x_1, \ldots, x_n]$ by the equivalent relation $\sim$ which is defined as $f \sim g$ if and only if $f(\mathbf{a}) = g(\mathbf{a})$ for any $\mathbf{a} \in \mathbb{R}^n$. The relation $\sim$ is a congruence on $\mathbb{T}[x_1.\dots,x_n]$, i.e. $\mathbb{T}[x_1,\dots,x_n]/\sim$ forms semiring via the induced addition and multiplication. Thus we call the quotient $\mathbb{T}[x_1,\dots,x_n]/\sim$ the *tropical polynomial function semiring.* An element of $\mathbb{T}[x_1,\dots,x_n]/\sim$ is called a *tropical polynomial function.* The class of $f \in \mathbb{T}[x_1,\dots,x_n]$ in $\mathbb{T}[x_1,\dots,x_n]/\sim$ is denoted by $\overline{f}$. By abuse of notation, we write $\overline{-\infty} = -\infty$. For $\varphi, \psi \in \mathbb{T}[x_1,\dots,x_n]/\sim$, we use the notation $\varphi \,|\, \psi$ to denote that ψ divides φ, i.e. there exists a tropical polynomial function $\zeta \in \mathbb{T}[x_1,\dots,x_n]/\sim$ such that $\psi = \varphi\zeta$.

Note that the relation $\sim$ is not trivial. For instance, we have $x^2 \oplus 0 \sim x^2 \oplus ax \oplus 0$ for any $a \leq 0$.

For a one variable tropical polynomial function $\varphi \in \mathbb{T}[x]/\sim$, the *degree* $\deg \varphi$ is defined as the maximum slope of the graph of $y = \varphi(x)$ in $\mathbb{R}^2$. We define $\deg(-\infty) = -\infty$.

Recall that the *support* of a tropical polynomial $f \in \mathbb{T}[x_1,\dots,x_n]$ is the set $\mathrm{Supp}(f) = \{\mathbf{u} \in \mathbb{Z}^n \mid [f]_{\mathbf{u}} \neq -\infty\}$. The *Newton polytope* $\mathrm{Newt}(f)$ of $f \in \mathbb{T}[x_1,\dots,x_n]$ is defined similarly to usual sense, namely, $\mathrm{Newt}(f) := \mathrm{conv}(\mathrm{Supp}(f)) \subset \mathbb{R}^n$. We also consider the polytope $\mathrm{Newt}^*(f) := \mathrm{conv}\{(\mathbf{u}, t) \in \mathbb{R}^n \times \mathbb{R} \mid [f]_{\mathbf{u}} \neq -\infty \text{ and } t \leq [f]_{\mathbf{u}}\}$. A face of $\mathrm{Newt}^*(f)$ is called an *upper face* if it has an outer normal vector whose last coordinate is positive. The union of upper faces of $\mathrm{Newt}^*(f)$ is denoted by $\mathrm{Newt}^u(f)$.

Proposition 4.1.1. *Let $f, g \in \mathbb{T}[x_1,\dots,x_n]$ be two tropical polynomials. Then $f \sim g$ if and only if $\mathrm{Newt}^u(f) = \mathrm{Newt}^u(g)$.*

Proof. For the if direction, it is sufficient to show that the function $\overline{f}$ is determined depend only on $\mathrm{Newt}^u(f)$. Fix $\mathbf{w} \in \mathbb{R}^n$. We have

$$\begin{aligned} f(\mathbf{w}) &= \max_{\mathbf{u}}([f]_{\mathbf{u}} + \mathbf{u} \cdot \mathbf{w}) \\ &= \max_{\mathbf{u}}((\mathbf{u}, [f]_{\mathbf{u}}) \cdot (\mathbf{w}, 1)), \end{aligned}$$

where $\cdot$ means the standard inner product. Consider the function $p_{\mathbf{w}} : \mathbb{R}^{n+1} \to \mathbb{R}, \mathbf{v} \mapsto \mathbf{v} \cdot (\mathbf{w}, 1)$. The value $f(\mathbf{w})$ is the maximum of $p_{\mathbf{w}}(\mathbf{v})$, where $\mathbf{v}$ varies in $\{(\mathbf{u}, [f]_{\mathbf{u}}) \in \mathbb{R}^n \times \mathbb{R} \mid [f]_{\mathbf{u}} \neq -\infty\}$. That maximum does not change if we vary $\mathbf{v}$ in $\mathrm{Newt}^*(f)$. Moreover, since the last coordinate of $(\mathbf{w}, 1)$ is positive, that maximum is attained at one or more points in $\mathrm{Newt}^u(f)$. Thus we have

$$f(\mathbf{w}) = \max\{p_{\mathbf{w}}(\mathbf{v}) \mid \mathbf{v} \in \mathrm{Newt}^u(f)\} \quad (*).$$

Hence $\overline{f}$ is depend only on $\mathrm{Newt}^u(f)$.

Next we prove the only if direction. Assume that $f \sim g$. It is sufficient to show that every vertex of $\mathrm{Newt}^u(f)$ is that of $\mathrm{Newt}^u(g)$. Let $P = (\mathbf{u}, u_{n+1})$ be a vertex of $\mathrm{Newt}^u(f)$. Then there is an affine half space $H(\mathbf{w}, b) := \{\mathbf{v} \in \mathbb{R}^{n+1} \mid \mathbf{v} \cdot (\mathbf{w}, 1) \geq b\}$ for some $\mathbf{w} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $H(\mathbf{w}, b) \cap \mathrm{Newt}^u(f) = \{P\}$. This means that the function $p_{\mathbf{w}}$ restricted to $\mathrm{Newt}^u(f)$ takes its maximum only at P. On the other hand, that maximum must be taken at a point in $\{(\mathbf{u}, [f]_{\mathbf{u}}) \in \mathbb{R}^n \times \mathbb{R} \mid [f]_{\mathbf{u}} \neq -\infty\}$. Hence we have $u_{n+1} = [f]_{\mathbf{u}}$ and

$$f(\mathbf{w}) = (\mathbf{u}, [f]_{\mathbf{u}}) \cdot (\mathbf{w}, 1) = \mathbf{u} \cdot \mathbf{w} + [f]_{\mathbf{u}}.$$

For any vector ϵ with $|\epsilon|$ sufficiently small, there is also an affine half space $H(\mathbf{w} + \epsilon, b')$ such that $H(\mathbf{w} + \epsilon, b') \cap \mathrm{Newt}^u(f) = \{P\}$. Hence $f(\mathbf{w} + \epsilon) = \mathbf{u} \cdot (\mathbf{w} + \epsilon) + [f]_{\mathbf{u}}$. This shows that f is differentiable at $\mathbf{w}$ and $\mathrm{grad}(f)(\mathbf{w}) = \mathbf{u}$. By the assumption that $f \sim g$, g is also differentiable at $\mathbf{w}$ and $\mathrm{grad}(g)(\mathbf{w}) = \mathbf{u}$. Then the maximum of $g(\mathbf{w}) = \max_{\mathbf{v}}([g]_{\mathbf{v}} + \mathbf{v} \cdot \mathbf{w})$ is attained only at $\mathbf{v} = \mathbf{u}$. Equivalently, the maximum of $\max\{p_{\mathbf{w}}(\mathbf{v}) \mid \mathbf{v} \in \mathrm{Newt}^u(g)\}$ is attained only at $(\mathbf{u}, [g]_{\mathbf{u}}) =: Q$. Hence we have $H(\mathbf{w}, g(\mathbf{w})) \cap \mathrm{Newt}^u(g) = \{Q\}$, which means that Q is a vertex of $\mathrm{Newt}^u(g)$. We show that $P = Q$. Since $f \sim g$, we have $f(\mathbf{w}) = g(\mathbf{w})$, which means that $[f]_{\mathbf{u}} + \mathbf{u} \cdot \mathbf{w} = [g]_{\mathbf{u}} + \mathbf{u} \cdot \mathbf{w}$. Hence $[f]_{\mathbf{u}} = [g]_{\mathbf{u}}$. Therefore $P = Q$. Thus P is a vertex of $\mathrm{Newt}^u(g)$. $\square$

For a tropical polynomial function $\varphi \in \mathbb{T}[x_1, \dots, x_n]/\sim$, we construct the *maximum representation* $\varphi^{\max}$ of φ as follows.

Let $f \in \mathbb{T}[x_1, \dots, x_n]$ be any representation of φ. For any monomial $\mathbf{x}^{\mathbf{v}}$ in $\mathrm{Newt}(f)$, there is a unique real number $r_{\mathbf{v}}$ such that the point $(\mathbf{x}^{\mathbf{v}}, r_{\mathbf{v}})$ is in $\mathrm{Newt}^u(f)$. Then the coefficient of $\mathbf{x}^{\mathbf{v}}$ in $\varphi^{\max}$ is $r_{\mathbf{v}}$. For a monomial $\mathbf{x}^{\mathbf{v}} \notin \mathrm{Newt}(f)$, we set $[\varphi^{\max}]_{\mathbf{v}} = -\infty$.

Obviously, the maximum representation $\varphi^{\max}$ is the unique representation of φ satisfying that $[\varphi^{\max}]_{\mathbf{u}} \geq [f]_{\mathbf{u}}$ for any representation f of φ and any monomial $\mathbf{x}^{\mathbf{u}}$. Note that, for any two tropical polynomial functions $\varphi, \psi \in \mathbb{T}[x_1, \dots, x_n]/\sim$, we have

$$\overline{\varphi^{\max} \oplus \psi^{\max}} = \overline{\varphi^{\max}} \oplus \overline{\psi^{\max}} = \varphi \oplus \psi = \overline{(\varphi \oplus \psi)^{\max}}.$$

For a tropical polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$, the maximum representation of $\overline{f}$ is denoted by $f^{\max}$ for simplicity. By Proposition 4.1.1, we have the following lemma.

Lemma 4.1.2. *Let $f \in \mathbb{T}[x_1, \ldots, x_n]$ be a tropical polynomial. Then the point $(\mathbf{u}, [f^{\max}]_{\mathbf{u}})$ is a vertex of* $\mathrm{Newt}^u(f)$ *if and only if $f \not\sim f_{\check{\mathbf{u}}}$, where $f_{\check{\mathbf{u}}}$ denotes the tropical polynomial $\bigoplus_{\boldsymbol{v} \neq \boldsymbol{u}} [f]_{\boldsymbol{v}} \mathbf{x}^{\mathbf{v}}$. Moreover, if those equivalent conditions hold, the coefficient $[f]_{\mathbf{u}}$ is equal to $[f^{\max}]_{\mathbf{u}}$.*

Now we define tropical ideals in tropical polynomial function semirings. An *ideal* I of $\mathbb{T}[x_1, \ldots, x_n]/\sim$ is a subset satisfying that (1) if $\varphi, \psi \in I$, then $\varphi \oplus \psi \in I$, (2) $-\infty \in I$, and (3) if $\varphi \in I$ and $\psi \in R$, then $\varphi\psi \in I$.

Definition 4.1.3. An ideal $I \subset \mathbb{T}[x_1, \ldots, x_n]/\sim$ is a *tropical ideal* if for any $\varphi_1, \varphi_2 \in I$, and $\mathbf{u} \in \mathbb{Z}^n_{\geq 0}$ with $[\varphi_1^{\max}]_{\mathbf{u}} = [\varphi_2^{\max}]_{\mathbf{u}} \neq -\infty$, there is a tropical polynomial h such that $\overline{h} \in I$, $[h]_{\mathbf{u}} = -\infty$ and $[h]_{\mathbf{v}} \leq [\varphi_1^{\max}]_{\mathbf{v}} \oplus [\varphi_2^{\max}]_{\mathbf{v}}$ for all $\mathbf{v}$, with the equality holding whenever $[\varphi_1^{\max}]_{\mathbf{v}} \neq [\varphi_2^{\max}]_{\mathbf{v}}$.

This definition is an analogy of the definition of the tropical ideals in $\mathbb{T}[x_1, \ldots, x_n]$.

Lemma 4.1.4. *Let $\pi : \mathbb{T}[x_1, \ldots, x_n] \to \mathbb{T}[x_1, \ldots, x_n]/\sim$ be the canonical surjection and $I \subset \mathbb{T}[x_1, \ldots, x_n]/\sim$ be an ideal. If $\pi^{-1}(I)$ is a tropical ideal in $\mathbb{T}[x_1, \ldots, x_n]$, then I is a tropical ideal in $\mathbb{T}[x_1, \ldots, x_n]/\sim$.*

Proof. This immediately follows from the definitions of tropical ideals in $\mathbb{T}[x_1, \ldots, x_n]$ and $\mathbb{T}[x_1, \ldots, x_n]/\sim$. □

Recall that the *tropical variety* $V(f) \subset \mathbb{T}^n$ defined by a tropical polynomial $f \in \mathbb{T}[x_1, \ldots, x_n]$ is the set

$$\{\mathbf{w} \in \mathbb{T}^n \mid f \text{ is not differentiable at } \mathbf{w} \in \mathbb{R}^n \text{ or } f(\mathbf{w}) = -\infty\}.$$

By the definition, if $f \sim g$, we have $V(f) = V(g)$. Hence we can define the tropical variety $V(\varphi)$ for a tropical polynomial function $\varphi \in \mathbb{T}[x]/\sim$.

Example 4.1.5. Fix a point $P \in \mathbb{T}^n$. Let I_P be a set $\{\varphi \in \mathbb{T}[x_1, \ldots, x_n]/\sim \mid P \in V(\varphi)\}$. By the examples in [14, Section 2], the inverse image $\pi^{-1}(I_P)$ is a tropical ideal in $\mathbb{T}[x_1, \ldots, x_n]$. Thus I_P is a tropical ideal in $\mathbb{T}[x_1, \ldots, x_n]/\sim$.

4.2 One variable tropical polynomial function semiring

In this section, we develop a theory of one variable tropical polynomial function semiring $\mathbb{T}[x]/\sim$. The results in this section are used in the proofs of our main theorems in the next section. We start with a tropical version of the fundamental theorem of algebra.

Theorem 4.2.1 (Grigg, Manwaring [6])**.** *Let $\varphi \in \mathbb{T}[x]/\sim$ be a tropical polynomial function with $\deg\varphi \geq 1$ and let $d_i = [\varphi^{\max}]_{i-1} - [\varphi^{\max}]_i$ for $i = 1, \ldots, \deg\varphi$ with the convention that $(-\infty)-(-\infty) = -\infty$. Then*

(1) *φ is uniquely factorized as $\varphi = \overline{a(x \oplus d_1)(x \oplus d_2)\cdots(x \oplus d_n)}$ for some $a \in \mathbb{R}$, and*

(2) *$\varphi(d_i) = [\varphi^{\max}]_i + i \cdot d_i = [\varphi^{\max}]_{i-1} + (i-1) \cdot d_i$, i.e. $\varphi^{\max}(d_i)$ attains its maximum on the terms of degree i and $i-1$.*

In the notation of Theorem 4.2.1, we call each d_i a *root* of φ. The root $d_i \neq -\infty$ coincides with the negative of the slope of the line segment in $\mathrm{Newt}^u(\varphi^{\max})$ connecting $(i-1, [\varphi^{\max}]_{i-1})$ and $(i, [\varphi^{\max}]_i)$. In the rest of this paper, we always assume that $(-\infty) - (-\infty) = -\infty$.

In [6], the following two propositions are also proved.

Proposition 4.2.2 ([6, Lemma 3.4])**.** *Let $f \in \mathbb{T}[x]$ be a tropical polynomial and let $d_i := [f]_{i-1} - [f]_i$. Then $f = f^{\max}$ if and only if $d_1 \leq d_2 \leq \cdots \leq d_n$.*

Proposition 4.2.3 ([6, Lemma 3.3])**.** *Let $f \in \mathbb{T}[x]$ be a tropical polynomial. Then the coefficients of $f^{\max}$ are obtained by*

$$[f^{\max}]_j = \max\left(\{[f]_j\} \cup \left\{ \frac{[f]_i \cdot (k-j) + [f]_k \cdot (j-i)}{k-i} \;\middle|\; i < j < k \right\}\right).$$

By the construction of maximum representation and theorem 4.2.1, we also have the following proposition.

Proposition 4.2.4. *Let $f \in \mathbb{T}[x]$ be a tropical polynomial.*

(1) *The coefficients $[f]_j$ and $[f^{\max}]_j$ coincides if and only if the following inequality holds;*

$$\min_{i<j}\left(\frac{[f]_j - [f]_i}{j-i}\right) \geq \max_{k>j}\left(\frac{[f]_k - [f]_j}{k-j}\right). \quad (*)$$

(2) *Suppose that the equivalent conditions in* (1) *hold. Let* $d_i := [f^{\max}]_{i-1} - [f^{\max}]_i$ *for any* i. *Let* i_0 *and* k_0 *be the minimum degree* i *and the maximum degree* k *which attains the minimum and the maximum of each side of* $(*)$, *respectively. Thus* i_0 *(resp.* k_0*) coincides with the minimum* i *(resp. the maximum* k*) such that*

$$d_j = d_{j-1} = \cdots = d_{i+1} = -\min_{i<j}\left(\frac{[f]_j - [f]_i}{j-i}\right)$$

$$\left(\textit{resp. } d_{j+1} = d_{j+2} = \cdots = d_k = -\max_{k>j}\left(\frac{[f]_k - [f]_j}{k-j}\right)\right).$$

The following lemma is used in the proof of Theorem 1.2.5.

Lemma 4.2.5. *Let* $\varphi, \psi \in \mathbb{T}[x]/\sim$ *be two tropical polynomial functions and* x^u *be a monomial. Suppose that* $[\varphi^{\max}]_u = [\psi^{\max}]_u$. *Then* $(\varphi^{\max} \oplus \psi^{\max})_{\check{u}} \sim ((\varphi \oplus \psi)^{\max})_{\check{u}}$ *holds.*

Proof. For simplicity, we write $f = \varphi^{\max} \oplus \psi^{\max}$. We show that the maximum representation of $f_{\check{u}}$ coincides with that of $((\varphi \oplus \psi)^{\max})_{\check{u}}$. Thus we check that

$$[(f_{\check{u}})^{\max}]_v = [(((\varphi \oplus \psi)^{\max})_{\check{u}})^{\max}]_v \qquad (*)$$

for all v.

First we consider the case that $v < u$. In general, for a tropical polynomial function ζ, we have $[((\zeta^{\max})_{\check{u}})^{\max}]_v = [\zeta^{\max}]_v$ by the construction of the maximal representation. Then the right hand side of $(*)$ is $[(\varphi \oplus \psi)^{\max}]_v$. Since f is a representation of $\varphi \oplus \psi$, by Proposition 4.2.3, we have

$$[(\varphi \oplus \psi)^{\max}]_v = [f]_v \oplus \max_{i<v<k}\left(\frac{[f]_i \cdot (k-v) + [f]_k \cdot (v-i)}{k-i}\right) \qquad (**),$$

while the left hand side of $(*)$ is

$$[(f_{\check{u}})^{\max}]_v = [f]_v \oplus \max_{\substack{i<v<k \\ k \neq u}}\left(\frac{[f]_i \cdot (k-v) + [f]_k \cdot (v-i)}{k-i}\right).$$

Suppose that the value of the right hand side of $(**)$ is that of the second term with $i = i_0$ and $k = u$ for some i_0. We may assume that $[\varphi^{\max}]_{i_0} \geq [\psi^{\max}]_{i_0}$. By the assumption $[\varphi^{\max}]_u = [\psi^{\max}]_u$, the value of right hand side of $(**)$ is

$$\begin{aligned}\frac{[f]_{i_0} \cdot (u-v) + [f]_u \cdot (v-i_0)}{u-i_0} &= \frac{[\varphi^{\max}]_{i_0} \cdot (u-v) + [\varphi^{\max}]_u \cdot (v-i_0)}{u-i_0} \\ &\leq [\varphi^{\max}]_v \leq [f]_v,\end{aligned}$$

where we use Proposition 4.2.3 for $\varphi^{\max}$. Then the value of right hand side of $(**)$ is not only the second term with $i = i_0$ and $k = u$, but also $[f]_v$. Hence $(*)$ holds.

The case that $v > u$ is similar.

If $v = u$, we show the both directions of inequality. We can easily see $\left[(f_{\check{u}})^{\max}\right]_u \leq \left[(((\varphi \oplus \psi)^{\max})_{\check{u}})^{\max}\right]_u$; since f is a representation of $\varphi \oplus \psi$, we have $\left[f_{\check{u}}\right]_v \leq \left[((\varphi \oplus \psi)^{\max})_{\check{u}}\right]_v$ for all v, hence $\left[(f_{\check{u}})^{\max}\right]_u \leq \left[(((\varphi \oplus \psi)^{\max})_{\check{u}})^{\max}\right]_u$ by Proposition 4.2.3. For the reverse direction, first we have

$$\left[(((\varphi \oplus \psi)^{\max})_{\check{u}})^{\max}\right]_u = \max_{i<u<k} \left(\frac{[f^{\max}]_i \cdot (k-u) + [f^{\max}]_k \cdot (u-i)}{k-i} \right).$$

Let i_0 and k_0 be the minimum i and the maximum k at which the maximum of right hand side is attained, respectively. In $\mathrm{Newt}^*(f)$, the points $(i_0, [f^{\max}]_{i_0})$ and $(k_0, [f^{\max}]_{k_0})$ are the vertices, which means that $[f^{\max}]_{i_0} = [f]_{i_0}$ and $[f^{\max}]_{k_0} = [f]_{k_0}$. Then we have

$$\begin{aligned}\left[(((\varphi \oplus \psi)^{\max})_{\check{u}})^{\max}\right]_u &= \left(\frac{[f^{\max}]_{i_0} \cdot (k_0-u) + [f^{\max}]_{k_0} \cdot (u-i_0)}{k_0-i_0} \right) \\ &= \left(\frac{[f]_{i_0} \cdot (k_0-u) + [f]_{k_0} \cdot (u-i_0)}{k_0-i_0} \right) \\ &\leq \max_{i<u<k} \left(\frac{[f]_i \cdot (k-u) + [f]_k \cdot (u-i)}{k-i} \right) \\ &= [(f_{\check{u}})^{\max}]_u.\end{aligned}$$

Hence we have the desired equality. □

We recall the definition of the *initial form* of a tropical polynomial.

Definition 4.2.6. The *initial form* $\mathrm{in}_{\mathbf{w}}(f)$ of a tropical polynomial $f \in \mathbb{T}[x_1, \ldots, x_n]$ with respect to a vector $\mathbf{w} \in \mathbb{R}^n$ is given by

$$\mathrm{in}_{\mathbf{w}}(f) = \begin{cases} \bigoplus_{[f]_{\mathbf{u}} + \mathbf{u} \cdot \mathbf{w} = f(\mathbf{w})} \mathbf{x}^{\mathbf{u}} & \text{if } f(\mathbf{w}) > -\infty, \\ -\infty & \text{otherwise.} \end{cases}$$

For a tropical polynomial function $\varphi \in \mathbb{T}[x]/\sim$ and a real number $a \in \mathbb{R}$, we denote by $S(\varphi, a) := \mathrm{Supp}(\mathrm{in}_a(\varphi^{\max}))$, which is the set of the degrees of the terms of $\varphi^{\max}$ at which the maximum of $\varphi^{\max}(a)$ is attained. $S(\varphi, a)$ consists of a single integer or two or more number of consecutive integers.

For a tropical polynomial $f \in \mathbb{T}[x]$ we have $S(\overline{f}, a) = \{j, j+1, \ldots, k\}$ where j (resp. k) is the minimum (resp. maximum) degree i of the term of f such that $[f]_i + i \cdot a = f(a)$. Thus we have the following lemma, which is used in the proof of Theorem 1.2.5.

Lemma 4.2.7. *Let $\varphi, \psi \in \mathbb{T}[x]/\sim$ be two tropical polynomial functions and $a \in \mathbb{R}$ be a real number. Then*

(1) $S(\varphi \oplus \psi, a) = S(\varphi, a)$ if $\varphi(a) > \psi(a)$,

(2) $S(\varphi \oplus \psi, a) = S(\psi, a)$ if $\varphi(a) < \psi(a)$ and

(3) $S(\varphi \oplus \psi, a) = \{j, j+1, \ldots, k\}$ if $\varphi(a) = \psi(a)$, where $j = \min(S(\varphi, a) \cup S(\psi, a))$ and $k = \max(S(\varphi, a) \cup S(\psi, a))$.

Definition 4.2.8. The *multiplicity* $\mathrm{mult}(\varphi; a)$ of a tropical polynomial function φ at $a \in \mathbb{T}$ is the maximum integer m satisfying $(x \oplus a)^m | \varphi$, where we define $(x \oplus a)^0 = 0$ for any a.

Lemma 4.2.9. *For a tropical polynomial function $\varphi \in \mathbb{T}[x]/\sim$ and a real number $a \in \mathbb{R}$, we have* $\mathrm{mult}(\varphi; a) = \#S(\varphi, a) - 1$.

The following lemma shows that, under some assumptions, we can obtain the product of two tropical polynomial function by addition.

Lemma 4.2.10. *Let $\varphi, \psi \in \mathbb{T}[x]/\sim$ be two tropical polynomial functions. Suppose that any root of φ is less than or equal to any root of ψ, and that $[\varphi^{\max}]_{\deg \varphi} = [\psi^{\max}]_0$. Then the product $\varphi\psi$ coincides with the sum $\varphi \oplus x^{\deg \varphi}\psi$ up to constant factors. More precisely, $\varphi\psi$ coincides with $[\psi^{\max}]_0(\varphi \oplus x^{\deg \varphi}\psi)$.*

Proof. Let $d = \deg \varphi$ and $d' = \deg \psi$. Let $\alpha_i = [\varphi^{\max}]_{i-1} - [\varphi^{\max}]_i$ $i = 1, \ldots, d$ be the roots of φ and $\beta_i = [\psi^{\max}]_{i-1} - [\psi^{\max}]_i$ $i = 1, \ldots, d'$ be the roots of ψ. The differences between the adjacent coefficients in $\varphi^{\max} \oplus (x^d\psi)^{\max}$ are

$$[\varphi^{\max} \oplus (x^d\psi)^{\max}]_{i-1} - [\varphi^{\max} \oplus (x^d\psi)^{\max}]_i = \begin{cases} \alpha_i & \text{if } i \leq d, \\ \beta_{i-d} & \text{if } i > d. \end{cases}$$

Hence, by Proposition 4.2.2, the tropical polynomial $\varphi^{\max} \oplus (x^d\psi)^{\max}$ is the maximum representation of its class. Moreover, by Theorem 4.2.1, we can factorize

$$\begin{aligned}\varphi \oplus x^d\psi &= \overline{a(x \oplus \alpha_1) \cdots (x \oplus \alpha_d)(x \oplus \beta_1) \cdots (x \oplus \beta_{d'})} \\ &= b\varphi\psi\end{aligned}$$

for some $a, b \in \mathbb{R}$. Since the constant terms of $[\psi^{\max}]_0(\varphi \oplus x^d\psi)$ and $\varphi\psi$ are equal, we obtain that $[\psi^{\max}]_0(\varphi \oplus x^d\psi) = \varphi\psi$. $\square$

For a tropical polynomial $f \in \mathbb{T}[x]$ and integers $u \leq v$, we denote by

$$f_{\leq u} := \bigoplus_{i \leq u} [f]_i x^i, \ f_{\geq u} := \bigoplus_{i \geq u} [f]_i x^{i-u} \ \text{ and } \ f_{u \leq v} := \bigoplus_{u \leq i \leq v} [f]_i x^{i-u}.$$

We conclude this section with the following lemma, which gives the key construction used in the proof of Theorem 1.2.5.

Lemma 4.2.11. *Let $\varphi \in \mathbb{T}[x]/\sim$ be a tropical polynomial function of degree $d \geq 1$.*

(a) *Assume that $d \geq 2$. Fix an integer $u \in \{1, \ldots, d-1\}$. Let α and β be the minimum root and the maximum root of φ, respectively. Let $\psi_1 \in \mathbb{T}[x]/\sim$ be the unique tropical polynomial function which satisfies*

$$\varphi = \overline{(x \oplus \alpha)} \odot \psi_1 \odot \overline{(x \oplus \beta)}.$$

Then there exists a tropical polynomial $h \in \mathbb{T}[x]$ such that

- (a–1) $\psi_1 \mid \overline{h}$,
- (a–2) $[h]_u = -\infty$,
- (a–3) $[h]_v \leq [\varphi^{\max}]_v$ *for all monomials* x^v,
- (a–4) $[h]_0 = [\varphi^{\max}]_0$, $[h]_d = [\varphi^{\max}]_d$, *and*
- (a–5) *any root* γ *of* $\overline{h}$ *satisfies* $\alpha \leq \gamma \leq \beta$.

(b) *Fix an integer $u \in \{0, \ldots, d-1\}$. Let β be the maximum root of φ. Let $\psi_2 \in \mathbb{T}[x]/\sim$ be the unique tropical polynomial function which satisfies*

$$\varphi = \psi_2 \odot \overline{(x \oplus \beta)}.$$

Then there exists a tropical polynomial $h \in \mathbb{T}[x]$ such that (*b*–1) $\psi_2 \,|\, \overline{h}$,

(*b*–2) $[h]_u = -\infty$,

(*b*–3) $[h]_v \leq [\varphi^{\max}]_v$ *for all monomials* x^v,

(*b*–4) $[h]_d = [\varphi^{\max}]_d$, *and*

(*b*–5) *any root* γ *of* h *satisfies* $\gamma \leq \beta$.

(*c*) *Fix an integer* $u \in \{1, \ldots, d\}$. *Let* α *be the minimum root of* φ. *Let* $\psi_3 \in \mathbb{T}[x]/\sim$ *be the unique tropical polynomial function which satisfies*

$$\varphi = \overline{(x \oplus \alpha)} \odot \psi_3.$$

Then there exists a tropical polynomial $h \in \mathbb{T}[x]$ *such that*

(*c*–1) $\psi_3 \,|\, \overline{h}$,

(*c*–2) $[h]_u = -\infty$,

(*c*–3) $[h]_v \leq [\varphi^{\max}]_v$ *for all monomials* x^v,

(*c*–4) $[h]_0 = [\varphi^{\max}]_0$, *and*

(*c*–5) *any root* γ *of* h *satisfies* $\alpha \leq \gamma$.

Proof. (*a*) We define four tropical polynomials h_1, h_1', h_2 and h_2' as follows;

$$h_1 = (\varphi^{\max})_{1 \leq u}, \quad h_1' = (\varphi^{\max})_{1 \leq u-1},$$
$$h_2 = (\varphi^{\max})_{u \leq d-1}, \quad h_2' = (\varphi^{\max})_{u+1 \leq d-1}.$$

Note that the product $\overline{h_1 h_2}$ coincides with ψ_1 up to constant factors. Let $\gamma_1 = [\varphi^{\max}]_{u-1} - [\varphi^{\max}]_u$ and $\gamma_2 = [\varphi^{\max}]_u - [\varphi^{\max}]_{u+1}$ be two roots of φ. We have that $\overline{h_1}$ (resp. $\overline{h_2}$) coincides with $\overline{(x \oplus \gamma_1) h_1'}$ (resp. $\overline{(x \oplus \gamma_2) h_2'}$) up to constant factors. Now we define a tropical polynomial h as

$$h = \alpha(h_1 \oplus (-\gamma_2) x^{u+1} h_2') \oplus (-\beta)(\gamma_1 x h_1' \oplus x^{u+1} h_2).$$

We show that this h satisfies the condition (*a*–1)–(*a*–5).

(*a*–1) Let $f = h_1 \oplus (-\gamma_2) x^{u+1} h_2'$. First we factorize $\overline{f}$ by using Lemma 4.2.10. By the construction of h_1 and h_2', we have $h_1^{\max} = h_1$, $h_2'^{\max} = h_2'$, and

$$(\text{ any root of } h_1) \leq \gamma_1 \leq \gamma_2 \leq (\text{ any root of } h_2').$$

Also we have $[f]_{u-1} = [h_1]_{u-1} = [\varphi^{\max}]_u$ and $[f]_{u+1} = [(-\gamma_2)x^{u+1}h_2']_{u+1} = -\gamma_2 + [\varphi^{\max}]_{u+1} = [\varphi^{\max}]_u - 2 \cdot \gamma_2$. Thus the slope of the line segment connecting the points $(u-1, [f]_{u-1})$ and $(u+1, [f]_{u+1})$ is $-\gamma_2$. Then, by applying Lemma 4.2.10 for $\overline{h_1}$, $\overline{[f]_{u-1}x^2 \oplus [f]_{u+1}}$ and $\overline{(-\gamma_2)h_2'}$, $\overline{f}$ is factorized as

$$
\begin{aligned}
\overline{f} &= \overline{ah_1(x \oplus \gamma_2)^2 h_2'} \\
&= \overline{ah_1(x \oplus \gamma_2)} \odot \overline{(x \oplus \gamma_2)h_2'} \\
&= \overline{bh_1(x \oplus \gamma_2)h_2}
\end{aligned}
$$

for some $a, b \in \mathbb{T}$. This factorization shows that the function $\overline{f}$ can be divided by $\overline{h_1h_2}$, hence by ψ_1.
Similarly, let $g = \gamma_1 x h_1' \oplus x^{u+1}h_2$, and we can show that $g \sim cxh_1(x \oplus \gamma_1)h_2$ for some $c \in \mathbb{T}$, hence $\psi_1 \mid \overline{g}$. Therefore $\psi_1 \mid \overline{h}$.

(a–2) Clear.

(a–3) If $v < u$, let us calculate the coefficient $[h]_v$;

$$
\begin{aligned}
[h]_v &= [\alpha(h_1 \oplus (-\gamma_2)x^{u+1}h_2') \oplus (-\beta)(\gamma_1 x h_1' \oplus x^{u+1}h_2)]_v \\
&= [\alpha h_1]_v \oplus [(-\beta + \gamma_1)xh_1']_v \\
&= (\alpha + [\varphi^{\max}]_{v+1}) \oplus (-\beta + \gamma_1 + [\varphi^{\max}]_v).
\end{aligned}
$$

We estimate the two terms of the last form;

$$
\begin{aligned}
\alpha + [\varphi^{\max}]_{v+1} &= ([\varphi^{\max}]_0 - [\varphi^{\max}]_1) + [\varphi^{\max}]_{v+1} \\
&\leq ([\varphi^{\max}]_v - [\varphi^{\max}]_{v+1}) + [\varphi^{\max}]_{v+1} = [\varphi^{\max}]_v, \\
-\beta + \gamma_1 + [\varphi^{\max}]_v &\leq [\varphi^{\max}]_v,
\end{aligned}
$$

where the both inequality is followed by Lemma 4.2.2. Hence we have $[h]_v \leq [\varphi^{\max}]_v$.
If $v > u$, again we calculate the coefficient $[h]_v$;

$$
\begin{aligned}
[h]_v &= [\alpha(h_1 \oplus (-\gamma_2)x^{u+1}h_2') \oplus (-\beta)(\gamma_1 x h_1' \oplus x^{u+1}h_2)]_v \\
&= [(\alpha - \gamma_2)x^{u+1}h_2']_v \oplus [(-\beta)x^{u+1}h_2]_v \\
&= ((\alpha - \gamma_2) + [\varphi^{\max}]_v) \oplus (-\beta + [\varphi^{\max}]_{v-1}).
\end{aligned}
$$

Then we can show $[h]_v \leq [\varphi^{\max}]_v$ in a similar way to the above case.

(a–4) By the calculation in the proof of (a–3), we have

$$\begin{aligned}[h]_0 &= (\alpha + [\varphi^{\max}]_1) \oplus (-\beta + \gamma_1 + [\varphi^{\max}]_0) \\ &= \max\{([\varphi^{\max}]_0 - [\varphi^{\max}]_1) + [\varphi^{\max}]_1, -\beta + \gamma_1 + [\varphi^{\max}]_0\} \\ &= \max\{[\varphi^{\max}]_0, -\beta + \gamma_1 + [\varphi^{\max}]_0\} \\ &= [\varphi^{\max}]_0,\end{aligned}$$

and

$$\begin{aligned}[h]_d &= (\alpha - \gamma_2 + [\varphi^{\max}]_d) \oplus (-\beta + [\varphi^{\max}]_{d-1}) \\ &= \max\{\alpha - \gamma_2 + [\varphi^{\max}]_d, -([\varphi^{\max}]_{d-1} - [\varphi^{\max}]_d) + [\varphi^{\max}]_{d-1}\} \\ &= \max\{\alpha - \gamma_2 + [\varphi^{\max}]_d, [\varphi^{\max}]_d\} \\ &= [\varphi^{\max}]_d.\end{aligned}$$

(a–5) By the calculation in the proof of (a–1), we have

$$h = \alpha f \oplus (-\beta) g \sim h_1 h_2 (b\alpha(x \oplus \gamma_2) \oplus c(-\beta)x(x \oplus \gamma_1)).$$

Thus it is sufficient to prove that any root γ of the function $b\alpha(x \oplus \gamma_2) \oplus c(-\beta)x(x \oplus \gamma_1)$ satisfies $\alpha \leq \gamma \leq \beta$. Let us calculate b and c. Since $f \sim bh_1(x \oplus \gamma_2)h_2$, their constant terms are equal, namely,

$$[h_1 \oplus (-\gamma_2)x^{u+1}h_2']_0 = [bh_1(x \oplus \gamma_2)h_2]_0.$$

The left hand side is $[h_1]_0$, while the right hand side is $b + [h_1]_0 + \gamma_2 + [h_2]_0$, so we have $b = -\gamma_2 - [h_2]_0$. Similarly, since $g \sim cxh_1(x \oplus \gamma_1)h_2$, we have

$$[\gamma_1 x h_1' \oplus x^{u+1} h_2]_1 = [cxh_1(x \oplus \gamma_1)h_2]_1.$$

The left hand side is $\gamma_1 + [h_1]_0$, while the right hand side is $c + [h_1]_0 + \gamma_1 + [h_2]_0$, which lead $c = -[h_2]_0$. Hence we have

$$\begin{aligned}&b\alpha(x \oplus \gamma_2) \oplus c(-\beta)x(x \oplus \gamma_1) \\ &\qquad = (-\gamma_2 - [h_2]_0 + \alpha)(x \oplus \gamma_2) \oplus (-[h_2]_0 - \beta)x(x \oplus \gamma_1) \\ &\qquad = (-[h_2]_0)\{(-\gamma_2 + \alpha)(x \oplus \gamma_2) \oplus (-\beta)x(x \oplus \gamma_1)\} \\ &\qquad = (-[h_2]_0)((-\beta)x^2 \oplus \max\{-\gamma_2 + \alpha, -\beta + \gamma_1\}x \oplus \alpha).\end{aligned}$$

Let $h_3 = (-\beta)x^2 \oplus \max\{-\gamma_2 + \alpha, -\beta + \gamma_1\}x \oplus \alpha$ and let γ be a root of h_3. γ is the negative of the slope of a line segment in $\mathrm{Newt}^u(h_3)$. Explicitly, γ is either $[h_3]_0 - [h_3]_1$, $[h_3]_1 - [h_3]_2$, or $([h_3]_0 - [h_3]_2)/2$. Hence γ is one of the following five numbers:

- $\alpha - (-\gamma_2 + \alpha) = \gamma_2$,
- $\alpha - (-\beta + \gamma_1) = \alpha + \beta - \gamma_1$,
- $(-\gamma_2 + \alpha) - (-\beta) = \alpha + \beta - \gamma_2$,
- $(-\beta + \gamma_1) - (-\beta) = \gamma_1$,
- $(\alpha + \beta)/2$.

If $\gamma = \gamma_1, \gamma_2$ or $(\alpha + \beta)/2$, then $\alpha \leq \gamma \leq \beta$ is clear. If $\gamma = \alpha + \beta - \gamma_l$ $(l = 1, 2)$, since $\alpha \leq \gamma_l \leq \beta$, we have $\gamma = \alpha + (\beta - \gamma_l) \geq \alpha$ and $\gamma = \beta + (\alpha - \gamma_l) \leq \beta$.

(b) Similar to (a), we define two tropical polynomials h_1' and h_2 as follows;

$$h_1' = (\varphi^{\max})_{0 \leq u-1}, \quad h_2 = (\varphi^{\max})_{u \leq d-1}.$$

where we define $h_1' = -\infty$ if $u = 0$. Let $\gamma_1 = [\varphi^{\max}]_{u-1} - [\varphi^{\max}]_u$ be a root of φ, where we define $\gamma_1 = -\infty$ if $u = 0$. We have that ψ_2 coincides with the product $\overline{h_1'(x \oplus \gamma_1)h_2}$ up to constant factors. We define a tropical polynomial h as

$$h = (-\beta)(\gamma_1 h_1' \oplus x^{u+1} h_2).$$

We show that this h satisfies the condition (b–1)–(b–5). The argument is similar to (a).

(b–1) Let $g = \gamma_1 h_1' \oplus x^{u+1} h_2$. We factorize $\overline{g}$ by using Lemma 4.2.10. By the construction of h_1' and h_2, we have ${h_1'}^{\max} = h_1'$, $h_2^{\max} = h_2$, and

$$(\text{ any root of } h_1') \leq \gamma_1 \leq (\text{ any root of } h_2').$$

Also we have $[g]_{u-1} = [\gamma_1 h_1']_{u-1} = \gamma_1 + [\varphi^{\max}]_{u-1} = 2 \cdot \gamma_1 + [\varphi^{\max}]_u$ and $[g]_{u+1} = [x^{u+1} h_2]_{u+1} = [\varphi^{\max}]_u$. Thus the slope of the line segment connecting the points $(u-1, [g]_{u-1})$ and $(u+1, [g]_{u+1})$ is $-\gamma_1$. Then, by applying Lemma 4.2.10 for $\overline{\gamma_1 h_1'}$, $\overline{[g]_{u-1} x^2 \oplus [g]_{u+1}}$ and $\overline{h_2}$, $\overline{g}$ is factorized as

$$\overline{g} = \overline{a h_1'(x \oplus \gamma_1)^2 h_2}$$

for some $a \in \mathbb{T}$. This factorization shows that the function $\overline{g}$ can be divided by $\overline{h_1'(x \oplus \gamma_1)h_2}$, hence by ψ_2. Therefore, we have $\psi_2 \,|\, \overline{h}$.

(b–2) Clear.

(b–3) If $v < u$, let us estimate the coefficient $[h]_v$;

$$\begin{aligned}[h]_v &= [(-\beta)(\gamma_1 h_1' \oplus x^{u+1}h_2)]_v \\ &= [(-\beta+\gamma_1)h_1']_v \\ &= -\beta+\gamma_1+[\varphi^{\max}]_v \leq [\varphi^{\max}]_v,\end{aligned}$$

where the inequality follows from Lemma 4.2.2.

If $v > u$, similarly we have

$$\begin{aligned}[h]_v &= [(-\beta)(\gamma_1 h_1' \oplus x^{u+1}h_2)]_v \\ &= [(-\beta)x^{u+1}h_2]_v \\ &= -\beta+[\varphi^{\max}]_{v-1} \\ &\leq -\gamma_1+[\varphi^{\max}]_{v-1} = [\varphi^{\max}]_v.\end{aligned}$$

(b–4) By the calculation in the proof of (b–3), we have

$$\begin{aligned}[h]_d &= -\beta+[\varphi^{\max}]_{d-1} \\ &= -([\varphi^{\max}]_{d-1}-[\varphi^{\max}]_d)+[\varphi^{\max}]_{d-1} = [\varphi^{\max}]_d.\end{aligned}$$

(b–5) By the calculation in the proof of (b–1), we have

$$\overline{h} = \overline{(-\beta)g} = \overline{(-\beta)ah_1'(x\oplus\gamma_1)^2h_2} = \overline{(-\beta)(x\oplus\gamma_1)}\psi_2.$$

Thus the desired condition holds obviously.

(c) Similar to (a), we define two tropical polynomials h_1 and h_2' as

$$h_1 = (\varphi^{\max})_{1\leq u}, \quad h_2' = (\varphi^{\max})_{u+1\leq d}.$$

where we define $h_2' = -\infty$ if $u = d$. Let $\gamma_2 = [\varphi^{\max}]_u - [\varphi^{\max}]_{u+1}$ be a root of φ, where we define $\gamma_2 = \infty$ if $u = d$. We define a tropical polynomial h as

$$h = \alpha(h_1 \oplus (-\gamma_2)x^{u+1}h_2').$$

By the similar argument to (b), we can see that h satisfies (c–1)–(c–5).

□

4.3 Tropical ideals in one variable tropical polynomial function semiring

In this section, we show our main theorem. These theorems say that the tropical ideals in one variable tropical polynomial function semiring are very like the ideals in PID.

Theorem 4.3.1. *For any tropical polynomial function $\varphi \in \mathbb{T}[x]/\sim$, the set $\varphi \odot (\mathbb{T}[x]/\sim) := \{\varphi \odot \psi \mid \psi \in \mathbb{T}[x]/\sim\}$ is a tropical ideal in $\mathbb{T}[x]/\sim$.*

Proof. Let $I = \varphi \odot (\mathbb{T}[x]/\sim)$. If $\deg \varphi = -\infty$ or $\deg \varphi = 0$, then $I = \{-\infty\}$ or $I = \mathbb{T}[x]/\sim$, respectively. Thus in those cases, the theorem immediately holds.

We assume that $\deg \varphi \geq 1$. Let ψ_1 and ψ_2 be elements of I and x^u be a monomial with $[\psi_1^{\max}]_u = [\psi_2^{\max}]_u \neq -\infty$. We will construct a tropical polynomial h such that $\overline{h} \in I$, $[h]_u = -\infty$ and $[h]_v \leq [\psi_1^{\max}]_v \oplus [\psi_2^{\max}]_v$ for all v, with the equality holding whenever $[\psi_1^{\max}]_v \neq [\psi_2^{\max}]_v$.

If the function $\overline{(\psi_1^{\max} \oplus \psi_2^{\max})_{\check{u}}}$ is in I, then $h = (\psi_1^{\max} \oplus \psi_2^{\max})_{\check{u}}$ satisfies the desired condition.

Suppose that $\overline{(\psi_1^{\max} \oplus \psi_2^{\max})_{\check{u}}} \notin I$. Since $(\psi_1^{\max} \oplus \psi_2^{\max})_{\check{u}} \not\sim \psi_1^{\max} \oplus \psi_2^{\max}$, the point $(u, [\psi_1^{\max} \oplus \psi_2^{\max}]_u)$ is a vertex of $\mathrm{Newt}^u(\psi_1^{\max} \oplus \psi_2^{\max})$. Hence we have

$$[(\psi_1 \oplus \psi_2)^{\max}]_u = [\psi_1^{\max} \oplus \psi_2^{\max}]_u = [\psi_1^{\max}]_u = [\psi_2^{\max}]_u.$$

We denote the roots of ψ_l $(l = 1, 2)$ by $\beta_{i,l} := [\psi_l^{\max}]_{i-1} - [\psi_l^{\max}]_i$ for $i = 1, \ldots, \deg \psi_l$, and the roots of $\psi_1 \oplus \psi_2$ by $\gamma_i := [(\psi_1 \oplus \psi_2)^{\max}]_{i-1} - [(\psi_1 \oplus \psi_2)^{\max}]_i$ for $i = 1, \ldots, \deg(\psi_1 \oplus \psi_2)$.

Claim 1. Either γ_u or γ_{u+1} is a root of φ. Moreover, at least one of such root γ satisfies $\mathrm{mult}(\psi_1 \oplus \psi_2; \gamma) = \mathrm{mult}(\varphi; \gamma)$.

Proof. By Lemma 4.2.5, we have $\overline{(\psi_1^{\max} \oplus \psi_2^{\max})_{\check{u}}} \sim \overline{((\psi_1 \oplus \psi_2)^{\max})_{\check{u}}}$. Since $\varphi \mid (\psi_1 \oplus \psi_2)$ and $\varphi \nmid \overline{(\psi_1^{\max} \oplus \psi_2^{\max})_{\check{u}}} = \overline{((\psi_1 \oplus \psi_2)^{\max})_{\check{u}}}$, there exists a root α of φ such that

$$\mathrm{mult}(\overline{((\psi_1 \oplus \psi_2)^{\max})_{\check{u}}}; \alpha) < \mathrm{mult}(\varphi; \alpha) \leq \mathrm{mult}(\psi_1 \oplus \psi_2; \alpha). \quad (*)$$

On the other hand, the roots of $\psi_1 \oplus \psi_2$ and $\overline{((\psi_1 \oplus \psi_2)^{\max})_{\check{u}}}$ coincide except for γ_u and γ_{u+1}. Then we have $\alpha = \gamma_u$ or $\alpha = \gamma_{u+1}$.

The difference between $\mathrm{mult}(\psi_1 \oplus \psi_2; \alpha)$ and $\mathrm{mult}(\overline{(\psi_1 \oplus \psi_2)^{\max}_{\check{u}}}; \alpha)$ is at most 2. Suppose it is 2, then $\alpha = \gamma_u = \gamma_{u+1}$, which mean that the point $(u, [(\psi_1 \oplus \psi_2)^{\max}]_u)$ is not a vertex of $\mathrm{Newt}((\psi_1 \oplus \psi_2)^{\max})$. This leads to $\overline{\psi_1 \oplus \psi_2} = \overline{((\psi_1 \oplus \psi_2)^{\max})_{\check{u}}}$, which contradicts to the assumption $\overline{((\psi_1 \oplus \psi_2)^{\max})_{\check{u}}} \notin I$. Hence we have $\mathrm{mult}(\psi_1 \oplus \psi_2; \alpha) - \mathrm{mult}(\overline{((\psi_1 \oplus \psi_2)^{\max})_{\check{u}}}; \alpha) = 1$, and then the equality in $(*)$ holds. The proof of Claim 1 is completed. □

This proof also shows that $\gamma_u \neq \gamma_{u+1}$.

Claim 2. Fix an integer $0 \le v \le \deg(\psi_1 \oplus \psi_2)$. Suppose that $[\psi_1^{\max}]_v = [\psi_2^{\max}]_v = [(\psi_1 \oplus \psi_2)^{\max}]_v$, and that one of γ_v and γ_{v+1} (which we write γ) is a root of φ which satisfies $\mathrm{mult}(\psi_1 \oplus \psi_2; \gamma) = \mathrm{mult}(\varphi; \gamma)$. Then we have the followings:

(1) $\psi_1(\gamma) = \psi_2(\gamma)$.

(2) If $\gamma_v \neq \gamma_{v+1}$, then

$$\begin{cases} [\psi_1^{\max}]_{v-j} = [\psi_2^{\max}]_{v-j} = [(\psi_1 \oplus \psi_2)^{\max}]_{v-j} & \text{if } \gamma = \gamma_v, \\ [\psi_1^{\max}]_{v+j} = [\psi_2^{\max}]_{v+j} = [(\psi_1 \oplus \psi_2)^{\max}]_{v+j} & \text{if } \gamma = \gamma_{v+1} \end{cases}$$

for $j = 1, \dots, m$, where $m = \mathrm{mult}(\varphi; \gamma)$.

Proof. (1) First we prove in the case that $\gamma \neq -\infty$. We may assume that $\psi_1(\gamma) \ge \psi_2(\gamma)$. Thus by Lemma 4.2.7, we have $S(\psi_1, \gamma) \subset S(\psi_1 \oplus \psi_2, \gamma)$, and then

$$\mathrm{mult}(\psi_1; \gamma) \le \mathrm{mult}(\psi_1 \oplus \psi_2; \gamma).$$

On the other hand, by $\varphi \,|\, \psi_1$, we also have

$$\mathrm{mult}(\psi_1; \gamma) \ge \mathrm{mult}(\varphi; \gamma) = \mathrm{mult}(\psi_1 \oplus \psi_2; \gamma).$$

Hence $\mathrm{mult}(\psi_1; \gamma) = \mathrm{mult}(\psi_1 \oplus \psi_2; \gamma)$. This means that $S(\psi_1, \gamma) = S(\psi_1 \oplus \psi_2, \gamma)$. By Lemma 4.2.1 (2), we have $v \in S(\psi_1 \oplus \psi_2, \gamma) = S(\psi_1, \gamma)$, and then

$$\psi_1(\gamma) = [\psi_1^{\max}]_v + v \cdot \gamma = [\psi_2^{\max}]_v + v \cdot \gamma \le \psi_2(\gamma) \le \psi_1(\gamma),$$

which shows that $\psi_1(\gamma) = \psi_2(\gamma)$.

If $\gamma = -\infty$, since $-\infty$ is a root of both ψ_1 and ψ_2, we have $\psi_1(-\infty) = \psi_2(-\infty) = -\infty$.

(2) First we assume that $\gamma \neq -\infty$. We only prove in the case that $\gamma = \gamma_{v+1}$. The other case is similar. Since $\psi_1(\gamma) = \psi_2(\gamma)$, by repeating the above argument, we obtain $S(\psi_1, \gamma) = S(\psi_2, \gamma) = S(\psi_1 \oplus \psi_2, \gamma)$. As $\gamma_v \neq \gamma_{v+1}$, these sets coincide with the set $\{v, v+1, \dots u+m\}$. Thus, for $j = 1, \dots, m$, we have $\beta_{v+j,1} = \beta_{v+j,2} = \gamma_{v+j} = \gamma$, hence $[\psi_1^{\max}]_{v+j-1} - [\psi_1^{\max}]_{v+j} = [\psi_2^{\max}]_{v+j-1} - [\psi_2^{\max}]_{v+j} = [(\psi_1 \oplus \psi_2)^{\max}]_{v+j-1} - [(\psi_1 \oplus \psi_2)^{\max}]_{v+j}$. By these equalities and the

assumption $[\psi_1^{\max}]_v = [\psi_2^{\max}]_v = [(\psi_1 \oplus \psi_2)^{\max}]_v$, we have $[\psi_1^{\max}]_{v+j} = [\psi_2^{\max}]_{v+j} = [(\psi_1 \oplus \psi_2)^{\max}]_{v+j}$ for $j = 1, \ldots, m$.

If $\gamma = -\infty$, then $\gamma = \gamma_v$ since $\gamma_v < \gamma_{v+1}$. In this case, we have $[(\psi_1 \oplus \psi_2)^{\max}]_{v-1} = -\infty$, and hence $[(\psi_1 \oplus \psi_2)^{\max}]_w = -\infty$ for any $w < v$. Since $[\psi_1^{\max}]_w, [\psi_2^{\max}]_w \leq [(\psi_1 \oplus \psi_2)^{\max}]_w$, we have $[\psi_1^{\max}]_w = [\psi_2^{\max}]_w = [(\psi_1 \oplus \psi_2)^{\max}]_w = -\infty$. This shows the statement. The proof of Claim 2 is completed. $\square$

Now let us construct a sequence $\{m_k\}_{k=n}^N$ of positive integers for suitable n and N defined below. First we define m_1 and m_0. If γ_{u+1} exists and is a root of φ which satisfies $\mathrm{mult}(\psi_1 \oplus \psi_2; \gamma_{u+1}) = \mathrm{mult}(\varphi; \gamma_{u+1})$, we set $m_1 = \mathrm{mult}(\varphi; \gamma_{u+1})$. Otherwise, we do not define m_1 and set $N = 0$. Similarly, if γ_u exists and is a root of φ which satisfies $\mathrm{mult}(\psi_1 \oplus \psi_2; \gamma_u) = \mathrm{mult}(\varphi; \gamma_u)$, we set $m_0 = \mathrm{mult}(\varphi; \gamma_u)$. Otherwise, we do not define m_0 and set $n = 1$. By Claim 1, at least one of m_0 and m_1 is defined. In the case that m_1 is defined, we define N and $m_2, \ldots, m_N$ by the following iterative construction ($k \geq 1$):

After m_k is defined, if $\gamma := \gamma_{u+m_1+\cdots+m_k+1}$ exists and is a root of φ which satisfies $\mathrm{mult}(\psi_1 \oplus \psi_2; \gamma) = \mathrm{mult}(\varphi; \gamma)$, then we set $m_{k+1} := \mathrm{mult}(\varphi; \gamma)$. Otherwise, we do not define m_{k+1} and set $N = k$.

Similarly, in the case that m_0 is defined, we define n and $m_{-1}, \ldots, m_n$ by the following iterative construction ($k \leq 0$):

After m_k is defined, if $\gamma := \gamma_{u-m_0-\cdots-m_k-1}$ exists and is a root of φ which satisfies $\mathrm{mult}(\psi_1 \oplus \psi_2; \gamma) = \mathrm{mult}(\varphi; \gamma)$, then we set $m_{k-1} := \mathrm{mult}(\varphi; \gamma)$. Otherwise, we do not define m_{k-1} and set $n = k$.

We denote by

$$M_k = \begin{cases} m_1 + m_2 + \cdots + m_k & \text{if } k \geq 1, \\ m_0 + m_{-1} + \cdots + m_k & \text{if } k \leq 0. \end{cases}$$

By using Claim 2(2) repeatedly, we have that $[\psi_1^{\max}]_{u+j} = [\psi_2^{\max}]_{u+j} = [(\psi_1 \oplus \psi_2)^{\max}]_{u+j}$ for $-M_n \leq j \leq M_N$. By the construction of $\{m_k\}_{k=n}^N$, we can factorize φ as

$$\varphi = \varphi_- \odot \overline{(x \oplus \gamma_{u-M_{n+1}})^{m_n} (x \oplus \gamma_{u-M_{n+2}})^{m_{n+1}} \cdots (x \oplus \gamma_u)^{m_0}}$$
$$\overline{(x \oplus \gamma_{u+M_1})^{m_1} (x \oplus \gamma_{u+M_2})^{m_2} \cdots (x \oplus \gamma_{u+M_N})^{m_N}} \odot \varphi_+, \quad (**)$$

where we take the factor φ_- and φ_+ so that any root of φ_- (resp. φ_+) is less (resp. greater) than $\gamma_{u-M_{n+1}}$ (resp. γ_{u+M_N}).

Now we consider the four cases.

<u>*Case 1.*</u> If $u - M_n \neq 0$ and $u + M_N \neq \deg(\psi_1^{\max} \oplus \psi_2^{\max})$, we define three tropical polynomials h_1, h_2 and h_3 as follows;

$$\begin{aligned} h_1 &= (\psi_1^{\max} \oplus \psi_2^{\max})_{\leq u-M_n-1}, \\ h_2 &= (\psi_1^{\max} \oplus \psi_2^{\max})_{u-M_n-1 \leq u+M_N+1}, \\ h_3 &= (\psi_1^{\max} \oplus \psi_2^{\max})_{\geq u+M_N+1}. \end{aligned}$$

Claim 3. $\varphi_- \,|\, \overline{h_1}$ and $\varphi_+ \mid \overline{h_3}$.

Proof. First we have $\varphi \,|\, \psi_1$, then $\varphi_- \,|\, \psi_1$. Any root of φ_- is less than $\gamma_{u-M_{n+1}} = \gamma_{u-M_n+1}$. On the other hand, since $[\psi_1^{\max}]_{u-M_n} = [(\psi_1 \oplus \psi_2)^{\max}]_{u-M_n}$ and $[\psi_1^{\max}]_{u-M_n+1} = [(\psi_1 \oplus \psi_2)^{\max}]_{u-M_n+1}$, we have

$$\begin{aligned} &\beta_{1,u-M_n+1} = [\psi_1^{\max}]_{u-M_n} - [\psi_1^{\max}]_{u-M_n+1} \\ &= [(\psi_1 \oplus \psi_2)^{\max}]_{u-M_n} - [(\psi_1 \oplus \psi_2)^{\max}]_{u-M_n+1} = \gamma_{u-M_n+1} > \gamma_{u-M_n}. \end{aligned}$$

Thus any root of φ_- is one of $\beta_{1,1}, \beta_{1,2}, \ldots, \beta_{1,u-M_n}$, which are the roots of $\overline{(\psi_1^{\max})_{\leq u-M_n}}$. Hence $\varphi_- \mid \overline{(\psi_1^{\max})_{\leq u-M_n}}$. In a similar way, we also have $\varphi_- \mid \overline{(\psi_2^{\max})_{\leq u-M_n}}$, hence $\varphi_- \mid \overline{(\psi_1^{\max} \oplus \psi_2^{\max})_{\leq u-M_n}}$.

To see that $\varphi_- \mid \overline{h_1}$, we show that $(\psi_1^{\max} \oplus \psi_2^{\max})_{\leq u-M_n} \sim a(x \oplus \gamma_{u-M_n})h_1$ for some $a \in \mathbb{R}$. Consider $\mathrm{Newt}^u(\psi_1^{\max} \oplus \psi_2^{\max})$. Since $[(\psi_1 \oplus \psi_2)^{\max}]_{u-M_n} = [\psi_1^{\max} \oplus \psi_2^{\max}]_{u-M_n}$, the point $P = (u - M_n, [\psi_1^{\max} \oplus \psi_2^{\max}]_{u-M_n})$ is in $\mathrm{Newt}^u(\psi_1^{\max} \oplus \psi_2^{\max})$. By $\gamma_{u-M_{n+1}} \neq \gamma_{u-M_n}$, P is a vertex of $\mathrm{Newt}^u(\psi_1^{\max} \oplus \psi_2^{\max})$. Let $Q = (v, [(\psi_1 \oplus \psi_2)^{\max}]_v)$ be the vertex of $\mathrm{Newt}^u(\psi_1^{\max} \oplus \psi_2^{\max})$ which adjacent to P with $v < u - M_n$. The slope of the line segment PQ is equal to $-\gamma_{u-M_n}$. By Lemma 4.1.2, we have $[(\psi_1 \oplus \psi_2)^{\max}]_v = [\psi_1^{\max} \oplus \psi_2^{\max}]_v = \max\{[\psi_1^{\max}]_v, [\psi_2^{\max}]_v\}$. We may assume $[(\psi_1 \oplus \psi_2)^{\max}]_v = [\psi_1^{\max}]_v$. Then the line segment PQ is included in $\mathrm{Newt}^u(\psi_1^{\max})$. By the construction of maximum representations, the point $(u - M_n - 1, [\psi_1^{\max}]_{u-M_n-1})$ is on PQ. It means that $[(\psi_1 \oplus \psi_2)^{\max}]_{u-M_n-1} = [\psi_1^{\max}]_{u-M_n-1} = [\psi_1^{\max} \oplus \psi_2^{\max}]_{u-M_n-1}$, and then $[\psi_1^{\max}]_{u-M_n-1} - [\psi_1^{\max}]_{u-M_n} = \gamma_{u-M_n}$. Hence, by applying Lemma 4.2.10 for $\overline{h_1}$ and $\overline{[\psi_1^{\max}]_{u-M_n} x \oplus [\psi_1^{\max}]_{u-M_n-1}}$, we have $(\psi_1^{\max} \oplus \psi_2^{\max})_{\leq u-M_n} \sim a(x \oplus \gamma_{u-M_n})h_1$ for some $a \in \mathbb{R}$.

By the construction of $\{m_k\}_{k=n}^{N}$, we have

$$\mathrm{mult}(\varphi; \gamma_{u-M_n}) < \mathrm{mult}(\psi_1 \oplus \psi_2; \gamma_{u-M_n}).$$

Therefore, $\varphi_- \mid \overline{h_1}$.

We can show $\varphi_+ \mid \overline{h_3}$ in a similar way. The proof of Claim 3 is completed. □

As for h_2, we can factorize $\overline{h_2}$ as

$$\begin{aligned}\overline{h_2} = \overline{b(x \oplus \gamma_{u-M_n})(x \oplus \gamma_{u-M_{n+1}})^{m_n} \cdots (x \oplus \gamma_u)^{m_0}} \\ \overline{(x \oplus \gamma_{u+M_1})^{m_1} \cdots (x \oplus \gamma_{u+M_N})^{m_N} (x \oplus \gamma_{u+M_N+1})}\end{aligned}$$

for some $b \in \mathbb{R}$. By applying Lemma 4.2.11(a) for $\overline{h_2}$, we obtain a tropical polynomial h_2' such that

(1) $\overline{h_2'}$ is divided by the function

$$\overline{(x \oplus \gamma_{u-M_{n+1}})^{m_n} \cdots (x \oplus \gamma_u)^{m_0} (x \oplus \gamma_{u+M_1})^{m_1} \cdots (x \oplus \gamma_{u+M_N})^{m_N}},$$

(2) $[h_2']_{M_n+1} = -\infty$,

(3) $[h_2']_v \leq [h_2]_v$ for all v,

(4) $[h_2']_0 = [h_2]_0$, $[h_2']_{M_N+M_n+2} = [h_2]_{M_N+M_n+2}$, and

(5) any root γ of h_2' satisfies $\gamma_{u-M_n} \leq \gamma \leq \gamma_{u+M_N+1}$.

Finally, we define a tropical polynomial h as $h = h_1 \oplus x^{u-M_n-1} h_2' \oplus x^{u+M_N+1} h_3$. Let us check that h satisfies the desired condition.

By the property (4), (5) and Lemma 4.2.10, $\overline{h}$ is factorized as $\overline{h} = \overline{c h_1 h_2' h_3}$ for some $c \in \mathbb{R}$. By the factorization $(**)$, Claim 3, and the property (1), we have $\varphi \mid \overline{h}$. Hence $\overline{h} \in I$. The coefficient $[h]_u$ is equal to $[x^{u-M_n-1} h_2']_u = [h_2']_{M_n+1} = -\infty$ by the property (2). If $v \leq u - M_n - 1$, or $v \geq u + M_N + 1$, we have $[h]_v = [\psi_1^{\max}]_v \oplus [\psi_2^{\max}]_v$. If $u - M_n \leq v \leq u + M_N$, since $[\psi_1^{\max}]_v = [\psi_2^{\max}]_v$, it is sufficient to show the inequality $[h]_v \leq [\psi_1^{\max}]_v \oplus [\psi_2^{\max}]_v$, which follows from the property (3).

<u>*Case 2.*</u> If $u - M_n = 0$ and $u + M_N \neq \deg(\psi_1^{\max} \oplus \psi_2^{\max})$, we define two tropical polynomials h_2 and h_3 as follows;

$$h_2 = (\psi_1^{\max} \oplus \psi_2^{\max})_{\leq u+M_N+1},$$
$$h_3 = (\psi_1^{\max} \oplus \psi_2^{\max})_{\geq u+M_N+1}.$$

We can show $\varphi_+ \mid \overline{h_3}$ in a similar way to the previous case. Note that in this case, φ_- is a constant function. We can factorize $\overline{h_2}$ as

$$\overline{h_2} = \overline{b(x \oplus \gamma_{u-M_{n+1}})^{m_n} \cdots (x \oplus \gamma_u)^{m_0}} \\ \overline{(x \oplus \gamma_{u+M_1})^{m_1} \cdots (x \oplus \gamma_{u+M_N})^{m_N} (x \oplus \gamma_{u+M_N+1})}$$

for some $b \in \mathbb{R}$. By applying Lemma 4.2.11(b) for $\overline{h_2}$, we obtain a tropical polynomial h'_2 such that

(1) $\overline{h'_2}$ is divided by the function

$$\overline{(x \oplus \gamma_{u-M_{n+1}})^{m_n} \cdots (x \oplus \gamma_u)^{m_0} (x \oplus \gamma_{u+M_1})^{m_1} \cdots (x \oplus \gamma_{u+M_N})^{m_N}},$$

(2) $[h'_2]_0 = -\infty$,

(3) $[h'_2]_v \leq [h_2]_v$ for all v,

(4) $[h'_2]_{M_N+M_n+1} = [h_2]_{M_N+M_n+1}$, and

(5) any root γ of h'_2 satisfies $\gamma \leq \gamma_{u+M_N+1}$.

Finally, we define a tropical polynomial h as $h = h'_2 \oplus x^{u+M_N+1} h_3$. We can prove that h satisfies the desired condition by the same argument to Case 1.

Case 3. Suppose that $u - M_n \neq 0$ and $u + M_N = \deg(\psi_1^{\max} \oplus \psi_2^{\max})$. This case is similar to Case 2.

Case 4. Suppose that $u - M_n = 0$ and $u + M_N = \deg(\psi_1^{\max} \oplus \psi_2^{\max})$. In this case, both φ_- and φ_+ are constant functions, then $\deg \varphi = M_N + M_n = \deg(\psi_1^{\max} \oplus \psi_2^{\max})$. Since $\varphi \mid \psi_1 \oplus \psi_2$, we have that $\psi_1 \oplus \psi_2$ coincides with φ up to constant factors. Moreover, for $l = 1, 2$, we have $\deg \varphi \leq \deg \psi_l \leq \deg(\psi_1^{\max} \oplus \psi_2^{\max})$ and $\varphi \mid \psi_l$, hence ψ_l also coincides with φ up to constant factors. By the assumption $[\psi_1^{\max}]_u = [\psi_2^{\max}]_u$, we have $\psi_1 = \psi_2$. Now we define $h = -\infty$, which clearly satisfies the desired conditions. □

A tropical polynomial function φ is called *monic* if $[\varphi^{\max}]_{\deg \varphi} = 0$.

Theorem 4.3.2. *Every tropical ideal in $\mathbb{T}[x]/\sim$ is of the form $\varphi \odot (\mathbb{T}[x]/\sim)$ for some $\varphi \in \mathbb{T}[x]/\sim$.*

Proof. Let $I \subset \mathbb{T}[x]/\sim$ be a tropical ideal. If $I = \{-\infty\}$ or $I = \mathbb{T}[x]/\sim$, the statement is clear. Suppose that $I \neq \{-\infty\}$ and $I \neq \mathbb{T}[x]/\sim$. Fix a monic tropical polynomial function $\varphi \in I \setminus \{-\infty\}$

which has the smallest degree in $I \smallsetminus \{-\infty\}$, We show that $I = \varphi \odot (\mathbb{T}[x]/{\sim})$. By the assumption, $d := \deg \varphi$ is positive.

Take an element $\psi \in I \smallsetminus \{-\infty\}$. We prove that $\varphi \mid \psi$ by an induction on $\deg \psi$. It is sufficient to show in the case that ψ is monic. By the definition of φ, we have $\deg \psi \geq d$.

If $\deg \psi = d$, by using the definition of tropical ideal for φ, ψ and a monomial x^d, there exists a tropical polynomial $h \in \mathbb{T}[x]$ such that $\overline{h} \in I$, $[h]_d = -\infty$ and $[h]_v \leq [\varphi^{\max}]_v \oplus [\psi^{\max}]_v$ for all v, with the equality holding whenever $[\varphi^{\max}]_v \neq [\psi^{\max}]_v$. This h satisfies $\deg h < d$, so $h = -\infty$ by the minimality of d. Assume that $\psi \neq \varphi$. Then $[\psi^{\max}]_v \neq [\varphi^{\max}]_v$ for some v. Therefore, we have $[\psi^{\max}]_v \oplus [\varphi^{\max}]_v = [h]_v = -\infty$. This means that $[\psi^{\max}]_v = [\varphi^{\max}]_v = -\infty$, which is a contradiction. Hence we have $\psi = \varphi$, then $\varphi \mid \psi$.

Suppose that $\deg \psi =: n > d$. Fix a tropical polynomial $h \in \mathbb{T}[x]$ which is obtained by using the definition of tropical ideal for $x^{n-d}\varphi$, ψ and a monomial x^n. h satisfies that $\overline{h} \in I$, $[h]_n = -\infty$ and $[h]_v \leq [\varphi^{\max}]_{v-(n-d)} \oplus [\psi^{\max}]_v$ for all v, with the equality holding whenever $[\varphi^{\max}]_{v-(n-d)} \neq [\psi^{\max}]_v$. By the induction hypothesis, $\overline{h}$ can be divided by φ.

We denote the roots of φ by $\alpha_i := [\varphi^{\max}]_{i-1} - [\varphi^{\max}]_i$ for $i = 1, \ldots, d$, and the roots of ψ by $\beta_i := [\psi^{\max}]_{i-1} - [\psi^{\max}]_i$ for $i = 1, \ldots, n$. Since φ is monic, we have

$$[\varphi^{\max}]_k = \sum_{i=k+1}^{d} ([\varphi^{\max}]_{i-1} - [\varphi^{\max}]_i) = \sum_{i=k+1}^{d} \alpha_i$$

for any k. Similarly, we have $[\psi^{\max}]_k = \sum_{i=k+1}^{n} \beta_i$.

We show that $\beta_k \geq \alpha_{k-(n-d)}$ for $k = n, n-1, \ldots n-d+1$. Assume that $\beta_k < \alpha_{k-(n-d)}$ for some k, and let k_0 be the maximum of such k's. We estimate the coefficients of h. If $k_0 \leq k < n$, we have $[\psi^{\max}]_k = \sum_{i=k+1}^{n} \beta_i \geq \sum_{i=k+1}^{n} \alpha_{i-(n-d)} = [\varphi^{\max}]_{k-(n-d)}$, hence

$$[h]_k \leq [\varphi^{\max}]_{k-(n-d)} \oplus [\psi^{\max}]_k = [\psi^{\max}]_k.$$

The other coefficients of h are clearly estimated as

$$\begin{aligned} &[h]_k \leq [\varphi^{\max}]_{k-(n-d)} \oplus [\psi^{\max}]_k && \text{if } n-d \leq k < k_0, \\ &[h]_k = [\psi^{\max}]_k && \text{if } k < n-d. \end{aligned}$$

Now, we consider the following two cases.

Case 1. Suppose that $k_0 < n$ and at least one of the equalities $\beta_n = \alpha_d$, $\beta_{n-1} = \alpha_{d-1}$, $\ldots$, $\beta_{k_0+1} = \alpha_{k_0-(n-d)+1}$ does not hold. This assumption leads the inequality $[\psi^{\max}]_{k_0} > [\varphi^{\max}]_{k_0-(n-d)}$, then we have $[h]_{k_0} = [\psi^{\max}]_{k_0}$. We will apply Proposition 4.2.4 for $[h]_{k_0}$, so let us check that the inequality

$$\min_{j<k_0}\left(\frac{[h]_{k_0} - [h]_j}{k_0 - j}\right) \geq \max_{k>k_0}\left(\frac{[h]_k - [h]_{k_0}}{k - k_0}\right).$$

holds.

If $n - d \leq j < k_0$, then

$$\begin{aligned}
&\frac{[h]_{k_0} - [h]_j}{k_0 - j} \\
&\geq \frac{[\psi^{\max}]_{k_0} - ([\varphi^{\max}]_{j-(n-d)} \oplus [\psi^{\max}]_j)}{k_0 - j} \\
&= \frac{1}{k_0 - j} \cdot \left(\sum_{i=k_0+1}^{n} \beta_i - \max\left(\sum_{i=j+1}^{n} \alpha_{i-(n-d)}, \sum_{i=j+1}^{n} \beta_i\right)\right) \\
&= \frac{1}{k_0 - j} \cdot \min\left(\sum_{i=k_0+1}^{n} \left(\beta_i - \alpha_{i-(n-d)}\right) - \sum_{i=j+1}^{k_0} \alpha_{i-(n-d)}, -\sum_{i=j+1}^{k_0} \beta_i\right) \\
&> \frac{1}{k_0 - j} \cdot (k_0 - j) \cdot (-\alpha_{k_0-(n-d)}) \\
&= -\alpha_{k_0-(n-d)},
\end{aligned}$$

where we use the inequalities $\beta_i \geq \alpha_{i-(n-d)}$ for any $i > k_0$, $\beta_i > \alpha_{i-(n-d)}$ for some $i > k_0$, $\alpha_{k_0-(n-d)} \geq \alpha_{i-(n-d)}$ for any $i \leq k_0$, and $\alpha_{k_0-(n-d)} > \beta_{k_0} \geq \beta_i$ for any $i \leq k_0$. Similarly, if $j < n - d$, then

$$\begin{aligned}
\frac{[h]_{k_0} - [h]_j}{k_0 - j} &= \frac{[\psi^{\max}]_{k_0} - [\psi^{\max}]_j}{k_0 - j} \\
&= \frac{1}{k_0 - j} \cdot \left(\sum_{i=k_0+1}^{n} \beta_i - \sum_{i=j+1}^{n} \beta_i\right) \\
&= \frac{1}{k_0 - j} \cdot \left(-\sum_{i=j+1}^{k_0} \beta_i\right) \\
&> \frac{1}{k_0 - j} \cdot (k_0 - j) \cdot (-\alpha_{k_0-(n-d)}) \\
&= -\alpha_{k_0-(n-d)}.
\end{aligned}$$

Also, for $k > k_0$, we have

$$\begin{aligned}
\frac{[h]_k - [h]_{k_0}}{k - k_0} &\leq \frac{[\psi^{\max}]_k - [\psi^{\max}]_{k_0}}{k - k_0} \\
&= \frac{1}{k - k_0} \cdot \left(\sum_{i=k+1}^{n} \beta_i - \sum_{i=k_0+1}^{n} \beta_i \right) \\
&= \frac{1}{k - k_0} \cdot \left(- \sum_{i=k_0+1}^{k} \beta_i \right) \\
&\leq \frac{1}{k - k_0} \cdot (k - k_0) \cdot (-\beta_{k_0+1}) \\
&= -\beta_{k_0+1}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\min_{j<k_0} \left(\frac{[h]_{k_0} - [h]_j}{k_0 - j} \right) > -\alpha_{k_0-(n-d)} \\
&\qquad \geq -\alpha_{k_0+1-(n-d)} \geq -\beta_{k_0+1} \geq \max_{k>k_0} \left(\frac{[h]_k - [h]_{k_0}}{k - k_0} \right),
\end{aligned}$$

which is the desired inequality. Hence by Proposition 4.2.4 and the above calculation, we have

$$-\min_{j<k_0} \left(\frac{[h]_{k_0} - [h]_j}{k_0 - j} \right) = [h^{\max}]_{k_0-1} - [h^{\max}]_{k_0} < \alpha_{k_0-(n-d)}.$$

On the other hand, $\alpha_{k_0-(n-d)}$ is the $(n - k_0 + 1)$-th largest root of $\underline{\varphi}$ and $[h^{\max}]_{k_0-1} - [h^{\max}]_{k_0}$ is the at most $(n - k_0)$-th largest root of $\overline{h}$. Since $\varphi \mid \overline{h}$, we have $[h^{\max}]_{k_0-1} - [h^{\max}]_{k_0} \geq \alpha_{k_0-(n-d)}$, which is a contradiction.

<u>*Case 2.*</u> Suppose that $k_0 = n$, or $k_0 < n$ and $\beta_n = \alpha_d$, $\beta_{n-1} = \alpha_{d-1}$, …, $\beta_{k_0+1} = \alpha_{k_0-(n-d)+1}$. In this case, we have the inequality $[\psi^{\max}]_{k_0-1} < [\varphi^{\max}]_{k_0-(n-d)-1}$, hence $[h]_{k_0-1} = [\varphi^{\max}]_{k_0-(n-d)-1}$. We will apply Proposition 4.2.4 for $[h]_{k_0-1}$, so let us check that the inequality

$$\min_{j<k_0-1} \left(\frac{[h]_{k_0-1} - [h]_j}{k_0 - 1 - j} \right) \geq \max_{k>k_0-1} \left(\frac{[h]_k - [h]_{k_0-1}}{k - (k_0 - 1)} \right).$$

holds.

If $n-d \le j < k_0 - 1$, then

$$
\begin{aligned}
&\frac{[h]_{k_0-1} - [h]_j}{k_0 - 1 - j} \\
&\ge \frac{[\varphi^{\max}]_{k_0-(n-d)-1} - ([\varphi^{\max}]_{j-(n-d)} \oplus [\psi^{\max}]_j)}{k_0 - 1 - j} \quad (\star) \\
&= \frac{1}{k_0 - 1 - j} \cdot \left(\sum_{i=k_0}^{n} \alpha_{i-(n-d)} - \max\left(\sum_{i=j+1}^{n} \alpha_{i-(n-d)}, \sum_{i=j+1}^{n} \beta_i \right) \right) \\
&= \frac{1}{k_0 - 1 - j} \cdot \min\left(- \sum_{i=j+1}^{k_0-1} \alpha_{i-(n-d)}, \alpha_{k_0-(n-d)} - \beta_{k_0} - \sum_{i=j+1}^{k_0-1} \beta_i \right) \\
&\ge \frac{1}{k_0 - 1 - j} \cdot (k_0 - 1 - j) \cdot (-\alpha_{k_0-(n-d)}) \quad (\star\star) \\
&= -\alpha_{k_0-(n-d)},
\end{aligned}
$$

where we use $\beta_i = \alpha_{i-(n-d)}$ for any $i > k_0$, $\alpha_{k_0-(n-d)} \ge \alpha_{i-(n-d)}$ for any $i \le k_0$, and $\alpha_{k_0-(n-d)} > \beta_{k_0} \ge \beta_i$ for any $i \le k_0$. Note that the inequality in $(\star\star)$ is equality if and only if $\alpha_{k_0-(n-d)} = \alpha_{i-(n-d)}$ for all $i = j+1, \ldots, k_0 - 1$. Moreover, if those equivalent conditions hold, then the inequality in $(\star)$ is also equality. Indeed, in such case, we have

$$
\begin{aligned}
[\varphi^{\max}]_{j-(n-d)} &= (k_0 - 1 - j) \cdot \alpha_{k_0-(n-d)} + \sum_{i=k_0+1}^{n} \alpha_{i-(n-d)} \\
&> (k_0 - 1 - j) \cdot \beta_{k_0} + \sum_{i=k_0+1}^{n} \beta_i \\
&\ge \sum_{i=j+1}^{n} \beta_i = [\psi^{\max}]_j,
\end{aligned}
$$

and then $[h]_j = [\varphi^{\max}]_{j-(n-d)}$.

If $j < n-d$, we have

$$\begin{aligned}
\frac{[h]_{k_0-1}-[h]_j}{k_0-1-j} &= \frac{[\varphi^{\max}]_{k_0-(n-d)-1}-[\psi^{\max}]_j}{k_0-1-j} \\
&= \frac{1}{k_0-1-j}\cdot\left(\sum_{i=k_0}^{n}\alpha_i-\sum_{i=j+1}^{n}\beta_i\right) \\
&= \frac{1}{k_0-1-j}\cdot\left(\alpha_{k_0-(n-d)}-\beta_{k_0}-\sum_{i=j+1}^{k_0-1}\beta_i\right) \\
&> \frac{1}{k_0-1-j}\cdot(k_0-1-j)\cdot(-\alpha_{k_0-(n-d)}) \\
&= -\alpha_{k_0-(n-d)}.
\end{aligned}$$

Also, for $k > k_0$, we have

$$\begin{aligned}
\frac{[h]_k-[h]_{k_0-1}}{k-(k_0-1)} &\le \frac{[\psi^{\max}]_k-[\varphi^{\max}]_{k_0-(n-d)-1}}{k-k_0+1} \\
&= \frac{1}{k-k_0+1}\cdot\left(\sum_{i=k+1}^{n}\beta_i-\sum_{i=k_0}^{n}\alpha_{i-(n-d)}\right) \\
&= \frac{1}{k-k_0+1}\cdot\left(-\sum_{i=k_0}^{k}\alpha_{i-(n-d)}\right) \\
&\le \frac{1}{k-k_0+1}\cdot(k-k_0+1)\cdot(-\alpha_{k_0-(n-d)}) \\
&= -\alpha_{k_0-(n-d)}.
\end{aligned}$$

Therefore we obtain

$$\min_{j<k_0-1}\left(\frac{[h]_{k_0-1}-[h]_j}{k_0-1-j}\right) \ge -\alpha_{k_0-(n-d)-1} \ge \max_{k>k_0-1}\left(\frac{[h]_k-[h]_{k_0-1}}{k-(k_0-1)}\right),$$

which is the desired inequality.

We denote the roots of $\overline{h}$ by $\gamma_i := [h^{\max}]_{i-1}-[h^{\max}]_i$ for $i = 1,\dots,\deg h$. By Proposition 4.2.4 and the above calculation, we have $\gamma_{k_0-1} \le \alpha_{k_0-(n-d)}$. On the other hand, similar to Case 1, we have $\gamma_{k_0-1} \ge \alpha_{k_0-(n-d)}$, hence $\gamma_{k_0-1} = \alpha_{k_0-(n-d)}$. By this equality, there exists j with $n-d \le j < k_0-1$ such that the inequality in ($\star\star$) is in fact equality. Let j_0 be the minimum of such j. Then we have $\alpha_{k_0-(n-d)} = \alpha_{i-(n-d)}$ for $i = j_0+1,\dots,k_0-1$. $\alpha_{j_0+1-(n-d)}$

is the $(n - j_0)$-th largest root of φ and γ_{j_0} is the at most $(n - j_0)$-th root of $\overline{h}$. Since $\varphi \,|\, \overline{h}$, then $\alpha_{k_0-(n-d)} \leq \gamma_{j_0}$. On the other hand, we have $\gamma_{j_0} \leq \gamma_{j_0+1} \leq \cdots \leq \gamma_{k_0-1} = \alpha_{k_0-(n-d)}$, hence $\gamma_{j_0} = \gamma_{j_0+1} = \cdots = \gamma_{k_0-1}$. Thus by Proposition 4.2.4(2), the minimum of $\min_{j<k_0-1}\left(\frac{[h]_{k_0-1}-[h]_j}{k_0-1-j}\right)$ is attained at $j = j_0 - 1$. This means that the inequality in $(\star\star)$ is equality for $j = j_0 - 1$, which contradicts to the minimality of j_0.

Next we show that $\psi^{\max} = x^{n-d}\varphi^{\max} \oplus h$. To do this, we check $[\psi^{\max}]_k = [x^{n-d}\varphi^{\max} \oplus h]_k$ for all k. If $k < n - d$, we have $[h]_k = [\psi^{\max}]$ and $[x^{n-d}\varphi^{\max}]_k = -\infty$, so $[\psi^{\max}]_k = [x^{n-d}\varphi^{\max} \oplus h]_k$. If $k \geq n - d$, we consider the following two cases. If $k = n$ or $k < n$ and $\beta_n = \alpha_d, \beta_{n-1} = \alpha_{d-1}, \ldots$, and $\beta_{k+1} = \alpha_{k-(n-d)+1}$, then we have

$$[x^{n-d}\varphi^{\max}]_k = [\varphi^{\max}]_{k-(n-d)} = \sum_{i=k-(n-d)+1}^{d} \alpha_i = \sum_{i=k+1}^{n} \beta_i = [\psi^{\max}]_k$$

and $[h]_k \leq [\varphi^{\max}]_{k-(n-d)} \oplus [\psi^{\max}]_k = [\varphi^{\max}]_{k-(n-d)}$, hence $[x^{n-d}\varphi^{\max} \oplus h]_k = [\varphi^{\max}]_{k-(n-d)} = [\psi^{\max}]_k$. Otherwise, we have $[x^{n-d}\varphi^{\max}]_k < [\psi^{\max}]_k$ similarly, then $[h]_k = [\psi^{\max}]_k$, hence $[x^{n-d}\varphi^{\max} \oplus h]_k = [h]_k = [\psi^{\max}]_k$.

Finally, we have $\psi = \overline{\psi^{\max}} = \overline{x^{n-d}\varphi^{\max} \oplus h} = \overline{x^{n-d}}\varphi \oplus \overline{h}$. By the induction hypothesis, we have $\varphi \,|\, \overline{h}$, hence $\varphi \,|\, \psi$. $\square$

As applications of the main theorems, we now show a number of results analogous to classical algebraic geometry. Remark that, by Theorem 4.2.1, we can naturally define the *least common multiple* $\mathrm{lcm}_\lambda(\varphi_\lambda)$ and the *greatest common divisor* $\gcd_\lambda(\varphi_\lambda)$ of a set of tropical polynomial functions $\{\varphi_\lambda\}_\lambda$ uniquely up to constant factors. Note that if there is no tropical polynomial function $\psi \neq -\infty$ which satisfies $\varphi_\lambda \,|\, \psi$ for all λ, the least common multiple $\mathrm{lcm}_\lambda(\varphi_\lambda)$ is $-\infty$.

Corollary 4.3.3. *Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a set of tropical ideals in $\mathbb{T}[x]/\sim$. Then $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a tropical ideal in $\mathbb{T}[x]/\sim$.*

Proof. For each λ, there is a tropical polynomial function φ_λ such that $I_\lambda = \varphi_\lambda \odot (\mathbb{T}[x]/\!\sim)$. Then we have

$$\begin{aligned}\bigcap_{\lambda\in\Lambda} I_\lambda &= \bigcap_{\lambda\in\Lambda} \{\psi \in \mathbb{T}[x]/\!\sim \mid \varphi_\lambda \text{ divides } \psi\} \\ &= \{\psi \in \mathbb{T}[x]/\!\sim \mid \mathrm{lcm}_\lambda(\varphi_\lambda) \text{ divides } \psi\}, \\ &= \mathrm{lcm}_\lambda(\varphi_\lambda) \odot (\mathbb{T}[x]/\!\sim).\end{aligned}$$

Thus $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a tropical ideal. $\square$

Let $S \subset \mathbb{T}[x]/\!\sim$ be an arbitrary set of tropical polynomial functions. Then, by Corollary 4.3.3, there is the minimum tropical ideal containing S, which we denote by $\langle S\rangle$. We call $\langle S\rangle$ the tropical ideal *generated by* S. Any tropical ideal in $\mathbb{T}[x]/\!\sim$ can be written as $\langle\varphi\rangle := \langle\{\varphi\}\rangle$ for some $\varphi \in \mathbb{T}[x]/\!\sim$. Obviously, we have $V(\langle\varphi\rangle) = V(\varphi)$.

For tropical ideals I and J, we define the *sum* $I \oplus J$ and the *product* IJ as

$$I \oplus J = \langle\{\varphi \oplus \psi \mid \varphi \in I, \psi \in J\}\rangle\,,$$

$$IJ = \langle\{\varphi\psi \mid \varphi \in I, \psi \in J\}\rangle\,.$$

We may also write $I \oplus J = \langle I \cup J\rangle$.

Proposition 4.3.4. *Let $I = \langle\varphi\rangle$ and $J = \langle\psi\rangle$ be two tropical ideals in $\mathbb{T}[x]/\!\sim$. Then we have $I \oplus J = \langle\gcd(\varphi,\psi)\rangle$ and $IJ = \langle\varphi\psi\rangle$. Hence*

$$V(I \oplus J) = V(I) \cap V(J),$$

$$V(IJ) = V(I) \cup V(J).$$

Proof. We show only $I \oplus J = \langle\gcd(\varphi,\psi)\rangle$. The others are easy. By Theorem 4.3.2, we may write $I \oplus J = \langle\zeta\rangle$ for some $\zeta \in \mathbb{T}[x]/\!\sim$. Since $\varphi, \psi \in \langle\zeta\rangle$, we have $\zeta \mid \varphi$ and $\zeta \mid \psi$, and hence $\zeta \mid \gcd(\varphi,\psi)$. Thus we have $I \oplus J \supset \langle\gcd(\varphi,\psi)\rangle$. On the other hand, since $\gcd(\varphi,\psi) \mid \varphi$ and $\gcd(\varphi,\psi) \mid \psi$, we have $I \cup J \subset \langle\gcd(\varphi,\psi)\rangle$. Thus by the minimality of $I \oplus J$, we have $I \oplus J \subset \langle\gcd(\varphi,\psi)\rangle$. $\square$

Obviously, the tropical ideals in $\mathbb{T}[x]/\!\sim$ satisfy the ascending chain condition:

Corollary 4.3.5 (Ascending chain condition)**.** *There is no infinite ascending chain $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ of tropical ideals in $\mathbb{T}[x]/\!\sim$.*

We conclude this section with *Tropical Nullstellensatz* for our tropical ideals. For an ideal $I \subset \mathbb{T}[x]/\sim$, we define

$$\sqrt{I} := \{\varphi \in \mathbb{T}[x]/\sim \ | \ \varphi^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

For any subset $X \subset \mathbb{T}$, we define

$$I(X) := \bigcap_{P \in X} I_P \subset \mathbb{T}[x]/\sim,$$

where I_P is the tropical ideal defined in Example 4.1.5. By Corollary 4.3.3, $I(X)$ is a tropical ideal.

Corollary 4.3.6 (Tropical Nullstellensatz)**.** *For any tropical ideal $I \subset \mathbb{T}[x]/\sim$, we have*

$$I(V(I)) = \sqrt{I}.$$

Proof. Let $I = \langle \varphi \rangle$. We can factorize φ as $\overline{a(x \oplus \alpha_1)^{m_1} \cdots (x \oplus \alpha_n)^{m_n}}$ with α_i's distinct. Let $\psi = \overline{(x \oplus \alpha_1) \cdots (x \oplus \alpha_n)}$. Then both $I(V(I))$ and $\sqrt{I}$ are equal to $\langle \psi \rangle$. $\square$

4.4 Unexpected examples of tropical ideals in tropical polynomial semirings

First note that, in this section, we treat only tropical polynomials (not tropical polynomial functions). Hence we refer the Maclagan and Rincón's definition of tropical ideals. This section is independent of the other sections.

The purpose of this section is giving examples of the followings:

- two tropical ideals $I, J \subset \mathbb{T}[x]$ such that $I \cap J$ is not a tropical ideal,
- a tropical polynomial $f \in \mathbb{T}[x]$ such that there is no minimum tropical ideal containing f,
- two tropical ideals $I, J \subset \mathbb{T}[x]$ such that $I \oplus J$ is not a tropical ideal, and
- two tropical ideals $I, J \subset \mathbb{T}[x]$ such that IJ is not a tropical ideal.

Though we already give the Maclagan and Rincón's definition of tropical ideals in Definition 1.2.4, we will recall another definition in terms of valuated matroids. For simplicity, we consider only the valuated matroids with the values in $\mathbb{T}$.

Definition 4.4.1. (valuated matroids) A *valuated matroid* is the system $\mathcal{M} = (E, r, p)$ of a finite set E, a nonnegative integer $r \leq |E|$ and a map $p : \binom{E}{r} \to \mathbb{T}$ such that

(1) there exists $B \in \binom{E}{r}$ such that $p(B) \neq -\infty$,

(2) for $A, B \in \binom{E}{r}$ and $a \in A - B$, there exists $v \in B - A$ such that

$$p(A) + p(B) \leq p(A \cup \{b\} - \{a\}) + p(B \cup \{a\} - \{b\}).$$

Let $\mathcal{M} = (E, r, p)$ be a valuated matroid. For $B \in \binom{E}{r}$ with $p(B) \neq -\infty$ and $e \in E \setminus B$, the *fundamental circuit* $H(B, e)$ of $\mathcal{M}$ is the element of $\mathbb{T}^E$ whose coordinate are given by

$$H(B, e)_{e'} := p(B \cup \{e\} - \{e'\}) - p(B) \text{ for any } e' \in E,$$

where $p(B') = -\infty$ if $|B'| > r$. The set of fundamental circuits of $\mathcal{M}$ is denoted by $\mathcal{H}(\mathcal{M})$.

A *vector* of a valuated matroid $\mathcal{M}$ on the ground set E is an element of $\mathbb{T}^E$ of the form

$$\lambda_1 \odot H_1 \oplus \cdots \oplus \lambda_n \odot H_n$$

for some $\lambda_i \in \mathbb{T}$ and fundamental circuits $H_i \in \mathcal{H}(\mathcal{M})$, where we define $\lambda \odot \mathbf{v} = (\lambda + v_1, \ldots, \lambda + v_m)$ for $\lambda \in \mathbb{T}$ and $\mathbf{v} = (v_1, \ldots, v_m) \in \mathbb{T}^m$. The set of vectors of $\mathcal{M}$ is denoted by $\mathcal{V}(\mathcal{M})$.

For simplicity, we consider only tropical polynomials of one variable. We denote by $\mathrm{Mon}_{\leq d}$ the set of monomials of degree at most d in $\mathbb{T}[x]$. We identify each element of $\mathbb{T}^{\mathrm{Mon}_{\leq d}}$ with a tropical polynomial in $\mathbb{T}[x]$ of degree at most d.

Definition 4.4.2. (tropical ideals) An ideal $I \subset \mathbb{T}[x]$ is a *tropical ideal* if for each degree $d \geq 0$ the set $I_{\leq d} := \{f \in I \mid \deg f \leq d\}$ is the set of vectors of a valuated matroid on $\mathrm{Mon}_{\leq d}$.

Definition 4.4.3. (compatible sequences) Let $S = \{\mathcal{M}_d\}_{d \geq 0} = \{(\mathrm{Mon}_{\leq d}, r_d, p_d)\}_{d \geq 0}$ be a sequence of valuated matroids. The sequence $\mathcal{S}$ is called a *compatible sequence* if $\mathcal{V}(\mathcal{M}_{d+1}) \cap \mathbb{T}^{\mathrm{Mon}_{\leq d}} =$

$\mathcal{V}(\mathcal{M}_d)$ for each d as subsets of $\mathbb{T}[x]$ and $\bigcup_{d\geq 0}\mathcal{V}(\mathcal{M}_d)$ is an ideal of $\mathbb{T}[x]$. In other words, $\mathcal{S}$ is compatible if, for each $d \geq 0$, $\mathcal{V}(\mathcal{M}_{d+1}) \cap \mathbb{T}^{\mathrm{Mon}_{\leq d}} = \mathcal{V}(\mathcal{M}_d)$ and $x \odot \mathcal{V}(\mathcal{M}_d) := \{x \odot H \mid H \in \mathcal{V}(\mathcal{M}_d)\} \subset \mathcal{V}(\mathcal{M}_{d+1})$.

Remark 4.4.4. Since the Maclagan and Rincón's definition of compatible sequence in [14] is the homogeneous version, we rewrite the definition to fit for our settings.

In order to construct the promised examples, we give two compatible sequences.

Firstly, we define the sequence of valuated matroids $\{\mathcal{M}_d\}_{d\geq 0} = \{(\mathrm{Mon}_{\leq d}, r_d, p_d)\}_{d\geq 0}$ as $r_d = \min\{d+1, 2\}$ and $p_d(B) = 0$ for any $B \in \binom{\mathrm{Mon}_{\leq d}}{r_d}$. Obviously, each $\mathcal{M}_d$ is a valuated matroid. The set of fundamental circuits $\mathcal{H}(\mathcal{M}_d)$ of $\mathcal{M}_d$ is

$$\left\{ f_C \;\middle|\; C \in \binom{\mathrm{Mon}_{\leq d}}{3} \right\},$$

where $f_C := \bigoplus_{x^i \in C} x^i \in \mathbb{T}[x]$. Note that $\mathcal{H}(\mathcal{M}_0) = \mathcal{H}(\mathcal{M}_1) = \varnothing$. We can also write $\mathcal{H}(\mathcal{M}_d) = \{x^i \oplus x^j \oplus x^k \mid 0 \leq i < j < k \leq d\}$ if $d \geq 2$. Thus the set of vectors $\mathcal{V}(\mathcal{M}_d)$ of $\mathcal{M}_d$ is

$$\left\{ f \in \mathbb{T}[x] \;\middle|\; \begin{array}{c} \deg f \leq d, \text{and the maximum of the} \\ \text{coefficients of } f \text{ is attained at least three times} \end{array} \right\}.$$

Note that $\mathcal{V}(\mathcal{M}_0) = \mathcal{V}(\mathcal{M}_1) = \{-\infty\}$. We can easily check that $\{\mathcal{M}_d\}_{d\geq 0}$ is a compatible sequence.

Secondly, we define the sequence of valuated matroids $\{\mathcal{N}_d\}_{d\geq 0} = \{(\mathrm{Mon}_{\leq d}, r'_d, p'_d)\}_{d\geq 0}$ as

- $r'_0 = 1$ and $p'_0(\{x^0\}) = 0$,
- for $d \geq 1$, $r'_d = 2$ and $p'_d(\{x^i, x^j\}) = \begin{cases} 0 & \text{if } i \not\equiv j \mod 3, \\ -\infty & \text{otherwise.} \end{cases}$

We show that each $\mathcal{N}_d$ is a valuated matroid. For $d = 0$, it is clear. Suppose that $d \geq 1$. Let $A = \{x^{i_1}, x^{j_1}\}, B = \{x^{i_2}, x^{j_2}\}$ be distinct elements of $\binom{\mathrm{Mon}_{\leq d}}{2}$. By the pigeonhole principle, at least two of i_1, j_1, i_2 and j_2 are congruent modulo 3. If $i_1 \equiv j_1 \mod 3$, then we have

$$p'_d(A) + p'_d(B) = -\infty,$$

which implies that these A and B satisfy the condition (2) in the definition of valuated matroids. If $i_1 \not\equiv j_1 \mod 3$, we may assume that $i_1 \equiv i_2 \mod 3$. Thus we have

$$p'_d(A \cup \{x^{i_2}\} - \{x^{i_1}\}) + p'_d(B \cup \{x^{i_1}\} - \{x^{i_2}\}) = p'_d(A) + p'_d(B),$$
$$p'_d(A \cup \{x^{j_2}\} - \{x^{j_1}\}) + p'_d(B \cup \{x^{j_1}\} - \{x^{j_2}\}) = p'_d(B) + p'_d(A),$$

which implies that these A, B also satisfy the condition (2) in the definition of valuated matroids.

The set $\mathcal{H}(\mathcal{N}_d)$ of fundamental circuits of $\mathcal{N}_d$ is

$$\{x^i \oplus x^j \mid 0 \leq i < j \leq d \text{ and } i \equiv j \mod 3\}$$
$$\cup\{x^i \oplus x^j \oplus x^k \mid 0 \leq i < j < k \leq d \text{ and } i, j \text{ and } k \text{ are distinct modulo } 3\}.$$

We can easily check that $\{\mathcal{N}_d\}_{d \geq 0}$ is a compatible sequence.

Now, we give the promised examples.

Example 4.4.5. Let $I = \bigcup_{d \geq 0} \mathcal{V}(\mathcal{M}_d)$ and $J = \bigcup_{d \geq 0} \mathcal{V}(\mathcal{N}_d)$ be two tropical ideals in $\mathbb{T}[x]$. First we have $x^2 \oplus x \oplus 0 \in I \cap J$. We show that any subset $K \subset \mathbb{T}[x]$ such that $x^2 \oplus x \oplus 0 \in K \subset I \cap J$ is not a tropical ideal. This leads the following two facts;

- $I \cap J$ is not a tropical ideal,
- there is no minimum tropical ideal containing f.

Suppose that K is a tropical ideal. Since $x^2 \oplus x \oplus 0 \in K$, we have $x^3 \oplus x^2 \oplus x \in K$. By applying Definition 1.2.4 for two tropical polynomials $x^2 \oplus x \oplus 0, x^3 \oplus x^2 \oplus x$ and a monomial x^2, we obtain an element of K of the form $x^3 \oplus cx \oplus 0$ for $c \leq 0$. Since $h \in K \subset I$, we can write

$$x^3 \oplus cx \oplus 0 = a_1(x^2 \oplus x \oplus 0) \oplus a_2(x^3 \oplus x \oplus 0) \oplus a_3(x^3 \oplus x^2 \oplus 0) \oplus a_4(x^3 \oplus x^2 \oplus x)$$

for some $a_1, a_2, a_3, a_4 \in \mathbb{T}$ by the construction of I. Comparing the coefficients of x^2, we have $a_1 = a_3 = a_4 = -\infty$, and then $a_2 = c = 0$. Thus we have $x^3 \oplus x \oplus 0 \in K \subset J$. Hence we can write

$$x^3 \oplus x \oplus 0 = b_1(x^3 \oplus 0) \oplus b_2(x^2 \oplus x \oplus 0) \oplus b_3(x^3 \oplus x^2 \oplus x)$$

for some $b_1, b_2, b_3 \in \mathbb{T}$. Comparing the coefficients of x^2, we have $b_2 = b_3 = -\infty$. Then there is no $b_1 \in \mathbb{T}$ which makes the equality holds. It is a contradiction.

Example 4.4.6. We give two tropical ideals $I, J \subset \mathbb{T}[x]$ such that $I \oplus J$ is not a tropical ideal. Note that $I \oplus J$ is defined as $I \oplus J = \{f \oplus g \mid f \in I, g \in J\}$.

Let $I = \{f \in \mathbb{T}[x] \mid 0 \in V(f)\}$ and $J = \{f \in \mathbb{T}[x] \mid 1 \in V(f)\}$, which are examples of tropical ideals in [14, Section 2]. Suppose that $I \oplus J$ is a tropical ideal. Neither I nor J include any tropical polynomial of degree 0. Since there is no cancellation in $\mathbb{T}$, $I \oplus J$ also does not have any tropical polynomial of degree 0. On the other hand, since $x \oplus 0 \in I$ and $x \oplus 1 \in J$, we have $x \oplus 0, x \oplus 1 \in I \oplus J$. Thus by using Definition 1.2.4, we have $1 \in I \oplus J$, which is a contradiction.

Example 4.4.7. We give two tropical ideals $I, J \subset \mathbb{T}[x]$ such that IJ is not a tropical ideal. Note that $I \oplus J$ is defined as $IJ = \{f_1 g_1 \oplus \cdots \oplus f_n g_n \mid n \geq 0, f_i \in I, g_i \in J\}$.

Let $I = \{f \in \mathbb{T}[x] \mid 0 \in V(f)\}$ and $J = \{f \in \mathbb{T}[x] \mid 1 \in V(f)\}$ be the tropical ideals same to the previous example. Suppose that IJ is a tropical ideal. We denote by $I_d := \{f \in I \mid \deg f = d\}$ and $J_d := \{f \in J \mid \deg f = d\}$. Explicitly, I_1, I_2, J_1 and J_2 are

$$I_1 = \{ax \oplus a \mid a \in \mathbb{R}\},$$

$$I_2 = \left\{ ax^2 \oplus bx \oplus c \;\middle|\; \begin{array}{c} a, b, c \in \mathbb{T}, a \neq -\infty, \text{ and the maximum of} \\ \{a, b, c\} \text{ is attained at least twice} \end{array} \right\},$$

$$J_1 = \{ax \oplus (a+1) \mid a \in \mathbb{R}\} \text{ and}$$

$$J_2 = \left\{ ax^2 \oplus bx \oplus c \;\middle|\; \begin{array}{c} a, b, c \in \mathbb{T}, c \neq -\infty, \text{ and the maximum of} \\ \{a+2, b+1, c\} \text{ is attained at least twice} \end{array} \right\}.$$

Since $x \oplus 0 \in I$ and $x \oplus 1, x^2 \oplus 2 \in J$, we have $(x \oplus 0)(x \oplus 1) = x^2 \oplus 1x \oplus 1 \in IJ$ and $(x \oplus 0)(x^2 \oplus 2) = x^3 \oplus x^2 \oplus 2x \oplus 2 \in IJ$. By applying Definition 1.2.4 for two tropical polynomials $x^2 \oplus 1x \oplus 1, x^3 \oplus x^2 \oplus 2x \oplus 2$ and a monomial x^2, we have $x^3 \oplus 2x \oplus 2 \in IJ$. Since there is no cancellation in $\mathbb{T}$, we can write

$$x^3 \oplus 2x \oplus 2 = \bigoplus_{i=1}^{m} f_{1i} g_{2i} \oplus \bigoplus_{i=1}^{n} f_{2i} g_{1i}, \quad (*)$$

where $f_{li} \in I_l, g_{li} \in J_l$ $(l = 1, 2)$.

On the other hand, for any $f \in I_1$ and $g \in J_2$, we have $[f]_0 \neq -\infty$ and $[g]_2 \neq -\infty$, hence $[fg]_2 \neq -\infty$. Similarly, for any $f \in I_2$ and $g \in J_1$, we have $[fg]_2 \neq -\infty$. Thus the coefficient of x^2 of the right hand side of $(*)$ cannot be $-\infty$. It is a contradiction.

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