Complex Interpolation of Generalized Morrey Spaces, Local Block Spaces, and Grand Lebesgue Spaces

Denny Ivanal Hakim

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論 文 名

一般化されたモレー空間，局所ブロック空間およびグランドルベーグ空間の複素補間　（英文）

著　者　デニー・イバナル・ハキム

審査担当者

主　査

委　員

委　員

委　員

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AUTHOR : Denny Ivanal Hakim

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Examiner:

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OF SCIENCE AND ENGINEERING
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Dean

Date

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Denny Ivanal Hakim

DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES,
TOKYO METROPOLITAN UNIVERSITY

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Abstract

The main aim of this thesis is to present a theory about the complex interpolation of some function spaces related to Morrey spaces. This thesis consists of six chapters. In Chapter 1, we recall the definition of Morrey spaces and generalized Morrey spaces and we also mention inclusion between Morrey spaces and the results on the boundedness of some classical integral operators in Morrey spaces. In addition, we recall a known result and a counterexample on interpolation of linear operators on Morrey spaces in this chapter. In Chapter 2, we recall the complex interpolation method and some useful lemmas on this method. We present our results about the complex interpolation of generalized Morrey spaces in Chapter 3. We obtain the description of the first and second complex interpolation of generalized Morrey spaces. We show that the first complex interpolation of generalized Morrey spaces can be described as a proper closed subspace of generalized Morrey spaces. Meanwhile, the second complex interpolation of generalized Morrey spaces yields generalized Morrey spaces. We also give a description of complex interpolation between generalized Morrey spaces and L^∞. Our results in this chapter can be viewed as an extension of the results in [12, 35, 36]. In Chapter 4, we discuss the complex interpolation of some closed subspaces of Morrey spaces. These subspaces arise naturally in some papers about Morrey spaces, for instance [51, 55]. We show that the first and second complex interpolation of these subspaces yield different spaces. In Chapter 5, we discuss local Morrey type spaces, local block spaces, and the first complex interpolation of local block spaces. We show that local block spaces behave well under the first complex interpolation method. Lastly, we discuss the first and second complex interpolation of grand Lebesgue spaces in Chapter 6.

Notation

We use the following notation:

1. We denote by $B(x,r)$ the *ball centered at* x *of radius* r. Namely, we write

$$B(x,r) := \{y \in \mathbb{R}^n \,:\, |x-y| < r\}$$

when $x \in \mathbb{R}^n$ and $r > 0$. Given a ball B, we denote by $c(B)$ its *center* and by $r(B)$ its *radius*. We write $B(r)$ instead of $B(o,r)$, where $o := (0,0,\dots,0)$.

2. Given a ball B and $k > 0$, we denote by $k\,B$ the *ball concentric to* B *with radius* $k\,r(B)$.

3. Let E be a measurable set. Then we denote its *characteristic function* by χ_E and $|E|$ denotes the *volume of* E.

4. The set $\mathcal{I}(\mathbb{R})$ denotes the set of all open intervals in $\mathbb{R}$.

5. The constants C and c denote positive constants that may change from one occurrence to another. The two constant c being different, the inequality $0 < 2c < c$ is by no means a contradiction. When we add a subscript p and α, for example, this means that the constant c depends upon the parameter. It can happen that the constants with subscript differ according to the above rule. In particular, we prefer to use c_n, various constants that depend on n, when we do not want to specify its precise value.

6. Let $A, B \ge 0$. Then $A \lesssim B$ and $B \gtrsim A$ mean that there exists a constant $C > 0$ such that $A \le CB$, where C depends only on the parameters of importance. The symbol $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ happen simultaneously. While $A \simeq B$ means that there exists a constant $C > 0$ such that $A = CB$.

7. We define

$$\mathbb{N} := \{1,2,\dots\}, \quad \mathbb{Z} := \{0,\pm1,\pm2,\dots\}, \quad \mathbb{N}_0 := \{0,1,\dots\}. \tag{1}$$

8. Let X be a Banach space. We denote its norm by $\|\cdot\|_X$.

9. Let Ω be an open set in $\mathbb{R}^n$. Then $C_c^\infty(\Omega)$ denotes the set of smooth function with compact support in Ω.

10. The space C denotes the set of all continuous functions on $\mathbb{R}^n$.

11. The space $\mathrm{BC}(\mathbb{R}^n)$ denotes the set of all bounded continuous functions on $\mathbb{R}^n$.

12. Occasionally we identify the value of functions with functions. For example $\sin x$ denotes the function on $\mathbb{R}$ defined by $x \mapsto \sin x$.

13. Given a Banach space X, we denote by X^* its dual space.

14. When two normed spaces X and Y are isomorphic, we write $X \approx Y$.

15. When A and B are sets, $A \subset B$ stands for the inclusion of sets. If, in addition, both A and B are topological spaces, and if the natural embedding mapping $A \to B$ is continuous, we write $A \hookrightarrow B$ in the sense of continuous embedding.

Contents

Chapter 1

Introduction

We recall here the definition of Morrey spaces and generalized Morrey spaces. We also give a summary of several previous results about interpolation of linear operators on Morrey spaces.

1.1 Morrey spaces and generalized Morrey spaces

Morrey spaces were first introduced by C.B. Morrey in [37] based on the study of the solution of certain elliptic partial differential equations. For $1 \leq q \leq p < \infty$, the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ such that

$$\sup_{x \in \mathbb{R}^n, r>0} |B(x,r)|^{\frac{q}{p}-1} \int_{B(x,r)} |f(y)|^q \, dy < \infty. \tag{1.1}$$

The norm on $\mathcal{M}^p_q(\mathbb{R}^n)$ is defined by

$$\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(B(x,r))}. \tag{1.2}$$

For simplicity, we shall write $\mathcal{M}^p_q$ instead of $\mathcal{M}^p_q(\mathbb{R}^n)$. Remark that there is also another notation for Morrey spaces, namely $L^{p,\lambda}$, where $1 \leq p < \infty$, $0 \leq \lambda < n$, and $\| \cdot \|_{L^{p,\lambda}}$ is defined by

$$\|f\|_{L^{p,\lambda}} := \sup_{x \in \mathbb{R}^n, r>0} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

We shall use the notation $\mathcal{M}^p_q$ throughout this thesis. Note that if $p = q$, then $\mathcal{M}^p_q = L^p$, so Morrey spaces can be seen as an extension of Lebesgue spaces. Moreover, when $1 \le q \le p < \infty$, we have

$$L^p \hookrightarrow \mathcal{M}^p_q. \tag{1.3}$$

Inclusion (1.3) is a consequence of the Hölder inequality. Furthermore, if $1 \le q < p < \infty$, then inclusion (1.3) is proper, since the function $f(x) := |x|^{-n/p}$ belongs to $\mathcal{M}^p_q \setminus L^p$. Inclusion (1.3) can be viewed as a special case of

$$\mathcal{M}^p_{q_2} \hookrightarrow \mathcal{M}^p_{q_1} \tag{1.4}$$

where $1 \le q_1 \le q_2 \le p < \infty$. We refer the reader to [45, 47] for (1.4). A generalization of (1.4) to weighted Morrey spaces can be seen in [32].

Note that the function $r \in (0,\infty) \mapsto r^{n/p} \in (0,\infty)$ in the definition of the $\mathcal{M}^p_q$-norm can be generalized to a suitable function $\varphi : (0,\infty) \to (0,\infty)$ to define the generalized Morrey space $\mathcal{M}^\varphi_q = \mathcal{M}^\varphi_q(\mathbb{R}^n)$ whose norm is given by

$$\|f\|_{\mathcal{M}^\varphi_q} := \sup_{x\in\mathbb{R}^n, r>0} \frac{\varphi(r)}{|B(x,r)|^{1/q}} \|f\|_{L^q(B(x,r))} < \infty. \tag{1.5}$$

The space $\mathcal{M}^\varphi_q$ was introduced by Nakai in [38]. Here, we may assume that $\varphi \in \mathcal{G}_q$, that is, φ is increasing and $r \in (0,\infty) \mapsto r^{-n/q}\varphi(r) \in (0,\infty)$ is decreasing (see [39, p. 446]). Remark that, when $\varphi(r) = r^{\frac{n}{p}}$ and $\psi(r) = 1$, we have $\mathcal{M}^\varphi_q = \mathcal{M}^p_q$ and $\mathcal{M}^\psi_q = L^\infty$ with identical norms (see [42, Proposition 3.3]), respectively. Recently, there are also various extension of generalized Morrey spaces to Orlicz-Morrey type spaces (see [41, 49]). Inclusion between generalized Morrey spaces, Orlicz-Morrey spaces and related spaces can be seen in [22, 42].

Let us now recall the boundedness results of some classical integral operators such as the Hardy-Littlewood maximal operator and the fractional integral operators in Morrey spaces and their generalization. The boundedness of the fractional integral operators in Morrey spaces were proved in [1, 44]. Meanwhile, F. Chiarenza and M. Frasca proved the boundedness of the Hardy-Littlewood maximal operator in Morrey spaces in [11]. Furthermore, they also reproved the result in [1] by using a Hedberg type inequality. In [38], E. Nakai proved the boundedness of the Hardy-Littlewood maximal operator and fractional integral operators on generalized Morrey spaces. He also introduced generalized fractional integral operators and proved the boundedness of these operators in [40]. For further results about the generalized fractional integral opertors on generalized Morrey spaces, we refer the reader to [14, 20, 21, 25, 48] and reference therein. A necessary and sufficient condition for the boundedness of the Hardy-Littlewood maximal operator on Orlicz-Morrey spaces is given in [42]. The boundedness result of the fractional maximal operators on generalized Morrey spaces can be seen in [28].

1.2 Interpolation of linear operators on Morrey spaces

First let us recall the Riesz-Thorin interpolation theorem.

Theorem 1.2.1. [3, p.2] *Let $\theta \in (0,1)$ and $1 \le p_0, p_1, r_0, r_1 \le \infty$. Let p and r be defined by*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{r} := \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

Suppose that T is a bounded linear operator from L^{p_0} to L^{r_0} and L^{p_1} to L^{r_1}. Then T is bounded from L^p to L^r.

One may inquire whether Lebesgue spaces in Theorem 1.2.1 can be replaced by Morrey spaces. When the domain of the operator T is the Lebesgue space $L^{p_0} + L^{p_1}$, an extension of Theorem 1.2.1 to Morrey spaces was obtained by G. Stampacchia [53].

Theorem 1.2.2. [53] *Let $\theta \in (0,1)$, $1 \le p_0, p_1 < \infty$, $1 \le s_0 \le r_0 < \infty$, and $1 \le s_1 \le r_1 < \infty$. Define p, r, s by*

$$\left(\frac{1}{p}, \frac{1}{r}, \frac{1}{s}\right) := (1-\theta)\left(\frac{1}{p_0}, \frac{1}{r_0}, \frac{1}{s_0}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{r_1}, \frac{1}{s_1}\right).$$

If T is a bounded linear operator from L^{p_0} to $\mathcal{M}^{r_0}_{s_0}$ and from L^{p_1} to $\mathcal{M}^{r_1}_{s_1}$, then T is bounded from L^p to $\mathcal{M}^r_s$.

Unfortunately, if the domain of the operator T is Morrey spaces, there are some counterexamples given by A. Ruiz and L. Vega [46] for the case $n > 1$ and by O. Blasco et al. in [5] for the case $n = 1$. Let us recall the result in [5].

Theorem 1.2.3. [5] *Let $n = 1$, $\theta \in (0,1)$, and $1 < q_1 < q_0$. Define*

$$\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \; r_0 := \frac{2}{\min(\frac{1}{q_0} + \frac{2}{q_1}.2)}, \; r_1 := q_1, \; \text{and} \; \frac{1}{r} := \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

Then there exists a bounded linear operator T from $L^{q_0} = \mathcal{M}^{q_0}_{q_0}$ to L^{r_0} and from $\mathcal{M}^{q_0}_{q_1}$ to L^{r_1} such that T is not bounded from $\mathcal{M}^{q_0}_{q}$ to L^r.

Proof. According to the definition of q, we know that $q_1 < q < q_0$. Hence, we may choose

$$\beta > \frac{\frac{1}{q_0}}{\frac{1}{q} - \frac{1}{q_0}}. \tag{1.6}$$

Let $N_0 \in \mathbb{N}$ be such that

$$\frac{\beta+1}{\log 2} < \frac{N+1}{\log N}, \tag{1.7}$$

for every $N \in \mathbb{N} \cap [N_0, \infty)$. Let $N \in \mathbb{N} \cap [N_0, \infty)$ be fixed. We define

$$I_j^N := [N! + jN^\beta, N! + jN^\beta + 1]$$

where $j = 0, 1, \ldots, N-1$ and set $E_N := \cup_{j=0}^{N-1} I_j^N$. Observe that the choice of β allows $\{E_N\}_{N=1}^\infty$ to be disjoint. Note that $r_0 < r_1$, so $r_0 < r < r_1$. Therefore, we may choose

$$\gamma \in \left(\frac{2}{r_1}, \frac{2}{r}\right). \tag{1.8}$$

With this choice of γ, we construct an operator T by the formula

$$Tf(x) := \sum_{N=N_0}^{\infty} N^{-\gamma} \chi_{E_N}(x) f(x)$$

for every measurable function f. By the Hölder inequality, for every $f \in L^{q_0}$ we have

$$\begin{aligned} \|Tf\|_{L^{r_0}} &\leq \left(\sum_{N=N_0}^{\infty} N^{-\gamma r_0} |E_N|^{1-\frac{r_0}{q_0}} \left(\int_{E_N} |f(x)|^{q_0}\right)^{\frac{r_0}{q_0}}\right)^{\frac{1}{r_0}} \\ &\leq \left(\sum_{N=N_0}^{\infty} N^{-\gamma r_0 + 1 - \frac{r_0}{q_0}}\right)^{\frac{1}{r_0}} \|f\|_{L^{q_0}}. \end{aligned}$$

It follows from (1.8) that

$$-\gamma r_0 + 1 - \frac{r_0}{q_0} < -\frac{2r_0}{r_1} + 1 - \frac{r_0}{q_0} = 1 - r_0\left(\frac{2}{q_1} + \frac{1}{q_0}\right) \leq -1.$$

Consequently,

$$\|Tf\|_{L^{r_0}} \leq C_0 \|f\|_{L^{q_0}}$$

for some constant $C_0 > 0$. We now show that

$$\|Tf\|_{L^{r_1}} \leq C_1 \|f\|_{\mathcal{M}_{q_1}^{q_0}} \tag{1.9}$$

for some $C_1 > 0$ and for every $f \in \mathcal{M}_{q_1}^{q_0}$. Since $\{E_N\}_{N=N_0}^\infty$ is a collection of disjoint sets and $q_1 = r_1$, we get

$$\|Tf\|_{L^{r_1}} \leq \left(\sum_{N=N_0}^{\infty} N^{-\gamma r_1} \sum_{j=0}^{N-1} \int_{I_j^N} |f(x)|^{r_1}\, dx\right)^{\frac{1}{r_1}}. \tag{1.10}$$

Combining (1.10) and

$$\int_{I_j^N} |f(x)|^{q_1}\, dx \leq |I_j^N|^{1-\frac{q_1}{q_0}} \|f\|_{\mathcal{M}_{q_1}^{q_0}}^{q_1} = \|f\|_{\mathcal{M}_{q_1}^{q_0}}^{q_1},$$

for each $j = 0, 1, \ldots, N-1$, we get

$$\|Tf\|_{L^{r_1}} \leq \left(\sum_{N=N_0}^{\infty} N^{1-\gamma r_1}\right)^{\frac{1}{r_1}} \|f\|_{\mathcal{M}_{q_1}^{q_0}}.$$

According to (1.8), we have

$$1 - \gamma r_1 < 1 - \frac{2}{r_1} r_1 = -1,$$

so

$$\sum_{N=N_0}^{\infty} N^{1-\gamma r_1} < \infty.$$

This implies (1.9). The proof of the unboundedness of T from $\mathcal{M}_q^{q_0}$ to L^r goes as follows. Define

$$f_0 := \sum_{N=N_0}^{\infty} \chi_{E_N}.$$

Note that, for every $N \in \mathbb{N}$, we have

$$\begin{aligned}
\|\chi_{E_N}\|_{\mathcal{M}_1^{\frac{q_0}{q}}} &= \sup_{I \subseteq \mathbb{R}} |I|^{\frac{q}{q_0}} \frac{|I \cap E_N|}{|I|} \\
&\lesssim \left((N-1)N^\beta + 1\right)^{\frac{q}{q_0}} \frac{N}{(N-1)N^\beta + 1} \\
&\lesssim \left(N^{\beta+1}\right)^{\frac{q}{q_0}} \frac{1}{N^\beta} = N^{\frac{q(\beta+1)}{q_0} - \beta}.
\end{aligned}$$

Let $J_N := (N_0!, N! + (N-1)N^\beta + 1)$ for every $N \in \mathbb{N} \cap [N_0, \infty)$. Since

$$\|f_0\|_{\mathcal{M}_q^{q_0}}^q = \|f_0^q\|_{\mathcal{M}_1^{\frac{q_0}{q}}} = \|f_0\|_{\mathcal{M}_1^{\frac{q_0}{q}}},$$

we have

$$\begin{aligned}
\|f_0\|_{\mathcal{M}_q^{q_0}}^q &= \sup_{I \in \mathcal{I}(\mathbb{R})} |I|^{\frac{q}{q_0} - 1} \int_I \sum_{N=N_0}^{\infty} \chi_{E_N}(y)\, dy \\
&\lesssim \max_{M \in \mathbb{N}} \left\{ |J_M|^{\frac{q}{q_0} - 1} \int_{J_M} \sum_{N=N_0}^{M} \chi_{E_N}(y)\, dy, \|\chi_{E_M}\|_{\mathcal{M}_1^{\frac{q_0}{q}}} \right\} \\
&\lesssim \max_{M \in \mathbb{N}} \left\{ \frac{M^2}{(M! + (M-1)M^\beta + 1 - N_0!)^{1-\frac{q}{q_0}}}, M^{\frac{q(\beta+1)}{q_0} - \beta} \right\}.
\end{aligned}$$

It follows from (1.6) that $\frac{q(\beta+1)}{q_0} - \beta < 0$. This implies

$$\|f_0\|_{\mathcal{M}^{q_0}_q} < \infty.$$

On the other hand, we claim

$$\|Tf_0\|_{L^r} = \infty. \tag{1.11}$$

Indeed, (1.11) follows from

$$\|Tf_0\|_{L^r} = \left(\sum_{N=N_0}^{\infty} N^{-\gamma r} |E_N| \right)^{\frac{1}{r}} = \left(\sum_{N=N_0}^{\infty} N^{1-\gamma r} \right)^{\frac{1}{r}}$$

and $1 - \gamma r > -1$. This ends the proof of Theorem 1.2.3. □

In view of Theorem 1.2.3, the Riesz-Thorin theorem can not be generalized to Morrey spaces. However, by adding some mild assumptions, there are recent researches about complex interpolation interpolation of Morrey spaces (see [12, 35, 36]). We shall recall the complex interpolation method and these results in Chapters 2 and 3, respectively.

Chapter 2

Complex interpolation method—Preliminaries

In this chapter we recall the complex interpolation method introduced by Calderón in [9]. We follow the terminology and presentation in [3, 9]. In Sections 2.1 and 2.2, we recall the definition of Calderón's first and second complex interpolation method. For the proof of our results in the next chapter, we shall discuss the Calderón product of Banach spaces in Section 2.3.

2.1 The first complex interpolation method

A pair (X_0, X_1) is said to be a compatible couple of Banach spaces if there exists a Hausdorff topological vector space Z such that X_0 and X_1 are subspaces of Z. From now on, let $\overline{S} := \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$ and S be its interior.

Definition 2.1.1 (Calderón's first complex interpolation functor)**.** Let (X_0, X_1) be a compatible couple of Banach spaces. Define $\mathcal{F}(X_0, X_1)$ as the set of all continuous functions $F : \overline{S} \to X_0 + X_1$ such that

1. $\sup_{z \in \overline{S}} \|F(z)\|_{X_0+X_1} < \infty$,
2. F is holomorphic on S,
3. the functions $t \in \mathbb{R} \mapsto F(j + it) \in X_j$ are bounded and continuous on $\mathbb{R}$ for $j = 0, 1$.

The norm on $\mathcal{F}(X_0, X_1)$ is defined by

$$\|F\|_{\mathcal{F}(X_0,X_1)} := \max\left\{\sup_{t\in\mathbb{R}} \|F(it)\|_{X_0},\ \sup_{t\in\mathbb{R}} \|F(1+it)\|_{X_1}\right\}.$$

Definition 2.1.2 (Calderón's first complex interpolation spaces)**.** Let $\theta \in (0,1)$ and (X_0, X_1) be a compatible couple of Banach spaces. The complex interpolation space $[X_0, X_1]_\theta$ with respect to (X_0, X_1) is defined by

$$[X_0, X_1]_\theta := \{f \in X_0 + X_1 : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\}$$

The norm on $[X_0, X_1]_\theta$ is defined by

$$\|f\|_{[X_0,X_1]_\theta} := \inf\{\|F\|_{\mathcal{F}(X_0,X_1)} : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\}.$$

The fact that $[X_0, X_1]_\theta$ is a Banach space can be seen in [9] and [3, Theorem 4.1.2]. When X_0 and X_1 are Lebesgue spaces, Calderón gave the following description of $[X_0, X_1]_\theta$.

Theorem 2.1.3. [9] *Let $\theta \in (0,1)$, $1 \le p_0 \le \infty$, and $1 \le p_1 \le \infty$. Then*

$$[L^{p_0}, L^{p_1}]_\theta = L^p$$

where p is defined by

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Note that the Riesz-Thorin complex interpolation theorem can be seen as a corollary of Theorem 2.1.3 and the following Calderón's result.

Theorem 2.1.4. [9] *Let $\theta \in (0,1)$. Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples of Banach spaces. If T is a bounded linear operator from X_j to Y_j for $j = 0, 1$, then T is bounded from $[X_0, X_1]_\theta$ to $[Y_0, Y_1]_\theta$.*

We also invoke the following useful lemmas.

Lemma 2.1.5. [9], [3, Theorem 4.2.2] *Let $\theta \in (0,1)$ and (X_0, X_1) be a compatible couple of Banach spaces. Then we have $X_0 \cap X_1$ is dense in $[X_0, X_1]_\theta$.*

Lemma 2.1.6. [3, Lemma 4.3.2] *Let $\theta \in (0,1)$ and $F \in \mathcal{F}(X_0, X_1)$. Then we have*

$$\|F(\theta)\|_{[X_0,X_1]_\theta} \le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|F(it)\|_{X_0} P_0(\theta,t)\ dt\right)^{1-\theta} \left(\frac{1}{\theta}\int_{\mathbb{R}} \|F(1+it)\|_{X_1} P_1(\theta,t)\ dt\right)^{\theta}, \quad (2.1)$$

where $P_0(\theta,t)$ and $P_1(\theta,t)$ are defined by

$$P_0(\theta,t) := \frac{\sin(\pi\theta)}{2(\cosh(\pi t) - \cos(\pi\theta))} \text{ and } P_1(\theta,t) := \frac{\sin(\pi\theta)}{2(\cosh(\pi t) + \cos(\pi\theta))}.$$

2.2 The second complex interpolation method

First let us recall the definition of Banach space-valued Lipschitz continuous functions. Let X be a Banach space. Denote by $\mathrm{Lip}(\mathbb{R}, X)$ the set of all functions $f : \mathbb{R} \to X$ such that

$$\|f\|_{\mathrm{Lip}(\mathbb{R},X)} := \sup_{-\infty<s<t<\infty} \frac{\|f(t) - f(s)\|_X}{|t - s|}$$

is finite.

Definition 2.2.1. [3, 9](Calderón's second complex interpolation functor) Let (X_0, X_1) be a compatible couple of Banach spaces. Denote by $\mathcal{G}(X_0, X_1)$ the set of all continuous functions $G : \bar{S} \to X_0 + X_1$ such that:

1. $\sup\limits_{z\in\bar{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{X_0+X_1} < \infty$,
2. G is holomorphic on S,
3. the functions
$$t \in \mathbb{R} \mapsto G(j + it) - G(j) \in X_j$$
are Lipschitz continuous on $\mathbb{R}$ for $j = 0, 1$.

The space $\mathcal{G}(X_0, X_1)$ is equipped with the norm

$$\|G\|_{\mathcal{G}(X_0,X_1)} := \max\left\{\|G(i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_0)},\ \|G(1 + i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_1)}\right\}. \qquad (2.2)$$

Definition 2.2.2. [3, 9](Calderón's second complex interpolation space) Let $\theta \in (0, 1)$. The second complex interpolation space $[X_0, X_1]^\theta$ with respect to (X_0, X_1) is defined to be the set of all $f \in X_0 + X_1$ such that $f = G'(\theta)$ for some $G \in \mathcal{G}(X_0, X_1)$. The norm on $[X_0, X_1]^\theta$ is defined by

$$\|f\|_{[X_0,X_1]^\theta} := \inf\{\|G\|_{\mathcal{G}(X_0,X_1)} : f = G'(\theta) \text{ for some } G \in \mathcal{G}(X_0, X_1)\}.$$

The relation between the inclusion and the second complex interpolation spaces is given as follows.

Lemma 2.2.3. [30, Lemma 2.8] *If $X_0 \hookrightarrow Y_0$ and $X_1 \hookrightarrow Y_1$, then*

$$[X_0, X_1]^\theta \hookrightarrow [Y_0, Y_1]^\theta.$$

Proof. Let $f \in [X_0, X_1]^\theta$. Then $f = G'(\theta)$ for some $G \in \mathcal{G}(X_0, X_1)$. By using the following inequalities

$$\|x_0\|_{Y_0} \lesssim \|x_0\|_{X_0}, \quad \|x_1\|_{Y_1} \lesssim \|x_1\|_{X_1}, \quad \text{and} \quad \|x\|_{Y_0+Y_1} \lesssim \|x\|_{X_0+X_1},$$

for every $x_0 \in X_0$, $x_1 \in X_1$, and $x \in X_0 + X_1$, we can show that $G \in \mathcal{G}(Y_0, Y_1)$. Thus, $f \in [Y_0, Y_1]^\theta$. □

The relation between the first and second complex interpolation functors is given in the following lemma:

Lemma 2.2.4. [27, Lemma 2.4] *For $G \in \mathcal{G}(X_0, X_1)$, $z \in \overline{S}$, and $k \in \mathbb{N}$, define*

$$H_k(z) := \frac{G(z + 2^{-k}i) - G(z)}{2^{-k}i}. \tag{2.3}$$

Then we have $H_k(\theta) \in [X_0, X_1]_\theta$.

Proof. We give a simplified proof of [27, Lemma 2.4]. The proof is adapted from [30]. The continuity and holomorphicity of H_k is a consequence of the corresponding property of G. Let $j \in \{0, 1\}$ be fixed. Since $t \in \mathbb{R} \mapsto G(j + it) \in X_j$ is Lipschitz-continuous, we see that $t \in \mathbb{R} \mapsto H_k(j + it) \in X_j$ is bounded and continuous on $\mathbb{R}$. Therefore, $H_k \in \mathcal{F}(X_0, X_1)$. Moreover,

$$\begin{aligned}\|H_k(\theta)\|_{[X_0,X_1]_\theta} &\le \|H_k\|_{\mathcal{F}(X_0,X_1)} \\ &= \max_{j=0,1} \sup_{t\in\mathbb{R}} \left\| \frac{G(j + i(t + 2^{-k})) - G(j + it)}{2^{-k}i} \right\|_{X_j} \\ &\le \|G\|_{\mathcal{G}(X_0,X_1)} < \infty,\end{aligned}$$

as desired. □

We shall also use the following useful connection between the first and second complex interpolation, obtained by Bergh [4].

Lemma 2.2.5. [4] *Let (X_0, X_1) be a compatible couple and $\theta \in (0, 1)$. Then we have*

$$[X_0, X_1]_\theta = \overline{X_0 \cap X_1}^{[X_0,X_1]^\theta}. \tag{2.4}$$

2.3 Calderón product

In order to obtain the description of the first complex interpolation spaces, sometimes it is easier to calculate the Calderón product of Banach lattices and applying the result of Sestakov in [52]. The definition of the Calderón product and Sestakov's lemma are given as follows.

Definition 2.3.1. Let $\theta \in (0,1)$ and (X_0, X_1) be a compatible couple of Banach spaces of measurable functions in $\mathbb{R}^n$. The Calderón product ${X_0}^{1-\theta}{X_1}^{\theta}$ of X_0 and X_1 is defined by

$$ {X_0}^{1-\theta}{X_1}^{\theta} := \bigcup_{f_0 \in X_0, f_1 \in X_1} \{ f : \mathbb{R}^n \to \mathbb{C} \, : \, |f(x)| \le |f_0(x)|^{1-\theta} |f_1(x)|^{\theta} \text{ a.e. } x \in \mathbb{R}^n \}. $$

For $f \in {X_0}^{1-\theta}{X_1}^{\theta}$, we define

$$ \|f\|_{{X_0}^{1-\theta}{X_1}^{\theta}} \\ := \inf\{ \|f_0\|_{X_0}^{1-\theta} \|f_1\|_{X_1}^{\theta} : f_0 \in X_0, f_1 \in X_1, |f(x)| \le |f_0(x)|^{1-\theta} |f_1(x)|^{\theta} \text{ a.e. } x \in \mathbb{R}^n \}. $$

By virtue of the Hölder inequality and factorization, for $1 \le p_0, p_1 \le \infty$

$$ (L^{p_0})^{1-\theta} (L^{p_1})^{\theta} = L^p, $$

where p is defined by $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. We now recall the following result by Sestakov.

Lemma 2.3.2. [52] *Let (X_0, X_1) be a compatible couple of Banach spaces of measurable functions in $\mathbb{R}^n$. Then for every $\theta \in (0,1)$, we have*

$$ [X_0, X_1]_\theta = \overline{X_0 \cap X_1}^{X_0^{1-\theta} X_1^{\theta}}. $$

Chapter 3

Complex interpolation of generalized Morrey spaces

3.1 Previous results about complex interpolation of Morrey spaces

The first result about the description of the first complex interpolation of Morrey spaces was given by Cobos et al. [12].

Theorem 3.1.1. [12] *Let $\theta \in (0,1)$, $1 \le q_0 \le p_0 < \infty$, and $1 \le q_1 \le p_1 < \infty$. Define p and q by*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{3.1}$$

respectively. Then

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q. \tag{3.2}$$

Assuming $\frac{p_0}{q_0} = \frac{p_1}{q_1}$, Lu et al. [36] improved the description of $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$ in Theorem 3.1.1. Morever, their result are in the setting of Morrey spaces over metric measure space.

Theorem 3.1.2. [36] *Let $\theta \in (0,1)$, $1 \le q_0 \le p_0 < \infty$, and $1 \le q_1 \le p_1 < \infty$. Assume that $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Then*

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}, \tag{3.3}$$

where p and q are defined by (3.1).

The key parts of the proof of Theorem 3.1.2 are Lemma 2.3.2 and the calculation of the Calderón product between $\mathcal{M}^{p_0}_{q_0}$ and $\mathcal{M}^{p_1}_{q_1}$. We shall see in Section 3.2 that Theorem 3.1.2 can be seen as a special case of the first complex interpolation of generalized Morrey spaces. Another decription of the first complex interpolation of Morrey spaces was given by Yuan, Sickel, and Yang [55] in term of the space $\mathcal{M}^{p_0,p_1,\theta}_{q_0,q_1}$. The definition of this space is given as follows.

Definition 3.1.3. Keep the same assumption as in Theorem 3.1.2. The space $\mathcal{M}^{p_0,p_1,\theta}_{q_0,q_1}$ is defined to be the set of all functions f for which

$$\max_{j=0,1} \sup_{x\in\mathbb{R}^n, r\geq 1} |B(x,r)|^{\frac{1}{p_j}-\frac{1}{q_j}} \left(\int_{B(x,r)} |f(y)|^{q_j} \, dy \right)^{\frac{1}{q_j}} < \infty,$$

$$\sup_{x\in\mathbb{R}^n, 0<r\leq 1} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^{q} \, dy \right)^{\frac{1}{q}} < \infty,$$

and

$$\lim_{r\to 0^+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^{q} \, dy \right)^{\frac{1}{q}} = 0,$$

uniformly in $x \in \mathbb{R}^n$.

Let us recall the description of the first complex interpolation of Morrey spaces in [55].

Theorem 3.1.4. [55] *Keep the same assumption as in Theorem 3.1.2. Then*

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \mathcal{M}^{p_0,p_1,\theta}_{q_0,q_1}. \tag{3.4}$$

The description of the right-hand side of (3.3) and can be refined as follows.

Theorem 3.1.5. [27] *Keep the same assumption as in Theorem 3.1.2 and assume also that $q_0 \neq q_1$. Then we have*

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \left\{ f \in \mathcal{M}^p_q : \lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N}\leq|f|\leq N\}} f\|_{\mathcal{M}^p_q} = 0 \right\}. \tag{3.5}$$

Note that Theorem 3.1.5 is an improvement of Theorems 3.1.2 and 3.1.4, in the sense that, $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$ is now written in term of the parameters p and q only and this description is more explicit than the right-hand side of (3.3). We shall prove Theorem 3.1.5 as a corollary of the corresponding result for generalized Morrey spaces.

Observe that the function $f(x) := |x|^{-n/p}$ does not belong to the set in the right-hand side of (3.5), but this function is in $\mathcal{M}^p_q$. From this observation, one may inquire whether we can interpolate Morrey spaces and the result is also Morrey spaces. The affirmative answer was given by Lemarié-Rieusset [35]. He proved the following result about the second complex interpolation of Morrey spaces.

Theorem 3.1.6. [35] *Keep the same asssumption as in Theorem 3.1.2. Then*

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta = \mathcal{M}^p_q.$$

It is written in the book [3, p. 90] that the first complex interpolation space is the main interest in this book and the second complex interpolation method is considered as a technical tool. Hence, Theorem 3.1.6 can be seen as an example of the importance of the second complex interpolation method. We shall give a generalization of Theorem 3.1.6 to the setting of generalized Morrey spaces in Section 3.3.

3.2 The first complex interpolation of generalized Morrey spaces

In this section we give a description of the first complex interpolation of generalized Morrey spaces. Our proof uses Lemma 2.3.2 and the following result about the Calderón product of generalized Morrey spaces.

Proposition 3.2.1. *Let $\theta \in (0,1)$, $q_0, q_1 \in (1,\infty)$, $\varphi_0 \in \mathcal{G}_{q_0}$, and $\varphi_1 \in \mathcal{G}_{q_1}$. Assume that φ_0 and φ_1 satisfy*

$$\varphi_0^{q_0} = \varphi_1^{q_1}. \tag{3.6}$$

Define φ and q by

$$\varphi := \varphi_0^{1-\theta}\varphi_1^\theta \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{3.7}$$

respectively. Then

$$(\mathcal{M}^{\varphi_0}_{q_0})^{1-\theta}(\mathcal{M}^{\varphi_1}_{q_1})^\theta = \mathcal{M}^\varphi_q. \tag{3.8}$$

Proof. Let $B = B(a,r)$ be any ball in $\mathbb{R}^n$ and $\varepsilon > 0$. Let $f \in (\mathcal{M}^{\varphi_0}_{q_0})^{1-\theta}(\mathcal{M}^{\varphi_1}_{q_1})^\theta$. Then, there exist some functions $f_0 \in \mathcal{M}^{\varphi_0}_{q_0}$ and $f_1 \in \mathcal{M}^{\varphi_1}_{q_1}$ such that

$$|f(x)| \le |f_0(x)|^{1-\theta}|f_1(x)|^\theta, \text{ a.e. } x \in \mathbb{R}^n \tag{3.9}$$

and

$$\|f_0\|_{\mathcal{M}^{\varphi_0}_{q_0}}^{1-\theta}\|f_1\|_{\mathcal{M}^{\varphi_1}_{q_1}}^{\theta} \leq (1+\varepsilon)\|f\|_{(\mathcal{M}^{\varphi_0}_{q_0})^{1-\theta}(\mathcal{M}^{\varphi_1}_{q_1})^{\theta}}. \tag{3.10}$$

By using Hölder's inequality and (3.9), we have

$$\begin{aligned}\left(\int_B |f(x)|^q \, dx\right)^{\frac{1}{q}} &\leq \left(\int_B |f_0(x)|^{q(1-\theta)}|f_1(x)|^{q\theta} \, dx\right)^{\frac{1}{q}} \\ &\leq \left(\int_B |f_0(x)|^{q_0} \, dx\right)^{\frac{1-\theta}{q_0}} \left(\int_B |f_1(x)|^{q_1} \, dx\right)^{\frac{\theta}{q_1}}. \end{aligned} \tag{3.11}$$

Combining (3.6)–(3.7) and inequalities (3.10)–(3.11), we obtain

$$\begin{aligned}\frac{\varphi(r)}{|B|^{\frac{1}{q}}}\left(\int_B |f(x)|^q dx\right)^{\frac{1}{q}} &\leq \frac{\varphi_0(r)^{1-\theta}\varphi_1(r)^{\theta}}{|B|^{\frac{1-\theta}{q_0}+\frac{\theta}{q_1}}}\|f_0\|_{L^{q_0}(B)}^{1-\theta}\|f_1\|_{L^{q_1}(B)}^{\theta} \\ &\leq \|f_0\|_{\mathcal{M}^{\varphi_0}_{q_0}}^{1-\theta}\|f_1\|_{\mathcal{M}^{\varphi_1}_{q_1}}^{\theta} \\ &\leq (1+\varepsilon)\|f\|_{\left(\mathcal{M}^{\varphi_0}_{q_0}\right)^{1-\theta}\left(\mathcal{M}^{\varphi_1}_{q_1}\right)^{\theta}}.\end{aligned}$$

Since ε is arbitary, we have $f \in \mathcal{M}^{\varphi}_q$ with $\|f\|_{\mathcal{M}^{\varphi}_q} \leq \|f\|_{(\mathcal{M}^{\varphi_0}_{q_0})^{1-\theta}(\mathcal{M}^{\varphi_1}_{q_1})^{\theta}}$. Thus,

$$(\mathcal{M}^{\varphi_0}_{q_0})^{1-\theta}(\mathcal{M}^{\varphi_1}_{q_1})^{\theta} \subseteq \mathcal{M}^{\varphi}_q.$$

Conversely, let $f \in \mathcal{M}^{\varphi}_q$. Define $\tilde{f}_j := |f|^{\frac{q}{q_j}}$ where $j \in \{0,1\}$. It follows from (3.6)–(3.7) that

$$\varphi^q = \varphi_0^{q_0} = \varphi_1^{q_1}. \tag{3.12}$$

Then $\tilde{f}_j \in \mathcal{M}^{\varphi_j}_{q_j}$ with $\|\tilde{f}_j\|_{\mathcal{M}^{\varphi_j}_{q_j}} = \|f\|_{\mathcal{M}^{\varphi}_q}^{\frac{q}{q_j}}$ for $j = 0, 1$. Observe that, we have

$$|\tilde{f}_0|^{1-\theta}|\tilde{f}_1|^{\theta} = |f|^{\frac{q(1-\theta)}{q_0}}|f|^{\frac{q\theta}{q_1}} = |f| \tag{3.13}$$

and

$$\|f\|_{\left(\mathcal{M}^{\varphi_0}_{q_0}\right)^{1-\theta}\left(\mathcal{M}^{\varphi_1}_{q_1}\right)^{\theta}} \leq \|\tilde{f}_0\|_{\mathcal{M}^{\varphi_0}_{q_0}}^{1-\theta}\|\tilde{f}_1\|_{\mathcal{M}^{\varphi_1}_{q_1}}^{\theta} = \|f\|_{\mathcal{M}^{\varphi}_q}^{\frac{q(1-\theta)}{q_0}+\frac{q\theta}{q_1}} = \|f\|_{\mathcal{M}^{\varphi}_q} < \infty. \tag{3.14}$$

Consequently, $f \in \left(\mathcal{M}^{\varphi_0}_{q_0}\right)^{1-\theta}\left(\mathcal{M}^{\varphi_1}_{q_1}\right)^{\theta}$. Therefore, $\mathcal{M}^{\varphi}_q \subseteq (\mathcal{M}^{\varphi_0}_{q_0})^{1-\theta}(\mathcal{M}^{\varphi_1}_{q_1})^{\theta}$. Thus, we have proved (3.8). □

Combining Lemma 2.3.2 and Proposition 3.2.1, we have the following generalization of Theorem 3.1.2.

Theorem 3.2.2. [24] *Keep the same assumption as in Propostion 3.2.1. Then*

$$[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]_{\theta} = \overline{\mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1}}^{\mathcal{M}^{\varphi}_q}.$$

Note that we can recover Theorem 3.1.2 by taking $\varphi_0(t) := t^{\frac{n}{p_0}}$ and $\varphi_1(t) := t^{\frac{n}{p_1}}$. We now prove the following generalization of Theorem 3.1.5.

Theorem 3.2.3. [27] *Keep the same assumption of Proposition 3.2.1 and assume also that $q_0 \neq q_1$. Then*

$$[\mathcal{M}_{q_0}^{\varphi_0}, \mathcal{M}_{q_1}^{\varphi_1}]_\theta = \left\{ f \in \mathcal{M}_q^\varphi : \lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N}\leq|f|\leq N\}} f\|_{\mathcal{M}_q^\varphi} = 0 \right\}. \tag{3.15}$$

Remark 3.2.4. If $\varphi_j(t) = t^{\frac{n}{p_j}}$ where $j = 0, 1$, then we can recover Theorem 3.1.5.

In order to prove Theorem 3.2.3, we need two lemmas. The first one is the fact that the set in the right-hand side of (3.15) is closed. The second lemma tells us that this set contains $\mathcal{M}_{q_0}^{\varphi_0} \cap \mathcal{M}_{q_1}^{\varphi_1}$.

Lemma 3.2.5. *Let $1 \leq q < \infty$ and $\varphi \in \mathcal{G}_q$. Then the set*

$$A := \left\{ f \in \mathcal{M}_q^\varphi : \lim_{N\to\infty} \left\| f - \chi_{\{\frac{1}{N}\leq|f|\leq N\}} f \right\|_{\mathcal{M}_q^\varphi} = 0 \right\} \tag{3.16}$$

is a closed subset $\mathcal{M}_q^\varphi$.

Proof. Let $\{f_j\}_{j=1}^\infty \subset A$ be such that f_j converges to f in $\mathcal{M}_q^\varphi$. Fix $j \in \mathbb{N}$. For every $N \in \mathbb{N}$, we have

$$\left\|\chi_{\{|f|<\frac{1}{N}\}} f\right\|_{\mathcal{M}_q^\varphi} \leq \|f - f_j\|_{\mathcal{M}_q^\varphi} + \left\|\chi_{\left\{|f|<\frac{1}{N}\right\}\cap\left\{|f_j|\geq\frac{2}{N}\right\}} f_j\right\|_{\mathcal{M}_q^\varphi} + \left\|\chi_{\left\{|f_j|<\frac{2}{N}\right\}} f_j\right\|_{\mathcal{M}_q^\varphi}$$

and

$$\left\|\chi_{\{|f|>N\}} f\right\|_{\mathcal{M}_q^\varphi} \leq \|f - f_j\|_{\mathcal{M}_q^\varphi} + \left\|\chi_{\left\{|f|>N\right\}\cap\left\{|f_j|\leq\frac{N}{2}\right\}} f_j\right\|_{\mathcal{M}_q^\varphi} + \left\|\chi_{\left\{|f_j|>\frac{N}{2}\right\}} f_j\right\|_{\mathcal{M}_q^\varphi}.$$

On the set $\left\{|f| < \frac{1}{N}\right\} \cap \left\{|f_j| \geq \frac{2}{N}\right\}$, we have

$$|f_j| \leq |f_j - f| + |f| < |f_j - f| + \frac{1}{N} \leq |f_j - f| + \frac{1}{2}|f_j|,$$

and hence $|f_j| \leq 2|f - f_j|$. Consequently,

$$\left\|\chi_{\{|f|<\frac{1}{N}\}} f\right\|_{\mathcal{M}_q^\varphi} \leq 3\|f - f_j\|_{\mathcal{M}_q^\varphi} + \left\|\chi_{\left\{|f_j|<\frac{2}{N}\right\}} f_j\right\|_{\mathcal{M}_q^\varphi}. \tag{3.17}$$

Meanwhile, on the set $\{|f| > N\} \cap \{|f_j| \leq \frac{N}{2}\}$, we have

$$|f_j| \leq \frac{N}{2} < \frac{|f|}{2} \leq \frac{|f - f_j|}{2} + \frac{|f_j|}{2},$$

and hence, $|f_j| \le |f - f_j|$. Therefore,

$$\|\chi_{\{|f|>N\}} f\|_{\mathcal{M}^{\varphi}_q} \le 2\|f - f_j\|_{\mathcal{M}^{\varphi}_q} + \|\chi_{\{|f_j|>\frac{N}{2}\}} f_j\|_{\mathcal{M}^{\varphi}_q}. \tag{3.18}$$

By combining (3.17) and (3.18), we get

$$\begin{aligned}\left\|f - \chi_{\left\{\frac{1}{N}\le|f|\le N\right\}} f\right\|_{\mathcal{M}^{\varphi}_q} &\le \left\|\chi_{\left\{|f|<\frac{1}{N}\right\}} f\right\|_{\mathcal{M}^{\varphi}_q} + \|\chi_{\{|f|>N\}} f\|_{\mathcal{M}^{\varphi}_q} \\ &\le 5\|f - f_j\|_{\mathcal{M}^{\varphi}_q} + \left\|\chi_{\left\{|f_j|<\frac{2}{N}\right\}} f_j\right\|_{\mathcal{M}^{\varphi}_q} + \left\|\chi_{\left\{|f_j|>\frac{N}{2}\right\}} f_j\right\|_{\mathcal{M}^{\varphi}_q}.\end{aligned}$$

Since $f_j \in A$, we have

$$\limsup_{N\to\infty} \|f - \chi_{\{\frac{1}{N}\le|f|\le N\}} f\|_{\mathcal{M}^{\varphi}_q} \le 5\|f - f_j\|_{\mathcal{M}^{\varphi}_q}.$$

By taking $j \to \infty$, we have $\lim\limits_{N\to\infty} \|f - \chi_{\{\frac{1}{N}\le|f|\le N\}} f\|_{\mathcal{M}^{\varphi}_q} = 0$, and hence, $f \in A$. $\square$

Lemma 3.2.6. [27] *Maintain the same conditions as Proposition* 3.2.1 *and let A be defined by* (3.16). *Then*

$$\mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1} \subseteq A.$$

Proof. Without loss of generality, we assume that $q_1 > q_0$. Then, $q_1 > q > q_0$. Consequently, for every $f \in \mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1}$, we have

$$\begin{aligned}\left\|f - \chi_{\{\frac{1}{N}\le|f|\le N\}} f\right\|_{\mathcal{M}^{\varphi}_q} &\le \|\chi_{\{|f|<\frac{1}{N}\}} |f|^{1-\frac{q_0}{q}} |f|^{\frac{q_0}{q}}\|_{\mathcal{M}^{\varphi}_q} + \|\chi_{\{|f|>N\}} |f|^{1-\frac{q_1}{q}} |f|^{\frac{q_1}{q}}\|_{\mathcal{M}^{\varphi}_q} \\ &\le N^{\frac{q_0-q}{q}} \left\||f|^{\frac{q_0}{q}}\right\|_{\mathcal{M}^{\varphi}_q} + N^{\frac{q-q_1}{q}} \left\||f|^{\frac{q_1}{q}}\right\|_{\mathcal{M}^{\varphi}_q} \\ &= N^{\frac{q_0-q}{q}} \|f\|^{\frac{q_0}{q}}_{\mathcal{M}^{\varphi_0}_{q_0}} + N^{\frac{q-q_1}{q}} \|f\|^{\frac{q_1}{q}}_{\mathcal{M}^{\varphi_1}_{q_1}} \to 0\end{aligned}$$

as $N \to \infty$, which implies $f \in A$. $\square$

Now we are ready to prove Theorem 3.2.3.

Proof of Theorem 3.2.3. By virtue of Theorem 3.2.2 and Lemmas 3.2.5 and 3.2.6, we have

$$[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]_\theta = \overline{\mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1}}^{\mathcal{M}^{\varphi}_q} \subseteq A.$$

Conversely, let $f \in A$. For every $N \in \mathbb{N}$, define $f_N := \chi_{\{\frac{1}{N}\le|f|\le N\}} f$. As in the proof of Lemma 3.2.6, we may assume that $q_0 < q_1$. Then $q_0 < q < q_1$. This implies

$$\|f_N\|_{\mathcal{M}^{\varphi_0}_{q_0}} \le \left\|\chi_{\{\frac{1}{N}\le|f|\}} |f|^{1-\frac{q}{q_0}} |f|^{\frac{q}{q_0}}\right\|_{\mathcal{M}^{\varphi_0}_{q_0}} \le N^{\frac{q-q_0}{q_0}} \|f\|^{q/q_0}_{\mathcal{M}^{\varphi}_q} < \infty$$

and

$$\|f_N\|_{\mathcal{M}^{\varphi_1}_{q_1}} \le \left\| \chi_{\{|f|<N\}} |f|^{1-\frac{q}{q_1}} |f|^{\frac{q}{q_1}} \right\|_{\mathcal{M}^{\varphi_1}_{q_1}} \le N^{\frac{q_1-q}{q_1}} \|f\|^{q/q_1}_{\mathcal{M}^{\varphi}_q} < \infty.$$

Therefore, $f \in \overline{\mathcal{M}^{\varphi_0}_{q_0} \cap \mathcal{M}^{\varphi_1}_{q_1}}^{\mathcal{M}^{\varphi}_q}$ by the definition of A. According to Theorem 3.2.2, we have $f \in [\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]_\theta$ as desired. □

3.3 The second complex interpolation of generalized Morrey spaces

We prove a generalization of Theorem 3.1.6 in the setting of generalized Morrey spaces. First we prove the following lemmas about the construction of the second complex interpolation functor.

Lemma 3.3.1. [24, Lemma 4] *Let $q_0 > q_1$ and $f \in L^0$. Define $q : \overline{S} \to \mathbb{C}$, $F : \overline{S} \to L^0$ and $G : \overline{S} \to L^0$ by:*

$$\frac{1}{q(z)} = \frac{1-z}{q_0} + \frac{z}{q_1}, \tag{3.19}$$

$$F(z) := \operatorname{sgn}(f) \exp\left(\frac{q}{q(z)} \log |f| \right) \quad (z \in \overline{S}), \tag{3.20}$$

and

$$G(z) := (z-\theta) \int_0^1 F(\theta + (z-\theta)t)\, dt \quad (z \in \overline{S}), \tag{3.21}$$

respectively. Define $F_0, F_1, G_0, G_1 : \overline{S} \to L^0$ by:

$$F_0(z) := F(z)\chi_{\{|f|\le 1\}}, \quad F_1(z) := F(z)\chi_{\{|f|>1\}}, \tag{3.22}$$

and

$$G_0(z) := G(z)\chi_{\{|f|\le 1\}}, \quad G_1(z) := G(z)\chi_{\{|f|>1\}}. \tag{3.23}$$

Then, for any $z \in \overline{S}$, we have

$$|G(z)| \le (1+|z|)(|f|^{q/q_0} + |f|^{q/q_1}). \tag{3.24}$$

For any $z \in \mathbb{C}$ with $\varepsilon < \mathrm{Re}(z) < 1-\varepsilon$ and $w \in \mathbb{C}$ with $|w| \ll 1$, we have

$$\left| \frac{G_0(z+w) - G_0(z)}{w} - F_0(z) \right| \le C_\varepsilon |w| \cdot |f|^{\frac{q}{q_0}}, \tag{3.25}$$

$$\left| \frac{G_1(z+w) - G_1(z)}{w} - F_1(z) \right| \le C_\varepsilon |w| \cdot |f|^{\frac{q}{q_1}}, \tag{3.26}$$

where the constant C_ε depending only on $\varepsilon \in (0, 1/2)$.

Proof. For $t \in [0,1]$, define $v := (z-\theta)t + \theta$. Since $\mathrm{Re}(v) \in [0,1]$, we have

$$
\begin{aligned}
|F(v)| &\le |f|^{\frac{q}{q_0}(1-\mathrm{Re}(v)) + \frac{q}{q_1}\mathrm{Re}(v)} \\
&\le (1-\mathrm{Re}(v))|f|^{\frac{q}{q_0}} + \mathrm{Re}(v)|f|^{\frac{q}{q_1}} \le |f|^{\frac{q}{q_0}} + |f|^{\frac{q}{q_1}}. \qquad (3.27)
\end{aligned}
$$

By the triangle inequality, we have

$$
|G(z)| \le |z-\theta| \left(|f|^{\frac{q}{q_0}} + |f|^{\frac{q}{q_1}} \right) \le (1+|z|) \left(|f|^{\frac{q}{q_0}} + |f|^{\frac{q}{q_1}} \right).
$$

Writing out the definitions in full, we obtain

$$
\begin{aligned}
&\left| \frac{G_0(z+w) - G_0(z)}{w} - F_0(z) \right| \\
&= |F_0(\mathrm{Re}(z))| \left| \frac{\exp\left[q \left(\frac{-w}{q_0} + \frac{w}{q_1} \right) \log|f| \right] - 1}{w(\frac{q}{q_1} - \frac{q}{q_0}) \log|f|} - 1 \right|.
\end{aligned}
$$

Since $q_0 > q_1$, we have

$$
\begin{aligned}
&\left| \frac{G_0(z+w) - G_0(z)}{w} - F_0(z) \right| \\
&= \chi_{\{|f|\le 1\}} |f|^{\frac{q}{q_0}(1-\mathrm{Re}(z)) + \frac{q}{q_1}\mathrm{Re}(z)} \left| \frac{\exp\left[q \left(\frac{-w}{q_0} + \frac{w}{q_1} \right) \log|f| \right] - 1}{w(\frac{q}{q_1} - \frac{q}{q_0}) \log|f|} - 1 \right| \\
&\le \chi_{\{|f|\le 1\}} |f|^{\frac{q}{q_0}} \cdot |f|^{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\varepsilon} \left| \frac{\exp\left[q \left(\frac{-w}{q_0} + \frac{w}{q_1} \right) \log|f| \right] - 1}{w(\frac{q}{q_1} - \frac{q}{q_0}) \log|f|} - 1 \right| \\
&\le |f|^{\frac{q}{q_0}} \sup_{0<t\le 1} t^{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\varepsilon} \left| \frac{\exp\left[q \left(\frac{-w}{q_0} + \frac{w}{q_1} \right) \log t \right] - 1}{w(\frac{q}{q_1} - \frac{q}{q_0}) \log t} - 1 \right| \le C_\varepsilon |w| \cdot |f|^{\frac{q}{q_0}}.
\end{aligned}
$$

By a similar argument, we also have

$$
\begin{aligned}
&\left| \frac{G_1(z+w) - G_1(z)}{w} - F_1(z) \right| \\
&= \chi_{\{|f|>1\}} |f|^{\frac{q}{q_1}} \cdot |f|^{\left(\frac{q}{q_0} - \frac{q}{q_1}\right)(1-\mathrm{Re}(z))} \left| \frac{\exp\left[q \left(\frac{-w}{q_0} + \frac{w}{q_1} \right) \log|f| \right] - 1}{w(\frac{q}{q_1} - \frac{q}{q_0}) \log|f|} - 1 \right| \\
&\le \chi_{\{|f|>1\}} |f|^{\frac{q}{q_1}} \cdot |f|^{\left(\frac{q}{q_0} - \frac{q}{q_1}\right)\varepsilon} \left| \frac{\exp\left[q \left(\frac{-w}{q_0} + \frac{w}{q_1} \right) \log|f| \right] - 1}{w(\frac{q}{q_1} - \frac{q}{q_0}) \log|f|} - 1 \right| \\
&\le |f|^{\frac{q}{q_1}} \sup_{t\ge 1} t^{\left(\frac{q}{q_0} - \frac{q}{q_1}\right)\varepsilon} \left| \frac{\exp\left[q \left(\frac{-w}{q_0} + \frac{w}{q_1} \right) \log t \right] - 1}{w(\frac{q}{q_1} - \frac{q}{q_0}) \log t} - 1 \right| \le C_\varepsilon |w| \cdot |f|^{\frac{q}{q_1}}
\end{aligned}
$$

as desired. □

Lemma 3.3.2. [24, Lemma 12] *Let $f \in \mathcal{M}^{\varphi}_{q}$. Via (3.19) define $F : \overline{S} \to \mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$ and $G : \overline{S} \to \mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$ by (3.20) and (3.21), respectively. Then, the function G belongs to $\mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1})$.*

Proof. It follows from (3.24) that $G(z) \in \mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$ and

$$\sup_{z \in \overline{S}} \frac{\|G(z)\|_{\mathcal{M}^{\varphi_0}_{q_0}+\mathcal{M}^{\varphi_1}_{q_1}}}{1+|z|} \leq \|f\|^{q/q_0}_{\mathcal{M}^{\varphi}_{q}} + \|f\|^{q/q_1}_{\mathcal{M}^{\varphi}_{q}}.$$

Now let $z_1, z_2 \in \overline{S}$. Then, by inequality (3.27), we get

$$\|G(z_1) - G(z_2)\|_{\mathcal{M}^{\varphi_0}_{q_0}+\mathcal{M}^{\varphi_1}_{q_1}} \leq |z_1 - z_2| \left(\|f\|^{q/q_0}_{\mathcal{M}^{\varphi}_{q}} + \|f\|^{q/q_1}_{\mathcal{M}^{\varphi}_{q}} \right).$$

This shows the continuity of $G : \overline{S} \to \mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$. The proof of holomorphicity of $G : S \to \mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$ goes as follows. Let $\varepsilon \in (0, \frac{1}{2})$ and define

$$S_\varepsilon := \{z \in S : \varepsilon < \mathrm{Re}(z) < 1 - \varepsilon\}.$$

According to (3.25) and (3.26), we have

$$\begin{aligned}
&\left\| \frac{G(z+w) - G(z)}{w} - F(z) \right\|_{\mathcal{M}^{\varphi_0}_{q_0}+\mathcal{M}^{\varphi_1}_{q_1}} \\
&\leq \left\| \frac{G_0(z+w) - G_0(z)}{w} - F_0(z) \right\|_{\mathcal{M}^{\varphi_0}_{q_0}} + \left\| \frac{G_1(z+w) - G_1(z)}{w} - F_1(z) \right\|_{\mathcal{M}^{\varphi_1}_{q_1}} \\
&\leq C_\varepsilon |w| \left(\|f\|^{q/q_0}_{\mathcal{M}^{\varphi}_{q}} + \|f\|^{q/q_1}_{\mathcal{M}^{\varphi}_{q}} \right).
\end{aligned}$$

Taking $w \to 0$, we see that $G : S_\varepsilon \to \mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$ is holomorphic. Since $\varepsilon > 0$ is arbitrary, we conclude that $G : S \to \mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$ is holomorphic.

We now verify that $G(j + it_1) - G(j + it_2) \in \mathcal{M}^{\varphi_j}_{q_j}$ for every $t_1, t_2 \in \mathbb{R}$ and $j \in \{0, 1\}$ and also

$$\|G(j + i\cdot)\|_{\mathrm{Lip}(\mathbb{R}, \mathcal{M}^{\varphi_j}_{q_j})} \leq (\|f\|_{\mathcal{M}^{\varphi}_{q}})^{q/q_j}. \tag{3.28}$$

for every $j \in \{0, 1\}$. Combining $|F(j + it)| = |f|^{\frac{q}{q_j}}$ and

$$G(j + it_1) - G(j + it_2) = -i \int_{t_1}^{t_2} F(j + it)\, dt,$$

we get

$$\|G(j + it_1) - G(j + it_2)\|_{\mathcal{M}^{\varphi_j}_{q_j}} \leq |t_1 - t_2| \|f\|^{\frac{q}{q_j}}_{\mathcal{M}^{\varphi}_{q}}.$$

This implies (3.29). Thus, $G \in \mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1})$ with

$$\|G\|_{\mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1})} \leq \|f\|^{q/q_j}_{\mathcal{M}^{\varphi}_q}, \tag{3.29}$$

as desired. □

Note that we can not use the function F defined by (3.20) as the first complex interpolation functor because F does not belong to $\mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$ when $f(x) := |x|^{-n/p}$. This fact is a consequence of the following proposition.

Proposition 3.3.3. [24, Proposition 4] *Let $f(x) := |x|^{-n/p}$ and define F by (3.20). Then the mapping $t \in \mathbb{R} \mapsto F(it) \in \mathcal{M}^{p_0}_{q_0}$ is not continuous at $t = 0$.*

Proof. Assume that $p_0 > p_1$ and define $Q := \frac{1}{p_1} - \frac{1}{p_0}$. Using $\frac{p_0}{q_0} = \frac{p}{q} = \frac{p_1}{q_1}$, for every $0 < t < \frac{1}{Q}$, we have

$$|F(it) - F(0)| = |x|^{-\frac{n}{p_0}} \left||x|^{-Qit} - 1\right| = 2|x|^{-\frac{n}{p_0}} \left|\sin\left(\frac{Qt\log|x|}{2}\right)\right|. \tag{3.30}$$

Using (3.30) and letting $R := \exp((Qt)^{-1})$, we get

$$\begin{aligned} &\|F(it) - F(0)\|_{\mathcal{M}^{p_0}_{q_0}} \\ &\geq 2|B(0,2R)|^{\frac{1}{p_0}-\frac{1}{q_0}} \left(\int_{B(0,2R)\setminus B(0,R)} |x|^{-\frac{nq_0}{p_0}} \left|\sin\left(\frac{Qt\log|x|}{2}\right)\right|^{q_0} dx\right)^{\frac{1}{q_0}} \\ &\gtrsim R^{\frac{n}{p_0}-\frac{n}{q_0}} \left(\int_{B(0,2R)\setminus B(0,R)} |x|^{-\frac{nq_0}{p_0}} dx\right)^{\frac{1}{q_0}} \gtrsim 1, \end{aligned} \tag{3.31}$$

where we use

$$\left|\sin\left(\frac{Qt\log|x|}{2}\right)\right| > \sin(1/2)$$

for every $R < |x| < 2R$. Thus, (3.31) implies

$$\lim_{t\to 0^+} \|F(it) - F(0)\|_{\mathcal{M}^{p_0}_{q_0}} \neq 0,$$

as desired. □

Now we arrive at our main result in this section.

Theorem 3.3.4. [24, p. 316] *Keep the same assumption as in Proposition 3.2.1. Then*

$$[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^{\theta} = \mathcal{M}^{\varphi}_q. \tag{3.32}$$

Remark 3.3.5. Taking $\varphi_0(t) := t^{\frac{n}{p_0}}$ and $\varphi_1(t) := t^{\frac{n}{p_1}}$, we recover Theorem 3.1.6.

Proof of Theorem 3.3.4. Let $f \in \mathcal{M}^\varphi_q$. By a normalization, we may suppose $\|f\|_{\mathcal{M}^\varphi_q} = 1$, for the purpose of proving $f \in [\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta$. For every $z \in \overline{S}$, define $F(z)$ and $G(z)$ as we did in Lemma 3.3.1. Thanks to Lemma 3.3.2, we have $G \in \mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1})$. Since $G'(\theta) = F(\theta) = f$, we have

$$\|f\|_{[\mathcal{M}^{\varphi_0}_{q_0},\mathcal{M}^{\varphi_1}_{q_1}]^\theta} \le \|G\|_{\mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0},\mathcal{M}^{\varphi_1}_{q_1})} = \max_{j=0,1} \|G(j+i\cdot)\|_{\mathrm{Lip}\left(\mathbb{R},\mathcal{M}^{\varphi_j}_{q_j}\right)} = 1.$$

This shows that $[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta \supset \mathcal{M}^\varphi_q$. Conversely, let $f \in [\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta$ with

$$\|f\|_{[\mathcal{M}^{\varphi_0}_{q_0},\mathcal{M}^{\varphi_1}_{q_1}]^\theta} = 1.$$

Suppose f is realized as $G'(\theta)$, where $G \in \mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1})$ and $\|G\|_{\mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0},\mathcal{M}^{\varphi_1}_{q_1})} \le 2$. For every $k \in \mathbb{N}$ and $z \in \overline{S}$, we define $H_k(z)$ by (2.3). According to Lemma 2.2.4 and Theorem 3.2.3, we obtain

$$\|H_k(\theta)\|_{\mathcal{M}^\varphi_q} \lesssim \|H_k(\theta)\|_{[\mathcal{M}^{\varphi_0}_{q_0},\mathcal{M}^{\varphi_1}_{q_1}]_\theta} \le \|G\|_{\mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0},\mathcal{M}^{\varphi_1}_{q_1})} \le 2. \tag{3.33}$$

Meanwhile, since $f = G'(\theta) = \lim_{k\to\infty} H_k(\theta)$ in $\mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$, there exists a subsequence $\{H_{k_j}\}_{j=1}^\infty$ such that $f(x) = \lim_{j\to\infty} H_{k_j}(\theta)(x)$ for almost every $x \in \mathbb{R}^n$. Consequently, by virtue of the Fatou lemma and the inequality (3.33), we have

$$\|f\|_{\mathcal{M}^\varphi_q} \lesssim \liminf_{j\to\infty} \|H_{k_j}(\theta)\|_{\mathcal{M}^\varphi_q} \le 2.$$

This implies $[\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]^\theta \hookrightarrow \mathcal{M}^\varphi_q$. □

3.4 Complex interpolation between L^∞ and the generalized Morrey space $\mathcal{M}^\varphi_q$

Note that, when p_0 and p_1 are finite, Theorem 2.1.3 is a special case of Theorems 3.1.2 and 3.2.2. In order to recover Theorem 2.1.3 for the case $p_0 = \infty$, we give the following supplement of Theorem 3.2.2.

Theorem 3.4.1. [30] *Let $\theta \in (0,1)$, $1 \le q < \infty$, and $\varphi \in \mathcal{G}_q$. Then*

$$[L^\infty, \mathcal{M}^\varphi_q]_\theta = \left\{ f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \lim_{N\to\infty} \|f\chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = 0 \right\}. \tag{3.34}$$

As in Section 3.2, we first prove the Calderón product between L^∞ and $\mathcal{M}^\varphi_q$.

Lemma 3.4.2. [30] *Let $\theta \in (0,1)$, $1 \le q < \infty$, and $\varphi \in \mathcal{G}_q$. Then we have*

$$(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta = \mathcal{M}_{q/\theta}^{\varphi^\theta}. \tag{3.35}$$

Proof. Let $f \in \mathcal{M}_{q/\theta}^{\varphi^\theta}$ and define $f_0 := 1$ and $f_1 := |f|^{1/\theta}$. Since

$$\|f_0\|_{L^\infty} = 1, \quad \|f_1\|_{\mathcal{M}_q^\varphi} = \|f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}}^{1/\theta} < \infty, \quad \text{and} \quad |f_0|^{1-\theta}|f_1|^\theta = |f|,$$

we have $f \in (L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta$ and $\|f\|_{(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta} \le \|f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}}$. Consequently,

$$\mathcal{M}_{q/\theta}^{\varphi^\theta} \hookrightarrow (L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta$$

with embedding constant 1.

Conversely, for $f \in (L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta$ and $\varepsilon > 0$, choose $f_0 \in L^\infty$ and $f_1 \in \mathcal{M}_q^\varphi$ such that

$$|f| \le |f_0|^{1-\theta}|f_1|^\theta \quad \text{and} \quad \|f_0\|_{L^\infty}^{1-\theta}\|f_1\|_{\mathcal{M}_q^\varphi}^\theta \le (1+\varepsilon)\|f\|_{(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta}. \tag{3.36}$$

Let $x \in \mathbb{R}^n$ and $r > 0$. As a consequence of (3.36), we get

$$\begin{aligned}
\frac{\varphi(r)^\theta}{|B(x,r)|^{\frac{\theta}{q}}}\left(\int_{B(x,r)} |f(y)|^{q/\theta}\, dy\right)^{\frac{\theta}{q}} &\le \frac{\varphi(r)^\theta}{|B(x,r)|^{\frac{\theta}{q}}}\left(\int_{B(x,r)} |f_0(y)|^{\frac{q(1-\theta)}{\theta}}|f_1(y)|^q\, dy\right)^{\frac{\theta}{q}} \\
&\le \|f_0\|_{L^\infty}^{1-\theta}\|f_1\|_{\mathcal{M}_q^\varphi}^\theta \\
&\le (1+\varepsilon)\|f\|_{(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta},
\end{aligned}$$

and hence, $f \in \mathcal{M}_{q/\theta}^{\varphi^\theta}$ with $\|f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} \le \|f\|_{(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta}$. Therefore,

$$(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta \hookrightarrow \mathcal{M}_{q/\theta}^{\varphi^\theta}.$$

Thus, (3.35) holds. $\square$

The proof of the first complex interpolation of L^∞ and $\mathcal{M}_q^\varphi$ is given as follows.

Proof of Theorem 3.4.1. We combine Lemmas 2.3.2 and 3.4.2 to obtain

$$[L^\infty, \mathcal{M}_q^\varphi]_\theta = \overline{L^\infty \cap \mathcal{M}_q^\varphi}^{(L^\infty)^{1-\theta}(\mathcal{M}_q^\varphi)^\theta} = \overline{L^\infty \cap \mathcal{M}_q^\varphi}^{\mathcal{M}_{q/\theta}^{\varphi^\theta}}. \tag{3.37}$$

Let $f \in [L^\infty, \mathcal{M}_q^\varphi]_\theta$. As a consequence of (3.37), for each $\varepsilon > 0$, there exists $g = g_\varepsilon \in L^\infty \cap \mathcal{M}_q^\varphi$ such that

$$\|f - g\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} < \frac{\varepsilon}{6}. \tag{3.38}$$

For each $N \in \mathbb{N}$, we have

$$\begin{aligned} |f\chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}| &\le |f\chi_{\{|f|<\frac{1}{N}\}}| + |f\chi_{\{|f|>N\}}| \\ &\le 2|f-g| + |g\chi_{\{|f|<\frac{1}{N}\}\cap\{|g|>\frac{2}{N}\}}| + |g\chi_{\{|g|\le\frac{2}{N}\}}| \\ &\quad + |f\chi_{\{|f|>N\}\cap\{|g|\le\frac{N}{2}\}}| + |g\chi_{\{|g|>\frac{N}{2}\}}|. \end{aligned} \tag{3.39}$$

Observe that, on the set $\{|f| < \frac{1}{N}\} \cap \{|g| > \frac{2}{N}\}$, we have

$$|g| \le |f-g| + |f| < |f-g| + \frac{1}{N} < |f-g| + \frac{|g|}{2}.$$

Therefore,

$$|g\chi_{\{|f|<\frac{1}{N}\}\cap\{|g|>\frac{2}{N}\}}| \le 2|f-g|. \tag{3.40}$$

Meanwhile, on the set $\{|f| > N\} \cap \{|g| \le \frac{N}{2}\}$, we have

$$|f| \le |f-g| + |g| \le |f-g| + \frac{N}{2} < |f-g| + \frac{|f|}{2},$$

and hence,

$$|f\chi_{\{|f|>N\}\cap\{|g|\le\frac{N}{2}\}}| \le 2|f-g|. \tag{3.41}$$

By combining (3.39)-(3.41), for

$$N > 2\max\left\{\|g\|_{L^\infty}, \left(\frac{1+\|g\|_{\mathcal{M}_q^\varphi}^\theta}{\varepsilon}\right)^{\frac{1}{1-\theta}}\right\}, \tag{3.42}$$

we have

$$\begin{aligned} |f\chi_{\{|f|<1/N\}\cup\{|f|>N\}}| &\le 6|f-g| + |g\chi_{\{|g|\le\frac{2}{N}\}}| + |g\chi_{\{|g|>\frac{N}{2}\}}| \\ &\le 6|f-g| + \left(\frac{2}{N}\right)^{1-\theta}|g|^\theta + |g\chi_{\{|g|>\frac{N}{2}\}}| \\ &\le 6|f-g| + \frac{\varepsilon}{1+\|g\|_{\mathcal{M}_q^\varphi}^\theta}|g|^\theta. \end{aligned}$$

We combine the last inequality and (3.38) to obtain

$$\begin{aligned} \|f\chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} &\le 6\|f-g\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} + \frac{\varepsilon}{1+\|g\|_{\mathcal{M}_q^\varphi}^\theta}\||g|^\theta\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} \\ &< \varepsilon + \frac{\varepsilon}{1+\|g\|_{\mathcal{M}_q^\varphi}^\theta}\|g\|_{\mathcal{M}_q^\varphi}^\theta < 2\varepsilon. \end{aligned}$$

This shows that $\lim_{N\to\infty} \|f\chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = 0.$

Conversely, let $f \in \mathcal{M}^{\varphi^\theta}_{q/\theta}$ be such that

$$\lim_{N\to\infty} \|f\chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = 0. \tag{3.43}$$

For every $N \in \mathbb{N}$, define $f_N := f\chi_{\{\frac{1}{N}\le|f|\le N\}}$. Since $f_N \in L^\infty$,

$$\|f_N\|_{\mathcal{M}^\varphi_q} \le (1/N)^{1-\frac{1}{\theta}}\||f|^{1/\theta}\|_{\mathcal{M}^\varphi_q} = N^{\frac{1}{\theta}-1}\|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} < \infty,$$

and

$$\|f - f_N\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = \|f\chi_{\{|f|<\frac{1}{N}\}\cup\{|f|>N\}}\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \to 0$$

as $N \to \infty$, we see that $f \in \overline{L^\infty \cap \mathcal{M}^\varphi_q}^{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = [L^\infty, \mathcal{M}^\varphi_q]_\theta$. □

Similar to Theorem 3.4.1, we also give a description of the second complex interpolation between L^∞ and $\mathcal{M}^\varphi_q$.

Theorem 3.4.3. [30] *Let $\theta \in (0,1)$, $1 \le q < \infty$, and $\varphi \in \mathcal{G}_q$. Then we have*

$$[L^\infty, \mathcal{M}^\varphi_q]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}. \tag{3.44}$$

Proof. Our proof of (3.44) combines (3.34) and Lemma 2.2.4. Let $f \in [L^\infty, \mathcal{M}^\varphi_q]^\theta$ and $\varepsilon > 0$. Then, we can choose $G \in \mathcal{G}(L^\infty, \mathcal{M}^\varphi_q)$ such that

$$G'(\theta) = f \quad \text{and} \quad \|G\|_{\mathcal{G}(L^\infty,\mathcal{M}^\varphi_q)} \le (1+\varepsilon)\|f\|_{[L^\infty,\mathcal{M}^\varphi_q]^\theta}.$$

For every $z \in \overline{S}$ and $k \in \mathbb{N}$, define $H_k(z)$ by (2.3). By virtue of Lemma 2.2.4, we have $H_k(\theta) \in [L^\infty, \mathcal{M}^\varphi_q]_\theta$ with

$$\|H_k(\theta)\|_{[L^\infty,\mathcal{M}^\varphi_q]_\theta} \le (1+\varepsilon)\|f\|_{[L^\infty,\mathcal{M}^\varphi_q]^\theta}. \tag{3.45}$$

Combining (3.45) and (3.34), we get

$$\|H_k(\theta)\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le (1+\varepsilon)\|f\|_{[L^\infty,\mathcal{M}^\varphi_q]^\theta}. \tag{3.46}$$

Since $\lim_{k\to\infty} H_k(\theta) = G'(\theta) = f$ in $L^\infty + \mathcal{M}^\varphi_q$, we can find a subsequence $\{H_{k_j}(\theta)\}_{j=1}^\infty \subseteq \{H_k(\theta)\}_{k=1}^\infty$ such that

$$\lim_{j\to\infty} H_{k_j}(\theta)(x) = f(x) \quad \text{a.e.}$$

By virtue of Fatou's lemma and (3.46), we get

$$\|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \leq \liminf_{j\to\infty} \|H_{k_j}(\theta)\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \leq (1+\varepsilon)\|f\|_{[L^\infty,\mathcal{M}^\varphi_q]^\theta}.$$

Since $\varepsilon > 0$ is arbitrary, we have $\|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \leq \|f\|_{[L^\infty,\mathcal{M}^\varphi_q]^\theta}$.

Conversely, let us assume that $f \in \mathcal{M}^{\varphi^\theta}_{q/\theta}$. For every $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{\frac{z}{\theta}} \quad \text{and} \quad G(z) := (z-\theta)\int_0^1 F(\theta + (z-\theta)t)\ dt. \tag{3.47}$$

Let $F_0(z) := \chi_{\{|f|\leq 1\}}F(z)$, $F_1(z) := F(z) - F_0(z)$, $G_0(z) := \chi_{\{|f|\leq 1\}}G(z)$, and $G_1(z) := G(z) - G_0(z)$. Let $u \in \overline{S}$. Since $\operatorname{Re}(u) \in [0,1]$, we have

$$|F_0(u)| = \chi_{\{|f|\leq 1\}}|f|^{\frac{\operatorname{Re}(u)}{\theta}} \leq 1 \quad \text{and} \quad |F_1(u)| = \chi_{\{|f|>1\}}|f|^{\frac{\operatorname{Re}(u)}{\theta}} \leq |f|^{\frac{1}{\theta}}.$$

Consequently,

$$\|F(u)\|_{L^\infty+\mathcal{M}^\varphi_q} \leq \|F_0(u)\|_{L^\infty} + \|F_1(u)\|_{\mathcal{M}^\varphi_q} \leq 1 + \||f|^{1/\theta}\|_{\mathcal{M}^\varphi_q} = 1 + \|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}, \tag{3.48}$$

$$\|G_0(z)\|_{L^\infty} = \left\|\int_\theta^z F_0(u)\ du\right\|_{L^\infty} \leq |z-\theta| \leq (1+|z|) \tag{3.49}$$

and

$$\|G_1(z)\|_{\mathcal{M}^\varphi_q} = \left\|\int_\theta^z F_1(u)\ du\right\|_{\mathcal{M}^\varphi_q} \leq |z-\theta|\||f|^{\frac{1}{\theta}}\|_{\mathcal{M}^\varphi_q} \leq (1+|z|)\,\|f\|^{\frac{1}{\theta}}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} < \infty. \tag{3.50}$$

This implies $G(z) \in L^\infty + \mathcal{M}^\varphi_q$ and

$$\sup_{z\in\overline{S}} \left\|\frac{G(z)}{1+|z|}\right\|_{L^\infty+\mathcal{M}^\varphi_q} \leq 1 + \|f\|^{\frac{1}{\theta}}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} < \infty. \tag{3.51}$$

Fix $0 < \varepsilon \ll 1$. Let $z \in S$ with $\varepsilon < \operatorname{Re}(z) < 1-\varepsilon$ and $w \in \mathbb{C}$ with $|w| < \frac{\varepsilon}{2}$. Since

$$G(z+w) - G(z) = \int_z^{z+w} F(u)\ du = \frac{F(z+w)-F(z)}{\log(|f|^{1/\theta})},$$

we have

$$\begin{aligned}
\left|\frac{G_0(z+w)-G_0(z)}{w}-F_0(z)\right| &= |F_0(z)|\left|\frac{\exp(w\log(|f|^{1/\theta}))-1}{w\log(|f|^{1/\theta})}-1\right| \\
&= \chi_{\{|f|\le 1\}}|f|^{\frac{\mathrm{Re}(z)}{\theta}}\left|\frac{\exp(w\log(|f|^{1/\theta}))-1}{w\log(|f|^{1/\theta})}-1\right| \\
&\le \chi_{\{|f|\le 1\}}|f|^{\frac{\varepsilon}{\theta}}\left|\frac{\exp(w\log(|f|^{1/\theta}))-1}{w\log(|f|^{1/\theta})}-1\right| \\
&\le \sup_{0<t\le 1} t^{\varepsilon}\left|\frac{\exp(w\log t)-1}{w\log t}-1\right|.
\end{aligned}$$

Observe that, for every $t\in(0,1)$, we have

$$\begin{aligned}
t^{\varepsilon}\left|\frac{\exp(w\log t)-1}{w\log t}-1\right| &= t^{\varepsilon}\left|\sum_{k=2}^{\infty}\frac{(w\log t)^{k-1}}{k!}\right| \\
&\le -t^{\varepsilon}|w|(\log t)\sum_{k=2}^{\infty}\frac{(-|w|\log t)^{k-2}}{(k-2)!} \\
&\le -t^{\varepsilon}|w|(\log t)\exp(-|w|\log t) \\
&\le -t^{\varepsilon}|w|(\log t)\exp(-\frac{\varepsilon}{2}\log t) \\
&= -t^{\frac{\varepsilon}{2}}(\log t)|w| \le \frac{2}{\varepsilon e}|w|.
\end{aligned}$$

Consequently,

$$\left\|\frac{G_0(z+w)-G_0(z)}{w}-F_0(z)\right\|_{L^\infty} \le \frac{2}{\varepsilon e}|w|. \tag{3.52}$$

By a similar argument, we also have

$$\left\|\frac{G_1(z+w)-G_1(z)}{w}-F_1(z)\right\|_{\mathcal{M}_q^{\varphi}} \le \frac{2}{\varepsilon e}|w|\||f|^{\frac{1}{\theta}}\|_{\mathcal{M}_q^{\varphi}} = \frac{2}{\varepsilon e}|w|\|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}. \tag{3.53}$$

Combining (3.52) and (3.53), we get

$$\left\|\frac{G(z+w)-G(z)}{w}-F(z)\right\|_{L^\infty+\mathcal{M}_q^{\varphi}} \le \frac{2}{\varepsilon e}\left(1+\|f\|^{\frac{1}{\theta}}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}\right)|w|\to 0 \tag{3.54}$$

as $w\to 0$. According to (3.48) and (3.54), we have $G: S_\varepsilon \to L^\infty+\mathcal{M}_q^{\varphi}$ is a holomorphic function. Since ε is arbitrary, we conclude that $G: S\to L^\infty+\mathcal{M}_q^{\varphi}$ is holomorphic.

Observe that, for $j=0,1$ and $t_1,t_2\in\mathbb{R}$, we have

$$G(j+it_2)-G(j+it_1) = i\int_{t_1}^{t_2} F(j+it)\ dt. \tag{3.55}$$

Combining (3.55), $|F(it)| = 1$, and $|F(1+it)| = |f|^{\frac{1}{\theta}}$, we have

$$\|G(it_2) - G(it_1)\|_{L^\infty} \leq |t_2 - t_1|$$

and

$$\|G(1+it_2) - G(1+it_1)\|_{\mathcal{M}^{\varphi}_{q}} \leq |t_2 - t_1| \|f\|^{\frac{1}{\theta}}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}},$$

which verify Lipschitz-continuity of the functions $t \in \mathbb{R} \mapsto G(it) - G(0) \in L^\infty$ and $t \in \mathbb{R} \mapsto G(1+it) - G(1) \in \mathcal{M}^{\varphi}_{q}$. Thus, we have $G \in \mathcal{G}(L^\infty, \mathcal{M}^{\varphi}_{q})$. Since $f = F(\theta) = G'(\theta)$, we conclude that $f \in [L^\infty, \mathcal{M}^{\varphi}_{q}]^{\theta}$ as desired. $\square$

Chapter 4

Complex interpolation of some closed subspaces of generalized Morrey spaces

Let $1 \le q < p < \infty$ and define $f(x) := |x|^{-n/p}$. Observe that $f \in \mathcal{M}^p_q$ and for any $R > 0$, we have

$$\|f - \chi_{B(0,R)} f\|_{\mathcal{M}^p_q} = \|f \chi_{B(0,R)}\|_{\mathcal{M}^p_q} = \|f\|_{\mathcal{M}^p_q}. \tag{4.1}$$

This shows the difficulty of approximating functions in the Morrey space $\mathcal{M}^p_q$ by compactly supported functions [50, p. 1744]. Recently, the description of the closure in $\mathcal{M}^p_q$ of L^∞_c is given in [24, Lemma 7]. For the next discussion, we use the following notation:

Definition 4.0.1. Let $1 \le q < \infty$, $\varphi \in \mathcal{G}_q$, and $L^0_{\rm c}$ be the set of compactly supported functions. The spaces $\widetilde{\mathcal{M}}^\varphi_q$, $\overset{*}{\mathcal{M}}{}^\varphi_q$, and $\overline{\mathcal{M}}^\varphi_q$ denote the closure in $\mathcal{M}^\varphi_q$ of $L^\infty_{\rm c}$, $L^0_{\rm c} \cap \mathcal{M}^\varphi_q$, and $L^\infty \cap \mathcal{M}^\varphi_q$, respectively. We also write $\widetilde{L^\infty}$ for the closure of L^∞_c in L^∞. If $\varphi(t) := t^{n/p}$, then we write $\widetilde{\mathcal{M}}^p_q$, $\overset{*}{\mathcal{M}}{}^p_q$, and $\overline{\mathcal{M}}^p_q$ for the corresponding closed subspaces of Morrey spaces.

Our results on the characterization of $\widetilde{\mathcal{M}}^\varphi_q$, $\overset{*}{\mathcal{M}}{}^\varphi_q$, and $\overline{\mathcal{M}}^\varphi_q$ are given as follows:

Theorem 4.0.2. [27, 30] *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Then we have*

$$\widetilde{\mathcal{M}}^\varphi_q = \{f \in \mathcal{M}^\varphi_q : \lim_{R \to \infty} \|\chi_{\{|f|>R\} \cup (\mathbb{R}^n \setminus B(0,R))} f\|_{\mathcal{M}^\varphi_q} = 0\}, \tag{4.2}$$

$$\overset{*}{\mathcal{M}}{}^{\varphi}_{q} = \left\{ f \in \mathcal{M}^{\varphi}_{q} : \lim_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(0,R)} f\|_{\mathcal{M}^{\varphi}_{q}} = 0 \right\}, \tag{4.3}$$

and

$$\overline{\mathcal{M}}{}^{\varphi}_{q} = \left\{ f \in \mathcal{M}^{\varphi}_{q} : \lim_{R\to\infty} \|\chi_{\{|f|>R\}} f\|_{\mathcal{M}^{\varphi}_{q}} = 0 \right\}. \tag{4.4}$$

Remark 4.0.3. Note that, the identity (4.2) for the case $\inf \varphi = 0$ can be seen in [24, Lemma 15]. We also remark that another characterization of $\overset{*}{\mathcal{M}}{}^{p}_{q}$ was given by Yuan et al. in [55, Lemma 2.33]. Meanwhile, the description of the space $\overline{\mathcal{M}}{}^{p}_{q}$ and $\overline{\mathcal{M}}{}^{\varphi}_{q}$ was given in [10, Lemma 3.1] and [27, Lemma 2.6], respectively.

The proof of Theorem 4.0.2 will be given in Section 4.1. By using (4.2), (4.3), and (4.4), we can verify the examples of $\widetilde{\mathcal{M}}{}^{p}_{q} \subsetneq \overset{*}{\mathcal{M}}{}^{p}_{q} \subsetneq \mathcal{M}^{p}_{q}$ and $\overline{\mathcal{M}^{p}_{q}} \subsetneq \mathcal{M}^{p}_{q}$ as follows:

Example 4.0.4. Let $1 \le q < p < \infty$. For $x \in \mathbb{R}^n$, define $f(x) := |x|^{-n/p}$, $g(x) := f(x)\chi_{\mathbb{R}^n\setminus B(0,1)}(x)$, and $h(x) := f(x)\chi_{B(0,1)}(x)$. Then $f \in \mathcal{M}^{p}_{q}\setminus(\overline{\mathcal{M}}{}^{p}_{q} \cup \overset{*}{\mathcal{M}}{}^{p}_{q})$, $g \in \overline{\mathcal{M}}{}^{p}_{q} \setminus \overset{*}{\mathcal{M}}{}^{p}_{q}$, and $h \in \overset{*}{\mathcal{M}}{}^{p}_{q} \setminus \widetilde{\mathcal{M}}{}^{p}_{q}$.

The sets $L^{\infty}_{\rm c}$, $L^{0}_{\rm c}$, and L^{∞} are the model cases of the following closed subspace of Morrey spaces.

Definition 4.0.5. Assume that a linear subspace $U \subset L^0$ enjoys the lattice property: $g \in U$ whenever $f \in U$ and $|g| \le |f|$. For $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$, define

$$U\mathcal{M}^{\varphi}_{q} := \overline{U \cap \mathcal{M}^{\varphi}_{q}}^{\mathcal{M}^{\varphi}_{q}}. \tag{4.5}$$

Taking $U = L^{\infty}_{\rm c}, L^{0}_{\rm c}, L^{\infty}$, we get $U\mathcal{M}^{\varphi}_{q} = \widetilde{\mathcal{M}}{}^{\varphi}_{q}, \overset{*}{\mathcal{M}}{}^{\varphi}_{q}, \overline{\mathcal{M}}{}^{\varphi}_{q}$. Another example of $U = L^0(\Omega)$, namely the set of measurable functions f vanishing outside bounded domain Ω. We also define

$$U \bowtie \mathcal{M}^{\varphi}_{q} := \{ f \in \mathcal{M}^{\varphi}_{q} : \chi_{\{a\le|f|\le b\}} f \in U\mathcal{M}^{\varphi}_{q} \text{ for all } 0 < a < b < \infty \}. \tag{4.6}$$

The first result on the complex interpolation of closed subspaces of Morrey spaces was given by Yang, Yuan, and Zhuo [54]. The authors gave a description of the first complex interpolation space $[\overset{\circ}{\mathcal{M}}{}^{p_0}_{q_0}, \overset{\circ}{\mathcal{M}}{}^{p_1}_{q_1}]_\theta$, where $\overset{\circ}{\mathcal{M}}{}^{p}_{q}$ denotes the closure in $\mathcal{M}^{p}_{q}$. Let us recall their result as follows.

Theorem 4.0.6. *Let $\theta \in (0,1)$, $1 < q_0 \le p_0 < \infty$, and $1 < q_1 \le p_1 < \infty$. Assume that $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Define p and q by*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then

$$[\overset{\circ}{\mathcal{M}}{}^{p_0}_{q_0}, \overset{\circ}{\mathcal{M}}{}^{p_1}_{q_1}]_\theta = [\mathcal{M}^{p_0}_{q_0}, \overset{\circ}{\mathcal{M}}{}^{p_1}_{q_1}]_\theta = [\overset{\circ}{\mathcal{M}}{}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \overset{\circ}{\mathcal{M}}{}^{p}_{q}.$$

In this thesis, we investigate the first and second complex interpolation of closed subspaces of generalized Morrey spaces $U\mathcal{M}^{\varphi_0}_{q_0}$ and $U\mathcal{M}^{\varphi_1}_{q_1}$, where U satisfies the condition in Definition 4.0.5. We shall discuss these results in Sections 4.2 and 4.3. Remark that a description of complex interpolation of the diamond spaces can be seen in [28].

4.1 Closed subspaces of generalized Morrey spaces satisfying the lattice property

Our proof of Theorem 4.0.2 (4.2) utilizes the information about the level sets of the functions in $\widetilde{\mathcal{M}^\varphi_q}$ and $L^\infty_c \cap \mathcal{M}^\varphi_q$. The proof of (4.2) is given as follows:

Proof of Theorem 4.0.2 (4.2). Let $f \in \widetilde{\mathcal{M}^\varphi_q}$ and $\varepsilon > 0$. Choose $g \in L^\infty_c$ such that

$$\|f - g\|_{\mathcal{M}^\varphi_q} < \frac{\varepsilon}{2}.$$

Choose $R_\varepsilon > 0$ such that $R_\varepsilon \ge 2\|g\|_{L^\infty}$ and $\operatorname{supp}(g) \subseteq B(0, R_\varepsilon)$. For every $R > R_\varepsilon$, we have

$$\begin{aligned} |\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f| &\le |f-g| + |\chi_{\{|f|>R\}} g| + |\chi_{\mathbb{R}^n\setminus B(0,R)} g| \\ &\le |f-g| + \chi_{\{|f|>R\}} \frac{R}{2} \\ &\le |f-g| + \chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} \frac{|f|}{2}. \end{aligned} \tag{4.7}$$

Therefore, for every $R > R_\varepsilon$, we have

$$|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f| \le 2|f-g|,$$

and hence

$$\|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f\|_{\mathcal{M}^\varphi_q} \le 2\|f-g\|_{\mathcal{M}^\varphi_q} < \varepsilon.$$

This shows that $\lim_{R\to\infty} \|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f\|_{\mathcal{M}_q^\varphi} = 0$.

Conversely, let $f \in \mathcal{M}_q^\varphi$ be such that

$$\lim_{R\to\infty} \|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f\|_{\mathcal{M}_q^\varphi} = 0.$$

For every $R > 0$, define $f_R := \chi_{\{|f|\le R\}\cap B(0,R)} f$. Note that $|f_R| \le R$ and $\operatorname{supp}(f_R) \subseteq B(0,R)$. Hence, $f_R \in L_{\mathrm{c}}^\infty$. Since

$$\lim_{R\to\infty} \|f - f_R\|_{\mathcal{M}_q^\varphi} = \lim_{R\to\infty} \|\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} f\|_{\mathcal{M}_q^\varphi} = 0$$

and $f_R \in L_{\mathrm{c}}^\infty$, we see that $f \in \widetilde{\mathcal{M}_q^\varphi}$. □

We now give the proof of Theorem 4.0.2 (4.3):

Proof of Theorem 4.0.2 (4.3). Let $f \in \overset{*}{\mathcal{M}}{}_q^\varphi$ and $\varepsilon > 0$. Then, there exists $g_\varepsilon \in L_{\mathrm{c}}^0 \cap \mathcal{M}_q^\varphi$ such that

$$\|f - g_\varepsilon\|_{\mathcal{M}_q^\varphi} < \varepsilon. \tag{4.8}$$

For any $R > 0$, we have

$$|\chi_{\mathbb{R}^n\setminus B(0,R)} f| \le |\chi_{\mathbb{R}^n\setminus B(0,R)} g_\varepsilon| + |\chi_{\mathbb{R}^n\setminus B(0,R)}(f - g_\varepsilon)| \le |\chi_{\mathbb{R}^n\setminus B(0,R)} g_\varepsilon| + |f - g_\varepsilon|.$$

Choose $R_\varepsilon > 0$ such that $\operatorname{supp}(g_\varepsilon) \subset B(0,R_\varepsilon)$. Then, for all $R > R_\varepsilon$, we have

$$|\chi_{\mathbb{R}^n\setminus B(0,R)} f| \le |f - g_\varepsilon|.$$

Consequently, for all $R > R_\varepsilon$, we have

$$\|\chi_{\mathbb{R}^n\setminus B(0,R)} f\|_{\mathcal{M}_q^\varphi} \le \|f - g_\varepsilon\|_{\mathcal{M}_q^\varphi} < \varepsilon.$$

This shows that $\lim_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(0,R)} f\|_{\mathcal{M}_q^\varphi} = 0$.

Conversely, assume that $f \in \mathcal{M}_q^\varphi$ and that

$$\lim_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(0,R)} f\|_{\mathcal{M}_q^\varphi} = 0.$$

For every $R > 0$, define $f_R := \chi_{B(0,R)} f$. Then $f_R \in L_{\mathrm{c}}^0 \cap \mathcal{M}_q^\varphi$, and it follows that

$$\lim_{R\to\infty} \|f - f_R\|_{\mathcal{M}_q^\varphi} = 0,$$

so then $f \in \overset{*}{\mathcal{M}}{}_q^\varphi$.

□

The proof of (4.4) is given as follows.

Proof of Theorem 4.0.2 (4.4). Let $f \in \mathcal{M}_q^\varphi$ be such that $\lim_{R\to\infty} \|\chi_{\{|f|>R\}} f\|_{\mathcal{M}_q^\varphi} = 0$. Define $f_R := \chi_{\{|f|\le R\}} f$ for every $R > 0$. Since $f_R \in L^\infty \cap \mathcal{M}_q^\varphi$ and

$$\|f - f_R\|_{\mathcal{M}_q^\varphi} = \|\chi_{\{|f|>R\}} f\|_{\mathcal{M}_q^\varphi} \to 0$$

as $R \to \infty$, we see that $f \in \overline{\mathcal{M}_q^\varphi}$.

Conversely, let $f \in \overline{\mathcal{M}}_q^\varphi$ and $\varepsilon > 0$. Choose $g \in L^\infty \cap \mathcal{M}_q^\varphi$ be such that

$$\|f - g\|_{\mathcal{M}_q^\varphi} < \frac{\varepsilon}{2}.$$

Let $R_\varepsilon := 2\|g\|_{L^\infty}$. Then, for every $R > R_\varepsilon$, we have

$$\begin{aligned} |\chi_{\{|f|>R\}} f| &\le |\chi_{\{|f|>R\}}(f-g)| + |\chi_{\{|f|>R\}} g| \\ &\le |f-g| + \chi_{\{|f|>R\}} \frac{R}{2} \\ &\le |f-g| + \chi_{\{|f|>R\}} \frac{|f|}{2}, \end{aligned}$$

so $|\chi_{\{|f|>R\}} f| \le 2|f-g|$. Consequently, for every $R > R_\varepsilon$, we have

$$\|\chi_{\{|f|>R\}} f\|_{\mathcal{M}_q^\varphi} \le 2\|f-g\|_{\mathcal{M}_q^\varphi} < \varepsilon.$$

This shows that $\lim_{R\to\infty} \|\chi_{\{|f|>R\}} f\|_{\mathcal{M}_q^\varphi} = 0$, as desired. □

As a corollary of Theorem 4.0.2, we show that $\widetilde{\mathcal{M}}_q^\varphi$ is the intersection of $\overset{*}{\mathcal{M}}_q^\varphi$ and $\overline{\mathcal{M}}_q^\varphi$.

Corollary 4.1.1. [30] *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Then, $\widetilde{\mathcal{M}}_q^\varphi = \overset{*}{\mathcal{M}}_q^\varphi \cap \overline{\mathcal{M}}_q^\varphi$.*

Proof. The inclusion $\widetilde{\mathcal{M}}_q^\varphi \subseteq \overset{*}{\mathcal{M}}_q^\varphi \cap \overline{\mathcal{M}}_q^\varphi$ follows from $\widetilde{\mathcal{M}}_q^\varphi \subseteq \overset{*}{\mathcal{M}}_q^\varphi$ and $\widetilde{\mathcal{M}}_q^\varphi \subseteq \overline{\mathcal{M}}_q^\varphi$. Conversely, let $f \in \overset{*}{\mathcal{M}}_q^\varphi \cap \overline{\mathcal{M}}_q^\varphi$. Define $A_R := \{|f| > R\} \cup (\mathbb{R}^n \setminus B(0,R))$. Then

$$\|\chi_{A_R} f\|_{\mathcal{M}_q^\varphi} \le \|\chi_{\mathbb{R}^n \setminus B(0,R)} f\|_{\mathcal{M}_q^\varphi} + \|\chi_{\{|f|>R\}} f\|_{\mathcal{M}_q^\varphi}. \tag{4.9}$$

Since $f \in \overset{*}{\mathcal{M}}_q^\varphi$ and $f \in \overline{\mathcal{M}}_q^\varphi$, by combining Theorem 4.0.2 and (4.9), we have

$$\lim_{R\to\infty} \|\chi_{A_R} f\|_{\mathcal{M}_q^\varphi} = 0, \tag{4.10}$$

and hence, $f \in \widetilde{\mathcal{M}}_q^\varphi$. This shows that $\overset{*}{\mathcal{M}}_q^\varphi \cap \overline{\mathcal{M}}_q^\varphi \subseteq \widetilde{\mathcal{M}}_q^\varphi$. □

4.2 The first complex interpolation of some closed subspaces of generalized Morrey spaces

We obtain the following description of $[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta$.

Theorem 4.2.1. [27] *Let $\theta \in (0,1)$. Suppose that the parameters $1 \le q_0 < \infty$, $\varphi_0 \in \mathcal{G}_{q_0}$, $1 \le q_1 < \infty$, $\varphi_1 \in \mathcal{G}_{q_1}$ satisfy $q_0 \ne q_1$ and $\varphi_0^{q_0} = \varphi_1^{q_1}$. Define*

$$\varphi := \varphi_0^{1-\theta}\varphi_1^{\theta} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then we have

$$\begin{aligned}[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta &= U\mathcal{M}^{\varphi}_{q} \cap [\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]_\theta \\ &= \left\{ f \in U\mathcal{M}^{\varphi}_{q} : \lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}^{\varphi}_{q}} = 0 \right\}.\end{aligned}$$

As a special case of Theorem 4.2.1, we have the following corollary:

Corollary 4.2.2. [27] *Suppose that $\theta \in (0,1)$, $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, and $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. Define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then we have

$$[\widetilde{\mathcal{M}}^{p_0}_{q_0}, \widetilde{\mathcal{M}}^{p_1}_{q_1}]_\theta = \widetilde{\mathcal{M}}^{p}_{q} = [\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}, \overset{*}{\mathcal{M}}{}^{p_1}_{q_1}]_\theta.$$

In order to prove Theorem 4.2.1, we need to prove the following lemmas:

Lemma 4.2.3. [24, Lemma 4.2] *Assume the same paramaters as in Theorem 4.2.1. Let E be a measurable set such that $\chi_E \in U\mathcal{M}^{\varphi}_{q}$. Then*

$$\chi_E \in U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}.$$

Proof. Let $\chi_E \in U\mathcal{M}^{\varphi}_{q}$ and choose $\{g_k\}_{k=1}^\infty \subseteq U \cap \mathcal{M}^{\varphi}_{q}$ for which

$$\lim_{k\to\infty} \|\chi_E - g_k\|_{\mathcal{M}^{\varphi}_{q}} = 0.$$

Define $h_k := \chi_{\{g_k \ne 0\} \cap E}$. Then, for each $k = 0, 1$, we have

$$\|\chi_E - h_k\|_{\mathcal{M}^{\varphi_j}_{q_j}} = \|\chi_E - h_k\|^{q/q_j}_{\mathcal{M}^{\varphi}_{q}} \le \|\chi_E - g_k\|^{q/q_j}_{\mathcal{M}^{\varphi}_{q}} \to 0$$

as $k \to \infty$. Thus, $\chi_E \in U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}$. □

Lemma 4.2.4. [24, Lemma 4.1] *Assume the same paramaters as in Theorem 4.2.1. Then* $U\mathcal{M}_{q_0}^{\varphi_0} \cap U\mathcal{M}_{q_1}^{\varphi_1} \subseteq U\mathcal{M}_q^{\varphi}$.

Proof. Without loss of generality assume that $q_1 > q_0$. Let $f \in U\mathcal{M}_{q_0}^{\varphi_0} \cap U\mathcal{M}_{q_1}^{\varphi_1}$. In view of Lemma 3.2.6, we may assume $f = \chi_{\{1/N \le |f| \le N\}} f$ for some $N \in \mathbb{N}$. By the lattice property of the spaces $U\mathcal{M}_{q_0}^{\varphi_0}$, $U\mathcal{M}_{q_1}^{\varphi_1}$ and $U\mathcal{M}_q^{\varphi}$, we may assume $f = \chi_E$ for some measurable set E. Choose a sequence $\{g_j\}_{j=1}^{\infty} \subseteq U \cap \mathcal{M}_{q_1}^{\varphi_1}$ such that

$$\lim_{j\to\infty} \|f - g_j\|_{\mathcal{M}_{q_1}^{\varphi_1}} = 0.$$

Define $F_j := \{g_j \neq 0\} \cap E$. Hence $|f - \chi_{F_j}| \le 2$ and $|f - \chi_{F_j}| \le |f - g_j|$. Consequently,

$$\|f - \chi_{F_j}\|_{\mathcal{M}_q^{\varphi}} = \left\| \, |f - \chi_{F_j}|^{1-\frac{q_1}{q}} |f - \chi_{F_j}|^{\frac{q_1}{q}} \, \right\|_{\mathcal{M}_q^{\varphi}} \le 2^{1-\frac{q_1}{q}} \|f - g_j\|_{\mathcal{M}_{q_1}^{\varphi_1}}^{\frac{q_1}{q}}.$$

This shows that $f \in U\mathcal{M}_q^{\varphi}$. □

The proof of Theorem 4.2.1 is given as follows:

Proof of Theorem 4.2.1. We assume that $q_1 > q_0$. By using Lemma 4.2.4, the inclusions $[U\mathcal{M}_{q_0}^{\varphi_0}, U\mathcal{M}_{q_1}^{\varphi_1}]_\theta \subseteq [\mathcal{M}_{q_0}^{\varphi_0}, \mathcal{M}_{q_1}^{\varphi_1}]_\theta \subseteq \mathcal{M}_q^p$, and the fact that $X_0 \cap X_1$ is a dense subset of $[X_0, X_1]_\theta$, we have $[U\mathcal{M}_{q_0}^{\varphi_0}, U\mathcal{M}_{q_1}^{\varphi_1}]_\theta \subseteq U\mathcal{M}_q^p$. Consequently,

$$[U\mathcal{M}_{q_0}^{\varphi_0}, U\mathcal{M}_{q_1}^{\varphi_1}]_\theta \subseteq U\mathcal{M}_q^p \cap [\mathcal{M}_{q_0}^{\varphi_0}, \mathcal{M}_{q_1}^{\varphi_1}]_\theta.$$

Conversely, let $f \in U\mathcal{M}_q^p \cap [\mathcal{M}_{q_0}^{\varphi_0}, \mathcal{M}_{q_1}^{\varphi_1}]_\theta$. Note that, for any $0 < b < c < \infty$, we have a pointwise estimate:

$$\chi_{\{b \le |f| \le c\}} \le \frac{1}{b}\chi_{\{b \le |f| \le c\}}|f| \le \frac{|f|}{b}, \tag{4.11}$$

so $\chi_{\{b \le |f| \le c\}} \in U\mathcal{M}_q^p$. From Lemma 4.2.3, it follows that $\chi_{\{b \le |f| \le c\}} \in U\mathcal{M}_{q_0}^{\varphi_0} \cap U\mathcal{M}_{q_1}^{\varphi_1}$. For every $N \in \mathbb{N}$ and $z \in \overline{S}$, define

$$F_N(z) = \operatorname{sgn}(f)|f|^{q\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right)} \chi_{\{\frac{1}{N} \le |f| \le N\}}.$$

Decompose $F_N(z) := F_{N,0}(z) + F_{N,1}(z)$ where $F_{N,0}(z) := F_N(z)\chi_{\{|f| \le 1\}}$. Since

$$|F_{N,0}(z)| \le \chi_{\{\frac{1}{N} \le |f| \le 1\}} \quad \text{and} \quad |F_{N,1}(z)| \le \left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}}\right) \chi_{\{1 \le |f| \le N\}},$$

we have $F_N(z) = F_{N,0}(z) + F_{N,1}(z) \in U\mathcal{M}_{q_0}^{\varphi_0} + U\mathcal{M}_{q_1}^{\varphi_1}$. Moreover, we also have

$$\sup_{z \in \overline{S}} \|F_N(z)\|_{U\mathcal{M}_{q_0}^{\varphi_0} + U\mathcal{M}_{q_1}^{\varphi_1}} \le \|\chi_{\{\frac{1}{N} \le |f| \le 1\}}\|_{U\mathcal{M}_{q_0}^{\varphi_0}} + \left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}}\right) \|\chi_{\{1 \le |f| \le N\}}\|_{U\mathcal{M}_{q_1}^{\varphi_1}}.$$

Observe that for every $w \in \overline{S}$, we have

$$|F_N'(w)| \le \left(\frac{q}{q_0} - \frac{q}{q_1}\right)\left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}}\right)(\log N) \times \chi_{\{\frac{1}{N} \le |f| \le N\}}. \tag{4.12}$$

Then we have

$$\begin{aligned}
&\|F_N(z) - F_N(z')\|_{U\mathcal{M}^{\varphi_0}_{q_0} + U\mathcal{M}^{\varphi_1}_{q_1}} \\
&= \left\| \int_{z'}^{z} F_N'(w)\, dw \right\|_{U\mathcal{M}^{\varphi_0}_{q_0} + U\mathcal{M}^{\varphi_1}_{q_1}} \\
&\le \left(\frac{q}{q_0} - \frac{q}{q_1}\right)\left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}}\right)(\log N) \times \left(\|\chi_{\{\frac{1}{N} \le |f| \le N\}}\|_{U\mathcal{M}^{\varphi_0}_{q_0} + U\mathcal{M}^{\varphi_1}_{q_1}}\right)|z - z'| \\
&\le \left(\frac{q}{q_0} - \frac{q}{q_1}\right)\left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}}\right)\log N \\
&\quad \times \left(\|\chi_{\{\frac{1}{N} \le |f| \le 1\}}\|_{U\mathcal{M}^{\varphi_0}_{q_0}} + \|\chi_{\{1 < |f| \le N\}}\|_{U\mathcal{M}^{\varphi_1}_{q_1}}\right)|z - z'|
\end{aligned}$$

for all $z, z' \in \overline{S}$. Thus, $F_N : \overline{S} \to U\mathcal{M}^{\varphi_0}_{q_0} + U\mathcal{M}^{\varphi_1}_{q_1}$ is a continuous function. Likewise we can check that $F_N|_S : S \to U\mathcal{M}^{\varphi_0}_{q_0} + U\mathcal{M}^{\varphi_1}_{q_1}$ is a holomorphic function. Note that, for all $t \in \mathbb{R}$ and $j = 0, 1$, we have

$$|F_N(j + it)| = |f|^{\frac{q}{q_j}} \chi_{\{\frac{1}{N} \le |f| \le N\}} \le N^{\frac{q}{q_j}} \chi_{\{\frac{1}{N} \le |f| \le N\}},$$

so, $F_N(j + it) \in U\mathcal{M}^{\varphi_j}_{q_j}$. Furthermore, by using (4.12), we get

$$\begin{aligned}
\|F_N(j + it) - F_N(j + it')\|_{U\mathcal{M}^{\varphi_j}_{q_j}} &= \left\| \int_{j+it'}^{j+it} F_N'(w)\, dw \right\|_{U\mathcal{M}^{\varphi_j}_{q_j}} \\
&\le \left(\frac{q}{q_0} - \frac{q}{q_1}\right)\left(N^{\frac{q}{q_0}} + N^{\frac{q}{q_1}}\right)\log N \\
&\quad \times \|\chi_{\{1/N \le |f| \le N\}}\|_{U\mathcal{M}^{\varphi_j}_{q_j}} |t - t'|
\end{aligned}$$

for all $t, t' \in \mathbb{R}$. This shows that $t \in \mathbb{R} \mapsto F_N(j + it) \in U\mathcal{M}^{\varphi_j}_{q_j}$ are continuous functions. In total, we have showed that $F_N \in \mathcal{F}(U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1})$. Note that, for $M, N \in \mathbb{N}$ with $N < M$, we have

$$\begin{aligned}
\|F_M(\theta) - F_N(\theta)\|_{[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta} &\le \|F_M - F_N\|_{\mathcal{F}(U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1})} \\
&= \max_{j=0,1} \sup_{t \in \mathbb{R}} \|F_M(j + it) - F_N(j + it)\|_{U\mathcal{M}^{\varphi_j}_{q_j}} \\
&= \max_{j=0,1} \sup_{t \in \mathbb{R}} \left\| |f|^{q/q_j} \chi_{\{\frac{1}{M} \le |f| \le \frac{1}{N}\} \cup \{N \le |f| \le M\}} \right\|_{\mathcal{M}^{\varphi_j}_{q_j}} \\
&= \max_{j=0,1} \|f \chi_{\{\frac{1}{M} \le |f| \le \frac{1}{N}\} \cup \{N \le |f| \le M\}}\|_{\mathcal{M}^{\varphi}_{q}}^{q/q_j} \\
&\le \max_{j=0,1} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}^{\varphi}_{q}}^{q/q_j}.
\end{aligned}$$

Since $\lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N}\le|f|\le N\}} f\|_{\mathcal{M}^p_q} = 0$, we see that

$$\|F_M(\theta) - F_N(\theta)\|_{[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta} \to 0$$

whenever $M, N \to \infty$. Thus, $F_N(\theta)$ converges to $g \in [U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta$. Hence, $\lim_{N\to\infty} F_N(\theta) = g$ in $\mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$. Meanwhile, by combining $\mathcal{M}^{\varphi}_{q} \subseteq \mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$ and

$$\lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N}\le|f|\le N\}} f\|_{\mathcal{M}^\varphi_q} = 0,$$

we have $\lim_{N\to\infty} F_N(\theta) = f$ in $\mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$, which implies $f = g$. Thus, $f \in [U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta$ as desired. □

4.3 The second complex interpolation of some closed subspaces of generalized Morrey spaces

Our main result in this section is the following theorem.

Theorem 4.3.1. [27] *Suppose that $\theta \in (0,1)$, $1 \le q_0 < \infty$, $1 \le q_1 < \infty$, and $\varphi_0^{q_0} = \varphi_1^{q_1}$. Define*

$$\varphi := \varphi_0^{1-\theta}\varphi_1^{\theta} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then we have

$$[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]^\theta = U \bowtie \mathcal{M}^{\varphi}_{q}. \tag{4.13}$$

As a special case of Theorem 4.3.1, we have the following results:

Corollary 4.3.2. [27, Theorems 5.2 and 5.12] *Suppose that $\theta \in (0,1)$, $1 \le q_0 < \infty$, $1 \le q_1 < \infty$, and $\varphi_0^{q_0} = \varphi_1^{q_1}$. Define $\varphi := \varphi_0^{1-\theta}\varphi_1^{\theta}$ and $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then, the description of the second interpolation functor of these closed subspaces is as follows:*

$$[\widetilde{\mathcal{M}^{\varphi_0}_{q_0}}, \widetilde{\mathcal{M}^{\varphi_1}_{q_1}}]^\theta = [\overset{*}{\mathcal{M}}{}^{\varphi_0}_{q_0}, \overset{*}{\mathcal{M}}{}^{\varphi_1}_{q_1}]^\theta = \bigcap_{0<a<b<\infty} \{f \in \mathcal{M}^\varphi_q : \chi_{\{a\le|f|\le b\}} f \in \widetilde{\mathcal{M}}^{\varphi}_{q}\}, \tag{4.14}$$

and

$$[\overline{\mathcal{M}}^{\varphi_0}_{q_0}, \overline{\mathcal{M}}^{\varphi_1}_{q_1}]^\theta = \mathcal{M}^{\varphi}_{q}. \tag{4.15}$$

From now on, we shall always use the assumption of Theorem 4.3.1. To prove Theorem 4.3.1, we shall invoke and prove several lemmas.

Lemma 4.3.3. [27] *Keep the assumption in Theorem 4.3.1. Then*

$$U \bowtie \mathcal{M}_q^{\varphi} \subseteq [U\mathcal{M}_{q_0}^{\varphi_0}, U\mathcal{M}_{q_1}^{\varphi_1}]^{\theta}. \tag{4.16}$$

Proof. Without loss of generality, assume that $q_0 > q_1$. Let $f \in U \bowtie \mathcal{M}_q^p$. Since $\chi_{\{a \le |f| \le b\}} \le \frac{1}{a}\chi_{\{a \le |f| \le b\}}|f|$, we have $\chi_{\{a \le |f| \le b\}} \in U\mathcal{M}_q^p$. From Lemma 4.2.3, we have $\chi_{\{a \le |f| \le b\}} \in U\mathcal{M}_{q_0}^{p_0} \cap U\mathcal{M}_{q_1}^{p_1}$. For $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{\frac{qz}{q_0} + \frac{q(1-z)}{q_1}} \text{ and } G(z) := (z-\theta)\int_0^1 F(\theta + (z-\theta)t)\, dt. \tag{4.17}$$

Decompose $G(z) = G_0(z) + G_1(z)$ where $G_0(z) := \chi_{\{|f| \le 1\}}G(z)$. Let $0 < \varepsilon < 1$. Since $\chi_{\{\varepsilon \le |f| \le 1\}} \in U\mathcal{M}_{q_0}^{p_0}$ and

$$\chi_{\{\varepsilon \le |f| \le 1\}}|G_0(z)| \le (1+|z|)(|f|^{q/q_0} + |f|^{q/q_1})\chi_{\{\varepsilon \le |f| \le 1\}} \le 2(1+|z|)\chi_{\{\varepsilon \le |f| \le 1\}}, \tag{4.18}$$

we have $\chi_{\{\varepsilon \le |f| \le 1\}}G_0(z) \in U\mathcal{M}_{q_0}^{p_0}$. Observe that

$$\begin{aligned} \|G_0(z) - \chi_{\{\varepsilon \le |f| \le 1\}}G_0(z)\|_{\mathcal{M}_{q_0}^{p_0}} &= \left\| \chi_{\{|f| \le \varepsilon\}} \frac{F(z) - F(\theta)}{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\log|f|} \right\|_{\mathcal{M}_{q_0}^{\varphi_0}} \\ &\le \left\| \frac{2|f|^{q/q_0}}{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\log(\varepsilon^{-1})} \right\|_{\mathcal{M}_{q_0}^{\varphi_0}} \\ &= \frac{2\|f\|_{\mathcal{M}_q^p}^{q/q_0}}{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\log \varepsilon^{-1}} \to 0 \end{aligned} \tag{4.19}$$

as $\varepsilon \to 0^+$. Hence $G_0(z) \in U\mathcal{M}_{q_0}^{p_0}$. Similarly, $G_1(z) \in U\mathcal{M}_{q_1}^{p_1}$. Thus $G(z) \in U\mathcal{M}_{q_0}^{p_0} + U\mathcal{M}_{q_1}^{p_1}$. Let $t \in \mathbb{R}$ and $R > 1$. Since $\chi_{\{R^{-1} \le |f| \le R\}} \in U\mathcal{M}_{q_0}^{p_0}$ and

$$|(G(it) - G(0))|\chi_{\{R^{-1} \le |f| \le R\}} \le (2+|t|)(R^{q/q_0} + R^{q/q_1})\chi_{\{R^{-1} \le |f| \le R\}}, \tag{4.20}$$

we have $[G(it) - G(0)]\chi_{\{R^{-1} \le |f| \le R\}} \in U\mathcal{M}_{q_0}^{p_0}$. Note that

$$\|[G(it) - G(0)]\chi_{\mathbb{R}^n \setminus \{R^{-1} \le |f| \le R\}}\|_{\mathcal{M}_{q_0}^{p_0}} \le \frac{2\|f\|_{\mathcal{M}_q^p}^{q/q_0}}{\left(\frac{q}{q_1} - \frac{q}{q_0}\right)\log R} \to 0 \tag{4.21}$$

as $R \to \infty$. Thus $G(it) - G(0) \in U\mathcal{M}_{q_0}^{p_0}$. Similarly, $G(1+it) - G(1) \in U\mathcal{M}_{q_1}^{p_1}$. Since $G \in \mathcal{G}(\mathcal{M}_{q_0}^{p_0}, \mathcal{M}_{q_1}^{p_1})$, we have $G \in \mathcal{G}(U\mathcal{M}_{q_0}^{p_0}, U\mathcal{M}_{q_1}^{p_1})$. From $f = G'(\theta)$, it follows that $f \in [U\mathcal{M}_{q_0}^{p_0}, U\mathcal{M}_{q_1}^{p_1}]^{\theta}$. □

Lemma 4.3.4. [27] *Let* $G \in \mathcal{G}(\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1})$ *and* $\theta \in (0,1)$*. For* $z \in \overline{S}$ *and* $k \in \mathbb{N}$*, define* $H_k(z)$ *by* (2.3)*. Then* $H_k(\theta) \in \overline{U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}}^{\mathcal{M}^{\varphi}_q}$.

Proof. It follows from Lemma 2.2.4, that $H_k(\theta) \in [U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta$. Let $\varepsilon > 0$. Since $U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}$ is dense in $[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta$, we can find $J_k(\theta) \in U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}$ such that

$$\|H_k(\theta) - J_k(\theta)\|_{[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta} < \varepsilon.$$

Since $[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta \subseteq [\mathcal{M}^{\varphi_0}_{q_0}, \mathcal{M}^{\varphi_1}_{q_1}]_\theta \subseteq \mathcal{M}^{\varphi}_q$, we have

$$\|H_k(\theta) - J_k(\theta)\|_{\mathcal{M}^{\varphi}_q} \lesssim \|H_k(\theta) - J_k(\theta)\|_{[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]_\theta} < \varepsilon.$$

This shows that $H_k(\theta) \in \overline{U\mathcal{M}^{\varphi_0}_{q_0} \cap U\mathcal{M}^{\varphi_1}_{q_1}}^{\mathcal{M}^{\varphi}_q}$. □

Lemma 4.3.5. [27] *We use the assumption of Theorem 4.3.1. Then we have*

$$\mathcal{M}^{\varphi}_q \cap \overline{U\mathcal{M}^{\varphi}_q}^{\mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}} \subseteq U \bowtie \mathcal{M}^{\varphi}_q.$$

Proof. Let $f \in \mathcal{M}^{\varphi}_q \cap \overline{U\mathcal{M}^{\varphi}_q}^{\mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}}$. Choose $\{f_j\}_{j=1}^\infty \subseteq U\mathcal{M}^{\varphi}_q$ such that

$$\lim_{j\to\infty} \|f - f_j\|_{\mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}} = 0.$$

Then, we can find $\{k_j\}_{j=1}^\infty \subset \widetilde{\mathcal{M}^{p_0}_{q_0}}$ and $\{h_j\}_{j=1}^\infty \subset \widetilde{\mathcal{M}^{p_1}_{q_1}}$ convergent to 0 in $\widetilde{\mathcal{M}^{p_0}_{q_0}}$ and $\widetilde{\mathcal{M}^{p_1}_{q_1}}$, respectively, such that $f - f_j = k_j + h_j$ for all j. Assume $0 < a < 1 < b < \infty$ as before. Let $\Theta \in C_c(\mathbb{R})$ be a piecewise linear function defined by

$$\Theta'(t) := \frac{2}{a}\chi_{(a/2,a)}(t) - \frac{1}{b}\chi_{(b,2b)}(t) \tag{4.22}$$

except at $t = \frac{a}{2}, a, b, 2b$. Let $C_{a,b} = \frac{2}{a} + \frac{1}{b}$. Since

$$|\Theta(t) - \Theta(s)| \le C_{a,b}|t - s| \text{ and } |\Theta(t) - \Theta(s)| \le 2,$$

we have

$$|\Theta(|f|) - \Theta(|f_j|)| \le C_{a,b}\min(1, ||f| - |f_j||) \le C_{a,b}\min(1, |f - f_j|).$$

Let $B = B(x_0, r)$ be any ball in $\mathbb{R}^n$. Then,

$$\int_B \chi_{[a,b]}(|f(x)|)|\Theta(|f(x)|) - \Theta(|f_j(x)|)|^q \, dx \lesssim \int_B \chi_{[a,b]}(|f(x)|)\min(1, |f(x) - f_j(x)|^q)\, dx.$$

By using the decomposition $f = f_j + k_j + h_j$, we obtain

$$\begin{aligned}
&\int_B \chi_{[a,b]}(|f(x)|)|\Theta(|f(x)|) - \Theta(|f_j(x)|)|^q \, dx \\
&\lesssim \int_B \chi_{[a,b]}(|f(x)|) \min(1, |k_j(x)|^q) \, dx + \int_B \chi_{[a,b]}(|f(x)|) \min(1, |h_j(x)|^q) \, dx.
\end{aligned}$$

By keeping in mind, $q_0 > q > q_1$ and $\frac{q_0}{p_0} = \frac{q_1}{p_1} = \frac{q}{p}$, we obtain

$$\begin{aligned}
&\frac{\varphi(r)^q}{|B|} \int_B \chi_{[a,b]}(|f(x)|)|\Theta(|f(x)|) - \Theta(|f_j(x)|)|^q \, dx \\
&\lesssim \frac{\varphi(r)^q}{|B|} \int_B |h_j(x)|^{q_1} \, dx \\
&\quad + \frac{\varphi(r)^q}{|B|} \left(\int_B \chi_{[a,b]}(|f(x)|) \, dx \right)^{1-\frac{q}{q_0}} \left(\int_B \min(1, |k_j(x)|^{q_0}) \, dx \right)^{\frac{q}{q_0}} \\
&\lesssim (\|h_j\|_{\mathcal{M}^{\varphi_1}_{q_1}})^{q_1} + \left(\frac{\varphi(r)^q}{|B|} \int_B |f(x)|^q \, dx \right)^{1-\frac{q}{q_0}} \left(\|k_j\|_{\mathcal{M}^{\varphi_0}_{q_0}} \right)^q \\
&\lesssim (\|h_j\|_{\mathcal{M}^{\varphi_1}_{q_1}})^{q_1} + \left(\frac{\varphi(r)^q}{|B|} \int_B |f(x)|^q \, dx \right)^{1-\frac{q}{q_0}} \left(\|k_j\|_{\mathcal{M}^{\varphi_0}_{q_0}} \right)^q \\
&\lesssim (\|h_j\|_{\mathcal{M}^{\varphi_1}_{q_1}})^{q_1} + \left(\|f\|_{\mathcal{M}^{\varphi}_{q}} \right)^{q-\frac{q^2}{q_0}} \left(\|k_j\|_{\mathcal{M}^{\varphi_0}_{q_0}} \right)^q .
\end{aligned}$$

Thus, it follows that

$$\lim_{j\to\infty} \|\chi_{\{a\le|f|\le b\}}\Theta(|f_j|) - \chi_{\{a\le|f|\le b\}}\Theta(|f|)\|_{\mathcal{M}^{\varphi}_{q}} = 0.$$

Since $\chi_{\{a\le|f|\le b\}}\Theta(|f_j|) \le a^{-1}|f_j|$, we have $\chi_{\{a\le|f|\le b\}}\Theta(|f|) \in U\mathcal{M}^{\varphi}_{q}$. From the equality

$$\chi_{\{a\le|f|\le b\}}|f| = b\chi_{\{a\le|f|\le b\}}\Theta(|f|),$$

it follows that $\chi_{\{a\le|f|\le b\}} f \in U\mathcal{M}^{\varphi}_{q}$. □

Now, we are ready to prove Theorem 4.3.1.

Proof of (4.13). In view of Lemma 4.3.3, we only need to show that

$$[U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]^{\theta} \subseteq U \bowtie \mathcal{M}^{\varphi}_{q}.$$

Let $f \in [U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1}]^{\theta}$. Then there exists $G \in \mathcal{G}(U\mathcal{M}^{\varphi_0}_{q_0}, U\mathcal{M}^{\varphi_1}_{q_1})$ such that $G'(\theta) = f$. For $z \in \overline{S}$ and $k \in \mathbb{N}$, define $H_k(z)$ by (2.3). By virtue of Lemmas 4.2.4 and 4.3.4, we have $H_k(\theta) \in U\mathcal{M}^{\varphi}_{q}$. Since $H_k(\theta)$ converges to $G'(\theta) = f$ in $\mathcal{M}^{\varphi_0}_{q_0} + \mathcal{M}^{\varphi_1}_{q_1}$, by Lemma 4.3.5, it follows that $f \in U \bowtie \mathcal{M}^{\varphi}_{q}$. □

4.4 Complex interpolation between L^∞ and some closed subspaces of Morrey spaces

We also consider the complex interpolation between L^∞ and each of the spaces $\widetilde{\mathcal{M}}^\varphi_q$, $\overset{*}{\mathcal{M}}{}^\varphi_q$, and $\overline{\mathcal{M}}^\varphi_q$. First, we prove the following lemma.

Lemma 4.4.1. [30] *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Then, for every $f \in L^\infty \cap \widetilde{\mathcal{M}}^\varphi_q$, we have*

$$\|f\|_{[L^\infty, \widetilde{\mathcal{M}}^\varphi_q]^\theta} \sim \|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}.$$

Proof. Since $[L^\infty, \widetilde{\mathcal{M}}^\varphi_q]^\theta \subseteq [L^\infty, \mathcal{M}^\varphi_q]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}$, we have

$$\|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \lesssim \|f\|_{[L^\infty, \widetilde{\mathcal{M}}^\varphi_q]^\theta}.$$

Assume that $\|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = 1$. For every $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{\frac{z}{\theta}},\ G(z) := \int_\theta^z F(u)\ du, \text{ and } G_1(z) := \chi_{\{|f|>1\}} G(z).$$

For every $u \in \overline{S}$, we have

$$|\chi_{\{|f|>1\}} F(u)| = \chi_{\{|f|>1\}} |f|^{\frac{\mathrm{Re} u}{\theta}} \le |f|^{\frac{1}{\theta}} \le \|f\|_{L^\infty}^{\frac{1}{\theta}-1} |f|,$$

so $|G_1(z)| \le (1+|z|)\|f\|_{L^\infty}^{\frac{1-\theta}{\theta}} |f|$. Since $f \in \widetilde{\mathcal{M}}^\varphi_q$, we see that $G_1(z) \in \widetilde{\mathcal{M}}^\varphi_q$. Let $t_1, t_2 \in \mathbb{R}$. Since $f \in L^\infty \cap \widetilde{\mathcal{M}}^\varphi_q$ and

$$|G(1+it_2) - G(1+it_1)| = \left| i \int_{t_1}^{t_2} F(1+it)\ dt \right| \le |t_2 - t_1| |f|^{1/\theta} \le |t_2 - t_1| \|f\|_{L^\infty}^{\frac{1-\theta}{\theta}} |f|,$$

we have $G(1+it_2) - G(1+it_1) \in \widetilde{\mathcal{M}}^\varphi_q$. Combining $G_1(z) \in \widetilde{\mathcal{M}}^\varphi_q$, $G(1+it_2) - G(1+it_1) \in \widetilde{\mathcal{M}}^\varphi_q$, and $G \in \mathcal{G}(L^\infty, \mathcal{M}^\varphi_q)$, we have $G \in \mathcal{G}(L^\infty, \widetilde{\mathcal{M}}^\varphi_q)$. Moreover,

$$\begin{aligned}
\|G\|_{\mathcal{G}(L^\infty, \widetilde{\mathcal{M}}^\varphi_q)} &= \max\left(\sup_{t<s} \left\| \frac{G(it) - G(is)}{t-s} \right\|_{L^\infty}, \sup_{t<s} \left\| \frac{G(1+it) - G(1+is)}{t-s} \right\|_{\widetilde{\mathcal{M}}^\varphi_q} \right) \\
&= \max\left(\sup_{t<s} \left\| \frac{G(it) - G(is)}{t-s} \right\|_{L^\infty}, \sup_{t<s} \left\| \frac{G(1+it) - G(1+is)}{t-s} \right\|_{\mathcal{M}^\varphi_q} \right) \\
&\le \max(1, \||f|^{1/\theta}\|_{\mathcal{M}^\varphi_q}) \\
&= \max(1, \|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}) = 1 = \|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}.
\end{aligned}$$

Since $f = G'(\theta)$, we have

$$\|f\|_{[L^\infty, \mathcal{M}_q^\varphi]^\theta} \le \|G\|_{\mathcal{G}(L^\infty, \widetilde{\mathcal{M}_q^\varphi})} \le \|f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}},$$

as desired. $\square$

One of our main results in this section is the following theorem:

Theorem 4.4.2. [30] *Let $\theta \in (0,1)$, $1 \le q < \infty$, and $\varphi \in \mathcal{G}_q$. Then we have*

$$[L^\infty, \widetilde{\mathcal{M}_q^\varphi}]_\theta = [L^\infty, \overset{*}{\mathcal{M}}{}_q^\varphi]_\theta = \widetilde{\mathcal{M}_{q/\theta}^{\varphi^\theta}}. \tag{4.23}$$

Proof. For $f \in [L^\infty, \widetilde{\mathcal{M}_q^\varphi}]_\theta$, choose $F \in \mathcal{F}(L^\infty, \widetilde{\mathcal{M}_q^\varphi})$ such that $f = F(\theta)$. Combining Lemma 2.1.6 and Theorem 3.4.1, we have

$$\begin{aligned}\|\chi_{\mathbb{R}^n\setminus B(0,R)} f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} &\le \|\chi_{\mathbb{R}^n\setminus B(0,R)} F(\theta)\|_{[L^\infty, \mathcal{M}_q^\varphi]_\theta} \\ &\le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|F(it)\|_{L^\infty} P_0(\theta,t)\, dt\right)^{1-\theta} \\ &\quad \times \left(\frac{1}{\theta}\int_{\mathbb{R}} \|\chi_{\mathbb{R}^n\setminus B(0,R)} F(1+it)\|_{\mathcal{M}_q^\varphi} P_1(\theta,t)\, dt\right)^{\theta}. \end{aligned} \tag{4.24}$$

From $F(1+it) \in \widetilde{\mathcal{M}_q^\varphi} \subseteq \overset{*}{\mathcal{M}}{}_q^\varphi$, we see that

$$\lim_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(0,R)} F(1+it)\|_{\mathcal{M}_q^\varphi} = 0. \tag{4.25}$$

We combine (4.24), (4.25), and the dominated convergence theorem to obtain

$$\lim_{R\to\infty} \|\chi_{\mathbb{R}^n\setminus B(0,R)} f\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} = 0.$$

According to (4.3), we have $f \in \overset{*}{\mathcal{M}}{}_{q/\theta}^{\varphi^\theta}$. Since

$$[L^\infty, \widetilde{\mathcal{M}_q^\varphi}]_\theta \subseteq [L^\infty, \mathcal{M}_q^\varphi]_\theta = \overline{L^\infty \cap \mathcal{M}_q^\varphi}^{\mathcal{M}_{q/\theta}^{\varphi^\theta}} \subseteq \overline{\mathcal{M}}{}_{q/\theta}^{\varphi^\theta},$$

we see that $f \in \overset{*}{\mathcal{M}}{}_{q/\theta}^{\varphi^\theta} \cap \overline{\mathcal{M}}{}_{q/\theta}^{\varphi^\theta} = \widetilde{\mathcal{M}_{q/\theta}^{\varphi^\theta}}$, as desired.

Now, let $f \in \widetilde{\mathcal{M}_{q/\theta}^{\varphi^\theta}}$. We shall show that $f \in [L^\infty, \widetilde{\mathcal{M}_q^\varphi}]_\theta$. Since $L^\infty_{\rm c} \subseteq \widetilde{\mathcal{M}_q^\varphi}$, we have $f \in \overline{L^\infty \cap \widetilde{\mathcal{M}_q^\varphi}}^{\mathcal{M}_{q/\theta}^{\varphi^\theta}}$. Then, there exists a sequence $\{f_j\}_{j=1}^\infty \subseteq L^\infty \cap \widetilde{\mathcal{M}_q^\varphi}$ such that

$$\|f - f_j\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} \le \frac{1}{j}. \tag{4.26}$$

Therefore, for every $j, k \in \mathbb{N}$ with $j > k$, we have

$$\|f_j - f_k\|_{[L^\infty, \widetilde{\mathcal{M}_q^\varphi}]^\theta} \sim \|f_j - f_k\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} \le \frac{1}{j} + \frac{1}{k} < \frac{2}{k},$$

so $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $[L^\infty, \widetilde{\mathcal{M}_q^\varphi}]^\theta$. By completeness of $[L^\infty, \widetilde{\mathcal{M}_q^\varphi}]^\theta$, there exists $g \in [L^\infty, \widetilde{\mathcal{M}_q^\varphi}]^\theta$ such that

$$\lim_{j\to\infty} \|f_j - g\|_{[L^\infty, \widetilde{\mathcal{M}_q^\varphi}]^\theta} = 0. \tag{4.27}$$

Combining $\mathcal{M}_{q/\theta}^{\varphi^\theta} \subseteq L^\infty + \mathcal{M}_q^\varphi$, $[L^\infty, \mathcal{M}_q^\varphi]^\theta \subseteq L^\infty + \mathcal{M}_q^\varphi$, (4.26), and (4.27), we get $f = g \in \overline{L^\infty \cap \widetilde{\mathcal{M}_q^\varphi}}^{[L^\infty, \widetilde{\mathcal{M}_q^\varphi}]^\theta}$. Finally, by using (2.4), we have $f \in [L^\infty, \widetilde{\mathcal{M}_q^\varphi}]_\theta$, as desired.

We shall show that $[L^\infty, \overset{*}{\mathcal{M}}{}_q^\varphi]_\theta = \widetilde{\mathcal{M}_{q/\theta}^{\varphi^\theta}}$. Let $f \in [L^\infty, \overset{*}{\mathcal{M}}{}_q^\varphi]_\theta$. By virtue of (3.34), we have $[L^\infty, \overset{*}{\mathcal{M}}{}_q^\varphi]_\theta \subseteq [L^\infty, \mathcal{M}_q^\varphi]_\theta \subseteq \overline{\mathcal{M}}{}_{q/\theta}^{\varphi^\theta}$, so $f \in \overline{\mathcal{M}}{}_{q/\theta}^{\varphi^\theta}$. By Lemma 2.1.5, for each $\varepsilon > 0$, there exists $g \in L^\infty \cap \overset{*}{\mathcal{M}}{}_q^\varphi$ such that

$$\|f - g\|_{[L^\infty, \mathcal{M}_q^\varphi]_\theta} < \varepsilon.$$

Since $L^\infty \cap \overset{*}{\mathcal{M}}{}_q^\varphi \subseteq \overline{\mathcal{M}}{}_q^\varphi \cap \overset{*}{\mathcal{M}}{}_q^\varphi = \widetilde{\mathcal{M}_q^\varphi}$, we have $g \in \widetilde{\mathcal{M}_q^\varphi}$. Therefore, by virtue of Theorem 4.0.2, we have

$$\|\chi_{\{|g|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} g\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} \le \|g\|_{L^\infty}^{1-\theta} \|\chi_{\{|g|>R\}\cup(\mathbb{R}^n\setminus B(0,R))} g\|_{\mathcal{M}_q^\varphi}^\theta \to 0$$

as $R \to \infty$. Consequently, $g \in \widetilde{\mathcal{M}_{q/\theta}^{\varphi^\theta}}$. Since $[L^\infty, \overset{*}{\mathcal{M}}{}_q^\varphi]_\theta \subseteq [L^\infty, \mathcal{M}_q^\varphi]_\theta \subseteq \mathcal{M}_{q/\theta}^{\varphi^\theta}$, we have

$$\|f - g\|_{\mathcal{M}_{q/\theta}^{\varphi^\theta}} \lesssim \varepsilon.$$

This implies $f \in \widetilde{\mathcal{M}_{q/\theta}^{\varphi^\theta}}$. Thus, $[L^\infty, \overset{*}{\mathcal{M}}{}_q^\varphi]_\theta \subseteq \widetilde{\mathcal{M}_{q/\theta}^{\varphi^\theta}}$. Meanwhile, the inclusion $\widetilde{\mathcal{M}_{q/\theta}^{\varphi^\theta}} \subseteq [L^\infty, \overset{*}{\mathcal{M}}{}_q^\varphi]_\theta$ follows from $[L^\infty, \widetilde{\mathcal{M}_q^\varphi}]_\theta \subseteq [L^\infty, \overset{*}{\mathcal{M}}{}_q^\varphi]_\theta$ and $[L^\infty, \widetilde{\mathcal{M}_q^\varphi}]_\theta = \widetilde{\mathcal{M}_{q/\theta}^{\varphi^\theta}}$. □

Next, we move on to the description of the spaces $[L^\infty, \overline{\mathcal{M}}{}_q^\varphi]^\theta$, $[L^\infty, \overline{\mathcal{M}}{}_q^\varphi]_\theta$, $[L^\infty, \widetilde{\mathcal{M}_q^\varphi}]^\theta$, and $[L^\infty, \overset{*}{\mathcal{M}}{}_q^\varphi]^\theta$. First, we prove the following lemma:

Lemma 4.4.3. [27] *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Then we have*

$$\overline{\overset{*}{\mathcal{M}}{}_{q/\theta}^{\varphi^\theta}}^{L^\infty + \mathcal{M}_q^\varphi} \cap \mathcal{M}_{q/\theta}^{\varphi^\theta} \subseteq \bigcap_{0<a<b<\infty} \left\{ f \in \mathcal{M}_{q/\theta}^{\varphi^\theta} : \chi_{\{a \le |f| \le b\}} \in \overset{*}{\mathcal{M}}{}_{q/\theta}^{\varphi^\theta} \right\}.$$

Proof. Let $\overline{\overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}}^{L^\infty+\mathcal{M}^\varphi_q} \cap \mathcal{M}^{\varphi^\theta}_{q/\theta}$ and $0 < a < b < \infty$. For every $t \geq 0$, define

$$\psi_{a,b}(t) := \chi_{(\frac{a}{2},2b)}(t)(t-a/2)^2(t-2b)^2$$

Since

$$\chi_{\{a\leq|f|\leq b\}} \leq \frac{1}{a}\chi_{\{a\leq|f|\leq b\}}|f| \leq \frac{b}{a}\chi_{\{a\leq|f|\leq b\}} \leq C_{a,b}\chi_{\{a\leq|f|\leq b\}}\psi_{a,b}(|f|), \tag{4.28}$$

we only need to show that $\chi_{\{a\leq|f|\leq b\}}\psi_{a,b}(|f|) \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$. Let $\{f_j\}_{j=1}^\infty$ be such that $\lim_{j\to\infty}\|f-f_j\|_{L^\infty+\mathcal{M}^\varphi_q} = 0$. Choose $\{g_j\}_{j=1}^\infty \subseteq L^\infty$ and $\{h_j\}_{j=1}^\infty \subseteq \mathcal{M}^\varphi_q$ such that

$$f - f_j = g_j + h_j, \ \lim_{j\to\infty}\|g_j\|_{L^\infty} = 0, \text{ and } \lim_{j\to\infty}\|h_j\|_{\mathcal{M}^\varphi_q} = 0. \tag{4.29}$$

Since $\psi_{a,b} \in C^1(\mathbb{R})$ and $\psi_{a,b}, \psi'_{a,b} \in L^\infty(\mathbb{R})$, we have

$$\begin{aligned}|\chi_{\{a\leq|f|\leq b\}}\psi_{a,b}(|f|) - \chi_{\{a\leq|f|\leq b\}}\psi_{a,b}(|f_j|)| &\lesssim \chi_{\{a\leq|f|\leq b\}}\min(1,|f-f_j|)\\ &\lesssim \chi_{\{a\leq|f|\leq b\}}(\min(1,|g_j|)+\min(1,|h_j|)).\end{aligned}$$

Since $\min(1,|h_j|) \leq |h_j|^\theta$, we have

$$\|\min(1,|h_j|)\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \leq \||h_j|^\theta\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = \|h_j\|^\theta_{\mathcal{M}^\varphi_q}. \tag{4.30}$$

Meanwhile,

$$\|\chi_{\{a\leq|f|\leq b\}}\min(1,|g_j|)\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \leq \frac{1}{a}\|g_j\|_{L^\infty}\|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}. \tag{4.31}$$

By combining (4.30) and (4.31), we get

$$\|\chi_{\{a\leq|f|\leq b\}}\psi_{a,b}(|f|) - \chi_{\{a\leq|f|\leq b\}}\psi_{a,b}(|f_j|)\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \leq \frac{1}{a}\|g_j\|_{L^\infty}\|f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} + \|h_j\|^\theta_{\mathcal{M}^\varphi_q}.$$

According to (4.29), we have $\lim_{j\to\infty}\|\chi_{\{a\leq|f|\leq b\}}\psi_{a,b}(|f|) - \chi_{\{a\leq|f|\leq b\}}\psi_{a,b}(|f_j|)\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = 0$. Since $\chi_{\{a\leq|f|\leq b\}}\psi_{a,b}(|f_j|) \lesssim |f_j|$, we have $\chi_{\{a\leq|f|\leq b\}}\psi_{a,b}(|f_j|) \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$, and hence, $\chi_{\{a\leq|f|\leq b\}}\psi_{a,b}(|f|) \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$. As a consequence of (4.28), we conclude that $\chi_{\{a\leq|f|\leq b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$. □

We describe the spaces $[L^\infty, \overline{\mathcal{M}}{}^\varphi_q]^\theta$, $[L^\infty, \overline{\mathcal{M}}{}^\varphi_q]_\theta$, $[L^\infty, \widetilde{\mathcal{M}}{}^\varphi_q]^\theta$, and $[L^\infty, \overset{*}{\mathcal{M}}{}^\varphi_q]^\theta$ as follows:

Theorem 4.4.4. [27] *Let $1 \leq q < \infty$ and $\varphi \in \mathcal{G}_q$. Then we have*

(i) $[L^\infty, \overline{\mathcal{M}}^\varphi_q]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}$,

(ii) $[L^\infty, \overline{\mathcal{M}}^\varphi_q]_\theta = \overline{\mathcal{M}}^{\varphi^\theta}_{q/\theta}$,

(iii) $[L^\infty, \widetilde{\mathcal{M}}^\varphi_q]^\theta = [L^\infty, \overset{*}{\mathcal{M}}^\varphi_q]^\theta = \bigcap_{0<a<b<\infty} \{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a \le |f| \le b\}} \in \overset{*}{\mathcal{M}}^{\varphi^\theta}_{q/\theta}\}$.

Proof. Note that $[L^\infty, \overline{\mathcal{M}}^\varphi_q]^\theta \subseteq [L^\infty, \mathcal{M}^\varphi_q]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}$. Now, let $f \in \mathcal{M}^{\varphi^\theta}_{q/\theta}$. Define $F(z) := \operatorname{sgn}(f)|f|^{\frac{z}{\theta}}$ and $G(z) := \int_\theta^z F(w)\, dw$. In the proof of (3.44), we know that $G \in \mathcal{G}(L^\infty, \mathcal{M}^\varphi_q)$, so it suffices to show that

1. $G_1(z) := \chi_{\{|f|>1\}} G(z) \in \overline{\mathcal{M}}^\varphi_q$ for every $z \in \overline{S}$;
2. $G(1+it) - G(1) \in \overline{\mathcal{M}}^\varphi_q$ for every $t \in \mathbb{R}$.

From the inequalities

$$|G_1(z)| = \left|\chi_{\{|f|>1\}} \frac{F(z) - F(\theta)}{\log |f|^{1/\theta}}\right| \lesssim \chi_{\{|f|>1\}} \frac{|f|^{1/\theta}}{\log |f|^{1/\theta}}$$

and $|G_1(z)| \le (1+|z|)|f|^{1/\theta}$, it follows that

$$\begin{aligned} \|\chi_{\{|G_1(z)|>R\}} G_1(z)\|_{\mathcal{M}^\varphi_q} &\lesssim \left\| \chi_{\{|f|^{1/\theta} > \frac{R}{1+|z|}\}} \frac{|f|^{1/\theta}}{\log |f|^{1/\theta}} \right\|_{\mathcal{M}^\varphi_q} \\ &\lesssim \frac{1}{\log(R/(1+|z|))} \|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \to 0 \end{aligned}$$

as $R \to \infty$. Therefore, $G_1(z) \in \overline{\mathcal{M}}^\varphi_q$. Similarly, for every $t \in \mathbb{R}$, we have

$$\|\chi_{\{|G(1+it)-G(1)|>R\}}(G(1+it) - G(1))\|_{\mathcal{M}^\varphi_q} \lesssim \frac{\|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}}{\log(R/(1+|t|))} \to 0 \ (R \to \infty),$$

so $G(1+it) - G(1) \in \overline{\mathcal{M}}^\varphi_q$. Hence, $G \in \mathcal{G}(L^\infty, \overline{\mathcal{M}}^\varphi_q)$ and $f = G'(\theta) \in [L^\infty, \overline{\mathcal{M}}^\varphi_q]^\theta$.

We now move on to the proof of (ii). Let $f \in [L^\infty, \overline{\mathcal{M}}^\varphi_q]_\theta$. By virtue of Lemma 2.1.5 and $[L^\infty, \overline{\mathcal{M}}^\varphi_q]_\theta \subseteq \mathcal{M}^{\varphi^\theta}_{q/\theta}$, for each $\varepsilon > 0$, there exists $g \in L^\infty \cap \mathcal{M}^\varphi_q$ such that

$$\|f - g\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \lesssim \varepsilon. \tag{4.32}$$

By combining (4.32) and $\|g\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \|g\|_{L^\infty}^{1-\theta}\|g\|_{\mathcal{M}^\varphi_q}^\theta < \infty$, we see that $f \in \overline{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. Meanwhile, by virtue of Theorem 4.4.4 (i) and (2.4), we have

$$\overline{\mathcal{M}}^{\varphi^\theta}_{q/\theta} \subseteq \overline{L^\infty \cap \overline{\mathcal{M}}^\varphi_q}^{[L^\infty, \overline{\mathcal{M}}^\varphi_q]^\theta} = [L^\infty, \overline{\mathcal{M}}^\varphi_q]_\theta,$$

as desired.

Finally, let us prove (iii). Let $f \in \mathcal{M}^{\varphi^\theta}_{q/\theta}$ be such that $\chi_{\{a \le |f| \le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$ for every $0 < a < b < \infty$. Since

$$\|\chi_{\mathbb{R}^n \setminus B(0,R)}\chi_{\{a \le |f| \le b\}}\|_{\mathcal{M}^\varphi_q} = \|\chi_{\mathbb{R}^n \setminus B(0,R)}\chi_{\{a \le |f| \le b\}}\|^{\frac{1}{\theta}}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \to 0$$

as $R \to \infty$, we have $\chi_{\{a \le |f| \le b\}} \in \overset{*}{\mathcal{M}}{}^\varphi_q$. For every $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{\frac{z}{\theta}} \text{ and } G(z) := \int_\theta^z F(w)\, dw.$$

In the proof of Theorem 4.4.4 (i), we know that $G \in \mathcal{G}(L^\infty, \overline{\mathcal{M}}^\varphi_q)$. Hence, in order to prove that $G \in \mathcal{G}(L^\infty, \widetilde{\mathcal{M}}^\varphi_q)$, we only need to show that

$$G_1(z) := \chi_{\{|f|>1\}}G(z) \in \overset{*}{\mathcal{M}}{}^\varphi_q \text{ and } G(1+it) - G(1) \in \overset{*}{\mathcal{M}}{}^\varphi_q$$

for each $z \in \overline{S}$ and $t \in \mathbb{R}$. For every $R > 0$, we have

$$|\chi_{\{|f| \le R\}}G_1(z)| \le (1+|z|)R^{1/\theta}\chi_{\{1 \le |f| \le R\}},$$

so $\chi_{\{|f| \le R\}}G_1(z) \in \overset{*}{\mathcal{M}}{}^\varphi_q$. Since

$$\|G_1(z) - \chi_{\{|f| \le R\}}G_1(z)\|_{\mathcal{M}^\varphi_q} \lesssim \frac{1}{\log(R/(1+|z|))}\|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \to 0$$

as $R \to \infty$, we have $G_1(z) \in \overset{*}{\mathcal{M}}{}^\varphi_q$. For every $t \in \mathbb{R}$ and $R > 1$, we have

$$|G(1+it) - G(1)|\chi_{\{\frac{1}{R} \le |f| \le R\}} \le (1+|t|)R^{1/\theta}\chi_{\{\frac{1}{R} \le |f| \le R\}},$$

so $(G(1+it) - G(1))\chi_{\{\frac{1}{R} \le |f| \le R\}} \in \overset{*}{\mathcal{M}}{}^\varphi_q$. Meanwhile,

$$\|(G(1+it) - G(1))\chi_{\mathbb{R}^n \setminus \{\frac{1}{R} \le |f| \le R\}}\|_{\mathcal{M}^\varphi_q} \lesssim \frac{\theta}{\log R}\|f\|^{1/\theta}_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \to 0$$

as $R \to \infty$, so $G(1+it) - G(1) \in \overset{*}{\mathcal{M}}{}^\varphi_q$. Since $G \in \mathcal{G}(L^\infty, \widetilde{\mathcal{M}}^\varphi_q)$ and $f = G'(\theta)$, we conclude that $f \in [L^\infty, \widetilde{\mathcal{M}}^\varphi_q]^\theta$. Combining with $[L^\infty, \widetilde{\mathcal{M}}^\varphi_q]^\theta \subseteq [L^\infty, \overset{*}{\mathcal{M}}{}^\varphi_q]^\theta$, we have

$$\bigcap_{0<a<b<\infty} \{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a \le |f| \le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}\} \subseteq [L^\infty, \widetilde{\mathcal{M}}^\varphi_q]^\theta \subseteq [L^\infty, \overset{*}{\mathcal{M}}{}^\varphi_q]^\theta. \tag{4.33}$$

Let $f \in [L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]^\theta$. From $[L^\infty, \mathcal{M}^{\varphi}_q]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}$, it follows that $f \in \mathcal{M}^{\varphi^\theta}_{q/\theta}$. Choose $G \in \mathcal{G}(L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q)$ such that $f = G'(\theta)$. For each $z \in \overline{S}$, define

$$h_k(z) := \frac{G(z + 2^{-k}i) - G(z)}{2^{-k}i}.$$

By Lemma 2.2.4 and Theorem 4.4.2, we have $h_k(\theta) \in [L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \subseteq \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$. Since $\lim_{k\to\infty} h_k(\theta) = f$ in $L^\infty + \mathcal{M}^{\varphi}_q$, we have $f \in \overline{\overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}}^{L^\infty + \mathcal{M}^{\varphi}_q} \cap \mathcal{M}^{\varphi^\theta}_{q/\theta}$. By virtue of Lemma 4.4.3, we conclude that $\chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$, for every $0 < a < b < \infty$. Hence,

$$[L^\infty, \overset{*}{\mathcal{M}}{}^{\varphi}_q]^\theta \subseteq \bigcap_{0<a<b<\infty} \{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}\}. \tag{4.34}$$

As a consequence of (4.33) and (4.34), we have Theorem 4.4.4 (iii). □

Finally, we also consider the complex interpolation between $\widetilde{L^\infty}$ and closed subspaces of Morrey spaces. Recall that $\widetilde{L^\infty}$ denotes the closure of $L^\infty_{\rm c}$ in L^∞.

Theorem 4.4.5. [30] *Let $1 \le q < \infty$ and $\varphi \in \mathcal{G}_q$. Then we have*

(i) $[\widetilde{L^\infty}, \mathcal{M}^{\varphi}_q]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$,

(ii)

$$\begin{aligned}\bigcap_{0<a<b<\infty} &\{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a\le|f|\le b\}} \in \widetilde{L^\infty}\} \\ &\subseteq [\widetilde{L^\infty}, \mathcal{M}^{\varphi}_q]^\theta \\ &\subseteq \bigcap_{0<a<b<\infty} \{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}\}\end{aligned}$$

(iii) If $\inf\varphi > 0$*, then*

$$[\widetilde{L^\infty}, \mathcal{M}^{\varphi}_q]^\theta = \bigcap_{0<a<b<\infty} \{f \in \mathcal{M}^{\varphi^\theta}_{q/\theta} : \chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}\}. \tag{4.35}$$

(iv) $[\widetilde{L^\infty}, \widetilde{\mathcal{M}}{}^{\varphi}_q]_\theta = [\widetilde{L^\infty}, \overset{*}{\mathcal{M}}{}^{\varphi}_q]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$.

Remark 4.4.6. In (ii), we only prove an inclusion relation for $[\widetilde{L^\infty}, \mathcal{M}^{\varphi}_q]^\theta$. The complete description of this space is an open problem for the future research.

Proof of Theorem 4.4.5. Let $f \in \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. Since $L^\infty_{\rm c} \subseteq \widetilde{L^\infty}$, we have

$$f \in \overline{\widetilde{L^\infty} \cap \mathcal{M}^\varphi_q}^{\mathcal{M}^{\varphi^\theta}_{q/\theta}}.$$

Therefore, there exists a sequence $\{f_j\}_{j=1}^\infty \subseteq \widetilde{L^\infty} \cap \mathcal{M}^\varphi_q$ such that

$$\|f - f_j\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \frac{1}{j}. \tag{4.36}$$

By using a similar argument as in the proof of Lemma 4.4.1, we have

$$\|f_j\|_{[\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta} \sim \|f_j\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}}.$$

Therefore, for every $j, k \in \mathbb{N}$ with $j > k$, we have

$$\|f_j - f_k\|_{[\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta} \sim \|f_j - f_k\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \frac{1}{j} + \frac{1}{k} < \frac{2}{k},$$

so $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $[\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta$. By completeness of $[\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta$, there exists $g \in [\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta$ such that

$$\lim_{j \to \infty} \|f_j - g\|_{[L^\infty, \widetilde{\mathcal{M}^\varphi_q}]^\theta} = 0. \tag{4.37}$$

Combining $\mathcal{M}^{\varphi^\theta}_{q/\theta} \subseteq L^\infty + \mathcal{M}^\varphi_q$, $[L^\infty, \mathcal{M}^\varphi_q]^\theta \subseteq L^\infty + \mathcal{M}^\varphi_q$, (4.36), and (4.37), we get $f = g \in \overline{\widetilde{L^\infty} \cap \mathcal{M}^\varphi_q}^{[\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta}$. As a consequence of (2.4), we have $f \in [\widetilde{L^\infty}, \mathcal{M}^\varphi_q]_\theta$.

Conversely, let $f \in [\widetilde{L^\infty}, \mathcal{M}^\varphi_q]_\theta$ and choose $F \in \mathcal{F}(\widetilde{L^\infty}, \mathcal{M}^\varphi_q)$ such that $f = F(\theta)$. Since $F(it) \in \widetilde{L^\infty}$, we have

$$\lim_{R \to \infty} \|\chi_{\mathbb{R}^n \setminus B(0,R)} F(it)\|_{L^\infty} = 0. \tag{4.38}$$

By Lemma 2.1.6, we have

$$\begin{aligned} \|\chi_{\mathbb{R}^n \setminus B(0,R)} f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \le \left(\frac{1}{1-\theta} \int_{\mathbb{R}} \|\chi_{\mathbb{R}^n \setminus B(0,R)} F(it)\|_{L^\infty} P_0(\theta, t)\ dt \right)^{1-\theta} \\ \times \|F\|_{\mathcal{F}(\widetilde{L^\infty}, \mathcal{M}^\varphi_q)}. \end{aligned} \tag{4.39}$$

By virtue of the dominated convergence theorem, (4.38), and (4.39), we have

$$\lim_{R \to \infty} \|\chi_{\mathbb{R}^n \setminus B(0,R)} f\|_{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = 0,$$

so $f \in \overset{*}{\mathcal{M}}^{\varphi^\theta}_{q/\theta}$. Since $[\widetilde{L^\infty}, \mathcal{M}^\varphi_q]_\theta \subseteq [L^\infty, \mathcal{M}^\varphi_q]_\theta = \overline{L^\infty \cap \mathcal{M}^\varphi_q}^{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \subseteq \overline{L^\infty \cap \mathcal{M}^{\varphi^\theta}_{q/\theta}}^{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$, we have

$$f \in \overline{\mathcal{M}}^{\varphi^\theta}_{q/\theta} \cap \overset{*}{\mathcal{M}}^{\varphi^\theta}_{q/\theta} = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}},$$

as desired.

The proof of (ii) goes as follows. Let $f \in \mathcal{M}^{\varphi^\theta}_{q/\theta}$ be such that $\chi_{\{a \le |f| \le b\}} \in \widetilde{L^\infty}$ for every $0 < a < b < \infty$. For each $z \in \overline{S}$, define

$$F(z) := \operatorname{sgn}(f)|f|^{z/\theta} \text{ and } G(z) := \int_\theta^z F(w)\, dw.$$

Since $G \in \mathcal{G}(L^\infty, \mathcal{M}^\varphi_q)$, we shall show that $G_0(z) := \chi_{\{|f| \le 1\}} G(z) \in \widetilde{L^\infty}$ for every $z \in \overline{S}$ and $G(it) - G(0) \in \widetilde{L^\infty}$ for every $t \in \mathbb{R}$. For each $N \in \mathbb{N}$, we have

$$|G_0(z)\chi_{\{|f| > \frac{1}{N}\}}| \le (1 + |z|)\chi_{\{\frac{1}{N} < |f| \le 1\}},$$

so $G_0(z)\chi_{\{|f| > \frac{1}{N}\}} \in \widetilde{L^\infty}$. Meanwhile,

$$\begin{aligned} \left\| G_0(z) - G_0(z)\chi_{\{|f| > 1/N\}} \right\|_{L^\infty} &= \left\| \theta \frac{\operatorname{sgn}(f)|f|^{z/\theta} - \operatorname{sgn}(f)|f|}{\log |f|} \chi_{\{1/N \le |f| \le 1\}} \right\|_{L^\infty} \\ &\le \frac{2\theta}{\log N} \to 0 \end{aligned}$$

as $N \to \infty$. Therefore, $G_0(z) \in \widetilde{L^\infty}$.

Next, for all $N \in \mathbb{N}$ and $t \in \mathbb{R}$, we have

$$|G(it) - G(0)|\chi_{\{1/N \le |f| \le N\}} \le (1 + |t|)\chi_{\{1/N \le |f| \le N\}},$$

so $(G(it) - G(0))\chi_{\{1/N \le |f| \le N\}} \in \widetilde{L^\infty}$. Since $|F(it)| = 1$ for every $t \in \mathbb{R}$, we have

$$\begin{aligned} \|(G(it) - G(0))\chi_{\mathbb{R}^n \setminus \{1/N \le |f| \le N\}}\|_{L^\infty} &= \left\| \theta \frac{F(it) - F(0)}{\log |f|} \chi_{\{|f| < 1/N\} \cup \{|f| > N\}} \right\|_{L^\infty} \\ &\le \frac{2\theta}{\log N} \to 0, \end{aligned}$$

as $N \to \infty$. Therefore, $G(it) - G(0) \in \widetilde{L^\infty}$. In total, $G \in \mathcal{G}(\widetilde{L^\infty}, \mathcal{M}^\varphi_q)$. Since $f = G'(\theta)$, we see that $f \in [\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta$.

Now, let $f \in [\widetilde{L^\infty}, \mathcal{M}^\varphi_q]^\theta$. Since $[L^\infty, \mathcal{M}^\varphi_q]^\theta = \mathcal{M}^{\varphi^\theta}_{q/\theta}$, we have $f \in \mathcal{M}^{\varphi^\theta}_{q/\theta}$. Let $G \in \mathcal{G}(\widetilde{L^\infty}, \mathcal{M}^\varphi_q)$ be such that $f = G'(\theta)$. For each $k \in \mathbb{N}$ and $z \in \overline{S}$, define $H_k(z)$

by (2.3). As a consequence of Lemma 2.2.4 and Theorem 4.4.5(i), we have $H_k(\theta) \in \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. Since $\lim_{k\to\infty} H_k(\theta) = f$ in $L^\infty + \mathcal{M}^\varphi_q$, we have $f \in \overline{\overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}}^{L^\infty+\mathcal{M}^\varphi_q} \cap \mathcal{M}^{\varphi^\theta}_{q/\theta}$. By virtue of Lemma 4.4.3, we conclude that $\chi_{\{a\le|f|\le b\}} \in \overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta}$, as desired.

Finally, let us prove (*iii*) and (*iv*). Recall that, when $\inf\varphi > 0$, we have $\mathcal{M}^{\varphi^\theta}_{q/\theta} \subseteq L^\infty$; see [42, Proposition 3.3]. Therefore, $\overset{*}{\mathcal{M}}{}^{\varphi^\theta}_{q/\theta} \subseteq \widetilde{L^\infty}$. Combining this fact with Theorem 4.4.5 (ii), we get (4.35). From Theorem 4.4.5 (i), it follows that $[\widetilde{L^\infty}, \widetilde{\mathcal{M}^\varphi_q}]_\theta \subseteq \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$. By the same argument as in the proof of Theorem 4.4.5 (i), we have

$$\widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \subseteq \overline{\widetilde{L^\infty} \cap \widetilde{\mathcal{M}^\varphi_q}}^{\mathcal{M}^{\varphi^\theta}_{q/\theta}} \subseteq \overline{\widetilde{L^\infty} \cap \widetilde{\mathcal{M}^\varphi_q}}^{[\widetilde{L^\infty}, \widetilde{\mathcal{M}^\varphi_q}]^\theta} = [\widetilde{L^\infty}, \widetilde{\mathcal{M}^\varphi_q}]_\theta.$$

By combining $[\widetilde{L^\infty}, \overset{*}{\mathcal{M}}{}^\varphi_q]_\theta \subseteq [\widetilde{L^\infty}, \mathcal{M}^\varphi_q]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$ and $\widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}} = [\widetilde{L^\infty}, \widetilde{\mathcal{M}^\varphi_q}]_\theta \subseteq [\widetilde{L^\infty}, \overset{*}{\mathcal{M}}{}^\varphi_q]_\theta$, we have $[\widetilde{L^\infty}, \overset{*}{\mathcal{M}}{}^\varphi_q]_\theta = \widetilde{\mathcal{M}^{\varphi^\theta}_{q/\theta}}$.

□

Chapter 5

Complex interpolation of local block spaces

In this chapter, we discuss the first complex interpolation of local block spaces, which are known to be a predual of local Morrey spaces. We prove that local block spaces behave well under the first complex interpolation. To prove this result, we show that the associate space of general local Morrey-type spaces can be realized as certain block spaces.

5.1 Local Morrey-type spaces and local block spaces

Let $0 < p \leq \infty$ and $0 \leq \lambda \leq \frac{n}{p}$. The local Morrey space $L\mathcal{M}_p^\lambda = L\mathcal{M}_p^\lambda(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{L\mathcal{M}_p^\lambda} := \sup_{r>0} r^{-\lambda}\|f\chi_{B(r)}\|_{L^p} < \infty,$$

where $B(r)$ denotes the open ball centered at the origin of radius $r > 0$. The local Morrey spaces behaves well with respect to the real interpolation method as shown in [6] and the references therein. Moreover, the results in [6] were generalized to general local Morrey-type spaces and B^u_w-spaces in [8, 43]. Interpolation of B^u_w-spaces by complex method can be found in [29]. The definition of general local Morrey-type spaces and general global Morrey-type spaces is given as follows.

Definition 5.1.1. Let $1 < p, q \leq \infty$ and w be a non-negative measurable function on $(0, \infty)$. The general local Morrey-type space $L\mathcal{M}_{pq,w} = L\mathcal{M}_{pq,w}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{L\mathcal{M}_{pq,w}} := \left\| w(r) \|f\chi_{B(r)}\|_{L^p} \right\|_{L^q(0,\infty)} < \infty.$$

The global Morrey-type space $G\mathcal{M}_{pq,w} = G\mathcal{M}_{pq,w}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{G\mathcal{M}_{pq,w}} := \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{L\mathcal{M}_{pq,w}} < \infty.$$

Note that, if $w(r) = 1$, then $L\mathcal{M}_{p\infty,w} = G\mathcal{M}_{p\infty,w} = L^p$. We shall assume the following condition on w so that $L\mathcal{M}_{pq,w}$ contains non-zero function,

Definition 5.1.2. Let $1 < p, q \le \infty$. We define

$$\Omega_q := \{w : (0,\infty) \to (0,\infty) : \|w\|_{L^q(t_0,\infty)} < \infty \text{ for some } t_0 > 0\}$$

and

$$\Omega_{pq} := \{w : (0,\infty) \to (0,\infty) : \|r^{n/p} w(r)\|_{L^q(0,t)} < \infty \text{ and } \|w\|_{L^q(t,\infty)} < \infty\}.$$

Remark 5.1.3. It is known in [7, Lemma 1] that $w \in \Omega_q$ if and only if $L\mathcal{M}_{pq,w}$ is not equal to the set of all functions equivalent to zero function. In addition, $w \in \Omega_{pq}$ is a necessary and sufficient condition for $G\mathcal{M}_{pq,w}$ to be non-trivial.

It was proved in [19](see Theorem 5.1.5 below) that the predual of $L\mathcal{M}_{pq,\tilde{w}}$ can be characterized as the local block space $LH_{p'q',w}$ where $1 < p < \infty$, $1 < q \le \infty$, $\frac{1}{p'} := 1 - \frac{1}{p}$, $\frac{1}{q'} := 1 - \frac{1}{q}$, $\tilde{w}(t) = t^{-1/q} w(t)$, and w satisfies the following doubling condition; there exists a constant $C > 1$ such that $C^{-1}w(r) \le w(s) \le Cw(r)$ for every r, s satisfying $\frac{r}{s} \in (\frac{1}{2}, 2)$. Let us recall the definition of $LH_{p'q',w}$.

Definition 5.1.4. Let $1 < p < \infty$, $1 < q \le \infty$, $w \in \Omega_q$, and $r \in (0,\infty)$. We define $p' := \frac{p}{p-1}$ and $q' := \frac{q}{q-1}$. A measurable function A is called a (p', w, r)-block if $\operatorname{supp}(A) \subseteq B(r)$ and $\|A\|_{L^{p'}} \le w(r)$. We define

$$\dot{\mathcal{A}}_w(L^{p'}) := \{\{(A_j, 2^j)\}_{j=-\infty}^{\infty} : A_j \text{ is a } (p', w, 2^j)\text{−block}\}.$$

The local block space $LH_{p'q',w}$ is defined by

$$LH_{p'q',w} := \left\{ \sum_{j=-\infty}^{\infty} \lambda_j A_j : \{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'} \text{ and } \{(A_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{\mathcal{A}}_w(L^{p'}) \right\}.$$

The norm on $LH_{p'q',w}$ is defined by

$$\|f\|_{LH_{p'q',w}} := \inf \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'} \right)^{1/q'}, \tag{5.1}$$

where the infimum is taken over all decompositions $f = \sum_{j=-\infty}^{\infty} \lambda_j A_j$, $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$, and $\{(A_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{\mathcal{A}}_w(L^{p'})$.

We now recall the result in [19] about a characterization of a predual of general local Morrey type space as certain local block spaces.

Theorem 5.1.5. [19, Theorem 4.1] *Let $1 < p < \infty$, $1 < q \le \infty$, and $w \in \Omega_{pq}$. Assume that w satisfies the doubling condition. Define*

$$\tilde{w}(t) := \begin{cases} t^{-1/q} w(t), & q < \infty, \\ w(t), & q = \infty. \end{cases}$$

Then $(LH_{p'q',w})^* \approx L\mathcal{M}_{pq,\tilde{w}}$ in the following sense:

1. *Let $g \in L\mathcal{M}_{pq,\tilde{w}}$. Then, for every $f \in LH_{p'q',w}$, we have $fg \in L^1$ and the mapping L_g defined by*

$$L_g(f) := \int_{\mathbb{R}^n} f(x)g(x)\, dx$$

is a bounded linear functional on $LH_{p'q',w}$.

2. *For every $L \in (LH_{p'q',w})^*$, there exists $g \in L\mathcal{M}_{pq,\tilde{w}}$ such that*

$$L = L_g \text{ and } \|L\|_{(LH_{p'q',w})^*} \sim \|g\|_{L\mathcal{M}_{pq,\tilde{w}}}.$$

5.2 Some basic properties of local block spaces

A non-trivial member of the space $LH_{p'q',w}$ is an $L^{p'}$-function supported on the ball B of radius 2^j for some $j \in \mathbb{N}$.

Lemma 5.2.1. [31] *Let $1 < p < \infty$, $1 < q \le \infty$, and $w \in \Omega_q$. If $A \in L^{p'}$ and* $\operatorname{supp}(A) \subseteq B(2^j)$ *for some $j \in \mathbb{Z}$, then $A \in LH_{p'q',w}$ and*

$$\|A\|_{LH_{p'q',w}} \le \frac{\|A\|_{L^{p'}}}{w(2^j)}. \tag{5.2}$$

Proof. If $\|A\|_{L^{p'}} = 0$, then $A = 0$, so (5.2) is trivial. Hence, we may assume that $\|A\|_{L^{p'}} \neq 0$. Define $\tilde{A} := \frac{w(2^j)A}{\|A\|_{L^{p'}}}$. Since $\operatorname{supp}(\tilde{A}) \subseteq B(2^j)$ and $\|\tilde{A}\|_{L^{p'}} = w(2^j)$, we see that $\tilde{A}$ is a $(p', w, 2^j)$-block. Moreover, (5.2) follows from $A = \frac{\|A\|_{L^{p'}}}{w(2^j)}\tilde{A}$. □

We prove the following basic properties of $LH_{p'q',w}$. The first one is the lattice property of $LH_{p'q',w}$.

Lemma 5.2.2. [31] *Let* $1 < p < \infty$, $1 < q \leq \infty$, *and* $w \in \Omega_{pq}$. *If* $0 \leq f(x) \leq g(x)$ *and* $g \in LH_{p'q',w}$, *then* $f \in LH_{p'q',w}$ *and*

$$\|f\|_{LH_{p'q',w}} \leq \|g\|_{LH_{p'q',w}}. \tag{5.3}$$

Proof. Given $\varepsilon > 0$. Let $g = \sum_{j=-\infty}^{\infty} \lambda_j A_j$ where $\{(A_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$ and $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$ satisfy

$$\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'}\right)^{1/q'} \leq (1+\varepsilon)\|g\|_{LH_{p'q',w}}. \tag{5.4}$$

For each $j \in \mathbb{Z}$, define $B_j := \chi_{\{g\neq 0\}} \frac{f}{g} A_j$. Then, $\{(B_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$ and

$$f = \chi_{\{g\neq 0\}} \frac{f}{g} \sum_{j=-\infty}^{\infty} \lambda_j A_j = \sum_{j=-\infty}^{\infty} \lambda_j B_j.$$

Consequently, $f \in LH_{p'q',w}$. From (5.4), it follows that

$$\|f\|_{LH_{p'q',w}} \leq (1+\varepsilon)\|g\|_{LH_{p'q',w}}. \tag{5.5}$$

By taking $\varepsilon \to 0^+$, we get (5.3). $\square$

Lemma 5.2.3. *Let* $1 < p < \infty$, $1 < q \leq \infty$, *and* $w \in \Omega_{pq}$. *If* $f \in LH_{p'q',w}$, *then* $|f| \in LH_{p'q',w}$ *with*

$$\||f|\|_{LH_{p'q',w}} = \|f\|_{LH_{p'q',w}}. \tag{5.6}$$

Proof. Let $\delta > 0$. Then there exist $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$ and $\{(A_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$ such that

$$f = \sum_{j=-\infty}^{\infty} \lambda_j A_j \text{ and } \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'}\right)^{1/q'} \leq (1+\delta)\|f\|_{LH_{p'q',w}}.$$

Since $|f| \leq \sum_{j=-\infty}^{\infty} |\lambda_j||A_j|$, $\{|\lambda_j|\}_{j=-\infty}^{\infty} \in \ell^{q'}$, and $\{(|A_j|, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$, by Lemma 5.2.2, we have $|f| \in LH_{p'q',w}$ and

$$\||f|\|_{LH_{p'q',w}} \leq \left\|\sum_{j=-\infty}^{\infty} |\lambda_j||A_j|\right\|_{LH_{p'q',w}} \leq \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'}\right)^{1/q'} \leq (1+\delta)\|f\|_{LH_{p'q',w}}.$$

By letting $\delta \to 0^+$, we get

$$\||f|\|_{LH_{p'q',w}} \leq \|f\|_{LH_{p'q',w}}. \tag{5.7}$$

Since $|f| \in LH_{p'q',w}$, for any $\varepsilon > 0$, we can find $\{\alpha_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$ and $\{(B_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$ such that

$$|f| = \sum_{j=\infty}^{\infty} \alpha_j B_j \text{ and } \left(\sum_{j=-\infty}^{\infty} |\alpha_j|^{q'} \right)^{1/q'} \leq (1+\varepsilon) \| |f| \|_{LH_{p'q',w}}.$$

It follows from $f = \sum_{j=-\infty}^{\infty} \alpha_j (\mathrm{sgn}(f) B_j)$ and $\{(\mathrm{sgn}(f) B_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{A}_w(L^{p'})$ that

$$\|f\|_{LH_{p'q',w}} \leq \left(\sum_{j=-\infty}^{\infty} |\alpha_j|^{q'} \right)^{1/q'} \leq (1+\varepsilon) \| |f| \|_{LH_{p'q',w}}.$$

Since ε is arbitrary, we have

$$\|f\|_{LH_{p'q',w}} \leq \| |f| \|_{LH_{p'q',w}}. \tag{5.8}$$

Thus, (5.6) follows from (5.7) and (5.8). □

5.3 The Fatou property of local block spaces

Next, we prove the Fatou property of $LH_{p'q',w}$. Note that the Fatou property of local block spaces is not trivial. In fact, our proof uses a quite delicate argument, inspired by [34].

Proposition 5.3.1. [31] *Let $1 < p < \infty$, $1 < q \leq \infty$, and $w \in \Omega_{pq}$. For $q = \infty$ only, assume also that $\lim_{t\to\infty} w(t) = 0$. Suppose that $\{f_k\}_{k=1}^{\infty} \subseteq LH_{p'q',w}$ satisfies $0 \leq f_k(x) \leq f_{k+1}(x)$ for a.e. $x \in \mathbb{R}^n$ and for every $k \in \mathbb{N}$. If*

$$\sup_{k\in\mathbb{N}} \|f_k\|_{LH_{p'q',w}} < \infty,$$

then $f := \lim_{k\to\infty} f_k \in LH_{p'q',w}$ and

$$\|f\|_{LH_{p'q',w}} = \sup_{k\in\mathbb{N}} \|f_k\|_{LH_{p'q',w}}. \tag{5.9}$$

Proof. Define $M := \sup_{k\in\mathbb{N}} \|f_k\|_{LH_{p'q',w}}$ and let $\varepsilon > 0$. Then, for every $k \in \mathbb{N}$, there exist $\{\lambda_{j,k}\}_{j=-\infty}^{\infty} \in \ell^{q'}$ and $\{(A_{j,k}, 2^j)\}_{j=\infty}^{\infty} \in \dot{\mathcal{A}}_w(L^{p'})$ such that

$$f_k = \sum_{j=-\infty}^{\infty} \lambda_{j,k} A_{j,k} \quad \text{and} \quad \left(\sum_{j=-\infty}^{\infty} |\lambda_{j,k}|^{q'} \right)^{1/q'} \leq (1+\varepsilon) \|f_k\|_{LH_{p'q',w}}.$$

Therefore, for each $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have

$$|\lambda_{j,k}| \le (1+\varepsilon)M$$

and

$$\|A_{j,k}\|_{L^{p'}} \le w(2^j). \tag{5.10}$$

Consequently, there exist $\{\lambda_{j,k_\ell}\}_{\ell=1}^{\infty} \subseteq \{\lambda_{j,k}\}_{k=1}^{\infty}$, $\{A_{j,k_\ell}\}_{\ell=1}^{\infty} \subseteq \{A_{j,k}\}_{k=1}^{\infty}$, $\lambda_j \in \mathbb{C}$, and $A_j \in L^{p'}$ such that $\lim_{\ell\to\infty} \lambda_{j,k_\ell} = \lambda_j$ and

$$\lim_{\ell\to\infty} \int_{\mathbb{R}^n} A_{j,k_\ell}(x)h(x)\ dx = \int_{\mathbb{R}^n} A_j(x)h(x)\ dx \tag{5.11}$$

for every $h \in L^p$. Moreover,

$$\|\{\lambda_j\}_{j\in\mathbb{Z}}\|_{\ell^{q'}} = \left(\sum_{j=-\infty}^{\infty} \lim_{\ell\to\infty} |\lambda_{j,k_\ell}|^{q'}\right)^{\frac{1}{q'}} \le \liminf_{\ell\to\infty} \left(\sum_{j=-\infty}^{\infty} |\lambda_{j,k_\ell}|^{q'}\right)^{\frac{1}{q'}} \le (1+\epsilon)M, \tag{5.12}$$

$\operatorname{supp}(A_j) \subseteq B(2^j)$, and

$$\|A_j\|_{L^{p'}} = \sup_{\|\tilde{g}\|_{L^p}\le 1} \lim_{\ell\to\infty} \int_{\mathbb{R}^n} A_{j,k_\ell}(y)\tilde{g}(y)\ dy \le \limsup_{\ell\to\infty} \|A_{j,k_\ell}\|_{L^{p'}} \le w(2^j). \tag{5.13}$$

Define $g := \sum_{j=-\infty}^{\infty} \lambda_j A_j$. From (5.12) and (5.13), it follows that $g \in LH_{p'q',w}$. Therefore, if we can prove that

$$f(x) = g(x) \text{ a.e. } x \in \mathbb{R}^n, \tag{5.14}$$

then $f \in LH_{p'q',w}$. The proof of (5.14) goes as follows. Let $x \in \mathbb{R}^n \setminus \{0\}$. Once we can show that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} g(y)\ dy = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\ dy \tag{5.15}$$

for all balls $B(x,r)$ which satisfy $0 \notin B(x,2r)$, we have (5.14), by virtue of the Lebesgue differentiation theorem. By substituting $h(y) := \chi_{B(x,r)}(y)$ to (5.11) and using the fact that $B(x,r) \cap B(2^j) = \emptyset$ for every $j < \log_2 r$, we have

$$\int_{B(x,r)} g(y)\ dy = \int_{B(x,r)} \sum_{j=-\infty}^{\infty} \lambda_j A_j(y)\ dy = \int_{B(x,r)} \sum_{j\in\mathbb{Z}, j\ge\log_2 r} \lambda_j A_j(y)\ dy.$$

As a consequence of (5.12) and (5.13), we have

$$\int_{B(x,r)} \sum_{j\in\mathbb{Z},j\geq\log_2 r} |\lambda_j A_j(y)|\ dy \leq |B(x,r)|^{\frac{1}{p}} \sum_{j\in\mathbb{Z},j\geq\log_2 r} \|\lambda_j A_j\|_{L^{p'}}$$

$$\leq |B(x,r)|^{\frac{1}{p}} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'}\right)^{\frac{1}{q'}} \left(\sum_{j\in\mathbb{Z},j\geq\log_2 r} \|A_j\|_{L^{p'}}^q\right)^{\frac{1}{q}}$$

$$\leq |B(x,r)|^{\frac{1}{p}}(1+\varepsilon)M\left(\sum_{j\in\mathbb{Z},j\geq\log_2 r} w(2^j)^q\right)^{\frac{1}{q}},$$

so, by using the doubling condition and $w \in \Omega_{pq}$, we get

$$\int_{B(x,r)} \sum_{j\in\mathbb{Z},j\geq\log_2 r} |\lambda_j A_j(y)|\ dy \lesssim |B(x,r)|^{\frac{1}{p}}(1+\varepsilon)M\left(\sum_{j\in\mathbb{Z},j\geq\log_2 r} \int_{2^j}^{2^{j+1}} \frac{w(t)^q}{t}\ dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|B(x,r)|^{\frac{1}{p}}(1+\varepsilon)M}{r^{\frac{1}{q}}} \left(\int_r^\infty w(t)^q\ dt\right)^{\frac{1}{q}} < \infty.$$

Consequently, by virtue of the dominated convergence theorem, we have

$$\int_{B(x,r)} g(y)\ dy = \sum_{j\in\mathbb{Z},j\geq\log_2 r} \int_{B(x,r)} \lambda_j A_j(y)\ dy.$$

Therefore, once we prove that

$$\sum_{j\in\mathbb{Z}:j\geq\log_2 r} \lambda_j \int_{B(x,r)} A_j(y)\ dy = \lim_{\ell\to\infty} \sum_{j\in\mathbb{Z}:j\geq\log_2 r} \lambda_{j,k_\ell} \int_{B(x,r)} A_{j,k_\ell}(y)\ dy, \tag{5.16}$$

we have

$$\int_{B(x,r)} g(y)\ dy = \lim_{\ell\to\infty} \sum_{j\in\mathbb{Z}:j\geq\log_2 r} \int_{B(x,r)} \lambda_{j,k_\ell} A_{j,k_\ell}(y)\ dy$$

$$= \lim_{\ell\to\infty} \int_{B(x,r)} \sum_{j=-\infty}^{\infty} \lambda_{j,k_\ell} A_{j,k_\ell}(y)\ dy$$

$$= \lim_{\ell\to\infty} \int_{B(x,r)} f_{k_\ell}(y)\ dy = \int_{B(x,r)} f(y)\ dy,$$

so we arrive at (5.15). Hence, we only need to verify (5.16).

Now, assume that $q < \infty$. Since $\int_1^\infty \frac{w(t)^q}{t}\ dt \leq \int_1^\infty w(t)^q\ dt < \infty$, for every $\delta > 0$, there exists $J \in \mathbb{N} \cap (\log_2 r, \infty)$ such that

$$\left(\int_{2^J-1}^\infty \frac{w(t)^q}{t}\ dt\right)^{1/q} < \delta. \tag{5.17}$$

Therefore, for every $\ell \in \mathbb{N}$, we have

$$\begin{aligned}
&\left| \sum_{j\in\mathbb{Z}:j\geq \log_2 r} \int_{B(x,r)} \lambda_j A_j(y)\, dy - \sum_{j\in\mathbb{Z}:j\geq \log_2 r} \int_{B(x,r)} \lambda_{j,k_\ell} A_{j,k_\ell}(y)\, dy \right| \\
&\quad \leq \sum_{j\in\mathbb{Z}:j\geq J} \int_{B(x,r)} |\lambda_j A_j(y)|\, dy + \sum_{j\in\mathbb{Z}:j\geq J} \int_{B(x,r)} |\lambda_{j,k_\ell} A_{j,k_\ell}(y)|\, dy \\
&\qquad + \sum_{j\in Z:\log_2 r\leq j\leq J} \left| \lambda_j \int_{B(x,r)} A_j(y)\, dy - \lambda_{j,k_\ell} \int_{B(x,r)} A_{j,k_\ell}(y)\, dy \right|. \qquad (5.18)
\end{aligned}$$

Let I_1, I_2, and I_3 be the first, second, and third term in the right-hand side of (5.18), respectively. By using Hölder inequality and (5.17), we have

$$\begin{aligned}
I_1 &\leq \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'} \right)^{1/q'} \left(\sum_{j=J}^{\infty} \left(\int_{B(x,r)} |A_j(y)|\, dy \right)^q \right)^{1/q} \\
&\leq (1+\varepsilon) M \left(\sum_{j=J}^{\infty} (\|A_j\|_{L^{p'}} |B(x,r)|^{1/p})^q \right)^{1/q} \\
&\leq (1+\varepsilon) M |B(x,r)|^{1/p} \left(\sum_{j=J}^{\infty} w(2^j)^q \right)^{1/q} \\
&\leq (1+\varepsilon) M |B(x,r)|^{1/p} \left(\int_{2^J-1}^{\infty} \frac{w(t)^q}{t}\, dt \right)^{1/q} \leq (1+\varepsilon) M |B(x,r)|^{1/p} \delta. \qquad (5.19)
\end{aligned}$$

Likewise,

$$I_2 \leq (1+\varepsilon) M |B(x,r)|^{1/p} \delta. \qquad (5.20)$$

Meanwhile, from $\lim_{\ell\to\infty} \lambda_{j,k_\ell} = \lambda_j$ and (5.11), it follows that

$$\lim_{\ell\to\infty} I_3 = \sum_{j\in\mathbb{Z}:\log_2 r\leq j\leq J} \lim_{\ell\to\infty} \left| \lambda_j \int_{B(x,r)} A_j(y)\, dy - \lambda_{j,k_\ell} \int_{B(x,r)} A_{j,k_\ell}(y)\, dy \right| = 0. \qquad (5.21)$$

By combining (5.18)-(5.21), we obtain

$$\begin{aligned}
\limsup_{\ell\to\infty} &\left| \sum_{j\in\mathbb{Z}:j\geq \log_2 r} \int_{B(x,r)} \lambda_j A_j(y)\, dy - \sum_{j\in\mathbb{Z}:j\geq \log_2 r} \int_{B(x,r)} \lambda_{j,k_\ell} A_{j,k_\ell}(y)\, dy \right| \\
&\leq 2(1+\varepsilon) M |B(x,r)|^{1/p} \delta. \qquad (5.22)
\end{aligned}$$

so by taking $\delta \to 0^+$, we get (5.16) when $q < \infty$.

Now, we consider the case $q=\infty$. Since $\lim_{t\to\infty} w(t)=0$, for each $\delta>0$, there exists $K\in\mathbb{N}\cap(\log_2 r,\infty)$ such that

$$w(2^K)<\delta.$$

For every $\ell\in\mathbb{N}$, we have

$$\left|\sum_{j\in\mathbb{Z}:j\geq\log_2 r}\int_{B(x,r)}\lambda_j A_j(y)\ dy-\sum_{j\in\mathbb{Z}:j\geq\log_2 r}\int_{B(x,r)}\lambda_{j,k_\ell}A_{j,k_\ell}(y)\ dy\right|\leq I_4+I_5+I_6 \tag{5.23}$$

where

$$I_4:=\sum_{j=K}^{\infty}|\lambda_j|\int_{B(x,r)}|A_j(y)|\ dy,\ \ I_5:=\sum_{j=K}^{\infty}|\lambda_{j,k_\ell}|\int_{B(x,r)}|A_{j,k_\ell}(y)|\ dy,$$

and

$$I_6:=\left|\sum_{j\in\mathbb{Z}:\log_2 r\leq j<K}\lambda_j\int_{B(x,r)}A_j(y)\ dy-\sum_{j\in\mathbb{Z}:\log_2 r\leq j<K}\lambda_{j,k_\ell}\int_{B(x,r)}A_{j,k_\ell}(y)\ dy\right|.$$

By Hölder's inequality, we get

$$\begin{aligned}I_4&\leq\sum_{j=K}^{\infty}|\lambda_j|\|A_j\|_{L^{p'}}|B(x,r)|^{1/p}\\&\leq|B(x,r)|^{1/p}\sum_{j=K}^{\infty}|\lambda_j|w(2^j)\\&\leq|B(x,r)|^{1/p}w(2^K)\sum_{j=-\infty}^{\infty}|\lambda_j|\leq|B(x,r)|^{1/p}(1+\varepsilon)M\delta.\end{aligned}\tag{5.24}$$

Similarly,

$$I_5\leq|B(x,r)|^{1/p}(1+\varepsilon)M\delta. \tag{5.25}$$

From $\lim_{\ell\to\infty}\lambda_{j,k_\ell}=\lambda_j$ and (5.11), it follows that $\lim_{\ell\to\infty}I_6=0$. We combine this and (5.23)–(5.25) to obtain

$$\begin{aligned}&\limsup_{\ell\to\infty}\left|\sum_{j\in\mathbb{Z}:j\geq\log_2 r}\int_{B(x,r)}\lambda_j A_j(y)\ dy-\sum_{j\in\mathbb{Z}:j\geq\log_2 r}\int_{B(x,r)}\lambda_{j,k_\ell}A_{j,k_\ell}(y)\ dy\right|\\&\qquad\leq 2|B(x,r)|^{1/p}(1+\varepsilon)M\delta.\end{aligned}$$

Therefore, by taking $\delta\to 0^+$, we have (5.16) for $q=\infty$.

Finally, we prove (5.9). From $f = \sum_{j=-\infty}^{\infty} \lambda_j A_j$ and (5.12), it follows that

$$\|f\|_{LH_{p'q',w}} \le (1+\varepsilon)M.$$

Since ε is arbitrary, we have

$$\|f\|_{LH_{p'q',w}} \le M. \tag{5.26}$$

On the other hand, by virtue of Lemma 5.2.2 and $0 \le f_k(x) \le f(x)$ for all $k \in \mathbb{N}$, we have

$$\|f_k\|_{LH_{p'q',w}} \le \|f\|_{LH_{p'q',w}},$$

and hence

$$M \le \|f\|_{LH_{p'q',w}}. \tag{5.27}$$

Thus, (5.9) follows from (5.26) and (5.27). □

5.4 A characterization of the associate space of local Morrey-type spaces

In this section we prove that the associate space of general local Morrey-type spaces can be realized as certain block spaces. First, we recall the definition of the associate space (see [2, Chapter 1]).

Definition 5.4.1. Let X be a Banach space of measurable functions on $\mathbb{R}^n$. The associate space of X, denoted by X', is defined to be the set of all measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{X'} := \sup_{\|g\|_X \le 1} \int_{\mathbb{R}^n} |f(x)g(x)|\, dx$$

is finite.

We remark that a different characterization of the associate space of local Morrey type space can be seen in [17, Theorem 4.3].

Proposition 5.4.2. [31] *Let $1 < p < \infty$ and $1 < q \le \infty$. Assume that $w \in \Omega_{pq}$ satisfies the doubling condition. If $q = \infty$, assume also that $\lim_{t\to\infty} w(t) = 0$. For every $t > 0$, define* $\tilde{w}(t) := \begin{cases} t^{-1/q} w(t), & q < \infty, \\ w(t), & q = \infty. \end{cases}$ *Then $(L\mathcal{M}_{pq,\tilde{w}})' = LH_{p'q',w}$.*

Proof. Let $f \in LH_{p'q',w}$. By virtue of Theorem 5.1.5, for every $g \in L\mathcal{M}_{pq,\tilde{w}}$ with $\|g\|_{L\mathcal{M}_{pq,\tilde{w}}} \le 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)g(x)|\,dx &= \int_{\mathbb{R}^n} \frac{f(x)}{\operatorname{sgn}(f(x))}|g(x)|\,dx \\ &\lesssim \|g\|_{L\mathcal{M}_{pq,\tilde{w}}}\|f\|_{LH_{p'q',w}} \le \|f\|_{LH_{p'q',w}} < \infty.\end{aligned}$$

Therefore, $f \in (L\mathcal{M}_{pq,\tilde{w}})'$. Hence, $LH_{p'q',w} \subseteq (L\mathcal{M}_{pq,\tilde{w}})'$.

Let $f \in (L\mathcal{M}_{pq,\tilde{w}})'$ of norm 1. Then

$$\sup_{\|g\|_{L\mathcal{M}_{pq,\tilde{w}}}\le 1} \int_{\mathbb{R}^n} |f(x)g(x)|\,dx = 1. \tag{5.28}$$

Without loss of generality, we may assume that $f \ge 0$. For each $k \in \mathbb{N}$, define

$$g_k(x) := 2^k \chi_{B(2^k)}(x) \text{ and } f_k(x) := \min(f(x), g_k(x)).$$

Since $\operatorname{supp}(g_k) = B(2^k)$ and $g_k \in L^{p'}$, by virtue of Lemma 5.2.1, we have $g_k \in LH_{p'q',w}$. From $0 \le f_k(x) \le g_k(x)$ and Lemma 5.2.2, it follows that $f_k \in LH_{p'q',w}$. By virtue of the Hahn-Banach theorem, there exists $L \in (LH_{p'q',w})^*$ such that $\|L\|_{(LH_{p'q',w})^*} = 1$ and

$$\|f_k\|_{LH_{p'q',w}} = |L(f_k)|. \tag{5.29}$$

According to Theorem 5.1.5, there exist $h \in L\mathcal{M}_{pq,\tilde{w}}$ such that $\|h\|_{L\mathcal{M}_{pq,\tilde{w}}} \sim 1$ and $L = L_h$. Consequently, by combining (5.28) and (5.29), we get

$$\|f_k\|_{LH_{p'q',w}} = |L_h(f_k)| \le \int_{\mathbb{R}^n} f_k(x)|h(x)|\,dx \le \|h\|_{L\mathcal{M}_{pq,\tilde{w}}} \sim 1,$$

so, $\sup_{k\in\mathbb{N}} \|f_k\|_{LH_{p'q',w}} \lesssim 1$. Since $\{f_k\}_{k=1}^{\infty}$ is increasing and $\lim_{k\to\infty} f_k(x) = f(x)$, by virtue of Proposition 5.3.1, we have $f \in LH_{p'q',w}$ and

$$\|f\|_{LH_{p'q',w}} \lesssim 1.$$

Hence, $(L\mathcal{M}_{pq,w})' \subseteq LH_{p'q',w}$. This completes the proof. □

We shall use the following corollary.

Corollary 5.4.3. [31] *Let $1 < p_0, p_1 < \infty$, $1 < q_0, q_1 \le \infty$, $w_0 \in \Omega_{p_0q_0}$, and $w_1 \in \Omega_{p_1q_1}$. Assume that w_0 and w_1 satisfy the doubling condition. If $q_0 = \infty$ and $q_1 = \infty$, assume also that $\lim_{t\to\infty} w_0(t) = 0$ and $\lim_{t\to\infty} w_1(t) = 0$. Define*

$$\tilde{w}_0(t) := \begin{cases} t^{-1/q_0} w_0(t), & q_0 < \infty, \\ w_0(t), & q_0 = \infty, \end{cases} \text{ and } \tilde{w}_1(t) := \begin{cases} t^{-1/q_1} w_1(t), & q_1 < \infty, \\ w_1(t), & q_1 = \infty. \end{cases}$$

If $f \in LH_{p_0'q_0',w_0} + LH_{p_1'q_1',w_1}$ *and* $g \in L\mathcal{M}_{p_0q_0,\tilde{w}_0} \cap L\mathcal{M}_{p_1q_1,\tilde{w}_1}$*, then* $fg \in L^1$ *and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \lesssim \|f\|_{LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}} \|g\|_{L\mathcal{M}_{p_0q_0,\tilde{w}_0}\cap L\mathcal{M}_{p_1q_1,\tilde{w}_1}}. \tag{5.30}$$

Proof. Let $f_0 \in LH_{p_0'q_0',w_0}$ and $f_1 \in LH_{p_1'q_1',w_1}$ be such that $f = f_0 + f_1$ and

$$\|f_0\|_{LH_{p_0'q_0',w_0}} + \|f_1\|_{LH_{p_1'q_1',w_1}} \lesssim \|f\|_{LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}}. \tag{5.31}$$

Then, by virtue of Proposition 5.4.2, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)g(x)|\, dx &\le \int_{\mathbb{R}^n} |f_0(x)g(x)|\, dx + \int_{\mathbb{R}^n} |f_1(x)g(x)|\, dx \\ &\le \|f_0\|_{(L\mathcal{M}^{p_0q_0,\tilde{w}_0})'} \|g\|_{L\mathcal{M}^{p_0q_0,\tilde{w}_0}} + \|f_1\|_{(L\mathcal{M}^{p_1q_1,\tilde{w}_1})'} \|g\|_{L\mathcal{M}^{p_1q_1,\tilde{w}_1}} \\ &\lesssim (\|f_0\|_{LH_{p_0'q_0',w_0}} + \|f_1\|_{LH_{p_1'q_1',w_1}}) \|g\|_{L\mathcal{M}_{p_0q_0,\tilde{w}_0}\cap L\mathcal{M}_{p_1q_1,\tilde{w}_1}}.\end{aligned} \tag{5.32}$$

Thus, (5.30) follows from (5.31) and (5.32). □

5.5 Interpolation of local block spaces

Our main results are given in the following theorems.

Theorem 5.5.1. [31] *Let* $\theta \in (0,1)$, $1 < p_0, p_1 < \infty$, $w_0 \in \Omega_{p_0\infty}$, *and* $w_1 \in \Omega_{p_1\infty}$. *Assume that* w_0 *and* w_1 *satisfy the doubling condition,* $\lim_{t\to\infty} w_0(t) = 0$, *and* $\lim_{t\to\infty} w_1(t) = 0$. *In addition, assume that* $w_0(t)^{p_0} = w_1(t)^{p_1}$. *Define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } w := w_0^{1-\theta} w_1^{\theta}.$$

Then

$$[LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta = LH_{p'1,w}.$$

Theorem 5.5.2. [31] *Let* $\theta \in (0,1)$, $1 < p_0, p_1 < \infty$, $1 < q_0, q_1 < \infty$, $w_0 \in \Omega_{p_0q_0}$, *and* $w_1 \in \Omega_{p_1q_1}$. *Assume that* w_0 *and* w_1 *satisfy the doubling condition. In addition, assume that* $\frac{p_0}{q_0} = \frac{p_1}{q_1}$ *and* $w_0(t)^{q_0} = w_1(t)^{q_1}$. *Define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1},\ \text{and } w := w_0^{1-\theta} w_1^{\theta}.$$

Then

$$[LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta = LH_{p'q',w}.$$

First we give the proof of Theorem 5.5.1.

Proof of Theorem 5.5.1. Without loss of generality, we may assume that $p_0 > p_1$. Since $\lim_{t\to\infty} w_0(t) = \lim_{t\to\infty} w_1(t) = 0$, we have $\lim_{t\to\infty} w(t) = 0$.

Let $f \in [LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta$. We shall show that $f \in LH_{p'1,w}$. According to Proposition 5.4.2, it suffices to show that

$$\sup_{\|g\|_{L\mathcal{M}_{p\infty,w}}\leq 1} \int_{\mathbb{R}^n} |f(x)g(x)|\ dx \lesssim \|f\|_{[LH_{p_0'1,w_0},LH_{p_1'1,w_1}]_\theta}. \tag{5.33}$$

Choose $F \in \mathcal{F}(LH_{p_0'1,w_0}, LH_{p_1'1,w_1})$ such that $f = F(\theta, \cdot)$ and

$$\|F\|_{\mathcal{F}(LH_{p_0'1,w_0},LH_{p_1'1,w_1})} \lesssim \|f\|_{[LH_{p_0'1,w_0},LH_{p_1'1,w_1}]_\theta}. \tag{5.34}$$

Let $g \in L\mathcal{M}_{p\infty,w}$ with $\|g\|_{L\mathcal{M}_{p\infty,w}} \leq 1$ and $M := \sup_{z\in\overline{S}} \|F(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}$. For $k \in \mathbb{N}$ and $z \in \overline{S}$, define

$$H_k(z,x) := \chi_{\{f\neq 0\}}(x)\frac{|f(x)|}{f(x)}|g(x)|^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)}\chi_{\left\{\frac{1}{k}\leq|g|\leq k\right\}}(x) \quad (x \in \mathbb{R}^n)$$

and

$$\phi_k(z) := \int_{\mathbb{R}^n} F(z,x)H_k(z,x)\ dx. \tag{5.35}$$

Since

$$\phi_k(\theta) = \int_{\mathbb{R}^n} f(x)H_k(\theta,x)\ dx = \int_{\mathbb{R}^n} |f(x)g(x)|\chi_{\{\frac{1}{k}\leq|g|\leq k\}}(x)\ dx.$$

we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|\ dx = \lim_{k\to\infty} \phi_k(\theta).$$

Hence, in order to obtain (5.33), we only need to prove that

$$\phi_k(\theta) \lesssim \|f\|_{[LH_{p_0'1,w_0},LH_{p_1'1,w_1}]_\theta}, \tag{5.36}$$

for all $k \in \mathbb{N}$.

We now prove (5.36). For every $z \in \overline{S}$, we have

$$|H_k(z,x)| \leq |g(x)|^{\frac{p}{p_0}}|g(x)|^{\left(\frac{p}{p_1}-\frac{p}{p_0}\right)\mathrm{Re}(z)}\chi_{\{\frac{1}{k}\leq|g|\leq k\}}(x) \leq k^{\frac{p}{p_1}-\frac{p}{p_0}}|g(x)|^{\frac{p}{p_0}} \tag{5.37}$$

and

$$|H_k(z,x)| \le |g(x)|^{\frac{p}{p_1}} |g(x)|^{-\left(\frac{p}{p_1}-\frac{p}{p_0}\right)(1-\mathrm{Re}(z))} \chi_{\{\frac{1}{k}\le |g| \le k\}}(x) \le k^{\frac{p}{p_1}-\frac{p}{p_0}} |g(x)|^{\frac{p}{p_1}}. \tag{5.38}$$

Note that, our assumptions imply $w_0(r)^{p_0} = w_1(r)^{p_1} = w(r)^p$. Therefore,

$$\left\| |g|^{\frac{p}{p_0}} \right\|_{L\mathcal{M}_{p_0\infty,w_0}} = \sup_{r>0} w(r)^{\frac{p}{p_0}} \|g\|_{L^p(B(r))}^{\frac{p}{p_0}} = \|g\|_{L\mathcal{M}_{p\infty,w}}^{\frac{p}{p_0}} \le 1 \tag{5.39}$$

and

$$\left\| |g|^{\frac{p}{p_1}} \right\|_{L\mathcal{M}_{p_1\infty,w_1}} = \sup_{r>0} w(r)^{\frac{p}{p_1}} \|g\|_{L^p(B(r))}^{\frac{p}{p_1}} = \|g\|_{L\mathcal{M}_{p\infty,w}}^{\frac{p}{p_1}} \le 1. \tag{5.40}$$

By combining (5.37)-(5.40), we have $H_k(z) \in L\mathcal{M}_{p_0\infty,w_0} \cap L\mathcal{M}_{p_1\infty,w_1}$ with

$$\|H_k(z,\cdot)\|_{L\mathcal{M}_{p_0\infty,w_0}\cap L\mathcal{M}_{p_1\infty,w_1}} \le k^{\frac{p}{p_1}-\frac{p}{p_0}}. \tag{5.41}$$

It folllows from the last inequality and Corollary 5.4.3, that

$$|\phi_k(z)| \lesssim k^{\frac{p}{p_1}-\frac{p}{p_0}} \|F(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}. \tag{5.42}$$

Therefore,

$$\sup_{z\in\overline{S}} |\phi_k(z)| \lesssim k^{\frac{p}{p_1}-\frac{p}{p_0}} M < \infty. \tag{5.43}$$

Next we estimate $|\phi_k(z)|$ on the boundary of $\overline{S}$. Let $t\in\mathbb{R}$. As a consequence of (5.39) and $|H_k(it,x)| \le |g(x)|^{\frac{p}{p_0}}$, we have

$$\|H_k(it,\cdot)\|_{L\mathcal{M}_{p_0\infty,w_0}} \le 1.$$

From the last inequality and Proposition 5.4.2, it follows that

$$\begin{aligned} |\phi_k(it)| &\le \int_{\mathbb{R}^n} |F(it,x)H_k(it,x)|\, dx \\ &\le \|F(it,\cdot)\|_{(L\mathcal{M}_{p_0\infty,w_0})'} \lesssim \|F(it,\cdot)\|_{LH_{p_0'1,w_0}} \le \|F\|_{\mathcal{F}(LH_{p_0'1,w_0},LH_{p_1'1,w_1})}. \end{aligned} \tag{5.44}$$

By a similar argument, we also have

$$|\phi_k(1+it)| \lesssim \|F\|_{\mathcal{F}(LH_{p_0'1,w_0},LH_{p_1'1,w_1})}. \tag{5.45}$$

In view of Lemma 5.6.1 and (5.43), we may use the three-lines lemma together with (5.34), (5.44), and (5.45) to obtain

$$\begin{aligned} \phi_k(\theta) &\le \left(\sup_{t\in\mathbb{R}} |\phi_k(it)|\right)^{1-\theta} \left(\sup_{t\in\mathbb{R}} |\phi_k(1+it)|\right)^{\theta} \\ &\le \|F\|_{\mathcal{F}(LH_{p_0'1,w_0},LH_{p_1'1,w_1})} \lesssim \|f\|_{[LH_{p_0'1,w_0},LH_{p_1'1,w_1}]_\theta}. \end{aligned}$$

Thus, the proof of $[LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta \subseteq LH_{p'1,w}$ is complete.

Conversely, we shall show that $LH_{p'1,w} \subseteq [LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta$. Let $f \in LH_{p'1,w}$. Then, there exist $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^1$ and $\{(A_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{\mathcal{A}}_w(L^{p'})$ such that

$$f = \sum_{j=-\infty}^{\infty} \lambda_j A_j \text{ and } \sum_{j=-\infty}^{\infty} |\lambda_j| \lesssim \|f\|_{LH_{p'1,w}}. \tag{5.46}$$

For every $J \in \mathbb{N}$ and $z \in \overline{S}$, define

$$F_J(z,\cdot) := \sum_{j=-J}^{J} \lambda_j \frac{w_0(2^j)^{1-z} w_1(2^j)^z}{w(2^j)^{p'\left(\frac{1-z}{p_0'}+\frac{z}{p_1'}\right)}} \operatorname{sgn}(A_j(\cdot))|A_j(\cdot)|^{p'\left(\frac{1-z}{p_0'}+\frac{z}{p_1'}\right)}. \tag{5.47}$$

We claim that

$$F_J(\theta,\cdot) \in [LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta \tag{5.48}$$

and

$$\lim_{\substack{J,K\to\infty \\ J>K}} \|F_J(\theta,\cdot) - F_K(\theta,\cdot)\|_{[LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta} = 0. \tag{5.49}$$

The proof of (5.48) and (5.49) will be given in Section 5.1. As a consequence of (5.49), there exists $g \in [LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta$ such that

$$\lim_{J\to\infty} \|F_J(\theta,\cdot) - g\|_{[LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta} = 0. \tag{5.50}$$

From (5.50) and $[LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta \subseteq LH_{p'1,w}$, it follows that

$$\lim_{J\to\infty} \|F_J(\theta,\cdot) - g\|_{LH_{p'1,w}} = 0. \tag{5.51}$$

Since $f - F_J(\theta,\cdot) = \sum_{j\in\mathbb{Z},|j|>J} \lambda_j A_j(\cdot)$, we have

$$\|f - F_J(\theta,\cdot)\|_{LH_{p'1,w}} \le \sum_{j\in\mathbb{Z},|j|>J} |\lambda_j| \to 0.$$

Therefore,

$$\lim_{J\to\infty} \|f - F_J(\theta,\cdot)\|_{LH_{p'1,w}} = 0. \tag{5.52}$$

Combining (5.51) and (5.52), we have $f = g$. Hence, $f \in [LH_{p_0'1,w_0}, LH_{p_1'1,w_1}]_\theta$. □

We now prove Theorem 5.5.2.

Proof of Theorem 5.5.2. Without loss of generality, assume that $p_0 > p_1$. First, we prove that

$$[LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta \subseteq LH_{p'q',w}. \tag{5.53}$$

Let $f \in [LH_{p_0'q_0,w_0}, LH_{p_1'q_1',w_1}]_\theta$. We shall show that $f \in LH_{p'q',w}$. Let $\tilde{w}(t) := t^{-\frac{1}{q}} w(t)$. By Proposition 5.4.2, we only need to show that

$$\sup_{\|g\|_{L\mathcal{M}_{pq,\tilde{w}}} \le 1} \int_{\mathbb{R}^n} |f(x)g(x)| \, dx \lesssim \|f\|_{[LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta}. \tag{5.54}$$

Let $g \in L\mathcal{M}_{pq,\tilde{w}}$ with $\|g\|_{L\mathcal{M}_{pq,\tilde{w}}} \le 1$. By the definition of $[LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta$, there exists $F \in \mathcal{F}(LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1})$ such that $f = F(\theta, \cdot)$ and that

$$\|F\|_{\mathcal{F}(LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1})} \lesssim \|f\|_{[LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta}. \tag{5.55}$$

For $k \in \mathbb{N}$ and $z \in \overline{S}$, define

$$H_k(z,x) := \chi_{\{f \neq 0\}}(x) \frac{|f(x)|}{f(x)} |g(x)|^{p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right)} \chi_{\left\{\frac{1}{k} \le |g| \le k\right\}}(x)$$

and

$$\phi_k(z) := \int_{\mathbb{R}^n} F(z,x) H_k(z,x) \, dx.$$

Since

$$\begin{aligned} \lim_{k\to\infty} \phi_k(\theta) &= \lim_{k\to\infty} \int_{\mathbb{R}^n} f(x) H_k(\theta, x) \, dx \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^n} |f(x)g(x)| \chi_{\{\frac{1}{k} \le |g| \le k\}}(x) \, dx = \int_{\mathbb{R}^n} |f(x)g(x)| \, dx, \end{aligned}$$

the inequality (5.54) can be obtained if we can show that

$$\phi_k(\theta) \lesssim \|f\|_{[LH_{p_0'q_0',w_0}, LH_{p_1'q_1',w_1}]_\theta}, \tag{5.56}$$

for all $k \in \mathbb{N}$.

The proof of (5.56) goes as follows. Let $z \in \overline{S}$. Let $\tilde{w}_0(t) := t^{-\frac{1}{q_0}} w_0(t)$ and $\tilde{w}_1(t) := t^{-\frac{1}{q_1}} w_1(t)$. From $w_0(t)^{q_0} = w_1(t)^{q_1}$, $w(t) = w_0(t)^{1-\theta} w_1(t)^\theta$, and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_0}$, it follows that

$$w_0(t)^{q_0} = w_1(t)^{q_1} = w(t)^q,$$

which yields

$$\tilde{w}_0(t)^{q_0} = \tilde{w}_1(t)^{q_1} = \tilde{w}(t)^q.$$

By using this identity, for every $j \in \{0,1\}$, we have

$$\left\| |g|^{\frac{p}{p_j}} \right\|_{L\mathcal{M}_{p_j q_j, \tilde{w}_j}} = \left\| \tilde{w}(r)^{\frac{q}{q_j}} \left\| |g|^{\frac{p}{p_j}} \chi_{B(r)} \right\|_{L^{p_j}(B(r))} \right\|_{L^{q_j}(0,\infty)}$$
$$= \left\| \left(\tilde{w}(r) \| g \chi_{B(r)} \|_{L^p(B(r))} \right)^{\frac{p}{p_j}} \right\|_{L^{q_j}(0,\infty)} = \|g\|_{L\mathcal{M}_{pq,\tilde{w}}}^{\frac{p}{p_j}} \leq 1, \quad (5.57)$$

where we used $\frac{p_0}{q_0} = \frac{p_1}{q_1} = \frac{p}{q}$. From inequalities (5.37), (5.38), and (5.57), it follows that $H_k(z, \cdot) \in L\mathcal{M}_{p_0 q_0, w_0} \cap L\mathcal{M}_{p_1 q_1, w_1}$ and

$$\|H_k(z, \cdot)\|_{L\mathcal{M}_{p_0 q_0, w_0} \cap L\mathcal{M}_{p_1 q_1, w_1}} \leq k^{\frac{p}{p_0} - \frac{p}{p_1}}. \quad (5.58)$$

Therefore, by combining (5.30) and (5.58), we have

$$\sup_{z \in \overline{S}} |\phi_k(z)| \lesssim k^{\frac{p}{p_1} - \frac{p}{p_0}} \sup_{z \in \overline{S}} \|F(z)\|_{LH_{p_0' 1, w_0} + LH_{p_1' 1, w_1}} < \infty. \quad (5.59)$$

Next, we estimate $|\phi_k(j+it)|$ for every $j \in \{0,1\}$ and $t \in \mathbb{R}$. Since $|H_k(j+it, \cdot)| \leq |g|^{p/p_j}$, by virtue of (5.57), we have

$$\|H_k(j+it, \cdot)\|_{L\mathcal{M}_{p_0 q_0, \tilde{w}_0}} \leq 1. \quad (5.60)$$

Consequently, by combining (5.55), (5.60), and Proposition 5.4.2, we see that

$$\begin{aligned} |\phi_k(j+it)| &\leq \int_{\mathbb{R}^n} |F(j+it, x) H_k(j+it, x)| \, dx \\ &\leq \|F(j+it, \cdot)\|_{(L\mathcal{M}_{p_0 q_0, \tilde{w}_0})'} \\ &\sim \|F(j+it, \cdot)\|_{LH_{p_0' q_0', w_0}} \\ &\leq \|F\|_{\mathcal{F}(LH_{p_0' q_0', w_0}, LH_{p_1' q_1', w_1})} \lesssim \|f\|_{[LH_{p_0' q_0', w_0}, LH_{p_1' q_1', w_1}]_\theta}. \quad (5.61) \end{aligned}$$

By using an analogous argument as in the proof of Lemma 5.6.1, we have $\phi_k(z)$ is continuous on $\overline{S}$ and holomorphic in S. Since (5.59) holds. we may use the three-lines lemma and estimate (5.61) to obtain

$$\phi_k(\theta) \lesssim \|f\|_{[LH_{p_0' q_0', w_0}, LH_{p_1' q_1', w_1}]_\theta},$$

as desired. Thus, the proof of (5.53) is complete.

Now, suppose that $f \in LH_{p'q',w}$. We shall show that $f \in [LH_{p_0' q_0', w_0}, LH_{p_1' q_1', w_1}]_\theta$. Write

$$f = \sum_{j=-\infty}^{\infty} \lambda_j A_j \quad (5.62)$$

for some $\{(A_j, 2^j)\}_{j=-\infty}^{\infty} \in \dot{\mathcal{A}}_w(L^{p'})$ and $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$ satisfying

$$\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^{q'}\right)^{\frac{1}{q'}} \lesssim \|f\|_{LH_{p'q',w}}. \tag{5.63}$$

For $J \in \mathbb{N}$ and $z \in \overline{S}$, define

$$F_J(z,\cdot) := \sum_{j=-J}^{J} \operatorname{sgn}(\lambda_j)|\lambda_j|^{q'\left(\frac{1-z}{q'_0}+\frac{z}{q'_1}\right)} \frac{w_0(2^j)^{1-z} w_1(2^j)^z}{w(2^j)^{p'\left(\frac{1-z}{p'_0}+\frac{z}{p'_1}\right)}} \operatorname{sgn}(A_j(\cdot))|A_j(\cdot)|^{p'\left(\frac{1-z}{p'_0}+\frac{z}{p'_1}\right)}. \tag{5.64}$$

We claim that

$$F_J(\theta,\cdot) \in [LH_{p'_0q'_0,w_0}, LH_{p'_1q'_1,w_1}]_\theta \tag{5.65}$$

and

$$\lim_{\substack{J,K\to\infty\\ J>K}} \|F_J(\theta,\cdot) - F_K(\theta,\cdot)\|_{[LH_{p'_0q'_0,w_0},LH_{p'_1q'_1,w_1}]_\theta} = 0. \tag{5.66}$$

We postpone the proof of (5.65) and (5.66) to Section 5.2. As a consequence of (5.65) and (5.66), there exists $g \in [LH_{p'_0q'_0,w_0}, LH_{p'_1q'_1,w_1}]_\theta$ such that

$$\lim_{J\to\infty} \|F_J(\theta,\cdot) - g\|_{[LH_{p'_0q'_0,w_0},LH_{p'_1q'_1,w_1}]_\theta} = 0. \tag{5.67}$$

By combining (5.67) and $[LH_{p'_0q'_0,w_0}, LH_{p'_1q'_1,w_1}]_\theta \subseteq LH_{p'q',w}$, we have

$$\lim_{J\to\infty} \|F_J(\theta,\cdot) - g\|_{LH_{p'q',w}} = 0. \tag{5.68}$$

From $f - F_J(\theta,\cdot) = \sum_{j\in\mathbb{Z},|j|>J} \lambda_j A_j(\cdot)$, (5.68), and $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$, it follows that

$$\begin{aligned}\|f-g\|_{LH_{p'q',w}} &\le \|f - F_J(\theta,\cdot)\|_{LH_{p'q',w}} + \|F_J(\theta,\cdot) - g\|_{LH_{p'q',w}} \\ &\le \left(\sum_{j\in\mathbb{Z},|j|>J} |\lambda_j|^{q'}\right)^{\frac{1}{q'}} + \|F_J(\theta,\cdot) - g\|_{LH_{p'q',w}} \to 0\end{aligned}$$

as $J \to \infty$. Consequently, $f = g$. Hence, $f \in [LH_{p'_01,w_0}, LH_{p'_11,w_1}]_\theta$. Thus, the proof of Theorem 5.5.2 is complete. □

5.6 Some lemmas about the first complex interpolation functor

In this section we provide the proof of continuity and holomorphicity of the function $\phi_k(z)$ in Subsection 4.1. We also prove (5.48), (5.49), (5.65) and (5.66).

Lemma 5.6.1. [31] *Let* $\phi_k(z)$ *be defined by* (5.35). *Then* $\phi_k(z)$ *is continuous on* $\overline{S}$ *and holomorphic in* S.

Proof. Let $z_1, z_2 \in \overline{S}$, $P := \frac{p}{p_1} - \frac{p}{p_0}$ and $M := \sup_{z\in\overline{S}} \|F(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}$. By combining (5.30) and (5.41) and

$$|H_k(z_2,x) - H_k(z_1,x)| \le |H_k(z_1,x)| \left(|g(x)|^{P|z_2-z_1|} - 1\right) \le |H_k(z_1,x)| \left(k^{P|z_2-z_1|} - 1\right),$$

we obtain

$$\begin{aligned}
&|\phi_k(z_2) - \phi_k(z_1)| \\
&\le \int_{\mathbb{R}^n} |F(z_2,x) - F(z_1,x)||H_k(z_2,x)|\, dx \\
&\quad + \int_{\mathbb{R}^n} |F(z_1,x)(H_k(z_2,x) - H_k(z_1,x))|\, dx \\
&\le k^P \|F(z_2,\cdot) - F(z_1,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} + k^P M \left(k^{P|z_2-z_1|} - 1\right). \qquad (5.69)
\end{aligned}$$

Hence, the continuity of $\phi_k(z)$ on $\overline{S}$ follows from (5.69) and the continuity of $F : \overline{S} \to LH_{p_0'1,w_0} + L\mathcal{M}_{p_1'1,w_1}$.

Now, we prove the holomorphicity of $\phi_k(z)$. Let $z \in S$. Then there exists $F'(z,\cdot) \in LH_{p_0'1,w_0} + LH_{p_1'1,w_1}$ such that

$$\lim_{\substack{h\to 0\\ z+h\in\overline{S}}} \left\| \frac{F(z,\cdot) - F(z,\cdot)}{h} - F'(z,\cdot) \right\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} = 0. \qquad (5.70)$$

Define $H_k'(z,\cdot) = P \cdot H_k(z,\cdot) \log|g|$ and

$$\phi_k'(z) := \int_{\mathbb{R}^n} (F'(z,x)H_k(z,x) + F(z,x)H_k'(z,x))\, dx.$$

Let $h \in \mathbb{C}$ be such that $z + h \in \overline{S}$. From (5.30), it follows that

$$\left| \frac{\phi_k(z+h) - \phi_k(z)}{h} - \phi_k'(z) \right| \le I_1 + I_2 + I_3 \qquad (5.71)$$

where

$$I_1 := \|H_k(z,\cdot)\|_{L\mathcal{M}_{p_0\infty,w_0}\cap L\mathcal{M}_{p_0\infty,w_0}} \times \left\| \frac{F(z,\cdot)-F(z,\cdot)}{h} - F'(z,\cdot) \right\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}, \tag{5.72}$$

$$I_2 := \|F(z+h,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} \times \left\| \frac{H_k(z+h,\cdot)-H_k(z,\cdot)}{h} - H_k'(z,\cdot) \right\|_{L\mathcal{M}_{p_0\infty,w_0}\cap L\mathcal{M}_{p_1\infty,w_1}}, \tag{5.73}$$

and

$$I_3 := \|H_k'(z,\cdot)\|_{L\mathcal{M}_{p_0\infty,w_0}\cap L\mathcal{M}_{p_1\infty,w_1}} \|F(z+h,\cdot)-F(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}. \tag{5.74}$$

Combining (5.41), (5.71)-(5.74), and

$$\left| \frac{H_k(z+h,x)-H_k(z,x)}{h} - H_k'(z,x) \right| \le |H_k(z,x)| \left(k^{P|h|}-1\right) P \log k,$$

we get

$$\begin{aligned} &\left| \frac{\phi_k(z+h)-\phi_k(z)}{h} - \phi_k'(z) \right| \\ &\le k^P \left\| \frac{F(z,\cdot)-F(z,\cdot)}{h} - F'(z,\cdot) \right\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} + MC_{k,P}(k^{P|h|}-1) \\ &\quad + Pk^P(\log k)\|F(z+h,\cdot)-F(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}. \end{aligned} \tag{5.75}$$

By virtue of continuity of $F(z,\cdot)$ and (5.70), the right-hand side of (5.75) tends to zero as $h \to 0$, so $\phi_k'(z)$ is the derivative of $\phi_k(z)$. Thus, $\phi_k(z)$ is holomorphic.

□

Proof of (5.48) and (5.49)

In this subsection, we suppose that f is any function in $LH_{p'1,w}$ such that (5.46) holds and $F_J(z)$ is defined by (5.47). For each $j \in [-J,J]\cap\mathbb{Z}$, we define

$$u_j := \frac{w_0(2^j)^{-1}w_1(2^j)}{w(2^j)^{\frac{p'}{p_1'}-\frac{p'}{p_0'}}} |A_j|^{\frac{p'}{p_1'}-\frac{p'}{p_0'}}. \tag{5.76}$$

For each $j \in [-J, J] \cap \mathbb{Z}$ and $z \in \overline{S}$, define

$$G_j(z,\cdot) := \frac{w_0(2^j)^{1-z} w_1(2^j)^z}{w(2^j)^{p'\left(\frac{1-z}{p_0'}+\frac{z}{p_1'}\right)}} \operatorname{sgn}(A_j(\cdot))|A_j(\cdot)|^{p'\left(\frac{1-z}{p_0'}+\frac{z}{p_1'}\right)}, \tag{5.77}$$

$$G_{j,0}(z,\cdot) := G_j(z,\cdot)\chi_{\{u_j \le 1\}}, \quad \text{and} \quad G_{j,1}(z,\cdot) := G_j(z,\cdot) - G_{j,0}(z,\cdot). \tag{5.78}$$

We prove (5.48) by checking the conditions given in Definition 2.1.1. We shall use the following calculation of the norm of some blocks.

Lemma 5.6.2. [31] *Let $k \in \{0,1\}$. Then, for each $j \in \mathbb{Z} \cap [-J, J]$*

$$\left\| \frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p_k'}}} |A_j|^{\frac{p'}{p_k'}} \right\|_{LH_{p_k' 1, w_k}} \le 1. \tag{5.79}$$

Proof. Inequality (5.79) follows from Lemma 5.2.1, $\||A_j|^{p'/p_k'}\|_{L^{p_k'}} = \|A_j\|_{L^{p'}}^{p'/p_k'}$, and $\|A_j\|_{L^{p'}} \le w(2^j)$. □

Lemma 5.6.3. [31] *For every $z \in \overline{S}$, we have $F_J(z,\cdot) \in LH_{p_0' 1, w_0} + LH_{p_1' 1, w_1}$. Moreover, $\sup_{z \in \overline{S}} \|F_J(z,\cdot)\|_{LH_{p_0' 1, w_0} + LH_{p_1' 1, w_1}} < \infty$.*

Proof. We decompose $F_J(z,\cdot) = F_{J,0}(z,\cdot) + F_{J,1}(z,\cdot)$ where

$$F_{J,0}(z,\cdot) := \sum_{j=-J}^{J} \lambda_j G_{j,0}(z,\cdot) \quad \text{and} \quad F_{J,1}(z,\cdot) := \sum_{j=-J}^{J} \lambda_j G_{j,1}(z,\cdot). \tag{5.80}$$

Combining Lemma 5.6.2 and

$$|G_{j,0}(z,\cdot)| = \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p_0'}}} |A_j(\cdot)|^{\frac{p'}{p_0'}} u_j^{\mathrm{Re}(z)} \chi_{\{u_j \le 1\}} \le \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p_0'}}} |A_j(\cdot)|^{\frac{p'}{p_0'}},$$

we have

$$\|F_{J,0}(z,\cdot)\|_{LH_{p_0' 1, w_0}} \le \sum_{j=-J}^{J} |\lambda_j| \|G_{j,0}\|_{LH_{p_0' 1, w_0}} \le \sum_{j=-J}^{J} |\lambda_j| \lesssim \|f\|_{LH_{p' 1, w}} < \infty.$$

Therefore, $F_{J,0}(z,\cdot) \in LH_{p_0' 1, w_0}$. Similarly, we also have $F_{J,1}(z) \in LH_{p_1' 1, w_1}$ with

$$\|F_{J,1}(z,\cdot)\|_{LH_{p_1' 1, w_1}} \lesssim \|f\|_{LH_{p' 1, w}}.$$

Since $F_J(z,\cdot)=F_{J,1}(z,\cdot)+F_{J,2}(z,\cdot)$, we have $F_J(z)\in LH_{p_0'1,w_0}+LH_{p_1'1,w_1}$ and

$$\begin{aligned}\|F_J(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} &\le \|F_{J,0}(z,\cdot)\|_{LH_{p_0'1,w_0}}+\|F_{J,1}(z,\cdot)\|_{LH_{p_1'1,w_1}}\\ &\lesssim \|f\|_{LH_{p'1,w}}.\end{aligned} \tag{5.81}$$

Thus, $\sup\limits_{z\in\overline{S}}\|F_J(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}\lesssim\|f\|_{LH_{p'1,w}}<\infty$. □

Lemma 5.6.4. [31] *The function $F_J:\overline{S}\to LH_{p_0'1,w_0}+LH_{p_1'1,w_1}$ is continuous.*

Proof. Let $z\in\overline{S}$. We shall show that

$$\lim_{\substack{h\to 0\\ z+h\in\overline{S}}}\|F_J(z+h,\cdot)-F_J(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}=0. \tag{5.82}$$

Let $F_{J,0}$ and $F_{J,1}$ be defined by (5.80). For every $h\in\mathbb{C}$ satisfying $z+h\in\mathbb{C}$, we have

$$\begin{aligned}\|F_{J,0}(z+h,\cdot)-F_{J,0}(z,\cdot)\|_{LH_{p_0'1,w_0}} &\le \sum_{j=-J}^{J}|\lambda_j|\|G_{j,0}(z+h,\cdot)-G_{j,0}(z,\cdot)\|_{LH_{p_0'1,w_0}}\\ &\le \sum_{j=-J}^{J}\frac{|\lambda_j|}{w_0(2^j)}\|G_{j,0}(z+h,\cdot)-G_{j,0}(z,\cdot)\|_{L^{p_0'}}.\end{aligned}$$

Since

$$\lim_{h\to 0}(G_{j,0}(z+h,x)-G_{j,0}(z,x))=G_{j,0}(z,x)\lim_{h\to 0}(u_j(x)^h-1)=0,\ x\in\mathbb{R}^n,$$

we have

$$\begin{aligned}|G_{j,0}(z+h,\cdot)-G_{j,0}(z,\cdot)| &\le \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p_0'}}}|A_j(\cdot)|^{\frac{p'}{p_0'}}\left(u_j^{\mathrm{Re}(z)}\chi_{\{u_j\le 1\}}+u_j^{\mathrm{Re}(z)}\chi_{\{u_j\le 1\}}\right)\\ &\le 2\frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p_0'}}}|A_j(\cdot)|^{\frac{p'}{p_0'}},\end{aligned}$$

and $|A_j|^{\frac{p'}{p_0'}}\in L^{p_0'}$, by virtue of the dominated convergence theorem, we have

$$\lim_{h\to 0}\|G_{j,0}(z+h,\cdot)-G_{j,0}(z,\cdot)\|_{L^{p_0'}}=0.$$

Consequently,

$$\lim_{h\to 0}\|F_{J,0}(z+h,\cdot)-F_{J,0}(z,\cdot)\|_{LH_{p_0'1,w_0}}=0. \tag{5.83}$$

By a similar argument, we also have

$$\lim_{h\to 0}\|F_{J,1}(z+h,\cdot)-F_{J,1}(z,\cdot)\|_{LH_{p_1'1,w_0}}=0. \tag{5.84}$$

Combining (5.83), (5.84), and

$$\begin{aligned}\|F_J(z+h,\cdot)-F_J(z,\cdot)\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}} &\le \|F_{J,0}(z+h,\cdot)-F_{J,0}(z,\cdot)\|_{LH_{p_0'1,w_0}}\\ &\quad+\|F_{J,1}(z+h,\cdot)-F_{J,1}(z,\cdot)\|_{LH_{p_1'1,w_1}},\end{aligned}$$

we get (5.82). □

Lemma 5.6.5. [31] *The function $F_J: S\to LH_{p_0'1,w_0}+LH_{p_1'1,w_1}$ is holomorphic.*

Proof. Let $\varepsilon\in(0,1/2)$. It suffices to show that

$$F_J'(z):=\sum_{j=-J}^{J}\lambda_j G_j(z,\cdot)\log(u_j(\cdot))\in LH_{p_0'1,w_0}+LH_{p_1'1,w_1} \tag{5.85}$$

for every $z\in S$ and

$$\lim_{\substack{h\to 0\\ z+h\in S}}\left\|\frac{F_J(z+h)-F_J(z)}{h}-F_J(z)\right\|_{LH_{p_0'1,w_0}+LH_{p_1'1,w_1}}=0 \tag{5.86}$$

for every $z\in S_\varepsilon:=\{\tilde z\in S:\varepsilon<\mathrm{Re}(\tilde z)<1-\varepsilon\}$. We define

$$F_{J,0}'(z,\cdot):=\sum_{j=-J}^{J}\lambda_j G_{j,0}(z,\cdot)\log(u_j(\cdot)) \text{ and } F_{J,1}'(z,\cdot):=\sum_{j=-J}^{J}\lambda_j G_{j,1}(z,\cdot)\log(u_j(\cdot)). \tag{5.87}$$

Since

$$\begin{aligned}|G_{j,0}(z,\cdot)\log(u_j(\cdot))| &= \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p_0'}}}|A_j(\cdot)|^{\frac{p'}{p_0'}}u_j(\cdot)^{\mathrm{Re}(z)}|\log(u_j(\cdot))|\chi_{\{u_j\le 1\}}\\ &\le \frac{w_0(2^j)}{e\mathrm{Re}(z)w(2^j)^{\frac{p'}{p_0'}}}|A_j(\cdot)|^{\frac{p'}{p_0'}},\end{aligned}$$

by virtue of Lemma 5.6.2, we have

$$\begin{aligned}\|F_{J,0}'(z,\cdot)\|_{LH_{p_0'1,w_0}} &\le \sum_{j=-J}^{J}|\lambda_j|\left\|\frac{w_0(2^j)}{e\mathrm{Re}(z)w(2^j)^{\frac{p'}{p_0'}}}|A_j|^{p'/p_0'}\right\|_{LH_{p_0'1,w_0}}\\ &\le \frac{1}{e\mathrm{Re}(z)}\sum_{j=-J}^{J}|\lambda_j|\lesssim\|f\|_{LH_{p'1,w}}<\infty,\end{aligned}$$

so $F'_{J,0}(z,\cdot) \in LH_{p'_0 1, w_0}$. By a similar argument, we have $F'_{J,1}(z,\cdot) \in LH_{p'_1 1, w_1}$. Since $F'_J(z,\cdot) = F'_{J,0}(z,\cdot) + F'_{J,1}(z,\cdot)$, we conclude that $F'_J(z,\cdot) \in LH_{p'_0 1, w_0} + LH_{p'_1 1, w_1}$.

Now, we prove (5.86). Let $z \in S_\varepsilon$ and $h \in C$ be such that $z + h \in S$ and $|h| < \frac{\varepsilon}{2}$. Since

$$|G_{j,0}(z)| = \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}} u_j(\cdot)^{\mathrm{Re}(z)} \chi_{\{u_j \le 1\}} \le \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}} \chi_{\{u_j \le 1\}},$$

we have

$$\begin{aligned}
&\left| \frac{G_{j,0}(z+h,\cdot) - G_{j,0}(z,\cdot)}{h} - G_{j,0}(z,\cdot)\log(u_j(\cdot)) \right| \\
&\quad = |G_{j,0}(z,\cdot)| \left| \frac{u_j(\cdot)^h - 1 - h\log(u_j(\cdot))}{h} \right| \\
&\quad \le \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}} u_j(\cdot)^{\mathrm{Re}(z)} \chi_{\{u_j \le 1\}} |h| (\log(u_j(\cdot)))^2 \sum_{k=2}^{\infty} \frac{|h\log(u_j(\cdot))|^{k-2}}{k!} \\
&\quad \le \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}} u_j(\cdot)^{\varepsilon} \chi_{\{u_j \le 1\}} |h| (\log(u_j(\cdot)))^2 e^{-|h|\log(u_j(\cdot))}. \qquad (5.88)
\end{aligned}$$

Combining (5.88) and $\sup_{0<t\le 1} t^{\varepsilon/2}(\log t)^2 = \frac{16}{e^2\varepsilon^2}$, we get

$$\begin{aligned}
&\left| \frac{G_{j,0}(z+h,\cdot) - G_{j,0}(z,\cdot)}{h} - G_{j,0}(z,\cdot)\log(u_j(\cdot)) \right| \\
&\qquad \le \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}} u_j(\cdot)^{\varepsilon/2} \chi_{\{u_j \le 1\}} |h| (\log(u_j(\cdot)))^2 \\
&\qquad \le C_\varepsilon |h| \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}}. \qquad (5.89)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\| \frac{F_{J,0}(z+h,\cdot) - F_{J,0}(z,\cdot)}{h} - F'_{J,0}(z,\cdot) \right\|_{LH_{p'_0 1, w_0}} \\
&\quad \le C_\varepsilon |h| \sum_{j=-J}^{J} |\lambda_j| \left\| \frac{w_0(2^j)|A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}} \right\|_{LH_{p'_0 1, w_0}} \le C_\varepsilon |h| \sum_{j=-J}^{J} |\lambda_j|.
\end{aligned}$$

Consequently,

$$\lim_{h\to 0}\left\|\frac{F_{J,0}(z+h,\cdot)-F_{J,0}(z,\cdot)}{h}-F'_{J,0}(z,\cdot)\right\|_{LH_{p'_0 1,w_0}}=0. \tag{5.90}$$

Likewise,

$$\lim_{h\to 0}\left\|\frac{F_{J,1}(z+h,\cdot)-F_{J,1}(z,\cdot)}{h}-F'_{J,1}(z,\cdot)\right\|_{LH_{p'_1 1,w_1}}=0. \tag{5.91}$$

Thus, (5.86) follows from (5.90) and (5.91). □

Lemma 5.6.6. [31] *For every $k\in\{0,1\}$, the function $t\in\mathbb{R}\mapsto F_J(k+it)\in LH_{p'_k 1,w_k}$ is bounded and continuous.*

Proof. First, we shall show that

$$\max_{k=0,1}\sup_{t\in\mathbb{R}}\|F_J(k+it)\|_{LH_{p'_k 1,w_k}}<\infty. \tag{5.92}$$

For each $k\in\{0,1\}$, $t\in\mathbb{R}$, and $j\in[-J,J]\cap\mathbb{Z}$, we have

$$|G_j(k+it)|=\frac{w_k(2^j)}{w(2^j)^{p'/p'_k}}|u_j^{it}||A_j|^{\frac{p'}{p'_k}}=\frac{w_k(2^j)}{w(2^j)^{p'/p'_k}}|A_j|^{\frac{p'}{p'_k}}, \tag{5.93}$$

so, by virtue of Lemma 5.6.2, we have

$$\|F_J(k+it)\|_{LH_{p'_k 1,w_k}}\le\sum_{j=-J}^{J}|\lambda_j|\lesssim\|f\|_{LH_{p'_k 1,w_k}}.$$

Therefore,

$$\max_{k=0,1}\sup_{t\in\mathbb{R}}\|F_J(k+it)\|_{LH_{p'1,w}}\lesssim\|f\|_{LH_{p'1,w}}<\infty.$$

Now, we prove the continuity of $t\in\mathbb{R}\mapsto F_J(k+it)$. Let $t_0\in\mathbb{R}$. By virtue of Lemma 5.2.1, we have

$$\begin{aligned}\|F_J(k+it,\cdot)-&F_J(k+it_0,\cdot)\|_{LH_{p'_k 1,w_k}}\\&\le\sum_{j=-J}^{J}\|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)\|_{LH_{p'_k 1,w_k}}\\&\le\sum_{j=-J}^{J}\frac{\|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)\|_{L^{p'_k}}}{w_k(2^j)}.\end{aligned} \tag{5.94}$$

Since $|A_j|^{\frac{p'}{p'_k}} \in L^{p'_k}$, $|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)| \le 2\frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p'_k}}}|A_j|^{\frac{p'}{p'_k}}$, and

$$\lim_{t\to t_0}(G_j(k+it,x)-G_j(k+it_0,x)) = G_j(k+it_0,x)\lim_{t\to t_0} u_j(x)^{t-t_0}-1=0$$

for every $x\in\mathbb{R}^n$, by virtue of the dominated convergence theorem, we have

$$\lim_{t\to t_0}\|G_j(k+it)-G_j(k+it_0)\|_{L^{p'_k}}=0. \tag{5.95}$$

By combining (5.94) and (5.95), we get

$$\lim_{t\to t_0}\|F_J(k+it,\cdot)-F_J(k+it,\cdot)\|_{LH_{p'_k1,w_k}}=0,$$

as desired. □

From Lemmas 5.6.3-5.6.6, it follows that $F_J\in\mathcal{F}(LH_{p'_01,w_0},LH_{p'_11,w_1})$. Consequently, $F_J(\theta)\in[LH_{p'_01,w_0},LH_{p'_11,w_1}]_\theta$. Thus, it remains to prove (5.49).

Proof of (5.49). Let $J,K\in\mathbb{N}$ with $J>K$. From (5.93) and Lemma 5.6.2, it follows that

$$\begin{aligned}
&\|F_J(\theta,\cdot)-F_K(\theta,\cdot)\|_{[LH_{p'_01,w_0},LH_{p'_11,w_1}]_\theta}\\
&\quad\le\max_{k=0,1}\sup_{t\in\mathbb{R}}\|F_J(k+it,\cdot)-F_K(k+it,\cdot)\|_{LH_{p'_k1,w_k}}\\
&\quad\le\max_{k=0,1}\sup_{t\in\mathbb{R}}\sum_{\substack{j\in\mathbb{Z}\\ J\ge|j|>K}}|\lambda_j|\,\|G_j(k+it,\cdot)\|_{LH_{p'_k1,w_k}}\\
&\quad=\max_{k=0,1}\sup_{t\in\mathbb{R}}\sum_{\substack{j\in\mathbb{Z}\\ J\ge|j|>K}}|\lambda_j|\left\|\frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p'_k}}}|A_j(\cdot)|^{\frac{p'}{p'_k}}\right\|_{LH_{p'_k1,w_k}}\le\sum_{\substack{j\in\mathbb{Z}\\ J\ge|j|>K}}|\lambda_j|.
\end{aligned}\tag{5.96}$$

Since $\{\lambda_j\}_{j=-\infty}^{\infty}\in\ell^1$, we see that

$$\lim_{\substack{J,K\to\infty\\ J>K}}\sum_{\substack{j\in\mathbb{Z}\\ J\ge|j|>K}}|\lambda_j|=0. \tag{5.97}$$

Thus, (5.49) follows from (5.96) and (5.97). □

Proof of (5.65) and (5.66)

Let $F_J(z)$ be defined by (5.64). For each $j \in [-J, J] \cap \mathbb{Z}$, we define

$$u_j := |\lambda_j|^{\frac{q'}{q'_1}-\frac{q'}{q'_0}} \frac{w_0(2^j)^{-1}w_1(2^j)}{w(2^j)^{\frac{p'}{p'_1}-\frac{p'}{p'_0}}} |A_j|^{\frac{p'}{p'_1}-\frac{p'}{p'_0}}. \tag{5.98}$$

For each $j \in [-J, J] \cap \mathbb{Z}$ and $z \in \overline{S}$, define

$$G_j(z, \cdot) := \operatorname{sgn}(\lambda_j)|\lambda_j|^{q'\left(\frac{1-z}{q'_0}+\frac{z}{q'_1}\right)} \frac{w_0(2^j)^{1-z}w_1(2^j)^z}{w(2^j)^{p'\left(\frac{1-z}{p'_0}+\frac{z}{p'_1}\right)}} \operatorname{sgn}(A_j(\cdot))|A_j(\cdot)|^{p'\left(\frac{1-z}{p'_0}+\frac{z}{p'_1}\right)}. \tag{5.99}$$

Lemma 5.6.7. [31] *For every $z \in \overline{S}$ and $j \in [-J, J] \cap \mathbb{Z}$, define*

$$G_{j,0}(z, \cdot) := G_j(z, \cdot)\chi_{\{u_j \le 1\}}, \quad \text{and} \quad G_{j,1}(z, \cdot) := G_j(z, \cdot) - G_{j,0}(z, \cdot). \tag{5.100}$$

Then, for each $k \in \{0, 1\}$, we have

$$|G_{j,k}(z, \cdot)| \le |\lambda_j|^{\frac{q'}{q'_k}} \frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p'_0}}} |A_j(\cdot)|^{\frac{p'}{p'_0}}. \tag{5.101}$$

Proof. We prove (5.101) only for $k = 0$. We leave to the reader the case $k = 1$ because there are no differences. Since $\operatorname{Re}(z) \ge 0$, we have

$$|G_{j,1}(z, \cdot)| = |\lambda_j|^{\frac{q'}{q'_0}} \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p'_0}}} |A_j(\cdot)|^{\frac{p'}{p'_0}} u_j^{\operatorname{Re}(z)} \chi_{\{u_j \le 1\}} \le |\lambda_j|^{\frac{q'}{q'_0}} \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p'_0}}} |A_j(\cdot)|^{\frac{p'}{p'_0}},$$

as desired. □

Lemma 5.6.8. [31] *Let $k \in \{0, 1\}$ and $j \in [-J, J] \cap \mathbb{Z}$. Then $\frac{w_k(2^j)}{w(2^j)^{p'/p'_k}}|A_j|^{p'/p'_k}$ is a $(p'_k, w_k, 2^j)$-block.*

Proof. Since $\operatorname{supp}(A_j) \subseteq B(2^j)$, we have $\operatorname{supp}\left(\frac{w_k(2^j)}{w(2^j)^{p'/p'_k}}|A_j|^{p'/p'_k}\right) \subseteq B(2^j)$. Moreover, from $\|A_j\|_{L^{p'}} \le w(2^j)$ it follows that

$$\left\| \frac{w_k(2^j)}{w(2^j)^{p'/p'_k}} |A_j|^{p'/p'_k} \right\|_{L^{p'_k}} = \frac{w_k(2^j)}{w(2^j)^{p'/p'_k}} \|A_j\|_{L^{p'}}^{p'/p'_k} \le w_k(2^j).$$

Thus, $\frac{w_k(2^j)}{w(2^j)^{p'/p'_k}}|A_j|^{p'/p'_k}$ is a $(p'_k, w_k, 2^j)$-block. □

Lemma 5.6.9. [31] *For every* $z \in \overline{S}$*, we have* $F_J(z, \cdot) \in LH_{p_0'q_0',w_0} + LH_{p_1'q_1',w_1}$*. Moreover,* $\sup_{z\in\overline{S}} \|F_J(z, \cdot)\|_{LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}} < \infty$*.*

Proof. We define

$$F_{J,0}(z, \cdot) := \sum_{j=-J}^{J} G_{j,0}(z, \cdot) \quad \text{and} \quad F_{J,1}(z, \cdot) := \sum_{j=-J}^{J} G_{j,1}(z, \cdot). \tag{5.102}$$

We use (5.3), (5.63), and Lemma 5.6.8 to obtain

$$\begin{aligned}\|F_{J,0}(z, \cdot)\|_{LH_{p_0'q_0',w_0}} &\le \left\| \sum_{j=-J}^{J} |G_{j,0}(z, \cdot)| \right\|_{LH_{p_0'q_0',w_0}} \\ &\le \left\| \sum_{j=-J}^{J} |\lambda_j|^{\frac{q'}{q_0'}} \frac{w_0(2^j)}{w(2^j)^{p'/p_0'}} |A_j|^{p'/p_0'} \right\|_{LH_{p_0'1,w_0}} \\ &\le \left(\sum_{j=-J}^{J} (|\lambda_j|^{q'/q_0'})^{q_0'} \right)^{1/q_0'} \lesssim \|f\|_{LH_{p'q',w}}^{q'/q_0'} < \infty.\end{aligned}$$

Therefore, $F_{J,0}(z, \cdot) \in LH_{p_0'q_0',w_0}$. Likewise, $F_{J,1}(z) \in LH_{p_1'q_1',w_1}$ with

$$\|F_{J,1}(z, \cdot)\|_{LH_{p_1'q_1',w_1}} \lesssim \|f\|_{LH_{p'q',w}}^{q'/q_1'}.$$

Consequently, $F_J(z, \cdot) \in LH_{p_0'q_0',w_0} + LH_{p_1'q_1',w_1}$ and

$$\begin{aligned}\|F_J(z, \cdot)\|_{LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}} &\le \|F_{J,0}(z, \cdot)\|_{LH_{p_0'q_0',w_0}} + \|F_{J,1}(z, \cdot)\|_{LH_{p_1'q_1',w_1}} \\ &\lesssim \|f\|_{LH_{p'q',w}}^{q'/q_0'} + \|f\|_{LH_{p'q'.w}}^{q'/q_1'}.\end{aligned} \tag{5.103}$$

Thus, $\sup_{z\in\overline{S}} \|F_J(z, \cdot)\|_{LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}} \lesssim \|f\|_{LH_{p'q',w}}^{q'/q_0'} + \|f\|_{LH_{p'q'.w}}^{q'/q_1'} < \infty.$ □

Lemma 5.6.10. [31] *The function* $F_J : \overline{S} \to LH_{p_0'q_0',w_0} + LH_{p_1'q_1',w_1}$ *is continuous.*

Proof. Let $z \in \overline{S}$. For $h \in \mathbb{C}$ with $z + h \in \overline{S}$, we have

$$\begin{aligned}&\|F_J(z + h, \cdot) - F_J(z, \cdot)\|_{LH_{p_0'q_0',w_0}+LH_{p_1'q_1',w_1}} \\ &\quad \le \|F_{J,0}(z + h, \cdot) - F_{J,0}(z, \cdot)\|_{LH_{p_0'q_0',w_0}} + \|F_{J,1}(z + h, \cdot) - F_{J,1}(z, \cdot)\|_{LH_{p_1'q_1',w_1}}.\end{aligned} \tag{5.104}$$

Hence, it suffices to show that

$$\lim_{\substack{h\to 0\\ z+h\in\overline{S}}} \|F_{J,k}(z+h,\cdot)-F_{J,k}(z,\cdot)\|_{LH_{p'_kq'_k,w_k}} = 0 \tag{5.105}$$

for each $k \in \{0,1\}$. By virtue of Lemma 5.2.1, we have

$$\begin{aligned}\|F_{J,k}(z+h,\cdot)-F_{J,k}(z,\cdot)\|_{LH_{p'_kq'_k,w_k}} &\le \sum_{j=-J}^{J} \|G_{j,k}(z+h,\cdot)-G_{j,k}(z,\cdot)\|_{LH_{p'_kq'_k,w_k}} \\ &\le \sum_{j=-J}^{J} \frac{\|G_{j,k}(z+h,\cdot)-G_{j,k}(z,\cdot)\|_{L^{p'_k}}}{w_k(2^j)}.\end{aligned} \tag{5.106}$$

According to Lemma 5.6.7, we have

$$\begin{aligned}|G_{j,k}(z+h,\cdot)-G_{j,k}(z,\cdot)| &\le |G_{j,k}(z+h,\cdot)|+|G_{j,k}(z,\cdot)| \\ &\le 2|\lambda_j|^{\frac{q'}{q'_k}} \frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p'_k}}} |A_j(\cdot)|^{\frac{p'}{p'_k}}.\end{aligned}$$

Since

$$\lim_{h\to 0}(G_{j,k}(z+h,x)-G_{j,k}(z,x)) = G_{j,k}(z,x)\lim_{h\to 0}(u_j(x)^h-1) = 0$$

for every $x \in \mathbb{R}^n$ and $|A_j|^{\frac{p'}{p'_k}} \in L^{p'_k}$, by virtue of the dominated convergence theorem, we have

$$\lim_{h\to 0} \|G_{j,k}(z+h,\cdot)-G_{j,k}(z,\cdot)\|_{L^{p'_k}} = 0. \tag{5.107}$$

Thus, we obtain (5.105) by combining (5.106) and (5.107). □

Lemma 5.6.11. [31] *The function $F_J : S \to LH_{p'_0q'_0,w_0}+LH_{p'_1q'_1,w_1}$ is holomorphic.*

Proof. Let $\varepsilon \in (0,1/2)$ and $S_\varepsilon := \{z \in S : \varepsilon < \mathrm{Re}(z) < 1-\varepsilon\}$. We shall show that

$$F'_J(z) := \sum_{j=-J}^{J} G_j(z,\cdot)\log(u_j(\cdot)) \in LH_{p'_0q'_0,w_0}+LH_{p'_1q'_1,w_1}. \tag{5.108}$$

for every $z \in S$ and

$$\lim_{\substack{h\to 0\\ z+h\in S}} \left\| \frac{F_J(z+h)-F_J(z)}{h} - F'_J(z) \right\|_{LH_{p'_0q'_0,w_0}+LH_{p'_1q'_1,w_1}} = 0 \tag{5.109}$$

for every $z \in S_\varepsilon$. We define

$$F'_{J,0}(z,\cdot) := \sum_{j=-J}^{J} \lambda_j G_{j,0}(z,\cdot)\log(u_j(\cdot)) \text{ and } F'_{J,1}(z,\cdot) := \sum_{j=-J}^{J} \lambda_j G_{j,1}(z,\cdot)\log(u_j(\cdot)). \tag{5.110}$$

Since

$$\begin{aligned} |G_{j,0}(z,\cdot)\log(u_j(\cdot))| &= |\lambda_j|^{\frac{q'}{q'_0}} \frac{w_0(2^j)}{w(2^j)^{\frac{p'}{p'_0}}} |A_j(\cdot)|^{\frac{p'}{p'_0}} u_j(\cdot)^{\mathrm{Re}(z)} |\log(u_j(\cdot))| \chi_{\{u_j \le 1\}} \\ &\le |\lambda_j|^{\frac{q'}{q'_0}} \frac{w_0(2^j)}{e\mathrm{Re}(z) w(2^j)^{\frac{p'}{p'_0}}} |A_j(\cdot)|^{\frac{p'}{p'_0}} \end{aligned}$$

by virtue of Lemma 5.6.8, we have

$$\|F'_{J,0}(z,\cdot)\|_{LH_{p'_0 q'_0, w_0}} \le \frac{1}{e\mathrm{Re}(z)} \left(\sum_{j=-J}^{J} |\lambda_j|^{q'} \right)^{1/q'_0} \lesssim \frac{\|f\|_{LH_{p'q',w}}^{q'/q'_0}}{e\mathrm{Re}(z)} < \infty,$$

so $F'_{J,0}(z,\cdot) \in LH_{p'_0 q'_0, w_0}$. Similarly, $F'_{J,1}(z,\cdot) \in LH_{p'_1 q'_1, w_1}$. Since

$$F'_J(z,\cdot) = F'_{J,0}(z,\cdot) + F'_{J,1}(z,\cdot),$$

we see that $F'_J(z,\cdot) \in LH_{p'_0 q'_0, w_0} + LH_{p'_1 q'_1, w_1}$.

The proof of (5.109) is obtained in a similar way as (5.86). Let $z \in S_\varepsilon$ and $h \in C$ be such that $z + h \in S$ and $|h| < \frac{\varepsilon}{2}$. By a similar calculation as in (5.89), we obtain

$$\left| \frac{G_{j,0}(z+h,\cdot) - G_{j,0}(z,\cdot)}{h} - G_{j,0}(z,\cdot)\log(u_j(\cdot)) \right| \le C_\varepsilon |h| |\lambda_j|^{\frac{q'}{q'_0}} \frac{w_0(2^j) |A_j(\cdot)|^{\frac{p'}{p'_0}}}{w(2^j)^{\frac{p'}{p'_0}}}.$$

The last inequality, (5.63), and Lemma 5.6.8 imply

$$\begin{aligned} \left\| \frac{F_{J,0}(z+h,\cdot) - F_{J,0}(z,\cdot)}{h} - F'_{J,0}(z,\cdot) \right\|_{LH_{p'_0 q'_0, w_0}} &\le C_\varepsilon |h| \left(\sum_{j=-J}^{J} |\lambda_j|^{q'} \right)^{1/q'_0} \\ &\lesssim C_\varepsilon |h| \|f\|_{LH_{p'q',w}}^{\frac{q'}{q'_0}}. \end{aligned}$$

Consequently,

$$\lim_{h \to 0} \left\| \frac{F_{J,0}(z+h,\cdot) - F_{J,0}(z,\cdot)}{h} - F'_{J,0}(z,\cdot) \right\|_{LH_{p'_0 q'_0, w_0}} = 0. \tag{5.111}$$

Similarly,

$$\lim_{h\to 0}\left\|\frac{F_{J,1}(z+h,\cdot)-F_{J,1}(z,\cdot)}{h}-F'_{J,1}(z,\cdot)\right\|_{LH_{p_1'q_1',w_1}}=0. \tag{5.112}$$

Thus, (5.109) follows from (5.111) and (5.112). □

Lemma 5.6.12. [31] *For every $k\in\{0,1\}$, the function $t\in\mathbb{R}\mapsto F_J(k+it)\in LH_{p_k'q_k',w_k}$ is bounded and continuous.*

Proof. First, we shall prove that

$$\max_{k=0,1}\sup_{t\in\mathbb{R}}\|F_J(k+it)\|_{LH_{p_k'q_k',w_k}}<\infty. \tag{5.113}$$

Let $k\in\{0,1\}$, $t\in\mathbb{R}$, and $j\in[-J,J]\cap\mathbb{Z}$. By virtue of Lemma 5.6.8 and

$$|G_j(k+it)|=|\lambda_j|^{\frac{q'}{q_k'}}\frac{w_k(2^j)}{w(2^j)^{p'/p_k'}}|u_j^{it}||A_j|^{\frac{p'}{p_k'}}=|\lambda_j|^{\frac{q'}{q_k'}}\frac{w_k(2^j)}{w(2^j)^{p'/p_k'}}|A_j|^{\frac{p'}{p_k'}}, \tag{5.114}$$

we have

$$\|F_J(k+it)\|_{LH_{p_k'q_k',w_k}}\le\left(\sum_{j=-J}^{J}|\lambda_j|^{q'}\right)^{1/q_k'}\lesssim\|f\|_{LH_{p'q',w}}^{q'/q_k'}.$$

Therefore,

$$\max_{k=0,1}\sup_{t\in\mathbb{R}}\|F_J(k+it)\|_{LH_{p_k'q_k',w_k}}\lesssim\|f\|_{LH_{p'q',w}}^{q'/q_k'}<\infty.$$

The proof of the continuity of $t\in\mathbb{R}\mapsto F_J(k+it)$ goes as follows. Let $t_0\in\mathbb{R}$. According to Lemma 5.2.1, we have

$$\begin{aligned}\|F_J(k+it,\cdot)-&F_J(k+it_0,\cdot)\|_{LH_{p_k'q_k',w_k}}\\&\le\sum_{j=-J}^{J}|\lambda_j|\|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)\|_{LH_{p_k'q_k',w_k}}\\&\le\sum_{j=-J}^{J}|\lambda_j|\frac{\|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)\|_{L^{p_k'}}}{w_k(2^j)}.\end{aligned} \tag{5.115}$$

Since $|A_j|^{\frac{p'}{p_k'}}\in L^{p_k'}$, $|G_j(k+it,\cdot)-G_j(k+it_0,\cdot)|\le 2|\lambda_j|^{\frac{q'}{q_k'}}\frac{w_k(2^j)}{w(2^j)^{\frac{p'}{p_k'}}}|A_j|^{\frac{p'}{p_k'}}$, and

$$\lim_{t\to t_0}G_j(k+it,x)-G_j(k+it,x)=G_j(k+it_0,x)\lim_{t\to t_0}\left(u_j(x)^{t-t_0}-1\right)=0,$$

by virtue the dominated convergence theorem, we have

$$\lim_{t\to t_0}\|G_j(k+it)-G_j(k+it_0)\|_{L^{p_k'}}=0. \tag{5.116}$$

By combining (5.115) and (5.116), we get

$$\lim_{t\to t_0} \|F_J(k+it,\cdot) - F_J(k+it,\cdot)\|_{LH_{p'_k q'_k, w_k}} = 0,$$

as desired. □

According to Lemmas 5.6.9-5.6.12, we have

$$F_J \in \mathcal{F}(LH_{p'_0 q'_0, w_0}, LH_{p'_1 q'_1, w_1}).$$

Therefore, $F_J(\theta,\cdot) \in [LH_{p'_0 q'_0, w_0}, LH_{p'_1 q'_1, w_1}]_\theta$. Thus, it remains to prove (5.66).

Proof of (5.66). Let $J, K \in \mathbb{N}$ with $J > K$. By virtue of (5.114) and Lemma 5.114, we have

$$\begin{aligned} &\|F_J(\theta,\cdot) - F_K(\theta,\cdot)\|_{[LH_{p'_0 q'_0, w_0}, LH_{p'_1 q'_1, w_1}]_\theta} \\ &\qquad \le \max_{k=0,1} \sup_{t\in\mathbb{R}} \|F_J(k+it,\cdot) - F_K(k+it,\cdot)\|_{LH_{p'_k q'_k, w_k}} \\ &\qquad = \max_{k=0,1} \sup_{t\in\mathbb{R}} \left\| \sum_{\substack{j\in\mathbb{Z} \\ J\ge|j|>K}} G_j(k+it,\cdot) \right\|_{LH_{p'_k q'_k, w_k}} \le \left(\sum_{\substack{j\in\mathbb{Z} \\ J\ge|j|>K}} |\lambda_j|^{q'} \right)^{\frac{1}{q'_k}}. \end{aligned} \tag{5.117}$$

Since $\{\lambda_j\}_{j=-\infty}^{\infty} \in \ell^{q'}$, the right-hand side of (5.117) tends to zero as $J, K \to \infty$, which implies (5.49). □

Chapter 6

Complex interpolation of grand Lebesgue spaces

In this chapter discuss interpolation of grand Lebesgue spaces by complex method. First let us explain the motivation of considering complex interpolation of grand Lebesgue spaces according to the result of complex interpolation of generalized Morrey spaces. Let Ω be a bounded measurable subset of $\mathbb{R}^n$ and define $\mathcal{M}_q^\varphi(\Omega)$ to be the set of all measurable functions f on Ω which belong to $\mathcal{M}_q^\varphi$, where it is understood tacitly that f is extended to be zero outside Ω. Substituting $U = L^0(\Omega)$ to Theorems 4.2.1 and 4.3.1 we get

Corollary 6.0.1. *Keep the same assumption as in Theorem 4.2.1. Then*

$$[\mathcal{M}_{q_0}^{\varphi_0}(\Omega), \mathcal{M}_{q_1}^{\varphi_1}(\Omega)]_\theta = \{f \in \mathcal{M}_q^\varphi(\Omega) : \lim_{N\to\infty} \|f\chi_{\{|f|>N\}}\|_{\mathcal{M}_q^\varphi(\Omega)} = 0\}$$

and

$$[\mathcal{M}_{q_0}^{\varphi_0}(\Omega), \mathcal{M}_{q_1}^{\varphi_1}(\Omega)]^\theta = \mathcal{M}_q^\varphi(\Omega).$$

It seems that Morrey spaces on the bounded domain behaves better than Morrey spaces on $\mathbb{R}^n$ under the first complex interpolation results. Hence, it is interesting to consider complex interpolation of some function spaces defined on set of finite measure other than Morrey space on the bounded domain, such as grand Lebesgue spaces.

Let us recall the definition of grand Lebesgue spaces. Let (Ω, μ) be a finite measure space, $1 < p < \infty$, and $\tau > 0$. The grand Lebesgue space $L^{p),\tau} =$

$L^{p),\tau}(\Omega,\mu)$ is the set of all measurable functions f for which

$$\|f\|_{L^{p),\tau}(\Omega,\mu)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\tau}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\Omega,\mu)} < \infty.$$

The space $L^{p),1}(\Omega,\mu)$ was first introduced by T. Iwaniec and C. Sbordone in their studies on the integrability condition of the Jacobian function on a domain $\Omega \subseteq \mathbb{R}^n$ (see [33]). Meanwhile, the definition of $L^{p),\tau}(\Omega,\mu)$ for general $\tau > 0$ can be found in [18]. An example of a member of $L^{p),\tau}(\Omega,\mu)$ is given as follows:

Example 6.0.2. [15, 26] Let $\Omega := (0,1)$, μ be the Lebesgue measure, and $f(x) := x^{-a}$. Then we can verify that

$$f \in L^{p),\tau} \Leftrightarrow \begin{cases} a < \frac{1}{p}, \\ a = \frac{1}{p}, \tau \geq 1. \end{cases}$$

It is known in [15, Remark 2] that the function $f(x) := x^{-1/p}$ is in $L^{p),\tau}(0,1)$ but it does not have absolutely continuous norm.

6.1 Basic properties of grand Lebesgue spaces

We recall the following inclusion of Lebesgue spaces on Ω to grand Lebesgue spaces on Ω which can be found in [16].

Lemma 6.1.1. [16, 26] *Let $1 < p < \infty$ and $r \in [p,\infty]$. Then we have*

$$L^r(\Omega,\mu) \subset L^{p),\tau}(\Omega,\mu) \subset \bigcap_{0<\varepsilon<p-1} L^{p-\varepsilon}(\Omega,\mu). \tag{6.1}$$

Proof. The inclusion (6.1) is stated in [16]. We give the proof of (6.1) for convenience of the reader. Let $f \in L^r(\Omega,\mu)$. If $r < \infty$, then for any $\varepsilon \in (0, p-1)$, we have

$$\begin{aligned} \varepsilon^{\frac{\tau}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\Omega,\mu)} &\leq \max(1,p-1)^\tau \mu(\Omega)^{\frac{1}{p-\varepsilon}-\frac{1}{r}} \|f\|_{L^r(\Omega,\mu)} \\ &\leq \frac{\max(1,p-1)^\tau \max(1,\mu(\Omega))}{\mu(\Omega)^{1/r}} \|f\|_{L^r(\Omega,\mu)}. \end{aligned}$$

Therefore,

$$\|f\|_{L^{p),\tau}(\Omega,\mu)} \leq \frac{\max(1,p-1)^\tau \max(1,\mu(\Omega))}{\mu(\Omega)^{1/r}} \|f\|_{L^r(\Omega,\mu)}.$$

Since $L^\infty(\Omega,\mu) \subseteq L^r(\Omega,\mu)$, $L^\infty(\Omega,\mu) \subseteq L^{p),\tau}(\Omega,\mu)$. If $f \in L^{p),\tau}(\Omega,\mu)$ and $\varepsilon \in (0, p-1)$, then

$$\|f\|_{L^{p-\varepsilon}(\Omega,\mu)} \leq \varepsilon^{-\frac{\tau}{p-\varepsilon}} \|f\|_{L^{p),\tau}(\Omega,\mu)} < \infty, \tag{6.2}$$

so $f \in \bigcap\limits_{0<\varepsilon<p-1} L^{p-\varepsilon}(\Omega,\mu)$ as desired. □

The next lemma is the scaling property of grand Lebesgue spaces:

Lemma 6.1.2. [26] *Let $0 < \tau < \infty$ and $1 < q \leq r \leq s < \infty$. Then there exists $C > 0$ such that for every $f \in L^{r),\tau}(\Omega,\mu)$, we have*

$$\||f|^{\frac{r}{s}}\|_{L^{s),\tau}(\Omega,\mu)} \leq C\|f\|^{\frac{r}{s}}_{L^{r),\tau}(\Omega,\mu)} \tag{6.3}$$

and

$$\left\||f|^{\frac{r}{q}}\right\|_{L^{q),\tau}(\Omega,\mu)} \leq \|f\|^{\frac{r}{q}}_{L^{r),\tau}(\Omega,\mu)}. \tag{6.4}$$

Proof. Let $\varepsilon \in (0, q-1)$ and $\delta := \frac{r}{q}\varepsilon$. Then we have

$$\varepsilon^{\frac{\tau}{q-\varepsilon}} \left\||f|^{\frac{r}{q}}\right\|_{L^{q-\varepsilon}(\Omega,\mu)} = \varepsilon^{\frac{\tau}{q-\varepsilon}} \|f\|^{\frac{r}{q}}_{L^{r-\frac{r\varepsilon}{q}}(\Omega,\mu)} = \left[\left(\frac{q}{r}\delta\right)^{\frac{\tau}{r-\delta}} \|f\|_{L^{r-\delta}(\Omega,\mu)}\right]^{\frac{r}{q}}.$$

Since $0 < \delta = \frac{r}{q}\varepsilon < r - \frac{r}{q} < r-1$, we have

$$\varepsilon^{\frac{\tau}{q-\varepsilon}} \left\||f|^{\frac{r}{q}}\right\|_{L^{q-\varepsilon}(\Omega,\mu)} \leq \left[\delta^{\frac{\tau}{r-\delta}} \|f\|_{L^{r-\delta}(\Omega,\mu)}\right]^{\frac{r}{q}} \leq \|f\|^{\frac{r}{q}}_{L^{r),\tau}(\Omega,\mu)},$$

and hence (6.4) follows. Let $\varepsilon \in (0, r-1)$. Write $\delta := \frac{r}{s}\varepsilon$. Since $0 < \delta < s-1$, we have

$$\begin{aligned}
\varepsilon^{\frac{\tau}{s-\varepsilon}} \||f|^{\frac{r}{s}}\|_{L^{s-\varepsilon}(\Omega,\mu)} &= \varepsilon^{\frac{\tau r}{s(r-\delta)}} \|f\|^{r/s}_{L^{\frac{r}{s}(s-\varepsilon)}(\Omega,\mu)} \\
&= \left[\left(\frac{s}{r}\delta\right)^{\frac{\tau}{r-\delta}} \|f\|_{L^{r-\delta}(\Omega,\mu)}\right]^{\frac{r}{s}} \\
&\leq \left(\frac{s}{r}\right)^{\frac{\tau r}{s}} \left(\delta^{\frac{\tau}{r-\delta}} \|f\|_{L^{r-\delta}(\Omega,\mu)}\right)^{\frac{r}{s}} \\
&\leq \left(\frac{s}{r}\right)^{\frac{\tau r}{s}} \|f\|^{\frac{r}{s}}_{L^{r),\tau}(\Omega,\mu)}.
\end{aligned} \tag{6.5}$$

Now, let $\varepsilon \in (r-1, s-1)$ and $\gamma := \frac{r(r-1)}{s}$. By using $0 < \gamma \leq \frac{r}{s}\varepsilon < r-1$, Hölder's inequality, and (6.2), we get

$$\varepsilon^{\frac{\tau}{s-\varepsilon}} \||f|^{\frac{r}{s}}\|_{L^{s-\varepsilon}(\Omega,\mu)} \leq \max(1, (s-1)^\tau)\|f\|^{\frac{r}{s}}_{L^{r-\frac{r}{s}\varepsilon}(\Omega,\mu)} \lesssim \|f\|^{\frac{r}{s}}_{L^{r-\gamma}(\Omega,\mu)} \lesssim \|f\|^{\frac{r}{s}}_{L^{r),\tau}(\Omega,\mu)}. \tag{6.6}$$

Combining (6.5) and (6.6), we have (6.3). □

6.2 The first complex interpolation of grand Lebesgue spaces

We give the following description of the first complex interpolation of grand Lebesgue spaces.

Theorem 6.2.1. [26, Theorem 1.1] *Let $\theta \in (0,1)$, $1 < p_0 < \infty$, $1 < p_1 < \infty$, and $\tau > 0$. Assume that $p_0 \neq p_1$ and define*

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{1}{p_1}. \tag{6.7}$$

Then

$$[L^{p_0),\tau}(\Omega,\mu), L^{p_1),\tau}(\Omega,\mu)]_\theta = \left\{ f \in L^{p),\tau}(\Omega,\mu) : \lim_{N\to\infty} \left\| \chi_{\{|f|>N\}} f \right\|_{L^{p),\tau}(\Omega,\mu)} = 0 \right\}.$$

Our proof of Theorem 6.2.1 uses Lemma 2.3.2 and the description of the Calderón product of grand Lebesgue spaces. To give a description of the Calderón product of grand Lebesgue spaces, we prove the following lemma.

Lemma 6.2.2. [26, Lemma 3.3] *Let $\theta \in (0,1)$, $0 < \tau < \infty$, and $1 < p_1 < p_0 < \infty$. Define p by (6.7). If $f_0 \in L^{p_0),\tau}$ and $f_1 \in L^{p_1),\tau}$, then*

$$\left\| |f_0|^{1-\theta} |f_1|^\theta \right\|_{L^{p),\tau}} \lesssim \|f_0\|_{L^{p_0),\tau}}^{1-\theta} \|f_1\|_{L^{p_1),\tau}}^{\theta}. \tag{6.8}$$

Remark 6.2.3. Note that (6.8) can be viewed as the Hölder inequality in grand Lebesgue spaces.

Proof of Lemma 6.2.2. Let $\varepsilon \in (0, p-1)$. By using Hölder's inequality, we have

$$\begin{aligned}
&\left\| |f_0|^{1-\theta} |f_1|^\theta \right\|_{L^{p-\varepsilon}(\Omega)} \\
&\le \left(\left[\int_\Omega |f_0(x)|^{\frac{p_0(p-\varepsilon)}{p}} d\mu(x) \right]^{\frac{(1-\theta)p}{p_0}} \left[\int_\Omega |f_1(x)|^{\frac{p_1(p-\varepsilon)}{p}} d\mu(x) \right]^{\frac{\theta p}{p_1}} \right)^{\frac{1}{p-\varepsilon}} \\
&= \left\| |f_0|^{p_0/p} \right\|_{L^{p-\varepsilon}(\Omega)}^{\frac{(1-\theta)p}{p_0}} \left\| |f_1|^{p_1/p} \right\|_{L^{p-\varepsilon}(\Omega)}^{\frac{(\theta)p}{p_1}}.
\end{aligned}$$

By virtue of Lemma 6.1.2, we get

$$\begin{aligned}
\varepsilon^{\frac{\tau}{p-\varepsilon}} \left\| |f_0|^{1-\theta} |f_1|^\theta \right\|_{L^{p-\varepsilon}(\mu)} &\le \left(\varepsilon^{\frac{\tau}{p-\varepsilon}} \left\| |f_0|^{p_0/p} \right\|_{L^{p-\varepsilon}(\Omega)} \right)^{\frac{(1-\theta)p}{p_0}} \left(\varepsilon^{\frac{\tau}{p-\varepsilon}} \left\| |f_1|^{p_1/p} \right\|_{L^{p-\varepsilon}(\Omega)} \right)^{\frac{\theta p}{p_1}} \\
&\le \left\| |f_0|^{p_0/p} \right\|_{L^{p),\tau}}^{\frac{(1-\theta)p}{p_0}} \left\| |f_1|^{p_1/p} \right\|_{L^{p),\tau}}^{\frac{\theta p}{p_1}} \\
&\lesssim \|f_0\|_{L^{p_0),\tau}}^{1-\theta} \|f_1\|_{L^{p_1),\tau}}^{\theta}.
\end{aligned}$$

Taking the supremum over any $\varepsilon \in (0, p-1)$, we get (6.8). □

We now prove the following description of the Calderón product for grand Lebesgue spaces.

Lemma 6.2.4. [26] *Let* $\theta \in (0,1)$, $0 < \tau < \infty$, *and* $1 < p_1 \le p_0 < \infty$. *Define* p *by (6.7). Then we have*

$$(L^{p_0),\tau}(\Omega,\mu))^{1-\theta}(L^{p_1),\tau}(\Omega,\mu))^{\theta} = L^{p),\tau}(\Omega,\mu).$$

Proof. Let $f \in L^{p),\tau}(\Omega,\mu)$. Define $f_0 := |f|^{p/p_0}$ and $f_1 := |f|^{p/p_1}$. By using Lemma 6.1.2, we have

$$\|f_0\|_{L^{p_0),\tau}(\Omega,\mu)} \lesssim \|f\|^{\frac{p}{p_0}}_{L^{p),\tau}(\Omega,\mu)} \quad \text{and} \quad \|f_1\|_{L^{p_1),\tau}(\Omega,\mu)} \le \|f\|^{\frac{p}{p_1}}_{L^{p),\tau}(\Omega,\mu)}.$$

Since $|f_0|^{1-\theta}|f_1|^{\theta} = |f|$ and

$$\begin{aligned}\|f\|_{(L^{p_0),\tau}(\Omega,\mu))^{1-\theta}(L^{p_1),\tau}(\Omega,\mu))^{\theta}} &\le \|f_0\|^{1-\theta}_{L^{p_0),\tau}(\Omega,\mu)}\|f_1\|^{\theta}_{L^{p_1),\tau}(\Omega,\mu)} \\ &\lesssim \|f\|^{\frac{p}{p_0}(1-\theta)\frac{p}{p_1}(\theta)}_{L^{p),\tau}(\Omega,\mu)} \\ &= \|f\|_{L^{p),\tau}(\Omega,\mu)},\end{aligned}$$

we have $f \in (L^{p_0),\tau}(\Omega,\mu))^{1-\theta}(L^{p_1),\tau}(\Omega,\mu))^{\theta}$. Therefore,

$$L^{p),\tau}(\Omega,\mu) \subseteq (L^{p_0),\tau}(\Omega,\mu))^{1-\theta}(L^{p_1),\tau}(\Omega,\mu))^{\theta}.$$

Conversely, let $g \in (L^{p_0),\tau}(\Omega,\mu))^{1-\theta}(L^{p_1),\tau}(\Omega,\mu))^{\theta}$. Choose $g_0 \in L^{p_0),\tau}(\Omega,\mu)$ and $g_1 \in L^{p_1),\tau}(\Omega,\mu)$ such that $|g| \le |g_0|^{1-\theta}|g_1|^{\theta}$ and

$$\|g_0\|^{1-\theta}_{L^{p_0),\tau}(\Omega,\mu)}\|g_1\|^{\theta}_{L^{p_1),\tau}(\Omega,\mu)} \le 2\|g\|_{(L^{p_0),\tau}(\Omega,\mu))^{1-\theta}(L^{p_1),\tau}(\Omega,\mu))^{\theta}}. \tag{6.9}$$

Combining (6.8) and (6.9), we get

$$\begin{aligned}\|g\|_{L^{p),\tau}(\Omega,\mu)} &\le \left\||g_0|^{1-\theta}|g_1|^{\theta}\right\|_{L^{p),\tau}(\Omega,\mu)} \\ &\lesssim \|g_0\|^{1-\theta}_{L^{p_0),\tau}(\Omega,\mu)}\|g_1\|^{\theta}_{L^{p_1),\tau}(\Omega,\mu)} \\ &\lesssim \|g\|_{(L^{p_0),\tau}(\Omega,\mu))^{1-\theta}(L^{p_1),\tau}(\Omega,\mu))^{\theta}}.\end{aligned}$$

Consequently, $g \in L^{p),\tau}(\Omega,\mu)$. Thus,

$$(L^{p_0),\tau}(\Omega,\mu))^{1-\theta}(L^{p_1),\tau}(\Omega,\mu))^{\theta} \subseteq L^{p),\tau}(\Omega,\mu),$$

as desired. □

The proof of Theorem 6.2.1 is given as follows.

Proof of Theorem 6.2.1. Combining Lemmas 2.3.2 and 6.2.4, we have

$$[L^{p_0),\tau}(\Omega,\mu), L^{p_1),\tau}(\Omega,\mu)]_\theta = \overline{L^{p_0),\tau}(\Omega,\mu) \cap L^{p_1),\tau}(\Omega,\mu)}^{L^{p),\tau}(\Omega,\mu)}. \tag{6.10}$$

Without loss of generality, we may assume that $p_0 > p > p_1$. Let $f \in L^{p),\tau}(\Omega,\mu)$ be such that

$$\lim_{N\to\infty} \|f\chi_{\{|f|>N\}}\|_{L^{p),\tau}(\Omega,\mu)} = 0. \tag{6.11}$$

Define $f_N := f\chi_{\{|f|\le N\}}$. Then, by virtue of Lemma 6.1.1, we have

$$f_N \in L^{p_0),\tau}(\Omega,\mu) \cap L^{p_1),\tau}(\Omega,\mu) \tag{6.12}$$

In view of (6.10) and (6.11), we have $f \in [L^{p_0),\tau}(\Omega,\mu), L^{p_1),\tau}(\Omega,\mu)]_\theta$. Thus, we have proved that

$$\left\{f \in L^{p),\tau}(\Omega,\mu) : \lim_{N\to\infty} \left\|\chi_{\{|f|>N\}} f\right\|_{L^{p),\tau}(\Omega,\mu)} = 0\right\} \subseteq [L^{p_0),\tau}(\Omega,\mu), L^{p_1),\tau}(\Omega,\mu)]_\theta.$$

Hence, to conclude the proof, it suffices to show that

$$\lim_{j\to\infty} \|f\chi_{\{|f|>j\}}\|_{L^{p),\tau}(\Omega,\mu)} = 0 \tag{6.13}$$

for every $f \in [L^{p_0),\tau}(\Omega,\mu), L^{p_1),\tau}(\Omega,\mu)]_\theta$. According (6.10), for every $\varepsilon > 0$, we can choose $g \in L^{p_0),\tau}(\Omega,\mu) \cap L^{p_1),\tau}(\Omega,\mu)$ such that

$$\|f - g\|_{L^{p),\tau}(\Omega,\mu)} < \varepsilon. \tag{6.14}$$

We now fix $j \in \mathbb{N}$. Note that

$$|f\chi_{\{|f|>j\}}| \le |f - g| + |g\chi_{\{|g|>j/2\}}| + |g\chi_{\{|f|>j\}\cap\{|g|\le j/2\}}|.$$

On the set $\{|f| > j\} \cap \{|g| \le j/2\}$, we have

$$|g| \le \frac{j}{2} < \frac{|f|}{2} \le \frac{|f-g|}{2} + \frac{|g|}{2},$$

so

$$\|f\chi_{\{|f|>j\}}\|_{L^{p),\tau}(\Omega,\mu)} \le 2\|f-g\|_{L^{p),\tau}(\Omega,\mu)} + \|g\chi_{\{|g|>j/2\}}\|_{L^{p),\tau}(\Omega,\mu)}. \tag{6.15}$$

By virtue of Lemma 6.1.2, we get

$$\begin{aligned}
\|g\chi_{\{|g|>j\}}\|_{L^{p),\tau}(\Omega,\mu)} &= \left\| |g|^{1-\frac{p_0}{p}} \chi_{\{|g|>j\}} |g|^{\frac{p_0}{p}} \right\|_{L^{p),\tau}(\Omega,\mu)} \\
&\le j^{1-\frac{p_0}{p}} \left\| |g|^{\frac{p_0}{p}} \right\|_{L^{p),\tau}(\Omega,\mu)} \\
&\lesssim j^{1-\frac{p_0}{p}} \|g\|_{L^{p_0),\tau}(\Omega,\mu)}^{\frac{p_0}{p}},
\end{aligned}$$

and hence

$$\lim_{j\to\infty} \|g\chi_{\{|g|>j\}}\|_{L^{p),\tau}(\Omega,\mu)} = 0. \tag{6.16}$$

Combining (6.14)-(6.16), we get

$$\limsup_{j\to\infty} \|f\chi_{\{|f|>j\}}\|_{L^{p),\tau}(\Omega,\mu)} \le 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have (6.13), as desired. □

6.3 The second complex interpolation of grand Lebesgue spaces

Our result on the second complex interpolation of grand Lebesgue spaces is as follows.

Theorem 6.3.1. [26] *Keep the same assumption as in Theorem 6.2.1. Then*

$$[L^{p_0),\tau}(\Omega,\mu), L^{p_1),\tau}(\Omega,\mu)]^\theta = L^{p),\tau}(\Omega,\mu).$$

In some particular cases, we may have a proper inclusion

$$[L^{p_0),\tau}(\Omega,\mu), L^{p_1),\tau}(\Omega,\mu)]_\theta \subsetneq [L^{p_0),\tau}(\Omega,\mu), L^{p_1),\tau}(\Omega,\mu)]^\theta. \tag{6.17}$$

For instance, let us consider $\Omega = (0,1)$, μ is the Lebesgue measure on Ω, and the function $f(x) := x^{-\frac{1}{p}},\ x \in \Omega$. Note that $f \in L^{p),1}(\Omega,\mu)$ and

$$\lim_{N\to\infty} \|\chi_{\{|f|>N\}} f\|_{L^{p),1}(\Omega,\mu)} \sim 1 \neq 0.$$

In view of Theorems 6.2.1 and 6.3.1, we see that

$$f \in [L^{p_0),1}(\Omega,\mu), L^{p_1),1}(\Omega,\mu)]^\theta \setminus [L^{p_0),1}(\Omega,\mu), L^{p_1),1}(\Omega,\mu)]_\theta.$$

This shows (6.17).

Our proof of Theorem 6.3.1 uses Lemma 2.2.4 and Theorem 6.2.1 and also the construction of the second complex interpolation functor given by (3.21).

Proposition 6.3.2. [26] *Keep the same assumption as in Theorem 6.2.1. Let $f \in L^{p),\tau}(\Omega,\mu)$ and define G by*

$$G(z) := (z-\theta)\int_0^1 F(\theta + (z-\theta)t)\,dt \quad (z \in \overline{S}), \tag{6.18}$$

where

$$\frac{1}{p(z)} := \frac{1-z}{p_0} + \frac{z}{p_1}, \tag{6.19}$$

and

$$F(z) := \operatorname{sgn}(f)\exp\left(\frac{p}{p(z)}\log|f|\right) \quad (z \in \overline{S}). \tag{6.20}$$

Then

$$G \in \mathcal{G}(L^{p_0),\tau}(\Omega,\mu), L^{p_1),\tau}(\Omega,\mu)).$$

Proof. The idea of the proof of Proposition 6.3.2 is similar to that of Lemma 3.3.2. We uses inequalities (3.24)–(3.27) with q_0, q_1, and q are replaced by p_0, p_1, and p, respectively. Combining these inequalities and Lemma 6.1.2, we get

$$\sup_{z\in\overline{S}} \frac{\|G(z)\|_{L^{p_0),\tau}(\Omega,\mu),L^{p_1),\tau}(\Omega,\mu)}}{1+|z|} \lesssim \|f\|_{L^{p),\tau}(\Omega,\mu)}^{p/p_0} + \|f\|_{L^{p),\tau}(\Omega,\mu)}^{p/p_1},$$

$$\begin{aligned}&\|G(z_1)-G(z_2)\|_{L^{p_0),\tau}(\Omega,\mu)+L^{p_1),\tau}(\Omega,\mu)}\\ &\quad\lesssim |z_1-z_2|\left(\|f\|_{L^{p),\tau}(\Omega,\mu)}^{p/p_0} + \|f\|_{L^{p),\tau}(\Omega,\mu)}^{p/p_1}\right),\ z_1,z_2\in\overline{S},\end{aligned}$$

and

$$\left\|\frac{G(z+w)-G(z)}{w} - F(z)\right\|_{L^{p_0),\tau}(\Omega,\mu)+L^{p_1),\tau}(\Omega,\mu)} \lesssim |w|(\|f\|_{L^{p),\tau}(\Omega,\mu)}^{p/p_0} + \|f\|_{L^{p),\tau}(\Omega,\mu)}^{p/p_1}),$$

whenever $z\in S$ satisfying $0<\varepsilon<\mathrm{Re}(z)<1-\varepsilon$ for $\varepsilon\in(0,\frac{1}{2})$ and $w\in\mathbb{C}$ with $|w|\ll 1$. These show the boundedness and continuity of G on $\overline{S}$ and also holomorphicity of G in S. Hence, we only need to verify that

$$G(j+it_1)-G(j+it_2)\in L^{p_j),\tau}(\Omega,\mu)$$

for every $t_1,t_2\in\mathbb{R}$ and $j\in\{0,1\}$ and also

$$\|G(j+i\cdot)\|_{\mathrm{Lip}(\mathbb{R},L^{p_j),\tau}(\Omega,\mu)} \le (\|f\|_{L^{p),\tau}(\Omega,\mu)})^{p/p_j}. \tag{6.21}$$

for every $j\in\{0,1\}$. Combining

$$G(j+it_1)-G(j+it_2) = -i\int_{t_1}^{t_2} F(j+it)\,dt$$

and $|F(j+it)| = |f|^{\frac{p}{p_j}}$, we get

$$\|G(j+it_1)-G(j+it_2)\|_{L^{p_j),\tau}(\Omega,\mu)} \le |t_1-t_2|\|f\|_{L^{p),\tau}(\Omega,\mu)}^{\frac{p}{p_j}}.$$

This implies (6.21). Thus, $G\in\mathcal{G}(L^{p_0),\tau}(\Omega,\mu),L^{p_1),\tau}(\Omega,\mu))$. □

Proof of Theorem 6.2.1. Let $f\in L^{p),\tau}(\Omega,\mu)$ and define G by (6.18). Then $G'(\theta)=f$ in $L^{p_0),\tau}(\Omega,\mu)+L^{p_1),\tau}(\Omega,\mu)$. This equality and Proposition 6.3.2 imply $f=G'(\theta)\in[L^{p_0),\tau}(\Omega,\mu),L^{p_1),\tau}(\Omega,\mu)]^\theta$.

Let $f\in[L^{p_0),\tau}(\Omega,\mu),L^{p_1),\tau}(\Omega,\mu)]^\theta$. Choose $G\in\mathcal{G}(L^{p_0),\tau}(\Omega,\mu),L^{p_1),\tau}(\Omega,\mu))$ such that $G'(\theta)=f$ and

$$\|G\|_{\mathcal{G}(L^{p_0),\tau}(\Omega,\mu),L^{p_1),\tau}(\Omega,\mu))} \lesssim \|f\|_{[L^{p_0),\tau},L^{p_1),\tau}]^\theta}. \tag{6.22}$$

For each $k \in \mathbb{N}$ and $z \in \overline{S}$, define $H_k(z)$ by (2.3). Combining Lemma 2.2.4, Theorem 6.2.1, and (6.22), we have

$$\|H_k(\theta)\|_{L^{p),\tau}(\Omega,\mu)} \lesssim \|f\|_{[L^{p_0),\tau}(\Omega,\mu),L^{p_1),\tau}(\Omega,\mu)]^\theta}.$$

Since

$$\lim_{k\to\infty} \|H_k(\theta)-f\|_{L^{p_0),\tau}(\Omega,\mu)+L^{p_1),\tau}(\Omega,\mu)} = \lim_{k\to\infty} \|H_k(\theta)-G'(\theta)\|_{L^{p_0),\tau}(\Omega,\mu)+L^{p_1),\tau}(\Omega,\mu)} = 0,$$

we can find a subsequence $\{H_{k_j}(\theta)\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} H_{k_j}(\theta)(x) = f(x)$$

a.e. $x \in \Omega$. Therefore, by the Fatou Lemma, we have

$$\|f\|_{L^{p),\tau}(\Omega,\mu)} \leq \liminf_{j\to\infty} \|H_{k_j}(\theta)\|_{L^{p),\tau}(\Omega,\mu)} \lesssim \|f\|_{[L^{p_0,\tau}(\Omega,\mu),L^{p_1,\tau}(\Omega,\mu)]^\theta}. \tag{6.23}$$

Hence, $f \in L^{p),\tau}(\Omega,\mu)$. Thus, we conclude that $[L^{p_0),\tau}(\Omega,\mu), L^{p_1),\tau}(\Omega,\mu)]^\theta \subseteq L^{p),\tau}(\Omega,\mu)$. □

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