Inflationary models in supergravity with inflaton in a vector multiplet, and spontaneous breaking of supersymmetry and R-symmetry after inflation

Yermek Aldabergenov

Department of Physics
Graduate School of Science and Engineering
Tokyo Metropolitan University

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Contents

Abstract

We consider inflationary model building in the framework of $N = 1$ supergravity, where the inflaton scalar field belongs to a massive vector multiplet, and supersymmetry (and R-symmetry) is spontaneously broken after inflation. We show that it is possible to obtain a Minkowski and de Sitter vacua that are stable. We also reformulate our models as the $U(1)$ gauge theories coupled to a Higgs chiral superfield, which in the minimal case corresponds to the standard $U(1)$ Higgs setup. Finally, we focus on a specific representative of our class of models (called Polonyi-Starobinsky supergravity), that leads to the Starobinsky inflationary potential. We discover that the simplest known way to obtain the Starobinsky potential leads to instability, and find a way to remove it by adding a Fayet-Iliopoulos term. This leads to a modification of the previously found Polonyi vacuum.

Throughout the thesis, various connections of the conducted research to the Standard Model (SM) of elementary particles, supersymmetry (SUSY) and supergravity, the Minimal Supersymmetric Standard Model (MSSM), the supersymmetric Grand Unified Theories (GUT), the Standard Cosmology (SC), cosmological inflation and superstrings are also discussed.

Introduction

The inflationary paradigm solves initial-condition problems of the pre-inflationary cosmology (like e.g. the flatness problem, the horizon problem, the monopole problem) and remarkably agrees with CMB observations (COBE, WMAP, PLANCK). On the other hand, supergravity (or SUGRA for short), as well as its flat-space-time limit – rigid supersymmetry (SUSY), is a well-motivated framework for building UV-extensions of the Standard Model. Moreover, it is a necessary step if one considers unification of the Standard Model and General Relativity in the only known consistent framework of quantum gravity - superstring theories. Supersymmetry, if exists, cannot be exact, it must be spontaneously broken at some high-enough scale in order to generate large masses of superpartners of the Standard Model particles, as we do not see them at presently available energies. One can build a theory with various numbers (N) of supersymmetries, that would result in several distinct superpartners of the same particle. For instance, in 4 space-time dimensions, maximum number of supersymmetries for a gauge theory (where particle spin is no higher than 1) is $N = 4$, while for supergravity (where maximal spin is 2), $N = 8$. However, $N > 1$ supersymmetric theories are non-chiral, and for that reason they cannot be used as immediate extensions of the Standard Model, which is a chiral theory.

One of the most promising candidates for beyond-the-Standard-Model theory is the Minimal Supersymmetric Standard Model, which implements $N = 1$ supersymmetry. This motivates us to consider inflationary model building in the framework of $N = 1$ supergravity. However, realising inflationary potentials in supergravity was met with difficulties. One of them, called the η-problem, is related to the dangerous exponential factor in the F-term potential, which leads to the large effective mass of the would-be inflaton, and ruins the slow-roll regime required for successful inflation. Another problem arises if we assume that inflation was caused by a chiral superfield. Since the lowest component of a chiral superfield is a complex scalar, it provides two real degrees of freedom, one of which should be stabilised while the other drives the inflation. These problems can be avoided in various ways, one of which is to identify the inflaton with the real scalar component of a massive vector multiplet. Thus, there is no need for stabilisation, and since the inflationary potential comes from the D-term, this resolves the η-problem.

In generic inflationary models, although supersymmetry is spontaneously broken during inflation (since either D- or F-term potentials must have non-vanishing effective values), in the end of the inflation it is restored, and thus must be broken again by some mechanism.

In Chapter 1 we briefly review the main features of the Standard Model of particle physics. In Chapter 2 we first introduce $N = 1$ supersymmetry, both global and local, then we show

how to apply global supersymmetry to the Standard Model. The resulting model is a very good candidate for beyond-the-Standard-Model theory, called the Minimal Supersymmetric Standard Model (MSSM). One of its features is the exact unification of its extrapolated coupling constants, which gives rise to the idea of Grand Unified Theories (GUT). We will discuss several candidate GUT models in Chapter 3.

The second half of the dissertation, devoted to cosmology, starts with the review of the Standard Cosmological Model (Chapter 4). In the end of Chapter 4 we review the problems of the pre-inflationary cosmology, and give motivation to introduce the idea of inflation. Chapter 5 is devoted to inflationary cosmology, where we review the simplest models, discuss particle production after inflation, and show the observational constraints on inflationary models. In Chapter 6 we consider embedding inflationary models in supergravity (local supersymmetry) by giving simple examples. Finally, in Chapter 7 we focus on the main goal of this dissertation, which is to (minimally) connect inflation, supergravity, and supersymmetry breaking after inflation, in a particular class of models. This is followed by conclusions where we summarise the main achievements of the research.

This research was conducted in collaboration with Associate Professor Sergei V. Ketov. The main results were published in [1, 2].

Chapter 1

The Standard Model

This chapter summarises basic information about Quantum Field Theory (QFT) and the Standard Model (SM) of elementary particles along the lines of textbooks [3, 4, 5, 6].

1.1 Quantum field theory overview

In classical field theory the basic objects are fields: functions defined over some region of space. Classical fields can be used to describe phenomena in classical physics, such as gravity or electromagnetism. However, to describe physics on subatomic scales, the need for the new class of theoretical framework arises, which is called quantum physics. We are particularly interested in its prominent representative called *quantum field theory* (QFT).

In QFT instead of classical fields one works with quantum fields, which are operator-valued functions. Quantum fields, in turn, act on a Fock space of all possible states, which is defined as a direct sum of $1, 2, ..., n$-particle Hilbert spaces

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ... \oplus \mathcal{H}_n \ . \tag{1.1}$$

A quantum field can be obtained by the quantization procedure of a classical field, depending on the type of that field.

1.1.1 Scalar field

Let us start with the simple example of a real massive scalar field. Quantization of such a field corresponds to promoting a classical scalar field $\phi(x)$ described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi^2(x) \ , \tag{1.2}$$

and obeying Klein-Gordon equation,

$$(\Box - m^2)\phi(x) = 0 \ , \tag{1.3}$$

to an operator, which can be decomposed as [1]

$$\phi(x) = \int \frac{d^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3}\left(a_{\mathbf{p}}e^{ipx} + a^{\dagger}_{\mathbf{p}}e^{-ipx}\right) \ , \tag{1.4}$$

where $px \equiv p_\mu x^\mu$, $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ is the energy, and $a_{\mathbf{p}}$ and $a^{\dagger}_{\mathbf{p}}$ are annihilation and creation operators, respectively, which create or annihilate spin-0 excitations (particles) with momentum $\mathbf{p}$ of the corresponding field at point x in spacetime. They satisfy commutation relations

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = [a^{\dagger}_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}] = 0 \ , \tag{1.5}$$

$$[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}] = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p} - \mathbf{q}) \ , \tag{1.6}$$

which come from equal-time canonical commutation relations for $\phi(x)$ and its conjugate momentum $\pi(x) \equiv \dot{\phi}(x)$:

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = [\pi(t,\mathbf{x}), \pi(t,\mathbf{y})] = 0 \ , \tag{1.7}$$

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}) \ . \tag{1.8}$$

The Hamiltonian

$$H = \frac{1}{2}\int d^3x \left(\pi^2(x) + \partial_i\phi(x)\partial^i\phi(x) + m^2\phi^2(x)\right) \tag{1.9}$$

in terms of creation and annihilation operators reads

$$H = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} E_{\mathbf{p}}\left(a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{p}}]\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left(a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}} + (2\pi)^3 E_{\mathbf{p}}\delta^3(\mathbf{p} - \mathbf{p})\right) \ . \tag{1.10}$$

The vacuum state is defined to be annihilated by $a_{\mathbf{p}}$,

$$a_{\mathbf{p}}|0\rangle = 0 \ , \tag{1.11}$$

for all $\mathbf{p}$. Thus, acting with the Hamiltonian on the vacuum gives

$$H|0\rangle = \frac{1}{2}\int d^3p E_{\mathbf{p}}\delta^3(0)|0\rangle \ , \tag{1.12}$$

which clearly contains infinity due to $\delta^3(0)$, which arises because we integrate over all space $\int_{-\infty}^{\infty} d^3x$. To regulate this divergence we use the finite volume trick, where we confine our integral to a box of volume V,

$$(2\pi)^3\tilde{\delta}^3(0) = \int_V d^3x = V \ , \tag{1.13}$$

[1] The factor $2E_{\mathbf{p}}$ in the denominator appears for normalization purposes, since for the Lorentz-invariance, delta function is multiplied by the same factor: $2E_{\mathbf{p}}\delta^3(a - b)$

where $\tilde{\delta}^3(0)$ is a "finite-volume" delta function. Then, to recover $\delta^3(0)$ we take the limit

$$\delta^3(0) = \lim_{V\to\infty} \tilde{\delta}^3(0) \ . \tag{1.14}$$

Then it is clear that we should consider energy density instead of total energy, i.e. divide (1.12) by V.

However, (1.12) is still divergent, because we integrate over arbitrarily high momenta/small distances, which means we are dealing with infinite number of zero-point-energy oscillators. The problem can be cured if we use the so-called normal-ordered Hamiltonian, i.e. if we move all annihilation operators in H to the right. Denoting the normal-ordered Hamiltonian as $:H:$ we have,

$$:H:= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \ , \tag{1.15}$$

which is exactly the difference $H - \langle H\rangle$. So normal ordering of H amounts to a subtraction of the infinity of vacuum oscillators. From now on we will drop :: since we will only be interested in normal-ordered Hamiltonians.

The excited states are constructed by acting with $a^\dagger_{\mathbf{p}}$ on the vacuum,

$$a^\dagger_{\mathbf{p}}|0\rangle = |\mathbf{p}\rangle \ , \tag{1.16}$$

where $|\mathbf{p}\rangle$ is one-particle state of momentum $\mathbf{p}$ and mass m, corresponding to the scalar field $\phi(x)$. Acting with Hamiltonian we recover its energy eigenvalues

$$H|\mathbf{p}\rangle = E_{\mathbf{p}}|\mathbf{p}\rangle \ . \tag{1.17}$$

Acting with n number of creation operators we get an n-particle state,

$$a^\dagger_{\mathbf{p}_1}...a^\dagger_{\mathbf{p}_n}|0\rangle = |\mathbf{p}_1...\mathbf{p}_n\rangle \ , \tag{1.18}$$

which is symmetric under permutations of $\mathbf{p}_i$, reflecting its bosonic nature. The n-particle Hilbert space (for the scalar field ϕ) is then nothing more than a collection of $|\mathbf{p}_1...\mathbf{p}_n\rangle$.

For a general complex scalar field $\phi^\dagger \neq \phi$, so there are two real degrees of freedom. ϕ and $\phi^\dagger$ independently obey Klein-Gordon equations, and can be decomposed as

$$\phi(x) = \int \frac{d^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3}\left(a_{\mathbf{p}}e^{ipx} + b^\dagger_{\mathbf{p}}e^{-ipx}\right) \ , \tag{1.19}$$

$$\phi^\dagger(x) = \int \frac{d^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3}\left(b_{\mathbf{p}}e^{ipx} + a^\dagger_{\mathbf{p}}e^{-ipx}\right) \ , \tag{1.20}$$

where there are two distinct sets of ladder operators, $a, a^\dagger$ and $b, b^\dagger$, one associated with particles and the other - with anti-particles.

1.1.2 Dirac spinor

We now proceed to quantization of a Dirac spinor. The corresponding Lagrangian

$$\mathcal{L} = -\bar{\psi}(\partial\!\!\!/ + m)\psi \tag{1.21}$$

leads to Dirac equation

$$(\partial\!\!\!/ + m)\psi = 0\ , \tag{1.22}$$

where $\bar{\psi} \equiv i\psi^\dagger\gamma^0$, and $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$ We choose the normalization of the Dirac matrices as $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ (with "mostly plus" metric), so that γ^0 is anti-Hermitian while γ^i are Hermitian. Dirac spinor also satisfies Klein-Gordon equation,

$$(\partial\!\!\!/ - m)(\partial\!\!\!/ + m)\psi = (\Box - m^2)\psi = 0\ , \tag{1.23}$$

and can be expanded as

$$\psi(x) = \sum_{s=\pm}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\left(c_{\mathbf{p}s}u_s(\mathbf{p})e^{-ipx} + d^\dagger_{\mathbf{p}s}v_s(\mathbf{p})e^{ipx}\right)\ , \tag{1.24}$$

$$\psi^\dagger(x) = \sum_{s=\pm}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\left(d_{\mathbf{p}s}v^\dagger_s(\mathbf{p})e^{-ipx} + c^\dagger_{\mathbf{p}s}u^\dagger_s(\mathbf{p})e^{ipx}\right)\ , \tag{1.25}$$

where $s = \pm$ are the two helicity states ± 1; $c, c^\dagger$ and $d, d^\dagger$ are ladder operators associated with spinor Fourier modes $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$, respectively [2]. Consistency requires spinor field operators to obey *anti*-commutation relations (as opposed to bosonic fields),

$$\{\psi(t,\mathbf{x}), \psi(t,\mathbf{y})\} = \delta^3(\mathbf{x}-\mathbf{y})\ , \tag{1.26}$$

or in terms of ladder operators,

$$\{c_{\mathbf{p}s}, c^\dagger_{\mathbf{q}r}\} = \{d_{\mathbf{p}s}, d^\dagger_{\mathbf{q}r}\} = 2E_{\mathbf{p}}(2\pi)^3\delta_{sr}\delta^3(\mathbf{p}-\mathbf{q})\ . \tag{1.27}$$

All other anti-commutators vanish.

(Normal-ordered) Hamiltonian is then

$$H = \frac{1}{2}\sum_{s=\pm}\int \frac{d^3p}{(2\pi)^3}(c^\dagger_{\mathbf{p}s}c_{\mathbf{p}s} + d^\dagger_{\mathbf{p}s}d_{\mathbf{p}s})\ , \tag{1.28}$$

and the excited states

$$c^\dagger_{\mathbf{p}_1 s_1}...c^\dagger_{\mathbf{p}_n s_n}|0\rangle = |\mathbf{p}_1 s_1, ..., \mathbf{p}_n s_n\rangle \tag{1.29}$$

are antisymmetric with respect to interchanging of any two particles.

[2] As in the case of complex scalars we interpret $c, c^\dagger$ and $d, d^\dagger$ as the operators creating and annihilating particles and anti-particles.

1.1.3 Abelian gauge field

Now we turn to the simplest example of a vector field in the gauge theory formulation - massless $U(1)$ abelian gauge field. The corresponding Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ , \tag{1.30}$$

where $F_{\mu\nu} \equiv F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ is the field strength, and $A_\mu(x)$ is the 4-potential - an abelian gauge field. The Lagrangian is invariant with respect to the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \omega(x) \ , \tag{1.31}$$

where $\omega(x)$ is a scalar function of spacetime. $F_{\mu\nu}$ by construction satisfies Bianchi identities

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0 \ . \tag{1.32}$$

The equations of motion are then exactly (free) Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0 \ . \tag{1.33}$$

If we try to naively impose the equal time commutation relations we will run into a problem because the relation

$$[A_0(t, \mathbf{x}), \pi_0(t, \mathbf{x})] = i\eta_{00}\delta^3(\mathbf{x} - \mathbf{y}) \tag{1.34}$$

is non-vanishing, which contradicts the fact that

$$\pi^0 \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \ , \tag{1.35}$$

i.e. the time component A_0 is non-dynamical.

A solution to the problem, that preserves explicit Lorentz covariance, uses the gauge freedom to add an extra (gauge fixing) term to the Lagrangian so that

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\xi}{2} (\partial_\mu A^\mu)^2 \ . \tag{1.36}$$

Then the Lagrange multiplier ξ can be treated as an independent gauge parameter, and its equation of motion can be used as the gauge fixing condition,

$$\partial_\mu A^\mu = 0 \ , \tag{1.37}$$

which is called the *Lorenz gauge*. However (1.37) cannot be understood as an operator equation as π^0 would still vanish in that case. Instead, after imposing canonical commutation relations we will interpret the Lorenz gauge condition as a relation for physical states.

Now with non-vanishing π^0 we are free to impose the commutation relations

$$[A_\mu(t, \mathbf{x}), \pi_\nu(t, \mathbf{y})] = i\eta_{\mu\nu}\delta^3(\mathbf{x} - \mathbf{y}) \ , \tag{1.38}$$

and expand the gauge field as

$$A_\mu(x) = \sum_{\lambda=0}^{3} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \epsilon_\mu^\rho(\mathbf{p}) \left(a_{\mathbf{p}\lambda} e^{-ipx} + a_{\mathbf{p}\rho}^\dagger e^{ipx} \right) \ , \tag{1.39}$$

where $\epsilon_\mu^\rho(\mathbf{p})$ is the polarization vector and $\rho = 0, ..., 3$ denote polarization states.

For ladder operators the commutation relations are

$$[a_{\mathbf{p}\rho}, a_{\mathbf{q}\sigma}^\dagger] = 2E_{\mathbf{p}}(2\pi)^3 \eta_{\rho\sigma} \delta^3(\mathbf{p} - \mathbf{q}) \ , \tag{1.40}$$

which is positive for η_{ij}, but since $\eta_{00} = -1$,

$$[a_{\mathbf{p}0}, a_{\mathbf{q}0}^\dagger] = -2E_{\mathbf{p}}(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \ , \tag{1.41}$$

The minus sign may seem problematic, since it leads to negative norm states

$$\langle 0 | a_{\mathbf{p}0} a_{\mathbf{p}0}^\dagger | 0 \rangle = \langle \mathbf{p}, 0 | \mathbf{p}, 0 \rangle < 0 \ , \tag{1.42}$$

if we consider the full Fock space $\mathcal{F}$, as we did before. But the rescue comes from the gauge condition (1.37) which we now properly introduce as

$$\langle \varphi_1 | \partial_\mu A^\mu | \varphi_2 \rangle = 0 \ , \tag{1.43}$$

where φ_1 and φ_2 are any two physical states. The condition (1.43) restricts the physical Fock space to a subspace $\mathcal{F}_{\text{phys}} \subset \mathcal{F}$, which has the positive definite norm.

This method of quantizing gauge fields is called Gupta-Bleuler formalism, developed in the works [7, 8]. It is suitable for abelian gauge theories, like QED, but is technically challenging to generalize to non-abelian theories because of self-interactions of the gauge fields. For this reason we shall introduce a more powerful framework - path integral quantization [9, 10, 11, 12, 13, 14].

1.1.4 Non-abelian gauge field

Gauge bosons of $SU(N)$ theory transform in the adjoint representation of the gauge group that has dimension $N^2 - 1$. Thus there are $N^2 - 1$ degrees of freedom associated with gauge bosons. Assigning the group index $a = 1, 2, ..., N^2 - 1$ to the gauge bosons, we write the Lagrangian as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \ , \tag{1.44}$$

where upper and lower gauge group indices are not distinguished, and summation over repeated indices is implied as usual. The non-abelian field strength in contrast to the abelian one has an additional term, when defined through the gauge field,

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \ , \tag{1.45}$$

where g is the gauge coupling, and f^{abc} are structure constants. This last term yields self-interaction of the gauge boson, which means it is charged with respect to the gauge group (abelian gauge fields, in contrast, are neutral). The Lagrangian (1.44) is invariant under the (infinitesimal) gauge transformations

$$A_\mu^a \to A_\mu^a + \partial_\mu \alpha^a + f^{abc} A_\mu^b \alpha^c \ , \tag{1.46}$$

where $\alpha^a(x)$ are $N^2 - 1$ arbitrary functions.

The equations of motion follow as

$$(D_\mu F^{\mu\nu})^a = \partial_\mu F^{a\mu\nu} + g f^{abc} A_\mu^b F^{c\mu\nu} = 0 \ , \tag{1.47}$$

where we have introduced the covariant derivative D_μ. We can also use this covariant derivative to define the field strength:

$$[D_\mu, D_\nu] = -ig F_{\mu\nu}^a T^a \ , \tag{1.48}$$

where T^a are generators of infinitesimal gauge transformations obeying

$$[T^a, T^b] = i f^{abc} T^c \ . \tag{1.49}$$

We will now quantize $SU(N)$ gauge theory (or Yang-Mills theory) in path integral formalism using the so-called Faddeev-Popov method. Consider the functional integral

$$Z = \int \mathcal{D} A_\mu^a \ e^{iS} \ , \tag{1.50}$$

where the (gauge invariant) measure $\mathcal{D} A_\mu^a$ represents integration over all possible field configurations of a non-abelian gauge field A_μ^a. Here the index a is a group index which for $SU(N)$ is $a = 1, 2, ..., N^2 - 1$. Since the integral (1.50) contains gauge redundancies, they should be eliminated. Following the standard procedure we insert into the integral a unity in the form [3]

$$1 = \int \mathcal{D}\omega \ \delta[G(A^\omega)] \Delta[A] \ , \tag{1.51}$$

where ω is an infinitesimal gauge transformation of A_μ^a

$$(A_\mu^a)^\omega = A_\mu^a + \partial_\mu \omega^a + f^{abc} A_\mu^b \omega^c \ . \tag{1.52}$$

Here $G(A^\omega)$ is a gauge fixing condition. As an example, we choose the Lorenz gauge $G(A) = (\partial_\mu A^\mu)^a = 0$, so that

$$G(A_\mu^a)^\omega = \Box \omega^a + f^{abc} A_b^\mu \partial_\mu \omega_c \ . \tag{1.53}$$

Then the Faddeev-Popov (FP) determinant is

$$\Delta[A] = \det \left| \frac{\delta G(A)}{\delta \omega} \right| = \det(\Box \delta^{ac} + f^{abc} A_b^\mu \partial_\mu) \ . \tag{1.54}$$

[3] From now on, for convenience we suppress spacetime and gauge indices when working with path integrals, so that $A \equiv A_\mu^a$, and write them explicitly when needed. Yang-Mills potential with no gauge group index should be understood as $A_\mu \equiv A_\mu^a T^a$.

Plugging (1.51) into the path integral (1.50) and changing the gauge field $A \to A^v$, we get

$$Z = \int \mathcal{D}\omega \mathcal{D}A^v \; e^{iS}\delta[G(A^{\omega v})]\Delta[A^v] \; . \tag{1.55}$$

Then, choosing $v = \omega^{-1}$ and using gauge invariance of the measure, action, and the FP determinant, we have

$$Z = \int \mathcal{D}\omega \mathcal{D}A \; e^{iS}\delta[G(A)]\Delta[A] = \left(\int \mathcal{D}\omega\right) \int \mathcal{D}A \; e^{iS}\delta[G(A)]\Delta[A] \; , \tag{1.56}$$

where the factorized quantity $\int \mathcal{D}\omega$ is the infinite (constant) volume of the gauge group, which we will hide in the normalisation.

We can represent the FP determinant as a Gaussian integral of Grassmann variables, using the formula

$$\Delta[A] = \int \mathcal{D}\bar{\eta}\mathcal{D}\eta \exp\left(-i \int d^4x \bar{\eta}^a M_{ac} \eta^c\right) \; , \tag{1.57}$$

where $M_{ac} \equiv \Box\delta_{ac} + f_{abc}A^{b\mu}\partial_\mu$, and Grassmann variables $\bar{\eta}$ and η are fermionic fields obeying bosonic statistics. Being unphysical, they are called (Faddeev-Popov) ghosts.

Next, changing the gauge condition as $G(A) = 0 \to G(A) = \alpha(x)$, and averaging over arbitrary functions $\alpha(x)$ with a properly normalized Gaussian weight, we have

$$Z = \int \mathcal{D}A\mathcal{D}\alpha \; e^{iS} e^{-i\int d^4x(\alpha^2/2\xi)}\delta[G - \alpha]\Delta[A] \; . \tag{1.58}$$

Integrating over α and using (1.57), we arrive at

$$Z = \int \mathcal{D}A\mathcal{D}\bar{\eta}\mathcal{D}\eta \; e^{i\int d^4x \mathcal{L}_G} \; , \tag{1.59}$$

where

$$\mathcal{L}_G = \mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}G^2 - \bar{\eta}^a(\Box\delta_{ac} + f_{abc}A^{b\mu}\partial_\mu)\eta^c\right] \tag{1.60}$$

is the total Lagrangian containing gauge-fixing (recall $G = \partial_\mu A^\mu$) and ghost terms. The parameter ξ determines choice of a gauge. For example, $\xi \to 0$ corresponds to the Lorenz gauge $\partial_\mu A^\mu = 0$, while $\xi = 1$ corresponds to the so-called Feynman-'t Hooft gauge which is more convenient for perturbative calculations.

1.1.5 Interactions and perturbation theory

When coupling constants are small ($g \ll 1$, which is true for electroweak interactions, QED, and some high-energy QCD processes), particle interactions can be treated using time-dependent perturbation theory, where we expand the "scattering" matrix, or S-matrix, in a small coupling constant and calculate approximate "scattering" amplitudes. We use quotation marks on the word "scattering" since in QFT particles can not only scatter, but also transform and decay into one-another, as far as conservation laws allow.

We introduce a small interaction term V as a perturbation to the (Schrödinger) Hamiltonian,

$$H = H_0 + V \ , \tag{1.61}$$

where H_0 is the unperturbed (free) Hamiltonian. In free QFT we prefer to work in Heisenberg picture where time dependence is assigned to operators, while state vectors are time-independent. The relation between the Schrödinger picture and Heisenberg picture states ($|\Omega_S(t)\rangle$ and $|\Omega_H\rangle$ respectively) is

$$|\Omega_S(t)\rangle = e^{-iHt}|\Omega_H\rangle \ , \quad \text{or} \quad |\Omega_H\rangle = e^{iHt}|\Omega_S(t)\rangle \ , \tag{1.62}$$

where e^{-iHt} is the unitary time-evolution operator. When we add interactions, it becomes convenient to work in the so-called interaction picture, where we introduce the (interaction picture) states $|\Omega_I(t)\rangle$. In analogy with (1.62) we express $|\Omega_I(t)\rangle$ in terms of $|\Omega_S(t)\rangle$:

$$|\Omega_I(t)\rangle = e^{iH_0t}|\Omega_S(t)\rangle \ , \tag{1.63}$$

Unlike the Heisenberg states, the interactions picture states are time-dependent. This is because we are not using the full Hamiltonian anymore, $H_0 \neq H$.

Taking time derivative of (1.63) we see that

$$i\frac{d}{dt}|\Omega_I(t)\rangle = i\frac{d}{dt}(e^{iH_0t}|\Omega_S(t)\rangle) = e^{iH_0t}\left(i\frac{d}{dt} - H_0\right)|\Omega_S(t)\rangle \ . \tag{1.64}$$

But from the Schrödinger equation we know that

$$i\frac{d}{dt}|\Omega_S(t)\rangle = H|\Omega_S(t)\rangle \ , \tag{1.65}$$

thus (1.64) reads (omitting (t) for simplicity)

$$i\frac{d}{dt}|\Omega_I\rangle = e^{iH_0t}(H - H_0)|\Omega_S\rangle = e^{iH_0t}V|\Omega_S\rangle \ . \tag{1.66}$$

Then, using (1.63) this becomes

$$i\frac{d}{dt}|\Omega_I\rangle = V(t)|\Omega_I\rangle \ , \tag{1.67}$$

where

$$V(t) = e^{iH_0t}Ve^{-iH_0t} \tag{1.68}$$

is a time-dependent perturbation in the interaction picture.

Next, we turn our attention to time evolution of the operators in the interaction picture. In analogy with the relation between Heisenberg and Schrodinger picture operators, i.e.

$$\phi_H(t, \mathbf{x}) = e^{-iHt}\phi_S(\mathbf{x})e^{iHt} \ , \quad (\text{here } H = H_0) \tag{1.69}$$

we express interaction picture operators as

$$\phi_I(t, \mathbf{x}) = e^{-iH_0 t}\phi_S(\mathbf{x})e^{iH_0 t} \ , \quad \text{(here } H = H_0 + V) \ . \tag{1.70}$$

We constructed the interaction picture in such a way that turning off interactions automatically takes us to the Heisenberg picture,

$$|\Omega_I\rangle|_{V=0} = |\Omega_H\rangle \ , \quad \phi_I(x)|_{V=0} = \phi_H(x) \ . \tag{1.71}$$

Remotely before and after an interaction, particles can be described by free asymptotic states

$$|\Omega\rangle_- \equiv |\Omega(t \to -\infty)\rangle \ , \quad |\Omega\rangle_+ \equiv |\Omega(t \to +\infty)\rangle \ , \tag{1.72}$$

and the transition between the two states is dictated by the S-operator, $\hat{S}$, as

$$|\Omega\rangle_+ = \hat{S}|\Omega\rangle_- \ , \tag{1.73}$$

where

$$\hat{S} = \prod_{i=1}^{n} \exp(-iV(t_i)\delta t_i) \ , \tag{1.74}$$

where we divided the timeline between the two asymptotic states into n segments, and transitions between the segments are achieved by $\exp(-iV(t_i)\delta t_i)$ operators. Time ordering of these transition operators in (1.74) does matter because two operators at different times, t_i, in general do not commute, and we cannot simply put

$$\prod_{i=1}^{n} \exp(-iV(t_i)\delta t_i) = \exp\left(-i\sum_{i=1}^{n} V(t_i)\delta t_i\right) \ . \tag{1.75}$$

We can, however, use the so-called time-ordering operator T which puts everything it acts on in the right order. Thus, we can write

$$\prod_{i=1}^{n} \exp(-iV(t_i)\delta t_i) = T\left\{\exp\left(-i\sum_{i=1}^{n} V(t_i)\delta t_i\right)\right\} \ , \tag{1.76}$$

or taking a continuous limit, $\delta t \to 0$ and $n \to \infty$,

$$\hat{S} = T\left\{\exp\left(-i\int_{-\infty}^{+\infty} V(t)dt\right)\right\} \ . \tag{1.77}$$

From (1.73) we infer the S-matrix which is built from the elements

$$S_{ba} \equiv {}_+\langle\Omega_b|\hat{S}|\Omega_a\rangle_- \ , \tag{1.78}$$

which encode the probability amplitudes of processes taking $|\Omega_a\rangle_-$ to $|\Omega_b\rangle_+$.

When $V(t)$ is small, we can expand the S-operator in Taylor series,

$$\hat{S} = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} T \left\{ \int V(t_1)dt_1 \int V(t_2)dt_2 ... \int V(t_n)dt_n \right\} \ . \tag{1.79}$$

In terms of Hamiltonian (or Lagrangian) density $\mathcal{H}$ ($\mathcal{L}$), the interaction Hamiltonian $V(t)$ is written as

$$V(t) = \int d^3x \mathcal{H}_I = -\int d^3x \mathcal{L}_I \ , \tag{1.80}$$

where $\mathcal{H}_I$ and $\mathcal{L}_I$ are interactions parts of Hamiltonian and Lagrangian densities, respectively.

It is convenient to define the M- and T-matrices as

$$S_{ba} = \delta_{ba} - iM_{ba}(2\pi)^4\delta^4(p_b - p_a) \ , \tag{1.81}$$

$$S_{ba} = \delta_{ba} - iT_{ba}2\pi\delta(E_b - E_a) \ , \tag{1.82}$$

where M_{ba} and T_{ba} are the probability amplitudes for the transition from distinct a to b states. In the first case the 4-momentum (p) conserving delta-function is factorized, while in the second case only the energy (E) conserving delta function is factorized.

Let us now use an example of the QED scattering process $e^+e^- \to e^+e^-$, to be more specific. The corresponding interaction Lagrangian is

$$\mathcal{L}_I = -e\bar{\psi}_e\gamma^\mu\psi_e A_\mu = -\mathcal{H}_I \ . \tag{1.83}$$

Then the initial ($|a\rangle$) and final ($|b\rangle$) states are (using the decomposition of a Dirac spinor (1.24)(1.25) but unpolarized)

$$|a\rangle = c_1^\dagger d_2^\dagger |0\rangle = |p_1, p_2\rangle \ , \tag{1.84}$$

$$|b\rangle = c_3^\dagger d_4^\dagger |0\rangle = |p_3, p_4\rangle \ , \tag{1.85}$$

where states are labeled by 4-momenta p_i, with $i = 1, 2, 3, 4$, of the initial and final particles. Labels $1, 2$ are assigned to the initial positron and electron, while $3, 4$ - final positron and electron, respectively. The 4-momentum conservation law yields

$$p_1 + p_2 = p_3 + p_4 \ . \tag{1.86}$$

We will be interested in the physical quantities called decay rates (or decay widths) and cross-sections of the process, which are closely related to each other. The decay rate is the probability of the process per unit time, and it is not Lorentz-invariant. The cross-section, on the other hand, is Lorentz-invariant, and is defined as

$$\text{Cross-section} = \frac{\text{decay rate}}{\text{incident flux of particles}} \ . \tag{1.87}$$

The cross-section is a function of the products $p_i \cdot p_j$, where $i \neq j$ because $p_i^2 = -m_i^2$ gives no information about kinematics of the process. Taking into account the 4-momentum conservation (1.86), which eliminates one degree of freedom, we can construct 3 independent scalars out of

p_i. It is customary to choose the following combinations,

$$s \equiv -(p_1 + p_2)^2 \ , \quad t \equiv -(p_1 - p_3)^2 \ , \quad u \equiv -(p_1 - p_4)^2 \ , \tag{1.88}$$

called Mandelstam variables.

When choosing a reference frame, there are two commonly used ones - center-of-mass (CM) frame, and the "lab" frame. CM frame is defined by $\mathbf{p}_1 = -\mathbf{p}_2$, while in the lab frame $\mathbf{p}_1 = 0$ and $E_1 = m_1$.

For two-body scattering in the CM frame, the differential decay rate $d\Gamma$ and differential cross-section $d\sigma$ are related to the amplitude M_{ba} as

$$d\Gamma(a \to b) = \frac{|M_{ba}|^2}{4E_1E_2V}(2\pi)^4\delta^4(p_a - p_b)\prod_b \frac{d^3p_b}{(2\pi)^3 2E_b} \ , \tag{1.89}$$

$$d\sigma(a \to b) = |M_{ba}|^2 \frac{(2\pi)^4\delta^4(p_a - p_b)}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}}\prod_b \frac{d^3p_b}{(2\pi)^3 2E_b} \ , \tag{1.90}$$

where the index b in the product takes values $b = 3, 4$ denoting the two final state particles, and V is the volume of the box in which the process takes place.

Now we are left with the calculation of scattering amplitude M_{ba} from S-matrix element S_{ba},

$$S_{ba} = \langle 0|d_4 c_3 \hat{S} c_1^\dagger d_2^\dagger|0\rangle \ . \tag{1.91}$$

Expanding S-operator (using (1.83)) and leaving only the leading term we have

$$\hat{S} = T\left(-ie\int d^4x A_\mu \bar{\psi}_e \gamma^\mu \psi_e\right)^2 + \mathcal{O}(e^4) \ , \tag{1.92}$$

where all the lower-order terms vanish since unpaired creation (annihilation) operators in (1.91) commute with everything on their left (right) side, and annihilate the vacuum. Omitting technical details (which can be found in [4, 3, 5], for example) the result in terms of M_{ba} reads

$$M_{ba} = M_{ba}(s) + M_{ba}(t) \ , \tag{1.93}$$

where

$$M_{ba}(s) = e^2(\bar{v}_1\gamma^\mu u_2)\frac{\eta_{\mu\nu}}{(p_1 + p_2)^2}(\bar{u}_4\gamma^\nu v_3) \ , \tag{1.94}$$

$$M_{ba}(t) = -e^2(\bar{v}_1\gamma^\mu v_3)\frac{\eta_{\mu\nu}}{(p_1 - p_3)^2}(\bar{u}_4\gamma^\nu u_2) \ . \tag{1.95}$$

In order to simplify calculations, Feynman introduced a technique of using diagrams to represent expansion terms in the amplitude. Each term is divided into several parts, each of which is assigned a line (including loops) or a vertex in the corresponding Feynman graph. External lines represent incoming and outgoing particles. Depending on the spin of a particle there is a corresponding factor as shown in Table 1.1. Possible types of internal lines, or propagators,

Particle	Feynman graph line	Incoming line	Outgoing line
Spin-0		1	1
Spin-$\frac{1}{2}$		$u(p, \lambda)$	$\bar{u}(p, \lambda)$
Spin-$\frac{1}{2}$ (antiparticle)		$\bar{v}(p, \lambda)$	$v(p, \lambda)$
Spin-1		$\epsilon(p, \lambda)$	$\epsilon^*(p, \lambda)$

Table 1.1: Expressions for external lines

Spin-0		$-i \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2 - i0}$
Spin-$\frac{1}{2}$		$-i \int \frac{d^4p}{(2\pi)^4} \frac{-i\not{p} + m}{p^2 + m^2 - i0}$
Spin-1 (R_ξ gauge)		$-i \int \frac{d^4p}{(2\pi)^4} \frac{\eta_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2+\xi m^2}}{p^2 + m^2 - i0}$

Table 1.2: Expressions for internal lines (propagators)

and their expressions are listed in Table 1.2, where the term $-i0$ in the denominator represents small imaginary shift to avoid poles during integration. The situation is a bit more complicated with vertices, since there are many different types of them in the Standard Model. For example, a vertex for the QED interaction $\gamma e^+ e^-$ contributes a factor of

$$- e\gamma^\mu (2\pi)^4 \delta^4(p_a - p_b) \ , \tag{1.96}$$

where p_a and p_b are incoming and outgoing 4-momenta respectively, so that the delta-function conserves the total 4-momentum of the system. For all possible vertices of the SM interactions see [5].

Going back to our example ($e^+e^- \to e^+e^-$), let us draw the two leading-order diagrams (Figures 1.1 and 1.2) for this process. The time axis conventionally goes from left to right, and while the arrows on the external lines of electrons coincide with the flow of time, those of positrons are often drawn pointing backwards in time (although if we label each line by the particles' names, we can ignore this convention and draw every arrow on external lines pointing towards future). So the top-left external line of both diagrams (1.1 and 1.2) represent incoming positron, while top-right lines represent outgoing positron. Similarly, bottom-left and -right lines stand for incoming and outgoing electron, respectively. If we are considering only QED, the wiggly line represents photon. But in the full Standard Model the same diagrams appear with Z boson propagator as well.

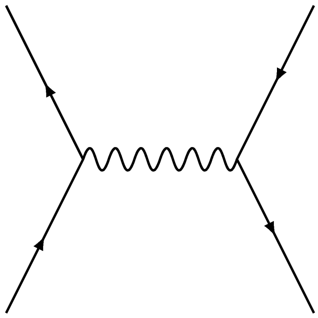

FIGURE 1.1: e^+e^- scattering s-channel diagram

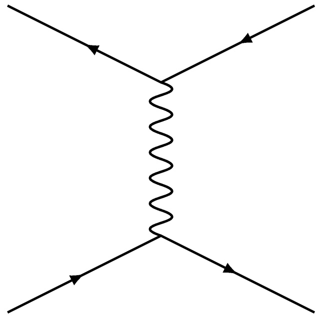

FIGURE 1.2: e^+e^- scattering t-channel diagram

Putting together the Feynman rules we listed above for the diagram in Figure 1.1, we obtain the S-matrix element (in Feynman-'t Hooft gauge, $\xi = 1$)

$$S_{ba}(s) = -ie^2 \int d^4k(2\pi)^4\delta^4(p_1 + p_2 - k)\delta^4(k - p_3 - p_4)(\bar{v}_1\gamma^\mu u_2)\frac{\eta_{\mu\nu}}{k^2 - i0}(\bar{u}_4\gamma^\nu v_3) \ , \tag{1.97}$$

where k is the photon 4-momentum. Then, performing the integration and using (1.81) we find

$$M_{ba}(s) = e^2(\bar{v}_1\gamma^\mu u_2)\frac{\eta_{\mu\nu}}{(p_1 + p_2)^2}(\bar{u}_4\gamma^\nu v_3) \ , \tag{1.98}$$

called the s-channel amplitude, and the corresponding diagram called the s-channel diagram, because the Mandelstam variable $s = -(p_1 + p_2)^2$. Similarly reading off the t-channel diagram in Figure 1.2, we obtain exactly (1.95). Again, the name t-channel follows from the Mandelstam variable $t = -(p_1 - p_3)^2$.

1.1.6 Renormalization

Renormalization is a reparametrization procedure of coupling constants of a theory, with the aim to eliminate the dependence of the physical quantities, like amplitudes, on the (ultraviolet, or UV) cut-off scale Λ. Naively, Λ can be taken arbitrarily large, however, it is inevitable that new physics will appear at some point (e.g. Grand Unification, quantum gravity), and so Λ should be taken as the corresponding scale. When couplings are renormalized they absorb the Λ-dependence of physical quantities, and become functions of the "running" scale - the scale at which the related physical process takes place. The couplings in the Lagrangian are referred to as bare couplings. The renormalized couplings are sums of the bare couplings and the infinity of loop contributions. If a coupling is small, it, of course, makes every subsequent term less and less significant.

As an example, consider the effective electromagnetic coupling e measured at the scale of the electron mass m_e, say, in a QED scattering process. For a tree-level Feynman diagram in Figure 1.3 (left), there is a one-loop diagram with a fermionic loop, called the self-energy diagram. So,

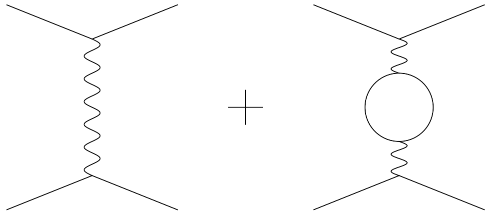

FIGURE 1.3

at one-loop order, the aforementioned coupling reads

$$e^2(m_e) = e_0^2 - \frac{e_0^4}{12\pi^2} \log\left(\frac{\Lambda^2}{m_e^2}\right) + O(e_0^6) \ , \tag{1.99}$$

where the first term on the RHS comes from the tree-level diagram (Figure 1.3, left), while the second term comes from the one-loop self-energy diagram (Figure 1.3, right).

Next, consider a similar scattering process but with a large momentum transfer $p^2 \gg m_e^2$. The amplitude for this process is proportional to

$$M \propto e_0^2 - \frac{e_0^4}{12\pi^2} \log\left(\frac{\Lambda^2}{p^2 e^{-5/3}}\right) + O(e_0^6) \ . \tag{1.100}$$

When substituting (1.99), and replacing e_0^4 with $e^4(m_e)$ (the difference is of higher order and can be neglected), Λ-dependence is cancelled because the logarithm term from (1.99) enters with the opposite sign.

We can generalize the expression (1.99) for e^2 at arbitrary ("running") scale μ,

$$e^2(\mu) = e_0^2 - \frac{e_0^4}{12\pi^2} \log\left(\frac{\Lambda^2}{\mu}\right) + O(e_0^6) \ , \tag{1.101}$$

and, in order to see how e^2 runs with μ, substitute e_0^2 from (1.99). We obtain

$$e^2(\mu) = e^2(m_e) - \frac{e^4(m_e)}{12\pi^2} \log\left(\frac{m_e^2}{\mu}\right) + O(e^6(m_e)) \ , \tag{1.102}$$

and further generalize it by differentiating with respect to μ^2 to get

$$\mu^2 \frac{de^2(\mu)}{\mu^2} = \frac{e^4(m_e)}{12\pi^2} + O(e^6(m_e)) \ . \tag{1.103}$$

The quantity $\mu^2 \frac{de^2(\mu)}{\mu^2} \equiv \beta(e)$ is called the renormalisation group beta function.

The beta functions for the SM interactions ($i = 1, 2, 3$ for $U(1)_Y$, $SU(2)_L$, $SU(3)_C$ respectively) read

$$\beta_i(g) = b_i \frac{g_i^4}{12\pi^2} \ , \tag{1.104}$$

G^a_μ	$(\mathbf{8}, \mathbf{1}, 0)$
W^i_μ	$(\mathbf{1}, \mathbf{3}, 0)$
B_μ	$(\mathbf{1}, \mathbf{1}, 0)$

Table 1.3: SM gauge bosons

with

$$b_1 = \frac{41}{10} \ , \quad b_2 = -\frac{19}{6} \ , \quad b_3 = -7 \ , \tag{1.105}$$

and g_i - coupling constants. Computation of b_i is rather technical, but note that b_1 is positive and b_2, b_3 are negative, which result in different behaviour of couplings: α_1 decreases at higher energies while α_2 and α_3 increase. This results in asymptotic freedom in QCD, in particular.

The general solution of (1.103), in terms of $\alpha_i \equiv g_i^2/4\pi$ and with m_e yet again generalized by μ_0, reads

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(\mu_0) + \frac{b_i}{3\pi} \log\left(\frac{\mu_0^2}{\mu^2}\right) \ , \tag{1.106}$$

and is referred to as Renormalization Group Equations. They define running of the couplings.

1.2 Standard Model particles

The three forces of the Standard Model (SM) - electromagnetic, weak, and strong - are described by a gauge theory based on the combined $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group, where $SU(3)_c$ corresponds to QCD (c for colour), and $SU(2)_L \times U(1)_Y$ corresponds to the electroweak interaction. Subscript L stands for "left", since only left-chiral fermions transform non-trivially under $SU(2)_L$. More precisely, they form $SU(2)$-doublets, while right-chiral fermions are $SU(2)$-singlets. Subscript Y denotes so-called hypercharge, to distinguish it from the electric charge. While $SU(3)_c$ symmetry is exact, electroweak symmetry is spontaneously broken as $SU(2)_L \times U(1)_Y \to U(1)_{em}$, where $U(1)_{em}$ is a gauge group of electromagnetic interaction, the coupling constant of which is the electric charge. $U(1)_{em}$ symmetry is a combination of $U(1)_Y$ and the $U(1)$ group sitting inside $SU(2)_L$.

Gauge boson content of the SM consists of 8 $SU(3)_c$ gauge bosons - gluons G^a_μ, transforming as octet under the corresponding gauge group; 3 $SU(2)_L$ (sometimes called "weak") gauge bosons W^i_μ, transforming as triplet; and the $U(1)_Y$ gauge boson B_μ. We use indices $a, b, c = 1, ..., 8$ for $SU(3)_c$ group, and $i, j, k = 1, 2, 3$ for $SU(2)_L$. Transformation properties of the gauge bosons under $SU(3)_c \times SU(2)_L \times U(1)_Y$ are summarized in Table 1.3. The first number in the parentheses stands for $SU(3)_c$ multiplicity, second one - for $SU(2)_L$ multiplicity, and the last one is hypercharge Y.

Fermionic content consists of "fundamental" spin-1/2 particles which can be divided into leptons and quarks. Leptons are defined as fermions that are $SU(3)_c$ singlets, i.e. that don't participate in strong interactions. They are: electron e, muon μ, tau lepton τ, and associated neutrinos ν_e, ν_μ, ν_τ. Quarks, on the other hand, are fermions that do carry colour charge and interact strongly. However, unlike leptons, at low energies they are only found in bound colour-neutral states - baryons (combination of 3 quarks) and mesons (combination of quark and anti-quark). There are six quark "flavours": up u, down d, strange s, charm c, top t, and bottom b. Each of them carry one of the three colour charges conventionally denoted r (red), g (green), and b (blue). Quarks and leptons can also be grouped into 3 generations, with each successive generation essentially being just a heavier version of the previous generation with the same quantum numbers. So, the three generations of leptons are e and ν_e, μ and ν_μ, τ and ν_τ. And for quarks - u and d, s and c, t and b. Interestingly, members of each generation of both quarks and leptons differ by one unit of electric charge. For example, e has an electric charge $Q(e) = -1$, while $Q(\nu_e) = 0$; similarly $Q(u) = +2/3$ and $Q(d) = -1/3$.

In the Standard Model it is convenient to use Weyl or Majorana spinors to represent left- and right-chiral [4] components of Dirac spinors, since they transform differently under $SU(2)_L$. The Dirac spinor of the electron (and its heavier cousins muon and tau) can be decomposed to left and right Weyl spinors as

$$e = \begin{pmatrix} e_L \\ e_R \end{pmatrix} . \tag{1.107}$$

Then to translate this into the language of Majorana spinors, we define (in the notation of [5])

$$\mathcal{E} = \begin{pmatrix} e_L \\ i\sigma_2 e_L^* \end{pmatrix} , \quad E = \begin{pmatrix} -i\sigma_2 e_R^* \\ e_R \end{pmatrix} , \tag{1.108}$$

where $\mathcal{E}$ and E are Majorana spinors containing left and right Weyl spinors respectively, and σ_2 is the second Pauli matrix. Now, using projection operators (see appendix) P_L and P_R, it is easy to see that

$$e = P_L\mathcal{E} + P_R E . \tag{1.109}$$

Left-chiral electron (mu, tau) $\mathcal{E}$ and the electron (mu, tau) neutrino ν (which has only left-chiral component) form an $SU(2)_L$ doublet L:

$$L_m = \begin{pmatrix} \nu \\ \mathcal{E} \end{pmatrix}_m , \tag{1.110}$$

where we use the index $m = 1, 2, 3$ to distinguish between the three generations. E is an $SU(2)_L$-singlet. For left-chiral quark doublet we have

$$Q_m = \begin{pmatrix} \mathcal{U} \\ \mathcal{D} \end{pmatrix}_m , \tag{1.111}$$

while the right-chiral singlets are U_m and D_m. Here U and $\mathcal{U}$ stand for up-type quarks u, c, t; D and $\mathcal{D}$ stand for down-type quarks d, s, b. Transformation properties of leptons and quarks are summarized in Table 1.4. Bar over a bold number means complex-conjugate representation. The electric charge is defined simply as a sum $Q = T_3 + Y$, where T_3 is the eigenvalue of the third $SU(2)_L$ generator called (third component of) isospin.

[4] We sometimes refer to left- and right-chiral (Weyl) spinors just as "left" and "right" for simplicity.

$P_L L_m$	$(\mathbf{1}, \mathbf{2}, -1/2)$	$P_R L_m$	$(\mathbf{1}, \mathbf{2}, +1/2)$
$P_L E_m$	$(\mathbf{1}, \mathbf{1}, +1)$	$P_R E_m$	$(\mathbf{1}, \mathbf{1}, -1)$
$P_L Q_m$	$(\mathbf{3}, \mathbf{2}, +1/6)$	$P_R Q_m$	$(\bar{\mathbf{3}}, \mathbf{2}, -1/6)$
$P_L U_m$	$(\bar{\mathbf{3}}, \mathbf{1}, -2/3)$	$P_R U_m$	$(\mathbf{3}, \mathbf{1}, +2/3)$
$P_L D_m$	$(\bar{\mathbf{3}}, \mathbf{1}, +1/3)$	$P_R D_m$	$(\mathbf{3}, \mathbf{1}, -1/3)$

TABLE 1.4: SM fermions

The last missing piece of the Standard Model that has been experimentally confirmed [15] is the Higgs boson - the only "fundamental" scalar in the SM. It transforms as $(\mathbf{1}, \mathbf{2}, +1/2)$ (while it's conjugate as $(\mathbf{1}, \mathbf{2}, -1/2)$), i.e. as an $SU(2)_L$-doublet,

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} , \tag{1.112}$$

with electrically charged component ϕ^+ and neutral component ϕ^0.

Now we are ready to write down the Standard Model Lagrangian,

$$\begin{aligned} \mathcal{L} = &-(D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{2}\bar{L}_m \not{D} L_m - \frac{1}{2}\bar{E}_m \not{D} E_m - \frac{1}{2}\bar{Q}_m \not{D} Q_m - \frac{1}{2}\bar{U}_m \not{D} U_m \\ &-\frac{1}{2}\bar{D}_m \not{D} D_m - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W^{i\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{g_3^2\theta_3}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}G^{a\mu\nu}G^{a\rho\sigma} \\ &-\frac{g_2^2\theta_2}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}W^{i\mu\nu}W^{i\rho\sigma} - \frac{g_1^2\theta_1}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}B^{\mu\nu}B^{\rho\sigma} - V(\phi,\phi^\dagger) \\ &-(y^e_{mn}\bar{L}_m P_R E_n \phi + y^u_{mn}\bar{Q}_m P_R U_n \tilde{\phi} + y^d_{mn}\bar{Q}_m P_R D_n \phi + \text{h.c.}) , \end{aligned} \tag{1.113}$$

where $\tilde{\phi} \equiv i\sigma_2\phi^*$; y^f_{mn}, with $f = e, u, d$, are Yukawa couplings (scalar-spinor), and $V(\phi, \phi^\dagger)$ is the Higgs potential,

$$V = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 , \tag{1.114}$$

with real parameters μ and λ satisfying $\mu^2 > 0$ (for spontaneous electroweak symmetry breaking) and $\lambda > 0$ (for stability). The covariant derivatives depend on the objects they act on as follows:

$$D_\mu \phi = \partial_\mu \phi - i g_2 W_\mu^i T^i \phi - \frac{i}{2} g_1 B_\mu \phi \ , \tag{1.115}$$

$$D_\mu L_m = \partial_\mu L_m + \left(-i g_2 W_\mu^i T^i + \frac{i}{2} g_1 B_\mu \right) P_L L_m + \left(i g_2 W_\mu^i T^{i*} - \frac{i}{2} g_1 B_\mu \right) P_L L_m \ , \tag{1.116}$$

$$D_\mu E_m = \partial_\mu E_m + i g_1 B_\mu P_R E_m - i g_1 B_\mu P_L E_m \ , \tag{1.117}$$

$$\begin{aligned} D_\mu Q_m = \partial_\mu Q_m &+ \left(-i g_3 G_\mu^a T^a - i g_2 W_\mu^i T^i - \frac{i}{6} g_1 B_\mu \right) P_L Q_m \\ &+ \left(i g_3 G_\mu^a T^{a*} + i g_2 W_\mu^i T^{i*} + \frac{i}{6} g_1 B_\mu \right) P_R Q_m \ , \end{aligned} \tag{1.118}$$

$$D_\mu U_m = \partial_\mu U_m + \left(-i g_3 G_\mu^a T^a - \frac{2i}{3} g_1 B_\mu \right) P_R U_m + \left(i g_3 G_\mu^a T^{a*} + \frac{2i}{3} g_1 B_\mu \right) P_L U_m \ , \tag{1.119}$$

$$D_\mu D_m = \partial_\mu D_m + \left(-i g_3 G_\mu^a T^a + \frac{i}{3} g_1 B_\mu \right) P_R D_m + \left(i g_3 G_\mu^a T^{a*} - \frac{i}{3} g_1 B_\mu \right) P_L D_m \ , \tag{1.120}$$

where $T^a = \lambda^a/2$ with Gell-Mann matrices λ^a, and $T^i = \sigma^i/2$ with Pauli matrices σ^i.

The so-called θ-terms (the terms including $\theta_1, \theta_2, \theta_3$) are total derivatives, and do not contribute to classical equations of motion. They are non-perturbative (topological) terms, which are important for CP violation [5].

The reason we do not introduce explicit mass terms for gauge bosons and fermions is that it would break gauge invariance. In the following section we introduce a mechanism, which can give masses to the aforementioned particles, via spontaneous symmetry breaking.

1.3 Spontaneous electroweak symmetry breaking

Mass terms for gauge bosons and fermions in the Lagrangian (1.113) are not allowed, as they would break gauge symmetry. But we know experimentally that the weak force is short-ranged (and does not exhibit confinement, unlike QCD), so that it must be mediated by a massive gauge boson. Furthermore, quarks and leptons are also found to be massive. A way to add masses to a theory while keeping the Lagrangian gauge invariant is to break gauge symmetry spontaneously, which means making the vacuum gauge variant by letting a certain field(s) acquire a non-zero vacuum expectation value(s) (VEV). Then in the SM the need for a fundamental scalar arises, because non-zero vev cannot be assigned to spinor and vector fields (that would break Lorentz symmetry.) With that purpose the Higgs complex scalar field, and the Higgs mechanism were introduced, by which certain particles acquire masses while preserving gauge symmetry of the Lagrangian.

We parametrize the Higgs field by choosing the unitary gauge where its upper (charged) component vanishes, while the lower (neutral) component is real,

[5] For more details on topological terms see e.g. [16], or Chapter 11 of [5].

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \upsilon + H \end{pmatrix} , \tag{1.121}$$

where υ is the (real constant) vev, and H is the redefined Higgs field with vanishing vev. Then in vacuum we have

$$\langle\phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \upsilon \end{pmatrix} , \tag{1.122}$$

The vacuum defined by this configuration has the residual gauge symmetry $SU(3)_c \times U(1)_{em}$.

We now examine the perturbative spectrum of the theory. After inserting (1.121) into the Lagrangian (1.113) we first consider the term

$$\mathcal{L} \supset -(D_\mu\phi)^\dagger D^\mu\phi = -\frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{4}g_2^2(\upsilon+H)^2 W_\mu^+ W^{-\mu} - \frac{1}{8}(g_1^2+g_2^2)(\upsilon+H)^2 Z_\mu Z^\mu , \tag{1.123}$$

where W_μ^1 and W_μ^2 combine as

$$W_\mu^\pm \equiv \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2) \tag{1.124}$$

with respective electric charges $Q = \pm 1$, and masses

$$M_W \equiv M_{W^\pm} = \frac{1}{2}g_2\upsilon . \tag{1.125}$$

Z_μ is another, electrically neutral, combination,

$$Z_\mu \equiv -\sin\theta_W B_\mu + \cos\theta_W W_\mu^3 , \tag{1.126}$$

where θ_W is the Weinberg angle defined as

$$\sin\theta_W \equiv \frac{g_2}{\sqrt{g_1^2+g_2^2}} , \qquad \cos\theta_W \equiv \frac{g_1}{\sqrt{g_1^2+g_2^2}} . \tag{1.127}$$

The mass of Z_μ reads

$$M_Z = \frac{1}{2}\sqrt{g_1^2+g_2^2}\,\upsilon = \frac{M_W}{\cos\theta_W} , \tag{1.128}$$

while the orthogonal field, the photon,

$$A_\mu \equiv \sin\theta_W B_\mu + \cos\theta_W W_\mu^3 \tag{1.129}$$

is massless.

The mass of the Higgs field (H) itself comes from the potential V which yields

$$M_H^2 = \mu^2 = \frac{1}{2}\lambda\upsilon^2 , \tag{1.130}$$

where the second equality comes from the vacuum condition

$$V_0 = -\frac{1}{2}\mu^2\upsilon^2 + \frac{1}{4}\lambda\upsilon^4 = 0 \ . \tag{1.131}$$

The fermion mass terms come from Yukawa couplings

$$\mathcal{L} \supset -\frac{\upsilon}{\sqrt{2}}(y^e_{mn}\bar{\mathcal{E}}_m P_R E_n + y^u_{mn}\bar{\mathcal{U}}_m P_R U_n + y^d_{mn}\bar{\mathcal{D}}_m P_R D_n + \text{h.c.}) \ , \tag{1.132}$$

where the Higgs VEV picks out specific components of left-chiral doublets. In particular this leaves neutrinos massless. One can add right-chiral (or right-handed) neutral heavy leptons by hand to introduce neutrino masses.

The Yukawa mass matrices $M^f = \upsilon y^f/\sqrt{2}$ in general are not diagonal, which they should be if we want to identify mass eigenstates. We can diagonalize them by six unitary matrices V^f_L and V^f_R as

$$\tilde{M}^f = \frac{\upsilon}{\sqrt{2}} V^{f\dagger}_L y^f V^f_R \ , \tag{1.133}$$

where $\tilde{M}^f$ is diagonal. These six unitary matrices are introduced by redefinition of the fermions as

$$\begin{aligned} P_L\mathcal{E}_m &= (V^e_L)_{mn} P_L \mathcal{E}'_n \ , \quad & P_R E_m &= (V^e_R)_{mn} P_R E'_n \ , \\ P_L\mathcal{U}_m &= (V^u_L)_{mn} P_L \mathcal{U}'_n \ , \quad & P_R U_m &= (V^u_R)_{mn} P_R U'_n \ , \\ P_L\mathcal{D}_m &= (V^d_L)_{mn} P_L \mathcal{D}'_n \ , \quad & P_R D_m &= (V^d_R)_{mn} P_R D'_n \ , \end{aligned} \tag{1.134}$$

This, in turn, has an interesting effect on the couplings between quarks and $W^\pm_\mu$,

$$\mathcal{L} \supset \frac{ig_2}{\sqrt{2}}[W^+_\mu \bar{\mathcal{U}}_m \gamma^\mu P_L \mathcal{D}_m + W^-_\mu \bar{\mathcal{D}}_m \gamma^\mu P_L \mathcal{U}_m] \ . \tag{1.135}$$

After the redefinitions [6] (1.134) the interaction terms (1.135) become

$$\frac{ig_2}{\sqrt{2}}[W^+_\mu \mathcal{V}_{mn} \bar{\mathcal{U}}'_m \gamma^\mu P_L \mathcal{D}'_n + W^-_\mu (\mathcal{V}^\dagger)_{mn} \bar{\mathcal{D}}'_m \gamma^\mu P_L \mathcal{U}'_n] \ , \tag{1.136}$$

where $\mathcal{V} \equiv (V^u_L)^\dagger V^d_L$ is known as Cabbibo-Kobayashi-Maskawa (CKM) matrix. It is a 3×3 unitary matrix responsible for mixing between different generations of quarks.

After adding neutrino masses to the SM, their mass terms undergo similar diagonalization procedure, and the generation-mixing matrix (analogous to CKM) can be defined. It is named Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix.

[6] For $\mathcal{U}$ and $\mathcal{D}$ we may use the redefinitions involving only V_L, since (1.135) involves only P_L projectors.

1.4 Problems of the Standard Model

The Standard Model of particle physics is a remarkably successful theory. It provides an extremely precise working model of particle interactions at presently available energies. However, the story does not end here, as the SM has a number of problems and unanswered questions. Let us mention some of these problems:

- One crucial thing the SM does not include is *gravity.* The difficulty arises if one tries to quantise General Relativity, due to the well-known fact that it is non-renormalisable. Of course one may not care about gravitational effects at scales well below the Planckian one ($M_P \equiv (8\pi G)^{-1/2} \sim 10^{18}$ GeV). But without the proper theory of quantum gravity, our picture of the Universe is not complete. There are at least two types of objects out there, to fully understand which, quantum gravity is necessary – black holes and the "Big Bang" singularity.

- *The hierarchy problem.* Why is there such an enormous gap between the electroweak scale and the Planck scale? The latter is $\sim 10^{16}$ times larger than the former! In the next chapter we are going to show that the hierarchy problem also leads to extreme fine tuning of the Higgs mass parameter.

- Another big questions in the SM is the *origin of quark and lepton masses* (or Yukawa couplings). In the SM the Yukawa couplings are input parameters, i.e. their values are put by hand. In the same category is the problem of neutrino masses which are absent in the SM. Although there is a possible solution to this problem (see-saw mechanism), it requires the introduction of a new heavy particle.

- Why are there *three generations* of leptons and quarks? These are basically three copies of the same particles that differ only in their masses.

- *Dark energy.* From astronomical observations we know that the visible matter constitutes only a fraction of the total energy density of the Universe. The total energy density is dominated by the dark sector which consists of dark energy and dark matter. Dark energy, in the simplest scenario, is identified with cosmological constant which, in turn, is assumed to be the vacuum energy. However, observations show that the cosmological constant (vacuum energy) is 120 *orders of magnitude* smaller than the expected value in the SM. This is a very large discrepancy!

- *Dark matter.* The existence of dark matter poses yet another challenge for the SM. Dark matter is made of particles of unknown origin, that do not interact with the SM particles by means other than gravity.

  In the next chapter we introduce SUSY and show that it helps resolve some of the mentioned problems.

Chapter 2

Supersymmetry and MSSM

This chapter summarises basic facts about supersymmetry, supergravity, and the Minimal Supersymmetric Standard Model (MSSM) along the lines of [17, 18, 19].

2.1 Rigid (global) supersymmetry

Supersymmetry originated as a tool for solving some of the problems of the Standard Model. It first of all addresses the fine-tuning problem. The essence of the problem is that the Higgs mass receives enormous quantum corrections proportional to the UV cut-off scale Λ. With the Higgs potential,

$$V = -\mu^2 \phi^\dagger \phi + \frac{\lambda}{4}(\phi^\dagger \phi)^2 \ , \tag{2.1}$$

to one-loop order, the measured value of the mass parameter μ is

$$\mu^2_{\text{meas}} = \mu^2 - \lambda \Lambda^2 \ , \tag{2.2}$$

where Λ is the UV cut-off. The correction $-\lambda\Lambda^2$ comes from the diagram (Fig. 2.1) representing the term proportional to

$$\lambda \int^{\Lambda} d^4k \frac{1}{k^2 - m_H^2} \ . \tag{2.3}$$

Here $m_H = \sqrt{2}\mu$ is the mass of the Higgs boson. When Λ is at the GUT scale ($\sim 10^{16}$ GeV) or at the Planck scale ($\sim 10^{18}$ GeV), the extremely precise cancellation is needed between the

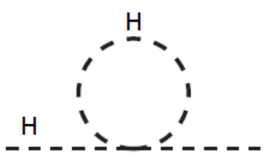

FIGURE 2.1: Higgs self-energy correction due to Higgs loop

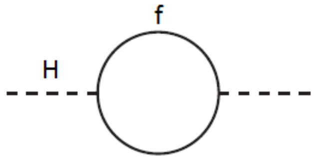

FIGURE 2.2: Higgs self-energy correction due to fermionic loop

Lagrangian parameter μ^2 and the correction $\lambda\Lambda^2$, in order to get the observed value μ_{meas} of order ~ 100 GeV.

Now, considering the contribution from the fermionic loop (Fig. 2.2), which is proportional to

$$-\lambda_f^2 \int^{\Lambda} d^4k \, \text{Tr}\left(\frac{1}{(\gamma^\mu k_\mu - m_f)^2}\right) \,, \tag{2.4}$$

one finds that if $\lambda_f^2 = \lambda$ and $m_f = m_H$, the bosonic and fermionic corrections cancel each other (and do so at all loops)! Thus, the problem can be solved by introducing the fermionic superpartner of the Higgs boson with the same coupling and mass.

This, of course, is not the only reason to be interested in SUSY. When considering Grand Unification, and embedding $SU(3) \times SU(2) \times U(1)$ in a larger simple Lie group, say, $SU(5)$, the couplings do not exactly unify with the SM beta functions, although they come very close to each other. If we consider the supersymmetric extension of the SM, the exact unification can be achieved, which, if coincidental, is quite miraculous.

It should also be mentioned, that supersymmetry is inevitable if one goes even further in unification, and considers the "Theory of Everything", in which case bosons and fermions should also be unified among other things. And indeed, the only known candidate for such theory (or rather, a framework) - superstring theory - requires the inclusion of supersymmetry for its consistency.

2.1.1 Wess-Zumino model

We begin with the simplest supersymmetric theory, the Wess-Zumino model, having only spinors and scalars.

The free Wess-Zumino Lagrangian is

$$\mathcal{L}_{free} = \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \bar{\psi}_i i \bar{\sigma}^\mu \partial_\mu \psi_i \,, \tag{2.5}$$

where the complex scalar field ϕ is partnered with the Weyl spinor ψ. $i = 1, ..., n$, where n represents the number of different fields, and the summation is implied over the repeated indices. On-shell both fields have 2 real degrees of freedom, but off-shell 2 additional spinor degrees of freedom will emerge, so we will match them by adding an auxiliary complex scalar field F (since we need 2 real bosonic degrees of freedom). The resulting Lagrangian

$$\mathcal{L}_{free} = \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \bar{\psi}_i i \bar{\sigma}^\mu \partial_\mu \psi_i + F_i^\dagger F_i \tag{2.6}$$

is invariant with respect to the SUSY transformations:

$$\delta_\xi \phi = \xi \psi \tag{2.7}$$
$$\delta_\xi \psi = -i \sigma^\mu \bar{\xi} \partial_\mu \phi + \xi F \tag{2.8}$$
$$\delta_\xi F = -i \bar{\xi} \bar{\sigma}^\mu \partial_\mu \psi \tag{2.9}$$

which can be obtained from dimensional analysis and Lorentz-invariance of the transformations. ξ is the constant parameter of SUSY transformations. On-shell $F_i = 0$ here.

To build a more general Lagrangian, we add interactions as

$$\mathcal{L}_{int} = -W_i F_i - \frac{1}{2} W_{ij} \psi_i \psi_j + h.c. \ , \tag{2.10}$$

where W_{ij} are some functions of ϕ_i and $\phi_i^\dagger$ (i represents flavour). Consistency requires SUSY transformation of W_{ij} to be proportional to $\delta\phi$:

$$-\frac{1}{2} \frac{\partial W_{ij}}{\partial \phi_k} (\xi \psi_k)(\psi_i \psi_j) - \frac{1}{2} \frac{\partial W_{ij}}{\partial \phi_k^\dagger} (\bar{\xi} \bar{\psi}_k)(\psi_i \psi_j) \ . \tag{2.11}$$

The first term vanishes due to the Fierz identity

$$\psi_k(\psi_i \psi_j) + \psi_i(\psi_j \psi_k) + \psi_j(\psi_k \psi_i) = 0 \ , \tag{2.12}$$

while for the second term there is no such identity and no other term can compensate it. This means if we want SUSY-invariant Lagrangian, W_{ij} cannot depend on $\phi^\dagger$ or ϕ^* (holomorphicity), and we have to introduce another scalar field in its place to generate the proper Yukawa sector (thus, in MSSM we will have 2 Higgs supermultiplets). The same goes for hermitian conjugate term, $W_{ij}^\dagger = W_{ij}^\dagger(\phi^\dagger)$.

SUSY-invariance requires W_i and W_{ij} to be of the form

$$W_i = \frac{\partial W}{\partial \phi_i}, \quad W_{ij} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \ . \tag{2.13}$$

Then, the general renormalisable superpotential takes the form

$$W = \text{const} + \frac{1}{2} m_{ij} \phi_i \phi_j + \frac{1}{6} y_{ijk} \phi_i \phi_j \phi_k \ , \tag{2.14}$$

where m_{ij} corresponds to the mass matrix elements, and y_{ijk} - to the Yukawa couplings, both totally symmetric. W is called the *superpotential*.

2.1.2 Superspace and Superfields

We could continue by adding new fields and adjusting the corresponding terms by hand for SUSY-invariance, but there is a more convenient and generic approach to build a supersymmetric theory - the *superspace formalism.*

In the superspace formalism space-time dimensions are complemented with four (in minimal case) anti-commuting fermionic degrees of freedom $\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2$ [1] (also known as Grassmann coordinates). As space-time symmetries constitute Poincare group with the corresponding Lie algebra, superspace symmetries constitute the so-called super-Poincare group. But the corresponding algebra is a *graded* Lie algebra, since fermionic generators obey anti-commutation relations. So, in $N = 1$ case the full algebra is

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = i\sigma_{\alpha\dot{\alpha}}{}^{\mu}\partial_\mu \equiv -\sigma_{\alpha\dot{\alpha}}{}^{\mu}P_\mu, \tag{2.15}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \tag{2.16}$$

$$[P_\mu, Q_\alpha] = \left[P_\mu, \bar{Q}_{\dot{\alpha}}\right] = 0 \ . \tag{2.17}$$

We identify the generators with differential operators in superspace as

$$Q_\alpha = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right), \tag{2.18}$$

$$\bar{Q}_{\dot{\alpha}} = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\right) \ . \tag{2.19}$$

However the choice is not unique, and there are different sets of coefficients in front of each term that will leave the algebra unchanged. Elements of the algebra act on the functions of superspace, called *superfields.* But what we are looking for are the chiral projections of superfields, which can be regarded as the SUSY extensions of left- and right- chiral Weyl spinors.

Chiral superfields

To find them, we first construct the algebra of "super-covariant derivatives" that generates super-translations in the opposite direction from (2.17), namely,

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -i\sigma_{\alpha\dot{\alpha}}{}^{\mu}\partial_\mu \ , \tag{2.20}$$

with

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \tag{2.21}$$

$$\bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \ . \tag{2.22}$$

[1] When dealing with supersymmetry in Weyl-spinor language, we use bar and dagger symbols interchangeably, e.g. $\phi^\dagger \equiv \bar{\phi}$.

Now we can define a *chiral* superfield by

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{2.23}$$

The most general solution to (2.23) is

$$\begin{aligned}\Phi = \phi(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x)\\ +\sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \theta\theta F\ ,\end{aligned} \tag{2.24}$$

or, if we make a transformation $x^\mu \to x'^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$, it reduces to

$$\Phi = \phi(x') + \sqrt{2}\theta\psi(x') - \theta\theta F(x')\ . \tag{2.25}$$

Conjugate of the chiral superfield (sometimes called anti-chiral superfield) satisfies

$$D_\alpha\bar{\Phi} = 0, \tag{2.26}$$

and has the form

$$\begin{aligned}\bar{\Phi} = \phi^\dagger(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^\dagger(x) - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi^\dagger(x)\\ +\sqrt{2}\bar{\theta}\bar{\psi}(x) - \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) - \bar{\theta}\bar{\theta}F^\dagger =\\ = \phi^\dagger(x'^+) + \sqrt{2}\bar{\theta}\bar{\psi}(x'^+) - \bar{\theta}\bar{\theta}F^\dagger(x'^+)\ ,\end{aligned} \tag{2.27}$$

where $(x'^\dagger)^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$.

Chiral superfields contain scalars and spinors, and can be shifted by a constant, since $\Phi = const$ is also a solution to (2.23). A collection of component fields of a chiral superfield is called chiral (or sometimes scalar) supermultiplet. All the SM fermions and the Higgs field (now two Higgs fields) are represented as the components of their respective chiral supermultiplets in the MSSM.

In the previous section we defined the superpotential $W(\phi)$ as a function of scalar fields only, and used it in the interaction terms of scalars and spinors. Generalizing the notion of the superpotential using chiral superfields,

$$\mathbb{W}(\Phi) = \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}y_{ijk}\Phi_i\Phi_j\Phi_k\ , \tag{2.28}$$

we can arrive at the Wess-Zumino Lagrangian:

$$\begin{aligned}\mathcal{L}_{WZ} = (\bar{\Phi}\Phi)|_{\theta\theta\bar{\theta}\bar{\theta}} + \mathbb{W}|_{\theta\theta} + \bar{\mathbb{W}}|_{\bar{\theta}\bar{\theta}} =\\ = \partial_\mu\phi_i^\dagger\partial^\mu\phi_i + \bar{\psi}_i i\bar{\sigma}^\mu\partial_\mu\psi_i + F_i^\dagger F_i - W_iF_i - F_i^\dagger W_i^\dagger - \frac{1}{2}W_{ij}\psi_i\psi_j - \frac{1}{2}(W_{ij}\psi_i\psi_j)^\dagger\ ,\end{aligned} \tag{2.29}$$

where $\mathcal{W}|_{\theta\theta}$ means $\theta\theta$ term of $\mathcal{W}$. We can equivalently use $\int d^2\theta W \equiv \mathcal{W}|_{\theta\theta}$. The kinetic term $\bar{\Phi}\Phi \equiv \mathcal{K}$ is also known as canonical Kähler potential.

Vector superfield

We have yet to generalize the gauge boson sector of the SM. So we define the vector superfield,

$$V = \bar{V}, \tag{2.30}$$

which generally takes the form

$$\begin{aligned} V = \bar{V} = C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{2}\theta\theta[M(x) + iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(x) - iN(x)] \\ -\theta\sigma^{\mu}\bar{\theta}A_{\mu}(x) + i\theta\theta\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\chi(x)\right) - i\bar{\theta}\bar{\theta}\theta\left(\lambda(x) + \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\chi}(x)\right) \\ +\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x) + \frac{1}{2}\Box C(x)\right) , \end{aligned} \tag{2.31}$$

where the vector field $A_\mu(x)$, and the scalar fields $C(x)$, $D(x)$, $M(x)$, $N(x)$ are all real valued. However, not all the fields are physical. Gauge invariance on superfield level introduces new redundant degrees of freedom which can be eliminated by appropriate gauge fixing. Gauge transformation of the vector superfield is

$$V' = V + \Lambda + \bar{\Lambda}, \tag{2.32}$$

where Λ and $\bar{\Lambda}$ are chiral and anti-chiral superfields which can be chosen specifically to cancel unphysical fields (the choice is called Wess-Zumino, or WZ gauge), leaving

$$V' = \theta\sigma^{\mu}\bar{\theta}A_{\mu}(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) . \tag{2.33}$$

Now it is clear, that $\lambda(x)$ is the superpartner of the gauge field $A_\mu(x)$, thus referred to as the *gaugino*. $D(x)$ is the auxiliary scalar field providing one off-shell bosonic degree of freedom. As one would expect, Wess-Zumino gauge fixing breaks supersymmetry (but preserves ordinary gauge invariance).

The superfield-generalized field strength is given by [2]

$$\mathcal{F}_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \overset{x\to x'}{=} -i\lambda_\alpha(x') + \theta_\alpha D(x') + \frac{i}{2}(\theta\sigma^\mu\bar{\sigma}^\nu)_\alpha F_{\mu\nu}(x') - \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda}(x'))_\alpha , \tag{2.34}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Since $\mathcal{F}_\alpha$ is chiral, $\bar{D}_{\dot{\alpha}}\mathcal{F}_\alpha = 0$, there is also the anti-chiral field strength,

$$\bar{\mathcal{F}}_\alpha = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V \overset{x\to x'^+}{=} i\bar{\lambda}_{\dot{\alpha}}(x'^+) + \bar{\theta}_{\dot{\alpha}}\bar{D}(x'^+) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\bar{\theta})_{\dot{\alpha}}F_{\mu\nu}(x'^+) - \bar{\theta}\bar{\theta}(\partial_\mu\lambda(x'^+)\sigma^\mu)_{\dot{\alpha}} , \tag{2.35}$$

[2] Sometimes, for the sake of simplicity, we omit (x) for the component fields, e.g. $A_\mu \equiv A_\mu(x)$. But the x'- and x'^+-dependence is always given explicitly.

satisfying $D_\alpha \bar{\mathcal{F}}_{\dot\alpha} = 0$. With these definitions we can write down the free vector supermultiplet Lagrangian:

$$\mathcal{L} = \frac{1}{4}(\mathcal{F}^\alpha \mathcal{F}_\alpha)|_{\theta\theta} + \frac{1}{4}(\bar{\mathcal{F}}_{\dot\alpha}\bar{\mathcal{F}}^{\dot\alpha})|_{\bar\theta\bar\theta} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\lambda i\bar\sigma^\mu\partial_\mu\lambda + \frac{1}{2}D^2 \ . \quad (2.36)$$

2.1.3 Supersymmetric abelian gauge theory

To construct gauge theory in terms of superfields, we have to find gauge invariant interactions between chiral and vector superfields. First, we introduce the $U(1)$ gauge transformation of the matter chiral superfield:

$$\Phi \to e^{i\Lambda}\Phi, \quad \Phi^\dagger \to \Phi^\dagger e^{-i\Lambda^\dagger} \ , \quad (2.37)$$

where Λ and $\Lambda^\dagger$ are chiral and anti-chiral superfields playing the roles of local parameters.

Vector superfields transform as in (2.32), but with proper normalization:

$$V \to V - \frac{i}{2g}(\Lambda - \Lambda^\dagger), \quad (2.38)$$

where g is the coupling constant of the theory. The Kähler potential $\Phi^\dagger\Phi$ is not invariant with respect to transformations (2.37), (2.38), so we replace it by $\Phi^\dagger e^{2gV}\Phi$.

In the abelian case there is another gauge invariant term, called a Fayet-Iliopoulos term,

$$2\xi g V|_{\theta\theta\bar\theta\bar\theta} = \xi g D, \quad (2.39)$$

with some real constant ξ. The superpotential and the field strength terms remain invariant, so the total $U(1)$ gauge theory Lagrangian will be

$$\mathcal{L} = \left(\Phi_i^\dagger e^{2gq_iV}\Phi_i + 2\xi g V\right)\Big|_{\theta\theta\bar\theta\bar\theta} + \left(\mathbb{W} + \frac{1}{4}\mathcal{F}^\alpha\mathcal{F}_\alpha\right)\Big|_{\theta\theta} + \left(\mathbb{W}^+ + \frac{1}{4}\bar{\mathcal{F}}_{\dot\alpha}\bar{\mathcal{F}}^{\dot\alpha}\right)\Big|_{\bar\theta\bar\theta} , \quad (2.40)$$

for arbitrary number of chiral superfields, where q_i is the $U(1)$ charge of the corresponding Φ_i. Rewriting it in terms of the component fields, we have

$$\begin{aligned}\mathcal{L} = {} & (D^\mu\phi_i)^\dagger D_\mu\phi_i + \bar\psi_i i\bar\sigma^\mu D_\mu\psi_i + F_i^\dagger F_i - W_iF_i - F_i^\dagger W_i^\dagger - \frac{1}{2}W_{ij}\psi_i\psi_j - \frac{1}{2}(W_{ij}\psi_i\psi_j)^\dagger \\ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\lambda i\bar\sigma^\mu\partial_\mu\lambda + \frac{1}{2}D^2 \\ & +gq_iD\phi_i^\dagger\phi_i + \sqrt{2}igq_i\phi_i^\dagger\lambda\psi_i - \sqrt{2}igq_i\bar\psi_i\bar\lambda\phi_i + \alpha g D, \end{aligned} \quad (2.41)$$

where $D_\mu = \partial_\mu + igA_\mu$ is the ordinary gauge covariant derivative, and $W_i \equiv \frac{\partial W}{\partial\phi_i}$; $W_{ij} \equiv \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}$ are as in (2.13). The first line in (2.41) coincides with the Wess-Zumino Lagrangian up to covariant derivative, the second line is the free vector supermultiplet Lagrangian, and the last line is due to supersymmetry.

2.1.4 Supersymmetric non-abelian gauge theory

In non-abelian gauge theories $V = V^a T^a$ and $\Lambda = \Lambda^a T^a$ are matrices, and the generators T^a belong to Lie algebra

$$\left[T^a, T^b\right] = i f_{abc} T^c. \tag{2.42}$$

Because V and Λ do not commute, the Kähler potential $(\Phi_i^\dagger e^{2gq_iV}\Phi_i)$ ceases to be gauge invariant. But this time, instead of changing it, we adjust gauge transformation of the vector superfield:

$$e^{2gV} \to e^{i\Lambda^\dagger} e^{2gV} e^{-i\Lambda}, \tag{2.43}$$

which is, in fact, the generalization of (2.38) to non-abelian cases. To make everything consistent, we also have to generalize the superfield strength $\mathcal{F}_\alpha$ to

$$\mathcal{F}_\alpha = \mathcal{F}_\alpha^a T^a = -\frac{1}{4}\bar{D}\bar{D}e^{-2gV} D_\alpha e^{2gV}. \tag{2.44}$$

The total Lagrangian of a non-abelian gauge theory is then

$$\mathcal{L} = \Phi_i^\dagger e^{2gV_{ij}}\Phi_j|_{\theta\theta\bar{\theta}\bar{\theta}} + \left(\mathbb{W} + \frac{1}{16g^2}Tr(\mathcal{F}^\alpha\mathcal{F}_\alpha)\right)\bigg|_{\theta\theta} + \left(\mathbb{W}^+ + \frac{1}{16g^2}Tr(\bar{\mathcal{F}}_{\dot{\alpha}}\bar{\mathcal{F}}^{\dot{\alpha}})\right)\bigg|_{\bar{\theta}\bar{\theta}}, \tag{2.45}$$

with no Fayet-Iliopoulos term, since it is not gauge invariant in non-abelian case. Rewriting (2.45) explicitly, we have

$$\begin{aligned}\mathcal{L} = (D^\mu\phi_i)^\dagger D_\mu\phi_i + \bar{\psi}_i i\bar{\sigma}^\mu D_\mu\psi_i + F_i^\dagger F_i - W_i F_i - F_i^\dagger W_i^\dagger - \frac{1}{2}W_{ij}\psi_i\psi_j - \frac{1}{2}(W_{ij}\psi_i\psi_j)^\dagger \\ -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \bar{\lambda}^a i\bar{\sigma}^\mu D_\mu\lambda^a + \frac{1}{2}D^a D^a \\ +gD^a\phi_i^\dagger T_{ij}^a\phi_j + \sqrt{2}ig\phi_i^\dagger\lambda^a T_{ij}^a\psi_j - \sqrt{2}ig\bar{\psi}_i\bar{\lambda}^a T_{ij}^a\phi_j.\end{aligned} \tag{2.46}$$

2.2 Supergravity

Up to this point we were considering rigid, or global, supersymmetry where the algebra (2.17) is the same at every point in space-time. This is legitimate when curvature of space-time is very small, or, in other words, gravitational effects can be neglected. When gravity comes into play, the translation generator becomes space-time dependent, $P_\mu = P_\mu(x)$, and space-time geometry becomes dynamical. Thus supergravity = local (or gauged) supersymmetry.

2.2.1 Superfields in curved superspace

There are two main approaches in formulating a supergravity theory. One of them uses superconformal calculus to build a Lagrangian with redundant degrees of freedom, and then fix them to obtain Poincare supergravity. Here we are going to use the second (equivalent) approach - the superspace formalism, just as we did in the case of rigid SUSY. However, superspace formulation of supergravity is more involved since Grassmann coordinates effectively become functions of space-time, $\theta = \theta(x)$, $\bar{\theta} = \bar{\theta}(x)$. The full treatment can be found in [19], here we will merely review the main results.

In analogy with how Einstein's gravity is obtained from space-time differential forms, Poincare supergravity can be obtained from differential forms in superspace. Derived in such a way components of superspace vielbein, torsion, and curvature have both tensor and spinor indices. E.g. components of the superspace vielbein are E_M^A, with $M, N = (m, \mu, \dot{\mu})$ and $A, B = (a, \alpha, \dot{\alpha})$. Here $m, n, ..$ are space-time Einstein indices, while $\mu, \dot{\mu}, ..$ are spinorial Einstein indices. $a, b, ..$ and $\alpha, \dot{\alpha}, ..$ are space-time and spinorial Lorentz indices, respectively.

The independent components of the super-vielbein, -torsion, and -curvature are determined by superspace Bianchi identities. These components form the off-shell supergravity multiplet. Physical components are the space-time vielbein, e_m^a, and the mediator of SUGRA interactions - the gravitino, ψ_m^α. The so-called old minimal set of auxiliary fields consists of the real vector b_m, and the complex scalar we call M. For the super-vielbein at $\theta = \bar{\theta} = 0$ (we denote $|_{\theta=\bar{\theta}=0} \equiv |$) we have

$$E_M^A(x, \theta, \bar{\theta})| = \begin{pmatrix} e_m^a(x) & \frac{1}{2}\psi_m^\alpha(x) & \frac{1}{2}\bar{\psi}_{m\dot{\alpha}}(x) \\ 0 & \delta_\mu^\alpha & 0 \\ 0 & 0 & \delta_{\dot{\alpha}}^{\dot{\mu}} \end{pmatrix} , \tag{2.47}$$

$$E_A^M(x, \theta, \bar{\theta})| = \begin{pmatrix} e_a^m(x) & -\frac{1}{2}\psi_a^\mu(x) & -\frac{1}{2}\bar{\psi}_{a\dot{\mu}}(x) \\ 0 & \delta_\alpha^\mu & 0 \\ 0 & 0 & \delta_{\dot{\mu}}^{\dot{\alpha}} \end{pmatrix} . \tag{2.48}$$

In supergravity it is convenient to define component fields of superfields by applying SUGRA analogs of the differential operators D_α and $\bar{D}^{\dot{\alpha}}$ (plus an additional operator with space-time index a), defined as

$$\mathcal{D}_\alpha = E_\alpha^m \mathcal{D}_m + E_\alpha^\mu \mathcal{D}_\mu + E_{\alpha\dot{\mu}} \bar{\mathcal{D}}^{\dot{\mu}} , \tag{2.49}$$

$$\bar{\mathcal{D}}^{\dot{\alpha}} = E^{\dot{\alpha}m} \mathcal{D}_m + E^{\dot{\alpha}\mu} \mathcal{D}_\mu + E_{\dot{\mu}}^{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\mu}} , \tag{2.50}$$

$$\mathcal{D}_a = E_a^m \mathcal{D}_m + E_a^\mu \mathcal{D}_\mu + E_{a\dot{\mu}} \bar{\mathcal{D}}^{\dot{\mu}} . \tag{2.51}$$

A chiral superfield Φ is then defined as

$$\bar{\mathcal{D}}_{\dot{\alpha}} \Phi = 0 , \tag{2.52}$$

and its components are

$$\begin{aligned}
&\Phi| = A \ , \\
&\mathcal{D}_\alpha \Phi| = \sqrt{2}\xi_\alpha \ , \\
&\mathcal{D}_\alpha \mathcal{D}_\beta \Phi| = -2\varepsilon_{\alpha\beta} F \ , \quad \bar{\mathcal{D}}_{\dot\alpha} \mathcal{D}_\alpha \Phi| = -2i\sigma_{\alpha\dot\alpha}{}^m (\partial_m A - \frac{1}{\sqrt{2}}\psi^\mu_m \xi_\mu) \ , \\
&\mathcal{D}_\alpha \mathcal{D}^2 \Phi| = -\frac{4}{3}\sqrt{2}\xi_\alpha \bar{M} \ , \\
&\mathcal{D}^2 \bar{\mathcal{D}}^2 \Phi| = 16 \mathcal{D}_m \hat{D}^m A - \frac{32}{3} i b^m \hat{D}_m A - 8\sqrt{2}\psi_m \hat{D}^m \xi + \frac{32}{3}\bar{M}\bar{F} + \frac{8}{3}\sqrt{2}\psi_{mn}\sigma^{mn}\xi \\
&\qquad -\frac{8}{3}\sqrt{2} i \psi_m \xi b^m + \frac{4}{3}\sqrt{2} i \psi_m \sigma^m \bar\sigma^n \xi b_n \ .
\end{aligned} \tag{2.53}$$

where A and ξ_α are physical complex scalar and spinor fields, while F is an auxiliary complex scalar. We define $\mathcal{D}^2 \equiv \mathcal{D}^\alpha \mathcal{D}_\alpha$, and $\bar{\mathcal{D}}^2 \equiv \bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}$ (see appendix). $\hat{D}_m A \equiv \partial_m A - \psi^\mu_m \xi_\mu/\sqrt{2}$ and $\hat{D}_m \xi_\alpha \equiv \mathcal{D}_m \xi_\alpha - \psi_{m\alpha} F/\sqrt{2} - i\bar\psi^{\dot\beta}_m \sigma^m_{\alpha\dot\beta}\hat{D}_m A$. Similarly, the vector superfield ($V = \bar{V}$) components are

$$\begin{aligned}
&V| = C \ , \\
&\mathcal{D}_\alpha V| = i\chi_\alpha \ , \quad \bar{\mathcal{D}}_{\dot\alpha} V| = -i\bar\chi_{\dot\alpha} \ , \\
&\mathcal{D}^2 V| = 2X \ , \quad \bar{\mathcal{D}}^2 V| = 2\bar{X} \ , \\
&\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\alpha V| = \sigma_{\alpha\dot\alpha}{}^m (B_m - i\partial_m C) \ , \\
&W_\alpha| \equiv -\frac{1}{4}(\bar{\mathcal{D}}^2 - 8\mathcal{R})\mathcal{D}^\beta V| = -i\lambda_\alpha \ , \quad \bar{W}_{\dot\alpha}| = i\bar\lambda_{\dot\alpha} \ , \\
&\mathcal{D}_\alpha W^\beta| = \frac{1}{2}\sigma^m_{\alpha\dot\alpha}\bar\sigma^{\dot\alpha\beta n}(\mathcal{D}_m\mathcal{D}_n C + iF_{mn}) + \delta^\beta_\alpha (D + \frac{1}{2}\Box C) \ ,
\end{aligned} \tag{2.54}$$

where $\Box \equiv \mathcal{D}_m \mathcal{D}^m$. As we saw earlier (for global SUSY), in the case of a massless vector multiplet, the real scalars C and D, spinor χ_α, and complex scalar X [3] are auxiliary (non-propagating) fields, whereas the spinor λ_α and vector B_m, whose field strength is

$$F_{mn} \equiv \partial_m B_n - \partial_n B_m - [B_m, B_n] \ ,$$

are the dynamical components. The auxiliary fields, except D, can be gauged away by choosing WZ gauge,

$$V| = \mathcal{D}_\alpha V| = \bar{\mathcal{D}}_{\dot\alpha} V| = \mathcal{D}_\alpha \mathcal{D}_\beta V| = \bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}_{\dot\beta} V| = 0 \ . \tag{2.55}$$

In the massive case we cannot use WZ gauge, and C and χ_α become physical, acquiring kinetic terms.

Expressions in (2.53) and (2.54) are not all independent, of course. Some of them are related by the commutation relations for the $\mathcal{D}$-operators,

$$\left[\mathcal{D}_A \mathcal{D}_B - (-1)^{ab}\mathcal{D}_B \mathcal{D}_A\right] V^C = (-1)^{d(a+b)} V^D R_{ABD}{}^C - T_{AB}{}^D \mathcal{D}_D V^C \ , \tag{2.56}$$

[3] Comparing this to (2.31), it can be seen that X and $\bar{X}$ here correspond to $N - iM$ and $N + iM$ in (2.31) where M, N are real

where a, b, c take the value one, whenever A, B, C are spinor indices, and zero, whenever A, B, C are tensor (vector) indices. The components of the curvature R and the torsion T can be found in [19]. One can also use (2.56) to compute higher-order terms of the superfields Φ and V.

2.2.2 Chiral theory

The Lagrangian for the pure supergravity reads

$$\mathcal{L}_{SG} = -6 \int d^2\Theta \mathcal{E} \mathcal{R} + \text{h.c.} \, , \tag{2.57}$$

where Θ is the redefined spinor variable having a Lorentz index (the old θ and $\bar{\theta}$ had Einstein indices). Thus, we can write

$$\int d^2\Theta = -\frac{1}{4}\mathcal{D}^2 \, . \tag{2.58}$$

Component-wise, we find

$$e^{-1}\mathcal{L}_{SG} = -\frac{1}{2}R - \frac{1}{3}\bar{M}M + \frac{1}{3}b_m b^m + \frac{1}{2}\varepsilon^{klmn}(\bar{\psi}_k\bar{\sigma}_l\hat{\mathcal{D}}_m\psi_n - \psi_k\sigma_l\hat{\mathcal{D}}_m\bar{\psi}_n) \, , \tag{2.59}$$

where spinor indices are supressed, and $\hat{\mathcal{D}}_m\psi_n \equiv \partial_m\psi_n^\alpha + \psi_n^\beta\omega_{n\beta}{}^\alpha$. Equations of motion eliminate b_m- and M-terms, leaving simply the Einstein-Hilbert plus Rarita-Schwinger action.

We add chiral superfields by introducing a real function, $K(\Phi_i, \overline{\Phi}_i)$, called a Kähler potential; and a holomorphic superpotential $\mathcal{W}(\Phi_i)$,

$$\mathcal{L} = \int d^2\Theta 2\mathcal{E}\left[\frac{3}{8}(\bar{\mathcal{D}}^2 - 8\mathcal{R})e^{-K/3} + \mathcal{W}\right] + \text{h.c.} \, . \tag{2.60}$$

The Kähler potential gives rise to the metric (here at the lowest order in $\theta, \bar{\theta}$) $K_{ij^*} \equiv \frac{\partial^2}{\partial A^i \partial \bar{A}^j}K$ on a so-called Kähler manifold, with complex coordinates $A^i, \bar{A}^i$. This metric is invariant with respect to Kähler transformations,

$$K(\Phi^i, \bar{\Phi}^j) \to K(\Phi^i, \bar{\Phi}^j) + f(\Phi^i) + \bar{f}(\bar{\Phi}^j) \, , \tag{2.61}$$

where f and $\bar{f}$ are holomorphic and antiholomorphic functions of the chiral superfields. Lagrangian (2.60) is invariant under a Kähler transformation, if it is accompanied by the Weyl transformation of spinors,

$$\xi^i \to \exp\left(\frac{i}{2}\text{Im}f\right)\xi^i \, , \quad \psi_m \to \exp\left(-\frac{i}{2}\text{Im}f\right)\psi_m \, , \tag{2.62}$$

and the transformation of the superpotential,

$$\mathcal{W} \to e^{-f}\mathcal{W} \, . \tag{2.63}$$

The invariance of the Lagrangian under these transformations implies the invariance under rigid isometries of the Kähler manifold. So, when dealing with non-trivial Kähler manifolds, the Kähler-Weyl invariance is required in order the Lagrangian be well defined globally.

The component expansion of (2.60) is in the Jordan frame, which means that the scalar curvature is multiplied by a function of scalar fields, specifically $\exp(-K/3)$ at $\theta = \bar{\theta} = 0$. The transition to the Einstein frame is achieved by the Weyl rescaling

$$e_m^a \to e^{K/6} e_m^a \ , \tag{2.64}$$

which leads to the transformation of the scalar curvature as

$$e e^{-K/3} R \to eR - \frac{1}{6} e \partial_m K \partial^m K \ . \tag{2.65}$$

Weyl rescaling is followed by the spinor redefinitions for canonical normalization,

$$\xi_i \to e^{-K/12} \xi_i \ , \quad \psi_m \to e^{K/12} \psi_m \ , \tag{2.66}$$

and a shift of the gravitino,

$$\psi_m \to \psi_m + \frac{i\sqrt{2}}{6} \sigma_m \bar{\xi}_i K_{i^*} \ . \tag{2.67}$$

Finally, after eliminating auxiliary fields and Weyl-rescaling, the Lagrangian (2.60) leads to

$$\begin{aligned}
e^{-1}\mathcal{L} = & -\frac{1}{2}R - K_{ij^*}\partial_m A_i \partial^m \bar{A}_j - iK_{ij^*}\bar{\xi}^j \bar{\sigma}^m \mathcal{D}_m \xi^i + \varepsilon^{klmn}\bar{\psi}_k \bar{\sigma}_l \tilde{\mathcal{D}}_m \psi_n \\
& -\frac{1}{\sqrt{2}} K_{ij^*}\partial_n \bar{A}^j \xi^i \sigma^m \bar{\sigma}^n \psi_m - \frac{1}{\sqrt{2}} K_{ij^*}\partial_n A^i \bar{\xi}^j \bar{\sigma}^m \sigma^n \bar{\psi}_m \\
& + \frac{1}{4} K_{ij^*}(i\varepsilon^{klmn}\psi_k \sigma_l \bar{\psi}_m + \psi_m \sigma^n \bar{\psi}^m)\xi^i \sigma_n \bar{\xi}^j - \frac{1}{8}(K_{ij^*}K_{kl^*} - 2R_{ij^*kl^*})\xi^i \xi^k \bar{\xi}^j \bar{\xi}^l \\
& - e^{K/2}\Big\{ \overline{W}\psi_a \sigma^{ab}\psi_b + W\bar{\psi}_a \bar{\sigma}^{ab}\bar{\psi}_b + \frac{i}{\sqrt{2}} D_i W \xi_i \sigma^a \bar{\psi}_a + \frac{i}{\sqrt{2}} D_{i^*}\overline{W}\bar{\xi}_i \bar{\sigma}^a \psi_a \\
& + \frac{1}{2}[W_{ij} + K_{ij}W + K_i D_j W + K_j D_i W - K_i K_j W - K^{kl^*}K_{ijl^*}D_k W]\xi_i \xi_j \\
& + \frac{1}{2}[\overline{W}_{ij} + K_{i^*j^*}\overline{W} + K_{i^*}D_{j^*}\overline{W} + K_{j^*}D_{i^*}\overline{W} - K_{i^*}K_{j^*}\overline{W} - K^{k^*l}K_{i^*j^*l}D_{k^*}\overline{W}]\bar{\xi}_i \bar{\xi}_j \Big\} \\
& - e^K (K^{ij^*}D_i W D_{j^*}\overline{W} - 3|W|^2) \ ,
\end{aligned} \tag{2.68}$$

where $D_i W \equiv W_i + K_i W$, $K^{ij^*} \equiv K^{-1}_{ij^*}$, and

$$\mathcal{D}_m \xi^i \equiv \partial_m \xi^i + \xi^i \omega_m + \Gamma^i_{jk}\partial_m A^j \xi^k - \frac{1}{4}(K_j \partial_m A^j - K_{j^*}\partial_m \bar{A}^j)\xi^i \ , \tag{2.69}$$

$$\tilde{\mathcal{D}}_m \psi_n \equiv \partial_m \psi_n + \psi_n \omega_m + \frac{1}{4}(K_j \partial_m A^j - K_{j^*}\partial_m \bar{A}^j)\psi_n \ , \tag{2.70}$$

with the space-time spin connection ω_m, the Christoffel symbols Γ^i_{jk}, and curvature tensor $R_{ij^*kl^*}$ of the Kähler manifold. The last line of (2.68) is the scalar potential,

$$V = e^K (K^{ij^*} D_i W D_{j^*} \overline{W} - 3|W|^2) \ . \tag{2.71}$$

Because of the Kähler invariance, only the invariant function

$$G \equiv K + \log W + \log \overline{W} \tag{2.72}$$

is relevant, in terms of which the scalar potential has the form

$$V = e^G (K^{ij^*} G_i G_{j^*} - 3) \ . \tag{2.73}$$

2.2.3 Gauge theory

A supersymmetric gauge theory is constructed by gauging holomorphic isometries on a given Kähler manifold. They are generated by Killing vectors,

$$Y^{\bar{a}} = Y^{i\bar{a}} \frac{\partial}{\partial A^i} \ , \quad \bar{Y}^{\bar{a}} = \bar{Y}^{i\bar{a}} \frac{\partial}{\partial \bar{A}^i} \ , \tag{2.74}$$

which can be defined through Killing potentials $D^{\bar{a}}$ - real scalar functions, as

$$K_{ij^*} Y^{i\bar{a}} = -i \frac{\partial D^{\bar{a}}}{\partial \bar{A}^j} \ , \quad K_{ij^*} \bar{Y}^{j\bar{a}} = i \frac{\partial D^{\bar{a}}}{\partial A^i} \ . \tag{2.75}$$

Here the "barred" indices $\bar{a}, \bar{b}, ..$ represent the isometry (gauge) group G.

Gauging the isometries of a Kähler manifold leads to an introduction of the gauge superfield in order to keep the Lagrangian invariant. Specifically, we add a counterterm $\Gamma(\Phi_i, \bar{\Phi}_i, V)$ which compensates the transformation of K, and the superfield strength term representing the vector (gauge) superfield V as

$$\mathcal{L} = \int d^2\Theta 2\mathcal{E} \left[\frac{3}{8} (\mathcal{D}^2 - 8\mathcal{R}) e^{-\frac{1}{3}(K+\Gamma)} + \frac{1}{16g^2} H_{\bar{a}\bar{b}} (W^\alpha)^{\bar{a}} (W_\alpha)^{\bar{b}} + \mathcal{W} \right] + \text{h.c.} \tag{2.76}$$

Here $H_{\bar{a}\bar{b}}$ is a gauge kinetic function, for which we use the simplest choice $H_{\bar{a}\bar{b}} = \delta_{\bar{a}\bar{b}}$.

The component expansion of the Lagrangian (2.76) in Einstein frame is

$$
\begin{aligned}
e^{-1}\mathcal{L} = & -\frac{1}{2}R - K_{ij^*}\tilde{\mathcal{D}}_m A^i \tilde{\mathcal{D}}^m \bar{A}^j - \frac{1}{2}g^2 D^{\bar{a}2} - \frac{1}{4}F^{\bar{a}}_{mn}F^{mn\bar{a}} - i\bar{\lambda}^{\bar{a}}\bar{\sigma}^m\tilde{\mathcal{D}}_m\lambda^{\bar{a}} \\
& -iK_{ij^*}\bar{\xi}^j\bar{\sigma}^m\tilde{\mathcal{D}}_m\xi^i + \varepsilon^{klmn}\bar{\psi}_k\bar{\sigma}_l\tilde{\mathcal{D}}_m\psi_n + \sqrt{2}gK_{ij^*}(\bar{Y}^{j\bar{a}}\xi^i\lambda^{\bar{a}} + Y^{i\bar{a}}\bar{\xi}^j\bar{\lambda}^{\bar{a}}) \\
& -\frac{1}{2}gD^{\bar{a}}(\psi_m\sigma^m\bar{\lambda}^{\bar{a}} - \bar{\psi}_m\bar{\sigma}^m\lambda^{\bar{a}}) - \frac{1}{\sqrt{2}}K_{ij^*}(\tilde{\mathcal{D}}_n\bar{A}^j\xi^i\sigma^m\bar{\sigma}^n\psi_m + \tilde{\mathcal{D}}_nA^i\bar{\xi}^j\bar{\sigma}^m\sigma^n\bar{\psi}_m) \\
& +\frac{i}{4}(\psi_m\sigma^{ab}\sigma^m\bar{\lambda}^{\bar{a}} + \bar{\psi}_m\bar{\sigma}^{ab}\bar{\sigma}^m\lambda^{\bar{a}})(F^{\bar{a}}_{ab} + \tilde{F}^{\bar{a}}_{ab}) + \frac{1}{4}K_{ij^*}(i\varepsilon^{klmn}\psi_k\sigma_l\bar{\psi}_m + \psi_m\sigma^n\bar{\psi}^m)\xi^i\sigma_n\bar{\xi}^j \\
& -\frac{1}{8}(K_{ij^*}K_{kl^*} - 2R_{ij^*kl^*})\xi^i\xi^k\bar{\xi}^j\bar{\xi}^l + \frac{1}{8}K_{ij^*}\bar{\xi}^j\bar{\sigma}^m\xi^i\bar{\lambda}^{\bar{a}}\bar{\sigma}_m\lambda^{\bar{a}} - \frac{3}{16}\lambda^{\bar{a}}\sigma^m\bar{\lambda}^{\bar{a}}\lambda^{\bar{b}}\sigma_m\bar{\lambda}^{\bar{b}} \\
& -e^{K/2}\Big\{\overline{W}\psi_a\sigma^{ab}\psi_b + W\bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b + \frac{i}{\sqrt{2}}D_iW\xi_i\sigma^a\bar{\psi}_a + \frac{i}{\sqrt{2}}D_{i^*}\overline{W}\bar{\xi}_i\bar{\sigma}^a\psi_a \\
& +\frac{1}{2}[W_{ij} + K_{ij}W + K_iD_jW + K_jD_iW - K_iK_jW - K^{kl^*}K_{ijl^*}D_kW]\xi_i\xi_j \\
& +\frac{1}{2}[\overline{W}_{ij} + K_{i^*j^*}\overline{W} + K_{i^*}D_{j^*}\overline{W} + K_{j^*}D_{i^*}\overline{W} - K_{i^*}K_{j^*}\overline{W} - K^{k^*l}K_{i^*j^*l}D_{k^*}\overline{W}]\bar{\xi}_i\bar{\xi}_j\Big\} \\
& -e^K(K^{ij^*}D_iWD_{j^*}\overline{W} - 3|W|^2)\ ,
\end{aligned}
\tag{2.77}
$$

where the field-dependent derivatives are

$$
\tilde{\mathcal{D}}_mA^i \equiv \partial_mA^i - gB^{\bar{a}}_mY^{i\bar{a}}\ , \tag{2.78}
$$

$$
\begin{aligned}
\tilde{\mathcal{D}}_m\xi^i \equiv & \ \partial_m\xi^i + \xi^i\omega_m + \Gamma^i_{jk}\tilde{\mathcal{D}}_mA^j\xi^k - \frac{1}{4}(K_j\tilde{\mathcal{D}}_mA^j - K_{j^*}\tilde{\mathcal{D}}_m\bar{A}^j)\xi^i \\
& -gB^{\bar{a}}_m\frac{\partial Y^{i\bar{a}}}{\partial A^j}\xi^j - \frac{i}{2}gB^{\bar{a}}_m\mathrm{Im}f^{\bar{a}}\xi^i\ ,
\end{aligned}
\tag{2.79}
$$

$$
\tilde{\mathcal{D}}_m\psi_n \equiv \partial_m\psi_n + \psi_n\omega_m + \frac{1}{4}(K_j\tilde{\mathcal{D}}_mA^j - K_{j^*}\tilde{\mathcal{D}}_m\bar{A}^j)\psi_n + \frac{i}{2}gB^{\bar{a}}_m\mathrm{Im}f^{\bar{a}}\psi_n\ , \tag{2.80}
$$

$$
\tilde{\mathcal{D}}_m\lambda^{\bar{a}} \equiv \partial_m\lambda^{\bar{a}} + \lambda^{\bar{a}}\omega_m + \frac{1}{4}(K_j\tilde{\mathcal{D}}_mA^j - K_{j^*}\tilde{\mathcal{D}}_m\bar{A}^j)\lambda^{\bar{a}} - gf^{abc}B^{\bar{b}}_m\lambda^{\bar{c}} + \frac{i}{2}gB^{\bar{b}}_m\mathrm{Im}f^{\bar{b}}\lambda^{\bar{a}}\ . \tag{2.81}
$$

f^{abc} are structure constants of the gauge group G, and $f^{\bar{a}} = f^{\bar{a}}(\Phi_i)$ are the parameters of Kähler transformations due to local isometries.

The full scalar potential in this case is a sum of the so-called F-term potential (the potential of chiral models),

$$
V_F = e^K(K^{ij^*}D_iWD_{j^*}\overline{W} - 3|W|^2)\ , \tag{2.82}
$$

and the D-term potential,

$$
V_D = \frac{1}{2}g^2D^{\bar{a}2}\ . \tag{2.83}
$$

2.3 Minimal Supersymmetric Standard Model

In this section we review the Minimal Supersymmetric Standard Model, or MSSM, which is obtained by applying $N = 1$ rigid supersymmetry to the Standard Model in a minimalistic way.

Chiral superfield	Scalar component	Spinor component	$SU(3)_C \times SU(2)_L \times U(1)_Y$ multiplicity
Q	$(\tilde{u}_{\rm L}, \tilde{d}_{\rm L})$	$(u_{\rm L}, d_{\rm L})$	$(\mathbf{3}, \mathbf{2}, \frac{1}{3})$
$\bar{u}$	$\tilde{\bar{u}}_{\rm L} \equiv \tilde{u}^\dagger_{\rm R}$	$\bar{u}_{\rm L} \equiv (u_{\rm R})^c$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{4}{3})$
$\bar{d}$	$\tilde{\bar{d}}_{\rm L} \equiv \tilde{d}^\dagger_{\rm R}$	$\bar{d}_{\rm L} \equiv (d_{\rm R})^c$	$(\bar{\mathbf{3}}, \mathbf{2}, \frac{2}{3})$
L	$(\tilde{\nu}_{\rm eL}, \tilde{e}_{\rm L})$	$(\nu_{\rm eL}, e_{\rm L})$	$(\mathbf{1}, \mathbf{2}, -1)$
$\bar{e}$	$\tilde{\bar{e}}_{\rm L} \equiv \tilde{e}^\dagger_{\rm R}$	$\bar{e}_{\rm L} \equiv (e_{\rm R})^c$	$(\mathbf{1}, \mathbf{1}, 2)$
H_u	(H_u^+, H_u^0)	$(\tilde{H}_u^+, \tilde{H}_u^0)$	$(\mathbf{1}, \mathbf{2}, 1)$
H_d	(H_d^0, H_d^-)	$(\tilde{H}_d^0, \tilde{H}_d^-)$	$(\mathbf{1}, \mathbf{2}, -1)$

TABLE 2.1: MSSM chiral superfields

Vector superfield	Spinor component	Vector component	$SU(3)_C \times SU(2)_L \times U(1)_Y$ multiplicity
$V_{\rm G}$	$\tilde{g}$	g	$(\mathbf{8}, \mathbf{1}, 0)$
$V_{\rm W}$	$\widetilde{W}^\pm$, $\widetilde{W}^0$	$W^\pm$, W^0	$(\mathbf{1}, \mathbf{3}, 0)$
$V_{\rm B}$	$\widetilde{B}$	B	$(\mathbf{1}, \mathbf{1}, 0)$

TABLE 2.2: MSSM vector (gauge) superfields

In the MSSM each Standard Model field is placed into a separate superfield, which roughly doubles the number of particles. The reason SM fermions and bosons cannot be partnered with each other is that superpartner fields must carry the same SM-charge and transform in the same representation of a given gauge group. Since this is not the case, and we have yet to observe superpartners of the SM particles, we conclude that SUSY is broken at higher energy scale (well beyond 100 GeV).

The chiral superfields of the MSSM are presented in the table 2.1. The bar notation of the component fields merely denotes anti-particle fields (not Dirac conjugation). Vector superfields and their components are presented in the table 2.2.

Different supersymmetric models are characterised by their superpotentials. The MSSM superpotential is

$$\mathbb{W} = Y_u^{ij} \bar{u}_i Q_j H_u + Y_d^{ij} \bar{d}_i Q_j H_d + Y_e^{ij} \bar{e}_i L_j H_d + \mu H_u H_d \ . \tag{2.84}$$

It could be further extended by renormalisable terms that violate baryon (B) and lepton (L) number conservation (they are not considered fundamental symmetries, and are violated non-perturbatively). However, the B and L violating terms in the superpotential would lead to the processes such as proton decay, that are not seen experimentally. To forbid such unwanted terms in the superpotential, we accommodate the new type of fundamental symmetries, called *R-parity*. It imposes the conservation of the following quantity:

$$R = (-1)^{3B+L+2s}, \tag{2.85}$$

where B and L are the baryon and lepton numbers, and s is the spin of the particle. Then, the SM particles have $R = +1$, while for their superpartners $R = -1$.

2.3.1 "Soft" SUSY breaking terms

There is no "standard" mechanism for spontaneous breaking of SUSY - different high-energy theories provide different answers. But regardless of that, we can always do phenomenology, that is, pick out all the possible explicit breaking terms to build the low-energy model. The reasonable requirement for these terms would be not to re-introduce divergences to the quantum theory. Such terms must have positive mass dimension, and are called "soft" SUSY breaking terms.

They include, but not limited to, gaugino masses,

$$M_1\tilde{B}\tilde{B} + M_2\tilde{W}^a\tilde{W}^a + M_3\tilde{g}^a\tilde{g}^a + h.c., \tag{2.86}$$

squark and slepton masses (3×3 matrices),

$$\tilde{Q}^\dagger m^2_{\tilde{q}}\tilde{Q} + \bar{\tilde{u}}^\dagger_{\mathrm{L}} m^2_{\tilde{u}}\bar{\tilde{u}}_{\mathrm{L}} + \bar{\tilde{d}}^\dagger_{\mathrm{L}} m^2_{\tilde{d}}\bar{\tilde{d}}_{\mathrm{L}} + \bar{\tilde{L}}^\dagger m^2_{\tilde{L}}\bar{\tilde{L}} + \bar{\tilde{e}}^\dagger_{\mathrm{L}} m^2_{\tilde{e}}\bar{\tilde{e}}_{\mathrm{L}}, \tag{2.87}$$

and Higgs scalar masses,

$$(m_{hu})^2 H^\dagger_u H_u + (m_{hd})^2 H^\dagger_d H_d + bH_uH_d + h.c. \tag{2.88}$$

What we have learned from this is that the masses of sparticles break SUSY explicitly, thus, should be generated by spontaneous SUSY breaking (except the higgsino masses which would break EW symmetry). Spontaneous breaking of the EW symmetry then gives masses to the ordinary SM particles.

The "soft" SUSY breaking terms introduce a number of new parameters that together with the existing ones reach $\gtrsim 100$. However, in the context of the high-energy unification models the number of the parameters is often vastly reduced.

2.3.2 Spontaneous electroweak symmetry breaking in MSSM

In the SM to spontaneously break EW symmetry, the Higgs potential

$$V = -\tilde{\mu}^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2 \tag{2.89}$$

is introduced, with the requirement $\tilde{\mu}^2 > 0$, $\lambda > 0$ (to avoid confusion, we mark the SM parameter μ with tilde). In the MSSM, the minus sign in front of μ^2 would break SUSY-invariance, hence, we cannot accommodate the standard Higgs potential and preserve SUSY at the same time. In order to spontaneously break the EW symmetry, one has to introduce explicit SUSY breaking terms (choosing from the "soft" terms).

The consistent MSSM Higgs potential is

$$\begin{aligned}\mathcal{V} &= (|\mu|^2 + m_{hu}^2)(|H_u^+|^2 + |H_u^0|^2) + (|\mu|^2 + m_{hd}^2)(|H_d^0|^2 + |H_d^-|^2)\\ &+b(H_u^+ H_d^- - H_u^0 H_d^0) + b(H_u^+ H_d^- - H_u^0 H_d^0)^\dagger\\ &+\frac{g^2 + g'^2}{8}(|H_u^+|^2 + |H_u^0|^2 - |H_d^0|^2 - |H_d^-|^2)^2 + \frac{g^2}{2}|H_u^+(H_d^0)^\dagger + H_u^0(H_d^-)^\dagger|^2.\end{aligned} \tag{2.90}$$

The parameter of the SUSY-invariant term, $|\mu|^2$ is positive, while the Higgs masses m_{hu}^2 and m_{hd}^2 can bear either sign, but it is required that either $(|\mu|^2 + m_{hu}^2)$ or $(|\mu|^2 + m_{hd}^2)$ is negative for the SSB to occur. The interesting point here is that the quartic self-coupling parameter of the Higgs scalars is the combination of electroweak couplings, namely, $\frac{1}{8}(g^2 + g'^2)$. In contrast, in the SM it is a free parameter (λ of (2.89)) [4].

Considering SSB, the charged components $H_{u,d}^+$ and $H_{u,d}^-$ cannot develop a VEV, since it would break electromagnetic $U(1)_{em}$ symmetry. That leaves us with $|H_u^0| = \upsilon_u$ and $|H_d^0| = \upsilon_d$ to define the Higgs vacuum. The VEVs combine to give masses to $W^\pm$ and Z gauge bosons:

$$\begin{aligned}m_Z^2 &= \frac{1}{2}(g^2 + g'^2)(\upsilon_u^2 + \upsilon_d^2),\\ m_W^2 &= \frac{1}{2}g^2(\upsilon_u^2 + \upsilon_d^2).\end{aligned} \tag{2.91}$$

2.3.3 Higgs mixing

In the Standard Model there are 4 real degrees of freedom of the Higgs field, 3 of which get "eaten" by $W^\pm$ and Z bosons, leaving one to become a massive excitation of the field - the Higgs boson. In the MSSM, Higgs scalars have $2 \times 4 = 8$ real degrees of freedom in total. 3 of them get absorbed into $W^\pm$ and Z as before, but the remaining degrees of freedom now count 5, which means there are 5 Higgs bosons. After the diagonalization of the mass matrices, the following mass eigenstates are extracted: first, the massless states, absorbed into the longitudinal modes of the electroweak bosons,

$$\sqrt{2}[\sin\beta \cdot \mathrm{Im}(H_u^0) - \cos\beta \cdot \mathrm{Im}(H_d^0)] \quad \rightarrow \quad Z \tag{2.92}$$

$$\sin\beta \cdot H_u^+ - \cos\beta \cdot (H_d^-)^\dagger \quad \rightarrow \quad W^+ \tag{2.93}$$

$$\sin\beta \cdot (H_u^+)^\dagger - \cos\beta \cdot H_d^- \quad \rightarrow \quad W^- \tag{2.94}$$

[4] It should be mentioned that renormalisation of λ drives it to a negative value beyond 10^{11} GeV. This is yet another SM problem that is overcome by SUSY!

where $\tan\beta \equiv \upsilon_u/\upsilon_d$; Next, the 5 massive states are

$$\begin{aligned} A^0 &= \sqrt{2}[\cos\beta \cdot \mathrm{Im}(H_u^0) + \sin\beta \cdot \mathrm{Im}(H_d^0)] &(2.95)\\ h^0 &= \mathrm{Re}(H_u^0) - \upsilon_u &(2.96)\\ H^0 &= \mathrm{Re}(H_d^0) - \upsilon_d &(2.97)\\ H^+ &= \cos\beta \cdot H_u^+ + \sin\beta \cdot (H_d^-)^\dagger &(2.98)\\ H^- &= \cos\beta \cdot (H_u^+)^\dagger + \sin\beta \cdot H_d^- . &(2.99) \end{aligned}$$

A^0, h^0 and H^0 are electrically neutral, and H^+ and H^- are the charged Higgs bosons. While the masses of A^0, H^0 and $H^\pm$ are unconstrained, the careful treatment [20] puts the upper limit to the h^0 mass $m_{h^0} \lesssim 140$ GeV, which is fully consistent with the SM Higgs boson mass $m_H \approx 125$ GeV. However, the recent search at LHC did not reveal any sign of SUSY yet. It implies that the MSSM may have to be modified by some non-minimal terms, and the SUSY breaking scale is higher than 10 TeV.

2.3.4 Sparticle mixing

Gluinos are the only gauginos that do not mix with other sparticles to form mass eigenstates due to the colour symmetry being unbroken. Their masses are coming purely from the "soft" SUSY breaking terms mentioned earlier. The electroweak gauginos, and higgsinos, on the other hand, mix among themselves, forming mass eigenstates of the same quantum numbers.

Electrically neutral mass eigenstates - the *neutralinos* (χ^0) - form from the mixing between $\tilde{B}$, $\tilde{W}^3$, $\tilde{H}_d^0$ and $\tilde{H}_u^0$ due to the diagonalization of the mass matrix,

$$\begin{pmatrix} M_1 & 0 & -m_Z c_\beta s_W & m_Z s_\beta s_W \\ 0 & M_2 & m_Z c_\beta c_W & -m_Z s_\beta c_W \\ -m_Z c_\beta s_W & m_Z c_\beta c_W & 0 & -\mu \\ m_Z s_\beta s_W & -m_Z s_\beta c_W & -\mu & 0 \end{pmatrix} \begin{pmatrix} \tilde{B} \\ \tilde{W}^3 \\ \tilde{H}_d^0 \\ \tilde{H}_u^0 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} M_{\chi_1^0} & & & \\ & M_{\chi_2^0} & & \\ & & M_{\chi_3^0} & \\ & & & M_{\chi_4^0} \end{pmatrix} \begin{pmatrix} \chi_1^0 \\ \chi_2^0 \\ \chi_3^0 \\ \chi_4^0 \end{pmatrix} \tag{2.100}$$

where $s_\beta \equiv \sin\beta$, $c_\beta \equiv \cos\beta$, $s_W \equiv \sin\theta_W$, $c_W \equiv \cos\theta_W$.

Similarly, electrically charged mass eigenstates - the *charginos* ($\chi^\pm$) - are formed by

$$\begin{pmatrix} M_2 & \sqrt{2} m_W s\beta \\ \sqrt{2} m_W c\beta & \mu \end{pmatrix} \begin{pmatrix} \tilde{W}^\pm \\ \tilde{H}^\pm \end{pmatrix} \Rightarrow \begin{pmatrix} M_{\chi_1} & \\ & M_{\chi_2} \end{pmatrix} \begin{pmatrix} \chi_1^\pm \\ \chi_2^\pm \end{pmatrix}. \tag{2.101}$$

So that χ_i^+ and χ_i^- have equal masses, M_{χ_i}. The mass hierarchies are $M_{\chi_1^0} < M_{\chi_2^0} < M_{\chi_3^0} < M_{\chi_4^0}$ for neutralinos, and $M_{\chi_1} < M_{\chi_2}$ for charginos. The neutralino χ_1^0 is the lightest (stable) supersymmetric particle (LSP) in the MSSM and a good candidate for WIMP dark matter particle.

For squarks and sleptons, the mixing is also possible, although is believed to be very small. Mixing occurs separately between up-type squarks $(\tilde{u}_{\rm L}, \tilde{c}_{\rm L}, \tilde{t}_{\rm L}, \tilde{u}_{\rm R}, \tilde{c}_{\rm R}, \tilde{t}_{\rm R})$, down-type squarks $(\tilde{d}_{\rm L}, \tilde{s}_{\rm L}, \tilde{b}_{\rm L}, \tilde{d}_{\rm R}, \tilde{s}_{\rm R}, \tilde{b}_{\rm R})$, charged sleptons $(\tilde{e}_{\rm L}, \tilde{\mu}_{\rm L}, \tilde{\tau}_{\rm L}, \tilde{e}_{\rm R}, \tilde{\mu}_{\rm R}, \tilde{\tau}_{\rm R})$, and sneutrinos $(\tilde{\nu}_e, \tilde{\nu}_\mu, \tilde{\nu}_\tau)$.

Chapter 3

Grand Unified Theories

In this chapter, supersymmetric Grand Unified Theories (GUTs) are briefly reviewed by following the references [21, 22, 23, 24, 25, 26]

3.1 $SU(5)$ unification

As we mentioned before, in the Standard Model, the couplings do not exactly unify in the simplest $SU(5)$ scenario. But this is "cured" in the MSSM, where they exactly unify at $M_U \sim 10^{16}$ GeV, as can be seen in Figure 3.1.

In the SUSY $SU(5)$ the matter (super)fields fit into two representations, $\overline{\mathbf{5}}$ and $\mathbf{10}$,

$$\overline{\mathbf{5}}_m = \begin{pmatrix} \bar{d}_r \\ \bar{d}_g \\ \bar{d}_b \\ e^- \\ \nu_e \end{pmatrix}_L , \quad \mathbf{10}_m = \begin{pmatrix} 0 & \bar{u}_b & -\bar{u}_g & -u_r & -d_r \\ -\bar{u}_b & 0 & \bar{u}_r & -u_g & -d_g \\ \bar{u}_g & -\bar{u}_r & 0 & -u_b & -d_b \\ u_r & u_g & u_b & 0 & -e^+ \\ d_r & d_g & d_b & e^+ & 0 \end{pmatrix}_L . \tag{3.1}$$

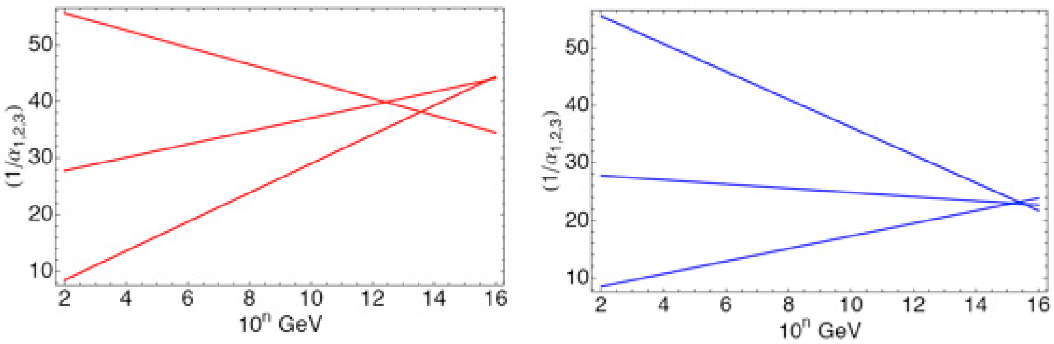

FIGURE 3.1: Running couplings in the SM (left), and those in the MSSM (right).

The Higgs sector contains $\mathbf{24}_H$ which breaks $SU(5)$ to the SM gauge group, and $\mathbf{5}_H(H_u)$ and $\overline{\mathbf{5}}_H(H_d)$ which break the EW symmetry. The part of the superpotential responsible for the breaking of $SU(5)$ is

$$\mathbb{W}_{SSB} = a\mathrm{Tr}(\mathbf{24}_H^2) + b\mathrm{Tr}(\mathbf{24}_H^3). \tag{3.2}$$

To obtain the unbroken $SU(3) \times SU(2) \times U(1)$, $\mathbf{24}_H$ must satisfy

$$\mathbf{24}_H = \mathrm{diag}\left(\frac{4a}{3b}, \frac{4a}{3b}, \frac{4a}{3b}, -\frac{2a}{b}, -\frac{2a}{b}\right) \tag{3.3}$$

In the SUSY $SU(5)$ the dominant channel of the proton decay is $p \to K^+\bar{\nu}$, due to Higgs color triplets in $\mathbf{5}_H$ and $\overline{\mathbf{5}}_H$. And to make a proton lifetime long enough ($\gtrsim 10^{33}$ years for the K^+ channel), those triplets have to have large masses $m_T \sim 10^{17}$ GeV, which results in a doublet-triplet splitting problem - the triplet Higgs becomes much heavier than the doublet. However this can be remedied in the *flipped* $SU(5)$ model [1] [2], as shown in [24].

3.2 Flipped $SU(5)$

The flipped $SU(5)$ model has the $SU(5) \times U(1)_X$ gauge group as the unification group which can be embedded into a larger GUT (i.e. $SO(10)$, E_6, or directly into stringy E_8). Originally, it was developed in [23] from superstrings and, in general, the flipped intermediate groups (those containing $U(1)_X$) are preferred by compactified superstrings.

The name "flipped" is due to the fermion assignment

$$\overline{\mathbf{5}}_m = \begin{pmatrix} \bar{u}_r \\ \bar{u}_g \\ \bar{u}_b \\ e^- \\ \nu_e \end{pmatrix}_L, \quad \mathbf{10}_m = \begin{pmatrix} 0 & \bar{d}_b & -\bar{d}_g & -u_r & -d_r \\ -\bar{d}_b & 0 & \bar{d}_r & -u_g & -d_g \\ \bar{d}_g & -\bar{d}_r & 0 & -u_b & -d_b \\ u_r & u_g & u_b & 0 & -\bar{\nu}_e \\ d_r & d_g & d_b & \bar{\nu}_e & 0 \end{pmatrix}_L; \quad \mathbf{1}_m = (e^+)_L; \tag{3.4}$$

where u-d quarks and ν_e-e^- are flipped compared to (3.1) in the ordinary $SU(5)$, and in addition we have an $SU(5)$-singlet positron. Another big difference is that the breaking of $SU(5)\times U(1)_X$ to the SM can be achieved by the 10-dimensional Higgs representations, $\mathbf{10}_H$ and $\overline{\mathbf{10}}_H$, with the corresponding terms of the superpotential:

$$\mathbb{W}_{SSB} = \lambda_1 \mathbf{10}_H \mathbf{10}_H \mathbf{5}_H + \lambda_2 \overline{\mathbf{10}}_H \overline{\mathbf{10}}_H \overline{\mathbf{5}}_H + \lambda_3 \mathbf{10}_m \overline{\mathbf{10}}_H \phi_m, \tag{3.5}$$

where ϕ_m is the additional $SU(5)$-singlet (for the see-saw mechanism, see [27]). The first two terms render the colour Higgs triplet heavy when $\mathbf{10}_H$ acquires a VEV $\langle \mathbf{10}_H \rangle \sim 10^{15}$ GeV.

[1] The flipped $SU(5)$ is mostly considered in the context of SUSY, so when writing "flipped $SU(5)$" we imply "flipped SUSY $SU(5)$".

[2] All flipped SUSY GUTs lead to the so-called "no-scale" $N = 1$ supergravity in four dimensions, as the low-energy effective action, which is characterised by independence of the Kähler potential upon some fields ("flat directions").

In the flipped $SU(5)$, at the GUT scale M_{GUT} ($\sim 10^{15}$ GeV) the gauge couplings α_2 and α_3 are unified into the $SU(5)$ coupling α_5:

$$\alpha_2(M_{GUT}) = \alpha_3(M_{GUT}) = \alpha_5(M_{GUT}), \tag{3.6}$$

and the coupling α_X of the $U(1)_X$ evolves separately as a combination

$$\frac{24}{\alpha_X(M_{GUT})} = \frac{25}{\alpha_1(M_{GUT})} - \frac{1}{\alpha_5(M_{GUT})}, \tag{3.7}$$

eventually unifying with α_5 at $\sim 10^{18}$ GeV for the stringy flipped $SU(5)$ [24].

3.3 $SO(10)$ models

Another example of GUT is based on $SO(10)$, and has the intermediate scale at $\sim 10^{12}$ GeV [25, 26]. With this particular scale, these models can naturally accommodate Majorana masses for neutrinos and the see-saw mechanism. The required right-handed neutrinos are already included in the three copies of 16-dimensional representation, along with all the other matter fields (three copies for three generations). The masses of the right-handed neutrinos (we collectively call them M_R) are assumed to be around the same scale as the intermediate scale $M_I \sim 10^{12}$ GeV (very roughly), in order to generate the observed small masses of the left-handed neutrinos.

There are several breaking patterns for $SO(10)$ GUTs with intermediate scale,

$$SO(10) \to SU(4)_c \times SU(2)_L \times SU(2)_R \to SM \ , \tag{3.8}$$
$$SO(10) \to SU(4)_c \times SU(2)_L \times U(1)_R \to SM \ , \tag{3.9}$$
$$SO(10) \to SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \to SM \ . \tag{3.10}$$

For the intermediate groups we use the notation

$$G_1 \equiv SU(4)_c \times SU(2)_L \times SU(2)_R \ ,$$
$$G_2 \equiv SU(4)_c \times SU(2)_L \times U(1)_R \ ,$$

$$\boxed{G_3 \equiv SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \ .}$$

At the one-loop order, the Renormalisation Group Equations describing the unification are

$$\alpha_i^{-1}(M_S) = \alpha_U^{-1}(M_U) + \frac{1}{3\pi} b_i \log\frac{M_I}{M_S} + \frac{1}{3\pi} b_i' \log\frac{M_U}{M_I} \ , \tag{3.12}$$

where $i = 1(Y), 2(L), 3(c)$; b_i' are the values taken at M_I, and M_S is the MSSM scale. The observed proton lifetime constrains $M_U > 10^{16}$ GeV, while $M_I \sim M_R$ can range from 10^{10} to 10^{14} GeV. Matter content at intermediate scale, M_I, can include the following multiplets of $SO(10)$: $\mathbf{10}, \mathbf{16}, \mathbf{45}, \mathbf{54}, \mathbf{120}, \mathbf{126}, \mathbf{210}$.

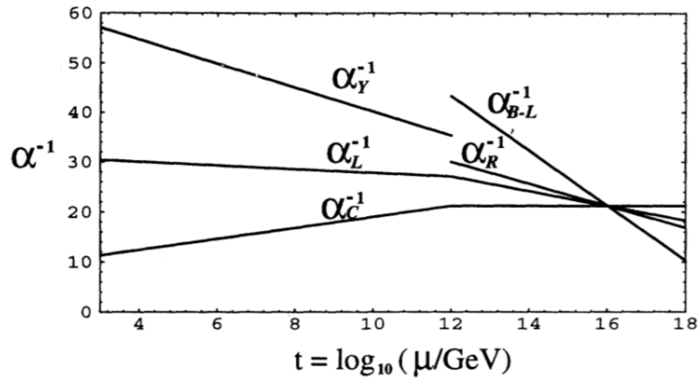

FIGURE 3.2: Unification of couplings in $SO(10)$, with G_3 as the intermediate group.

It is known that the most favoured intermediate group is G_3, assuming that M_R is obtained from a renormalizable Yukawa coupling. The corresponding evolution of the couplings is given in Figure 3.2, where the Higgs content is taken as

$$(\mathbf{1},\mathbf{2},\mathbf{2},0)\ \times 2\ , \tag{3.13}$$

$$(\mathbf{1},\mathbf{1},\mathbf{3},\pm 6)\ ,\quad (\mathbf{1},\mathbf{3},\mathbf{1},0)\ , \tag{3.14}$$

$$(\mathbf{1},\mathbf{2},\mathbf{1},\pm 3)\ ,\quad (\mathbf{8},\mathbf{1},\mathbf{1},0)\ , \tag{3.15}$$

where we have denoted each multiplet by its transformation law under G_3. The multiplets $(\mathbf{1},\mathbf{1},\mathbf{3},\pm 6)$ break $SO(10)$ to G_3 and give masses to the right-handed neutrinos. The two $(\mathbf{1},\mathbf{2},\mathbf{2},0)$ multiplets are the MSSM Higgs doublets. With this Higgs content, $\alpha_U^{-1}(M_U) \approx 21.2$, and $M_U \approx 2 \times 10^{16}$ GeV.

3.4 E_6 models

Under similar considerations, we can build a model based on an even larger group - E_6. In this case the number of intermediate group choices is vast, so we first present the maximal compact subgroups of E_6:

1) $SO(10) \times U(1)$
2) $SU(3) \times SU(3) \times SU(3)$
3) $SU(6) \times SU(2)$

In the group $SO(10) \times U(1)$ the $U(1)$ factor, often called $U(1)_X$ is orthogonal to $SO(10)$ and is irrelevant to the SM group. Thus the pattern is essentially the same as for $SO(10)$ models, but with some extra fields, since E_6 is larger.

In the case 2), one $SU(3)$ is identified with $SU(3)_c$, for the other two we can choose either $SU(3)_{L(R)}$ containing $SU(2)_{L(R)}$, or $SU(2)_{L(R)} \times U(1)_Z \subset SU(3)$. There is also a possibility of choosing $SU(2)_Z \times U(1)_R \subset SU(3)$. The hypercharge is then obtained as

$$Y = \frac{1}{6} Z - \frac{1}{2} T_R^3 \,, \tag{3.16}$$

where T_R^3 is the third $SU(2)_R$ generator.

In the case 3), $SU(6)$ can be decomposed as

$$\begin{aligned} &a)\ SU(5) \times U(1) \\ &b)\ SU(4) \times SU(2) \times U(1) \\ &c)\ SU(3) \times SU(3) \times U(1) \end{aligned}$$

where the second option (b) coincides with one of the $SO(10)$ models above. In the case a), since there is already an $SU(2)$ (outside of $SU(6)$) for $SU(2)_L$ role, the $SU(5)$ can be the extended color group $SU(5)_c$ containing $SU(3)_c$.

Among all these options it is found [Sato] that only the following intermediate groups of E_6 lead to the small unification coupling (in perturbative treatment):

$$SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \times U(1)_X \tag{3.17}$$

$$SU(3)_c \times SU(3)_R \times SU(2)_L \times U(1)_Z \tag{3.18}$$

$$SU(4)_c \times SU(2)_L \times U(1)_X \times [SU(2) \text{ or subgroups}] \tag{3.19}$$

$$SU(3)_c \times SU(3)_L \times U(1)_Z \times [SU(2) \text{ or subgroups}] \tag{3.20}$$

There is also a variety of Higgs and matter combinations that gives us too many options. More general treatment and extensive reviews of GUT models can be found e.g. in [22, 21, 28].

In the context of superstrings, E_6 can arise from one of the E_8-factors in the anomaly-free $E_8 \times E_8$ gauge group, in the context of Calabi-Yau compactification breaking $E_8 \to E_6 \times SU(3)$ [29].

Chapter 4

Standard Cosmology

In this chapter we move from elementary particles to theoretical cosmology that is another essential part of our investigation.

The Standard Cosmological Model is the simplest model describing all known cosmological observations. These include

- accelerated expansion of the universe;
- large-scale homogeneity and isotropy;
- current composition of the universe, in terms of abundances of light elements;
- existence of cosmic structures (galaxies and clusters);
- cosmic microwave background (CMB) radiation.

The model is based on General Relativity [1], but can be extended to include higher-curvature terms ($f(R)$ gravity), as at low curvatures a modified gravity theory can be practically indistinguishable from Einstein's gravity.

4.1 FLRW universe

An expanding universe can be described by a Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, which in spherical coordinates takes the form

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) , \tag{4.1}$$

[1]General Relativity was tested many times. Recently, two of its most important predictions - the existence of black holes and gravitational waves - was directly confirmed by LIGO and Virgo collaborations [30][31][32][33], by detecting gravitational waves coming from black hole and neutron star merging events.

where $a(t)$ is the cosmic scale factor describing spatial expansion. The topological parameter k defines the choice of one of the three symmetric spaces: $k = 1$ for spherical space (positive 3-curvature), $k = 0$ for flat space, and $k = -1$ for hyperbolic space (negative 3-curvature). The expansion rate of the universe is given by the Hubble function

$$H \equiv \frac{\dot{a}}{a} \ , \tag{4.2}$$

where the dot stands for the time derivative. According to the latest data, the present expansion rate is [34]

$$H = (67.8 \pm 0.9) \text{ km s}^{-1} \text{ Mpc}^{-1} \ . \tag{4.3}$$

Homogeneity and isotropy are reflected in the form of the matter stress-energy tensor (in the co-moving frame),

$$T_{\mu\nu} = \text{diag}(\rho, p, p, p) \ , \tag{4.4}$$

which is called the perfect fluid form. $\rho = \rho(t)$ is energy density, and $p = p(t)$ is pressure. Plugging (4.1) and (4.4) into the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \ , \tag{4.5}$$

the 00-component gives the first Friedmann equation,

$$H^2 + \frac{k}{a^2} = \frac{8}{3}\pi G\rho \ , \tag{4.6}$$

and the ij-components give the second Friedmann equation (also called Raychaudhuri equation),

$$2\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} = -8\pi G\rho \ . \tag{4.7}$$

Here $R_{\mu\nu}$ and R are Ricci tensor and scalar curvature, respectively, G is the gravitational constant.

4.1.1 Composition of the Universe

The (covariant) conservation law

$$\nabla_\mu T^{\mu\nu} = 0 \ , \tag{4.8}$$

for perfect fluid, for $\nu = 0$ yields

$$d(\rho a^3) + p d(a^3) = 0 \ . \tag{4.9}$$

Given an equation of state $p = \omega\rho$, with some constant ω, integrating (4.9) gives rise to

$$\rho \propto a^{-3(1+\omega)} \ . \tag{4.10}$$

Non-relativistic matter behaves like pressureless dust with $\omega = 0$, so the above equation gives $\rho \propto a^{-3}$. Ultra-relativistic matter (or radiation) has $\omega = 1/3$, and $\rho \propto a^{-4}$, thus its energy density dilutes more rapidly with expansion than that of non-relativistic matter. Substances with negative pressure, like dark energy (cosmological constant), have $\omega = -1$ and constant energy density $\rho \propto a^0$.

It is convenient to define the critical energy density,

$$\rho_c \equiv \frac{3H^2}{8\pi G} , \tag{4.11}$$

and the density ratio,

$$\Omega \equiv \frac{\rho}{\rho_c} , \tag{4.12}$$

when looking at the first Friedmann equation (4.6), it takes values $\Omega > 1$ for $k = 1$, $\Omega = 1$ for $k = 0$, and $\Omega < 1$ for $k = -1$. The density parameter can be broken down as

$$\Omega = \Omega_b + \Omega_{dm} + \Omega_{de} , \tag{4.13}$$

where Ω_b corresponds to baryonic matter, Ω_{dm} to cold dark matter, and Ω_Λ to dark energy. The present-day values are [34, 35, 36]

$$\Omega_b = 0.0486 \pm 0.0010 , \quad \Omega_{dm} = 0.2589 \pm 0.0057 , \quad \Omega_{de} = 0.6911 \pm 0.0062 , \tag{4.14}$$

so that $\Omega = 0.9986 \pm 0.0129$, and we conclude that the visible Universe is (almost) spatially flat.

However, spatial geometry does not determine the space-time geometry (nor does it work backwards). For the maximally symmetric space-times, space-time geometry can be classified as Minkowski, de Sitter, and anti-de Sitter. Minkowski space-time is well known from the Special Relativity courses, and it corresponds to zero 4-curvature case. De Sitter and anti-de Sitter space-times have positive and negative constant scalar curvature, respectively (in our notation).

4.1.2 Thermal history

As we look back into the cosmic history, the energy density becomes larger, but for different components it has different dependence on time.

As shown in Figure 4.1, we can divide the timeline into 3 stages:

I. The first stage is the radiation-dominated era, which lasted until t_{eq} (parametrised by a_{eq}). At this stage the scale factor behaves as $a \propto \sqrt{t}$ ($\omega = 1/3$ for radiation).

II. After the equilibrium at t_{eq}, where radiation and matter [2] energy densities meet, the matter-dominated era begins, where $a \propto t^{2/3}$.

III. Eventually, as ρ_m drops, since $\rho_\Lambda = \text{const}$, dark energy dominates onwards, with $a \propto e^t$.

[2] we refer as "matter" to baryons and cold dark matter together, $\rho_m = \rho_b + \rho_{dm}$.

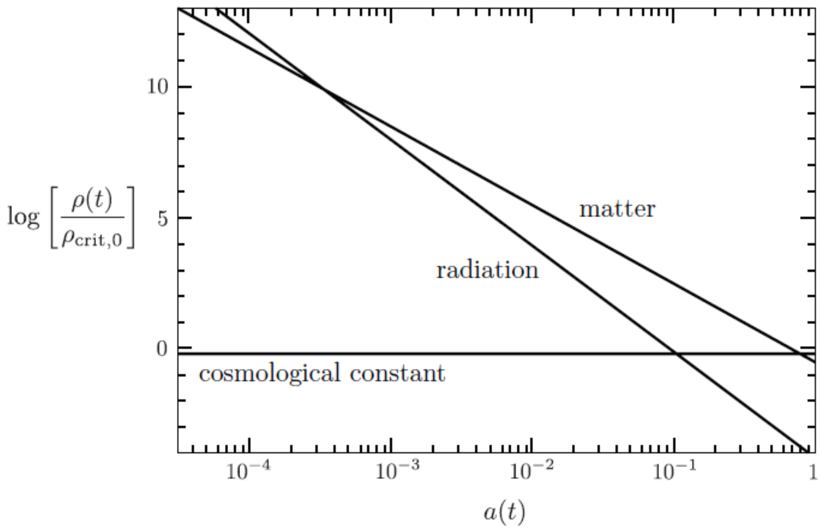

FIGURE 4.1: Evolution of the energy densities of radiation, matter, and dark energy

It turns out that the present time, t_0, is at the beginning of the stage III - the dark-energy-dominated era. This is implied by the CMB data [37, 34, 35, 36], which yields

$$\Omega_{\rm total} \approx \Omega_{\rm cr} \ . \tag{4.15}$$

To determine the thermal history of the Universe, we compare the interaction rate Γ with the expansion rate H at various stages of its evolution.

When $\Gamma \gg H$, the interaction rate is large enough to maintain thermal equilibrium. On the other hand, if $H \gg \Gamma$, i.e. the expansion rate is much larger, then the particles quickly fall out of equilibrium, or following the terminology, freeze out. When ultra-relativistic matter ($T \gg m$) freezes out, it is called a hot relic. When non-relativistic matter ($T \ll m$) freezes out, it is called a cold relic.

If equilibrium were maintained until today, the Universe would consist mostly of radiation, and in addition there would be equal amounts of matter and antimatter. Since this is not the case, we have to understand how freeze-out occurred for different particle species, and explain the present composition of the Universe.

The observed overabundance of matter over antimatter, and the baryon-to-photon ratio, $n_b/n_\gamma \sim 10^{-9}$, should be generated by some mechanism called *baryogenesis*. Along with freeze-out, baryogenesis requires B (baryon number) and CP violation. These three requirements carry the name of *Sakharov's conditions*. All three need to be satisfied for successful baryogenesis. CP violation is already present in the SM weak interactions [38, 39], while GUTs naturally provide the baryon number violation in the processes like proton decay. The exact mechanism is still an open question.

Let us summarise the thermal history in the energy scale order, by listing major events:

- Around 1 TeV: thermal equilibrium. Radiation-dominated era begins.

- 1 TeV – 100 GeV: EW phase transition and (presumably) baryogenesis occur.
- 100 MeV: quarks form bound states – hadrons.
- 1 MeV: neutrinos decouple.
- 0.1 MeV: Nucleosynthesis, helium-4 forms.
- 1 eV: matter-dominated era begins.
- 0.3 eV: recombination. Atoms form, and the universe becomes transparent to light.
- 10^{-3} eV: formation of galaxies and the present epoch.

4.1.3 Cosmological redshift

The light travelling through an expanding space undergoes a redshift. It is convenient to parametrise redshift by the parameter

$$z \equiv \frac{\Delta\lambda}{\lambda_i} = \frac{\lambda_f - \lambda_i}{\lambda_i} , \tag{4.16}$$

where λ_i and λ_f are the initial (emission) and final (observation) wavelengths of a photon. This can be recast in terms of a_i and a_f, using $\lambda_f/\lambda_i = a_f/a_i$, as

$$z = \frac{a_f}{a_i} - 1 , \tag{4.17}$$

The redshift parameter is in one-to-one correspondence with a_i, the cosmic scale factor at the time of the emission of photon.

The Hubble parameter can be rewritten in terms of z as

$$H(z) = -\frac{\dot{z}}{1+z} . \tag{4.18}$$

By measuring the redshift from, say, a distant star, we can tell the distance to that star, because

$$L = \int_{t_i}^{t_f} \frac{dt}{a(t)} = \frac{1}{a_0} \int_0^z \frac{dz}{H(z)} , \tag{4.19}$$

using (4.18). Here $a_0 \equiv a_f$.

4.1.4 Horizons

Due to finiteness of the speed of light and the age of the Universe, there are various types of cosmological horizons.

Light emitted at the earliest conceivable time travels a finite distance over the current age of the Universe, and marks the *particle horizon*. In practice, we can only receive the light emitted after the recombination, since before that the universe was opaque to photons. The distance that light can travel since recombination is called the *optical horizon*.

The cosmic *event horizon* is the maximal distance from which light (emitted at a given time) can reach the observer in the future.

The so-called *Hubble horizon*, thought not a horizon in a strict sense, is the curvature scale defined as

$$r_H = H^{-1}(t) \ . \tag{4.20}$$

4.2 Problems of Standard Cosmology

Before the inflationary paradigm was developed, the Standard Cosmology (SC) had several initial-conditions problems:

Horizon problem. First of all, the SC is unable to explain the observed large-scale homogeneity and isotropy of the universe. If we look back in time, according to the SC, right after the Big Bang the universe consisted of many causally disconnected regions. This raises the question – how did those regions evolve into such a homogeneous and isotropic universe we see today without causally connecting to each other? Unless they were already produced as such. This is called the horizon problem.

Flatness problem. It is known that the energy density of the universe is close to the critical density, which favours spatially flat universe. Since the total density departs from the crtitical value rather quickly at cosmic timescales, at early times it would be even closer to the critical value. In fact, it would need to be extremely fine-tuned in order to be able to evolve into its present-day value.

Monopole problem. Yet another class of problems arise if we go beyond the Standard Model. Grand Unified Theories predict various kinds of topological defects, such as monopoles, cosmic strings, and domain walls. These exotic objects should be abundantly produced in early, hot universe. But since we do not see them now, they should have been somehow diluted.

All these problems require unnaturally finely-tuned initial conditions for the universe. It becomes necessary to find a way to produce (at least) the observable universe from a single causally connected region. This demands a *cosmological inflation* - a period of rapid, quasi-exponential (or quasi-de Sitter) expansion of the Universe.

Last and not least, the SC does not allow any structure formation by its definition (as FLRW universe). Hence, the SC needs to be upgraded.

Chapter 5

Inflationary Cosmology

In this chapter we introduce inflationary cosmology by following [40, 41, 42].

As we showed earlier, expansion of space requires a substance with negative pressure and constant energy density. At the present epoch dark energy drives the accelerating expansion, which in the simplest case is a cosmological constant. However, the cosmological constant cannot be responsible for inflation, because, well, it is a constant. And a too small constant for that.

A more powerful way of obtaining an expanding universe is the use of scalar fields with potentials having local minima, also known as false vacua, which was first realised in the early 1970s [43, 44, 45], and was developed further in [46, 47, 48, 49]. However, such models (now called the *old inflation*) suffered from the *graceful exit* problem [50, 51] – the problem of successful reheating (particle production by decaying inflaton field) after inflation. The problem was later solved in the *new inflation* [52, 53, 54, 55], where it was realised that if one employs an inflationary potential with a (slightly tilted) approximately flat region, an inflaton field, slowly rolling down that plateau, causes an accelerating expansion of space. And once the slow-roll regime ends, under the right conditions one can realise the graceful exit. However, new inflation also suffered from certain problems related to the fact that it was assumed that the Universe was initially very large and very hot (and in thermal equilibrium), and inflation happened during the cosmological phase transitions.

Later, the first models of the so-called *chaotic inflation* were proposed [56], which solved all the prior problems. In chaotic inflaton scenarios, all the initial condition assumptions, such as thermal equilibrium and "hotness", were relaxed. Chaotic inflation in a broad sense represents any model where the potential has sufficiently flat region, so that the slow-roll regime lasts long enough (as $N_e \sim (50 \div 60)$ in terms of the e-foldings number N_e introduced below).

5.1 Chaotic inflation

The equation of motion for a scalar field in the FLRW space-time is

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \ , \tag{5.1}$$

where $H \equiv \dot{a}/a$ as before, $V = V(\phi)$ is the potential of the scalar field ϕ, and $V' \equiv \partial V/\partial\phi$. The second term above acts as a friction force, slowing down oscillations of the scalar field.

The corresponding Einstein equations yield (setting the Planck mass $M_P = (8\pi G)^{-1/2} = 1$, and using the perfect fluid stress-energy tensor)

$$H^2 + \frac{k}{a^2} = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right) \ , \tag{5.2}$$

where k is the 3d curvature parameter, which we set to 0 (spatially flat universe) from now on.

5.1.1 Slow-roll conditions

In order to realise inflation, we, first of all, make sure that the desired equation of state $p = -\rho$ is satisfied. For a perfect fluid with a scalar potential V, energy density and pressure take the form

$$\rho = \frac{1}{2}\dot{\phi}^2 + V \ , \tag{5.3}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V \ . \tag{5.4}$$

Thus, if

$$V \gg \frac{1}{2}\dot{\phi}^2 \ , \tag{5.5}$$

the desired equation of state is approximately satisfied, $p \approx -\rho$. We then require that the friction term is non-negligible (since, otherwise, there will be no inflation),

$$3H\dot{\phi} \gg \ddot{\phi} \ . \tag{5.6}$$

This, together with (5.5) leads to the slow-roll regime, where the equations (5.1) and (5.2) are simplified,

$$3H\dot{\phi} + V' = 0 \ , \tag{5.7}$$

$$H^2 = \frac{1}{3}V \ . \tag{5.8}$$

For more convenience we can recast the slow-roll conditions (5.5)(5.6) into the conditions for a potential only. To do this, we combine the equations (5.7)(5.8) to obtain

$$\dot{\phi}^2 = \frac{1}{3}\frac{V'^2}{V} \ , \tag{5.9}$$

and substitute ϕ from here into the condition (5.5). This yields

$$\frac{1}{2}\left(\frac{V'}{V}\right)^2 \equiv \epsilon \ll 1 \ , \tag{5.10}$$

where we have omitted the factor of 1/3 (as it does not affect the condition).

There is another condition on a potential, to find which, we take the time derivative of the condition (5.5), which leads to

$$V' \gg \ddot{\phi} \ , \tag{5.11}$$

assuming for simplicity positivity of both sides, and taking positive square root of (5.9). Substituting $\dot{\phi}$ from (5.9) we have

$$|\ddot{\phi}| = \frac{d\dot{\phi}}{dt} = \frac{d}{dt}\left(\frac{V'}{\sqrt{3V}}\right) = \frac{V''}{\sqrt{3V}}\dot{\phi} - \frac{3}{2}\frac{V'^2}{\sqrt{3V}^3}\dot{\phi} \ll V' \ . \tag{5.12}$$

Then, using (5.9) again, we arrive at

$$\frac{1}{3}\frac{V''}{V} - \frac{1}{6}\frac{V'^2}{V^2} \ll 1 \ . \tag{5.13}$$

Since the second term on the left-hand side is already negligible due to (5.10), we are left with

$$\left|\frac{V''}{V}\right| \equiv \eta \ll 1 \ , \tag{5.14}$$

where we have again omitted 1/3, and included modulus bars.

We introduced the parameters ϵ and η with the conditions (5.10)(5.14) that can be conveniently used to determine the slow-roll regime - whether or not it is possible to realise it (and thus inflation) depends upon a potential V.

5.1.2 $m^2\phi^2$-inflation

The simplest example of chaotic inflation is the model with the quadratic potential

$$V = \frac{1}{2}m^2\phi^2 \ , \tag{5.15}$$

where m is the mass of the scalar field ϕ.

The slow-roll parameters in this case are

$$\epsilon = \eta = 2\phi^{-2} \; , \tag{5.16}$$

which for $\phi \gg 1$ (in Planck mass units) are negligibly small, thus satisfying the slow-roll conditions down until the value of ϕ drops to ~ 1.

During the slow roll, the oscillator and Einstein equations for this model take the form

$$3H\dot{\phi} + m^2\phi = 0 \; , \tag{5.17}$$

$$H^2 = \frac{1}{6}m^2\phi^2 \; . \tag{5.18}$$

The solution of the system (5.17)(5.18) is

$$\phi(t) \approx \phi_i + \frac{m}{\sqrt{12\pi}}(t_i - t) \approx \frac{m}{\sqrt{12\pi}}(t_f - t) \; , \tag{5.19}$$

where t_i and t_f correspond to the beginning and the end of the slow-roll regime. $\phi_i \equiv \phi(t_i)$, while $\phi_f \equiv \phi(t_f)$ is negligible. However, it should be noted that the slow-roll regime ends before ϕ vanishes, so the domain of validity of the solution (5.19) is more limited than appears here. It follows from this solution that inflation lasts for $t_f - t_i \approx \sqrt{12\pi}\phi_i/m$.

To determine the behaviour of the cosmic scale factor during inflation, we plug the solution (5.19) into the Einstein equation (5.8) and find

$$a(t) \approx a_i \exp\left(\frac{1}{2}(H + H_i)(t - t_i)\right) \approx a_f \exp\left(-\frac{1}{6}m^2(t - t_f)^2\right) \; . \tag{5.20}$$

It is convenient to measure the duration of inflation by the number of e-foldings N_e,

$$\frac{a_f}{a_i} = e^{N_e} \; , \tag{5.21}$$

which, for (5.20), is $N_e \approx 2\pi\phi_i^2$.

5.1.3 Starobinsky inflation

Soon after chaotic inflation was introduced, it was realised that it is possible to obtain inflation from a higher-derivative gravity. In particular, consider the simplest $f(R)$ gravity due to Starobinsky [57], given by the Lagrangian ($M_P = 1$)

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(R + \frac{R^2}{6M^2}\right) \; , \tag{5.22}$$

where M is the mass parameter of the model. The dual scalar-tensor description has a potential for the scalaron field ϕ (which we identify with the inflaton), as [58]

$$V = \frac{3}{4}M^2\left(1 - e^{-\sqrt{\frac{2}{3}}\phi}\right)^2 \ . \tag{5.23}$$

The potential is non-negative, and has a plateau for large positive values of ϕ, since V in that case asymptotically approaches $3M^2/4$. Therefore, it can easily accommodate slow-roll inflation.

5.2 Graceful exit and reheating

After inflation, the scalar inflaton field begins to (coherently) oscillate around the minimum of the potential. This regime starts when $\phi \sim 1$ (in Planckian units). For simplicity, we go back to the $m^2\phi^2$-model where the field equations are

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \ , \tag{5.24}$$

$$H^2 + \frac{k}{a^2} = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2\right) \ . \tag{5.25}$$

Soon after the end of inflation, the solution to these equations asymptotically approaches

$$\phi(t) = \Phi(t)\cos(mt) \ , \tag{5.26}$$

where $\Phi(t) \equiv (\sqrt{3\pi}mt)^{-1}$ is the slowly decaying amplitude of oscillations (m plays the role of the frequency of oscillations). The solution (5.26) is valid as long as $mt \gg 1$. In this regime the energy density of ϕ behaves as $\rho \propto a^{-3}$, which means that the universe expands as effectively (non-relativistic) matter-dominated, with zero pressure. Thus, the coherently oscillating inflaton field can be thought of as a collection of heavy scalar condensates with mass m, and vanishing kinetic energy.

Once the regime $mt \gg 1$ is over, the amplitude Φ decays more rapidly, and the energy density of ϕ is transferred by inflaton decays to lighter particles, including ordinary matter. This process is called the *reheating*. The details of the reheating are very model-dependent, but we can still derive some general results.

Let us consider the simplest interaction between the inflaton ϕ, some real scalar χ, and a spinor ψ,

$$\mathcal{L}_{\text{int}} = -g\phi\chi^2 - h\phi\bar{\psi}\psi \ . \tag{5.27}$$

This yields the decay rates [59, 40]

$$\Gamma(\phi \to \chi\chi) = \frac{g^2}{8\pi m} \ , \quad \Gamma(\phi \to \psi\psi) = \frac{h^2 m}{8\pi} \ , \tag{5.28}$$

obtained from perturbation theory. Therefore, for $m \ll M_P$ the decay rate into scalar particles ($\Gamma_\chi \equiv \Gamma(\phi \to \chi\chi)$), is much larger than the decay rate into fermions. Taking this into account, the decay rate Γ_χ can be inserted into the effective equation

$$\ddot{\phi} + (3H + \Gamma_\chi)\dot{\phi} + m^2\phi = 0 \ , \tag{5.29}$$

where it serves as the additional friction term.

5.2.1 Parametric resonance

The problem of the perturbative treatment above is that it does not take into account Bose condensation effect, which dominates in the early stages of reheating. The essence of the effect is that the decay rate, Γ_χ, of ϕ into two χ particles is boosted, if there already exist χ particles with the same phase-space coordinates (momentum $\mathbf{k}$, and position $\mathbf{x}$). And the factor by which it is boosted is proportional to the occupation number of χ.

The dynamics of non-perturbative particle production, which is called preheating, can be determined by the mode equation for χ (in the case (5.27)),

$$\ddot{\chi}_k + [k^2 + m_\chi^2 + 2g\Phi\cos(mt)]\chi_k = 0 \ . \tag{5.30}$$

This equation describes an oscillator with periodically varying frequency ω. It can be conveniently transformed, after the redefinition $mt \equiv 2\tau$, into Mathieu equation

$$\chi_k''(\tau) + [A_k - 2q\cos(2\tau)]\chi_k(\tau) = 0 \ , \tag{5.31}$$

where we have introduced $A_k \equiv 4(k^2 + m_\chi^2)/m^2$, and $q \equiv -4g\Phi/m^2$. This equation is well-known, and the detailed study of its solutions can be found in [60], with its application to preheating in [59, 61, 40]. The main point is the existence of resonance bands of the solution, which leads to periodically explosive particle production.

The relevant solution of (5.31) has the form

$$\chi_k \propto \exp(\mu_k^{(n)}\tau) \ , \tag{5.32}$$

where the factor $\mu_k^{(n)}$ describes the exponential growth of n-th resonance band with the momentum spread $\Delta k^{(n)}$. It is related to the occupation number as

$$n_k \propto \exp(2\mu_k^{(n)}\tau) \ . \tag{5.33}$$

When $g\Phi \ll m^2$ the instability bands are thin (*narrow resonance*), and produced particles have small momentum spread, $\Delta k \ll m$. But if $g\Phi > m^2$ the momentum spread can be much larger. In this case we have a *broad resonance.*

We summarise that the realistic theory of reheating after inflation should incorporate both perturbative and non-perturbative treatments, as they yield different results. This is true not only for bosons, but for fermions also [61], although Pauli exclusion principle restricts the number density of fermions, $n_\psi \leq 1$. The interaction Lagrangian, in general, should also include quartic interactions, e.g. $-\lambda\chi^2\phi^2$, where λ is a coupling constant.

5.3 Inflation and cosmic perturbations

Although the Universe is apparently homogeneous and isotropic on cosmological scales (> 100 Mpc), there are inhomogeneities at smaller scales (< 100 Mpc) due to cosmic structures such as galaxies and stars. This can also be seen in the CMB data [34, 35, 36]. As long as those inhomogeneities are small, they can be treated by using perturbation theory.

5.3.1 Classification of perturbations

Let us consider small perturbations of the FLRW metric

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + \delta g_{\mu\nu} \ , \tag{5.34}$$

which then translate into matter perturbations by Einstein equations.

To simplify the treatment of perturbations, we decompose them into the scalar, vector, and tensor contributions. This can be done because Einstein equations at the linear order do not mix the three kinds of perturbations. The perturbed FLRW metric is then decomposed as

$$ds^2 = a^2(\tau)\left[(1+2A)d\tau^2 - 2B_i dx^i d\tau - (\delta_{ij} + H_{ij})dx^i dx^j\right] \ , \tag{5.35}$$

where A, B_i, H_{ij} are scalar, vector, and tensor perturbations, respectively (in 3d sense, since i, j run over spacial dimensions). This is not a complete decomposition though, as B_i and H_{ij} can be further decomposed as

$$B_i = \partial_i B + b_i \ , \tag{5.36}$$

$$H_{ij} = 2C\delta_{ij} + 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right)D + (\partial_i h_j - \partial_j h_i) + 2h_{ij} \ , \tag{5.37}$$

where Δ is the Laplacian operator. In total we have 4 scalars A, B, C, D; 2 (divergenceless) vectors b_i, h_i; and one (traceless) tensor h_{ij}. These are not uniquely defined, as we have general coordinate invariance. We can either reparametrise the perturbations to make them gauge-invariant, or we can work with specific gauge choices.

Scalar perturbations are important for structure formation, as they can lead to gravitational instabilities. Vector perturbations are not so interesting, as they decay very quickly. Tensor perturbations are (primordial) gravitational waves which at the linear order do not lead to instabilities, but are important since they are predicted by inflation, and should leave imprints in CMB.

An important quantity for observational cosmology is the curvature perturbation, defined as (in the so-called Newtonian gauge, $B = D = 0$)

$$\mathcal{R} = C + \frac{2M_P^2 H(C' + HC)}{a^2(\rho_0 + p_0)} \ , \tag{5.38}$$

where the prime stands for the derivative w.r.t. the conformal time τ; ρ_0 and p_0 are the unperturbed energy density and pressure, respectively.

During inflation, metric fluctuations (5.34) can be understood as originating from quantum fluctuations of the inflaton field (since they are related by Einstein equations),

$$\phi = \phi_0 + \delta\phi \ . \tag{5.39}$$

Of course, quantum fluctuations are originally very small, and were created inside the Hubble horizon. However, inflation soon stretches them to superhorizon scales. At superhorizon scales the fluctuations lose their quantum-mechanical nature and can be treated with classical methods (e.g., VEVs can be identified with average values of classical fields). Therefore, after the fluctuations re-enter the Hubble horizon, their observation can tell us about the inflationary epoch.

5.3.2 Scalar perturbations

At the horizon crossing, i.e. when the 3-momentum modulus $k = aH$, we can relate the curvature perturbation $\mathcal{R}$ and the inflaton perturbation $\delta\phi$, in the spatially flat gauge ($C = D = 0$),

$$\mathcal{R} = -\frac{H\delta\phi}{\phi_0'} \ . \tag{5.40}$$

Then, for their power spectra (quadratic deviations) $\Delta^2_{\mathcal{R}}(k) \equiv \langle|\mathcal{R}_k|^2\rangle$ and $\Delta^2_{\delta\phi}(k) \equiv \langle|\delta\phi_k|^2\rangle$ we have

$$\Delta^2_{\mathcal{R}}(k) = H^2\Delta^2_{\delta\phi}(k)\dot{\phi}^{-2} \approx \left.\frac{H^4}{4\pi^2\dot{\phi}^2}\right|_{k=aH} , \tag{5.41}$$

where in the last step we have used the approximated power spectrum of $\delta\phi$ at $k = aH$,

$$\Delta^2_{\delta\phi}(k) \approx \left.\frac{H^2}{4\pi^2}\right|_{k=aH} . \tag{5.42}$$

We can also relate $\Delta^2_{\mathcal{R}}$ with the shape of the inflationary potential,

$$\Delta^2_{\mathcal{R}} = \frac{1}{12\pi^2 M_P^6}\frac{V^3}{V'^2} \ . \tag{5.43}$$

It is convenient to parametrise the curvature power spectrum as

$$\Delta^2_{\mathcal{R}}(k) = A_s\left(\frac{k}{k^*}\right)^{n_s-1} , \tag{5.44}$$

where k^* is a reference scale, while n_s is called the *scalar spectral index*. Taking $k^* = k$ we have

$$n_s - 1 = \frac{d\log\Delta^2_{\mathcal{R}}}{d\log k} \ . \tag{5.45}$$

$n_s = 1$ corresponds to an exactly scale-symmetric scalar spectrum.

The right-hand side of (5.45) can be expressed in terms of the slow-roll parameters ϵ and η,

$$n_s - 1 = 2\eta - 6\epsilon \ . \tag{5.46}$$

This is one of the main observational parameters related to inflationary potentials.

5.3.3 Tensor perturbations

The tensor power spectrum can be written as

$$\Delta_t^2(k) = \frac{2}{\pi^2}\frac{H^2}{M_P^2}\bigg|_{k=aH} \ , \tag{5.47}$$

while, unlike the scalar spectrum (5.41), it does not depend on $\dot{\phi}^2$.

Similarly to the scalar spectrum, the tensor spectrum can be parametrised as

$$\Delta_t^2(k) = A_t \left(\frac{k}{k^*}\right)^{n_t - 1} \ , \tag{5.48}$$

where n_t is the tensor spectral index, and A_t is an amplitude of tensor perturbations. The value of A_t is subject to normalisation w.r.t. A_s, but we can use their ratio

$$r \equiv \frac{A_r}{A_s} \ , \tag{5.49}$$

which is normalisation-independent. r is called *tensor-to-scalar ratio*, and is another important parameter, together with n_s, connecting inflationary models with observations.

One can conveniently express r in terms of the slow-roll parameter ϵ as

$$r = 16\epsilon \ . \tag{5.50}$$

5.4 Observational constraints on inflationary models

Here we present the latest results from Planck collaboration [35, 62] on inflationary parameters n_s and r. As can be seen in Figure 5.1, the combined results of PLANCK, BICEP (BKP), and Baryonic Acoustic Oscillation (BAO) experiments favour low values of $r(k^* = 0.002) \lesssim 0.07$, and spectral index values $0.96 \lesssim n_s \lesssim 0.97$. The results, although not conclusive, strongly favour Starobinsky, or R^2, inflation with the number of e-foldings $N \sim 50 \div 60$. The quadratic, quartic, and power-law inflationary models are well outside the presented 99.7 % confidence-level region in the $n_s - r$ parameter space. Other models, like hilltop, hybrid, and natural inflation are in agreement with observations, but some of them have limited choices of their parameters.

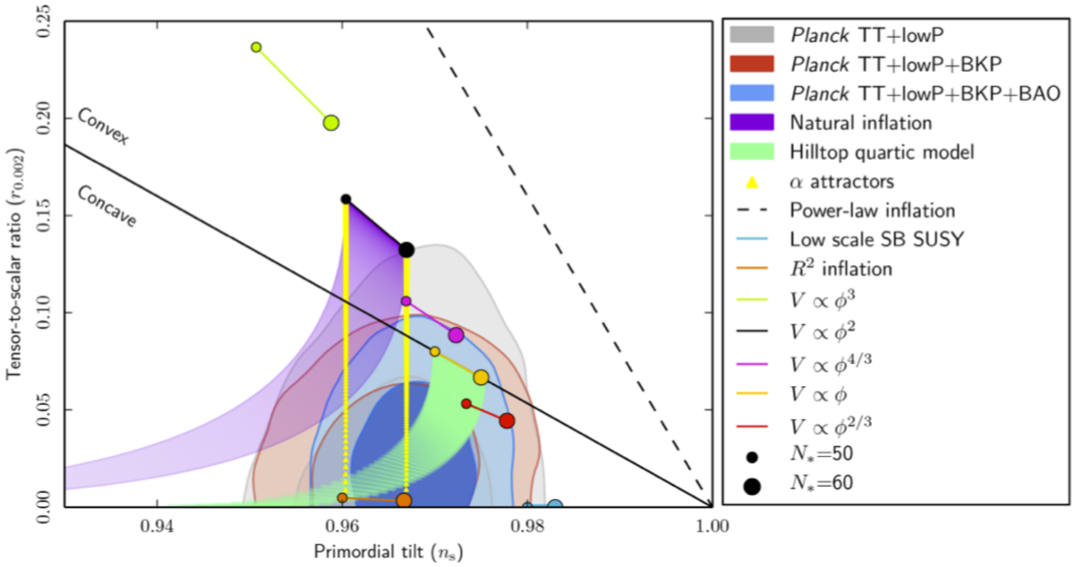

FIGURE 5.1: Observational constraints on the parameters n_s and r, and their predicted values from different inflationary models.

Chapter 6

Inflation in supergravity

6.1 Difficulties of embedding inflation into supergravity

As we already showed, supersymmetry is the very attractive idea that is used to extend the Standard Model, solving many of its problems. It is inevitable that at higher energies, when gravitational effects kick in, global supersymmetry should be replaced with local (or gauged) SUSY. This means that when we deal with the early-universe cosmology, particularly inflation, any realistic model should be reconsidered in the supergravity framework, and a supersymmetric model is more restrictive than its non-SUSY counterpart.

But it turns out that embedding inflationary models into supergravity is not an easy task. First of all, supersymmetrisation of inflationary models requires that the inflaton (real!) scalar field belongs to a supermultiplet. The simplest case then corresponds to a chiral supermultiplet that on-shell contains one complex scalar and one spinor. This means that the would-be inflaton is accompanied by another real scalar that can potentially spoil inflation. Thus, if we choose to deal with chiral multiplets, we have to deal with the problem of stabilisation of the extra real scalar during inflation.

Next, we recall the general form of a scalar potential in $N = 1$ chiral supergravity [19],

$$V = e^K \left(K^{ij^*} |W_i + K_i W|^2 - 3|W|^2 \right) \ , \tag{6.1}$$

where K is a (real) Kähler potential, W is a (holomorphic) superpotential. The problem is the factor e^K that can easily destroy the slow-roll regime. To illustrate this, let us choose the canonical Kähler potential $K = |A|^2$ for the complex scalar A, and assume that the inflaton is its real (or imaginary) component, $\phi \equiv \mathrm{Re}A$ ($\phi \equiv \mathrm{Im}A$). Then the slow-roll parameter η in this case is

$$\eta \equiv \frac{V''}{V} \gtrsim 2 \ , \tag{6.2}$$

where the primes denote the derivatives w.r.t. ϕ, and the 2 comes from taking the derivatives of e^K. So, one of the slow-roll conditions is violated. This is called the η-problem.

The problem can be avoided by various methods, e.g. considering non-canonical Kähler potentials and/or special kinds of superpotentials (this includes imposing certain symmetries on the potentials), or using quantum corrections. We are going to consider super-gauge theories, where inflation comes from a D-term instead of an F-term.

In addition, if a scalar potential leads to inflation, it should have a plateau of positive height. In other words, the quantity $W_i + K_i W$ should be non-vanishing during inflation. And since it is proportional to the F-term (as a result of Euler-Lagrange equations),

$$\bar{F}_j = -e^{K/3} K^{ij^*} (W_i + K_i W) \ , \tag{6.3}$$

the non-vanishing F_i ($\bar{F}_i$) break supersymmetry during that period of time. However, in the minimal models, supersymmetry is restored after inflation (in true vacuum). Since we don not see supersymmetry at low energies, it must be broken at a higher scale (perhaps not too far from the inflationary scale). This serves as a motivation to study possible connections between inflation and SUSY breaking.

In what follows, we give a few examples of the F-term, and D-term inflationary models in supergravity. In the next chapter, we consider D-term inflation and SUSY breaking together in a specific class of models.

6.2 F-term inflationary models

6.2.1 $m^2\phi^2$-inflation

As we already mentioned, one way to realise inflation in supergravity is to impose certain symmetries on the potentials of the model. Here we give an example of obtaining $m^2\phi^2$-inflation (from F-term) by imposing a symmetry on Kähler potential w.r.t. a shift of the (inflaton) chiral superfield as

$$\Phi \to \Phi + if \ , \tag{6.4}$$

where f is a real parameter. This symmetry requires that the Kähler potential is of the form $K = K(\Phi + \overline{\Phi})$. This means that the imaginary part of the scalar component of Φ vanishes from e^K, and can take values larger than one, whereas the real part of $\Phi|$ is bounded by the slow-roll conditions to be < 1. Because the exact shift symmetry makes the plateau completely flat, we need to break it, in order to introduce a small tilt. This is achieved by considering the superpotential

$$W = mX\Phi \ , \tag{6.5}$$

where X is a new chiral superfield. Then for $m \ll 1$ (in Planck units), the full Kähler potential can be approximated as

$$K \approx \frac{1}{2}(\Phi + \overline{\Phi})^2 + |X|^2 \ . \tag{6.6}$$

The model is invariant w.r.t. the sign flips of X and Φ, as well as the R-symmetry:

$$X \to e^{2i\alpha} X \ , \quad \Phi \to e^{i\alpha} \Phi \ , \tag{6.7}$$

where α is a real parameter. The corresponding kinetic terms of $X| = \chi$ and $\Phi|$ are canonical, while the scalar potential, parametrising $\Phi| = (\xi + i\phi)/\sqrt{2}$, takes the form

$$V = m^2 e^{\xi^2+|\chi|^2} \left\{ \frac{1}{2}(\xi^2+\phi^2)(1+|\chi|^4) + |\chi|^2 \left[1 - \frac{1}{2}(\xi^2+\phi^2) + 2\xi^2 \left(1 + \frac{\xi^2}{2} + \frac{\phi^2}{2} \right) \right] \right\} , \quad (6.8)$$

where we identify ϕ with the inflaton. The fields ξ and χ do not affect the inflation, and can be ignored, leaving effectively

$$V \approx \frac{1}{2} m^2 \phi^2 \ . \quad (6.9)$$

6.2.2 Hybrid inflation

Another example of the F-term inflation is the model called *hybrid inflation*, proposed in [63, 64, 65], which is attractive from the GUT perspective. Its supergravity extension [66] is defined by the potentials

$$K = |S|^2 + |\Psi|^2 + |\tilde{\Psi}|^2 \ , \quad (6.10)$$

$$W = \lambda S \Psi \tilde{\Psi} - \mu^2 S \ , \quad (6.11)$$

where Ψ and $\tilde{\Psi}$ are the conjugate representations of a gauge group G that can be part of a GUT group. λ and μ are the parameters of the model, both assumed to be smaller than one.

The model is invariant w.r.t. the R-symmetry transformations

$$S \to e^{2i\alpha} S \ , \quad \Psi \to e^{2i\alpha} \Psi \ , \quad \tilde{Psi} \to e^{-2i\alpha} \tilde{\Psi} \ . \quad (6.12)$$

The corresponding F-term scalar potential is

$$V_F = e^{|S|^2+|\Psi|^2+|\tilde{\Psi}|^2} \Big\{ (1 - |S|^2 + |S|^4) |\lambda \Psi \tilde{\Psi} - \mu^2|^2 + |S|^2 \left[|\lambda(1+|\Psi|^2)\tilde{\Psi} - \mu^2 \overline{\Psi}|^2 + |\lambda(1+|\tilde{\Psi}|^2)\Psi - \mu^2 \overline{\tilde{\Psi}}|^2 \right] \Big\} , \quad (6.13)$$

where the inflaton is defined as $\phi \equiv \sqrt{2}\mathrm{Re}S$. There is also the D-term scalar potential due to Killing potentials of the gauge group G.

From (6.13) it follows that the mass eigenstates are

$$Z_- = \frac{1}{\sqrt{2}}(\Psi - \overline{\tilde{\Psi}}) \ , \quad Z_+ \equiv \frac{1}{\sqrt{2}}(\Psi + \overline{\tilde{\Psi}}) \ , \quad (6.14)$$

with masses

$$M^2_{\mp} = \frac{1}{2}(\lambda^2 + \mu^4)\phi^2 \mp \lambda\mu^2 \ . \quad (6.15)$$

We consider the state Z_+ since M_+^2 is always positive. Then putting $Z_- = 0$, and assuming for simplicity $\mu \ll \lambda$, we arrive at

$$V \approx (\lambda|\Psi|^2 - \mu^2)^2 + \lambda^2\phi^2|\Psi|^2 + \frac{1}{8}\mu^4\phi^4 \ . \tag{6.16}$$

Due to SUSY breaking during inflation, which generates superpartner fermion masses, the above potential receives quantum corrections. At the one-loop order, the inflaton effective potential during inflation is given by

$$V = \mu^4 \left(1 + \frac{\lambda^2 d}{8\pi^2}\log\frac{\phi}{\phi_c} + \frac{1}{8}\phi^4\right) \ , \tag{6.17}$$

where d is the dimension of the representation of G, and $\phi_c \equiv \sqrt{2}\mu/\sqrt{\lambda}$ stands for the critical value of ϕ, below that the inflaton starts quickly rolling down to the minimum of the potential, ending the inflation.

In this model, the requirement of the sufficient number of e-foldings, $N_e \gtrsim 50$, sets the limit $\lambda\sqrt{d} \lesssim 0.2$. The spectral index and tensor-to-scalar ratio are given by

$$n_s - 1 \approx -\frac{1}{N} \ , \quad r \approx \frac{\lambda^2 d}{2\pi^2 N} \ . \tag{6.18}$$

6.3 D-term inflationary models

6.3.1 Quartic potential

Consider the following Kähler potential and superpotential [67, 68]:

$$K = |S|^2 + |X|^2 + |\tilde{X}|^2 \ , \quad W = \lambda(X\tilde{X} - \mu^2) \ , \tag{6.19}$$

where λ and μ are taken real and positive. The model has a $U(1)$ gauge symmetry and $U(1)$ rigid R-symmetry. Under the gauge $U(1)$ group the chiral superfields $S, X, \tilde{X}$ have charges $(0, 1, -1)$ and are minimally coupled to the $U(1)$ gauge superfield; while their R-charges are $(2, 0, 0)$, respectively.

Taking the simplest gauge-kinetic function, the corresponding F-term potential is

$$V_F = \lambda^2 e^K \Bigg\{ \left|X\tilde{X} - \mu^2\right|^2 (1 - |S|^2 + |S|^4) + |S|^2 \left[\left|\tilde{X} + \overline{X}(X\tilde{X} - \mu^2)\right|^2 + \left|X + \overline{\tilde{X}}(X\tilde{X} - \mu^2)\right|^2\right] \Bigg\} \ , \tag{6.20}$$

while the D-term potential reads

$$V_D = \frac{g^2}{2}(|X|^2 - |\tilde{X}|^2)^2 \ , \tag{6.21}$$

where g is the $U(1)$ coupling constant.

The global minimum of the full potential $V = V_F + V_D$ has

$$S = 0 \ , \quad X = \tilde{X} = \mu \ , \tag{6.22}$$

where we used the gauge symmetry to set the phase of $X, \tilde{X}$ to zero.

For inflation to take place, V_D must dominate, with $V_F \approx 0$. The vanishing of V_F requires

$$X\tilde{X} = \mu^2 \ , \quad S = 0 \ , \tag{6.23}$$

so that during inflation this condition should be approximately satisfied. But because of the factor e^K, in order to approximately satisfy (6.23), we need X to be much larger than one, $X \gg 1$. Then we can approximate the scalar potential as

$$V \approx \frac{1}{2} g^2 |X|^4 \ , \tag{6.24}$$

so that the inflaton can be identified with e.g. the real part of X.

As we showed, however, quartic models (actually, all power-law inflationary models) are disfavoured by PLANCK measurements of the tensor-to-scalar ratio $r < 0.07$.

6.3.2 D-term inflation with a massive vector multiplet

According to the end of the previous chapter, the Starobinsky model of inflation is favoured by the CMB observations. It is also one of the simplest models, as it is obtained from the $(R + R^2)$ gravity that is a purely gravitational theory, and no extra fields are needed solely for inflation. However, there is a problem with obtaining slow-roll inflation from supergravity extensions of $f(R)$ gravities. As was shown in [69, 70], a chiral (F-type) $F(R)$ supergravity cannot accommodate Starobinsky inflation. And a generic $F(R)$ supergravity (with both F- and D-type terms) leads to problems with stabilisation of additional scalars, and ghosts [71]. It is an open question which $F(R)$ (superspace) functions can lead to viable inflationary potentials. Fortunately, there is another way to embed the Starobinsky model of inflation into (N=1) supergrvity.

Consider a class of models first proposed in [72, 73], where $N = 1$ supergravity is non-minimally coupled to a *massive* vector superfield V represented by an arbitrary real function $J(V)$ with the coupling constant g [1]. Then, the bosonic part of the Lagrangian is given by

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{g^2}{2}J''\partial_\mu C\partial^\mu C - \frac{g^2}{2}J''B_\mu B^\mu - \mathcal{V} \ , \tag{6.25}$$

where the real scalar field C and the vector B_μ are the physical components of the massive vector superfield V (in fact, C is defined as $C \equiv V|$) along with two spinor fields. Thus, in (6.25) J is a function of the real scalar $J = J(C)$, while the prime stands for the derivative

[1] The general framework for massive vector multiplets in $N = 1$ supergravity was developed in [74].

w.r.t. C. As is clear from the kinetic term of C, the absence of ghosts requires $J'' > 0$ [2]. For the scalar potential $\mathcal{V}$ we have

$$\mathcal{V} = V_D = \frac{g^2}{2} J'^2 \ . \tag{6.26}$$

It is easy to realise chaotic inflation in this framework. As the simplest example, consider the function $J' = C$, coming from $J = C^2/2 + \text{const}$ ($J'' > 0$ is satisfied). With this choice we arrive at the quadratic potential

$$\mathcal{V} = \frac{g^2}{2} C^2 \ , \tag{6.27}$$

where g can be identified with the mass of the inflaton C.

It makes this class of models even more attractive that there is the possibility to obtain the Starobinsky inflation due to the freedom to choose the function J. Consider the following choice:

$$J = -\frac{3}{2}[\log(-C) + C] \ , \quad J' = -\frac{3}{2}(C^{-1} + 1) \ , \quad J'' = \frac{3}{2} C^{-2} \ . \tag{6.28}$$

In this case J'' yields a non-canonical kinetic term for C, so that we have to normalise it by redefining C as

$$C = -\exp\left(\sqrt{\frac{2}{3}}\phi\right) \ , \tag{6.29}$$

where ϕ is the canonical inflaton scalar.

The function (6.28), along with the normalisation (6.29), leads to the following potential:

$$\mathcal{V} = \frac{9}{8} g^2 \left(1 - e^{-\sqrt{\frac{2}{3}}\phi}\right)^2 \ , \tag{6.30}$$

that is exactly of the Starobinsky type.

This minimal class of models is just a starting point to consider more general setups that include matter chiral superfields, and possibly embed them into larger (SUSY GUT) gauge groups. But it also leaves supersymmetry exact after inflation. That is why we would like to consider minimal extensions of these models, including chiral superfields, where one of them would be responsible for spontaneous SUSY breaking (after inflation). We also discuss the equivalent formulation of these models, where the vector superfield V is massless and plays the role of the gauge field of a $U(1)$ gauge symmetry. The $U(1)$ (super-)gauge symmetry is then spontaneously broken by introduction of a Higgs chiral superfield, whose scalar component acquires a non-vanishing VEV.

[2] Our notation for J differs by the sign from that of [73].

Chapter 7

Inflation with inflaton in a vector multiplet and SUSY breaking

During inflation, supersymmetry is spontaneously broken, since either F-term or D-term acquires a non-vanishing VEV. But in the simplest models under consideration, after inflation SUSY is restored. Thus it should be spontaneously broken again, but now by a true VEV of F/D-terms. In this chapter we develop a class of inflationary models based on the earlier work [72, 73], where inflaton belongs to a massive vector multiplet [1]. We minimally extend those models by connecting them to a model of (F-term) SUSY breaking due to Polonyi [75], which gives rise to a non-minimal coupling between chiral and vector superfields.

This chapter is based on our original research results [1, 2, 76].

7.1 Non-minimal coupling of vector and chiral multiplets

Let us consider models with some chiral superfields Φ_i, represented by an arbitrary Kähler potential $K = K(\Phi_i, \overline{\Phi}_i)$ and a superpotential $\mathcal{W} = \mathcal{W}(\Phi_i)$, coupled to a massive vector superfield V, described by an arbitrary real function $J = J(V)$. Chiral superfields Φ_i are gauge singlets in our construction.

Our models are described by the Lagrangian ($M_P = g = 1$, where g is the coupling constant of the massive vector multiplet)

$$\mathcal{L} = \int d^2\theta 2\mathcal{E} \left\{ \frac{3}{8}(\overline{\mathcal{D}}\overline{\mathcal{D}} - 8\mathcal{R})e^{-\frac{1}{3}(K+2J)} + \frac{1}{4}W^\alpha W_\alpha + \mathcal{W} \right\} + \text{h.c.} \,, \tag{7.1}$$

where we have introduced the density superfield $2\mathcal{E}$, the scalar curvature superfield $\mathcal{R}$, and the vector superfield strength $W_\alpha \equiv -\frac{1}{4}(\overline{\mathcal{D}}\overline{\mathcal{D}} - 8\mathcal{R})\mathcal{D}_\alpha V$, all chiral.

[1] Supergravity with a massive vector multiplet in the non-minimal (arbitrary) setting was developed by Van Proeyen [74].

We focus on the bosonic part of our models, and set all fermions to zero. The (bosonic) components of supergravity multiplet are defined as

$$2\mathcal{E}| = e, \quad \mathcal{D}\mathcal{D}(2\mathcal{E})| = 4e\overline{M} \ ,$$
$$\mathcal{R}| = -\frac{1}{6}M, \quad \mathcal{D}\mathcal{D}\mathcal{R}| = -\frac{1}{3}R + \frac{4}{9}M\overline{M} + \frac{2}{9}b_m b^m - \frac{2}{3}i\mathcal{D}_m b^m \ ,$$

where $e \equiv \det e_m^a$ is the vierbein determinant, R is the space-time scalar curvature. We use the old-minimal set of the supergravity auxiliary fields: the complex scalar M and the real vector b_m. The vertical bars denote the leading ($\theta = \bar{\theta} = 0$) field components of a superfield.

The components of Φ_i and V are defined by

$$\Phi_i| = A_i \ , \quad \mathcal{D}_\alpha \mathcal{D}_\beta \Phi_i| = -2\varepsilon_{\alpha\beta}F_i \ , \quad \overline{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}_\alpha \Phi_i| = -2i\sigma_{\alpha\dot{\alpha}}{}^m \partial_m A_i \ ,$$
$$\overline{\mathcal{D}}\overline{\mathcal{D}}\mathcal{D}\mathcal{D}\Phi_i| = 16\Box A_i + \frac{32}{3}ib_a\partial^a A_i + \frac{32}{3}F_i M \ ,$$
$$V| = C \ , \quad \mathcal{D}_\alpha \mathcal{D}_\beta V| = \varepsilon_{\alpha\beta}X \ , \quad \overline{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}_\alpha V| = \sigma_{\alpha\dot{\alpha}}{}^m(B_m - i\partial_m C) \ ,$$
$$\mathcal{D}_\alpha W^\beta| \equiv -\frac{1}{4}\mathcal{D}_\alpha(\overline{\mathcal{D}}\overline{\mathcal{D}} - 8\mathcal{R})\mathcal{D}^\beta V = \frac{1}{2}\sigma_{\alpha\dot{\alpha}}{}^m \bar{\sigma}^{\dot{\alpha}\beta n}(\mathcal{D}_m\partial_n C + iF_{mn}) + \delta_\alpha{}^\beta(D + \frac{1}{2}\Box C) \ ,$$
$$\overline{\mathcal{D}}\overline{\mathcal{D}}\mathcal{D}\mathcal{D}V| = \frac{16}{3}b^m(B_m - i\partial_m C) + 8\Box C - \frac{16}{3}MX + 8D \ ,$$

in terms of the physical fields: complex scalars A_i, a real scalar C, and a vector B_m. The chiral auxiliary fields F_i and X are complex scalars, while the real auxiliary field D is a real scalar. $F_{mn} = \mathcal{D}_m B_n - \mathcal{D}_n B_m$ is the field strength of B_m.

Using these definitions, we find that the kinetic part of our Lagrangian is given by

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{kin.}} = e^{-\frac{1}{3}(K+2J)}\Big\{ & -\frac{1}{2}R - K_{ij*}\partial_m A_i \partial^m \bar{A}_j - \frac{1}{6}K_i K_j \partial_m A_i \partial^m A_j - \frac{1}{6}K_{i^*}K_{j^*}\partial_m \bar{A}_i \partial^m \bar{A}_j \\ & - \left(\frac{1}{3}J'^2 - \frac{1}{2}J''\right)\partial_m C \partial^m C + \left(\frac{1}{3}J'^2 - \frac{1}{2}J''\right)B_m B^m + J'\Box C + \frac{i}{3}J'B_m(K_{i^*}\partial^m \bar{A}_i - K_i \partial^m A_i) \\ & - \frac{1}{3}J'\partial_m C(K_{i^*}\partial^m \bar{A}_i + K_i\partial^m A_i)\Big\} - \frac{1}{4}F_{mn}F^{mn} \ , \end{aligned} \tag{7.2}$$

while its auxiliary part reads

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{aux.}} = e^{-\frac{1}{3}(K+2J)}\Big\{ & \frac{1}{3}b_m b^m + \frac{i}{3}b_m(K_{i^*}\partial^m \bar{A}_i - K_i\partial^m A_i) + \frac{2}{3}J'b_m B^m + J'D + K_{ij^*}F_i\overline{F}_j \\ & - \left(\frac{1}{3}J'^2 - \frac{1}{2}J''\right)X\overline{X} - \frac{1}{3}(M\overline{M} + K_i K_{j^*}F_i\overline{F}_j - J'K_{i^*}\overline{F}_i X - J'K_i F_i \overline{X} + K_{i^*}\overline{F}_i\overline{M} + K_i F_i M - J'MX \\ & - J'\overline{M}\overline{X})\Big\} + \frac{1}{2}D^2 + F_i\mathcal{W}_i + \overline{F}_i\overline{\mathcal{W}}_i - \overline{M}\mathcal{W} - M\overline{\mathcal{W}} \ . \end{aligned} \tag{7.3}$$

Here K, J, and $\mathcal{W}$ represent the leading components of the corresponding superfields, i.e. they are functions of the scalar fields A_i and C. We use the notation $K_i \equiv \frac{\partial K}{\partial A_i}$, $K_{i^*} \equiv \frac{\partial K}{\partial \bar{A}_i}$, $K_{ij^*} \equiv \frac{\partial^2 K}{\partial A_i \partial \bar{A}_j}$, $J' \equiv \frac{\partial J}{\partial C}$, $\mathcal{W}_i \equiv \frac{\partial \mathcal{W}}{\partial A_i}$, $\overline{\mathcal{W}}_i \equiv \frac{\partial \overline{\mathcal{W}}}{\partial \bar{A}_i}$.

In order to eliminate the auxiliary fields by their equations of motion, we first separate M, F_i and X from each other via the substitution

$$M = N + J'\overline{X} - K_{i^*}\overline{F}_i \ , \tag{7.4}$$
$$\overline{M} = \overline{N} + J'X - K_iF_i \ . \tag{7.5}$$

In terms of the new auxiliary fields N and $\overline{N}$, the auxiliary part of the Lagrangian reads

$$e^{-1}\mathcal{L}_{\text{aux.}} = e^{-\frac{1}{3}(K+2J)}\Big\{\frac{1}{3}b_mb^m + \frac{i}{3}b_m(K_{i^*}\partial^m\bar{A}_i - K_i\partial^mA_i) + \frac{2}{3}J'b_mB^m + J'D + K_{ij^*}F_i\overline{F}_j$$
$$+\frac{1}{2}J''X\overline{X} - \frac{1}{3}N\overline{N}\Big\} + \frac{1}{2}D^2 + F_i\mathcal{W}_i + \overline{F}_i\overline{\mathcal{W}}_i - \mathcal{W}(\overline{N} + J'X - K_iF_i) - \overline{\mathcal{W}}(N + J'\overline{X} - K_{i^*}\overline{F}_i) \ , \tag{7.6}$$

and Euler-Lagrange equations for the auxiliary fields are easily solved,

$$b_m = -J'B_m - \frac{i}{2}(K_{i^*}\partial_m\bar{A}_i - K_i\partial_mA_i) \ ,$$
$$D = -J'e^{-\frac{1}{3}(K+2J)}, \quad N = -3e^{\frac{1}{3}(K+2J)}\mathcal{W} \ ,$$
$$F_i = -e^{\frac{1}{3}(K+2J)}K^{-1}_{ij^*}(\overline{\mathcal{W}}_j + K_{j^*}\overline{\mathcal{W}}), \quad X = 2\frac{J'}{J''}e^{\frac{1}{3}(K+2J)}\overline{\mathcal{W}} \ .$$

Plugging these solutions back into the Lagrangian, we have

$$e^{-1}\mathcal{L} = e^{-\frac{1}{3}(K+2J)}\Big\{-\frac{1}{2}R - K_{ij*}\partial_mA_i\partial^m\bar{A}_j - \frac{1}{6}K_iK_{j^*}\partial_mA_i\partial^m\bar{A}_j - \frac{1}{12}K_iK_j\partial_mA_i\partial^mA_j$$
$$-\frac{1}{12}K_{i^*}K_{j^*}\partial_m\bar{A}_i\partial^m\bar{A}_j - \left(\frac{1}{3}J'^2 - \frac{1}{2}J''\right)\partial_mC\partial^mC + J'\Box C - \frac{1}{3}J'\partial_mC(K_{i^*}\partial^m\bar{A}_i + K_i\partial^mA_i)$$
$$-\frac{1}{2}J''B_mB^m\Big\} - \frac{1}{4}F_{mn}F^{mn} - \frac{1}{2}e^{-\frac{2}{3}(K+2J)}J'^2$$
$$-e^{\frac{1}{3}(K+2J)}\left[K^{-1}_{ij^*}(\mathcal{W}_i + K_i\mathcal{W})(\overline{\mathcal{W}}_j + K_{j^*}\overline{\mathcal{W}}) - \left(3 - 2\frac{J'^2}{J''}\right)\mathcal{W}\overline{\mathcal{W}}\right] \ . \tag{7.7}$$

This is in the Jordan frame. A transition to the Einstein frame is achieved by Weyl rescaling of spacetime metric,

$$g_{mn} \to e^{\Lambda}g_{mn} \ , \quad e \to e^{2\Lambda}e \ , \quad \text{with} \quad \Lambda = \frac{1}{3}(K + 2J) \ .$$

It leads to the transformation of the scalar curvature term,

$$-\frac{1}{2}ee^{-\frac{1}{3}(K+2J)}R \to -\frac{1}{2}eR + \frac{1}{12}e(\partial_mK + 2\partial_mJ)^2 \ . \tag{7.8}$$

This gives rise to the Lagrangian

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - K_{ij*}\partial_m A_i \partial^m \bar{A}_j - \frac{1}{4}F_{mn}F^{mn} - \frac{1}{2}J''\partial_m C\partial^m C - \frac{1}{2}J''B_m B^m - \mathcal{V}, \tag{7.9}$$

with the scalar potential

$$\mathcal{V} = \frac{1}{2}J'^2 + e^{K+2J}\left[K^{-1}_{ij^*}(\mathcal{W}_i + K_i\mathcal{W})(\overline{\mathcal{W}}_j + K_{j^*}\overline{\mathcal{W}}) - \left(3 - 2\frac{J'^2}{J''}\right)\mathcal{W}\overline{\mathcal{W}}\right]. \tag{7.10}$$

Our main results here are the equations (7.1), (7.9), and (7.10). If we drop the vector superfield V ($J = 0$), our Lagrangian coincides with the standard one, (2.60), for a chiral supergravity model. If we set $K(\Phi_i, \overline{\Phi}_i) = \mathcal{W}(\Phi_i) = 0$, our results coincide with the models proposed in [72, 73, 74].

As is clear from (7.9), the absence of ghosts (i.e. the negative sign of the kinetic term of C) requires $J''(C) > 0$.

7.2 Vacuum solution

In this Section we restrict ourselves to a single chiral superfield Φ having the canonical Kähler potential and the superpotential given by a sum of a linear term and a constant,

$$K = \Phi\overline{\Phi}\ , \qquad \mathcal{W} = \mu(\Phi + \beta)\ . \tag{7.11}$$

This particular choice is known in the literature as *Polonyi model* [75]. [2]

In accordance to the previous Section, it gives rise to the Lagrangian

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - \partial_m A\partial^m \bar{A} - \frac{1}{4}F_{mn}F^{mn} - \frac{1}{2}J''\partial_m C\partial^m C - \frac{1}{2}J''B_m B^m - \frac{1}{2}J'^2 \\ - \mu^2 e^{A\bar{A}+2J}\left[|1 + A\beta + A\bar{A}|^2 - \left(3 - 2\frac{J'^2}{J''}\right)|A+\beta|^2\right]\ . \tag{7.12}$$

The (Minkowski) vacuum conditions in this model are given by

$$V = \frac{1}{2}J'^2 + \mu^2 e^{A\bar{A}+2J}\left[|1 + A\beta + A\bar{A}|^2 - \left(3 - 2\frac{J'^2}{J''}\right)|A+\beta|^2\right] = 0\ , \tag{7.13}$$

$$\partial_{\bar{A}}V = A\tilde{V}_F + \mu^2 e^{A\bar{A}+2J}\left[A(1 + \bar{A}\beta + A\bar{A}) + (A+\beta)(1 + A\beta + A\bar{A}) - \left(3 - 2\frac{J'^2}{J''}\right)(A+\beta)\right] = 0\ , \tag{7.14}$$

$$\partial_C V = J'\left\{J'' + 2\mu^2 e^{A\bar{A}+2J}\left[|1 + A\beta + A\bar{A}|^2 - \left(1 - 2\frac{J'^2}{J''} + \frac{J'J'''}{J''^2}\right)|A+\beta|^2\right]\right\} = 0\ , \tag{7.15}$$

[2] It is worth mentioning that this choice is natural for a nilpotent (Akulov-Volkov) superfield, $\Phi^2 = 0$.

where we have introduced $\tilde{V}_F$ as the F-type scalar potential with the additional J-dependent term as

$$\tilde{V}_F = \mu^2 e^{A\bar{A}+2J}\left[|1 + A\beta + A\bar{A}|^2 - \left(3 - 2\frac{J'^2}{J''}\right)|A+\beta|^2\right]. \tag{7.16}$$

A simple solution to vacuum equations exists when $J' = 0$, which separates the Polonyi multiplet from the vector multiplet. The remaining vacuum equations allow a solution with the VEV $\langle A\rangle \equiv \alpha = (\sqrt{3}-1)$ and $\beta = 2 - \sqrt{3}$ [75]. This celebrated (Polonyi) solution describes a stable Minkowski vacuum with spontaneously broken SUSY since $\langle F\rangle = \mu$. Hence, the parameter μ defines the scale of SUSY breaking, that is *arbitrary* in this model.

The physical spectrum of this model, including fermions, consists of

- $\Phi \to A, \chi_\alpha$
- $V \to C, \xi_\alpha, \lambda_\alpha, B_m$
- $\mathcal{R} \to e^a_m, \psi^\alpha_m$

coming from the corresponding supermultiplets ($\mathcal{R}$ stands for the supergravity multiplet). Here the fermion χ_α is massless, while the fermions $\xi_\alpha, \lambda_\alpha$, belong to a massive multiplet with the mass $m_{\xi,\lambda} = \sqrt{J''}$. ψ^α_m is the gravitino with the mass $m_\psi = \mu e^{2-\sqrt{3}+\langle J\rangle}$.

It should be emphasized that the Polonyi field does not affect inflation driven by the scalar C as the inflaton belonging to the massive vector multiplet and having the D-type scalar potential $V(C) = \frac{1}{2}J'^2$ with arbitrary real J-function. Of course, the true inflaton field should be canonically normalized via the appropriate field redefinition of C.

When trying to get other patterns of SUSY breaking after inflation by demanding $J' \neq 0$ and $\alpha = \beta = 0$, we get two conditions on the J-function,

$$J'^2 = J'' \ , \tag{7.17}$$

$$J'' = -2\mu^2 e^{2J} \ . \tag{7.18}$$

The first equation is solved by $J = -\log C + const.$, then the second condition yields the consistency relation $const. = -\frac{1}{2}\log(-2\mu^2)$. Since both J and μ should be real, there is no solution. However, when allowing $\beta \neq 0$, the second equation (7.18) gets modified as

$$J'' = -2\mu^2 e^{2J}(1-\beta^2) \ , \tag{7.19}$$

so that the reality of J and μ requires $\beta > 1$. Then (7.17) reads $J'^2 = C^{-2}$ and is easily solvable. However, such scalar potential is not suitable for inflation (no slow roll).

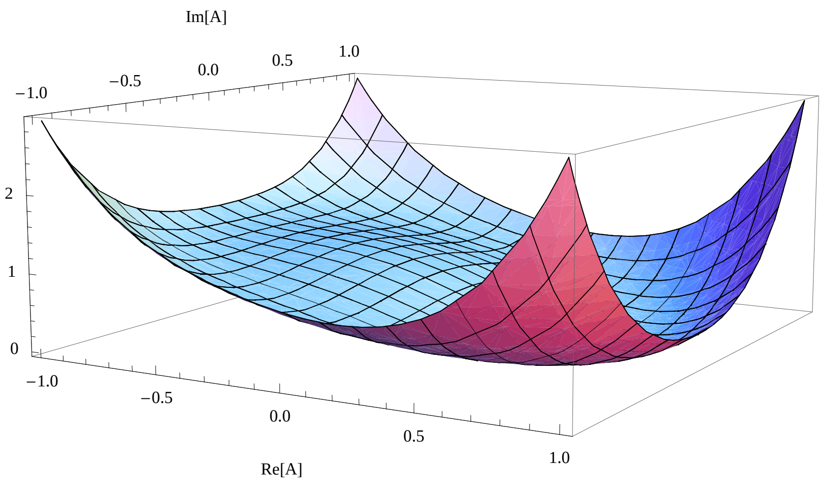

FIGURE 7.1: The scalar potential $\tilde{V} = \mu^{-2} e^{-A\bar{A}-2J} V$ as a function of Re(A) and Im(A) at $J' = 0$.

7.3 Stability of the vacuum

Here we examine stability of the vacuum in our models. For $|A| > 1$ the stability is guaranteed because the functions J'^2 and J'' enter the scalar potential

$$V = \frac{1}{2} J'^2 + \mu^2 e^{A\bar{A}+2J} \left[|1 + A\beta + A\bar{A}|^2 - \left(3 - 2\frac{J'^2}{J''}\right) |A + \beta|^2 \right] \tag{7.20}$$

with positive signs, while the function J'' is already positive (for ghost-freedom). On the other hand, the only term with a negative sign in this potential is $3|A + \beta|^2$ that grows slower than the positive quartic term $|1 + A\beta + A\bar{A}|^4$.

However, for $|A| < 1$, non-negativity of the potential (7.20) is not apparent, so we provide a plot in Figs. 7.1 and 7.2, where it is clear that the potential is non-negative for $|A| < 1$ as well.

7.4 Adding a cosmological constant

We can easily introduce a cosmological constant (dark energy) into our models without breaking any symmetries. It requires modification of the Polonyi VEV α, and the parameter β [3].

[3]The similar idea was used in Ref. [77].

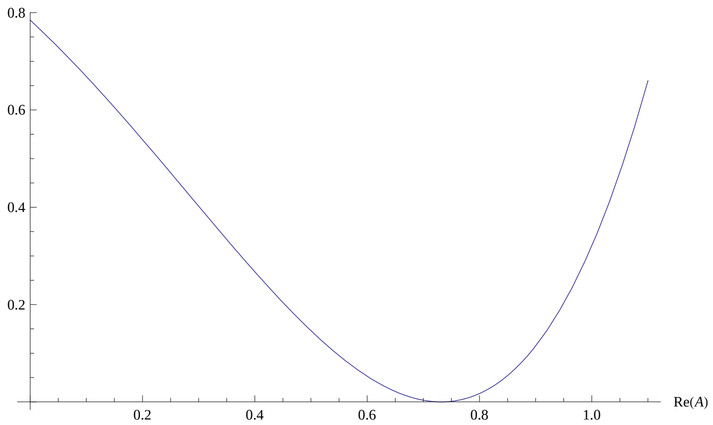

FIGURE 7.2: The real slice at $\text{Im}(A) = 0$ of Fig. 7.1 around the minimum of $\tilde{V}$.

Adding a (very) small positive constant δ, and assuming that at the minimum of the potential $J' = 0$, leads to the shift of the minimum of V,

$$V = \mu^2 e^{\alpha^2}\delta = m_{3/2}^2 \delta \ . \tag{7.21}$$

By comparing the condition (7.21) to the equation (7.13), we find the relation

$$(1 + \alpha\beta + \alpha^2)^2 - 3(\alpha + \beta)^2 = \delta \ . \tag{7.22}$$

The solution to the equations (7.22) and (7.14) with $V = m_{3/2}^2 \delta$ reads

$$\alpha = (\sqrt{3} - 1) + \frac{3 - 2\sqrt{3}}{3(\sqrt{3} - 1)}\delta + \mathcal{O}(\delta^2) \ , \quad \beta = (2 - \sqrt{3}) + \frac{\sqrt{3} - 3}{6(\sqrt{3} - 1)}\delta + \mathcal{O}(\delta^2) \ , \tag{7.23}$$

and describes a de Sitter vacuum with spontaneously broken SUSY.

Substituting the solution into the superpotential and ignoring the $\mathcal{O}(\delta^2)$-terms, we find

$$\langle \mathcal{W} \rangle = \mu(\alpha + \beta) = \mu(a + b - \frac{1}{2}\delta) \ , \tag{7.24}$$

where $a \equiv (\sqrt{3} - 1)$ and $b \equiv (2 - \sqrt{3})$ are the SUSY breaking vacuum solutions in the absence of a cosmological constant.

7.5 Massless vector multiplet and Higgs mechanism

The supergravity model (7.1) with a massive vector multiplet can also be considered as a supersymmetric (Abelian, Higgs-type) gauge theory where the Higgs superfield is gauged away (unitary gauge). In the formulation, where the $U(1)$ symmetry is restored, the vector superfield V becomes the $U(1)$ gauge superfield, gauge-coupled to the Higgs chiral superfield (responsible for spontaneous breaking of $U(1)$). We restore the gauge symmetry in the way that is consistent with local supersymmetry, and compare the results with the massive formulation (7.1).

We start with a Lagrangian that has essentially the same form as (7.1),

$$\mathcal{L} = \int d^2\theta 2\mathcal{E} \left\{ \frac{3}{8}(\overline{\mathcal{D}}\overline{\mathcal{D}} - 8\mathcal{R})e^{-\frac{1}{3}(K+2\tilde{J})} + \frac{1}{4}W^\alpha W_\alpha + \mathcal{W}(\Phi_i) \right\} + \text{h.c.} \ , \tag{7.25}$$

where $K = K(\Phi_i, \overline{\Phi}_j)$ and the indices i, j, k count the chiral superfields, *excluding* the Higgs chiral superfield, $H, \overline{H}$. In contrast to the massive formulation (7.1), the (real) function $\tilde{J}$ depends on the Higgs superfield, $\tilde{J} = \tilde{J}(He^{2V}\overline{H})$, and the vector superfield V is *massless*. The Lagrangian (7.25) is gauge-invariant under the supersymmetric $U(1)$,

$$H \to H' = e^{-iZ}H \ , \quad \overline{H} \to \overline{H}' = e^{i\overline{Z}}\overline{H} \ , \tag{7.26}$$

$$V \to V' = V + \frac{i}{2}(Z - \overline{Z}) \ , \tag{7.27}$$

where the gauge parameter Z itself is a chiral superfield.

Since the Lagrangian (7.25) has the $U(1)$ gauge symmetry, this allows us to choose the *Wess-Zumino* gauge, i.e. to "gauge away" the chiral and anti-chiral parts of the general vector superfield V by appropriately choosing the parameters Z and $\overline{Z}$,

$$\begin{aligned} V| = \mathcal{D}_\alpha \mathcal{D}_\beta V| = \overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}_{\dot\beta} V| = 0, \\ \overline{\mathcal{D}}_{\dot\alpha}\mathcal{D}_\alpha V| = \sigma_{\alpha\dot\alpha}{}^m B_m \ , \\ \mathcal{D}_\alpha W^\beta| = \frac{1}{4}\sigma_{\alpha\dot\alpha}{}^m \bar\sigma^{\dot\alpha\beta n}(2iF_{mn}) + \delta_\alpha{}^\beta D \ , \\ \overline{\mathcal{D}}\overline{\mathcal{D}}\mathcal{D}\mathcal{D}V| = \frac{16}{3}b^m B_m + 8D \ . \end{aligned}$$

A calculation of the bosonic part of the Lagrangian (7.25), after eliminating the auxiliary fields yields (in Einstein frame)

$$\begin{aligned} e^{-1}\mathcal{L} = -\frac{1}{2}R - K_{ij^*}\partial^m A_i \partial_m \bar{A}_j - \frac{1}{4}F_{mn}F^{mn} - 2J_{h\bar{h}}\partial_m h \partial^m \bar{h} - \frac{1}{2}J_{V^2}B_m B^m \\ + iB_m(J_{Vh}\partial^m h - J_{V\bar{h}}\partial^m \bar{h}) - \mathcal{V} \ , \end{aligned} \tag{7.28}$$

where h, $\bar{h}$ is the Higgs complex scalar field. We use the notation $J_{h\bar{h}} \equiv \frac{\partial^2 J}{\partial h \partial \bar{h}}|$, $J_{Vh} \equiv \frac{\partial^2 J}{\partial h \partial V}|$ and $J_{V^2} \equiv \frac{\partial^2 J}{\partial V^2}|$. For the scalar potential we have

$$\mathcal{V} = \frac{1}{2}J_V^2 + e^{K+2J}\left\{(K+2J)^{IJ^*}(W_I + (K+2J)_I W)(\overline{W}_{J^*} + (K+2J)_{J^*}\overline{W}) - 3W\overline{W}\right\} \ , \tag{7.29}$$

where the indices I, J collectively denote all chiral superfields (as well as their leading components), including the Higgs superfield.

The standard $U(1)$ Higgs mechanism can be obtained if we employ the canonical function $J = \frac{1}{2}he^{2V}\bar{h}$. Then, for the Higgs sector we have

$$e^{-1}\mathcal{L}_{Higgs} = -\partial_m h\partial^m\bar{h} + iB_m(\bar{h}\partial^m h - h\partial^m\bar{h}) - h\bar{h}B_m B^m - \mathcal{V} \ . \tag{7.30}$$

Parametrising h and $\bar{h}$ as

$$h = \frac{1}{\sqrt{2}}(\rho+\nu)e^{i\zeta}, \quad \bar{h} = \frac{1}{\sqrt{2}}(\rho+\nu)e^{-i\zeta} \ , \tag{7.31}$$

where ρ is the (real) Higgs boson, $\nu \equiv \langle h\rangle = \langle\bar{h}\rangle$ is the Higgs VEV, and ζ is the Goldstone boson, in the unitary gauge, $h \to h' = e^{-i\zeta}h$ and $B_m \to B'_m = B_m + \partial_m\zeta$, we reproduce the standard well-known result [78]

$$e^{-1}\mathcal{L}_{Higgs} = -\frac{1}{2}\partial_m\rho\partial^m\rho - \frac{1}{2}(\rho+\nu)^2 B_m B^m - \mathcal{V} \ . \tag{7.32}$$

The same result can also be achieved by considering the super-Higgs mechanism, where to get rid of the Goldstone mode, we use the super-gauge transformations (7.26)(7.27), and define the relevant components of Z and $i(Z-\overline{Z})$ as

$$Z| = \zeta + i\xi \ , \quad \frac{i}{2}\overline{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}_\alpha(Z-\overline{Z})| = \sigma^m_{\alpha\dot{\alpha}}\partial_m\zeta \ . \tag{7.33}$$

Looking at the lowest components of the transformation (7.26), it can be seen that the real part of $Z|$ and $\overline{Z}|$ cancels the Goldstone mode of (7.31). Similarly, applying $\overline{\mathcal{D}}_{\dot{\alpha}}$ and $\mathcal{D}_\alpha$ to (7.27), and taking their lowest components (recalling $\overline{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}_\alpha V| = \sigma^m_{\alpha\dot{\alpha}}B_m$), we conclude that the massive vector field indeed absorbs the Goldstone mode,

$$B'_m = B_m + \partial_m\zeta \ . \tag{7.34}$$

As for the physical spectrum of the model, we have

- $\Phi \to A, \chi_\alpha$
- $H \to h, \omega_\alpha$
- $V \to \lambda_\alpha, B_m$
- $\mathcal{R} \to e^a_m, \psi^\alpha_m$

where in contrast to the massive phase, we have an extra chiral multiplet - Higgs multiplet that includes massless Higgs complex scalar and fermion ω_α (Higgsino). The vector multiplet, being massless, includes only one (also massless) fermion λ, and no real scalars.

7.6 Polonyi-Starobinsky model

In this section we return to the massive formulation of our models with the Polonyi choice of K and $\mathcal{W}$,

$$K = \Phi\overline{\Phi} \ , \quad \mathcal{W} = \mu(\Phi + \beta) \ . \tag{7.35}$$

This gives rise to the component Lagrangian (7.12).

In order to obtain the Starobinsky inflationary potential we fix the real function $J(C)$ and redefine C (following [72, 73]) as

$$C = -\exp\left(\sqrt{2/3}\phi\right) \ , \quad J = -\frac{3}{2}[\log(-C) + C] \ , \quad J' = -\frac{3}{2}\left(C^{-1} + 1\right) \ , \quad J'' = \frac{3}{2}C^{-2} \ , \tag{7.36}$$

in terms of the canonical scalar ϕ playing the role of inflaton. The full scalar potential is a sum of the following D-type and F-type terms ($M_P = 1$):

$$V_D = \frac{9g^2}{8}\left(1 - e^{-\sqrt{2/3}\phi}\right)^2 \ , \tag{7.37}$$

$$V_F = \mu^2 \exp\left(\bar{A}A - \sqrt{6}\phi + 3e^{\sqrt{2/3}\phi}\right)\left\{|\bar{A}A + A\beta + 1|^2 - \left[3 - 3e^{\sqrt{8/3}\phi}\left(1 - e^{-\sqrt{2/3}\phi}\right)^2\right]|A + \beta|^2\right\} \ , \tag{7.38}$$

where V_D is the Starobinsky potential responsible for (large-field, slow-roll) inflation, while g is proportional to the inflaton mass that is fixed by CMB observations.

The phenomenological consequences of the PS model were studied in [79], where it was used to realise super-heavy gravitino dark matter scenario.

The Starobinsky inflationary potential V_D can be affected by the F-term potential, since V_F is also dependent upon ϕ. The exponential in front of the r.h.s. of (7.38) has the dangerous (fast growing) factor

$$\exp\left(3e^{\sqrt{2/3}\phi}\right) \tag{7.39}$$

that can destroy the slow-roll of inflaton ϕ. Indeed, for the Starobinsky inflation lasting for $N_e = 50$, the initial value of the inflaton is $\phi_{ini} \approx 5.1$ so that $e^{\sqrt{2/3}\phi} \approx 64.3$, and the factor (7.39) is extremely large. This poses a problem because the expression in curly braces in (7.38) cannot be suppressed for $\phi = \phi_{ini}$, and the potential V_F spoils the inflation.

Therefore, the Polonyi field should be strongly stabilised, and the double exponential factor in V_F should be removed. In order to stabilise the Polonyi field, we assume that it has a large mass (beyond the Hubble value), and, hence, a high-scale SUSY breaking dictated by the free parameter μ. As regards to the double exponential factor, we employ an FI term in order to be able to remove it.

7.7 Improved PS model with FI term

Our idea is to find a function J that would yield the Starobinsky inflationary potential for V_D, while keeping V_F suppressed against V_D. As we are going to show, this can be achieved with the help of a FI term.

To introduce a FI term, we consider the $U(1)$ gauge-invariant formulation (7.25) of our models, where the real function of the massless vector superfield depends upon the Higgs chiral superfield H as $J = J(\overline{H}e^{2V}H)$ [2]. Then we add a FI term with the real coupling constant ξ and its SUSY completion according to [80] [4]

$$\mathcal{L}_{\rm FI} = 8\xi \int d^4\theta E \frac{W^2\overline{W}^2}{\mathcal{D}^2W^2\overline{\mathcal{D}}^2\overline{W}^2}\mathcal{D}^\alpha W_\alpha \ . \tag{7.40}$$

Going back to the massive formulation (in the unitary gauge $H = 1$), it leads to the following D-type and F-type scalar potentials:

$$V_D = \frac{g^2}{2}\left(J' + \xi e^{\frac{1}{3}(K+2J)}\right)^2 \ , \tag{7.41}$$

$$V_F = \mu^2 e^{K+2J}\left[|\bar{A}A + A\beta + 1|^2 - \left(3 - 2\frac{J'^2}{J''}\right)|A+\beta|^2\right] \ , \tag{7.42}$$

where $K = \overline{\Phi}\Phi$, as before.

Equating (7.41) to the Starobinsky potential (in terms of $C = -e^{\sqrt{2/3}\phi}$), we get a first-order non-linear differential equation for J (we have to choose the negative square root sign on the r.h.s.),

$$J' + \tilde{\xi}e^{\frac{2}{3}J} = -\frac{3}{2}(C^{-1}+1) \ , \tag{7.43}$$

where we have introduced a "field-dependent" FI term $\tilde{\xi} \equiv \xi e^{K/3}$. We require that the Polonyi field A stays at its VEV during inflation so that $\tilde{\xi} = \xi e^{\langle K\rangle/3}$. During slow-roll, C takes large negative values ($|C^{-1}| \ll 1$), and the equation (7.43) can be approximately solved as

$$J(C) \approx J_\infty - \frac{3}{2}\log\left(1 - e^{C-C_0}\right) \ , \tag{7.44}$$

where C_0 is the integration constant, and we have introduced

$$J_\infty \equiv \frac{3}{2}\log\left(\frac{-3}{2\tilde{\xi}}\right) \ . \tag{7.45}$$

As is clear from (7.45), the existence of J_∞ requires $\tilde{\xi} < 0$ (thus, $\xi < 0$). Requiring the Starobinsky potential in V_D leads to the vanishing VEV of the auxiliary field D, which may result in the inconsistency of the fermionic terms multiplied by the negative powers of D.

[4] Ref. [80] introduces a new linearly-realized SUSY completion of a constant FI term, without gauging R-symmetry and allowing for a non-vanishing gravitino mass.

However, the problem can be cured if we uplift the Minkowski vacuum to a de Sitter vacuum (after inflation) via a minor modification of the function J by uplifting its minimum.

According to the equation (7.42) the stability of inflationary trajectories also requires that

$$\frac{(J')^2}{J''} \ll 1 \ . \tag{7.46}$$

Using the asymptotic solution (7.44), we find

$$\frac{(J')^2}{J''} \approx -\frac{3}{2}C^{-1} \ , \tag{7.47}$$

so that (7.46) is satisfied for $|C| \gg 1$.

The full scalar potential (during slow-roll inflation) of PS supergravity in the presence of FI term reads

$$\mathcal{V} \approx \frac{9g^2}{8}M_P^4\left(1 - e^{-\sqrt{2/3}\phi/M_P}\right)^2 + \mu^2 M_P^{-2}\exp(M_P^{-2}\bar{A}A + 2J_\infty)\times \tag{7.48}$$

$$\times\left\{|\bar{A}A + A\beta + M_P^2|^2 - 3M_P^2\left(1 - e^{-\sqrt{2/3}\phi/M_P}\right)|A+\beta|^2\right\} \ , \tag{7.49}$$

where we have restored the (reduced) Planck mass M_P. Here the first term (V_D) is exact, while the second term is approximate, as we have used the asymptotic solution (7.44).

Conclusion

Our main results begin with the original Lagrangian (7.9) and (7.10) that describes a new class of models suitable for inflationary model building that can accommodate SUSY breaking (along with R-symmetry breaking) after inflation. Our models are described by three arbitrary potentials K, $\mathcal{W}$ and J, providing flexibility and, perhaps, derivable from superstring theory. These models are *limited* in the sense that they provide the *minimal* extension of the inflationary models proposed in [72, 73] for the sake of spontaneous SUSY breaking in Minkowski vacuum after inflation.

We showed that considering the simple Polonyi setup (specific K, W, but general J-function) of SUSY breaking, we can obtain Minkowski and de Sitter vacua, both of which are stable. We also demonstrated that there is a gauge-invariant formulation of our models, which is intended for unification of inflation with super-GUTs in the context of supergravity. Unfortunately, physical applications of our model to SUSY GUTs and reheating appear to be highly model-dependent. Hence, a derivation of our supergravity model from superstrings would be very desirable because it would simultaneously fix those interactions and thus provide specific tools for a computation of reheating temperature, matter abundance, etc. after inflation, together with the low-energy predictions – see e.g., [81] for previous studies along these lines.

In the end of Chapter 7, we considered a specific choice of the J-function that leads to the Polonyi-Starobinsky supergravity model. This model can be part of a more general (and more realistic) theory including more matter and the hidden sector, suitable for phenomenological applications. We found an instability of inflation in the PS supergravity, and showed that it can be removed by adding a Fayet-Iliopoulos term to the model.

Our models can be extended in the gauge-sector by replacing Maxwell-type kinetic term with DBI-type one along the lines of [82, 83], providing further support towards their possible origin in compactified superstrings.

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