DERIVED FACTORIZATION CATEGORIES OF GAUGED LANDAU-GINZBURG MODELS

YUKI HIRANO

Abstract. In the first half of this thesis, for a given Fourier-Mukai equivalence of bounded derived categories of coherent sheaves on smooth quasi-projective varieties, we construct Fourier-Mukai equivalences of derived factorization categories of gauged Landau-Ginzburg (LG) models. As an application, we obtain some equivalences of derived factorization categories of K-equivalent gauged LG models. This result is an equivariant version of the result of Baranovsky and Pecharich, and it also gives a partial answer to Segal's conjecture. As another application, we prove that if the kernel of the Fourier-Mukai equivalence is linearizable with respect to a reductive affine algebraic group action, then the derived categories of equivariant coherent sheaves on the varieties are equivalent. This result is shown by Ploog for finite groups case.

In the second half, we prove a Knörrer periodicity type equivalence between derived factorization categories of gauged LG models, which is an analogy of a theorem proved by Shipman and Isik independently. As an application, we obtain a gauged LG version of Orlov's theorem describing a relationship between categories of graded matrix factorizations and derived categories of hypersurfaces in projective spaces, by combining the above Knörrer periodicity type equivalence and the theory of variations of GIT quotients due to Ballard, Favero and Katzarkov.

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1. Introduction

When X is a scheme, G is an affine algebraic group acting on X, $\chi : G \to \mathbb{G}_m$ is a character, and $W : X \to \mathbb{A}^1$ is a χ-semi invariant regular function, we call data $(X, \chi, W)^G$ a *gauged Landau-Ginzburg (LG) model.* Following Positselski [Pos1], [EP], we consider *the derived factorization category* of $(X, \chi, W)^G$, denoted by

$$\mathrm{Dcoh}_G(X, \chi, W).$$

Derived factorization categories are simultaneous generalizations of bounded derived categories of coherent sheaves on schemes, and of categories of (graded) matrix factorizations of (homogeneous) polynomials.

Derived factorization categories play an important role in Homological Mirror Symmetry for non-Calabi-Yau varieties [Orl4], and are useful to study derived categories of coherent sheaves on algebraic stacks. For example, studying windows in derived factorization categories gives a new technique to obtain some equivalences or semi-orthogonal decompositions of derived categories of algebraic stacks [Seg], [BFK2]. Moreover, by using derived factorization categories, we obtain new approach to Kuznetsov's homological projective duality [BDFIK2], [ADS], [ST], [Ren].

1.1. **Main results (Part I).**

1.1.1. *Background and motivation.* Since derived factorization categories are generalizations of the bounded derived category of coherent sheaves, it is natural to expect similarities between derived categories and derived factorization categories; such similarities are observed in [Vel], [BP], [LS], for example. In the present paper, we obtain equivalences between derived factorization categories of certain gauged LG models from equivalences between derived categories of smooth quasi-projective varieties.

1.1.2. *Statements.* Let X_1 and X_2 be smooth quasi-projective varieties over an algebraically closed field k of characteristic zero, and G be a reductive affine algebraic group acting on each X_i. Let $W_i : X_i \to \mathbb{A}^1$ be a χ-semi invariant regular function on X_i for some character $\chi : G \to \mathbb{G}_m$, and $\pi_i : X_1 \times X_2 \to X_i$ be the projections. Consider the fibre product

$$\begin{tikzcd}[column sep=small] & X_1 \times_{\mathbb{A}^1} X_2 \arrow[dl] \arrow[dr] & \\ X_1 \arrow[dr, "W_1"'] & & X_2 \arrow[dl, "W_2"] \\ & \mathbb{A}^1 & \end{tikzcd}$$

and let $j : X_1 \times_{\mathbb{A}^1} X_2 \hookrightarrow X_1 \times X_2$ be the embedding.

An object $P \in \mathrm{D^b}(\mathrm{coh} X_1 \times_{\mathbb{A}^1} X_2)$ whose support is proper over X_2 defines the integral functor

$$\Phi_{j_*(P)} : \mathrm{D^b}(\mathrm{coh} X_1) \to \mathrm{D^b}(\mathrm{coh} X_2) \qquad (-) \longmapsto \mathbf{R}\pi_{2*}(\pi_1^*(-) \otimes^{\mathbf{L}} j_*(P)).$$

On the other hand, the object P induces an object $\widetilde{P} \in \mathrm{Dcoh}(X_1 \times X_2, \pi_2^* W_2 - \pi_1^* W_1^*)$ and it defines the integral functor

$$\Phi_{\widetilde{P}} : \mathrm{Dcoh}(X_1, W_1) \to \mathrm{Dcoh}(X_2, W_2) \qquad (-) \longmapsto \mathbf{R}\pi_{2*}(\pi_1^*(-) \otimes^{\mathbf{L}} \widetilde{P}).$$

Furthermore, if the object P is G-linearizable, i.e. it is in the essential image of the forgetful functor

$$\Pi : \mathrm{D^b}(\mathrm{coh}_G X_1 \times_{\mathbb{A}^1} X_2) \to \mathrm{D^b}(\mathrm{coh} X_1 \times_{\mathbb{A}^1} X_2),$$

then the object P induces an object $\widetilde{P}_G \in \mathrm{Dcoh}_G(X_1 \times X_2, \pi_2^* W_2 - \pi_1^* W_1^*)$ and it defines the integral functor

$$\Phi_{\widetilde{P}_G} : \mathrm{Dcoh}_G(X_1, W_1) \to \mathrm{Dcoh}_G(X_2, W_2) \qquad (-) \longmapsto \mathbf{R}\pi_{2*}(\pi_1^*(-) \otimes^{\mathbf{L}} \widetilde{P}_G).$$

The main result of the present paper is the following:

Theorem 1.1 (Theorem 5.6)**.** *Let $P \in \mathrm{D^b}(\mathrm{coh} X_1 \times_{\mathbb{A}^1} X_2)$ be a G-linearizable object whose support is proper over X_1 and X_2. If the integral functor $\Phi_{j_*(P)} : \mathrm{D^b}(\mathrm{coh} X_1) \to \mathrm{D^b}(\mathrm{coh} X_2)$ is an equivalence (resp. fully faithful), then so is $\Phi_{\widetilde{P}_G} : \mathrm{Dcoh}_G(X_1, W_1) \to \mathrm{Dcoh}_G(X_2, W_2)$.*

This theorem is proved when the group G is trivial, the functions W_i are flat, and X_i are smooth Deligne-Mumford stacks, in [BP]. Combining Theorem 1.1 with the result in [Bri], we obtain the following:

Corollary 1.2. *Let X and X^+ be smooth quasi-projective threefolds, and let the diagram*

$$X \xrightarrow{f} Y \xleftarrow{f^+} X^+$$

be a flop. Let G be a reductive affine algebraic group acting on X, X^+ and Y with the morphisms f and f^+ equivariant. Take a semi invariant regular function $W_Y : Y \to \mathbb{A}^1$, and set $W := f^* W_Y$ and $W^+ := f^{+*} W_Y$. Then we have an equivalence

$$\mathrm{Dcoh}_G(X, W) \cong \mathrm{Dcoh}_G(X^+, W^+).$$

The gauged LG models $(X, W)^G$ and $(X^+, W^+)^G$ in Corollary 1.2 are *K-equivalent.* Here, K-equivalence means that there exists a common equivariant resolution of the varieties such that the pull-backs of the functions of LG models, and the classes of canonical divisors, coincide. We expect the following conjecture, which is a generalization of [Seg, Conjecture 2.15]:

Conjecture 1.3. *If two gauged LG models are K-equivalent, then their derived factorization categories are equivalent.*

Conjecture 1.3 for gauged LG models with trivial $\mathbb{G}_m$-actions and trivial functions is proposed in [Kaw].

As another corollary of Theorem 1.1, we obtain the following result.

Corollary 1.4. *Let $P \in \mathrm{D^b}(\mathrm{coh} X_1 \times X_2)$ be an G-linearizable object whose support is proper over X_1 and X_2. Let $P_G \in \mathrm{D^b}(\mathrm{coh}_G X_1 \times X_2)$ be an object with $\Pi(P_G) \cong P$, where Π is the forgetful functor. If the integral functor $\Phi_P : \mathrm{D^b}(\mathrm{coh} X_1) \to \mathrm{D^b}(\mathrm{coh} X_2)$ is an equivalence (resp. fully faithful), then so is $\Phi_{P_G} : \mathrm{D^b}(\mathrm{coh}_G X_1) \to \mathrm{D^b}(\mathrm{coh}_G X_2)$.*

Corollary 1.4 is obtained in the case of smooth projective varieties with finite group actions by [Plo, Lemma 5]; see also [KS]. We can also prove Corollary 1.4 for a finite group G by using [Ela2, Theorem 5.2].

1.2. **Main results (Part II).**

1.2.1. *Background and motivation.* Orlov proved the following semi-orthogonal decompositions between bounded derived categories of hypersurfaces in projective spaces and categories of graded matrix factorizations [Orl3].

Theorem 1.5 ([Orl3] Theorem 40). *Let $X \subset \mathbb{P}_k^{N-1}$ be the hypersurface defined by a section $f \in \Gamma(\mathbb{P}_k^{N-1}, \mathcal{O}(d))$. Denote by F the corresponding homogeneous polynomial.*
(1) *If $d < N$, there is a semi-orthogonal decomposition*

$$\mathrm{D^b}(\mathrm{coh} X) = \langle \mathcal{O}_X(d-N+1), ..., \mathcal{O}_X, \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N, \chi_d, F)\rangle.$$

(2) *If $d = N$, there is an equivalence*

$$\mathrm{D^b}(\mathrm{coh} X) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N, \chi_d, F).$$

(3) *If $d > N$, there is a semi-orthogonal decomposition*

$$\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N, \chi_d, F) = \langle k, ..., k(N-d+1), \mathrm{D^b}(\mathrm{coh} X)\rangle.$$

While Orlov's approach was algebraic, there are geometric approaches to the above theorem [Shi], [BFK2], [BDFIK3], where a version of Knörrer periodicity [Shi], [Isi] and homological variations of GIT quotients [Seg], [H-L], [BFK2] are the main tools. Combinations of Knörrer periodicity and the theory of variations of GIT quotients also imply homological projective dualities [BDFIK2], [ADS], [ST], [Ren].

In part II, we prove another version of Knörrer periodicity [Knö], which is a derived (or global) version, and we combine it with the theory of variations of GIT quotients by [BFK2] to obtain a gauged LG version of Orlov's theorem.

1.2.2. *Statements.* Let k be an algebraically closed field of characteristic zero. Let X be a smooth quasi-projective variety over k, and let G be a reductive affine algebraic group acting on X. Let $\mathcal{E}$ be a G-equivariant locally free sheaf of finite rank, and choose a G-invariant regular section $s \in \Gamma(X, \mathcal{E}^\vee)^G$. Denote by $Z \subset X$ the zero scheme of s. Let $\chi : G \to \mathbb{G}_m$ be a character of G, and set $\mathcal{E}(\chi) := \mathcal{E} \otimes \mathcal{O}(\chi)$, where $\mathcal{O}(\chi)$ is the G-equivariant invertible sheaf corresponding to χ. Then $\mathcal{E}(\chi)$ induces a vector bundle $\mathrm{V}(\mathcal{E}(\chi))$ over X with a G-action induced by the equivariant structure of $\mathcal{E}(\chi)$. Let $q : \mathrm{V}(\mathcal{E}(\chi)) \to X$ and $p : \mathrm{V}(\mathcal{E}(\chi))|_Z \to Z$ be natural projections, and let $i : \mathrm{V}(\mathcal{E}(\chi))|_Z \to \mathrm{V}(\mathcal{E}(\chi))$ be a natural inclusion. The regular section s induces a χ-semi invariant regular function $Q_s : \mathrm{V}(\mathcal{E}(\chi)) \to \mathbb{A}^1$. The first main result in this paper is the following:

Theorem 1.6. *Let $W : X \to \mathbb{A}^1$ be a χ-semi invariant regular function, such that the restricted function $W|_Z : Z \to \mathbb{A}^1$ is flat. Then there is an equivalence*

$$i_*p^* : \mathrm{Dcoh}_G(Z, \chi, W|_Z) \xrightarrow{\sim} \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, q^*W + Q_s).$$

The above result is an analogy of Shipman's and Isik's result, where they consider the case when $G = \mathbb{G}_m$, the G-action on X is trivial, $\chi = \mathrm{id}_{\mathbb{G}_m}$, and $W = 0$ [Shi], [Isi]. Furthermore, the above theorem can be considered as a generalization of Knörrer periodicity to a derived and G-equivariant version. The proof of the above theorem is quite different from Shipman's and Isik's proofs, and we consider relative singularity categories introduced in [EP], which are equivalent to derived factorization categories, and use results in [Orl2].

To state the next result, let S be a smooth quasi-projective variety over k with a $\mathbb{G}_m$-action, and let $W : S \to \mathbb{A}^1$ be a $\chi_1 := \mathrm{id}_{\mathbb{G}_m}$-semi invariant regular function which is flat. Let $d > 1$ and $N > 0$ be positive integers, and consider $\mathbb{G}_m$-actions on $\mathbb{A}^N_S := S \times \mathbb{A}^N_k$ and on $\mathbb{P}^{N-1}_S := S \times \mathbb{P}^{N-1}_k$ given by

$$\mathbb{G}_m \times \mathbb{A}^N_S \ni t \times (s, v_1, ..., v_N) \mapsto (t^d \cdot s, tv_1, ...tv_N) \in \mathbb{A}^N_S$$

$$\mathbb{G}_m \times \mathbb{P}^{N-1}_S \ni t \times (s, v_1 : ... : v_N) \mapsto (t \cdot s, v_1 : ... : v_N) \in \mathbb{P}^{N-1}_S.$$

Denote by the same notation $W : \mathbb{A}^N_S \to \mathbb{A}^1$ and $W : \mathbb{P}^{N-1}_S \to \mathbb{A}^1$ the pull-backs of $W : S \to \mathbb{A}^1$ by the natural projections respectively. Combining the above derived Knörrer periodicity with the theory of variations of GIT quotients, we obtain the following gauged LG version of the Orlov's theorem:

Theorem 1.7. *Let $X \subset \mathbb{P}^{N-1}_S$ be the hypersurface defined by a $\mathbb{G}_m$-invariant section $f \in \Gamma(\mathbb{P}^{N-1}_S, \mathcal{O}(d))^{\mathbb{G}_m}$, and assume that the morphism $W : \mathbb{P}^{N-1}_S \to \mathbb{A}^1$ is flat on X. Denote by $F : \mathbb{A}^N_S \to \mathbb{A}^1$ the regular function induced by f.*
(1) If $d < N$, there are fully faithful functors

$$\Phi : \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F) \to \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W)$$

$$\Upsilon : \mathrm{Dcoh}_{\mathbb{G}_m}(S, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W),$$

and there is a semi-orthogonal decomposition

$$\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) = \langle \Upsilon_{d-N+1}, ..., \Upsilon_0, \Phi(\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F))\rangle,$$

where Υ_i denotes the essential image of the composition $(-) \otimes \mathcal{O}(i) \circ \Upsilon$.

(2) If $d = N$, there is an equivalence

$$\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F).$$

(3) If $d > N$, there are fully faithful functors

$$\Psi : \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F)$$

$$\Upsilon : \mathrm{Dcoh}_{\mathbb{G}_m}(S, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F),$$

and there is a semi-orthogonal decomposition

$$\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F) = \langle \Upsilon_0, ..., \Upsilon_{N-d+1}, \Psi(\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W))\rangle,$$

where Υ_i denotes the essential image of the composition $(-) \otimes \mathcal{O}(\chi_i) \circ \Upsilon$.

Since we have an equivalence

$$\mathrm{D^b}(\mathrm{coh}X) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, 0),$$

where the $\mathbb{G}_m$-action on X is trivial, we can view Orlov's theorem as the case when $S = \mathrm{Spec}k$ and $W = 0$ in the above theorem.

1.3. **Notation and conventions.**

- For an integer $n \in \mathbb{Z}$, we denote by $\chi_n : \mathbb{G}_m \to \mathbb{G}_m$ the character of the 1-dimensional algebraic torus $\mathbb{G}_m$ defined by
$$\chi_n(t) := t^n.$$
- For a character $\chi : G \to \mathbb{G}_m$ of an algebraic group G, we denote by $\mathcal{O}(\chi)$ the G-equivariant invertible sheaf $(\mathcal{O}_X, \lambda : \pi^*\mathcal{O}_X \xrightarrow{\sim} \sigma^*\mathcal{O}_X)$ associated to χ, where $\pi : G \times X \to X$ and $\sigma : G \times X \to X$ are the projection and the morphism defining the G-action respectively. For any $g \in G$, $\lambda_g := \lambda|_{\{g\}\times X} : \mathcal{O}_X \xrightarrow{\sim} g^*\mathcal{O}_X$ is given as the composition
$$\mathcal{O}_X \xrightarrow{\chi(g)} \mathcal{O}_X \xrightarrow{\sim} g^*\mathcal{O}_X$$
of the multiplication by $\chi(g) \in \mathbb{G}_m$ and the natural isomorphism $\mathcal{O}_X \xrightarrow{\sim} g^*\mathcal{O}_X$. For a G-equivariant quasi-coherent sheaf F on a G-scheme, we set
$$F(\chi) := F \otimes \mathcal{O}(\chi).$$
- Throughout this article, unless stated otherwise, all schemes and categories are over an algebraically closed field k of characteristic zero.

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2. Comodules over comonads

Categories of comodules over comonads are main tools to obtain the main result. In this section, we recall the definitions of comonads and comodules over comonads, and provide basic properties of them, following [Ela2].

2.1. **Comodules over comonads.**

Let $\mathcal{C}$ be a category. We start by recalling the definitions of comonads on $\mathcal{C}$ and comodules over a comonad.

Definition 2.1. A **comonad** $\mathbb{T} = (T, \varepsilon, \delta)$ on the category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ and functor morphisms $\varepsilon : T \to \mathrm{id}_{\mathcal{C}}$ and $\delta : T \to T^2$ such that the following diagrams are commutative:

$$\begin{tikzcd} T \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, equal, "\mathrm{id}_T"] & T^2 \arrow[d, "T\varepsilon"] \\ T^2 \arrow[r, "\varepsilon T"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\delta"] \arrow[d, "\delta"'] & T^2 \arrow[d, "T\delta"] \\ T^2 \arrow[r, "\delta T"'] & T^3 \end{tikzcd}$$

Example 2.2. *Let $P = (P^* \dashv P_*)$ be an adjoint pair of functors $P^* : \mathcal{C} \to \mathcal{D}$ and $P_* : \mathcal{D} \to \mathcal{C}$, and let $\eta_P : \mathrm{id}_{\mathcal{C}} \to P_*P^*$ and $\varepsilon_P : P^*P_* \to \mathrm{id}_{\mathcal{D}}$ be the adjunction morphisms. Set $T_P := P^*P_*$ and $\delta_P := P^*\eta_P P_*$. Then $\mathbb{T}(P) := (T_P, \varepsilon_P, \delta_P)$ is a comonad on $\mathcal{D}$.*

Definition 2.3. Let $\mathbb{T} = (T, \varepsilon, \delta)$ be a comonad on $\mathcal{C}$. A **comodule** over $\mathbb{T}$ is a pair (C, θ_C) of an object $C \in \mathcal{C}$ and a morphism $\theta_C : C \to T(C)$ such that
(1) $\varepsilon(C) \circ \theta_C = \mathrm{id}_C$
(2) the following diagram is commutative:

$$\begin{tikzcd} C \arrow[r, "\theta_C"] \arrow[d, "\theta_C"'] & T(C) \arrow[d, "T(\theta_C)"] \\ T(C) \arrow[r, "\delta(C)"] & T^2(C). \end{tikzcd}$$

Given a comonad $\mathbb{T}$ on $\mathcal{C}$, we define the category $\mathcal{C}_{\mathbb{T}}$ of comodules over the comonad $\mathbb{T}$ as follows:

Definition 2.4. Let $\mathbb{T} = (T, \varepsilon, \delta)$ be a comonad on $\mathcal{C}$. The category $\mathcal{C}_{\mathbb{T}}$ of comodules over $\mathbb{T}$ on $\mathcal{C}$ is the category whose objects are comodules over $\mathbb{T}$ and whose sets of morphisms are defined as follows;

$$\mathrm{Hom}((C_1, \theta_{C_1}), (C_2, \theta_{C_2})) := \{f : C_1 \to C_2 \mid T(f) \circ \theta_{C_1} = \theta_{C_2} \circ f\}.$$

For a full subcategory $\mathcal{B} \subset \mathcal{C}$, we define the full subcategory $\mathcal{C}_{\mathbb{T}}^{\mathcal{B}} \subset \mathcal{C}_{\mathbb{T}}$ as

$$\mathrm{Ob}(\mathcal{C}_{\mathbb{T}}^{\mathcal{B}}) := \{(C, \theta_C) \in \mathrm{Ob}(\mathcal{C}_{\mathbb{T}}) \mid C \cong B \text{ for some } B \in \mathcal{B}\}.$$

Remark 2.5. Let $(C, \theta_C) \in \mathcal{C}_{\mathbb{T}}^{\mathcal{B}}$. By definition, there exist an object $B \in \mathcal{B}$ and an isomorphism $\varphi : C \xrightarrow{\sim} B$. If we set $\theta_B := T(\varphi)\theta_C\varphi^{-1}$, then the pair (B, θ_B) is an object of $\mathcal{C}_{\mathbb{T}}^{\mathcal{B}}$ and φ gives an isomorphism from (C, θ_C) to (B, θ_B) in $\mathcal{C}_{\mathbb{T}}^{\mathcal{B}}$.

For a comonad which is given by an adjoint pair $(P^* \dashv P_*)$, we have a canonical functor, called comparison functor, from the domain of P^* to the category of comodules over the comonad.

Definition 2.6. The notation is the same as in Example 2.2. For an adjoint pair $P = (P^* \dashv P_*)$, we define a functor

$$\Gamma_P : \mathcal{C} \to \mathcal{D}_{\mathbb{T}(P)}$$

as follows: For any $C \in \mathcal{C}$ and for any morphism f in $\mathcal{C}$

$$\Gamma_P(C) := (P^*(C), P^*(\eta_P(C))) \quad \text{and} \quad \Gamma_P(f) := P^*(f).$$

This functor is called **the comparison functor** of P. Restricting Γ_P to a full subcategory $\mathcal{B} \subset \mathcal{C}$, we have a restricted functor

$$\Gamma_P|_{\mathcal{B}} : \mathcal{B} \to \mathcal{D}_{\mathbb{T}(P)}^{P^*(\mathcal{B})}.$$

The following proposition gives sufficient conditions for a comparison functor to be fully faithful or an equivalence.

Proposition 2.7 ([Ela1] Theorem 3.9, Corollary 3.11)**.** *The notation is the same as in* Example 2.2.

(1) *If for any $C \in \mathcal{C}$, the morphism $\eta_P(C) : C \to P_*P^*(C)$ is a split mono, i.e. there is a morphism $\zeta_C : P_*P^*(C) \to C$ such that $\zeta \circ \eta_P(C) = \mathrm{id}_C$, then the comparison functor $\Gamma_P : \mathcal{C} \to \mathcal{D}_{\mathbb{T}(P)}$ is fully faithful.*
(2) *If $\mathcal{C}$ is idempotent complete and the functor morphism $\eta_P : \mathrm{id}_{\mathcal{C}} \to P_*P^*$ is split mono, i.e. there exists a functor morphism $\zeta : P_*P^* \to \mathrm{id}_{\mathcal{C}}$ such that $\zeta \circ \eta = \mathrm{id}$, then $\Gamma_P : \mathcal{C} \to \mathcal{D}_{\mathbb{T}(P)}$ is an equivalence.*

2.2. **Functors between categories of comodules.** We introduce the notion of linearizable functors which induce natural functors between categories of comodules. Let $\mathcal{A}$ (resp. $\mathcal{B}$ and $\mathcal{C}$) be a category and let $\mathbb{T}_{\mathcal{A}} = (T_{\mathcal{A}}, \varepsilon_{\mathcal{A}}, \delta_{\mathcal{A}})$ (resp. $\mathbb{T}_{\mathcal{B}} = (T_{\mathcal{B}}, \varepsilon_{\mathcal{B}}, \delta_{\mathcal{B}})$ and $\mathbb{T}_{\mathcal{C}} = (T_{\mathcal{C}}, \varepsilon_{\mathcal{C}}, \delta_{\mathcal{C}})$) be a comonad on $\mathcal{A}$ (resp. $\mathcal{B}$ and $\mathcal{C}$).

Definition 2.8. A functor $F : \mathcal{A} \to \mathcal{B}$ is called **linearizable** with respect to $\mathbb{T}_{\mathcal{A}}$ and $\mathbb{T}_{\mathcal{B}}$, or just **linearizable**, if there exists an isomorphism of functors

$$\Omega : FT_{\mathcal{A}} \xrightarrow{\sim} T_{\mathcal{B}}F$$

such that the following two diagrams of functor morphisms are commutative :

(1)
$$\begin{tikzcd} FT_{\mathcal{A}} \arrow[rr, "\Omega"] \arrow[dr, "F\varepsilon_{\mathcal{A}}"'] & & T_{\mathcal{B}}F \arrow[dl, "\varepsilon_{\mathcal{B}}F"] \\ & F & \end{tikzcd}$$

(2)
$$\begin{tikzcd} FT_{\mathcal{A}} \arrow[r, "\Omega"] \arrow[d, "F\delta_{\mathcal{A}}"'] & T_{\mathcal{B}}F \arrow[d, "\delta_{\mathcal{B}}F"] \\ FT_{\mathcal{A}}^2 \arrow[r, "T_{\mathcal{B}}\Omega \circ \Omega T_{\mathcal{A}}"] & T_{\mathcal{B}}^2F \end{tikzcd}$$

We call the pair (F, Ω) a **linearized functor** with respect to $\mathbb{T}_{\mathcal{A}}$ and $\mathbb{T}_{\mathcal{B}}$, and the isomorphism of functors Ω is called a **linearization** of F with respect to $\mathbb{T}_{\mathcal{A}}$ and $\mathbb{T}_{\mathcal{B}}$.

If $F : \mathcal{A} \to \mathcal{B}$ is a linearizable functor with a linearization $\Omega : FT_{\mathcal{A}} \xrightarrow{\sim} T_{\mathcal{B}}F$, we have an induced functor

$$F_{\Omega} : \mathcal{A}_{\mathbb{T}_{\mathcal{A}}} \to \mathcal{B}_{\mathbb{T}_{\mathcal{B}}}$$

defined by

$$F_{\Omega}(A, \theta_A) := (F(A), \Omega(A) \circ F(\theta_A)) \quad \text{and} \quad F_{\Omega}(f) := F(f).$$

Lemma 2.9. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be linearizable functors with linearizations $\Phi : FT_{\mathcal{A}} \xrightarrow{\sim} T_{\mathcal{B}}F$ and $\Psi : GT_{\mathcal{B}} \xrightarrow{\sim} T_{\mathcal{C}}G$ respectively. Then the composition GF is a linearizable functor with linearization $\Omega := \Psi F \circ G\Phi$ and $(GF)_{\Omega} = G_{\Psi}F_{\Phi}$.*

Proof. By definition it is sufficient to prove that

$$GF\varepsilon_{\mathcal{A}} = \varepsilon_{\mathcal{C}}GF \circ \Omega \quad \text{and} \quad T_{\mathcal{C}}\Omega \circ \Omega T_{\mathcal{A}} \circ GF\delta_{\mathcal{A}} = \delta_{\mathcal{C}}GF \circ \Omega.$$

The former one of the above equations follows from easy diagram chasing as follows.

$$GF\varepsilon_{\mathcal{A}} = G(\varepsilon_{\mathcal{B}}F \circ \Phi) = G\varepsilon_{\mathcal{B}}F \circ G\Phi = (\varepsilon_{\mathcal{C}}G \circ \Psi)F \circ G\Phi = \varepsilon_{\mathcal{C}}GF \circ \Omega,$$

where the first and third equations follow from the commutativity of the diagrams corresponding to (1) in Definition 2.8. The latter one is verified as follows;

$$\begin{aligned} & T_{\mathcal{C}}\Omega \circ \Omega T_{\mathcal{A}} \circ GF\delta_{\mathcal{A}} \\ &= T_{\mathcal{C}}\Psi F \circ T_{\mathcal{C}}G\Phi \circ \Psi FT_{\mathcal{A}} \circ G\Phi T_{\mathcal{A}} \circ GF\delta_{\mathcal{A}} \\ &= T_{\mathcal{C}}\Psi F \circ \Psi T_{\mathcal{B}}F \circ GT_{\mathcal{B}}\Phi \circ G\Phi T_{\mathcal{A}} \circ GF\delta_{\mathcal{A}} \\ &= T_{\mathcal{C}}\Psi F \circ \Psi T_{\mathcal{B}}F \circ G(T_{\mathcal{B}}\Phi \circ \Phi T_{\mathcal{A}} \circ F\delta_{\mathcal{A}}) \\ &= T_{\mathcal{C}}\Psi F \circ \Psi T_{\mathcal{B}}F \circ G(\delta_{\mathcal{B}}F \circ \Phi) \\ &= (T_{\mathcal{C}}\Psi \circ \Psi T_{\mathcal{B}} \circ G\delta_{\mathcal{B}})F \circ G\Phi \\ &= (\delta_{\mathcal{C}}G \circ \Psi)F \circ G\Phi \\ &= \delta_{\mathcal{C}}GF \circ \Omega, \end{aligned}$$

where the second equation follows from the functoriality of Ψ, and the fourth and the sixth equations follow from the commutativity of the diagrams corresponding to (2) in Definition 2.8. $\square$

The next proposition gives a sufficient condition for a restriction of the functor F_Ω associated with a linearized functor (F,Ω) to be fully faithful or an equivalence.

Proposition 2.10. *Let $F : \mathcal{A} \to \mathcal{B}$ be a linearizable functor with a linearization $\Omega : F \circ T_{\mathcal{A}} \xrightarrow{\sim} T_{\mathcal{B}} \circ F$. Let $\mathcal{C} \subset \mathcal{A}$ be a full subcategory of $\mathcal{A}$ and let $\mathcal{D} \subset \mathcal{B}$ be a full subcategory of $\mathcal{B}$ containing $F(\mathcal{C})$. Assume the following condition:*

- $(*)$: $\mathrm{Hom}(C, T^n_{\mathcal{A}}(C')) \xrightarrow{F} \mathrm{Hom}(F(C), F(T^n_{\mathcal{A}}(C')))$ *is an isomorphism for any $C, C' \in \mathcal{C}$ and $n = 1, 2$.*

If $F|_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ is fully faithful (resp. an equivalence), then the functor

$$F_\Omega|_{\mathcal{A}^{\mathcal{C}}_{\mathbb{T}_{\mathcal{A}}}} : \mathcal{A}^{\mathcal{C}}_{\mathbb{T}_{\mathcal{A}}} \to \mathcal{B}^{\mathcal{D}}_{\mathbb{T}_{\mathcal{B}}}$$

is also fully faithful (resp. an equivalence).

Proof. Assume $F|_{\mathcal{C}}$ is fully faithful. At first we show that F_Ω is fully faithful on $\mathcal{A}^{\mathcal{C}}_{\mathbb{T}_{\mathcal{A}}}$.

Let $\widetilde{C} := (C, \theta_C)$ and $\widetilde{C'} := (C', \theta_{C'})$ be objects of $\mathcal{A}^{\mathcal{C}}_{\mathbb{T}_{\mathcal{A}}}$. By Remark 2.5, we may assume that C and C' are objects of $\mathcal{C}$. For $f, g \in \mathrm{Hom}(\widetilde{C}, \widetilde{C'}) \subset \mathrm{Hom}_{\mathcal{A}}(C, C')$, if $F_\Omega(f) = F_\Omega(g)$, then $F(f) = F(g)$ as morphisms in $\mathcal{B}$. Since F is fully faithful on $\mathcal{C}$, this implies that $f = g$ as morphisms in $\mathcal{A}$, whence $f = g$ in $\mathcal{A}_{\mathbb{T}_{\mathcal{A}}}$. Hence F_Ω is faithful.

Take any morphism $h \in \mathrm{Hom}(F_\Omega(\widetilde{C}), F_\Omega(\widetilde{C'}))$. Since F is full on $\mathcal{C}$, there exists a morphism $f \in \mathrm{Hom}(C, C')$ such that $F(f) = h$, and we have the following commutative diagram:

$$\begin{array}{ccccc}
F(C) & \xrightarrow{F(\theta_C)} & F(T_{\mathcal{A}}(C)) & \xrightarrow{\Omega(C)} & T_{\mathcal{B}}(F(C)) \\
{\scriptstyle F(f)}\downarrow & & & & \downarrow{\scriptstyle T_{\mathcal{B}}(F(f))} \\
F(C') & \xrightarrow{F(\theta_{C'})} & F(T_{\mathcal{A}}(C')) & \xrightarrow{\Omega(C')} & T_{\mathcal{B}}(F(C'))
\end{array}$$

By the functoriality of Ω, the following diagram is commutative:

$$\begin{array}{ccc}
F(T_{\mathcal{A}}(C)) & \xrightarrow{\Omega(C)} & T_{\mathcal{B}}(F(C)) \\
{\scriptstyle F(T_{\mathcal{A}}(f))}\downarrow & & \downarrow{\scriptstyle T_{\mathcal{B}}(F(f))} \\
F(T_{\mathcal{A}}(C')) & \xrightarrow{\Omega(C')} & T_{\mathcal{B}}(F(C'))
\end{array}$$

Combining commutativity of the above diagrams, we have

$$F(T_{\mathcal{A}}(f) \circ \theta_C) = F(\theta'_C \circ f)$$

since $\Omega(C')$ is an isomorphism. By the condition $(*)$ in the assumption, we see that $T_{\mathcal{A}}(f) \circ \theta_C = \theta'_C \circ f$, which implies that f is a morphism in $\mathcal{A}^{\mathcal{C}}_{\mathbb{T}_{\mathcal{A}}}$. Hence F_Ω is full.

Assume $F|_{\mathcal{C}}$ is an equivalence. We verify that the functor $F_\Omega|_{\mathcal{A}^{\mathcal{C}}_{\mathbb{T}_{\mathcal{A}}}} : \mathcal{A}^{\mathcal{C}}_{\mathbb{T}_{\mathcal{A}}} \to \mathcal{B}^{\mathcal{D}}_{\mathbb{T}_{\mathcal{B}}}$ is essentially surjective. Since $F|_{\mathcal{C}}$ is an equivalence, it is sufficient to prove that for any object $(B, \theta_B) \in \mathcal{B}^{\mathcal{D}}_{\mathbb{T}_{\mathcal{B}}}$ with $B = F(C)$ for some $C \in \mathcal{C}$, there exists an object $(C, \theta_C) \in \mathcal{A}^{\mathcal{C}}_{\mathbb{T}_{\mathcal{A}}}$ such that $F_\Omega(C, \theta_C) = (B, \theta_B)$. By the condition $(*)$, we know that there exists a morphism $\theta_C : C \to T_{\mathcal{A}}(C)$ such that $F(\theta_C) = \Omega(C)^{-1} \circ \theta_{F(C)} : F(C) \to F(T_{\mathcal{A}}(C))$. To show that the pair (C, θ_C) is an object of $\mathcal{A}_{\mathbb{T}_{\mathcal{A}}}$, we check two conditions in Definition 2.3. Considering the following commutative diagram;

$$
\begin{tikzcd}
F(C) \arrow[rr, "F(\theta_C)"] \arrow[dr, "\theta_{F(C)}"] \arrow[ddr, equal] & & F(T_{\mathcal{A}}(C)) \arrow[dl, "\Omega(C)"'] \arrow[ddl, "F(\varepsilon_{\mathcal{A}}(C))"] \\
& T_{\mathcal{B}}(F(C)) \arrow[d, "\varepsilon_{\mathcal{B}}(F(C))"] & \\
& F(C) &
\end{tikzcd}
$$

we see that $F(\varepsilon_{\mathcal{A}}(C) \circ \theta_C) = \mathrm{id}_{F(C)}$. Since $F|_{\mathcal{C}}$ is fully faithful, we obtain

$$\varepsilon_{\mathcal{A}}(C) \circ \theta_C = \mathrm{id}_C,$$

which is the first condition in Definition 2.3. By the following commutative diagram;

$$
\begin{tikzcd}
& & & & F(T_{\mathcal{A}}(C)) \arrow[dl, "\Omega(C)"] \arrow[ddd, "F(T_{\mathcal{A}}(\theta_C))"] \\
& F(C) \arrow[urrr, "F(\theta_C)"] \arrow[rr, "\theta_{F(C)}"'] \arrow[d, "\theta_{F(C)}"] \arrow[ddl, "F(\theta_C)"'] & & T_{\mathcal{B}}(F(C)) \arrow[d, "T_{\mathcal{B}}(\theta_{F(C)})"'] & \\
& T_{\mathcal{B}}(F(C)) \arrow[rr, "\delta_{\mathcal{B}}(F(C))"'] & & T^2_{\mathcal{B}}(F(C)) & \\
F(T_{\mathcal{A}}(C)) \arrow[ur, "\Omega(C)"'] \arrow[rrrr, "F(\delta_{\mathcal{A}}(C))"'] & & & & F(T^2_{\mathcal{A}}(C)), \arrow[ul, "T_{\mathcal{B}}(\Omega(C))\circ\Omega(T_{\mathcal{A}}(C))"']
\end{tikzcd}
$$

we see that $F(\delta_{\mathcal{A}}(C) \circ \theta_C) = F(T_{\mathcal{A}}(\theta_C) \circ \theta_C)$. By the condition $(*)$, we obtain

$$\delta_{\mathcal{A}}(C) \circ \theta_C = T_{\mathcal{A}}(\theta_C) \circ \theta_C,$$

which is the second condition in Definition 2.3. Hence, the pair (C, θ_C) is a comodule over $\mathbb{T}_{\mathcal{A}}$, and we see that $F_\Omega(C, \theta_C) = (F(C), \theta_{F(C)})$ by the construction of (C, θ_C). $\square$

The following lemma gives a useful criteria for a functor to be linearizable with respect to comonads which are constructed from "compatible" adjoint pairs.

Lemma 2.11. *Assume that we have the following diagram of functors;*

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, bend left, "P_*"] & \mathcal{B} \arrow[d, bend left, "Q_*"] \\
\mathcal{A}' \arrow[r, "F'"] \arrow[u, bend left, "P^*"] & \mathcal{B}' \arrow[u, bend left, "Q^*"]
\end{tikzcd}
$$

where $P := (P^* \dashv P_*)$ and $Q := (Q^* \dashv Q_*)$ are adjoint pairs. Assume that we have two isomorphisms of functors $\Omega^* : FP^* \xrightarrow{\sim} Q^*F'$ and $\Omega_* : F'P_* \xrightarrow{\sim} Q_*F$. Let $\Omega : FT_P \xrightarrow{\sim} T_QF$ be the composition of functor morphisms $Q^*\Omega_* \circ \Omega^* P_*$. Consider the following two diagrams *of functor morphisms:*

(i)
$$\begin{array}{ccc} Q^*F'P_* & \xrightarrow{Q^*\Omega_*} & Q^*Q_*F \\ {\scriptstyle \Omega^*P_*}\uparrow & & \downarrow{\scriptstyle \varepsilon_Q F} \\ FP^*P_* & \xrightarrow{F\varepsilon_P} & F \end{array}$$

(ii)
$$\begin{array}{ccc} F' & \xrightarrow{\eta_Q F'} & Q_*Q^*F' \\ {\scriptstyle F'\eta_P}\downarrow & & \uparrow{\scriptstyle Q_*\Omega^*} \\ F'P_*P^* & \xrightarrow{\Omega_*P^*} & Q_*FP^* \end{array}$$

If the above two diagrams are commutative, then (F, Ω) *is a linearized functor with respect to* $\mathbb{T}(P)$ *and* $\mathbb{T}(Q)$*, and there exists an isomorphism of functors* $\Sigma : F_\Omega\Gamma_P \xrightarrow{\sim} \Gamma_Q F'$.

Proof. We verify that the diagrams corresponding to ones in Definition 2.8 are commutative. The commutativity of (i) immediately implies the commutativity of the diagram corresponding to (1) in Definition 2.8. We show that if the diagram of (ii) is commutative, the diagram corresponding to (2) in Definition 2.8 is commutative. By the functoriality of Ω^* and η_Q, the following diagrams of functor morphisms are commutative;

$$\begin{array}{ccc} FP^*P_* & \xrightarrow{\Omega^*P_*} & Q^*F'P_* \\ {\scriptstyle FP^*\eta_P P_*}\downarrow & & \downarrow{\scriptstyle Q^*F'\eta_P P_*} \\ FP^*P_*P^*P_* & \xrightarrow{\Omega^*P_*P^*P_*} & Q^*F'P_*P^*P_* \end{array}$$

and

$$\begin{array}{ccc} F'P_* & \xrightarrow{\eta_{P'}F'P_*} & Q_*Q^*F'P_* \\ {\scriptstyle \Omega_*}\downarrow & & \downarrow{\scriptstyle Q_*Q^*\Omega_*} \\ Q_*F(B) & \xrightarrow{\eta_Q Q_* F} & Q_*Q^*Q_*F. \end{array}$$

Hence, we have equations of functor morphisms

$$(a): \quad \Omega^*P_*P^*P_* \circ FP^*\eta_P P_* = Q^*F'\eta_P P_* \circ \Omega^*P_*$$

and

$$(b): \quad \eta_Q Q_* F \circ \Omega_* = Q_*Q^*\Omega_* \circ \eta_Q F'P_*.$$

We see that the diagram corresponding to (2) in Definition 2.8 is commutative as follows;

$$\begin{aligned}
& T_Q\Omega \circ \Omega T_P \circ F\delta_P \\
& = T_Q(Q^*\Omega_* \circ \Omega^*P_*) \circ (Q^*\Omega_* \circ \Omega^*P_*)T_P \circ F\delta_P \\
& = Q^*Q_*(Q^*\Omega_* \circ \Omega^*P_*) \circ (Q^*\Omega_* \circ \Omega^*P_*)P^*P_* \circ FP^*\eta_P P_* \\
& = Q^*Q_*Q^*\Omega_* \circ Q^*Q_*\Omega^*P_* \circ Q^*\Omega_*P^*P_* \circ (\Omega^*P_*P^*P_* \circ FP^*\eta_P P_*) \\
(a) \to & = Q^*Q_*Q^*\Omega_* \circ Q^*Q_*\Omega^*P_* \circ Q^*\Omega_*P^*P_* \circ (Q^*F'\eta_P P_* \circ \Omega^*P_*) \\
& = Q^*Q_*Q^*\Omega_* \circ (Q^*Q_*\Omega^*P_* \circ Q^*\Omega_*P^*P_* \circ Q^*F'\eta_P P_*) \circ \Omega^*P_* \\
& = Q^*Q_*Q^*\Omega_* \circ Q^*(Q_*\Omega^* \circ \Omega_*P^* \circ F'\eta_P)P_* \circ \Omega^*P_* \\
(\mathrm{ii}) \to & = Q^*Q_*Q^*\Omega_* \circ Q^*\eta_Q F'P_* \circ \Omega^*P_* \\
& = Q^*(Q_*Q^*\Omega_* \circ \eta_Q F'P_*) \circ \Omega^*P_* \\
(b) \to & = Q^*(\eta_Q Q_*F \circ \Omega_*) \circ \Omega^*P_* \\
& = Q^*\eta_Q Q_*F \circ Q^*\Omega_* \circ \Omega^*P_* \\
& = \delta_Q F \circ \Omega,
\end{aligned}$$

where the fourth, seventh and ninth equations follow from the above equation (a), the commutativity of (ii) and the above equation (b) respectively. Hence (F, Ω) is a linearized functor.

For any $A \in \mathcal{A}'$, let $\Sigma(A) := \Omega^*(A)$. By constructions, we have $F_\Omega \Gamma_P(A) = (FP^*(A), \Omega P^*(A) \circ FP^*\eta_P(A))$ and $\Gamma_Q F'(A) = (Q^*F'(A), Q^*\eta_Q F'(A))$. We show that $\Sigma(-)$ defines a functor morphism $\Sigma : F_\Omega \Gamma_P \to \Gamma_Q F'$. So we have to verify that $\Omega^*(A)$ is a morphism in $\mathcal{B}_{\mathbb{T}(Q)}$ for each $A \in \mathcal{A}'$, i.e., verify the following diagram is commutative:

$$\begin{tikzcd}
FP^*(A) \arrow[r, "\Omega P^*(A) \circ FP^*\eta_P(A)"] \arrow[d, "\Omega^*(A)"'] & T_Q(FP^*(A)) \arrow[d, "T_Q(\Omega^*(A))"] \\
Q^*F'(A) \arrow[r, "Q^*\eta_Q F'(A)"] & T_Q(Q^*F'(A))
\end{tikzcd}$$

By the functoriality of Ω^* and the commutativity of (ii), we see that the above diagram is commutative as follows:

$$\begin{aligned}
& T_Q(\Omega^*(A)) \circ \Omega P^*(A) \circ FP^*\eta_P(A) \\
&= Q^*Q_*\Omega^*(A) \circ Q^*\Omega_* P^*(A) \circ \Omega^* P_* P^*(A) \circ FP^*\eta_P(A) \\
&= Q^*Q_*\Omega^*(A) \circ Q^*\Omega_* P^*(A) \circ \{\Omega^*(P_*P^*(A)) \circ FP^*(\eta_P(A))\} \\
\text{functoriality of } \Omega^* \to &= Q^*Q_*\Omega^*(A) \circ Q^*\Omega_* P^*(A) \circ \{Q^*F'(\eta_P(A)) \circ \Omega^*(A)\} \\
&= Q^*\{Q_*\Omega^*(A) \circ \Omega_* P^*(A) \circ F'(\eta_P(A))\} \circ \Omega^*(A) \\
\text{(ii)} \to &= Q^*\eta_Q F'(A) \circ \Omega^*(A).
\end{aligned}$$

Hence $\Sigma(-)$ defines a functor morphism, and it is an isomorphism. $\square$

In the following, we give an important lemma to prove the main theorem. Notation is same as the above lemma. Let $G : \mathcal{B} \to \mathcal{A}$ and $G' : \mathcal{B}' \to \mathcal{A}'$ be functors. Let $\mathcal{C} \subset \mathcal{A}$, $\mathcal{D} \subset \mathcal{B}$, $\mathcal{C}' \subset \mathcal{A}'$ and $\mathcal{D}' \subset \mathcal{B}'$ be full subcategories with $F(\mathcal{A}) \subset \mathcal{D}$, $G(\mathcal{D}) \subset \mathcal{C}$, $P^*(\mathcal{C}') \subset \mathcal{C}$ and $Q^*(\mathcal{D}') \subset \mathcal{D}$. Now we have the following diagram of functors;

$$\begin{tikzcd}
\mathcal{C} \arrow[rrr, bend left=10, "F|_{\mathcal{C}}"] \arrow[dr, hook] & & & \mathcal{D} \arrow[lll, bend left=10, "G|_{\mathcal{D}}"] \arrow[dl, hook'] \\
& \mathcal{A} \arrow[r, bend left, "F"] \arrow[d, bend left, "P_*"] & \mathcal{B} \arrow[l, bend left, "G"] \arrow[d, bend left, "Q_*"] & \\
& \mathcal{A}' \arrow[r, bend left, "F'"] \arrow[u, bend left, "P^*"] & \mathcal{B}' \arrow[l, bend left, "G'"] \arrow[u, bend left, "Q^*"] & \\
\mathcal{C}' \arrow[uuu, "P^*|_{\mathcal{C}'}"] \arrow[ur, hook] \arrow[rrr, bend left=10, "F'|_{\mathcal{C}'}"] & & & \mathcal{D}' \arrow[uuu, "Q^*|_{\mathcal{D}'}"'] \arrow[ul, hook'] \arrow[lll, bend left=10, "G'|_{\mathcal{D}'}"]
\end{tikzcd}$$

Let $\Omega_F^* : FP^* \xrightarrow{\sim} Q^*F'$, $\Omega_{F*} : F'P_* \xrightarrow{\sim} Q_*F$, $\Omega_G^* : GQ^* \xrightarrow{\sim} P^*G'$ and $\Omega_{G*} : G'Q_* \xrightarrow{\sim} P_*G$ be isomorphisms of functors such that the diagrams corresponding to (i) and (ii) in Lemma 2.11, namely the following diagrams, are commutative.

$$\begin{tikzcd}
Q^*F'P_* \arrow[r, "Q^*\Omega_*"] & Q^*Q_*F \arrow[d, "\varepsilon_Q F"] \\
FP_*P^* \arrow[u, "\Omega^*P_*"] \arrow[r, "F\varepsilon_P"] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
F' \arrow[r, "\eta_Q F'"] \arrow[d, "F'\eta_P"'] & Q^*Q_*F' \\
F'P_*P^* \arrow[r, "\Omega_* P^*"] & Q_*FP^*, \arrow[u, "Q_*\Omega^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
P^*G'Q_* \arrow[r, "P^*\Omega_*"] & P^*P_*G \arrow[d, "\varepsilon_P G"] \\
GQ_*Q^* \arrow[u, "\Omega^*Q_*"] \arrow[r, "G\varepsilon_Q"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G' \arrow[r, "\eta_P G'"] \arrow[d, "G'\eta_Q"'] & P^*P_*G' \\
G'Q_*Q^* \arrow[r, "\Omega_*Q^*"] & P_*GQ^*, \arrow[u, "P_*\Omega^*"']
\end{tikzcd}$$

Set $\Omega_F := Q^*\Omega_{F*} \circ \Omega^*_F P_*$ and $\Omega_G := P^*\Omega_{G*} \circ \Omega^*_G Q_*$.

Lemma 2.12. *Notation is same as above. Assume that the adjunction morphisms $\eta_P : \mathrm{id} \to P_*P^*$ and $\eta_Q : \mathrm{id} \to Q_*Q^*$ are split mono, and for any $D \in \mathcal{D}$ and $A \in \mathcal{A}$ there is a natural isomorphism*

$$\Sigma(D, A) : \mathrm{Hom}_{\mathcal{B}}(D, F(A)) \cong \mathrm{Hom}_{\mathcal{A}}(G(D), A)$$

which is functorial in D and A. Then, if $F|_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ is fully faithful, so is $F'|_{\mathcal{C}'} : \mathcal{C}' \to \mathcal{D}'$. Moreover, if $F|_{\mathcal{C}}$ and $G'|_{\mathcal{D}'}$ are fully faithful and the following diagram $(\star)$ of functor morphisms is commutative, $F'|_{\mathcal{C}'}$ is an equivalence. Define a diagram by

$$(\star) : \quad \begin{tikzcd}
GFP^*|_{\mathcal{C}'} \arrow[r, "GFP^*\eta_P"] \arrow[d, "\omega P^*"'] & GFP^*P_*P^*|_{\mathcal{C}'} \arrow[r, "G\Omega_F P^*"] & GQ^*Q_*FP^*|_{\mathcal{C}'} \arrow[r, "\Omega_G FP^*"] & P^*P_*GFP^*|_{\mathcal{C}'} \arrow[d, "P^*P_*\omega P^*"] \\
P^*|_{\mathcal{C}'} \arrow[rrr, "P^*\eta_P"] & & & P^*P_*P^*|_{\mathcal{C}'},
\end{tikzcd}$$

where $\omega : GF|_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ is the adjunction morphism of the adjoint pair $(G|_{\mathcal{D}} \dashv F|_{\mathcal{C}})$ given by $\Sigma(-, *)$.

Proof. By the assumption and Lemma 2.11, (F, Ω_F) and (G, Ω_G) are linearized functor, and we have the following commutative diagram of functors

$$\begin{tikzcd}
\mathcal{A}^{\mathcal{C}}_{\mathbb{T}(P)} \arrow[r, "F_{\Omega_F}|_{\mathcal{A}^{\mathcal{C}}_{\mathbb{T}(P)}}"] & \mathcal{B}^{\mathcal{D}}_{\mathbb{T}(Q)} \\
\mathcal{C}' \arrow[u, "\Gamma_P|_{\mathcal{C}'}"] \arrow[r, "F'|_{\mathcal{C}'}"] & \mathcal{D}'. \arrow[u, "\Gamma_Q|_{\mathcal{D}'}"']
\end{tikzcd}$$

Since the adjunction morphisms η_P and η_Q are split mono, the comparison functors $\Gamma_P : \mathcal{A}' \to \mathcal{A}_{\mathbb{T}(P)}$ and $\Gamma_Q : \mathcal{B}' \to \mathcal{B}_{\mathbb{T}(Q)}$ are fully faithful functors by Proposition 2.7.

We show that if $F|_{\mathcal{C}}$ is fully faithful, then the condition $(*)$ in Proposition 2.10 is satisfied, i.e. the map $F : \mathrm{Hom}(C_1, T^n_P(C_2)) \to \mathrm{Hom}(F(C_1), F(T^n_P(C_2)))$ is bijective for any $C_i \in \mathcal{C}$ and $n = 1, 2$. Consider the following commutative diagram of maps

$$\begin{tikzcd}
\mathrm{Hom}(C_1, T^n_P(C_2)) \arrow[rr, "(-)\circ\omega(C_1)"] \arrow[dr, "F"'] & & \mathrm{Hom}(G(F(C_1)), T^n_P(C_2)) \\
& \mathrm{Hom}(F(C_1), F(T^n_P(C_2))) \arrow[ur, "\Sigma(F(C_1), T^n_P(C_2))"'] &
\end{tikzcd}$$

Since $F|_{\mathcal{C}}$ is fully faithful, $\omega(C_1)$ is an isomorphism, whence the maps in the above diagram except for F are bijective. Hence, the condition $(*)$ in Proposition 2.10 is satisfied, and we see that if $F|_{\mathcal{C}}$ is fully faithful, then $F'|_{\mathcal{C}'}$ is also fully faithful by Proposition 2.10.

Assume that $F|_{\mathcal{C}}$ and $G'|_{\mathcal{D}'}$ are fully faithful and that the diagram $(\star)$ is commutative. Since the diagram $(\star)$ is commutative, the functor morphism $\omega P^*|_{\mathcal{C}'} : GF|_{\mathcal{C}}P^*|_{\mathcal{C}'} \to P^*|_{\mathcal{C}'}$ induces a functor morphism $\omega' : G_{\Omega_G} \circ F_{\Omega_F} \circ \Gamma_P|_{\mathcal{C}'} \xrightarrow{\sim} \Gamma_P|_{\mathcal{C}'}$. Since $F|_{\mathcal{C}}$ is fully faithful, ω' is an isomorphism of functors. Since we have $G_{\Omega_G} \circ F_{\Omega_F} \circ \Gamma_P|_{\mathcal{C}'} \cong \Gamma_P \circ G' \circ F'|_{\mathcal{C}'}$ and Γ_P is fully faithful, the functor isomorphism ω' implies an isomorphism of functors $G'F'|_{\mathcal{C}'} \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}'}$. Hence $G'|_{\mathcal{D}'} : \mathcal{D}' \to \mathcal{C}'$ is an equivalence, and therefore, $F'|_{\mathcal{C}'}$ is also an equivalence. $\square$

3. Derived factorization categories

In this section, we give definitions and foundations of categories with potentials, and construct derived factorization categories of them. We also construct functors between factorization categories from cwp-functors.

3.1. **Factorization categories.**

Let $\mathcal{A}$ be an exact category in the sense of Quillen (see [Qui]). First of all, we define potentials on $\mathcal{A}$.

Definition 3.1. A **potential** of $\mathcal{A}$ is a pair (Φ, W) of an exact equivalence $\Phi : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ and a functor morphism $W : \mathrm{id}_{\mathcal{A}} \to \Phi$ such that $\Phi W = W\Phi$. The triple $(\mathcal{A}, \Phi, W)$ is called a **category with a potential**.

Let (Φ, W) be a potential of $\mathcal{A}$. A **factorization** of (Φ, W) is a sequence in $\mathcal{A}$

$$A = \left(A_1 \xrightarrow{\varphi_1^A} A_0 \xrightarrow{\varphi_0^A} \Phi(A_1)\right)$$

such that $\varphi_0^A \circ \varphi_1^A = W(A_1)$ and $\Phi(\varphi_1^A) \circ \varphi_0^A = W(A_0)$. Objects A_1 and A_0 in the above sequence are called **components** of the factorization A.

Definition 3.2. For a category with a potential $(\mathcal{A}, \Phi, W)$, we define a dg-category $\mathfrak{F}(\mathcal{A}, \Phi, W)$, whose objects are factorizations of (Φ, W), as follows. For two factorizations $A, B \in \mathfrak{F}(\mathcal{A}, \Phi, W)$, the set of morphisms $\mathrm{Hom}(A, B)$ is a complex

$$\mathrm{Hom}(A, B) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(A, B)^n$$

with a differential d on $\mathrm{Hom}(A, B)$ given by

$$d(f) := \varphi^B \circ f - (-1)^{\deg(f)} f \circ \varphi^A \quad \text{if } f \in \mathrm{Hom}(A, B)^{\deg(f)},$$

where

$$\mathrm{Hom}(A, B)^{2n} := \mathrm{Hom}(A_1, \Phi^n(B_1)) \oplus \mathrm{Hom}(A_0, \Phi^n(B_0))$$
$$\mathrm{Hom}(A, B)^{2n+1} := \mathrm{Hom}(A_1, \Phi^n(B_0)) \oplus \mathrm{Hom}(A_0, \Phi^{n+1}(B_1)).$$

We call $\mathfrak{F}(\mathcal{A}, \Phi, W)$ the **factorization category** of $(\mathcal{A}, \Phi, W)$.

For any dg-category $\mathcal{D}$, we define two categories $Z^0(\mathcal{D})$ and $H^0(\mathcal{D})$ whose objects are same as $\mathcal{D}$ and whose morphisms are defined as follows;

$$\mathrm{Hom}_{Z^0(\mathcal{D})}(A, B) := Z^0(\mathrm{Hom}_{\mathcal{D}}(A, B))$$
$$\mathrm{Hom}_{H^0(\mathcal{D})}(A, B) := H^0(\mathrm{Hom}_{\mathcal{D}}(A, B)),$$

where $\mathrm{Hom}_{\mathcal{D}}(A, B)$ in the right hand sides are considered as complexes.

Remark 3.3. The categories $Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$ and $H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$ are generalizations of categories of classical matrix factorizations introduced by Eisenbud [Eis].

Let A, B be objects in $Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$. Then the set of morphisms from A to B can be described as follows:

$$\mathrm{Hom}_{Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))}(A, B) \cong \{(f_1, f_0) \mid f_i : A_i \to B_i \text{ and the diagram } (\star) \text{ is commutative.}\}$$

$$(\star): \quad \begin{tikzcd} A_1 \arrow[r, "\varphi_1^A"] \arrow[d, "f_1"'] & A_0 \arrow[r, "\varphi_0^A"] \arrow[d, "f_0"] & \Phi(A_1) \arrow[d, "\Phi(f_1)"] \\ B_1 \arrow[r, "\varphi_1^B"] & B_0 \arrow[r, "\varphi_0^B"] & \Phi(B_1) \end{tikzcd}$$

The set of morphisms in the category $H^0(\mathfrak{F}(\mathcal{A},\Phi,W))$ can be described as the set of homotopy equivalence classes of $\mathrm{Hom}_{Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))}(A,B)$;

$$\mathrm{Hom}_{H^0(\mathfrak{F}(\mathcal{A},\Phi,W))}(A,B) \cong \mathrm{Hom}_{Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))}(A,B)/\sim .$$

Two morphisms $f=(f_1,f_0)$ and $g=(g_1,g_0)$ in $\mathrm{Hom}_{Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))}(A,B)$ are **homotopy equivalence** if there exist morphisms

$$h_0 : A_0 \to B_1 \quad \text{and} \quad h_1 : \Phi(A_1) \to B_0$$

such that $f_0 = \varphi_1^B h_0 + h_1\varphi_0^A$ and $\Phi(f_1) = \varphi_0^B h_1 + \Phi(h_0)\Phi(\varphi_1^A)$.

Definition 3.4. For each $i=0,1$, we have a natural exact functor

$$(-)_i : Z^0(\mathfrak{F}(\mathcal{A},\Phi,W)) \to \mathcal{A}$$

defined by $(A_1 \xrightarrow{\varphi_1^A} A_0 \xrightarrow{\varphi_0^A} \Phi(A_1))_i := A_i$. This functor extends to an exact functor of their derived categories,

$$(-)_i : \mathrm{D^b}(Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))) \to \mathrm{D^b}(\mathcal{A}).$$

Proposition 3.5. *The category $Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))$ is an exact category. Furthermore, if $\mathcal{A}$ is abelian category, then $Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))$ is an abelian category.*

Proof. Assume that $\mathcal{A}$ is abelian category. At first, we show that $Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))$ is an abelian category. For any morphism $f=(f_1,f_0): A\to B$ in $Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))$, let

$$k_i : K_i \hookrightarrow A_i$$

be the kernel of $f_i : A_i \to B_i$ for each $i=0,1$. By the universal property of kernels, there exist morphisms $\varphi_1^K : K_1 \to K_0$ and $\varphi_0^K : K_0 \to \Phi(K_1)$ such that the following diagram is commutative:

$$\begin{tikzcd} K_1 \arrow[r, "\varphi_1^K"] \arrow[d, "k_1"'] & K_0 \arrow[r, "\varphi_0^K"] \arrow[d, "k_0"] & \Phi(K_1) \arrow[d, "\Phi(k_1)"] \\ A_1 \arrow[r, "\varphi_1^A"] & A_0 \arrow[r, "\varphi_0^A"] & \Phi(A_1) \end{tikzcd}$$

Since we have an equality $\Phi(k_1)\circ(\varphi_0^K\circ\varphi_1^K) = \Phi(k_1)\circ W(K_1)$, and $\Phi(k_1)$ is injective, we have $\varphi_0^K\circ\varphi_1^K = W(K_1)$. Similarly, we see that $\Phi(\varphi_1^K)\circ\varphi_0^K = W(K_0)$. Hence,

$$K := (K_1 \xrightarrow{\varphi_1^K} K_0 \xrightarrow{\varphi_0^K} \Phi(K_1))$$

is an object of $Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))$. Since, K_i is the kernel of f_i, K is the kernel of f. Similarly, we see that $Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))$ admits cokernel of any morphism, and we obtain a natural isomorphism $\mathrm{Im}(f)\cong\mathrm{Coim}(f)$. Hence, $Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))$ is an abelian category.

Next, we show that $Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))$ is an exact category. Let $\overline{\mathcal{A}}$ be the category of left exact functors from $\mathcal{A}^{\mathrm{op}}$ to the category of abelian groups (in a fixed universe containing $\mathcal{A}$). By [Qui], the category $\overline{\mathcal{A}}$ is an abelian category, and we have a fully faithful functor

$$h : \mathcal{A} \to \overline{\mathcal{A}},$$

such that h embeds $\mathcal{A}$ as a full subcategory of $\overline{\mathcal{A}}$ closed under extensions, and a sequence

$$A' \to A \to A''$$

in $\mathcal{A}$ is exact if and only if h carries it into an exact sequence in $\overline{\mathcal{A}}$ (the category $\overline{\mathcal{A}}$ is called *the abelian envelope* of $\mathcal{A}$). We define an exact autoequivalence $\overline{\Phi} : \overline{\mathcal{A}} \to \overline{\mathcal{A}}$ and a functor morphism $\overline{W} : \mathrm{id}_{\overline{\mathcal{A}}} \to \overline{\Phi}$ as follows: For an object $F\in\overline{\mathcal{A}}$, we define $\overline{\Phi}(F) := F\circ(\Phi^{\mathrm{op}})^{-1} \in$

$\overline{\mathcal{A}}$ and $\overline{W}(F) := FW^{\mathrm{op}}(\Phi^{\mathrm{op}})^{-1} : F \to F \circ (\Phi^{\mathrm{op}})^{-1}$ where $W^{\mathrm{op}}(\Phi^{\mathrm{op}})^{-1} : \mathrm{id}_{\mathcal{A}^{\mathrm{op}}} \to (\Phi^{\mathrm{op}})^{-1}$ is the composition

$$\mathrm{id}_{\mathcal{A}^{\mathrm{op}}} \xrightarrow{\sim} \Phi^{\mathrm{op}} \circ (\Phi^{\mathrm{op}})^{-1} \xrightarrow{W^{\mathrm{op}}(\Phi^{\mathrm{op}})^{-1}} (\Phi^{\mathrm{op}})^{-1}$$

Since the functor h is compatible with potentials, it induces a fully faithful functor

$$Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W)) \to Z^0(\mathfrak{F}(\overline{\mathcal{A}}, \overline{\Phi}, \overline{W})).$$

By this embedding, we obtain a natural structure of exact category on $Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$. $\square$

For an object $A \in Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$, we can construct a twisted-periodic infinite sequence $\mathrm{Com}(A) = (\mathrm{Com}(A)^\bullet, d_A^\bullet)$ in $\mathcal{A}$ with $d_A^{i+1} \circ d_A^i = W(\mathrm{Com}(A)^i)$ as follows;

$$\mathrm{Com}(A)^{2i} := \Phi^i(A_0), \qquad \mathrm{Com}(A)^{2i-1} := \Phi^i(A_1),$$
$$d_A^{2i} := \Phi^i(\phi_0^A), \qquad d_A^{2i-1} := \Phi^i(\phi_1^A).$$

For a morphism $f = (f_1, f_0) \in \mathrm{Hom}_{Z^0(\mathfrak{F}(\mathcal{A},\Phi,W))}(A, B) \subset \mathrm{Hom}(A_1, B_1) \oplus \mathrm{Hom}(A_0, B_0)$, we define a morphism $\mathrm{Com}(f) = (\mathrm{Com}(f)^\bullet)$ from $\mathrm{Com}(A)$ to $\mathrm{Com}(B)$ as follows:

$$\mathrm{Com}(f)^{2i} := \Phi^i(f_0) \qquad \mathrm{Com}(f)^{2i-1} := \Phi^i(f_1)$$

Definition 3.6. Let $C^\bullet = (\cdots \to C^i \xrightarrow{\delta^i_{C^\bullet}} C^{i+1} \to \cdots)$ be a bounded complex of $Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$. We define the **totalization** of $C^\bullet$ as an object $\mathrm{Tot}(C^\bullet) \in Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$ in a similar way to construct the total complex of a double complex, i.e.,

$$\mathrm{Tot}(C^\bullet) := (T_1 \xrightarrow{t_1} T_0 \xrightarrow{t_0} \Phi(T_1)),$$

where

$$T_l := \bigoplus_{i+j=-l} \mathrm{Com}(C^i)^j,$$

$$t_l|_{\mathrm{Com}(C^i)^j} := \mathrm{Com}(\delta^i_{C^\bullet})^j + (-1)^i d^j_{C^i}.$$

Let $\varphi^\bullet : C^\bullet \to D^\bullet$ be a morphism of complexes of $Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$. We define a morphism $\mathrm{Tot}(\varphi^\bullet) : \mathrm{Tot}(C^\bullet) \to \mathrm{Tot}(D^\bullet)$ in $Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$ as

$$\mathrm{Tot}(\varphi^\bullet) := (\tau_1, \tau_0),$$

where

$$\tau_l|_{\mathrm{Com}(C^i)^j} := \mathrm{Com}(\varphi^i)^j.$$

Taking totalizations gives an exact functor

$$\mathrm{Tot} : \mathrm{Ch}^{\mathrm{b}}(Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))) \to Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W)).$$

In what follows, we will see that the category $H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$ has a structure of a triangulated category.

Definition 3.7. We define an automorphism T on $H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$, which is called **the shift functor**, as follows. For an object $A \in H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$, we define an object $T(A)$ as

$$T(A) := (A_0 \xrightarrow{-\varphi_0^A} \Phi(A_1) \xrightarrow{-\Phi(\varphi_1^A)} \Phi(A_0))$$

and for a morphism $f \in \mathrm{Hom}(A, B)$, a morphism $T(f) \in \mathrm{Hom}(T(A), T(B))$ is suitably defined. For any integer $n \in \mathbb{Z}$, denote by $(-)[n]$ the functor $T^n(-)$.

Definition 3.8. Let $f : A \to B$ be a morphism in $Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$. We define its **mapping cone** $\mathrm{Cone}(f)$ to be the totalization of the complex

$$(\cdots \to 0 \to A \xrightarrow{f} B \to 0 \to \cdots)$$

with B in degree zero.

A sequence in $H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$ of the form

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} A[1],$$

where i is the natural injection and p is the natural projection, is called a **standard triangle** and a sequence which is isomorphic to a standard triangle is called **distinguished triangle**.

Proposition 3.9. $H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$ *is a triangulated category with respect to its shift functor and its distinguished triangles.*

Proof. This follows from an argument similar to a proof showing that homotopy categories of exact categories are triangulated categories. □

Following Positselski (cf. [Pos1] or [EP]), we define derived factorization categories.

Definition 3.10. Denote by $\mathrm{Acycl}^{\mathrm{abs}}(\mathcal{A}, \Phi, W))$ the smallest thick subcategory of $H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$ containing all totalizations of short exact sequences in $Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W)$. $E \in H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$ is called **absolutely acyclic** if it lies in $\mathrm{Acycl}^{\mathrm{abs}}(\mathcal{A}, \Phi, W))$. The **absolute derived factorization category** of $(\mathcal{A}, \Phi, W)$ is the Verdier quotient

$$\mathrm{D}^{\mathrm{abs}}(\mathcal{A}, \Phi, W) := H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))/\mathrm{Acycl}^{\mathrm{abs}}(\mathcal{A}, \Phi, W)$$

Definition 3.11. Assume $\mathcal{A}$ admits small coproducts. Denote $\mathrm{Acycl}^{\mathrm{co}}(\mathcal{A}, \Phi, W))$ the smallest thick subcategory of $H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$ containing all totalizations of short exact sequences in $Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W)$ and closed under taking small coproducts. $E \in H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$ is called **co-acyclic** if it lies in $\mathrm{Acycl}^{\mathrm{co}}(\mathcal{A}, \Phi, W)$. The **co-derived factorization category** of $(\mathcal{A}, \Phi, W)$ is the Verdier quotient

$$\mathrm{D}^{\mathrm{co}}(\mathcal{A}, \Phi, W) := H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))/\mathrm{Acycl}^{\mathrm{co}}(\mathcal{A}, \Phi, W)$$

Remark 3.12. (1)Let $\mathcal{E}$ be an exact category, and take a complex $E^\bullet$ in $\mathcal{E}$;

$$E^\bullet = \cdots \to E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \to \cdots.$$

We say that the complex $E^\bullet$ is **exact** if all kernels and images of differentials exist, and for any $n \in \mathbb{Z}$, we have natural isomorphisms

$$\mathrm{Im}(d^{n-1}) \cong \mathrm{Ker}(d^n).$$

Let $\mathcal{B}$ be an abelian category, and let $\mathcal{C}$ be a strictly full additive subcategory of $\mathcal{B}$ which is closed under extensions. The category $\mathcal{C}$ has a natural structure of an exact category. If $\mathcal{C}$ admits either all kernels or all cokernels, then a bounded complex in $\mathcal{C}$ is exact in the above sense if and only if the complex is exact in $\mathcal{B}$.
(2)Note that in the definitions of $\mathrm{Acycl}^{\mathrm{abs}}(\mathcal{A}, \Phi, W)$ and $\mathrm{Acycl}^{\mathrm{co}}(\mathcal{A}, \Phi, W)$, we can replace the words "totalizations of short exact sequences" with "totalizations of bounded exact sequences".

By the next lemma, we see that the totalization functor

$$\mathrm{Tot} : \mathrm{Ch}^{\mathrm{b}}(Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))) \to Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$$

induces a functor

$$\mathrm{Tot} : \mathrm{K}^{\mathrm{b}}(Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))) \to H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$$

which is an exact functor of triangulated categories. This functor naturally induces an exact functor

$$\mathrm{Tot} : \mathrm{D}^{\mathrm{b}}(Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W))) \to \mathrm{D}^{\mathrm{abs}}(\mathcal{A}, \Phi, W).$$

Lemma 3.13. *Let $\varphi^\bullet : C^\bullet \to D^\bullet$ be a morphism in $\mathrm{Ch}^{\mathrm{b}}(Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W)))$. If $\varphi^\bullet$ is homotopic to zero, i.e. $\varphi^\bullet = 0$ in $\mathrm{K}^{\mathrm{b}}(Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W)))$, then $\mathrm{Tot}(\varphi^\bullet) = 0$ in $H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$.*

Proof. Let $\delta^i_{C^\bullet} : C^i \to C^{i+1}$ and $\delta^i_{D^\bullet} : D^i \to D^{i+1}$ be differentials of complexes $C^\bullet$ and $D^\bullet$, and set

$$S = (S_1 \xrightarrow{s_1} S_0 \xrightarrow{s_0} \Phi(S_1)) := \mathrm{Tot}(C^\bullet),$$

$$T = (T_1 \xrightarrow{t_1} T_0 \xrightarrow{t_0} \Phi(T_1)) := \mathrm{Tot}(D^\bullet)$$

and

$$\tau = (\tau_1, \tau_0) := \mathrm{Tot}(\varphi^\bullet),$$

where $\tau_l : S_l \to T_l$. If $\varphi^\bullet = 0$ in $\mathrm{K}^{\mathrm{b}}(Z^0(\mathfrak{F}(\mathcal{A}, \Phi, W)))$, then there exist morphisms $h^i : C^i \to D^{i-1}$ such that $\varphi^i = \delta^{i-1}_{D^\bullet} h^i + h^{i+1}\delta^i_{C^\bullet}$. We define two morphisms $\sigma_0 : S_0 \to T_1$ and $\sigma_1 : \Phi(S_1) \to T_0$ in $\mathcal{A}$ as

$$\sigma_l|_{\mathrm{Com}(C^i)^j} := \mathrm{Com}(h^i)^j$$

for each $l = 0, 1$. Then we have

$$\begin{aligned}
&\big(s_1\sigma_0 + \Phi(\sigma_1)t_0\big)|_{\mathrm{Com}(C^i)^j}\\
&= \big(\mathrm{Com}(\delta^{i-1}_{D^\bullet})^j + (-1)^{i-1} d^j_{D^{i-1}}\big)\mathrm{Com}(h^i)^j + \Phi(\sigma_1)\big(\mathrm{Com}(\delta^i_{C^\bullet})^j + (-1)^i d^j_{C^i}\big)\\
&= \mathrm{Com}(\delta^{i-1}_{D^\bullet})^j \mathrm{Com}(h^i)^j + (-1)^{i-1} d^j_{D^{i-1}}\mathrm{Com}(h^i)^j + \mathrm{Com}(h^{i+1})^j\mathrm{Com}(\delta^i_{C^\bullet})^j + (-1)^i\mathrm{Com}(h^i)^{j+1} d^j_{C^i}\\
&= \mathrm{Com}(\delta^{i-1}_{D^\bullet} h^i + h^{i+1}\delta^i_{C^\bullet})^j\\
&= \tau_0|_{\mathrm{Com}(C^i)^j},
\end{aligned}$$

where $d^j_{D^{i-1}}$ and $d^j_{C^i}$ are morphisms in the infinite sequences $\mathrm{Com}(D^{i-1})$ and $\mathrm{Com}(C^i)$ respectively. Hence, we have $\tau_0 = s_1\sigma_0 + \Phi(\sigma_1)t_0$. Similarly, we obtain $\Phi(\tau_1) = s_0\sigma_1 + \Phi(\sigma_0)\Phi(t_1)$. Hence, $\mathrm{Tot}(\varphi^\bullet) = 0$ in $H^0(\mathfrak{F}(\mathcal{A}, \Phi, W))$. □

Consider an exact functor of exact categories

$$\tau : \mathcal{A} \to Z^0(\mathfrak{F}(\mathcal{A}, \Phi, 0)),$$

which is defined by

$$\tau(A) := (0 \longrightarrow A \longrightarrow 0).$$

Then this functor induces an exact functor of triangulated categories

$$\tau : \mathrm{D}^{\mathrm{b}}(\mathcal{A}) \to \mathrm{D}^{\mathrm{b}}(Z^0(\mathfrak{F}(\mathcal{A}, \Phi, 0))).$$

Definition 3.14. We define an exact functor

$$\Upsilon : \mathrm{D}^{\mathrm{b}}(\mathcal{A}) \to \mathrm{D}^{\mathrm{abs}}(\mathcal{A}, \Phi, 0)$$

as the composition

$$\mathrm{D}^{\mathrm{b}}(\mathcal{A}) \xrightarrow{\tau} \mathrm{D}^{\mathrm{b}}(Z^0(\mathfrak{F}(\mathcal{A}, \Phi, 0))) \xrightarrow{\mathrm{Tot}} \mathrm{D}^{\mathrm{abs}}(\mathcal{A}, \Phi, 0).$$

3.2. **cwp-functors.** Let $(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})$, $(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$ and $(\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}})$ be categories with potentials.

Definition 3.15. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. F is **compatible with potentials** with respect to $(\Phi_{\mathcal{A}}, W_{\mathcal{A}})$ and $(\Phi_{\mathcal{B}}, W_{\mathcal{B}})$ if there exists a functor isomorphism $\sigma : F\Phi_{\mathcal{A}} \xrightarrow{\sim} \Phi_{\mathcal{B}} F$ such that $W_{\mathcal{B}} F = \sigma \circ F W_{\mathcal{A}}$. We call the pair (F, σ) a **cwp-functor** and write

$$(F, \sigma) : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}).$$

We just say F is a cwp-functor and write $F : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$ when there is no confusion about what σ is.

A cwp-functor $(F, \sigma) : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$ induces a natural dg-functor

$$\mathfrak{F}(F, \sigma) : \mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to \mathfrak{F}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$$

as follows. For objects $A, B \in \mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})$ and for a morphism $f \in \mathrm{Hom}(A_i, \Phi^n_{\mathcal{A}}(B_j))$, we define

$$\mathfrak{F}(F, \sigma)(A) := \Big(F(A_1) \xrightarrow{F(\varphi^A_1)} F(A_0) \xrightarrow{\sigma(A_1) \circ F(\varphi^A_0)} \Phi_{\mathcal{B}}(F(A_1)) \Big)$$

and

$$\mathfrak{F}(F, \sigma)(f) := \sigma^n(B_j) \circ F(f) \in \mathrm{Hom}(F(A_i), \Phi^n_{\mathcal{B}}(F(B_j))),$$

where $\sigma^n : F\Phi^n_{\mathcal{A}} \xrightarrow{\sim} \Phi^n_{\mathcal{B}} F$ is the functor isomorphism induced by σ. By the construction, we see that the morphism $\mathfrak{F}(F, \sigma) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathfrak{F}(F, \sigma)(A), \mathfrak{F}(F, \sigma)(B))$ preserves degrees of complexes and is compatible with differentials.

In the following lemma, we give fundamental properties of dg-functors give as $\mathfrak{F}(-)$. Since the proof is straightforward, we skip the proof.

Lemma 3.16. *Let $(F, \sigma) : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$ and $(G, \tau) : (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \to (\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}})$ be cwp-functors. Then we have*
(1) *$(G \circ F, \tau F \circ G\sigma)$ is a cwp-functor, and we have*

$$\mathfrak{F}(G \circ F, \tau F \circ G\sigma) = \mathfrak{F}(G, \tau) \circ \mathfrak{F}(F, \sigma).$$

(2) *If F is fully faithful, so is $\mathfrak{F}(F, \sigma)$.*
(3) *If F is an equivalence, so is $\mathfrak{F}(F, \sigma)$.*

Definition 3.17. Let $(F, \sigma), (F', \sigma') : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$ be cwp-functors and let $\alpha : F \to F'$ be functor morphism. We say that α is a **cwp-functor morphism** if the following diagram of functor morphisms is commutative.

$$\begin{tikzcd}
F\Phi_{\mathcal{A}} \arrow[r, "\sigma"] \arrow[d, "\alpha\Phi_{\mathcal{A}}"'] & \Phi_{\mathcal{B}} F \arrow[d, "\Phi_{\mathcal{B}}\alpha"] \\
F'\Phi_{\mathcal{A}} \arrow[r, "\sigma'"] & \Phi_{\mathcal{B}} F'
\end{tikzcd}$$

If α is a cwp-functor morphism, we write $\alpha : (F, \sigma) \to (F', \sigma')$.

A cwp-functor morphism $\alpha : (F, \sigma) \to (F', \sigma')$ induces a functor morphism

$$\mathfrak{F}(\alpha) : \mathfrak{F}(F, \sigma) \to \mathfrak{F}(F', \sigma')$$

defined by

$$\mathfrak{F}(\alpha)(A) := (\alpha(A_1), \alpha(A_0)) \in \mathrm{Hom}(F(A_1), F'(A_1)) \oplus \mathrm{Hom}(F(A_0), F'(A_0)).$$

Since α is a cwp-functor morphism, the following diagram is commutative,

$$\begin{tikzcd}
F(A_1) \arrow[r, "F(\varphi_1^A)"] \arrow[d, "\alpha(A_1)"'] & F(A_0) \arrow[r, "\sigma(A_1)\circ F(\varphi_0^A)"] \arrow[d, "\alpha(A_0)"'] & \Phi_{\mathcal{B}}(F(A_1)) \arrow[d, "\Phi_{\mathcal{B}}(\alpha(A_1))"] \\
F'(A_1) \arrow[r, "F'(\varphi_1^A)"] & F'(A_0) \arrow[r, "\sigma'(A_1)\circ F'(\varphi_0^A)"] & \Phi_{\mathcal{B}}(F'(A_1)),
\end{tikzcd}$$

which means that $\mathfrak{F}(\alpha)(A) \in Z^0(\mathrm{Hom}(F(A), F'(A)))$ for any $A \in \mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})$.

Definition 3.18. Let $(F, \sigma), (F', \sigma') : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$ be cwp-functors and let $\alpha : (F, \sigma) \to (F', \sigma')$ be a cwp-functor morphism. For a cwp-functor $(G, \mu) : (\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}}) \to (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})$, we define a cwp-functor morphism

$$\alpha(G, \mu) : (F \circ G, \mu F \circ G\sigma) \to (F' \circ G, \mu F' \circ G\sigma')$$

as $\alpha(G, \mu)(C) := \alpha(G(C))$ for any $C \in \mathcal{C}$. Similarly, for a cwp-functor $(H, \nu) : (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \to (\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}})$, we define a cwp-functor morphism

$$(H, \nu)\alpha : (H \circ F, \nu F \circ H\sigma) \to (H \circ F', \nu F' \circ H\sigma')$$

as $(H, \nu)\alpha(A) := H(\alpha(A))$ for any $A \in \mathcal{A}$.

The next lemma gives fundamental properties of functor morphisms given as $\mathfrak{F}(-)$. The proof is left to the reader.

Lemma 3.19. *Let (F, σ), (F', σ') and (F'', σ'') be cwp-functors from $(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})$ to $(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$ and let $\alpha : (F, \sigma) \to (F', \sigma')$ and $\beta : (F', \sigma') \to (F'', \sigma'')$ be cwp-functor morphisms. Then*

(1) *$\beta \circ \alpha : (F, \sigma) \to (F'', \sigma'')$ is a cwp-functor morphism, and we have*
$$\mathfrak{F}(\beta \circ \alpha) = \mathfrak{F}(\beta) \circ \mathfrak{F}(\alpha)$$
(2) *If α is an isomorphism of functors, so is $\mathfrak{F}(\alpha)$.*
(3) *For a cwp-functor $(G, \mu) : (\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}}) \to (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})$, we have*
$$\mathfrak{F}(\alpha(G, \mu)) = \mathfrak{F}(\alpha)\mathfrak{F}(G, \mu).$$
Similarly, for a cwp-functor $(H, \nu) : (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \to (\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}})$, we have
$$\mathfrak{F}((H, \nu)\alpha) = \mathfrak{F}(H, \nu)\mathfrak{F}(\alpha).$$

Next, we introduce the notion of cwp-adjunction of cwp-functors.

Definition 3.20. Let $(F, \sigma) : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$ and $(G, \tau) : (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \to (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})$ be cwp-functors. We say that (F, σ) is **left cwp-adjoint** to (G, τ), denoted by $(F, \sigma) \dashv (G, \tau)$, if F is left adjoint to G and adjunction morphisms are cwp-functor morphisms.

Lemma 3.21. *In the same notation as above, assume $(F, \sigma) \dashv (G, \tau)$ and let $\varepsilon : (FG, \sigma G \circ F\tau) \to \mathrm{id}_{\mathcal{B}}$ and $\eta : \mathrm{id}_{\mathcal{A}} \to (GF, \tau F \circ G\sigma)$ be adjunction morphisms which are cwp-functor morphisms. Then $\mathfrak{F}(F, \sigma) \dashv \mathfrak{F}(G, \tau)$ and*

$$\mathfrak{F}(\varepsilon) : \mathfrak{F}(F, \sigma) \circ \mathfrak{F}(G, \tau) \to \mathrm{id}_{\mathfrak{F}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})}$$

$$\mathfrak{F}(\eta) : \mathrm{id}_{\mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})} \to \mathfrak{F}(G, \tau) \circ \mathfrak{F}(F, \sigma)$$

are the adjunction morphisms of the adjoint pair $\mathfrak{F}(F, \sigma) \dashv \mathfrak{F}(G, \tau)$.

Proof. Since $\varepsilon : FG \to \mathrm{id}_{\mathcal{B}}$ and $\eta : \mathrm{id}_{\mathcal{A}} \to GF$ are adjunction morphisms of the adjoint pair $F \dashv G$, the following compositions are identities of functors;

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F \quad \text{and} \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G.$$

By Lemma 3.16 and Lemma 3.19, the following compositions are also identities of dg-functors;

$$\mathfrak{F}(F,\sigma) \xrightarrow{\mathfrak{F}(F,\sigma)\mathfrak{F}(\eta)} \mathfrak{F}(F,\sigma)\mathfrak{F}(G,\tau)\mathfrak{F}(F,\sigma) \xrightarrow{\mathfrak{F}(\varepsilon)\mathfrak{F}(F,\sigma)} \mathfrak{F}(F,\sigma)$$

and

$$\mathfrak{F}(G,\tau) \xrightarrow{\mathfrak{F}(\eta)\mathfrak{F}(G,\tau)} \mathfrak{F}(G,\tau)\mathfrak{F}(F,\sigma)\mathfrak{F}(G,\tau) \xrightarrow{\mathfrak{F}(G,\tau)\mathfrak{F}(\varepsilon)} \mathfrak{F}(G,\tau).$$

Hence, we have an adjunction $\mathfrak{F}(F,\sigma) \dashv \mathfrak{F}(G,\tau)$, and $\mathfrak{F}(\varepsilon)$ and $\mathfrak{F}(\eta)$ are adjunction morphisms. ☐

We give definitions of relative adjoint functors and basic properties of it after [Ulm].

Definition 3.22. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{D}$ be categories and let $F : \mathcal{C}_1 \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}_2$ and $J : \mathcal{C}_1 \to \mathcal{C}_2$ be functors. F is called **left J-relative adjoint to G** (or **J-left adjoint to G**) if for each $C \in \mathcal{C}_1, D \in \mathcal{D}$ there is an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(C), D) \cong \mathrm{Hom}_{\mathcal{C}_2}(J(C), G(D))$$

which is functorial in C and D.

Dually, F is called **right J-relative adjoint to G** (or **J-right adjoint to G**) if for each $C \in \mathcal{C}_1, D \in \mathcal{D}$ there is an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(D, F(C)) \cong \mathrm{Hom}_{\mathcal{C}_2}(G(D), J(C))$$

which is functorial in C and D.

Remark 3.23.
(1) Relative adjointness is not symmetric property, i.e. although F is J-left adjoint to G, G is not J-right adjoint to F in general.
(2) If F is J-left adjoint to G, there is a functor morphism

$$\mu : J \to GF$$

such that $\mu(C) : J(C) \to G(F(C))$ corresponds to $\mathrm{id}_{F(C)}$.

Similarly, if F is J-right adjoint to G, there is a functor morphism

$$\nu : GF \to J$$

such that $\nu(C) : G(F(C)) \to J(C)$ is corresponding to $\mathrm{id}_{F(C)}$.

The above functor morphisms $\mu : J \to GF$ and $\nu : GF \to J$ are called **the front adjunction**.

By the next lemma, we see that the existence of a front adjunction implies a relative adjunction.

Lemma 3.24 ([Ulm] Lemma 2.7)**.** *The notation is the same as in* Definition 3.22. *The functor F is J-left adjoint to G if and only if there exists a functor morphism $\mu : J \to GF$ such that for each $C \in \mathcal{C}_1$ and $D \in \mathcal{D}$ the composition of maps*

$$\mathrm{Hom}(F(C), D) \xrightarrow{G(-)} \mathrm{Hom}(G(F(C)), G(D)) \xrightarrow{\mathrm{Hom}(\mu(C), G(D))} \mathrm{Hom}(J(C), G(D))$$

is a bijection.

Similarly, F is J-right adjoint to G if and only if there exists a functor morphism $\nu : GF \to J$ such that for each $C \in \mathcal{C}_1$ and $D \in \mathcal{D}$ the composition of maps

$$\mathrm{Hom}(D, F(C)) \xrightarrow{G(-)} \mathrm{Hom}(G(D), G(F(C))) \xrightarrow{\mathrm{Hom}(G(D), \nu(C))} \mathrm{Hom}(G(D), J(C))$$

is bijective.

Similarly, the notion of relative cwp-adjunction is given in the following.

Definition 3.25. In the same notation as in Definition 3.22, let (Φ_i, W_i) and (Ψ, V) be potentials of $\mathcal{C}_i$ and $\mathcal{D}$ respectively. Let $(F,\sigma) : (\mathcal{C}_1, \Phi_1, W_1) \to (\mathcal{D}, \Psi, V)$, $(G,\tau) : (\mathcal{D}, \Psi, V) \to (\mathcal{C}_2, \Phi_2, W_2)$ and $(J,\eta) : (\mathcal{C}_1, \Phi_1, W_1) \to (\mathcal{C}_2, \Phi_2, W_2)$ be cwp-functors. (F,σ) is called (J,η)**-left cwp-adjoint to** (G,τ) if F is J-left adjoint to G and the front adjunction is cwp-functor morphism.

Dually, we say (F,σ) is (J,η)**-right cwp-adjoint to** (G,τ) if F is J-right adjoint to G and the front adjunction is cwp-functor morphism.

Lemma 3.26. *Notation is the same as in* Definition 3.25. *If (F,σ) is (J,η)-left cwp-adjoint to (G,τ) and $\mu : J \to GF$ is the front adjunction, then $\mathfrak{F}(F,\sigma)$ is $\mathfrak{F}(J,\eta)$-left adjoint to $\mathfrak{F}(G,\tau)$ and the front adjunction is $\mathfrak{F}(\mu) : \mathfrak{F}(J,\eta) \to \mathfrak{F}(G,\tau)\mathfrak{F}(F,\sigma)$.*

Similarly, if (F,σ) is (J,η)-right cwp-adjoint to (G,τ) and ν is the front adjunction, then $\mathfrak{F}(F,\sigma)$ is $\mathfrak{F}(J,\eta)$-right adjoint to $\mathfrak{F}(G,\tau)$ and the front adjunction is $\mathfrak{F}(\nu)$.

Proof. If (F,σ) is (J,η)-left cwp-adjoint to (G,τ), then the front adjunction $\mu : J \to GF$ is cwp-functor morphism, and the composition

$$\mathrm{Hom}(F(C), D) \xrightarrow{G(-)} \mathrm{Hom}(G(F(C)), G(D)) \xrightarrow{\mathrm{Hom}(\mu(C), G(D))} \mathrm{Hom}(J(C), G(D))$$

is a bijection. Hence, the composition of morphisms

$$\mathrm{Hom}(\mathfrak{F}(F,\sigma)(C), D) \xrightarrow{\mathfrak{F}(G,\tau)(-)} \mathrm{Hom}(\{\mathfrak{F}(G,\tau) \circ \mathfrak{F}(F,\sigma)\}(C), \mathfrak{F}(G,\tau)(D))$$

and

$$\mathrm{Hom}(\{\mathfrak{F}(G,\tau)\circ\mathfrak{F}(F,\sigma)\}(C), \mathfrak{F}(G,\tau)(D)) \xrightarrow{\mathrm{Hom}(\mathfrak{F}(\mu)(C), \mathfrak{F}(G,\tau)(D))} \mathrm{Hom}(\mathfrak{F}(J,\eta)(C), \mathfrak{F}(G,\tau)(D))$$

is also bijective. By Lemma 3.24, we see that $\mathfrak{F}(F,\sigma)$ is $\mathfrak{F}(J,\eta)$-left adjoint to $\mathfrak{F}(G,\tau)$, and the front adjunction is $\mathfrak{F}(\mu) : \mathfrak{F}(J,\eta) \to \mathfrak{F}(G,\tau)\mathfrak{F}(F,\sigma)$.

The latter statement can be proved in a similar way. $\square$

In what follows, we define cwp-bifunctors.

Definition 3.27. Let $P : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bifunctor. We say that P is **compatible with potentials** with respect to $(\Phi_{\mathcal{A}}, W_{\mathcal{A}})$, $(\Phi_{\mathcal{B}}, W_{\mathcal{B}})$ and $(\Phi_{\mathcal{C}}, W_{\mathcal{C}})$ if there are bifunctor isomorphisms $\sigma_{\mathcal{A}} : P(\Phi_{\mathcal{A}} \times \mathrm{id}_{\mathcal{B}}) \xrightarrow{\sim} \Phi_{\mathcal{C}} P$ and $\sigma_{\mathcal{B}} : P(\mathrm{id}_{\mathcal{A}} \times \Phi_{\mathcal{B}}) \xrightarrow{\sim} \Phi_{\mathcal{C}} P$ such that

$$\sigma_{\mathcal{A}}(A,B) \circ P(W_{\mathcal{A}}(A), B) + \sigma_{\mathcal{B}}(A,B) \circ P(A, W_{\mathcal{B}}(B)) = W_{\mathcal{C}}(P(A,B))$$

and

$$\Phi_{\mathcal{C}}(\sigma_{\mathcal{B}}(A,B)) \circ \sigma_{\mathcal{A}}(A, \Phi_{\mathcal{B}}(B)) = \Phi_{\mathcal{C}}(\sigma_{\mathcal{A}}(A,B)) \circ \sigma_{\mathcal{B}}(\Phi_{\mathcal{A}}(A), B)$$

for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$. By the latter equation above, $\sigma_{\mathcal{A}}$ and $\sigma_{\mathcal{B}}$ induce a natural functor isomorphism $\sigma^{m,n} : P(\Phi_{\mathcal{A}}^m \times \Phi_{\mathcal{B}}^n) \xrightarrow{\sim} \Phi^{m+n} P$ for any $m, n \in \mathbb{Z}$. The triple $(P, \sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})$ is called **cwp-bifunctor** and we write

$$(P, \sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}) : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \times (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \to (\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}}).$$

For a cwp-bifunctor $(P, \sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}) : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \times (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \to (\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}})$, we define a dg-bifunctor

$$\mathfrak{F}(P, \sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}) : \mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \otimes \mathfrak{F}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \to \mathfrak{F}(\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}})$$

as follows. For any object $A = (A_1 \xrightarrow{\varphi_1^A} A_0 \xrightarrow{\varphi_0^A} \Phi_{\mathcal{A}}(A_1)) \in \mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})$ and $B = (B_1 \xrightarrow{\psi_1^B} B_0 \xrightarrow{\psi_0^B} \Phi_{\mathcal{B}}(B_1)) \in \mathfrak{F}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$, we define the object $\mathfrak{F}(P, \sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})(A, B) \in \mathfrak{F}(\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}})$ as

$$\Big(P(A_1, B_0) \oplus P(A_0, B_1) \xrightarrow{\omega_1} P(A_0, B_0) \oplus \Phi_{\mathcal{C}}(P(A_1, B_1)) \xrightarrow{\omega_0} \Phi_{\mathcal{C}}(P(A_1, B_0)) \oplus \Phi_{\mathcal{C}}(P(A_0, B_1))\Big),$$

where

$$\omega_1 = \begin{pmatrix} P(\varphi_1^A, \mathrm{id}) & P(\mathrm{id}, \psi_1^B) \\ -\sigma_{\mathcal{B}}(A_1, B_1) \circ P(\mathrm{id}, \psi_0^B) & \sigma_{\mathcal{A}}(A_1, B_1) \circ P(\varphi_0^A, \mathrm{id}) \end{pmatrix}$$

and

$$\omega_0 = \begin{pmatrix} \sigma_{\mathcal{A}}(A_1, B_0) \circ P(\varphi_0^A, \mathrm{id}) & -\Phi_{\mathcal{C}}(P(\mathrm{id}, \psi_1^B)) \\ \sigma_{\mathcal{B}}(A_0, B_1) \circ P(\mathrm{id}, \psi_0^B) & \Phi_{\mathcal{C}}(P(\varphi_1^A, \mathrm{id})) \end{pmatrix}.$$

For a morphism $f : (A, B) \to (A', B')$ in $\mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \otimes \mathfrak{F}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$, we define the morphism $\mathfrak{F}(P, \sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})(f) : \mathfrak{F}(P, \sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})(A, B) \to \mathfrak{F}(P, \sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})(A', B')$ by the following rule,

$$\mathfrak{F}(P, \sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})(g_{i,j}^m \otimes h_{k,l}^n) := \begin{cases} (-1)^{\deg(h_{1,l}^n)} \Phi_{\mathcal{C}}(\sigma^{m,n}(A'_j, B'_l) \circ P(g_{1,j}^m, h_{1,l}^n)) & \text{if } i = k = 1 \\ (-1)^{i \deg(h_{k,l}^n)} \sigma^{m,n}(A'_j, B'_l) \circ P(g_{i,j}^m, h_{k,l}^n) & \text{otherwise} \end{cases},$$

where $g_{i,j}^m \in \mathrm{Hom}_{\mathcal{A}}(A_i, \Phi_{\mathcal{A}}^m(A'_j))$ and $h_{k,l}^n \in \mathrm{Hom}_{\mathcal{B}}(B_k, \Phi_{\mathcal{B}}^n(B'_l))$.

Definition 3.28. Let $Q : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{C}$ be a bifunctor. We say that Q is **compatible with potentials** with respect to $(\Phi_{\mathcal{A}}, W_{\mathcal{A}})$, $(\Phi_{\mathcal{B}}, W_{\mathcal{B}})$ and $(\Phi_{\mathcal{C}}, W_{\mathcal{C}})$ if there are bifunctor isomorphisms $\tau_{\mathcal{A}} : Q((\Phi_{\mathcal{A}}^{\mathrm{op}})^{-1} \times \mathrm{id}_{\mathcal{B}}) \xrightarrow{\sim} \Phi_{\mathcal{C}} Q$ and $\tau_{\mathcal{B}} : Q(\mathrm{id}_{\mathcal{A}} \times \Phi_{\mathcal{B}}) \xrightarrow{\sim} \Phi_{\mathcal{C}} Q$ such that

$$-\tau_{\mathcal{A}}(A, B) \circ Q((\Phi_{\mathcal{A}}^{\mathrm{op}})^{-1}(W_{\mathcal{A}}^{\mathrm{op}}(A)), B) + \tau_{\mathcal{B}}(A, B) \circ Q(A, W_{\mathcal{B}}(B)) = W_{\mathcal{C}}(Q(A, B))$$

and

$$\Phi_{\mathcal{C}}(\tau_{\mathcal{B}}(A, B)) \circ \tau_{\mathcal{A}}(A, \Phi_{\mathcal{B}}(B)) = \Phi_{\mathcal{C}}(\tau_{\mathcal{A}}(A, B)) \circ \tau_{\mathcal{B}}((\Phi_{\mathcal{A}}^{\mathrm{op}})^{-1}(A), B)$$

for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, where $\Phi_{\mathcal{A}}^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{A}^{\mathrm{op}}$ is the opposite equivalence of $\Phi_{\mathcal{A}}$ and $W_{\mathcal{A}}^{\mathrm{op}} : \Phi_{\mathcal{A}}^{\mathrm{op}} \to \mathrm{id}_{\mathcal{A}^{\mathrm{op}}}$ is the opposite functor morphism of $W_{\mathcal{A}}$. By the latter equation above, $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ induce a natural functor isomorphism $\tau^{m,n} : Q(\Phi_{\mathcal{A}}^m \times \Phi_{\mathcal{B}}^n) \xrightarrow{\sim} \Phi^{-m+n} Q$ for any $m, n \in \mathbb{Z}$. The triple $(Q, \tau_{\mathcal{A}}, \tau_{\mathcal{B}})$ is called **cwp-bifunctor** and we write

$$(Q, \tau_{\mathcal{A}}, \tau_{\mathcal{B}}) : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})^{\mathrm{op}} \times (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \to (\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}}).$$

For a cwp-bifunctor $(Q, \tau_{\mathcal{A}}, \tau_{\mathcal{B}}) : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})^{\mathrm{op}} \times (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \to (\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}})$, we define a dg-bifunctor

$$\mathfrak{F}(Q, \tau_{\mathcal{A}}, \tau_{\mathcal{B}}) : \mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})^{\mathrm{op}} \otimes \mathfrak{F}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \to \mathfrak{F}(\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}})$$

as follows. For any object $A = (A_1 \xrightarrow{\varphi_1^A} A_0 \xrightarrow{\varphi_0^A} \Phi_{\mathcal{A}}(A_1)) \in \mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})^{\mathrm{op}}$ and $B = (B_1 \xrightarrow{\psi_1^B} B_0 \xrightarrow{\psi_0^B} \Phi_{\mathcal{B}}(B_1)) \in \mathfrak{F}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$, we define the object $\mathfrak{F}(Q, \tau_{\mathcal{A}}, \tau_{\mathcal{B}})(A, B) \in \mathfrak{F}(\mathcal{C}, \Phi_{\mathcal{C}}, W_{\mathcal{C}})$ as

$$\Big(\Phi_{\mathcal{C}}^{-1}(Q(A_1, B_0)) \oplus Q(A_0, B_1) \xrightarrow{\omega_1} Q(A_0, B_0) \oplus Q(A_1, B_1)) \xrightarrow{\omega_0} Q(A_1, B_0) \oplus \Phi_{\mathcal{C}}(Q(A_0, B_1))\Big),$$

where

$$\omega_1 = \begin{pmatrix} Q(\varphi_0^A, \mathrm{id}) \circ ((\tau^{1,0})(A_1, B_0))^{-1} & Q(\mathrm{id}, \psi_1^B) \\ \Phi_{\mathcal{C}}^{-1}(\tau_{\mathcal{B}}(A_1, B_1) \circ Q(\mathrm{id}, \psi_0^B)) & Q(\varphi_1^A, \mathrm{id}) \end{pmatrix}$$

and

$$\omega_0 = \begin{pmatrix} -Q(\varphi_1^A, \mathrm{id}) & Q(\mathrm{id}, \psi_1^B) \\ \tau_{\mathcal{B}}(A_0, B_1) \circ Q(\mathrm{id}, \psi_0^B) & -\tau_{\mathcal{A}}(A_0, B_1) \circ (Q(\Phi_A^{-1}(\varphi_0^A), \mathrm{id})) \end{pmatrix}.$$

For a morphism $f : (A, B) \to (A', B')$ in $\mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})^{\mathrm{op}} \otimes \mathfrak{F}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$, we define the morphism $\mathfrak{F}(Q, \tau_{\mathcal{A}}, \tau_{\mathcal{B}})(f) : \mathfrak{F}(Q, \tau_{\mathcal{A}}, \tau_{\mathcal{B}})(A, B) \to \mathfrak{F}(Q, \tau_{\mathcal{A}}, \tau_{\mathcal{B}})(A', B')$ by the following rule,

$$\begin{aligned} &\mathfrak{F}(Q, \tau_{\mathcal{A}}, \tau_{\mathcal{B}})(g_{i,j}^m \otimes h_{k,l}^n) \\ := &\begin{cases} \Phi_{\mathcal{C}}^{m-1}(\tau^{0,n}(A'_j, B'_l) \circ Q(g_{1,j}^m, h_{0,l}^n) \circ \tau^{m,0}(A_1, B_0)^{-1}) & \text{if } i = 1 \;\; k = 0 \\ (-1)^{i-k+1}\Phi_{\mathcal{C}}^{m}(\tau^{0,n}(A'_j, B'_l) \circ Q(g_{i,j}^m, h_{k,l}^n) \circ \tau^{m,0}(A_i, B_k)^{-1}) & \text{otherwise} \end{cases} \end{aligned}$$

where $g_{i,j}^m \in \mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(\Phi_{\mathcal{A}}^m(A_i), A'_j)$ and $h_{k,l}^n \in \mathrm{Hom}(B_k, \Phi_{\mathcal{B}}^n(B'_l))$.

3.3. **ind/pro-categories and their factorization categories.** In this section, we recall the notion of ind-categories and pro-categories, and study factorization categories of ind/pro-categories. For the detail of ind/pro-categories, see [CP] or [Kas], for example.

At first, we recall the definition and the foundations of ind/pro-categories.

Definition 3.29. A small category $\mathcal{I}$ is called **filtering** if the following properties hold;

(1) For any objects $i, i' \in \mathcal{I}$, there exist an object $j \in \mathcal{I}$ and morphisms $i \to j$ and $i' \to j$.
(2) For two morphisms $u, v : k' \to k$ in $\mathcal{I}$, there exist an object $l \in \mathcal{I}$ and a morphism $w : k \to l$ such that $w \circ u = w \circ v$.

A small category $\mathcal{J}$ is called **cofiltering** if its opposite category $\mathcal{J}^{\mathrm{op}}$ is filtering.

Definition 3.30. Let $\mathcal{C}$ be a category.
(1) We define the **ind-category** of $\mathcal{C}$, denoted by $\mathrm{Ind}(\mathcal{C})$, as follows:

An object of $\mathrm{Ind}(\mathcal{C})$ is a functor $D : \mathcal{I} \to \mathcal{C}$ with $\mathcal{I}$ filtering. For two objects $D : \mathcal{I} \to \mathcal{C}$ and $E : \mathcal{J} \to \mathcal{C}$, we define the set of morphisms as

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(D, E) := \varprojlim_{i\in\mathcal{I}} \varinjlim_{j\in\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(D(i), E(j)).$$

(2) We define the **pro-category** of $\mathcal{C}$, denoted by $\mathrm{Pro}(\mathcal{C})$, by the following:

An object of $\mathrm{Pro}(\mathcal{C})$ is a functor $P : \mathcal{I} \to \mathcal{C}$ with $\mathcal{I}$ cofiltering. For two objects $P : \mathcal{I} \to \mathcal{C}$ and $Q : \mathcal{J} \to \mathcal{C}$, we define the space of morphisms as

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(P, Q) := \varprojlim_{j\in\mathcal{J}} \varinjlim_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(P(i), Q(j)).$$

Remark 3.31. (1) We have a natural equivalence

$$\mathrm{Pro}(\mathcal{C}) \cong \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}.$$

(2) Let $D : \mathcal{I} \to \mathcal{C}$ and $E : \mathcal{J} \to \mathcal{C}$ be objects of $\mathrm{Ind}(\mathcal{C})$. The set of morphisms $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(D, E)$ is interpreted as the set of equivalence classes of maps of systems defined as follows:

A **map of systems** from D to E is a pair $\varphi = (\{\varphi_i\}_{i\in\mathcal{I}}, \theta_\varphi)$ where $\theta_\varphi : \mathrm{Ob}(\mathcal{I}) \to \mathrm{Ob}(\mathcal{J})$ is a map from $\mathrm{Ob}(\mathcal{I})$ to $\mathrm{Ob}(\mathcal{J})$, and $\varphi_i \in \mathrm{Hom}_{\mathcal{C}}(D(i), E(\theta_\varphi(i)))$, such that for any morphism $v : i \to i'$ in $\mathcal{I}$ there are $j \in \mathcal{J}$, $u : \theta_\varphi(i) \to j$ and $u' : \theta_\varphi(i') \to j$ such that the following diagram is commutative:

$$\begin{tikzcd}
D(i) \arrow[r, "\varphi_i"] \arrow[dd, "D(v)"'] & E(\theta_\varphi(i)) \arrow[dr, "E(u)"] & \\
 & & E(j) \\
D(i') \arrow[r, "\varphi_{i'}"] & E(\theta_\varphi(i')) \arrow[ur, "E(u')"] &
\end{tikzcd}$$

Two maps of systems $\varphi = (\{\varphi_i\}_{i\in\mathcal{I}}, \theta_\varphi)$ and $\psi = (\{\psi_i\}_{i\in\mathcal{I}}, \theta_\psi)$ are **equivalent** if for each $i \in \mathcal{I}$, there exist $j \in \mathrm{Ob}(\mathcal{J})$, $u : \theta_\varphi(i) \to j$ and $v : \theta_\psi(i) \to j$ such that the following diagram commutes:

$$\begin{tikzcd}
 & E(\theta_\varphi(i)) \arrow[dr, "E(u)"] & \\
D(i) \arrow[ur, "\varphi_i"] \arrow[dr, "\psi_i"'] & & E(j) \\
 & E(\theta_\psi(i)) \arrow[ur, "E(v)"] &
\end{tikzcd}$$

We denote by $[\varphi]$ the morphism from D to E in $\mathrm{Ind}(\mathcal{C})$ corresponding to the equivalence class of a map of systems φ. With this notation one can easily write down the composition of $[\varphi] \in \mathrm{Hom}(D, E)$ and $[\psi] \in \mathrm{Hom}(E, H)$, where $H : \mathcal{K} \to \mathcal{C}$. The composition is given by

$$[\psi] \circ [\varphi] = [(\{\psi_{\theta_\varphi(i)} \circ \varphi_i\}_{i\in\mathcal{I}}, \theta_\psi \circ \theta_\varphi)].$$

(3) Let $P : \mathcal{I} \to \mathcal{C}$ and $Q : \mathcal{J} \to \mathcal{C}$ be objects of $\mathrm{Pro}(\mathcal{C})$. Similarly to (2), the set $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(P, Q)$ is interpreted as the set of equivalence classes of maps of systems defined by the following:

A **map of systems** from P to Q is a pair $\varphi = (\{\varphi_j\}_{j\in\mathcal{J}}, \theta_\varphi)$ where $\theta_\varphi : \mathrm{Ob}(\mathcal{J}) \to \mathrm{Ob}(\mathcal{I})$ is a map from $\mathrm{Ob}(\mathcal{J})$ to $\mathrm{Ob}(\mathcal{I})$ and $\varphi_j \in \mathrm{Hom}_{\mathcal{C}}(P(\theta_\varphi(j)), Q(j))$, such that for any morphism $v : j \to j'$ in $\mathcal{J}$ there are $i \in \mathcal{I}$, $u : i \to \theta_\varphi(j)$ and $u' : i \to \theta_\varphi(j')$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & P(\theta_\varphi(j')) \arrow[r, "\varphi_{j'}"] & Q(j') \\
P(i) \arrow[ur, "P(u')"] \arrow[dr, "P(u)"'] & & \\
 & P(\theta_\varphi(j)) \arrow[r, "\varphi_j"] & Q(j) \arrow[uu, "Q(v)"']
\end{tikzcd}$$

Two maps of systems $\varphi = (\{\varphi_j\}_{j\in\mathcal{J}}, \theta_\varphi)$ and $\psi = (\{\psi_j\}_{j\in\mathcal{J}}, \theta_\psi)$ are **equivalent** if for each $j \in \mathcal{J}$, there exist $i \in \mathrm{Ob}(\mathcal{I})$, $u : i \to \theta_\varphi(j)$ and $v : i \to \theta_\psi(j)$ such that the following diagram commutes:

$$\begin{tikzcd} & P(\theta_\varphi(j)) \arrow[dr, "\varphi_j"] & \\ P(i) \arrow[ur, "P(u)"] \arrow[dr, "P(v)"'] & & Q(j) \\ & P(\theta_\psi(j)) \arrow[ur, "\psi_j"'] & \end{tikzcd}$$

We denote by $[\varphi]$ the morphism from P to Q in $\mathrm{Pro}(\mathcal{C})$ corresponding to the equivalence class of a map of systems φ. Let $[\varphi] \in \mathrm{Hom}(P, Q)$ and $[\psi] \in \mathrm{Hom}(Q, R)$ be morphisms, where $R : \mathcal{K} \to \mathcal{C}$. The composition of $[\varphi]$ with $[\psi]$ is given by

$$[\psi] \circ [\varphi] = [(\{\psi_k \circ \varphi_{\theta_\psi(k)}\}_{k\in\mathcal{K}}, \theta_\varphi \circ \theta_\psi)].$$

Definition 3.32. For $C \in \mathcal{C}$, $\iota(C) : \widetilde{\{1\}} \to \mathcal{C}$ is the functor from the category $\widetilde{\{1\}}$ with a unique object, 1, and a unique morphism, id_1, defined by $\iota(C)(1) := C$. $\iota(-)$ defines natural functors

$$\iota_{\mathrm{Ind}} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$$
$$\iota_{\mathrm{Pro}} : \mathcal{C} \to \mathrm{Pro}(\mathcal{C}).$$

By the constructions, the functors ι_{Ind} and ι_{Pro} are fully faithful.

Remark 3.33. $\mathrm{Ind}(-)$ defines an endofunctor on the category of functors, i.e.
(a) A functor $F : \mathcal{C} \to \mathcal{D}$ induces a natural functor

$$\mathrm{Ind}(F) : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$$

as follows: For an object $D : \mathcal{I} \to \mathcal{C} \in \mathrm{Ind}(\mathcal{C})$, the object $\mathrm{Ind}(F)(D)$ is defined by $F \circ D : \mathcal{I} \to \mathcal{D}$. For another object $D' : \mathcal{I}' \to \mathcal{C}$ and for a morphism $[\varphi] : D \to D'$, $\mathrm{Ind}(F)([\varphi])$ is defined by $[(\{F(\varphi_i)\}_{i\in\mathcal{I}}, \theta_\varphi)]$. The following diagram is commutative.

$$\begin{tikzcd} \mathrm{Ind}(\mathcal{C}) \arrow[r, "\mathrm{Ind}(F)"] & \mathrm{Ind}(\mathcal{D}) \\ \mathcal{C} \arrow[u, "\iota_{\mathrm{Ind}}"] \arrow[r, "F"] & \mathcal{D} \arrow[u, "\iota_{\mathrm{Ind}}"'] \end{tikzcd}$$

(b) Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A functor morphism $\alpha : F \to G$ induces a natural functor morphism

$$\mathrm{Ind}(\alpha) : \mathrm{Ind}(F) \to \mathrm{Ind}(G)$$

as follows: For an object $D : \mathcal{I} \to \mathcal{C} \in \mathrm{Ind}(\mathcal{C})$, the morphism $\mathrm{Ind}(\alpha)(D)$ is defined by $[(\{\alpha D(i)\}_{i\in\mathcal{I}}, \mathrm{id}_{\mathcal{I}})]$.

Similarly, $\mathrm{Pro}(-)$ defines an endofunctor on the category of functors, i.e.
(a') A functor $F : \mathcal{C} \to \mathcal{D}$ induces a natural functor

$$\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$$

as follows: For an object $P : \mathcal{I} \to \mathcal{C} \in \mathrm{Pro}(\mathcal{C})$, the object $\mathrm{Pro}(F)(P)$ is defined by $(F \circ P : \mathcal{I} \to \mathcal{D})$. For another object $P' : \mathcal{I}' \to \mathcal{C}$ and for a morphism $[\varphi] : P \to P'$, $\mathrm{Pro}(F)([\varphi])$ is defined by $[((F(\varphi_{i'}))_{i'\in\mathcal{I}'}, \theta_\varphi)]$. The following diagram is commutative.

$$\begin{tikzcd} \mathrm{Pro}(\mathcal{C}) \arrow[r, "\mathrm{Pro}(F)"] & \mathrm{Pro}(\mathcal{D}) \\ \mathcal{C} \arrow[u, "\iota"] \arrow[r, "F"] & \mathcal{D} \arrow[u, "\iota"'] \end{tikzcd}$$

(b') Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A functor morphism $\alpha : F \to G$ induces a natural functor morphism

$$\mathrm{Pro}(\alpha) : \mathrm{Pro}(F) \to \mathrm{Pro}(G)$$

as follows: For an object $P : \mathcal{I} \to \mathcal{C} \in \mathrm{Pro}(\mathcal{C})$, the morphism $\mathrm{Pro}(\alpha)(P)$ is defined by $[(\alpha P(i))_{i \in \mathcal{I}}, \mathrm{id}_{\mathcal{I}})]$.

Proposition 3.34. *We have the following:*

(1) *If $\mathcal{C}$ is an abelian category, then the categories* $\mathrm{Ind}(\mathcal{C})$ *and* $\mathrm{Pro}(\mathcal{C})$ *are abelian categories.*
(2) *If $\mathcal{E}$ is an exact category, then the categories* $\mathrm{Ind}(\mathcal{E})$ *and* $\mathrm{Pro}(\mathcal{E})$ *are exact categories.*
(3) *If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor of exact categories, then the functors* $\mathrm{Ind}(F) : \mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B})$ *and* $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{A}) \to \mathrm{Pro}(\mathcal{B})$ *are exact functors.*

Proof. (1) This follows from [Kas, Theorem 8.6.5.]
(2) This is [Pre, Proposition 4.18.]
(3) Since we can take abelian envelopes of the exact categories $\mathcal{A}$ and $\mathcal{B}$, and extend the functor F to a functor between the abelian envelopes (see the proof of Proposition 3.5), we may assume that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories. Then we obtain the result by [Kas, Corollary 8.6.8.] $\square$

Let $(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})$ be a category with a potential. Then

$$\mathrm{Ind}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) := (\mathrm{Ind}(\mathcal{A}), \mathrm{Ind}(\Phi_{\mathcal{A}}), \mathrm{Ind}(W))$$

$$\mathrm{Pro}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) := (\mathrm{Pro}(\mathcal{A}), \mathrm{Pro}(\Phi_{\mathcal{A}}), \mathrm{Pro}(W))$$

are categories with potentials. Since the natural functor $\iota_{\mathrm{Ind}} : \mathcal{A} \to \mathrm{Ind}(\mathcal{A})$ (resp. $\iota_{\mathrm{Pro}} : \mathcal{A} \to \mathrm{Pro}(\mathcal{A})$) is compatible with potentials with respect to $(\Phi_{\mathcal{A}}, W_{\mathcal{A}})$ and $(\mathrm{Ind}(\Phi_{\mathcal{A}}), \mathrm{Ind}(W))$ (resp. $(\mathrm{Pro}(\Phi_{\mathcal{A}}), \mathrm{Pro}(W))$), it induces a natural fully faithful functor

$$\mathfrak{F}(\iota_{\mathrm{Ind}}) : \mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to \mathfrak{F}\mathrm{Ind}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}})$$

$$(\text{resp. } \ \mathfrak{F}(\iota_{\mathrm{Pro}}) : \mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to \mathfrak{F}\mathrm{Pro}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \ ).$$

Let $(F, \sigma) : (\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to (\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$ be a cwp-functor. Then

$$\mathrm{Ind}(F, \sigma) := (\mathrm{Ind}(F), \mathrm{Ind}(\sigma)) : \mathrm{Ind}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to \mathrm{Ind}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$$

$$\mathrm{Pro}(F, \sigma) := (\mathrm{Pro}(F), \mathrm{Pro}(\sigma)) : \mathrm{Pro}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \to \mathrm{Pro}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}})$$

are cwp-functors, and the following diagrams are commutative:

$$\begin{tikzcd}
\mathfrak{F}\mathrm{Ind}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \arrow[r, "\mathfrak{F}\mathrm{Ind}(F,\sigma)"] & \mathfrak{F}\mathrm{Ind}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \\
\mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \arrow[u, "\mathfrak{F}(\iota_{\mathrm{Ind}})"] \arrow[r, "\mathfrak{F}(F,\sigma)"] & \mathfrak{F}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \arrow[u, "\mathfrak{F}(\iota_{\mathrm{Ind}})"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{F}\mathrm{Pro}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \arrow[r, "\mathfrak{F}\mathrm{Pro}(F,\sigma)"] & \mathfrak{F}\mathrm{Pro}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \\
\mathfrak{F}(\mathcal{A}, \Phi_{\mathcal{A}}, W_{\mathcal{A}}) \arrow[u, "\mathfrak{F}(\iota_{\mathrm{Pro}})"] \arrow[r, "\mathfrak{F}(F,\sigma)"] & \mathfrak{F}(\mathcal{B}, \Phi_{\mathcal{B}}, W_{\mathcal{B}}) \arrow[u, "\mathfrak{F}(\iota_{\mathrm{Pro}})"']
\end{tikzcd}$$

4. Derived factorization categories of gauged LG models

Let X be a scheme and let G be an affine algebraic group acting on X over an algebraically closed field k of characteristic zero. Let $\sigma : G \times X \to X$ be the morphism defining the action, $\pi : G \times X \to X$ be a projection and $\iota : X \to G \times X$ be an embedding given by $x \mapsto (e, x)$, where $e \in G$ is the identity of group G.

4.1. **Equivariant sheaves and factorization categories of gauged LG models.**

Definition 4.1. A quasi-coherent (resp. coherent) G**-equivariant** sheaf is a pair $(\mathcal{F}, \theta)$ of a quasi-coherent (resp. coherent) sheaf $\mathcal{F}$ and an isomorphism $\theta : \pi^*\mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}$ such that

$$\iota^*\theta = \mathrm{id}_{\mathcal{F}} \quad \text{and} \quad \big((1_{\mathrm{G}} \times \sigma) \circ (s \times 1_{\mathrm{X}})\big)^*\theta \circ (1_{\mathrm{G}} \times \pi)^*\theta = (m \times 1_{\mathrm{X}})^*\theta,$$

where $m : G \times G \to G$ is the multiplication and $s : G \times G \to G \times G$ is the switch of two factors. A G**-invariant** morphism $\varphi : (\mathcal{F}_1, \theta_1) \to (\mathcal{F}_2, \theta_2)$ of equivariant sheaves is a morphism of sheaves $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ which is commutative with θ_i, i.e. $\sigma^*\varphi \circ \theta_1 = \theta_2 \circ \pi^*\varphi$.

We denote by $\mathrm{Qcoh}_G(X)$ (resp. $\mathrm{coh}_G(X)$) the category of quasi-coherent (resp. coherent) G-equivariant sheaves on X whose morphisms are G-invariant morphisms. And we denote by $\mathrm{Inj}_G(X)$, $\mathrm{LFr}_G(X)$ and $\mathrm{lfr}_G(X)$ the full subcategories of $\mathrm{Qcoh}_G(X)$ consisting of injective quasi-coherent equivariant sheaves, locally free equivariant sheaves and locally free equivariant sheaves of finite ranks.

Let $L \in \mathrm{Pic}_G(X)$ be a G-equivariant invertible sheaf on X and let $W \in H^0(X, L)^G$ be an invariant section of L.

Definition 4.2. We call the data $(X, L, W)^G$ a **gauged Landau-Ginzburg model** or **gauged LG model**, for short. We sometimes drop the script L from the notion $(X, L, W)^G$, and write $(X, W)^G$ if there is no confusion.

The pair $(L, W) := ((-) \otimes L, (-) \otimes W)$ is a potential of $\mathrm{Qcoh}_G(X)$, $\mathrm{coh}_G(X)$, $\mathrm{Inj}_G(X)$, $\mathrm{LFr}_G(X)$ and $\mathrm{lfr}_G(X)$, where W is considered as the morphism $W : \mathcal{O}_X \to L$ corresponding to the section of L.

Definition 4.3. We define factorization categories of $(X, L, W)^G$ as

$$\begin{aligned}
\mathrm{Qcoh}_G(X, L, W) &:= \mathfrak{F}(\mathrm{Qcoh}_G(X), L, W)\\
\mathrm{coh}_G(X, L, W) &:= \mathfrak{F}(\mathrm{coh}_G(X), L, W)\\
\mathrm{Inj}_G(X, L, W) &:= \mathfrak{F}(\mathrm{Inj}_G(X), L, W)\\
\mathrm{LFr}_G(X, L, W) &:= \mathfrak{F}(\mathrm{LFr}_G(X), L, W)\\
\mathrm{lfr}_G(X, L, W) &:= \mathfrak{F}(\mathrm{lfr}_G(X), L, W).
\end{aligned}$$

We define categories of acyclic factorizations as

$$\begin{aligned}
\mathrm{Acycl}_G(X, L, W) &:= \mathrm{Acycl}^{\mathrm{abs}}(\mathrm{Qcoh}_G(X), L, W)\\
\mathrm{Acycl}^{\mathrm{co}}_G(X, L, W) &:= \mathrm{Acycl}^{\mathrm{co}}(\mathrm{Qcoh}_G(X), L, W)
\end{aligned}$$

and derived factorization categories are defined as

$$\begin{aligned}
\mathrm{DQcoh}_G(X, L, W) &:= \mathrm{D}^{\mathrm{abs}}(\mathrm{Qcoh}_G(X), L, W)\\
\mathrm{Dcoh}_G(X, L, W) &:= \mathrm{D}^{\mathrm{abs}}(\mathrm{coh}_G(X), L, W)\\
\mathrm{DLFr}_G(X, L, W) &:= \mathrm{D}^{\mathrm{abs}}(\mathrm{LFr}_G(X), L, W)\\
\mathrm{Dlfr}_G(X, L, W) &:= \mathrm{D}^{\mathrm{abs}}(\mathrm{lfr}_G(X), L, W).
\end{aligned}$$

We call the category $\mathrm{Dcoh}_G(X, L, W)$ **the derived factorization category** of a gauged LG model $(X, L, W)^G$. For $E, F \in \mathrm{Qcoh}_G(X, L, W)$, we say E and F are **quasi-isomorphic** if E and F are isomorphic in $\mathrm{DQcoh}_G(X, L, W)$. We denote by $\mathrm{D_{coh}Qcoh}_G(X, L, W)$ the full subcategory of $\mathrm{DQcoh}_G(X, L, W)$ whose objects are quasi-isomorphic to objects in $\mathrm{Dcoh}_G(X, L, W)$.

Similarly, we consider coderived factorization categories;

$$\mathrm{D^{co}Qcoh}_G(X, \chi, W) := H^0(\mathrm{Qcoh}_G(X, \chi, W))/\mathrm{Acycl}^{\mathrm{co}}(\mathrm{Qcoh}_G(X, \chi, W))$$

$$\mathrm{D^{co}LFr}_G(X, \chi, W) := H^0(\mathrm{LFr}_G(X, \chi, W))/\mathrm{Acycl}^{\mathrm{co}}(\mathrm{LFr}_G(X, \chi, W)).$$

If G is trivial, we drop the subscript G in the above notations.

Remark 4.4. By Lemma 4.10, if X is smooth variety, then $\mathrm{Acycl}_G(X, L, W) = \mathrm{Acycl}^{\mathrm{co}}_G(X, L, W)$ and hence $\mathrm{DQcoh}_G(X, L, W) = \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W)$.

Definition 4.5. A gauged LG model $(X, \mathcal{O}(\chi), 0)^{G\times\mathbb{G}_m}$ such that the potential is zero, the character $\chi : G \times \mathbb{G}_m \to \mathbb{G}_m$ is projection, and the action of $\mathbb{G}_m$ is trivial, is called **of σ-type**. If G is trivial, the gauged LG model $(X, \mathcal{O}(\chi), 0)^{\mathbb{G}_m}$ of σ-type is called **of trivial σ-type**.

The derived factorization category of a gauged LG models of σ-type is equivalent to bounded derived category of coherent sheaves on some algebraic stack.

Proposition 4.6 (cf.[BFK2], Corollary 2.3.12)**.** *Let $(X, \mathcal{O}(\chi), 0)^{G\times\mathbb{G}_m}$ be a gauged LG model of σ-type. Then we have equivalences*

$$\mathrm{Dcoh}_{G\times\mathbb{G}_m}(X, \mathcal{O}(\chi), 0) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{coh}[X/G]).$$

The following lemma is necessary to replace objects of $\mathrm{DQcoh}_G(X, L, W)$ or $\mathrm{Dcoh}_G(X, L, W)$ to ones with injective (or locally free) components. These replacements ensure that we can define derived functors between derived factorization categories from exact functors between homotopy categories of factorization categories.

Lemma 4.7 (cf. [LS], Lemma 2.10.)**.** *Assume that X is a smooth variety. Then we have*

(1) *For any $F \in \mathrm{Qcoh}_G(X, L, W)$ there exists a bounded exact sequence $0 \to F \to I^0 \to \cdots \to I^n \to 0$ in $Z^0(\mathrm{Qcoh}_G(X, L, W))$ with all $I^m \in \mathrm{Inj}_G(X, L, W)$. In particular, there is an isomorphism $F \to \mathrm{Tot}(I^\bullet)$ in $\mathrm{DQcoh}_G(X, L, W)$.*

(2) *For any object F of $\mathrm{Qcoh}_G(X, L, W)$ (resp. $\mathrm{coh}_G(X, L, W)$) there exists a bounded exact sequence $0 \to P^n \to \cdots \to P^0 \to F \to 0$ in $Z^0(\mathrm{Qcoh}_G(X, L, W))$ (resp. $Z^0(\mathrm{coh}_G(X, L, W))$) with all P^m in $\mathrm{LFr}_G(X, L, W)$ (resp. $\mathrm{lfr}_G(X, L, W)$). In particular, we have an isomorphism $\mathrm{Tot}(P^\bullet) \to F$.*

Proof. This is an equivariant version of [LS, Lemma 2.10]. Since $\mathrm{Qcoh}_G(X, L, W)$ has enough injective objects and for any equivariant sheaf $E \in \mathrm{Qcoh}_G(X)$ there exist an equivariant locally free sheaf P and surjection $P \to E$ (see e.g. [CG, Proposition 5.1.26]), the exact sequences can be constructed in a similar way as in [LS, Lemma 2.10]. $\square$

Lemma 4.8 ([BFK1] Proposition 3.11)**.** *Assume X is a smooth variety. We have*

$$\mathrm{Hom}_{H^0(\mathrm{Qcoh}_G(X,L,W))}(A, I) = 0$$

for any $A \in \mathrm{Acycl}_G(X, L, W)$ and $I \in H^0(\mathrm{Inj}_G(X, L, W))$. Moreover, the following compositions are equivalences;

$$H^0(\mathrm{Inj}_G(X, L, W)) \to H^0(\mathrm{Qcoh}_G(X, L, W)) \to \mathrm{DQcoh}_G(X, L, W)$$
$$H^0(\mathrm{inj}_G(X, L, W)) \to H^0(\mathrm{Qcoh}_G(X, L, W)) \to \mathrm{Dcoh}_G(X, L, W),$$

where $\mathrm{inj}_G(X, L, W)$ is the dg-subcategory of $\mathrm{Inj}_G(X, L, W)$ consisting of factorizations which are quasi-isomorphic to factorizations with coherent components.

Since the embedding $H^0(\mathrm{inj}_G(X, L, W)) \to H^0(\mathrm{Inj}_G(X, L, W))$ is fully faithful, so is

$$\mathrm{Dcoh}_G(X, L, W) \to \mathrm{DQcoh}_G(X, L, W)$$

by the above lemma. Hence we have a natural equivalence,

$$\mathrm{Dcoh}_G(X, L, W) \xrightarrow{\sim} \mathrm{D}_{\mathrm{coh}}\mathrm{Qcoh}_G(X, L, W)$$

Lemma 4.9 ([BFK1] Proposition 3.14)**.** *Assume X is a smooth variety. The following natural functors are equivalences:*

$$\mathrm{DLFr}_G(X,L,W) \to \mathrm{DQcoh}_G(X,L,W)$$
$$\mathrm{Dlfr}_G(X,L,W) \to \mathrm{Dcoh}_G(X,L,W)$$

Lemma 4.10 (cf. [LS], Corollary 2.23.)**.** *Assume X is a smooth variety. The categories $H^0(\mathrm{Qcoh}_G(X,L,W))$, $H^0(\mathrm{Inj}_G(X,L,W))$, $\mathrm{Acycl}_G(X,L,W)$ and $\mathrm{DQcoh}_G(X,L,W)$ are closed under arbitrary direct sums and therefore idempotent complete.*

Proof. We can prove this in a similar way as in [LS, Corollary 2.23]. $\square$

We define the supports of factorizations and complexes of factorizations as follows:

Definition 4.11. Let $E \in Z^0(\mathrm{coh}_G(X,L,W))$. **The support** $\mathrm{Supp}(E)$ of E is defined as

$$\mathrm{Supp}(E) := \mathrm{Supp}(E_1) \cup \mathrm{Supp}(E_0).$$

For an object $E^\bullet \in \mathrm{D^b}(Z^0(\mathrm{coh}_G(X,L,W)))$, we define **the support** $\mathrm{Supp}(E^\bullet)$ of $E^\bullet$ as

$$\mathrm{Supp}(E^\bullet) := \bigcup_{i\in\mathbb{Z}} \mathrm{Supp}(H^i(E^\bullet)).$$

Remark 4.12. By definition the support of $E^\bullet \in \mathrm{D^b}(Z^0(\mathrm{coh}_G(X,L,W)))$ is the union of supports of objects $E_i^\bullet \in \mathrm{D^b}(X)$, i.e. $\mathrm{Supp}(E^\bullet) = \bigcup_{i=0,1} \mathrm{Supp}(E_i^\bullet)$, where the support of a complex in $\mathrm{D^b}(X)$ is defined by the union of supports of its cohomologies.

In the following, we define properness of the "support" of an object in $\mathrm{Dcoh}_G(X,L,W)$ by using totalization.

Definition 4.13. Let $f : X \to Y$ be a morphism of schemes. A closed subset Z of X is called **f-proper** if the composition $Z \hookrightarrow X \xrightarrow{f} Y$ is a proper morphism. We denote by $\mathrm{coh}^f_{\sqcap G}(X,L,W)$ the full subcategory of $\mathrm{coh}_G(X,L,W)$ consisting of objects whose supports are f-proper.

Let F be an object in $\mathrm{Dcoh}_G(X,L,W)$. We say F **has a f-proper support** if there exists an object $F^\bullet \in \mathrm{D^b}(Z^0(\mathrm{coh}_G(X,L,W)))$ such that $\mathrm{Tot}(F^\bullet)$ is isomorphic to F in $\mathrm{Dcoh}_G(X,L,W)$ and the closed subset $\mathrm{Supp}(F^\bullet)$ is f-proper.

We denote by $\mathrm{D}^f_{\sqcap}\mathrm{coh}_G(X,L,W)$ the full subcategory of $\mathrm{Dcoh}_G(X,L,W)$ consisting of objects which have f-proper supports.

Remark 4.14.
(1) $\mathrm{D}^f_{\sqcap}\mathrm{coh}_G(X,L,W)$ is strictly full subcategory, i.e. closed under isomorphisms in $\mathrm{Dcoh}_G(X,L,W)$.
(2) If f is proper morphism then $\mathrm{D}^f_{\sqcap}\mathrm{coh}_G(X,L,W) = \mathrm{Dcoh}_G(X,L,W)$.
(3) Let $g : Y \to Z$ be another morphism of quasi-projective varieties. If $F \in \mathrm{Dcoh}_G(X,L,W)$ has a $g \circ f$-proper support, then F has a f-proper support.
(4) An object $E \in \mathrm{Dcoh}_G(X,L,W)$ which is quasi-isomorphic to $F \in \mathrm{coh}^f_{\sqcap G}(X,L,W)$ has a f-proper support.

4.2. **Functors of factorization categories of gauged LG models.** Throughout this section, we assume X is a smooth variety. In what follows, we define exact functors between derived equivariant factorization categories.

4.2.1. *Derived functors between triangulated categories.* In this section, we recall definitions and generalities on derived functors of exact functors of triangulated categories after [Mur]. Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{C}$ be a full subcategory of $\mathcal{D}$ with Verdier quotient $Q : \mathcal{D} \to \mathcal{D}/\mathcal{C}$. Throughout this section, all functor morphisms of exact functors are assumed to be commutative with shift functors, i.e. if $\alpha : F \to G$ is a functor morphism between exact functors $F, G : \mathcal{T} \to \mathcal{T}'$ of triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ with shift functors $\Sigma : \mathcal{T} \to \mathcal{T}$ and $\Sigma' : \mathcal{T}' \to \mathcal{T}'$, then α satisfies the commutativity of the following diagram of functor morphisms,

$$\begin{tikzcd} F\Sigma \arrow[r, "\sim"] \arrow[d, "F\alpha"'] & \Sigma' F \arrow[d, "\Sigma'\alpha"] \\ G\Sigma \arrow[r, "\sim"] & \Sigma' G. \end{tikzcd}$$

Definition 4.15. Let $F : \mathcal{D} \to \mathcal{T}$ be an exact functor of triangulated categories. The **right derived functor of** F (with respect to $\mathcal{C}$) is a pair $(\mathbf{R}F, \zeta)$ of an exact functor $\mathbf{R}F : \mathcal{D}/\mathcal{C} \to \mathcal{T}$ and functor morphism $\zeta : F \to \mathbf{R}F \circ Q$ with the following universal property: for any exact functor $G : \mathcal{D}/\mathcal{C} \to \mathcal{T}$ and functor morphism $\rho : F \to G \circ Q$ there is a unique functor morphism $\eta : \mathbf{R}F \to G$ making the following diagram commute:

$$\begin{tikzcd} & F \arrow[dl, "\zeta"'] \arrow[dr, "\rho"] & \\ \mathbf{R}F \circ Q \arrow[rr, "\eta Q"] & & G \circ Q. \end{tikzcd}$$

We will often drop the subcategory $\mathcal{C}$ and ζ from the notation, and say simply that $\mathbf{R}F$ is right derived functor of F.

Remark 4.16. By the definition, if right derived functor exists, it is unique up to natural equivalence.

Definition 4.17. Let $F : \mathcal{D} \to \mathcal{T}$ be an exact functor. An object $A \in \mathcal{D}$ is **right F-acyclic** with respect to $\mathcal{C}$ if the following condition holds: if $s : A \to B$ is a morphism with cone in $\mathcal{C}$, there is a morphism $t : B \to C$ with cone in $\mathcal{C}$ such that $F(ts)$ is an isomorphism.

Remark 4.18. If $A \in \mathcal{D}$ is a right F-acyclic with respect to $\mathcal{C}$ and in $\mathcal{C}$, then $F(A) = 0$.

The following theorem will be applied several times in the following sections to construct exact functors between derived factorization categories.

Theorem 4.19 ([Mur] Theorem 116)**.** *Let $F : \mathcal{D} \to \mathcal{T}$ be an exact functor. Assume $\mathcal{C}$ is a thick subcategory of $\mathcal{D}$. Suppose that for each object $X \in \mathcal{D}$ there exists a right F-acyclic object A_X and a morphism $\eta_X : X \to A_X$ with cone in $\mathcal{C}$. Then F admits a right derived functor $(\mathbf{R}F, \zeta)$ with the following properties*

(1) *For any object $X \in \mathcal{D}$ we have $\mathbf{R}F(X) = F(A_X)$ and $\zeta(X) = F(\eta_X)$.*
(2) *An object $X \in \mathcal{D}$ is right F-acyclic if and only if $\zeta(X)$ is an isomorphism in $\mathcal{T}$.*

There are similar definitions and results for left derived functors. See [Mur] for the detail.

4.2.2. *Direct and inverse image.* Let Y be another smooth quasi-projective variety with an action of G, defined by $\tau : G \times Y \to Y$, and let $f : X \to Y$ be an equivariant morphism, i.e. $f \circ \sigma = \tau \circ (1_G \times f)$.

For the morphism f, the direct image $f_* : \mathrm{Qcoh}_G(X) \to \mathrm{Qcoh}_G(Y)$ and the inverse image $f^* : \mathrm{Qcoh}_G(Y) \to \mathrm{Qcoh}_G(X)$ are defined by

$$f_*(\mathcal{F}, \theta) := (f_*(\mathcal{F}), (1 \times f)_*\theta) \quad \text{and} \quad f^*(\mathcal{F}, \theta) := (f^*\mathcal{F}, (1 \times f)^*\theta).$$

Let $L \in \mathrm{Pic}_G(Y)$ be an equivariant invertible sheaf on Y and let $W \in H^0(Y, L)^G$ be an invariant section of L. Then we have potentials (f^*L, f^*W) and (L, W) of $\mathrm{Qcoh}_G(X)$ and $\mathrm{Qcoh}_G(Y)$ respectively. By the natural isomorphisms of functors $f_*((-)\otimes f^*L) \cong f_*(-)\otimes L$ and $f^*((-) \otimes L) \cong f^*(-) \otimes f^*L$, we see that the direct image f_* and inverse image f^* are compatible with potentials with respect to (f^*L, f^*W) and (L, W) (see Definition 3.15). So we have direct image f_* and inverse image f^*, denoted by the same notation as usual ones, between factorization categories

$$f_* : \mathrm{Qcoh}_G(X, f^*L, f^*W) \to \mathrm{Qcoh}_G(Y, L, W)$$
$$f^* : \mathrm{Qcoh}_G(Y, L, W) \to \mathrm{Qcoh}_G(X, f^*L, f^*W).$$

Taking $H^0(-)$ of these dg-functors, we have exact functors

$$f_* : H^0(\mathrm{Qcoh}_G(X, f^*L, f^*W)) \to H^0(\mathrm{Qcoh}_G(Y, L, W))$$
$$f^* : H^0(\mathrm{Qcoh}_G(Y, L, W)) \to H^0(\mathrm{Qcoh}_G(X, f^*L, f^*W)).$$

Since these exact functors don't send acyclic objects to acyclic ones in general, we need to take derived functors of them. In the following, we give a proposition that implies existences of derived functors and two lemmas about them, following [LS]. Since the proofs are same as [LS], we will omit proofs.

Denote the following compositions by same notation f_* and f^*,

$$f_* : H^0(\mathrm{Qcoh}_G(X, f^*L, f^*W)) \to H^0(\mathrm{Qcoh}_G(Y, L, W)) \to \mathrm{DQcoh}_G(Y, L, W)$$
$$f^* : H^0(\mathrm{Qcoh}_G(Y, L, W)) \to H^0(\mathrm{Qcoh}_G(X, f^*L, f^*W)) \to \mathrm{DQcoh}_G(X, f^*L, f^*W).$$

By Lemma 4.7 and Theorem 4.19, we have the following:

Proposition 4.20 (cf. [LS] Theorem 2.35)**.**

1. *The functor* $f_* : H^0(\mathrm{Qcoh}_G(X, f^*L, f^*W)) \to \mathrm{DQcoh}_G(Y, L, W)$ *admits a right derived functor* $\mathbf{R}f_* : \mathrm{DQcoh}_G(X, f^*L, f^*W) \to \mathrm{DQcoh}_G(Y, L, W)$ *with respect to* $\mathrm{Acycl}_G(X, f^*L, f^*W)$.
2. *The functor* $f^* : H^0(\mathrm{Qcoh}_G(Y, L, W)) \to \mathrm{DQcoh}_G(X, f^*L, f^*W)$ *has a left derived functor* $\mathbf{L}f^*\mathrm{DQcoh}_G(Y, L, W) \to \mathrm{DQcoh}_G(X, f^*L, f^*W)$ *with respect to* $\mathrm{Acycl}_G(Y, L, W)$. *This left derived functor maps to* $\mathrm{Dcoh}_G(Y, L, W)$ *to* $\mathrm{Dcoh}_G(X, f^*L, f^*W)$.

The right derived functor $\mathbf{R}f_*$ doesn't map an object $E \in \mathrm{Dcoh}_G(X, f^*L, f^*W)$ to an object in $\mathrm{Dcoh}_G(Y, L, W)$ in general. But the following Lemma 4.21 implies that if E has a f-proper support, then $\mathbf{R}f_*(E)$ is isomorphic to an object in $\mathrm{Dcoh}_G(Y, L, W)$. In particular, if f is proper morphism then $\mathbf{R}f_*$ maps an object in $\mathrm{Dcoh}_G(X, f^*L, f^*W)$ to an object which is isomorphic to an object in $\mathrm{Dcoh}_G(Y, L, W)$ and we also denote by $\mathbf{R}f_*$ the following composition

$$\mathrm{Dcoh}_G(X, f^*L, f^*W) \xrightarrow{\mathbf{R}f_*} \mathrm{D_{coh}Qcoh}_G(Y, L, W) \xrightarrow{\sim} \mathrm{Dcoh}_G(Y, L, W).$$

Lemma 4.21 ([LS] Lemma 2.40)**.** *Let* $F \in \mathrm{Ch}^{\mathrm{b}}(Z^0(\mathrm{Qcoh}_G(Y, L, W)))$. *If each* $H^i(F) \in \mathrm{DQcoh}_G(Y, L, W)$ *is isomorphic to an object in* $\mathrm{Dcoh}_G(Y, L, W)$, *then so is* $\mathrm{Tot}(F)$.

Lemma 4.22 ([LS] Lemma 2.38). *Let $E = (E_1 \to E_0 \to E_1 \otimes f^*L) \in H^0(\mathrm{Qcoh}_G(X, f^*L, f^*W))$ and assume that $\mathbf{R}^i f_*(E_n) = 0$ in $\mathrm{Qcoh}_G(Y)$ for any $i > 0$ and each $n = 0, 1$. Then E is right f_*-acyclic. In particular, if f is affine morphism then we have a canonical isomorphism of functors $f_* \xrightarrow{\sim} \mathbf{R}f_*$.*

Similarly, if $F = (F_1 \to F_0 \to F_1 \otimes L) \in H^0(\mathrm{Qcoh}_G(Y, L, W))$ and $\mathbf{L}^j f^*(F_m) = 0$ in $\mathrm{Qcoh}_G(X)$ for any $j > 0$ and each $m = 0, 1$, then F is left f^*-acyclic. In particular, if f is flat morphism then $\mathbf{L}f^* \xrightarrow{\sim} f^*$.

Since the direct image $f_* : \mathrm{Qcoh}_G(X) \to \mathrm{Qcoh}_G(Y)$ is right cwp-adjoint to the inverse image $f^* : \mathrm{Qcoh}_G(Y) \to \mathrm{Qcoh}_G(X)$ with respect to potentials (f^*L, f^*W) and (L, W), $f_* : \mathrm{Qcoh}_G(X, f^*L, f^*W) \to \mathrm{Qcoh}_G(Y, L, W)$ is right adjoint to $f^* : \mathrm{Qcoh}_G(Y, L, W) \to \mathrm{Qcoh}_G(X, f^*L, f^*W)$, whose adjunction morphisms are of degree zero. Taking $H^0(-)$, we see that $f_* : H^0(\mathrm{Qcoh}_G(X, f^*L, f^*W)) \to H^0(\mathrm{Qcoh}_G(Y, L, W))$ is right adjoint to $f^* : H^0(\mathrm{Qcoh}_G(Y, L, W)) \to H^0(\mathrm{Qcoh}_G(X, f^*L, f^*W))$. Thus, by [Mur, Theorem 122], we obtain the following adjoint pair:

$$\mathbf{L}f^* \dashv \mathbf{R}f_*$$

4.2.3. *Tensor product and local Hom.* Let $L \in \mathrm{Pic}_G(X)$ and $V, W \in H^0(X, L)^G$.

Taking tensor product gives a bifunctor $(-)\otimes(-) : \mathrm{Qcoh}_G(X) \times \mathrm{Qcoh}_G(X) \to \mathrm{Qcoh}_G(X)$. Note that this functor is compatible with potentials with respect to potentials (L, V), (L, W) and $(L, V + W)$ (see Definition 3.27). So it induces a dg-bifuctor

$$(-) \otimes (-) : \mathrm{Qcoh}_G(X, L, V) \otimes \mathrm{Qcoh}_G(X, L, W) \to \mathrm{Qcoh}_G(X, L, V + W).$$

If we fix an object $P \in \mathrm{Qcoh}_G(X, L, W)$, we have an exact functor

$$(-) \otimes P : H^0(\mathrm{Qcoh}_G(X, L, V)) \to \mathrm{DQcoh}_G(X, L, V + W).$$

Proposition 4.23. *The functor $(-) \otimes P : H^0(\mathrm{Qcoh}_G(X, L, V)) \to \mathrm{DQcoh}_G(X, L, V + W)$ has a left derived functor $(-) \otimes^{\mathbf{L}} P : \mathrm{DQcoh}_G(X, L, V) \to \mathrm{DQcoh}_G(X, L, V + W)$ with respect to $\mathrm{Acycl}_G(X, L, V)$. If $P \in \mathrm{coh}_G(X, L, W)$ then this left derived functor maps $\mathrm{Dcoh}_G(X, L, V)$ to $\mathrm{Dcoh}_G(X, L, V + W)$.*

Proof. The proof is very similar to the proof of [LS, Theorem 2.35 (b)], and the detail is left to the reader. □

Definition 4.24. For any complex $C^\bullet \in \mathrm{D^b}(\mathrm{Qcoh}_G(X))$, we define an exact functor

$$(-) \otimes^{\mathbf{L}} C^\bullet : \mathrm{DQcoh}_G(X, L, W) \to \mathrm{DQcoh}_G(X, L, W)$$

as

$$E \otimes^{\mathbf{L}} C^\bullet := E \otimes^{\mathbf{L}} \Upsilon(C^\bullet),$$

where $\Upsilon : \mathrm{D^b}(\mathrm{Qcoh}_G(X)) \to \mathrm{DQcoh}_G(X, L, 0)$ is the functor defined in Definition 3.14. We denote by $E \otimes C^\bullet$ if $E \otimes^{\mathbf{L}} \Upsilon(C^\bullet) \cong E \otimes \Upsilon(C^\bullet)$.

Taking local Hom gives a bifunctor $\mathcal{H}om(-,-) : \mathrm{coh}_G(X)^{\mathrm{op}} \times \mathrm{Qcoh}_G(X) \to \mathrm{Qcoh}_G(X)$. Note that this bifunctor is compatible with potentials with respect to potentials (L, V), (L, W) and $(L, W - V)$ (see Definition 3.28). So it induces a dg-bifunctor

$$\mathcal{H}om(-,-) : \mathrm{coh}_G(X, L, V) \otimes \mathrm{Qcoh}_G(X, L, W) \to \mathrm{Qcoh}_G(X, L, W - V).$$

If we fix an object $Q \in \mathrm{coh}_G(X, L, V)$, we have an exact functor

$$\mathcal{H}om(Q, -) : H^0(\mathrm{Qcoh}_G(X, L, W)) \to \mathrm{DQcoh}_G(X, L, W - V)$$

Proposition 4.25. *The functor* $\mathcal{H}om(Q,-) : H^0(\mathrm{Qcoh}_G(X,L,W)) \to \mathrm{DQcoh}_G(X,L,W-V)$ *has a right derived functor* $\mathbf{R}\mathcal{H}om(Q,-) : \mathrm{DQcoh}_G(X,L,W) \to \mathrm{DQcoh}_G(X,L,W-V)$ *with respect to* $\mathrm{Acycl}_G(X,L,W)$*.*

Proof. The proof is very similar to the proof of [LS, Theorem 2.35 (a)], and the detail is left to the reader. □

By Lemma 4.21, if $E \in \mathrm{Dcoh}_G(X,L,W)$, then $\mathbf{R}\mathcal{H}om(Q,E) \in \mathrm{D}_{\mathrm{coh}}\mathrm{Qcoh}_G(X,L,W-V)$. We use same notation $\mathbf{R}\mathcal{H}om(Q,-)$ for the composition

$$\mathrm{Dcoh}_G(X,L,W) \xrightarrow{\mathbf{R}\mathcal{H}om(Q,-)} \mathrm{D}_{\mathrm{coh}}\mathrm{Qcoh}_G(X,L,W-V) \xrightarrow{\sim} \mathrm{Dcoh}_G(X,L,W-V).$$

Lemma 4.26. *Let* $E = (E_1 \to E_0 \to E_1 \otimes L) \in H^0(\mathrm{Qcoh}_G(X,L,V))$ *and* $P = (P_1 \to P_0 \to P_1 \otimes L) \in \mathrm{Qcoh}_G(X,L,W)$*. If* $\mathcal{T}or^i(E_n,P_m) = 0$ *for any* $i > 0$ *and any* $n,m \in \{0,1\}$*, then* E *is* $(-)\otimes P$*-acyclic object. In particular, if* $P \in \mathrm{LFr}_G(X,L,W)$*, then there is an isomorphism of exact functors* $(-)\otimes^{\mathbf{L}} P \xrightarrow{\sim} (-)\otimes P$*.*

Let $F = (F_1 \to F_0 \to F_1 \otimes L) \in H^0(\mathrm{Qcoh}_G(X,L,W))$ *and* $Q = (Q_1 \to Q_0 \to Q_1 \otimes L) \in \mathrm{coh}_G(X,L,V)$*. If* $\mathcal{E}xt^i(Q_n,F_m) = 0$ *for each* $i > 0$ *and any* $n,m \in \{0,1\}$*, then* F *is* $\mathcal{H}om(Q,-)$*-acyclic object. In particular, if* $Q \in \mathrm{lfr}_G(X,L,V)$*, there is an isomorphism of exact functors* $\mathcal{H}om(Q,-) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om(Q,-)$*.*

Proof. The proof is similar to [LS, Lemma 2.38], and we leave the detail to the reader. □

Remark 4.27. In the above lemma, we can take P and Q as objects whose components are flat sheaves.

Proposition 4.28 ([BFK1] Proposition 3.27)**.** *Let* $R \in \mathrm{coh}_G(X,L,V)$*. Then* $\mathcal{H}om(R,-) : \mathrm{Qcoh}_G(X,L,W) \to \mathrm{Qcoh}_G(X,L,W-V)$ *is right adjoint to* $(-)\otimes R : \mathrm{Qcoh}_G(X,L,W-V) \to \mathrm{Qcoh}_G(X,L,W)$*.*

$\mathcal{H}om(R,-) : H^0(\mathrm{Qcoh}_G(X,L,W)) \to H^0(\mathrm{Qcoh}_G(X,L,W-V))$ is right adjoint to $(-)\otimes R : H^0(\mathrm{Qcoh}_G(X,L,W-V)) \to H^0(\mathrm{Qcoh}_G(X,L,W))$ by the above proposition. If $I \in \mathrm{Inj}_G(X,L,W)$, $J \in \mathrm{Inj}_G(X,L,W-V)$ and $F \in \mathrm{lfr}_G(X,L,V)$, then $\mathcal{H}om(F,I) \in \mathrm{Inj}_G(X,L,W-V)$ and $J \otimes R \in \mathrm{Inj}_G(X,L,W)$. Hence by Lemma 4.8 and Lemma 4.9 we obtain an adjoint pair,

$$(-)\otimes^{\mathbf{L}} R \dashv \mathbf{R}\mathcal{H}om(R,-).$$

Definition 4.29. Let $\mathcal{O}_X := (0 \to \mathcal{O}_X \to 0) \in \mathrm{coh}_G(X,L,0)$. Then we define functors

$$(-)^{\vee} := \mathcal{H}om(-,\mathcal{O}_X) : \mathrm{coh}_G(X,L,W)^{\mathrm{op}} \to \mathrm{coh}_G(X,L,-W)$$
$$(-)^{\mathbf{L}\vee} := \mathbf{R}\mathcal{H}om(-,\mathcal{O}_X) : \mathrm{Dcoh}_G(X,L,W)^{\mathrm{op}} \to \mathrm{Dcoh}_G(X,L,-W).$$

Lemma 4.30 ([BFK1] Lemma 3.30, 3.11)**.** *The functor,*

$$(-)^{\mathbf{L}\vee} : \mathrm{Dcoh}_G(X,L,W)^{\mathrm{op}} \to \mathrm{Dcoh}_G(X,L,-W)$$

is an equivalence.

For $F \in \mathrm{lfr}_G(X,L,W)$*, we have an isomorphism of functors,*

$$F^{\vee}\otimes(-) \cong \mathcal{H}om(F,-).$$

For $E \in \mathrm{Dcoh}_G(X,L,W)$*, there is an isomorphism of functors,*

$$E^{\mathbf{L}\vee}\otimes^{\mathbf{L}}(-) \cong \mathbf{R}\mathcal{H}om(E,-).$$

Lemma 4.31. *Let $E \in \mathrm{Dcoh}_G(X, L, V)$ and $F \in \mathrm{Dcoh}_G(X, L, W)$. Let Y be a smooth quasi-projective variety and let $f : X \to Y$ be a morphism. If E has a f-proper support, both of $E \otimes^{\mathbf{L}} F$ and $\mathbf{R}\mathcal{H}om(E, F)$ have f-proper supports. In particular, if E has a f-proper support, so is $E^{\mathbf{L}\vee}$.*

Proof. By the assumption, there exists an object $E^\bullet \in \mathrm{D^b}(Z^0(\mathrm{coh}_G(X, L, V)))$ such that $\mathrm{Tot}(E^\bullet) \cong E$ and the morphism $\mathrm{Supp}(E^\bullet) \to Y$ is proper. Since X is smooth and for any $M \in \mathrm{coh}_G(X)$ there exists a locally free equivariant sheaf P and a surjection $P \to M$, there exists an object $P^\bullet \in \mathrm{D^b}(Z^0(\mathrm{lfr}_G(X, L, V)))$ which is isomorphic to $E^\bullet$ in $\mathrm{D^b}(Z^0(\mathrm{coh}_G(X, L, V)))$. Then we have

$$E \otimes^{\mathbf{L}} F \cong \mathrm{Tot}(P^\bullet) \otimes F \cong \mathrm{Tot}(P^\bullet \otimes F)$$

and

$$\mathbf{R}\mathcal{H}om(E, F) \cong \mathcal{H}om(\mathrm{Tot}(P^\bullet), F) \cong \mathrm{Tot}(\mathcal{H}om(P^\bullet, F)).$$

Hence it is sufficient to prove that closed subsets $\mathrm{Supp}(P^\bullet \otimes F)$ and $\mathrm{Supp}(\mathcal{H}om(P^\bullet, F))$ are contained in $\mathrm{Supp}(P^\bullet)$. But this follows from equalities

$$\mathrm{Supp}(P^\bullet \otimes F) = \bigcup_{i,j=0,1} \mathrm{Supp}(P_i^\bullet \otimes F_j)$$

$$\mathrm{Supp}(\mathcal{H}om(P^\bullet, F)) = \bigcup_{k,l=0,1} \mathrm{Supp}(\mathcal{H}om(P_k^\bullet, F_l))$$

and the fact that for $A^\bullet, B^\bullet \in \mathrm{D^b}(X)$, we have $\mathrm{Supp}(A^\bullet \otimes^{\mathbf{L}} B^\bullet) \subset \mathrm{Supp}(A^\bullet)$ and $\mathrm{Supp}(\mathbf{R}\mathcal{H}om(A^\bullet, B^\bullet)) \subset \mathrm{Supp}(A^\bullet)$. □

4.2.4. *Projection formula, flat base change and Grothendieck duality.* Let X and Y be smooth quasi-projective varieties and let G be an affine algebraic group acting on X and Y. Let $f : X \to Y$ be an equivariant morphism. Take $L \in \mathrm{Pic}_G(Y)$ and $W \in H^0(Y, L)^G$.

The following proposition is a version of projection formula for factorization categories.

Proposition 4.32 ([BFK1] Lemma 3.38)**.** *For objects $E \in \mathrm{DQcoh}_G(Y, L, W)$ and $F \in \mathrm{DQcoh}_G(X, f^*L, f^*W)$, we have a natural isomorphism of exact functors,*

$$\mathbf{R}f_*F \otimes^{\mathbf{L}} E \cong \mathbf{R}f_*(F \otimes^{\mathbf{L}} \mathbf{L}f^*E).$$

Let Z be another smooth quasi-projective variety with G-action and let $u : Z \to Y$ be an equivariant flat morphism. Consider the fiber product $W := X \times_Y Z$,

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "u'"'] & Z \arrow[d, "u"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Lemma 4.33 (cf. [BFK1] Lemma 2.19)**.** *We have a natural isomorphism of functors between coherent sheaves,*

$$u^* \circ f_* \cong f'_* \circ u'^* : \mathrm{Qcoh}_G(X) \to \mathrm{Qcoh}_G(Z).$$

Note that the above natural isomorphism of functors is a cwp-functor morphism. By Lemma 3.19 (2), we have an induced isomorphism of functors between factorizations,

$$u^* \circ f_* \cong f'_* \circ u'^* : \mathrm{Qcoh}_G(X, f^*L, f^*W) \to \mathrm{Qcoh}_G(Z, u^*L, u^*W).$$

Since this isomorphism of dg-functors is of degree zero, there is a natural isomorphism of exact functors,

$$u^* \circ f_* \cong f'_* \circ u'^* : H^0(\mathrm{Qcoh}_G(X, f^*L, f^*W)) \to H^0(\mathrm{Qcoh}_G(Z, u^*L, u^*W)).$$

Since u and u' are flat, we have $\mathbf{L}u^* \cong u^*$ and $\mathbf{L}u'^* \cong u'^*$. For $E \in \mathrm{DQcoh}_G(X, f^*L, f^*W)$, let $I \in \mathrm{Inj}_G(X, f^*L, f^*W)$ be an object which is quasi-isomorphic to E. Then we have

$$u^* \circ \mathbf{R}f_*(E) \cong u^*(f_*(I)) \cong f'_*(u'^*(I)).$$

By the second property of right derived functor in Theorem 4.19 and Lemma 4.22, we see that $u'^*(I)$ is right f_*-acyclic, which implies $f'_*(u'^*(I)) \cong \mathbf{R}f'_*(u'^*(I))$. Hence we have the following:

Lemma 4.34. *We have a natural isomorphism of functors*

$$u^* \circ \mathbf{R}f_* \cong \mathbf{R}f'_* \circ u'^* : \mathrm{DQcoh}_G(X, f^*L, f^*W) \to \mathrm{DQcoh}_G(Z, u^*L, u^*W).$$

Definition 4.35. Let $\varphi : X_1 \to X_2$ be a equivariant morphism of smooth G-varieties. We define the **relative dualizing bundle** $\omega_\varphi \in \mathrm{Pic}_G(X_1)$ as

$$\omega_\varphi := \omega_{X_1} \otimes \varphi^* \omega_{X_2}^\vee,$$

where $\omega_{X_i} \in \mathrm{Pic}_G(X_i)$ is the canonical bundle on X_i with tautological equivariant structure.

In [EP], Positselski proved a version of Grothendieck duality for derived factorization categories. In the following we give an immediate consequence of the Positselski's result.

Theorem 4.36 (cf. [EP] Theorem 3.8)**.** *If f is proper,* $\mathbf{R}f_* : \mathrm{DQcoh}(X, f^*L, f^*W) \to \mathrm{DQcoh}(Y, L, W)$ *has a right adjoint functor* $f^! : \mathrm{DQcoh}(Y, L, W) \to \mathrm{DQcoh}(X, f^*L, f^*W)$. *An explicit form of the functor $f^!$ is the following:*

$$f^!(-) \cong \mathbf{L}f^*(-) \otimes \omega_f[\dim(X) - \dim(Y)],$$

where the tensor product on the right hand side is given by Definition 4.24.

Proof. Let $D_Y^\bullet$ be a dualizing complex on Y and write $D_X^\bullet := f^+ D_Y^\bullet$, where f^+ is a right adjoint functor of the direct image $\mathbf{R}f_* : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ of derived categories of coherent sheaves. By [EP, Theorem 3.8], for any object $E \in \mathrm{D^{co}Qcoh}(X, f^*L, f^*W)$ and an object $F \in \mathrm{D^{co}Qcoh}(Y, L, W)$ whose components F_i are flat sheaves, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{D^{co}Qcoh}(Y,L,W)}(\mathbf{R}f_*E, F \otimes D_Y^\bullet) \cong \mathrm{Hom}_{\mathrm{D^{co}Qcoh}(X,f^*L,f^*W)}(E, f^*(F) \otimes D_X^\bullet).$$

Since X and Y are smooth, co-derived factorization categories are equal to absolute derived factorization categories by Remark 4.4, and the structure sheaf $\mathcal{O}_Y$ is quasi-isomorphic to a dualizing complex. We have $f^+\mathcal{O}_Y \cong \omega_X \otimes f^*\omega_Y^{-1}[\dim(X) - \dim(Y)]$. Since for any object of $\mathrm{DQcoh}(Y, L, W)$ is isomorphic to an object whose components are locally free, in particular, flat, we obtain the theorem. $\square$

4.2.5. *Extension by zero.* In this section we construct a relative left adjoint functor $i_!$ of the inverse image i^* of an open immersion i.

Let U be an open subvariety of X and let $i : U \hookrightarrow X$ be the open immersion. In what follows we don't consider G-actions until the next section.

Definition 4.37. For $F \in \mathrm{coh}(U)$, let $\overline{F}$ be coherent sheaf on X such that $\overline{F}|_U \cong F$. Let $\widetilde{\mathbb{Z}_{\geq 0}}$ be the category such that $\mathrm{Ob}(\widetilde{\mathbb{Z}_{\geq 0}}) = \mathbb{Z}_{\geq 0}$ and whose sets of morphisms are defined as follows:

$$\mathrm{Hom}_{\widetilde{\mathbb{Z}_{\geq 0}}}(n, m) = \begin{cases} \emptyset & \text{if } n < m \\ \{\geq^n_m\} & \text{if } n \geq m \end{cases}$$

Then we define an object $i_!(F) \in \mathrm{Pro}(\mathrm{coh}(X))$ as a functor $i_!(F) : \widetilde{\mathbb{Z}_{\geq 0}} \to \mathrm{coh}(X)$ defined by

$$i_!(F)(n) := \mathcal{I}^n \overline{F},$$

where $\mathcal{I}$ is the ideal sheaf defining the complement $X \setminus U$. Since the object $i_!(F)$ doesn't depend on the choice of an extension $\overline{F}$ by the following Lemma 4.38, this gives an exact functor

$$i_! : \mathrm{coh}(U) \to \mathrm{Pro}(\mathrm{coh}(X)).$$

The functor $i_!$ is called **the extension by zero** of i. We also denote by $i_!$ the composition

$$\mathrm{coh}(U) \xrightarrow{i_!} \mathrm{Pro}(\mathrm{coh}(X)) \hookrightarrow \mathrm{Pro}(\mathrm{Qcoh}(X)).$$

Lemma 4.38. *Let $F \in \mathrm{coh}(U)$ be an coherent sheaf on U, and let $N \in \mathrm{coh}(X)$ and $M \in \mathrm{Qcoh}(X)$ be subsheaves of $i_*(F) \in \mathrm{Qcoh}(X)$. If $i^*(N)$ is contained in $i^*(M)$, then there is a positive integer n such that $\mathcal{I}^n N$ is contained in M.*

Proof. Since we can take finite affine covering, it is enough to prove it for the case $X = \mathrm{Spec}(A)$ and $U = \mathrm{Spec}(A_f)$ for some ring A and an element $f \in A$. Then $\mathcal{I}$ corresponds to the ideal $I = \langle f \rangle$ of A generated by f. We consider F, N and M as corresponding modules. Let $\{x_k\}_{1 \leq k \leq r} \subset N$ be a generator of N. Since $i^*(N) = N \otimes_A A_f$ is contained in $i^*(M) = M \otimes_A A_f$, for each k, there is an element $y_k \in M$ and $n_k \geq 0$ such that $x_k \otimes 1 = y_k \otimes 1/f^{n_k}$ in $i_*(F) \otimes A_f$. This implies that $f^{n_k} x_k = y_k \in M$, since $i_*(F) \otimes A_f \cong F$. Set $n := \max\{n_k | 1 \leq k \leq r\}$. Then we have $I^n N \subset M$. □

Deligne proved that the extension by zero $i_!$ is a relative left adjoint to the inverse image i^*.

Proposition 4.39 (cf. [Del] Proposition 4)**.** *For any $F \in \mathrm{coh}(U)$ and $(G : \mathcal{I} \to \mathrm{Qcoh}(X)) \in \mathrm{Pro}(\mathrm{Qcoh}(X))$, we have an isomorphism*

$$\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Qcoh}(X))}(i_!(F), G) \cong \mathrm{Hom}_{\mathrm{Pro}(\mathrm{Qcoh}(U))}(J(F), \mathrm{Pro}(i^*)(G)),$$

where $J : \mathrm{coh}(U) \to \mathrm{Pro}(\mathrm{Qcoh}(U))$ is the natural inclusion.

Proof. This is shown as follows;

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Pro}(\mathrm{Qcoh}(X))}(i_!(F), G) &= \varprojlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathrm{Pro}(\mathrm{Qcoh}(X))}(i_!(F), G(i)) \\ &\cong \varprojlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathrm{Qcoh}(U)}(F, i^* G(i)) \\ &= \mathrm{Hom}_{\mathrm{Pro}(\mathrm{Qcoh}(U))}(J(F), \mathrm{Pro}(i^*)(G)), \end{aligned}$$

where the isomorphism in the second line follows from [Del, Proposition 4]. □

Let $L \in \mathrm{Pic}(X)$ and let $W \in H^0(X, L)$. Then $(\mathrm{Pro}(L), \mathrm{Pro}(W))$ is a potential of $\mathrm{Pro}(\mathrm{Qcoh}(X))$ and $\mathrm{Pro}(\mathrm{coh}(X))$. We denote their factorization categories by

$$\mathrm{Qcoh}_{\mathrm{Pro}}(X, L, W) := \mathfrak{F}(\mathrm{Pro}(\mathrm{Qcoh}(X)), \mathrm{Pro}(L), \mathrm{Pro}(W))$$

$$\mathrm{coh}_{\mathrm{Pro}}(X, L, W) := \mathfrak{F}(\mathrm{Pro}(\mathrm{coh}(X)), \mathrm{Pro}(L), \mathrm{Pro}(W)).$$

The extension by zero $i_!$ is compatible with potentials with respect to $(L|_U, W|_U)$ and $(\mathrm{Pro}(L), \mathrm{Pro}(W))$. Hence the functor $i_!$ induces a dg-functor

$$i_! : \mathrm{coh}(U, L|_U, W|_U) \to \mathrm{coh}_{\mathrm{Pro}}(X, L, W).$$

Since $i_! : \mathrm{coh}(U) \to \mathrm{Pro}(\mathrm{coh}(X))$ is an exact functor of abelian categories, $i_!$ preserves acyclic objects. Hence $i_! : H^0(\mathrm{coh}(U, L|_U, W|_U)) \to H^0(\mathrm{coh}_{\mathrm{Pro}}(X, L, W))$ naturally induces an exact functor

$$i_! : \mathrm{Dcoh}(U, L|_U, W|_U) \to \mathrm{Dcoh}_{\mathrm{Pro}}(X, L, W).$$

On the other hand, there is a natural functor $\mathrm{Dcoh}_{\mathrm{Pro}}(X, L, W) \to \mathrm{Pro}(\mathrm{Dcoh}(X, L, W))$. Composing it with the embedding $\mathrm{Pro}(\mathrm{Dcoh}(X, L, W)) \to \mathrm{Pro}(\mathrm{DQcoh}(X, L, W))$ and $i_! : \mathrm{Dcoh}(U, L|_U, W|_U) \to \mathrm{Dcoh}_{\mathrm{Pro}}(X, L, W)$, we construct a functor

$$i_! : \mathrm{Dcoh}(U, L|_U, W|_U) \to \mathrm{Pro}(\mathrm{DQcoh}(X, L, W)),$$

which is also denoted by the same notation $i_!$.

Proposition 4.40. (1) *The dg-functor* $i_! : \mathrm{coh}(U, L|_U, W|_U) \to \mathrm{Qcoh}_{\mathrm{Pro}}(X, L, W)$ *is* J*-left adjoint to* $\mathrm{Pro}(i^*) : \mathrm{Qcoh}_{\mathrm{Pro}}(X, L, W) \to \mathrm{Qcoh}_{\mathrm{Pro}}(U, L|_U, W|_U)$*, where* J *is the natural embedding functor* $J : \mathrm{coh}(U, L|_U, W|_U) \to \mathrm{Qcoh}_{\mathrm{Pro}}(U, L|_U, W|_U)$*.*
(2) *For any* $E \in \mathrm{Dcoh}(U, L|_U, W|_U)$ *and* $F \in \mathrm{DQcoh}(X, L, W)$*, we have an isomorphism*

$$\mathrm{Hom}_{\mathrm{Pro}(\mathrm{DQcoh}(X,L,W))}(i_!(E), \iota_{\mathrm{Pro}}(F)) \cong \mathrm{Hom}_{\mathrm{DQcoh}(U,L|_U,W|_U)}(E, i^*(F)).$$

Proof. (1) Consider the following diagram,

$$\begin{tikzcd} \mathrm{coh}(U) \arrow[rr, "i_!"] \arrow[dr, "J"'] & & \mathrm{Pro}(\mathrm{Qcoh}(X)) \arrow[dl, "\mathrm{Pro}(i^*)"] \\ & \mathrm{Pro}(\mathrm{Qcoh}(U)) & \end{tikzcd}$$

where J is the natural embedding. Then Proposition 4.39 implies that $i_!$ is J-left adjoint to $\mathrm{Pro}(i^*)$ (see Definition 3.22). Hence, (1) holds since the front adjunction $J \to \mathrm{Pro}(i^*) \circ i_!$ is a cwp-functor morphism.
(2) Let $F \xrightarrow{\sim} J$ be an isomorphism in $\mathrm{DQcoh}(X, L, W)$ such that the components of I are injective quasi-coherent sheaves. Then $i^*(I)$ is an object whose components are injective quasi-coherent sheaves on U. By (1) and Lemma 4.8, the right hand side of the desired isomorphism is isomorphic to

$$H^0(\{\varinjlim_{n\in\mathbb{Z}_{\geq 0}} \mathrm{Hom}_{\mathrm{Qcoh}(X,L,W)}(\mathcal{I}^n\overline{E}, I)\}^\bullet).$$

Since taking direct limit is an exact functor, the above abelian group is isomorphic to

$$\varinjlim_{n\in\mathbb{Z}_{\geq 0}} H^0(\mathrm{Hom}_{\mathrm{Qcoh}(X,L,W)}(\mathcal{I}^n\overline{E}, I)^\bullet),$$

which is isomorphic to the left hand side of the desired isomorphism by Lemma 4.8 again. □

For later use, we will extend the extension by zero $i_! : \mathrm{coh}(U) \to \mathrm{Pro}(\mathrm{Qcoh}(X))$ to a functor defined on $\mathrm{Qcoh}(U)$. To do it, we need the following lemma.

Lemma 4.41 ([Del], Proposition 2)**.** *Let* Y *be a Noetherian scheme, and let* $F \in \mathrm{Qcoh}(Y)$ *be a quasi-coherent sheaf. Denote by* $\{F_k\}_{k\in K}$ *the family of all coherent subsheaves of* F*. Let* $\theta(F) \in \mathrm{Ind}(\mathrm{coh}(Y))$ *be a functor given by*

$$\begin{array}{ccc} K & \xrightarrow{\theta(F)} & \mathrm{coh}(Y) \\ \cup & & \cup \\ k & \longmapsto & F_k \end{array}$$

Then $\theta(-)$ *gives an exact equivalence*

$$\theta : \mathrm{Qcoh}(Y) \xrightarrow{\sim} \mathrm{Ind}(\mathrm{coh}(Y)).$$

Definition 4.42. We define an exact functor

$$i_{\#} : \mathrm{Qcoh}(U) \to \mathrm{Ind}(\mathrm{Pro}(\mathrm{Qcoh}(X)))$$

as the compositions

$$\mathrm{Qcoh}(U) \xrightarrow{\theta} \mathrm{Ind}(\mathrm{coh}(U)) \xrightarrow{\mathrm{Ind}(i_!)} \mathrm{Ind}(\mathrm{Pro}(\mathrm{coh}(X))) \hookrightarrow \mathrm{Ind}(\mathrm{Pro}(\mathrm{Qcoh}(X)))$$

Remark 4.43. By the construction of $i_{\#}$, we have a natural isomorphism of functors

$$i_{\#}|_{\mathrm{coh}(U)} \cong \iota_{\mathrm{Ind}} i_!.$$

The following lemma will be necessary to prove Lemma 5.2.

Lemma 4.44. *The notation is the same as above.*

(1) *We have a natural functor morphism*

$$\gamma_{\#} : i_{\#} \to \iota_{\mathrm{InPr}} i_*,$$

where $\iota_{\mathrm{InPr}} : \mathrm{Qcoh}(X) \to \mathrm{Ind}(\mathrm{Pro}(\mathrm{Qcoh}(X)))$. *Restricting* $\gamma_{\#}$, *we obtain a natural functor morphism*

$$\gamma_! : i_! \to \iota_{\mathrm{Pro}} i_*.$$

such that $\iota_{\mathrm{Ind}}\gamma_! = \gamma_{\#}|_{\mathrm{coh}(U)}$.

(2) *Consider the following cartesian square:*

$$\begin{tikzcd} V := U \times_X Y \arrow[r, "j"] \arrow[d, "q"'] & Y \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

We have a morphism between functors from $\mathrm{coh}(V)$ *to* $\mathrm{Ind}(\mathrm{Pro}(\mathrm{Qcoh}(Y)))$

$$\lambda : j_{\#}q^*q_* \to \iota_{\mathrm{Ind}}\mathrm{Pro}(p^*p_*)j_!$$

such that the following diagram is commutative:

$$\begin{tikzcd} \iota_{\mathrm{InPr}} j_* q^* q_* \arrow[r, "\iota_{\mathrm{InPr}}\delta"] & \iota_{\mathrm{InPr}} p^* p_* j_* \arrow[r, equal] & \iota_{\mathrm{Ind}}\mathrm{Pro}(p^*p_*)\iota_{\mathrm{Pro}} j_* \\ j_{\#} q^* q_* \arrow[u, "\gamma q^* q_*"] \arrow[rr, "\lambda"] & & \iota_{\mathrm{Ind}}\mathrm{Pro}(p^*p_*)j_! \arrow[u, "\iota_{\mathrm{Ind}}\mathrm{Pro}(p^*p_*)\gamma|_{\mathrm{coh}(V)}"'] \end{tikzcd}$$

where $\delta : j_*q^*q_* \xrightarrow{\sim} p^*p_*j_*$ *is a natural isomorphism of functors.*

Proof. (1) Let $F \in \mathrm{Qcoh}(U)$ be a quasi-coherent sheaf on U, and let $\{F_k\}_{k\in K}$ be the family of all coherent subsheaves of F. By definition, $i_{\#}(F) : K \to \mathrm{Pro}(\mathrm{Qcoh}(U))$ is a functor given by $i_{\#}(F)(k) = i_!(F_k)$, and the object $i_!(F_k) \in \mathrm{Pro}(\mathrm{Qcoh}(U))$ is the functor given by

$$\mathbb{Z}_{\geq 0} \ni n \mapsto \mathcal{I}^n\overline{F_k} \in \mathrm{coh}(U),$$

where $\overline{F_k}$ is a coherent subsheaf of $i_*(F_k)$ such that $i^*(\overline{F_k}) \cong F_k$. Hence, the natural inclusion $\overline{F_k} \hookrightarrow i_*(F)$ gives a morphism of functors

$$\gamma : i_{\#} \to \iota_{\mathrm{InPr}} i_*.$$

(2) For $F \in \mathrm{coh}(V)$, we will define a morphism $\lambda(F) : j_{\#}q^*q_*(F) \to \iota_{\mathrm{Ind}}\mathrm{Pro}(p^*p_*)j_!(F)$. Let $\{E_k\}_{k\in K}$ be the family of all coherent subsheaves of $q^*q_*(F)$. Then the object $j_{\#}q^*q_*(F) \in \mathrm{Ind}(\mathrm{Pro}(\mathrm{Qcoh}(Y)))$ is given by the following functor

$$\begin{array}{ccc} K & \longrightarrow & \mathrm{Pro}(\mathrm{Qcoh}(Y)) \\ \cup & & \cup \\ k & \longmapsto & j_!(E_k) \end{array}$$

In order to define a morphism $\lambda(F) : j_{\#}q^*q_*(F) \to \iota_{\mathrm{Ind}}\mathrm{Pro}(p^*p_*)j_!(F)$, it is enough to give a family of morphisms $\{\lambda(F)_k : j_!(E_k) \to \mathrm{Pro}(p^*p_*)j_!(F)\}_{k\in K}$ in $\mathrm{Pro}(\mathrm{Qcoh}(Y))$ such that for any inclusion $v : E_k \hookrightarrow E_l$, the equation $\lambda(F)_k = \lambda(F)_l j_!(v)$ holds. Let $\mathcal{J}$ be the ideal sheaf defining $Y \setminus V$, and let $\overline{E_k}$ and $\overline{F}$ be coherent subsheaves of $j_*(E_k)$ and $j_*(F)$ with $j^*(\overline{E_k}) \cong E_k$ and $j^*(\overline{F}) \cong F$ respectively. Then the object $j_!(E_k)$ and $\mathrm{Pro}(p^*p_*)j_!(F)$ are the following functors

$$\begin{array}{ccc} \mathbb{Z}_{\geq 0} & \xrightarrow{j_!(E_k)} & \mathrm{Qcoh}(Y) \\ \Psi & & \Psi \\ n & \longmapsto & \mathcal{J}^n\overline{E_k} \end{array}$$

$$\begin{array}{ccc} \mathbb{Z}_{\geq 0} & \xrightarrow{\mathrm{Pro}(p^*p_*)j_!(F)} & \mathrm{Qcoh}(Y) \\ \Psi & & \Psi \\ m & \longmapsto & \mathcal{J}^m p^*p_*(\overline{F}) \end{array}$$

$\overline{E_k}$ is contained in $j_*q^*q_*(F)$ and $p^*p_*(\overline{F})$ can be considered as a subsheaf of $j_*q^*q_*(F)$ via the isomorphism $\delta(F) : j_*q^*q_*(F) \xrightarrow{\sim} p^*p_*j_*(F)$. Since $j^*\overline{E_k} \cong E_k$ is contained in $j^*p^*p_*(\overline{F}) \cong q^*q_*(F)$, there is a positive integer N such that $\mathcal{J}^N\overline{E_k}$ is a subsheaf of $p^*p_*(\overline{F})$ by Lemma 4.38. Let $\theta_{\lambda(F)_k} : \mathbb{Z}_{\geq 0} \ni n \mapsto n + N \in \mathbb{Z}_{\geq 0}$ be a map, and let $\lambda(F)_k^n : j_!(E_k)(n+N) \to \mathrm{Pro}(p^*p_*)j_!(F)(n)$ be a morphism induced by the inclusion $\mathcal{J}^N\overline{E_k} \hookrightarrow p^*p_*(\overline{F})$. If we define a morphism $\lambda(F)_k : j_!(E_k) \to \mathrm{Pro}(p^*p_*)j_!(F)$ as a map of systems $(\{\lambda(F)_k^n\}_{n\in\mathbb{Z}_{\geq 0}}, \theta_{\lambda(F)_k})$ for each $k \in K$, then the family $\{\lambda(F)_k\}_{k\in K}$ defines a morphism $\lambda(F) : j_{\#}q^*q_*(F) \to \iota_{\mathrm{Ind}}\mathrm{Pro}(p^*p_*)j_!(F)$, and this gives a functor morphism

$$\lambda : j_{\#}q^*q_* \to \iota_{\mathrm{Ind}}\mathrm{Pro}(p^*p_*)j_!.$$

The commutativity of the diagram follows since γ is induced by natural inclusions, and λ is also induced by natural inclusions via δ. □

4.2.6. *Integral functor for factorization.* Let X_1 and X_2 be smooth quasi-project varieties with actions of affine algebraic group G. Take a character χ of G, and let $\mathcal{O}_i(\chi)$ be the corresponding equivariant line bundle on X_i. Let $W_i \in \mathrm{H}^0(X_i, \mathcal{O}_i(\chi))^G$ be a G-invariant section. Then the corresponding regular function $W_i : X_i \to \mathbb{A}^1$ is χ-semi invariant, i.e. $W(g\cdot x) = \chi(g)\cdot W(x)$ for any $g \in G$ and $x \in X_i$. Denote by $\pi_i : X_1 \times X_2 \to X_i$ the projection for each $i = 1, 2$.

Throughout this section 4.2.6, dropping the script L from notation, we write $\mathrm{Dcoh}_G(-,*)$ instead of $\mathrm{Dcoh}_G(-,L,*)$, because all equivariant line bundles in this section are the one corresponding to the character χ.

Definition 4.45. For $P \in \mathrm{DQcoh}_G(X_1 \times X_2, \pi_2^*W_2 - \pi_1^*W_1)$, we define **the integral functor Φ_P with kernel P** as

$$\Phi_P := \mathbf{R}\pi_{2*}(\pi_1^*(-) \otimes^{\mathbf{L}} P) : \mathrm{DQcoh}_G(X_1, W_1) \to \mathrm{DQcoh}_G(X_2, W_2).$$

Remark 4.46. If $Q \in \mathrm{Dcoh}_G(X_1 \times X_2, \pi_2^*W_2 - \pi_1^*W_1)$ has a π_2-proper support, then Φ_Q maps an object in $\mathrm{Dcoh}_G(X_1, W_1)$ to an object in $\mathrm{D_{coh}Qcoh}_G(X_2, W_2)$. We also denote by Φ_Q the following composition

$$\mathrm{Dcoh}_G(X_1, W_1) \xrightarrow{\Phi_Q} \mathrm{D_{coh}Qcoh}_G(X_2, W_2) \xrightarrow{\sim} \mathrm{Dcoh}_G(X_2, W_2).$$

For an object $P \in \mathrm{Dcoh}_G(X_1 \times X_2, \pi_2^*W_2 - \pi_1^*W_1)$, we define objects P_R and P_L in $\mathrm{Dcoh}_G(X_1 \times X_2, \pi_1^*W_1 - \pi_2^*W_2)$ as

$$P_R := P^{\mathbf{L}\vee} \otimes \pi_1^*\omega_{X_1}[\dim(X_1)]$$
$$P_L := P^{\mathbf{L}\vee} \otimes \pi_2^*\omega_{X_2}[\dim(X_2)].$$

If G is trivial, we see that there are relative adjoint pairs of integral functors.

Proposition 4.47. *Let $P \in \mathrm{Dcoh}(X_1 \times X_2, \pi_2^*W_2 - \pi_1^*W_1)$ be an object which has a π_1-proper support. Then for any objects $E \in \mathrm{DQcoh}(X_1, W_1)$ and $F \in \mathrm{Dcoh}(X_2, W_2)$, we have an isomorphism*

$$\mathrm{Hom}_{\mathrm{DQcoh}(X_2,W_2)}(F, \Phi_P(E)) \cong \mathrm{Hom}_{\mathrm{DQcoh}(X_1,W_1)}(\Phi_{P_L}(F), E).$$

In particular, if P has a π_2-proper support, then $\Phi_{P_L} : \mathrm{Dcoh}(X_2, W_2) \to \mathrm{Dcoh}(X_1, W_1)$ (resp. Φ_{P_R}) is a left (resp. right) adjoint functor of $\Phi_P : \mathrm{Dcoh}(X_1, W_1) \to \mathrm{Dcoh}(X_2, W_2)$.

Proof. Since we already have the adjunction $\pi_2^* \dashv \mathbf{R}\pi_{2*}$, it is enough to obtain the following isomorphism

$$\mathrm{Hom}_{\mathrm{DQcoh}(X_1\times X_2,\pi_2^*W_2)}(D, \pi_1^*E \otimes^{\mathbf{L}} P) \cong \mathrm{Hom}_{\mathrm{DQcoh}(X_1,W_1)}(\mathbf{R}\pi_{1*}(D \otimes^{\mathbf{L}} P_L), E)$$

for any objects $D \in \mathrm{Dcoh}(X_1 \times X_2, \pi_2^*W_2)$ and $E \in \mathrm{DQcoh}(X_1, W_1)$. This is proved in a similar way to the proof of [Log, Lemma 4]. Compactify X_2 and denote by $\overline{X_2}$ a smooth proper variety containing X_2 as an open subvariety. Let $\iota : X_1 \times X_2 \hookrightarrow X_1 \times \overline{X_2}$ be the open immersion, and let $\overline{\pi_1} : X_1 \times \overline{X_2} \to X_1$ be the projection. Then $\pi_1 = \overline{\pi_1} \circ \iota$ and $\overline{\pi_1}$ is a proper morphism. By Theorem 4.36 and Proposition 4.40, we obtain the following isomorphism:

$$\mathrm{Hom}_{\mathrm{DQcoh}(X_1\times X_2,\pi_2^*W_2)}(D, \pi_1^*E \otimes^{\mathbf{L}} P) \cong \mathrm{Hom}_{\mathrm{Pro}(\mathrm{DQcoh}(X_1,W_1))}(\mathbf{R}\overline{\pi_1}_*(\iota_!(D \otimes^{\mathbf{L}} P^{\mathbf{L}\vee}) \otimes \omega_{\overline{\pi_1}}[\dim(X_2)]), E)$$

Since the object $P^{\mathbf{L}\vee}$ has a π_1-proper support, there exists an object $P^\bullet \in \mathrm{D^b}(Z^0(\mathrm{coh}(X_1 \times X_2, \pi_1^*W_1 - \pi_2^*W_2)))$ such that $P^{\mathbf{L}\vee} \cong \mathrm{Tot}(P^\bullet)$ and $\mathrm{Supp}(P^\bullet)$ is π_1-proper, in particular, ι-proper. By a similar reasoning to one of [Log, Lemma 4], we see that there is an isomorphism

$$\iota_!((-) \otimes^{\mathbf{L}} P^\bullet) \xrightarrow{\sim} \iota_*((-) \otimes^{\mathbf{L}} P^\bullet)$$

of functors from $\mathrm{D^b}(Z^0(\mathrm{coh}(X_1 \times X_2, \pi_2^*W_2)))$ to $\mathrm{Pro}(\mathrm{D^b}(Z^0(\mathrm{Qcoh}(X_1 \times \overline{X_2}, \pi_1^*W_1))))$. By taking totalizations of the above isomorphism, we obtain an isomorphism of functors

$$\iota_!((-) \otimes^{\mathbf{L}} P^{\mathbf{L}\vee}) \xrightarrow{\sim} \iota_*((-) \otimes^{\mathbf{L}} P^{\mathbf{L}\vee}).$$

Hence, we have an isomorphism $\mathbf{R}\overline{\pi_1}_*(\iota_!(D \otimes^{\mathbf{L}} P^{\mathbf{L}\vee}) \otimes \omega_{\overline{\pi_1}}[\dim(X_2)]) \cong \mathbf{R}\pi_{1*}(D \otimes^{\mathbf{L}} P_L)$. If P has a π_2-proper support, the integral functor Φ_P maps $\mathrm{Dcoh}(X_1, W_1)$ to $\mathrm{Dcoh}(X_2, W_2)$, and Φ_{P_L} maps $\mathrm{Dcoh}(X_2, W_2)$ to $\mathrm{Dcoh}(X_1, W_1)$ since P_L has a π_1-proper support by Lemma 4.31. Hence we have $\Phi_{P_L} \dashv \Phi_P$. Since $(P_L)_R \cong P$, we obtain the other adjunction $\Phi_P \dashv \Phi_{P_R}$. □

We will show that the composition of integral functors is also an integral functor. Let X_3 be another smooth quasi-projective G-variety, $\mathcal{O}_3(\chi)$ be the equivariant line bundle corresponding to the character χ, and $W_3 \in H^0(X_3, \mathcal{O}_3(\chi))^G$ be an invariant section. We define morphisms of varieties by the following diagram;

$$\begin{tikzcd}
 & X_1 \times X_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X_1 & & X_2 \\
 & X_1 \times X_2 \times X_3 \arrow[uu, "\pi_{12}"] \arrow[dl, "\pi_{13}"'] \arrow[dr, "\pi_{23}"] & \\
X_1 \times X_3 \arrow[uu, "q_1"] \arrow[dr, "q_3"'] & & X_2 \times X_3 \arrow[uu, "p_2"'] \arrow[dl, "p_3"] \\
 & X_3 &
\end{tikzcd}$$

where all morphisms are projections. For two objects

$$P \in \mathrm{Dcoh}_G(X_1 \times X_2, \pi_2^* W_2 - \pi_1^* W_1)$$

$$Q \in \mathrm{Dcoh}_G(X_2 \times X_3, p_3^* W_3 - p_2^* W_2),$$

we set another object

$$P \star Q := \pi_{13*}(\pi_{12}^* P \otimes^{\mathbf{L}} \pi_{23}^* Q) \in \mathrm{Dcoh}_G(X_1 \times X_3, q_3^* W_3 - q_1^* W_1).$$

For two complexes $P^\bullet \in \mathrm{D^b}(\mathrm{coh} X_1 \times X_2)$ and $Q^\bullet \in \mathrm{D^b}(\mathrm{coh} X_2 \times X_3)$, we also define another object

$$P^\bullet \star Q^\bullet \in \mathrm{D^b}(\mathrm{coh} X_1 \times X_3)$$

in the same manner. Then we have the following:

Proposition 4.48. *The notation is the same as above. The composition of integral functors*

$$\mathrm{Dcoh}_G(X_1, W_1) \xrightarrow{\Phi_P} \mathrm{Dcoh}_G(X_2, W_2) \xrightarrow{\Phi_Q} \mathrm{Dcoh}_G(X_3, W_3)$$

is isomorphic to the following integral functor

$$\mathrm{Dcoh}_G(X_1, W_1) \xrightarrow{\Phi_{P \star Q}} \mathrm{Dcoh}_G(X_3, W_3).$$

The similar result holds for integral functors of derived categories of coherent sheaves.

Proof. For the proof of the result for derived categories of coherent sheaves, see [Huy, Proposition 5.10], for example. We can prove the result for derived factorization categories in the same way. □

4.3. **Support properties of factorizations.** Following [EP, Section 1.10], we consider set-theoretic supports of factorizations. In this section, X is a Noetherian scheme.

Definition 4.49. Let $(X, \chi, W)^G$ be a gauged LG model, and let $Z \subset X$ be a G-invariant closed subset of X. We say that a factorization $F \in \mathrm{Qcoh}_G(X, \chi, W)$ is **set-theoretically supported on** Z if the supports $\mathrm{Supp}(F_i)$ of components of F are contained in Z.

Denote by

$$\mathrm{Qcoh}_G(X, \chi, W)_Z$$

the dg subcategory of $\mathrm{Qcoh}_G(X, \chi, W)$ consisting of factorizations set-theoretically supported on Z. $H^0(\mathrm{Qcoh}_G(X, \chi, W)_Z)$ is a full triangulated subcategory of $H^0(\mathrm{Qcoh}_G(X, \chi, W))$. Denote by $\mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X, \chi, W)_Z)$ the smallest thick subcategory of $H^0(\mathrm{Qcoh}_G(X, \chi, W)_Z)$

which is closed under small direct sums and contains all totalizations of short exact sequences in $Z^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)$. Set

$$\mathrm{D^{co}Qcoh}_G(X,\chi,W)_Z := H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)/\mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X,\chi,W)_Z).$$

Similarly, denote by

$$\mathrm{coh}_G(X,\chi,W)_Z$$
$$\mathrm{Inj}_G(X,\chi,W)_Z$$

the dg subcategory of $\mathrm{coh}_G(X,\chi,W)$ and $\mathrm{Inj}_G(X,\chi,W)$, respectively, consisting of factorizations set-theoretically supported on Z. Let $\mathrm{Acycl}(\mathrm{coh}_G(X,\chi,W)_Z)$ be the smallest thick subcategory of $H^0(\mathrm{coh}_G(X,\chi,W)_Z)$ containing all totalizations of short exact sequences in $Z^0(\mathrm{coh}_G(X,\chi,W)_Z)$, and consider the Verdier quotient

$$\mathrm{Dcoh}_G(X,\chi,W)_Z := H^0(\mathrm{coh}_G(X,\chi,W)_Z)/\mathrm{Acycl}(\mathrm{coh}_G(X,\chi,W)_Z).$$

Lemma 4.50. *Let $A \in \mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X,\chi,W)_Z)$ and $I \in H^0(\mathrm{Inj}_G(X,\chi,W)_Z)$. Then we have*

$$\mathrm{Hom}_{H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)}(A,I) = 0.$$

Proof. Since arbitrary direct sums of short exact sequences are exact and the totalization functor commutes with arbitrary direct sums, it is enough to show that for a short exact sequence $A^\bullet : 0 \to A^1 \to A^2 \to A^3 \to 0$ in $Z^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)$, we have $\mathrm{Hom}_{H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)}(\mathrm{Tot}(A^\bullet),I) = 0$. This follows from a similar argument as in the proof of [LS, Lemma 2.13]. □

By the above Lemma, we see that every morphism from $\mathrm{Acycl^{co}}(\mathrm{Qcoh}_G(X,\chi,W)_Z)$ to $\mathrm{Inj}_G(X,\chi,W)_Z$ factors through the zero object in $H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)$. Hence, by [LS, Proposition B.2], we have the following lemma:

Lemma 4.51. *Let $F \in H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)$ and $I \in H^0(\mathrm{Inj}_G(X,\chi,W)_Z)$. Then the natural map*

$$\mathrm{Hom}_{H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)}(F,I) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D^{co}Qcoh}_G(X,\chi,W)_Z}(F,I)$$

is an isomorphism.

Furthermore, we have the following:

Lemma 4.52. *The natural functor*

$$H^0(\mathrm{Inj}_G(X,\chi,W)_Z) \to \mathrm{D^{co}Qcoh}_G(X,\chi,W)_Z$$

is an equivalence.

Proof. This follows from [BDFIK1, Cororally 2.25]. □

The following two propositions are G-equivariant versions of results in [EP, Section 1.10].

Proposition 4.53 (cf. [EP] Proposition 1.10)**.**
(1) *The natural functor*

$$\mathrm{Dcoh}_G(X,\chi,W)_Z \to \mathrm{D^{co}Qcoh}_G(X,\chi,W)_Z$$

is fully faithful, and its image is a set of compact generators.
(2) *The natural functor*

$$\iota_Z : \mathrm{D^{co}Qcoh}_G(X,\chi,W)_Z \to \mathrm{D^{co}Qcoh}_G(X,\chi,W)$$

induced by the embedding of dg categories $\iota_Z : \mathrm{Qcoh}_G(X,\chi,W)_Z \to \mathrm{Qcoh}_G(X,\chi,W)$ is fully faithful.
(3) *The functor*

$$\iota_Z : \mathrm{Dcoh}_G(X,\chi,W)_Z \to \mathrm{Dcoh}_G(X,\chi,W)$$

induced by the embedding of dg categories $\iota_Z : \mathrm{coh}_G(X,\chi,W)_Z \to \mathrm{coh}_G(X,\chi,W)$ *is fully faithful.*

Proof. (1) It is enough to prove that any morphism $F \to A$ in $H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)$ from $F \in H^0(\mathrm{coh}_G(X,\chi,W)_Z)$ to $A \in \mathrm{Acycl}^{\mathrm{co}}(\mathrm{Qcoh}_G(X,\chi,W)_Z)$ factors through some object in $\mathrm{Acycl}(\mathrm{coh}_G(X,\chi,W)_Z)$. This follows from a similar argument as in the proof of [LS, Lemma 2.15].

We show that $\mathrm{Dcoh}_G(X,\chi,W)_Z$ generates $\mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)_Z$ by using a similar discussion in the proof of [Pos1, Theorem 3.11.2]. By Lemma 4.51 and 4.52, it is enough to show that for an object $I \in H^0(\mathrm{Inj}_G(X,\chi,W)_Z)$ if

$$\mathrm{Hom}_{H^0(\mathrm{Qcoh}_G(X,\chi,W)_Z)}(F,I) = 0$$

for any $F \in \mathrm{coh}_G(X,\chi,W)_Z$, then $\mathrm{id}_I : I \to I$ is homotopic to zero. Consider the partially ordered set of pairs (M,h), where $M \subset I$ is a subfactorization of I and $h : M \to I$ is a contracting homotopy of the embedding $i : M \hookrightarrow I$, i.e. $d(h) = i$. By Zorn's lemma, the partially ordered set contains a maximal element. Hence it suffices to show that given (M,h) with $M \neq I$, there exists (M',h') with $M \subsetneq M'$ and $h'|_M = h$. Take a subfactorization $M' \subset I$ such that $M \subsetneq M'$ and $M'/M \in \mathrm{coh}_G(X,\chi,W)_Z$. Since the components of I are injective sheaves, the morphism $h : M \to I$ of degree -1 can be extended to a morphism $h'' : M' \to I$. Denote by $i : M \hookrightarrow I$ and $i' : M' \hookrightarrow I$ the embeddings. Since the map $i' - d(h'')$ is a closed degree zero morphism and vanishes on M, it induces a closed degree zero morphism $g : M'/M \to I$. By the assumption, g is homotopic to zero, i.e. there exists a homotopy $c : M'/M \to I$ such that $d(c) = g$. Then $h' = h'' + c \circ p : M' \to I$ is a contracting homotopy for i' extending h, where $p : M' \to M'/M$ is the natural projection.

The compactness of objects in $\mathrm{Dcoh}_G(X,\chi,W)_Z$ follows from Lemma 4.51 and 4.52.
(2) and (3) follows from Lemma 4.52 and (1). □

Proposition 4.54 (cf. [EP] Theorem 1.10)**.** *Let* $U := X \setminus Z$ *be the complement of* $Z \subset X$*, and let* $j : U \to X$ *be the open immersion.*
(1) *The restriction*

$$j^* : \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(U,\chi,W|_U)$$

is the Verdier localization by the thick subcategory $\mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)_Z$*.*
(2) *The restriction*

$$j^* : \mathrm{Dcoh}_G(X,\chi,W) \to \mathrm{Dcoh}_G(U,\chi,W|_U)$$

is the Verdier localization by the triangulated subcategory $\mathrm{Dcoh}_G(X,\chi,W)_Z$*. The kernel of* j^* *is the thick envelope of* $\mathrm{Dcoh}_G(X,\chi,W)_Z$ *in* $\mathrm{Dcoh}_G(X,\chi,W)$*.*

Proof. We can prove this by a standard discussion as in the proof of [EP, Theorem 1.10].

(1) Since j^* has a right adjoint $\mathbf{R}j_* : \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(U,\chi,W|_U) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)$ which is fully faithful, we see that j^* is the Verdier (Bousfield) localization by its kernel which is generated by cones of adjunctions $F \to \mathbf{R}j_*j^*F$ for any $F \in \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)$.

We show that $\mathrm{Ker}(j^*) = \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)_Z$. Since the inclusion $\mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)_Z \subset \mathrm{Ker}(j^*)$ is trivial, it is enough to show that the cone of the adjunction $F \to \mathbf{R}j_*j^*F$, for any $F \in \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)$, is contained in $\mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)_Z$. By Lemma 7.6, we may take F as an factorization whose components are injective quasi-coherent sheaves. Then the adjunction comes from a closed morphism $F \to j_*j^*F$ in $Z^0(\mathrm{Qcoh}_G(X,\chi,W))$. Since its kernel and cokernel are objects in $\mathrm{Qcoh}_G(X,\chi,W)_Z$, so is the cone of the adjunction by an equivariant version of [LS, Lemma 2.7.c].

(2) By Proposition 4.53 (1) and [Nee], we have a fully faithful functor

$$\overline{\mathrm{Dcoh}_G(X,\chi,W)}/\overline{\mathrm{Dcoh}_G(X,\chi,W)_Z} \longrightarrow \overline{\mathrm{Dcoh}_G(U,\chi,W|_U)},$$

where $\overline{(-)}$ denotes the idempotent completion of $(-)$. Since every morphism $D \to E$ from $D \in \mathrm{Dcoh}_G(X,\chi,W)$ to $E \in \overline{\mathrm{Dcoh}_G(X,\chi,W)_Z}$ factors through an object in $\mathrm{Dcoh}_G(X,\chi,W)_Z$, the natural functor

$$\mathrm{Dcoh}_G(X,\chi,W)/\mathrm{Dcoh}_G(X,\chi,W)_Z \to \overline{\mathrm{Dcoh}_G(X,\chi,W)}/\overline{\mathrm{Dcoh}_G(X,\chi,W)_Z}$$

is fully faithful. Hence we see that the natural functor

$$\mathrm{Dcoh}_G(X,\chi,W)/\mathrm{Dcoh}_G(X,\chi,W)_Z \to \mathrm{Dcoh}_G(U,\chi,W|_U)$$

is also fully faithful. This functor is essentially surjective since for every G-equivariant coherent $F \in \mathrm{coh}_G U$ there exists a G-equivariant coherent sheaf $\overline{F} \in \mathrm{coh}_G X$ such that $j^*\overline{F} \cong F$ and the coherent sheaves generate $\mathrm{Dcoh}_G(U,\chi,W|_U)$ by [BDFIK1, Corollary 2.29] $\square$

4.4. **Comonads induced by restriction and induction functors.** In this section X is a scheme. We construct restriction and induction functors and study comonads induced by these functors.

Let $G\times^l X$ and $G\times^d X$ be the varieties $G\times X$ with different G-actions which are defined as follows;

$$\begin{array}{ccc} G\times G\times^l X & \longrightarrow & G\times^l X \\ \Psi & & \Psi \\ (g,g',x) & \longmapsto & (gg',x) \end{array}$$

and

$$\begin{array}{ccc} G\times G\times^d X & \longrightarrow & G\times^d X \\ \Psi & & \Psi \\ (g,g',x) & \longmapsto & (gg',gx). \end{array}$$

Then the following morphisms

$$\begin{array}{ccc} \varphi : G\times^l X & \longrightarrow & G\times^d X \\ \Psi & & \Psi \\ (g,x) & \longmapsto & (g,gx) \end{array}$$

and

$$\begin{array}{ccc} \pi : G\times^d X & \longrightarrow & X \\ \Psi & & \Psi \\ (g,x) & \longmapsto & x \end{array}$$

are G-equivariant. The action $\sigma : G\times X \to X$ on X is the composition $\pi\circ\varphi$.

Let $\iota : X \to G\times X$ be a morphism defined by

$$X \ni x \longmapsto (e,x) \in G\times X.$$

We define an exact functor $\iota^* : \mathrm{Qcoh}_G(G\times^l X) \to \mathrm{Qcoh}X$ as

$$\begin{array}{ccc} \mathrm{Qcoh}_G(G\times^l X) & \longrightarrow & \mathrm{Qcoh}X \\ \Psi & & \Psi \\ (\mathcal{F},\theta) & \longmapsto & \iota^*\mathcal{F}. \end{array}$$

Lemma 4.55. (1) *The functor $\iota^* : \mathrm{Qcoh}_G(G\times^l X) \to \mathrm{Qcoh}X$ is an equivalence.*

(2) *The functors $\varphi^* : \mathrm{Qcoh}_G(G\times^d X) \to \mathrm{Qcoh}_G(G\times^l X)$ and $\varphi_* : \mathrm{Qcoh}_G(G\times^l X) \to \mathrm{Qcoh}_G(G\times^d X)$ are equivalences.*

(3) *The functors $\pi^* : \mathrm{Qcoh}_G(X) \to \mathrm{Qcoh}_G(G\times^d X)$ and $\pi_* : \mathrm{Qcoh}_G(G\times^d X) \to \mathrm{Qcoh}_G(X)$ are exact functors.*

Proof. (1)This is a special case of [Tho, Lemma 1.3.]
(2)The morphism φ is an isomorphism.
(3)Since π is smooth, in particular flat, and affine, π^* and π_* are exact functors. $\square$

Definition 4.56. We define **the restriction functor** $\mathrm{Res}_G : \mathrm{Qcoh}_G(X) \to \mathrm{Qcoh}X$ and **the induction functor** $\mathrm{Ind}_G : \mathrm{Qcoh}X \to \mathrm{Qcoh}_G(X)$ as

$$\mathrm{Res}_G := \iota^* \circ \sigma^* \quad \text{and} \quad \mathrm{Ind}_G := \sigma_* \circ (\iota^*)^{-1}.$$

Remark 4.57. Note that the restriction functor $\mathrm{Res}_G : \mathrm{Qcoh}_G(X) \to \mathrm{Qcoh}X$ is isomorphic to the forgetful functor, i.e. $\mathrm{Res}_G(\mathcal{F}, \theta) \cong \mathcal{F}$.

Let L be an invertible G-equivariant sheaf, and let W be an invariant section of L. Then the pair (L, W) defines potentials of $\mathrm{Qcoh}_G(X)$ and $\mathrm{Qcoh}X$. Since the functors Res_G and Ind_G are cwp-functors, these functors induce functors of factorization categories

$$\mathrm{Res}_G : \mathrm{Qcoh}_G(X, L, W) \to \mathrm{Qcoh}(X, L, W)$$

$$\mathrm{Ind}_G : \mathrm{Qcoh}(X, L, W) \to \mathrm{Qcoh}_G(X, L, W)$$

Since ι^* is an equivalence, the adjoint pair $\sigma^* \dashv \sigma_*$ induces the adjoint pair

$$\mathrm{Res}_G \dashv \mathrm{Ind}_G.$$

Since the functors Res_G and Ind_G are exact functors, we obtain the exact functor of derived factorization categories

$$\Pi_G^* := \mathrm{Res}_G : \mathrm{D^{co}Qcoh}_G(X, L, W) \to \mathrm{D^{co}Qcoh}(X, L, W)$$

$$\Pi_{G*} := \mathrm{Ind}_G : \mathrm{D^{co}Qcoh}(X, L, W) \to \mathrm{D^{co}Qcoh}_G(X, L, W),$$

and these defines an adjoint pair

$$\Pi_G := (\ \Pi_G^* \dashv \Pi_{G*}\ ).$$

Remark 4.58. The functor Π_G^* sends objects in $\mathrm{Dcoh}_G(X, L, W)$ to objects in $\mathrm{Dcoh}(X, L, W)$. But the functor Π_{G*} does not preserve coherentness of components of factorizations.

Definition 4.59. We define a comonad $\mathbb{T}_G$ on $\mathrm{D^{co}Qcoh}(X, L, W)$ induced by G-action as the one induced by the adjoint pair $\Pi_G := (\Pi_G^* \dashv \Pi_{G*})$;

$$\mathbb{T}_G := \mathbb{T}(\Pi_G),$$

where the notation is the same as in Example 2.2. Denote by Γ_G is the comparison functor of the adjoint pair $\Pi_G := (\Pi_G^* \dashv \Pi_{G*})$,

$$\Gamma_G : \mathrm{D^{co}Qcoh}_G(X, L, W) \to \mathrm{D^{co}Qcoh}(X, L, W)_{\mathbb{T}_G}.$$

We recall the definition of (linearly) reductiveness of algebraic groups.

Definition 4.60. Let H be an affine algebraic group over a field K.

(1) H is called **reductive** if the radical of H is a torus.
(2) H is called **linearly reductive** if every rational representations of H over K is completely reducible.

Proposition 4.61 ([MFK] Appendix A)**.** *Let H be an affine algebraic group over a field K of characteristic zero. Then H is reductive if and only if linearly reductive.*

Lemma 4.62. *Assume that G is linearly reductive.*

(1) The adjunction morphism

$$\varepsilon : \mathrm{id}_{\mathrm{Qcoh}_G X} \to \mathrm{Ind}_G \circ \mathrm{Res}_G$$

is a split mono i.e., there exists a functor morphism $\eta : \mathrm{Ind}_G \circ \mathrm{Res}_G \to \mathrm{id}_{\mathrm{Qcoh}_G X}$ such that $\eta \circ \varepsilon = \mathrm{id}$. The adjunction morphisms

$$\mathrm{id}_{\mathrm{Qcoh}_G(X,X,W)} \to \mathrm{Ind}_G \circ \mathrm{Res}_G$$

$$\mathrm{id}_{\mathrm{D^{co}Qcoh}(X,L,W)} \to \Pi_{G*}\Pi_G^*$$

are also split mono. In particular, the comparison functor $\Gamma_G : \mathrm{D^{co}Qcoh}_G(X, L, W) \to \mathrm{D^{co}Qcoh}(X, L, W)_{\mathbb{T}_G}$ is an equivalence.

(2) The restriction functors

$$\mathrm{Res}_G : \mathrm{D^b}(\mathrm{Qcoh}_G X) \to \mathrm{D^b}(\mathrm{Qcoh} X)$$

$$\mathrm{Res}_G : \mathrm{D^{co}Qcoh}_G(X, L, W) \to \mathrm{D^{co}Qcoh}(X, W)$$

are faithful.

Proof. (1) We show that $\mathrm{id}_{\mathrm{D^{co}Qcoh}(X,L,W)} \to \Pi_{G*}\Pi_G^*$ is a split mono. Since the adjunction morphism $\mathrm{id} \to \Pi_{G*}\Pi_G^*$ coincide with the adjunction morphism $\mathrm{id} \to \pi_*\pi^*$, and the morphism $E \to \pi_*\pi^*E$ is equal to the morphism $E \otimes (\mathcal{O}_X \to \pi_*\pi^*\mathcal{O}_X)$ via the projection formula, it is enough to show that $\mathcal{O}_X \to \pi_*\pi^*\mathcal{O}_X$ is split mono. Since G is linearly reductive, the homomorphism $k \to \mathcal{O}_G(G)$ of G-modules is split mono. This means that the adjunction $\mathcal{O}_{\mathrm{Spec}(k)} \to p_*p^*\mathcal{O}_{\mathrm{Spec}(k)}$ is split mono, where $p : G \to \mathrm{Spec}(k)$ is the morphism defining the base space. Hence by the cartesian square,

$$\begin{tikzcd} G \times X \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ G \arrow[r, "p"] & \mathrm{Spec}(k), \end{tikzcd}$$

we see that $\mathcal{O}_X \to \pi_*\pi^*\mathcal{O}_X$ is also a split mono. The latter statement follows from Proposition 2.7 and Lemma 4.10.

(2) We will prove that the upper functor $\mathrm{Res}_G : \mathrm{D^b}(\mathrm{Qcoh}_G X) \to \mathrm{D^b}(\mathrm{Qcoh} X)$ is faithful; the proof of the faithfulness of the lower functor is similar. The functor morphism $\eta : \mathrm{Ind}_G \circ \mathrm{Res}_G \to \mathrm{id}_{\mathrm{Qcoh}_G X}$ constructed in (1) naturally induces the functor morphism $\overline{\eta} : \mathrm{Ind}_G \circ \mathrm{Res}_G \to \mathrm{id}_{\mathrm{D^b}(\mathrm{Qcoh}_G X)}$ such that the composition with the adjunction morphism

$$\mathrm{id}_{\mathrm{D^b}(\mathrm{Qcoh}_G X)} \to \mathrm{Ind}_G \circ \mathrm{Res}_G \xrightarrow{\overline{\eta}} \mathrm{id}_{\mathrm{D^b}(\mathrm{Qcoh}_G X)}$$

is the identity. Hence any morphism f in $\mathrm{D^b}(\mathrm{Qcoh}_G X)$ factors through $\mathrm{Ind}_G \circ \mathrm{Res}_G(f)$, and so $f = 0$ if $\mathrm{Res}(f) = 0$. □

Lemma 4.63. *Let X' be another smooth quasi-projective variety with G-action and let $f : X' \to X$ be a G-equivariant morphism. Let $\mathbb{T}'_G = \mathbb{T}(\Pi'_G)$ be the comonad on $\mathrm{DQcoh}(X', f^*L, f^*W)$ induced by its G-action. Let $P \in \mathrm{Dcoh}_G(X, L, W)$ be an object.*

Then there exist functor isomorphisms $\Omega^* : \mathbf{R}f_*\Pi'^*_G \xrightarrow{\sim} \Pi_G^*\mathbf{R}f_*$ and $\Omega_* : \mathbf{R}f_*\Pi'_{G*} \xrightarrow{\sim} \Pi_{G*}\mathbf{R}f_*$ such that the following diagrams are commutative;

$$\begin{tikzcd} \Pi_G^*\mathbf{R}f_*\Pi'_{G*} \arrow[r, "\Pi_G^*\Omega_*"] & \Pi_G^*\Pi_{G*}\mathbf{R}f_* \arrow[d, "\varepsilon\mathbf{R}f_*"] \\ \mathbf{R}f_*\Pi'^*_G\Pi'_{G*} \arrow[u, "\Omega^*\Pi'_{G*}"] \arrow[r, "\mathbf{R}f_*\varepsilon'"] & \mathbf{R}f_* \end{tikzcd} \qquad \begin{tikzcd} \mathbf{R}f_* \arrow[r, "\eta\mathbf{R}f_*"] \arrow[d, "\mathbf{R}f_*\eta'"'] & \Pi_{G*}\Pi_G^*\mathbf{R}f_* \\ \mathbf{R}f_*\Pi'_{G*}\Pi'^*_G \arrow[r, "\Omega_*\Pi'^*_G"] & \Pi_{G*}\mathbf{R}f_*\Pi'^*_G, \arrow[u, "\Pi_{G*}\Omega^*"'] \end{tikzcd}$$

where ε, ε', η and η' are adjunction morphisms. In particular, the direct image $\mathbf{R}f_*$: $\mathrm{DQcoh}(X', f^*L, f^*W) \to \mathrm{DQcoh}(X, L, W)$ is a linearizable functor with respect to $\mathbb{T}'_G$ and $\mathbb{T}_G$ with a linearization $\Omega := \Pi_G^*\Omega_* \circ \Omega^*\Pi'_{G*}$, and the following diagram is commutative:

$$\begin{tikzcd} \mathrm{DQcoh}(X', f^*L, f^*W)_{\mathbb{T}'_G} \arrow[r, "\mathbf{R}f_{*\Omega}"] & \mathrm{DQcoh}(X, L, W)_{\mathbb{T}_G} \\ \mathrm{DQcoh}_G(X', f^*L, f^*W) \arrow[u, "\Gamma'_G"] \arrow[r, "\mathbf{R}f_*"] & \mathrm{DQcoh}_G(X, L, W). \arrow[u, "\Gamma_G"'] \end{tikzcd}$$

The similar results hold for the inverse image $\mathbf{L}f^* : \mathrm{DQcoh}(X, L, W) \to \mathrm{DQcoh}(X', f^*L, f^*W)$ and the tensor product $(-) \otimes^{\mathbf{L}} \Pi_G^*P : \mathrm{DQcoh}(X, L, V) \to \mathrm{DQcoh}(X, L, V + W)$.

Proof. We only give a proof for the case of the direct image. Let $\pi : G \times X \to X$ and $\pi' : G \times X' \to X'$ be natural projections, and set $\overline{f} := \mathrm{id}_G \times f : G \times X' \to G \times X$. By Lemma 4.55 (1) and (2), we have the following equivalences;

$$\Phi : \mathrm{DQcoh}_G(G \times^d X, \pi^*L, \pi^*W) \xrightarrow{\sim} \mathrm{DQcoh}(X, L, W)$$
$$\Phi' : \mathrm{DQcoh}_G(G \times^d X', \pi'^* f^*L, \pi'^* f^*W) \xrightarrow{\sim} \mathrm{DQcoh}(X', f^*L, f^*W)$$

such that $\Pi_G^* \cong \Phi\pi^*$, $\Pi_{G*} \cong \pi_*\Phi^{-1}$, ${\Pi'_G}^* \cong \Phi'\pi'^*$ and $\Pi'_{G*} \cong \pi'_*\Phi'^{-1}$. By the following cartesian square

$$\begin{tikzcd} G \times X' \arrow[r, "\overline{f}"] \arrow[d, "\pi'"'] & G \times X \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X, \end{tikzcd}$$

we have isomorphisms of functors between categories of quasi-coherent sheaves;

$$\overline{\omega}^* : \overline{f}_*\pi'^* \xrightarrow{\sim} \pi^* f_* \quad \text{and} \quad \overline{\omega}_* : f_*\pi'_* \xrightarrow{\sim} \pi_*\overline{f}_*.$$

By easy computation, we see that the following diagrams are commutative;

$$\begin{tikzcd} \pi^* f_*\pi'_* \arrow[r, "\pi^*\overline{\omega}_*"] & \pi^*\pi_*\overline{f}_* \arrow[d, "\varepsilon_\pi \overline{f}_*"] \\ \overline{f}_*\pi'^*\pi'_* \arrow[u, "\overline{\omega}^*\pi'_*"] \arrow[r, "\overline{f}_*\varepsilon_{\pi'}"] & \overline{f}_* \end{tikzcd} \qquad \begin{tikzcd} f_* \arrow[r, "\eta_\pi f_*"] \arrow[d, "f_*\eta_{\pi'}"'] & \pi_*\pi^* f_* \\ f_*\pi'_*\pi'^* \arrow[r, "\overline{\omega}_*\pi'^*"] & \pi_*\overline{f}_*\pi'^*. \arrow[u, "\pi_*\overline{\omega}^*"'] \end{tikzcd}$$

Since the functor morphisms in the above diagrams are cwp-functor morphisms, taking $H^0(\mathfrak{F}(-))$, we obtain similar isomorphisms of functors between homotopy categories of factorization categories, and similar commutative diagrams of morphisms of exact functors between homotopy categories of factorization categories. These isomorphisms of functors and commutative diagrams induce isomorphisms of functors between derived factorization categories

$$\overline{\Omega}^* : \mathbf{R}\overline{f}_*\pi'^* \xrightarrow{\sim} \pi^*\mathbf{R}f_* \quad \text{and} \quad \overline{\Omega}_* : \mathbf{R}f_*\pi'_* \xrightarrow{\sim} \pi_*\mathbf{R}\overline{f}_*$$

and the following commutative diagrams

$$\begin{tikzcd} \pi^*\mathbf{R}f_*\pi'_* \arrow[r, "\pi^*\overline{\Omega}_*"] & \pi^*\pi_*\mathbf{R}\overline{f}_* \arrow[d, "\varepsilon_\pi \mathbf{R}\overline{f}_*"] \\ \mathbf{R}\overline{f}_*\pi'^*\pi'_* \arrow[u, "\overline{\Omega}^*\pi'_*"] \arrow[r, "\mathbf{R}\overline{f}_*\varepsilon_{\pi'}"] & \mathbf{R}\overline{f}_* \end{tikzcd} \qquad \begin{tikzcd} \mathbf{R}f_* \arrow[r, "\eta_\pi \mathbf{R}f_*"] \arrow[d, "\mathbf{R}f_*\eta_{\pi'}"'] & \pi_*\pi^*\mathbf{R}f_* \\ \mathbf{R}f_*\pi'_*\pi'^* \arrow[r, "\overline{\Omega}_*\pi'^*"] & \pi_*\mathbf{R}\overline{f}_*\pi'^*. \arrow[u, "\pi_*\overline{\Omega}^*"'] \end{tikzcd}$$

Since $\mathbf{R}f_*\Phi' \cong \Phi\mathbf{R}\overline{f}_*$, applying the equivalences Φ and Φ'^{-1} to the above functor isomorphisms and commutative diagrams, we obtain the desired functor isomorphisms and commutative diagrams.

The results for the inverse image and the tensor product are proved similarly. $\square$

5. Main results (Part I)

At first, we prepare notation used throughout this section. Let X_1 and X_2 be smooth quasi-project varieties with actions of reductive affine algebraic group G over an algebraically closed field k of characteristic zero. For a character $\chi : G \to \mathbb{G}_m$ of G, take χ-semi invariant regular functions $W_i \in \mathrm{H}^0(X_i, \mathcal{O}_{X_i}(\chi))^G$ on X_i. Let $\pi_i : X_1 \times X_2 \to X_i$ and $q_i : X_1 \times_{\mathbb{A}^1} X_2 \to X_i$ be the projections and let $j : X_1 \times_{\mathbb{A}^1} X_2 \hookrightarrow X_1 \times X_2$ be the embedding. We have the following commutative diagram:

$$\begin{tikzcd}
 & X_1 \times X_2 \arrow[ddl, "\pi_1"'] \arrow[ddr, "\pi_2"] & \\
 & X_1 \times_{\mathbb{A}^1} X_2 \arrow[u, "j"] \arrow[dl, "q_1"] \arrow[dr, "q_2"'] & \\
X_1 \arrow[dr, "W_1"'] & & X_2 \arrow[dl, "W_2"] \\
 & \mathbb{A}^1 &
\end{tikzcd}$$

Abbreviating $\mathcal{O}_{X_i}(\chi)$, we write

$$\mathrm{DQcoh}_G(X_i, W_i) := \mathrm{DQcoh}_G(X_i, \mathcal{O}_{X_i}(\chi), W_i)$$

and

$$\mathrm{Dcoh}_G(X_i, W_i) := \mathrm{Dcoh}_G(X_i, \mathcal{O}_{X_i}(\chi), W_i).$$

5.1. **Equivariantization.** The action of G on X_i induces an adjoint pair

$$\Pi_i := (\ \Pi_i^* \dashv \Pi_{i*}\ ),$$

where the functor Π_i^* and Π_{i*} are given by restriction and induction functors respectively;

$$\Pi_i^* := \mathrm{Res}_G : \mathrm{DQcoh}_G(X_i, W_i) \to \mathrm{DQcoh}(X_i, W_i)$$

$$\Pi_{i*} := \mathrm{Ind}_G : \mathrm{DQcoh}(X_i, W_i) \to \mathrm{DQcoh}_G(X_i, W_i).$$

Denote by $\mathbb{T}_i$ be the comonad on $\mathrm{DQcoh}(X_i, W_i)$ induced by the adjoint pair $\Pi_i = (\Pi_i^* \dashv \Pi_{i*})$ and let Γ_i be the comparison functor of the adjoint pair Π_i,

$$\Gamma_i : \mathrm{DQcoh}_G(X_i, W_i) \to \mathrm{DQcoh}(X_i, W_i)_{\mathbb{T}_i}.$$

Theorem 5.1. *Let $P_G \in \mathrm{Dcoh}_G(X_1 \times X_2, \pi_2^* W_2 - \pi_1^* W_1)$ be an object and set $P := \mathrm{Res}_G(P_G) \in \mathrm{Dcoh}(X_1 \times X_2, \pi_2^* W_2 - \pi_1^* W_1)$. Assume that P has a π_i-proper support ($i = 1, 2$). If the integral functor $\Phi_P : \mathrm{Dcoh}(X_1, W_1) \to \mathrm{Dcoh}(X_2, W_2)$ is fully faithful (resp. equivalence), then the integral functor $\Phi_{P_G} : \mathrm{Dcoh}_G(X_1, W_1) \to \mathrm{Dcoh}_G(X_2, W_2)$ is fully faithful (resp. equivalence).*

Proof. Set $(P_L)_G := (P_G)^{\mathbf{L}\vee} \otimes \pi_2^* \omega_{X_2}[\dim(X_2)] \in \mathrm{Dcoh}_G(X_1 \times X_2, \pi_1^* W_1 - \pi_2^* W_2)$ and $P_L := \mathrm{Res}_G((P_L)_G) \in \mathrm{Dcoh}(X_1 \times X_2, \pi_1^* W_1 - \pi_2^* W_2)$. Then we have the following diagram.

$$
\begin{tikzcd}
\mathrm{Dcoh}(X_1,W_1) \arrow[rrr, shift left, "\Phi_P"] \arrow[dr, hook] & & & \mathrm{Dcoh}(X_2,W_2) \arrow[lll, shift left, "\Phi_{P_L}"] \arrow[dl, hook'] \\
& \mathrm{DQcoh}(X_1,W_1) \arrow[r, shift left, "\Phi_P"] \arrow[d, shift left, "\Pi_{1*}"] & \mathrm{DQcoh}(X_2,W_2) \arrow[l, shift left, "\Phi_{P_L}"] \arrow[d, shift left, "\Pi_{2*}"] & \\
& \mathrm{DQcoh}_G(X_1,W_1) \arrow[u, shift left, "\Pi_1^*"] \arrow[r, shift left, "\Phi_{P_G}"] & \mathrm{DQcoh}_G(X_2,W_2) \arrow[u, shift left, "\Pi_2^*"] \arrow[l, shift left, "\Phi_{(P_L)_G}"] & \\
\mathrm{Dcoh}_G(X_1,W_1) \arrow[uuu, "\Pi_1^*"] \arrow[ur, hook] \arrow[rrr, shift left, "\Phi_{P_G}"] & & & \mathrm{Dcoh}_G(X_2,W_2) \arrow[uuu, "\Pi_2^*"'] \arrow[ul, hook'] \arrow[lll, shift left, "\Phi_{(P_L)_G}"]
\end{tikzcd}
$$

By Lemma 4.63, there exist functor isomorphisms $\Omega^* : \Phi_P\Pi_1^* \xrightarrow{\sim} \Pi_2^*\Phi_{P_G}$, $\Omega_* : \Phi_{P_G}\Pi_{1*} \xrightarrow{\sim} \Pi_{2*}\Phi_P$, $\Omega_L^* : \Phi_{P_L}\Pi_2^* \xrightarrow{\sim} \Pi_1^*\Phi_{(P_L)_G}$ and $\Omega_{L*} : \Phi_{(P_L)_G}\Pi_{2*} \xrightarrow{\sim} \Pi_{1*}\Phi_{P_L}$ such that the diagrams corresponding to (i) and (ii) in Lemma 2.11, namely the following diagrams, are commutative.

$$
\begin{tikzcd}
\Pi_2^*\Phi_{P_G}\Pi_{1*} \arrow[r, "\Pi_2^*\Omega_*"] & \Pi_2^*\Pi_{2*}\Phi_P \arrow[d, "\varepsilon_2\Phi_P"] \\
\Phi_P\Pi_1^*\Pi_{1*} \arrow[u, "\Omega^*\Pi_{1*}"] \arrow[r, "\Phi_P\varepsilon_1"] & \Phi_P
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi_{P_G} \arrow[r, "\eta_2\Phi_{P_G}"] \arrow[d, "\Phi_{P_G}\eta_1"'] & \Pi_{2*}\Pi_2^*\Phi_{P_G} \\
\Phi_{P_G}\Pi_{1*}\Pi_1^* \arrow[r, "\Omega_*\Pi_1^*"] & \Pi_{2*}\Phi_P\Pi_1^*, \arrow[u, "\Pi_{2*}\Omega^*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Pi_1^*\Phi_{(P_L)_G}\Pi_{2*} \arrow[r, "\Pi_1^*\Omega_{L*}"] & \Pi_1^*\Pi_{1*}\Phi_{P_L} \arrow[d, "\varepsilon_1\Phi_{P_L}"] \\
\Phi_{P_L}\Pi_2^*\Pi_{2*} \arrow[u, "\Omega_L^*\Pi_{2*}"] \arrow[r, "\Phi_{P_L}\varepsilon_2"] & \Phi_{P_L}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi_{(P_L)_G} \arrow[r, "\eta_1\Phi_{(P_L)_G}"] \arrow[d, "\Phi_{(P_L)_G}\eta_2"'] & \Pi_{1*}\Pi_1^*\Phi_{(P_L)_G} \\
\Phi_{(P_L)_G}\Pi_{2*}\Pi_2^* \arrow[r, "\Omega_{L*}\Pi_2^*"] & \Pi_{1*}\Phi_{P_L}\Pi_2^*, \arrow[u, "\Pi_{1*}\Omega_L^*"']
\end{tikzcd}
$$

where ε_i and η_i are adjunction morphisms of the adjoint pair $(\Pi_i^* \dashv \Pi_{i*})$. Combining Lemma 2.12 with Proposition 4.47 and Lemma 4.62, we see that if $\Phi_P : \mathrm{Dcoh}(X_1,W_1) \to \mathrm{Dcoh}(X_2,W_2)$ is fully faithful, then $\Phi_{P_G} : \mathrm{Dcoh}_G(X_1,W_1) \to \mathrm{Dcoh}_G(X_2,W_2)$ is also fully faithful.

Assume Φ_P is an equivalence. Then, Φ_{P_L} is fully faithful functor. Applying the above argument to Φ_{P_L}, we see that $\Phi_{(P_L)_G}$ is also fully faithful. Set $\Omega := \Pi_2^*\Omega_* \circ \Omega^*\Pi_{1*}$ and $\Omega_L := \Pi_1^*\Omega_{L*} \circ \Omega_L^*\Pi_{2*}$. By Lemma 2.12, we see that Φ_{P_G} is an equivalence by the following Lemma 5.2. $\Box$

Lemma 5.2. *With notation same as above, the following diagram of functors from* $\mathrm{Dcoh}_G(X_1,W_1)$ *to* $\mathrm{DQcoh}(X_1,W_1)$ *is commutative;*

$$
(*) : \quad
\begin{tikzcd}[column sep=small]
\Phi_{P_L}\Phi_P\Pi_1^* \arrow[r, "\Phi_{P_L}\Phi_P\Pi_1^*\eta_1"] \arrow[d, "\omega\Pi_1^*"'] & \Phi_{P_L}\Phi_P\Pi_1^*\Pi_{1*}\Pi_1^* \arrow[r, "\Phi_{P_L}\Omega\Pi_1^*"] & \Phi_{P_L}\Pi_2^*\Pi_{2*}\Phi_P\Pi_1^* \arrow[r, "\Omega_L\Phi_P\Pi_1^*"] & \Pi_1^*\Pi_{1*}\Phi_{P_L}\Phi_P\Pi_1^* \arrow[d, "\Pi_1^*\Pi_{1*}\omega\Pi_1^*"] \\
\Pi_1^* \arrow[rrr, "\Pi_1^*\eta_1"] & & & \Pi_1^*\Pi_{1*}\Pi_1^*,
\end{tikzcd}
$$

where $\omega : \Phi_{P_L}\Phi_P \to \mathrm{id}_{\mathrm{Dcoh}(X_1,W_1)}$ *is the adjunction morphism of* $(\Phi_{P_L} \dashv \Phi_P)$. $\Box$

We will prove the above lemma in the next section.

5.2. **Proof of Lemma 5.2.** In what follows, we will prove the above Lemma 5.2. Since it seems difficult to verify the commutativity of the diagram $(*)$ directly, we will replace it with another diagram $(*)'$, and decompose the diagram $(*)'$ into several diagrams whose commutativity are easier to verify.

Take a smooth proper variety $\overline{X_2}$ containing X_2 as an open subvariety as in the proof of Lemma 4.47. Let $i : X_1 \times X_2 \hookrightarrow X_1 \times \overline{X_2}$ be the open immersion, and let $\overline{\pi_1} : X_1 \times \overline{X_2} \to X_1$ be the natural projection. Denote natural projections by $p_i : G \times X_i \to X_i$, $p_{12} : G \times X_1 \times X_2 \to X_1 \times X_2$ and $\overline{p_{12}} : G \times X_1 \times \overline{X_2} \to X_1 \times \overline{X_2}$, and set

$$\pi'_i := 1_G \times \pi_i : G \times X_1 \times X_2 \to G \times X_i$$

$$i' := 1_G \times i : G \times X_1 \times X_2 \to G \times X_1 \times \overline{X_2}$$

$$\overline{\pi_1}' := 1_G \times \overline{\pi_1} : G \times X_1 \times \overline{X_2} \to G \times X_1.$$

Then objects $Q_G := p_{12}^* P_G \in \mathrm{Dcoh}_G(G \times X_1 \times X_2, {\pi'_2}^* p_2^* W_2 - {\pi'_1}^* p_1^* W_1)$ and $(Q_L)_G := p_{12}^*(P_L)_G \in \mathrm{Dcoh}_G(G \times X_1 \times X_2, {\pi'_1}^* p_1^* W_1 - {\pi'_2}^* p_2^* W_2)$ define functors

$$\begin{array}{rcl} \Psi_{Q_G} : \mathrm{DQcoh}_G(G \times X_1, p_1^* W_1) & \longrightarrow & \mathrm{DQcoh}_G(G \times X_2, p_2^* W_2) \\ F & \longmapsto & \pi'_{2*}(\pi'^*_1(F) \otimes^{\mathbf{L}} Q_G) \end{array}$$

and

$$\begin{array}{rcl} \Psi_{(Q_L)_G} : \mathrm{DQcoh}_G(G \times X_2, p_2^* W_2) & \longrightarrow & \mathrm{DQcoh}_G(G \times X_1, p_1^* W_1) \\ E & \longmapsto & \pi'_{1*}(\pi'^*_2(E) \otimes^{\mathbf{L}} (Q_L)_G). \end{array}$$

Note that Q_G has a π'_i-proper support $(i = 1, 2)$. Hence the functors Ψ_{Q_G} and $\Psi_{(Q_L)_G}$ preserve coherent factorizations.

Similarly, the objects $Q := \mathrm{Res}_G(Q_G) \in \mathrm{Dcoh}(G \times X_1 \times X_2, {\pi'_2}^* p_2^* W_2 - {\pi'_1}^* p_1^* W_1)$ and $Q_L := \mathrm{Res}_G((Q_L)_G) \in \mathrm{Dcoh}(G \times X_1 \times X_2, {\pi'_1}^* p_1^* W_1 - {\pi'_2}^* p_2^* W_2)$ defines functors

$$\begin{array}{rcl} \Psi_Q : \mathrm{DQcoh}(G \times X_1, p_1^* W_1) & \longrightarrow & \mathrm{DQcoh}(G \times X_2, p_2^* W_2) \\ F & \longmapsto & \pi'_{2*}(\pi'^*_1(F) \otimes^{\mathbf{L}} Q) \end{array}$$

and

$$\begin{array}{rcl} \Psi_{Q_L} : \mathrm{DQcoh}(G \times X_2, p_2^* W_2) & \longrightarrow & \mathrm{DQcoh}(G \times X_1, p_1^* W_1) \\ E & \longmapsto & \pi'_{1*}(\pi'^*_2(E) \otimes^{\mathbf{L}} Q_L). \end{array}$$

By Lemma 4.55 the composition,

$$\iota^* \circ \varphi^* : \mathrm{DQcoh}_G(G \times X_1, p_1^* W_1) \xrightarrow{\sim} \mathrm{DQcoh}(X_1, W_1),$$

is an equivalence, and the following diagrams are commutative,

$$\begin{tikzcd} \mathrm{DQcoh}_G(G \times X_1, p_1^* W_1) \arrow[r, "\Psi_{Q_G}"] \arrow[d, "\iota^* \circ \varphi^*"'] & \mathrm{DQcoh}_G(G \times X_2, p_2^* W_2) \arrow[d, "\iota^* \circ \varphi^*"] \\ \mathrm{DQcoh}(X_1, W_1) \arrow[r, "\Phi_P"] & \mathrm{DQcoh}(X_2, W_2) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{DQcoh}_G(G \times X_2, p_2^* W_2) \arrow[r, "\Psi_{(Q_L)_G}"] \arrow[d, "\iota^* \circ \varphi^*"'] & \mathrm{DQcoh}_G(G \times X_1, p_1^* W_1) \arrow[d, "\iota^* \circ \varphi^*"] \\ \mathrm{DQcoh}(X_2, W_2) \arrow[r, "\Phi_P"] & \mathrm{DQcoh}(X_1, W_1). \end{tikzcd}$$

Let $\Omega' : \Psi_{Q_G} p_1^* p_{1*} \xrightarrow{\sim} p_2^* p_{2*} \Psi_{Q_G}$ and $\Omega'_L : \Psi_{(Q_L)_G} p_2^* p_{2*} \xrightarrow{\sim} \Psi_{(Q_L)_G} p_1^* p_{1*}$ be functor isomorphisms induced by the functor isomorphisms $\Omega : \Phi_P \Pi_1^* \Pi_{1*} \xrightarrow{\sim} \Pi_2^* \Pi_{2*} \Phi_P$ and

$\Omega' : \Phi_{P_L}\Pi_2^*\Pi_{2*} \xrightarrow{\sim} \Pi_1^*\Pi_{1*}\Phi_{P_L}$ via the equivalence $\iota^* \circ \varphi^*$ respectively. Via the equivalence $\iota^* \circ \varphi^*$, the diagram $(*)$ is commutative if and only if the following diagram is commutative;

$$\begin{tikzcd}
\Psi_{(Q_L)_G}\Psi_{Q_G}p_1^* \arrow[r, "\Psi_{(Q_L)_G}\Psi_{Q_G}p_1^*\eta_{p_1}"] \arrow[d, "\omega'_G p_1^*"'] & \Psi_{(Q_L)_G}\Psi_{Q_G}p_1^*p_{1*}p_1^* \arrow[r, "\Psi_{(Q_L)_G}\Omega' p_1^*"] & \Psi_{(Q_L)_G}p_2^*p_{2*}\Psi_{Q_G}p_1^* \arrow[r, "\Omega'_L\Psi_{Q_G}p_1^*"] & p_1^*p_{1*}\Psi_{(Q_L)_G}\Psi_{Q_G}p_1^* \arrow[d, "p_1^*p_{1*}\omega'_G p_1^*"] \\
p_1^* \arrow[rrr, "p_1^*\eta_{p_1}"] & & & p_1^*p_{1*}p_1^*,
\end{tikzcd}$$

where $\omega'_G : \Psi_{(Q_L)_G}\Psi_{Q_G} \to \mathrm{id}_{\mathrm{Dcoh}_G(G\times X_1, p_1^*W_1)}$ is the adjunction morphism of $(\Psi_{(Q_L)_G} \dashv \Psi_{Q_G})$. Furthermore, since the restriction functor

$$\mathrm{Res}_G : \mathrm{DQcoh}_G(G \times X_2, {\pi'_2}^* p_2^* W_2) \to \mathrm{DQcoh}(G \times X_2, {\pi'_2}^* p_2^* W_2)$$

is faithful functor, in order to prove that the above diagram is commutative, it is enough to show that the following diagram is commutative,

$$(*)' : \quad \begin{tikzcd}
\Psi_{Q_L}\Psi_Q p_1^* \arrow[r, "\Psi_{Q_L}\Psi_Q p_1^*\eta_{p_1}"] \arrow[d, "\omega' p_1^*"'] & \Psi_{Q_L}\Psi_Q p_1^*p_{1*}p_1^* \arrow[r, "\Psi_{Q_L}\Omega' p_1^*"] & \Psi_{Q_L}p_2^*p_{2*}\Psi_Q p_1^* \arrow[r, "\Omega'_L\Psi_Q p_1^*"] & p_1^*p_{1*}\Psi_{Q_L}\Psi_Q p_1^* \arrow[d, "p_1^*p_{1*}\omega' p_1^*"] \\
p_1^* \arrow[rrr, "p_1^*\eta_{p_1}"] & & & p_1^*p_{1*}p_1^*,
\end{tikzcd}$$

where $\omega' : \Psi_{Q_L}\Psi_Q \to \mathrm{id}_{\mathrm{Dcoh}(G\times X_1, p_1^*W_1)}$ is the adjunction morphism of $(\Psi_{Q_L} \dashv \Psi_Q)$.

To decompose the diagram $(*)'$, we give the following:

Lemma 5.3. *Given the following diagram of functors*

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, bend left, "F_1"] \arrow[d, bend left, "P_{1*}"] & \mathcal{A}_2 \arrow[l, bend left, "G_1"] \arrow[r, bend left, "F_2"] \arrow[d, bend left, "P_{2*}"] & \mathcal{A}_3 \arrow[l, bend left, "G_2"] \arrow[d, bend left, "P_{3*}"] \\
\mathcal{A}'_1 \arrow[u, bend left, "P_1^*"] \arrow[r, bend left, "F'_1"] & \mathcal{A}'_2 \arrow[u, bend left, "P_2^*"] \arrow[l, bend left, "G'_1"] \arrow[r, bend left, "F'_2"] & \mathcal{A}'_3 \arrow[u, bend left, "P_3^*"] \arrow[l, bend left, "G'_2"]
\end{tikzcd}$$

and isomorphisms of functors $\Omega_{F_i} : F_iP_i^*P_{i*} \xrightarrow{\sim} P_{i+1}^*P_{i+1*}F_i$ and $\Omega_{G_i} : G_iP_{i+1}^*P_{i+1*} \xrightarrow{\sim} P_i^*P_{i*}G_i$, assume the adjunction $(G_i \dashv F_i)$ and $(P_i^* \dashv P_{i*})$ for each $i = 1, 2$. Set $F := F_2 \circ F_1$ and $G := G_1 \circ G_2$, and denote by $\omega : GF \to \mathrm{id}$ the functor morphism given by the composition $GF = G_1G_2F_2F_1 \xrightarrow{G_1\omega_2F_1} G_1F_1 \xrightarrow{\omega_1} \mathrm{id}$, where $\omega_i : G_iF_i \to \mathrm{id}$ is the adjunction morphism. Let $\Omega_F : FP_1^*P_{1*} \to P_3^*P_{3*}F$ and $\Omega_G : GP_3^*P_{3*} \to P_1^*P_{1*}G$ be the functor isomorphisms induced by Ω_{F_i} and Ω_{G_i}, i.e. $\Omega_F := \Omega_{F_2}F_1 \circ F_2\Omega_{F_1}$ and $\Omega_G := \Omega_{G_1}G_2 \circ G_1\Omega_{G_2}$. For each $i = 1, 2$, consider the following diagrams of functor morphisms

$$(\diamondsuit)_i : \quad \begin{tikzcd}
G_iF_iP_i^* \arrow[r, "G_iF_iP_i^*\eta_i"] \arrow[d, "\omega_iP_i^*"'] & G_iF_iP_i^*P_{i*}P_i^* \arrow[r, "G_i\Omega_{F_i}P_i^*"] & G_iP_{i+1}^*P_{i+1*}F_iP_i^* \arrow[r, "\Omega_{G_i}F_iP_i^*"] & P_i^*P_{i*}G_iF_iP_i^* \arrow[d, "P_i^*P_{i*}\omega_iP_i^*"] \\
P_i^* \arrow[rrr, "P_i^*\eta_i"] & & & P_i^*P_{i*}P_i^*,
\end{tikzcd}$$

where $\eta_i : \mathrm{id} \to P_{i*}P_i^*$ is the adjunction.

If the above diagrams $(\diamondsuit)_1$ *and* $(\diamondsuit)_2$ *are commutative, and there exist isomorphisms of functors* $\mu : F_1' P_{1*} P_1^* \xrightarrow{\sim} P_{2*} P_2^* F_1'$ *and* $\nu : F_1 P_1^* \xrightarrow{\sim} P_2^* F_1'$ *with the following diagrams*

$$(\dagger): \quad \begin{tikzcd} F_1 P_1^* P_{1*} P_1^* \arrow[r, "\Omega_{F_1} P_1^*"] \arrow[d, "\nu P_{1*} P_1^*"'] & P_2^* P_{2*} F_1 P_1^* \arrow[d, "P_2^* P_{2*} \nu"] \\ P_2^* F_1' P_{1*} P_1^* \arrow[r, "P_2^* \mu"] & P_2^* P_{2*} P_2^* F_1' \end{tikzcd} \qquad (\dagger\dagger): \quad \begin{tikzcd} F_1' P_{1*} P_1^* \arrow[rr, "\mu"] & & P_{2*} P_2^* F_1' \\ & F_1' \arrow[ul, "F_1' \eta_1"] \arrow[ur, "\eta_2 F_1'"'] & \end{tikzcd}$$

commutative, then the following diagram $(\diamondsuit)$ *is also commutative.*

$$(\diamondsuit): \quad \begin{tikzcd} GFP_1^* \arrow[r, "GFP_1^* \eta_1"] \arrow[d, "\omega P_1^*"'] & GFP_1^* P_{1*} P_1^* \arrow[r, "G\Omega_F P_1^*"] & GP_3^* P_{3*} FP_1^* \arrow[r, "\Omega_G FP_1^*"] & P_1^* P_{1*} GFP_1^* \arrow[d, "P_1^* P_{1*} \omega P_1^*"] \\ P_1^* \arrow[rrr, "P_1^* \eta_1"] & & & P_1^* P_{1*} P_1^*. \end{tikzcd}$$

Proof. At first, we show that the following diagram is commutative;

$$(\clubsuit): \quad \begin{tikzcd} G_2 FP_1^* \arrow[r, "G_2 FP_1^* \eta_1"] \arrow[d, "\omega_2 F_1 P_1^*"'] & G_2 FT_1 P_1^* \arrow[r, "G_2 F_2 \Omega_{F_1} P_1^*"] & G_2 F_2 T_2 F_1 P_1^* \arrow[r, "G_2 \Omega_{F_2} F_1 P_1^*"] & G_2 T_3 FP_1^* \arrow[r, "\Omega_{G_2} FP_1^*"] & T_2 G_2 FP_1^* \arrow[d, "T_2 \omega_2 F_1 P_1^*"] \\ F_1 P_1^* \arrow[r, "F_1 P_1^* \eta_1"] & F_1 T_1 P_1^* \arrow[rrr, "\Omega_{F_1} P_1^*"] & & & T_2 F_1 P_1^*, \end{tikzcd}$$

where $T_i := P_i^* P_{i*}$ for $i = 1, 2, 3$. By the commutativity of the diagram $(\dagger)$, the following diagram is commutative;

$$\begin{tikzcd} F_1 P_1^* \arrow[r, "F_1 P_1^* \eta_1"] \arrow[d, "\nu"'] & F_1 P_1^* P_{1*} P_1^* \arrow[r, "\Omega_{F_1} P_1^*"] \arrow[d, "\nu P_{1*} P_1^*"] & P_2^* P_{2*} F_1 P_1^* \arrow[d, "P_2^* P_{2*} \nu"] \\ P_2^* F_1' \arrow[r, "P_2^* F_1' \eta_1"] & P_2^* F_1' P_{1*} P_1^* \arrow[r, "P_2^* \mu"] & P_2^* P_{2*} P_2^* F_1', \end{tikzcd}$$

and we have $P_2^* \mu \circ P_2^* F_1' \eta_1 = P_2^* \eta_2 F_1'$ by the commutativity of the diagram $(\dagger\dagger)$. Hence we see that, via the isomorphism of functors $\nu : F_1 P_1^* \xrightarrow{\sim} P_2^* F_1'$, the commutativity of the diagram $(\clubsuit)$ is equivalent to the commutativity of the following diagram

$$\begin{tikzcd} G_2 F_2 P_2^* F_1' \arrow[r, "G_2 F_2 P_2^* \eta_2 F_1'"] \arrow[d, "\omega_2 P_2^* F_1'"'] & G_2 F_2 P_2^* P_{2*} P_2^* F_1' \arrow[r, "G_2 \Omega_{F_2} P_2^* F_1'"] & G_2 P_3^* P_{3*} F_2 P_2^* F_1' \arrow[r, "\Omega_{G_2} F_2 P_2^* F_1'"] & P_2^* P_{2*} G_2 F_2 P_2^* F_1' \arrow[d, "P_2^* P_{2*} \omega_2 P_2^* F_1'"] \\ P_2^* F_1' \arrow[rrr, "P_2^* \eta_2 F_1'"] & & & P_2^* P_{2*} P_2^* F_1'. \end{tikzcd}$$

This diagram is commutative by the commutativity of the diagram $(\diamondsuit)_2$.

Now we see that the diagram $(\diamondsuit)$ is commutative as follows;

$$\begin{aligned} P_1^* \eta_1 \circ \omega P_1^* &= T_1 \omega_1 P_1^* \circ \Omega_{G_1} F_1 P_1^* \circ G_1 \Omega_{F_1} P_1^* \circ G_1 F_1 P_1^* \eta_1 \circ G_1 \omega_2 F_1 P_1^* \\ &= T_1 \omega_1 P_1^* \circ \Omega_{G_1} F_1 P_1^* \circ G_1 T_2 \omega_2 F_1 P_1^* \circ G_1 \Omega_{G_2} FP_1^* \circ G\Omega_{F_2} F_1 P_1^* \circ GF_2 \Omega_{F_1} P_1^* \circ GFP_1^* \eta_1 \\ &= T_1 \omega_1 P_1^* \circ T_1 G_1 \omega_2 F_1 P_1^* \circ \Omega_{G_1} G_2 FP_1^* \circ G_1 \Omega_{G_2} FP_1^* \circ G\Omega_{F_2} F_1 P_1^* \circ GF_2 \Omega_{F_1} P_1^* \circ GFP_1^* \eta_1 \\ &= T_1 \omega P_1^* \circ \Omega_G FP_1^* \circ G\Omega_F P_1^* \circ GFP_1^* \eta_1, \end{aligned}$$

where the first equation (resp. the second equation) follows from the commutativity of the diagram $(\diamondsuit)_1$ (resp. $(\clubsuit)$), and the third equation follows from the functoriality of the functor isomorphism Ω_{G_1}. $\square$

The adjoint pair

$$\Psi_{P_L} \dashv \Psi_Q$$

$$\begin{tikzcd} \mathrm{Dcoh}(G \times X_1, p_1^*W_1) \arrow[r, shift left, "\Psi_Q"] & \mathrm{Dcoh}(G \times X_2, p_2^*W_2) \arrow[l, shift left, "\Psi_{Q_L}"] \end{tikzcd}$$

is induced by the following three adjoint pairs

(1) : $\overline{\pi_1}'_! \dashv \overline{\pi_1}'^*$

$$\begin{tikzcd} \mathrm{Dcoh}(G \times X_1, p_1^*W_1) \arrow[r, shift left, "\overline{\pi_1}'^*"] & \mathrm{Dcoh}(G \times X_1 \times \overline{X_2}, \overline{\pi_1}'^* p_1^*W_1), \arrow[l, shift left, "\overline{\pi_1}'_!"] \end{tikzcd}$$

where $\overline{\pi_1}'_! := \mathbf{R}\overline{\pi}'_{1*}((-) \otimes \overline{p_{12}}^*\omega_{\overline{\pi_1}}[\dim(\overline{X_2})])$.

(2) : $i'_*((-) \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee}) \dashv i'^*(-) \otimes^{\mathbf{L}} Q$

$$\begin{tikzcd} \mathrm{Dcoh}(G \times X_1 \times \overline{X_2}, \overline{\pi_1}'^* p_1^*W_1) \arrow[r, shift left, "i'^*(-)\otimes^{\mathbf{L}}Q"] & \mathrm{Dcoh}(G \times X_1 \times X_2, \pi_2'^* p_2^*W_2) \arrow[l, shift left, "i'_*((-)\otimes^{\mathbf{L}}Q^{\mathbf{L}\vee})"] \end{tikzcd}$$

and

(3) : $\pi_2'^* \dashv \mathbf{R}\pi'_{2*}$

$$\begin{tikzcd} \mathrm{Dcoh}(G \times X_1 \times X_2, \pi_2'^* p_2^*W_2) \arrow[r, shift left, "\mathbf{R}\pi'_{2*}"] & \mathrm{Dcoh}(G \times X_2, p_2^*W_2). \arrow[l, shift left, "\pi_2'^*"] \end{tikzcd}$$

Hence the adjunction morphism $\omega' : \Psi_{Q_L}\Psi_Q \to \mathrm{id}_{\mathrm{Dcoh}(G\times X_1, p_1^*W_1)}$ in the diagram $(*)'$ is the composition

$$\begin{aligned} \Psi_{Q_L}\Psi_Q &= \mathbf{R}\overline{\pi}'_{1*}(i'_*(\pi_2'^*\mathbf{R}\pi'_{2*}(i'^*\overline{\pi_1}'^*(-) \otimes^{\mathbf{L}} Q) \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee}) \otimes \overline{p_{12}}^*\omega_{\overline{\pi_1}}[\dim(\overline{X_2})]) \\ &\xrightarrow{\zeta_3} \mathbf{R}\overline{\pi}'_{1*}(i'_*(i'^*\overline{\pi_1}'^*(-) \otimes^{\mathbf{L}} Q \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee}) \otimes \overline{p_{12}}^*\omega_{\overline{\pi_1}}[\dim(\overline{X_2})]) \\ &\xrightarrow{\zeta_2} \mathbf{R}\overline{\pi}'_{1*}(\overline{\pi_1}'^*(-) \otimes \overline{p_{12}}^*\omega_{\overline{\pi_1}}[\dim(\overline{X_2})]) \\ &\xrightarrow{\zeta_1} \mathrm{id}_{\mathrm{Dcoh}(G\times X_1, p_1^*W_1)}, \end{aligned}$$

where for each $i = 1, 2, 3$, ζ_i is the functor morphism induced by the adjunction morphism of the above adjunction pair (i). Hence, by Lemma 5.3 and Lemma 4.63, to prove that the diagram $(*)'$ is commutative, it is enough to prove that the following diagrams $(*)'_i$ are commutative;

$(*)'_1$:

$$\begin{tikzcd} \overline{\pi_1}'_!\overline{\pi_1}'^*p_1^* \arrow[r, "\overline{\pi_1}'_!\overline{\pi_1}'^*p_1^*\eta_{p_1}"] \arrow[d, "\omega'_1 p_1^*"'] & \overline{\pi_1}'_!\overline{\pi_1}'^*p_1^*p_{1*}p_1^* \arrow[r, "\overline{\pi_1}'_!\Omega_1 p_1^*"] & \overline{\pi_1}'_!\overline{p_{12}}^*\overline{p_{12}}_*\overline{\pi_1}'^*p_1^* \arrow[r, "\Omega_{L1}\overline{\pi_1}'^*p_1^*"] & p_1^*p_{1*}\overline{\pi_1}'_!\overline{\pi_1}'^*p_1^* \arrow[d, "p_1^*p_{1*}\omega'_1 p_1^*"] \\ p_1^* \arrow[rrr, "p_1^*\eta_{p_1}"] & & & p_1^*p_{1*}p_1^*, \end{tikzcd}$$

where $\Omega_1 : \overline{\pi_1}'^*p_1^*p_{1*} \xrightarrow{\sim} \overline{p_{12}}^*\overline{p_{12}}_*\overline{\pi_1}'^*$ and $\Omega_{L1} : \overline{\pi_1}'_!\overline{p_{12}}^*\overline{p_{12}}_* \xrightarrow{\sim} p_1^*p_{1*}\overline{\pi_1}'_!$ are the functor isomorphisms given by Lemma 4.63, and $\omega'_1 : \overline{\pi_1}'_!\overline{\pi_1}'^* \to \mathrm{id}$ is the adjunction morphism of the adjoint pair (1).

$(*)_2'$:

$$\begin{tikzcd}[column sep=small]
i'_{Q*}{i'_Q}^*\overline{p_{12}}^* \arrow[r, "i'_{Q*}{i'_Q}^*\overline{p_{12}}^*\eta_{\overline{p_{12}}}"] \arrow[d, "\omega_2'\overline{p_{12}}^*"'] & i'_{Q*}{i'_Q}^*\overline{p_{12}}^*\overline{p_{12}}_*\overline{p_{12}}^* \arrow[r, "i'_{Q*}\Omega_2\overline{p_{12}}^*"] & i'_{Q*}{p_{12}}^*p_{12*}{i'_Q}^*\overline{p_{12}}^* \arrow[r, "\Omega_{L2}{i'_Q}^*\overline{p_{12}}^*"] & \overline{p_{12}}^*\overline{p_{12}}_*i'_{Q*}{i'_Q}^*\overline{p_{12}}^* \arrow[d, "\overline{p_{12}}^*\overline{p_{12}}_*\omega_2'\overline{p_{12}}^*"] \\
\overline{p_{12}}^* \arrow[rrr, "\overline{p_{12}}^*\eta_{\overline{p_{12}}}"] & & & \overline{p_{12}}^*\overline{p_{12}}_*\overline{p_{12}}^*,
\end{tikzcd}$$

where $i'_{Q*}(-) := i'_*((-)\otimes^{\mathbf{L}} Q^{\mathbf{L}\vee})$ and ${i'_Q}^* := i'^*(-)\otimes^{\mathbf{L}} Q$, and $\Omega_2 : {i'_Q}^*\overline{p_{12}}^*\overline{p_{12}}_* \xrightarrow{\sim} {p_{12}}^*p_{12*}{i'_Q}^*$ and $\Omega_{L2} : i'_{Q*}{p_{12}}^*p_{12*} \xrightarrow{\sim} {p_{12}}^*\overline{p_{12}}_*i'_{Q*}$ are the functor isomorphisms given by Lemma 4.63, and $\omega_2' : i'_{Q*}{i'_Q}^* \to \mathrm{id}$ is the adjunction morphism of the adjoint pair (2).

$(*)_3'$:

$$\begin{tikzcd}[column sep=small]
{\pi_2'}^*\mathbf{R}\pi'_{2*}{p_{12}}^* \arrow[r, "{\pi_2'}^*\mathbf{R}\pi'_{2*}{p_{12}}^*\eta_{p_{12}}"] \arrow[d, "\omega_3'{p_{12}}^*"'] & {\pi_2'}^*\mathbf{R}\pi'_{2*}{p_{12}}^*p_{12*}{p_{12}}^* \arrow[r, "{\pi_2'}^*\Omega_3{p_{12}}^*"] & {\pi_2'}^*{p_2}^*p_{2*}\mathbf{R}\pi'_{2*}{p_{12}}^* \arrow[r, "\Omega_{L3}\mathbf{R}\pi'_{2*}{p_{12}}^*"] & {p_{12}}^*p_{12*}{\pi_2'}^*\mathbf{R}\pi'_{2*}{p_{12}}^* \arrow[d, "{p_{12}}^*p_{12*}\omega_3'{p_{12}}^*"] \\
{p_{12}}^* \arrow[rrr, "{p_{12}}^*\eta_{p_{12}}"] & & & {p_{12}}^*p_{12*}{p_{12}}^*,
\end{tikzcd}$$

where $\Omega_3 : \mathbf{R}\pi'_{2*}{p_{12}}^*p_{12*} \xrightarrow{\sim} {p_2}^*p_{2*}\mathbf{R}\pi'_{2*}$ and $\Omega_{L3} : {\pi_2'}^*{p_2}^*p_{2*} \xrightarrow{\sim} {p_{12}}^*p_{12*}{\pi_2'}^*$ are the functor isomorphisms given by Lemma 4.63, and $\omega_3' : {\pi_2'}^*\mathbf{R}\pi'_{2*} \to \mathrm{id}$ is the adjunction morphism of the adjoint pair (3).

In the following, for each $i = 1, 2, 3$, we will prove that the diagram $(*)_i'$ is commutative.

- **Proof of the commutativity of** $(*)_1'$

Since the adjunction morphism $\omega_1' : \overline{\pi_1}'_!\overline{\pi_1}'^* \to \mathrm{id}_{\mathrm{Dcoh}(G\times X_1, p_1^*W_1)}$ is a restriction of the adjunction morphism $\omega_1' : \overline{\pi_1}'_!\overline{\pi_1}'^* \to \mathrm{id}_{\mathrm{DQcoh}(G\times X_1, p_1^*W_1)}$ of the adjoint pair

$$\begin{tikzcd}
\mathrm{DQcoh}(G\times X_1, p_1^*W_1) \arrow[r, shift left, "\overline{\pi_1}'^*"] & \mathrm{DQcoh}(G\times X_1\times \overline{X_2}, \overline{\pi_1}'^*p_1^*W_1), \arrow[l, shift left, "\overline{\pi_1}'_!"]
\end{tikzcd}$$

we have the functor morphism

$$\omega_1'p_1^*p_{1*}p_1^* : \overline{\pi_1}'_!\overline{\pi_1}'^*p_1^*p_{1*}p_1^* \to p_1^*p_{1*}p_1^*.$$

By the functoriality of ω_1', to prove the commutativity of $(*)_1'$ it is enough to prove that the following diagram is commutative;

$$(*)_{1a}' : \quad \begin{tikzcd}
\overline{\pi_1}'_!\overline{\pi_1}'^*p_1^*p_{1*} \arrow[r, "\overline{\pi_1}'_!\Omega_1"] \arrow[dr, "\omega_1'p_1^*p_{1*}"'] & \overline{\pi_1}'_!\overline{p_{12}}^*\overline{p_{12}}_*\overline{\pi_1}'^* \arrow[r, "\Omega_{L1}\overline{\pi_1}'^*"] & p_1^*p_{1*}\overline{\pi_1}'_!\overline{\pi_1}'^* \arrow[dl, "p_1^*p_{1*}\omega_1'"] \\
& p_1^*p_{1*} &
\end{tikzcd}$$

The adjunction morphism

$$\omega_1' : \overline{\pi_1}'_!\overline{\pi_1}'^* \to \mathrm{id}$$

is given by the composition of the following functor morphisms;

$$\varphi : \overline{\pi_1}'_!\overline{\pi_1}'^*(-) = \mathbf{R}\overline{\pi_1}'_*(\overline{\pi_1}'^*(-)\otimes\overline{p_{12}}^*\omega_{\overline{\pi_1}}[d_2]) \longrightarrow (-)\otimes^{\mathbf{L}} p_1^*\mathbf{R}\overline{\pi_1}_*\omega_{\overline{\pi_1}}[d_2]$$

and

$$\psi : (-)\otimes^{\mathbf{L}} p_1^*\mathbf{R}\overline{\pi_1}_*\omega_{\overline{\pi_1}}[d_2] \longrightarrow (-),$$

where $d_2 := \dim(\overline{X_2})$, the functor morphism φ is given by the projection formula and an isomorphism $\mathbf{R}\overline{\pi_1}'_{*}\overline{p_{12}}^* \cong p_1^*\mathbf{R}\overline{\pi_1}_*$, whence φ is a functor *isomorphism*, and ψ is given as follows. Let

$$\sigma : \mathbf{R}\overline{\pi_1}_*\omega_{\overline{\pi_1}}[d_2] \longrightarrow \mathcal{O}_{X_1}$$

be the following composition of morphisms in $\mathrm{D^b}(X_1)$;

$$\mathbf{R}\overline{\pi_1}_*\omega_{\overline{\pi_1}}[d_2] \xrightarrow{\sim} \mathbf{R}\overline{\pi_1}_*\overline{\pi_1}^!(\mathcal{O}_{X_1}) \longrightarrow \mathcal{O}_{X_1},$$

where the morphism $\mathbf{R}\overline{\pi_1}_*\overline{\pi_1}^!(\mathcal{O}_{X_1}) \to \mathcal{O}_{X_1}$ is induced by the adjunction morphism of the adjoint pair,

$$\mathbf{R}\overline{\pi_1}_* \dashv \overline{\pi_1}^! \qquad \begin{tikzcd} \mathrm{D^b}(\mathrm{coh}X_1) \arrow[r, shift left, "\overline{\pi_1}^!"] & \mathrm{D^b}(\mathrm{coh}X_1 \times \overline{X_2}). \arrow[l, shift left, "\mathbf{R}\overline{\pi_1}_*"] \end{tikzcd}$$

Then the functor morphism ψ is given as

$$\psi := (-) \otimes p_1^*\Upsilon(\sigma),$$

where $\Upsilon : \mathrm{D^b}(X_1) \to \mathrm{Dcoh}(X_1, 0)$ is the functor defined in Definition 3.14. Hence it is enough to prove that for any object $F \in \mathrm{Dcoh}(G \times X_1, p_1^*W)$ the following two diagrams are commutative,

$$(*)'_{1b}: \quad \begin{tikzcd} \mathbf{R}\overline{\pi_1}'_*(\overline{\pi_1}'^* p_1^*p_{1*}(F) \otimes \overline{p_{12}}^*\omega_{\overline{\pi_1}}[d_2]) \arrow[r, "\varphi p_1^* p_{1*}"] \arrow[d, "\overline{\pi_1}'_{!}\Omega_1"] & p_1^*p_{1*}(F) \otimes^{\mathbf{L}} p_1^*\mathbf{R}\overline{\pi_1}_*\omega_{\overline{\pi_1}}[d_2] \arrow[d] \\ \mathbf{R}\overline{\pi_1}'_*(\overline{p_{12}}^*\overline{p_{12}}_*\overline{\pi_1}'^*(F) \otimes \overline{p_{12}}^*\omega_{\overline{\pi_1}}[d_2]) \arrow[d, "\Omega_{L1}\overline{\pi_1}'^*"] & p_1^*(p_{1*}(F) \otimes^{\mathbf{L}} \mathbf{R}\overline{\pi_1}_*\omega_{\overline{\pi_1}}[d_2]) \arrow[d] \\ p_1^*p_{1*}\mathbf{R}\overline{\pi_1}'_*(\overline{\pi_1}'^*(F) \otimes \overline{p_{12}}^*\omega_{\overline{\pi_1}}[d_2]) \arrow[r, "p_1^*p_{1*}\varphi"] & p_1^*p_{1*}((F) \otimes^{\mathbf{L}} p_1^*\mathbf{R}\overline{\pi_1}_*\omega_{\overline{\pi_1}}[d_2]) \end{tikzcd}$$

and

$$(*)'_{1c}: \quad \begin{tikzcd} p_1^*p_{1*}(F) \otimes^{\mathbf{L}} p_1^*\mathbf{R}\overline{\pi_1}_*\omega_{\overline{\pi_1}}[d_2] \arrow[r, "p_1^*p_{1*}(F)\otimes p_1^*\Upsilon(\sigma)"] \arrow[d] & p_1^*p_{1*}(F) \otimes p_1^*\mathcal{O}_{X_1} \arrow[d] \arrow[dr] & \\ p_1^*(p_{1*}(F) \otimes^{\mathbf{L}} \mathbf{R}\overline{\pi_1}_*\omega_{\overline{\pi_1}}[d_2]) \arrow[r, "p_1^*(p_{1*}(F)\otimes\Upsilon(\sigma))"] \arrow[d] & p_1^*(p_{1*}(F) \otimes \mathcal{O}_{X_1}) \arrow[d] & p_1^*p_{1*}(F) \\ p_1^*p_{1*}((F) \otimes^{\mathbf{L}} p_1^*\mathbf{R}\overline{\pi_1}_*\omega_{\overline{\pi_1}}[d_2]) \arrow[r, "p_1^*p_{1*}((F)\otimes p_1^*\Upsilon(\sigma))"] & p_1^*p_{1*}((F) \otimes p_1^*\mathcal{O}_{X_1}), \arrow[ur] & \end{tikzcd}$$

where arrows with no symbols are natural isomorphisms.

At first, we show the diagram $(*)'_{1b}$ is commutative. Since functor morphisms in the diagram $(*)'_{1b}$ are natural in F and $\omega_{\overline{\pi_1}}[d_2]$, we can replace the objects F and $\omega_{\overline{\pi_1}}[d_2]$ with objects $E \in \mathrm{Dlfr}(G \times X_1, p_1^*W)$ and $I \in \mathrm{DQcoh}(X_1 \times \overline{X_2}, \overline{\pi_1}^*W)$ whose components I_1 and I_0 are injective sheaves respectively. Then derived functors in $(*)'_{1b}$ are isomorphic to underived functors, since the derived functor in the lowest row on the right side in $(*)'_{1b}$ is isomorphic to underived functor, and the direct images p_{1*} and $\overline{p_{12}}_*$ maps locally free sheaves to locally free sheaves, and the projection formulae for p_1 and $\overline{p_{12}}$ hold in categories of quasi-coherent sheaves without assuming locally freeness of sheaves. So it is enough to prove that the commutativity of the similar diagram in the abelian category $\mathrm{Qcoh}(G \times X_1)$. But this is checked by easy computations.

Next, we show the diagram $(*)'_{1c}$ is commutative. The commutativity of two square diagrams on the left side follows automatically by the functoriality. So we have only to verify that the triangular diagram on the right side is commutative. But this is verified by easy computations, and the detail is left to the reader.

• Proof of the commutativity of $(*)'_2$

To decompose the diagram $(*)'_2$, we will embed the diagram $(*)'_2$ to a larger category. Before embedding it, we provide some functors and some functor morphisms.

Since the functor $i'_{\#} : \mathrm{Qcoh}(G \times X_1 \times X_2) \to \mathrm{Ind}(\mathrm{Pro}(\mathrm{Qcoh}(G \times X_1 \times \overline{X_2})))$, constructed in Definition 4.42, is exact and compatible with potentials, it induces a functor

$$i'_{\#} : \mathrm{DQcoh}(G \times X_1 \times X_2, {\pi'_1}^* p_1^* W_1) \to \mathrm{Ind}(\mathrm{Pro}(\mathrm{DQcoh}(G \times X_1 \times \overline{X_2}, \overline{\pi'_1}^* p_1^* W_1))).$$

Let $i'_! : \mathrm{Dcoh}(G \times X_1 \times X_2, {\pi'_1}^* p_1^* W_1) \to \mathrm{Pro}(\mathrm{DQcoh}(G \times X_1 \times \overline{X_2}, \overline{\pi'_1}^* p_1^* W_1))$ be the extension by zero, and set

$$i'_{Q!}(-) := i'_!((-) \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee}) \quad \text{ and } \quad i'_{Q\#}(-) := i'_{\#}((-) \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee}).$$

Functor morphisms constructed in Lemma 4.44 (1) induces functor isomorphism

$$\gamma_{Q!} : i'_{Q!} \xrightarrow{\sim} \iota_{\mathrm{Pro}} i'_{Q*}$$

and functor morphism

$$\gamma_{Q\#} : i'_{Q\#} \to \iota_{\mathrm{InPr}} i'_{Q*}.$$

with $\iota_{\mathrm{Ind}} \gamma_{Q!} = \gamma_{Q\#}|_{\mathrm{Dcoh}(G \times X_1 \times X_2, {\pi'_1}^* p_1^* W_1)}$. Let $\omega'_2 : i'_{Q*} i'^*_Q \to \mathrm{id}$ be the adjunction morphism. Then the morphism $\iota_{\mathrm{Pro}} \omega'_2 : \iota_{\mathrm{Pro}} i'_{Q*} i'^*_Q \to \iota_{\mathrm{Pro}}$ is decomposed into the following compositions

$$\iota_{\mathrm{Pro}} i'_{Q*} i'^*_Q \xrightarrow{\gamma_{Q!}^{-1}} i'_{Q!} i'^*_Q \xrightarrow{i'_! {i'}^* \omega_Q} i'_! i'^* \xrightarrow{\omega_{i_!}} \iota_{\mathrm{Pro}},$$

where $\omega_Q : (-) \otimes^{\mathbf{L}} Q \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee} \to (-)$ and $\omega_{i'_!} : i'_! {i'}^* \to \iota_{\mathrm{Pro}}$ are the adjunction morphisms. Furthermore, the functor morphism constructed in Lemma 4.44 (2) induces a functor morphism

$$\lambda : i'_{\#} p_{12}^* p_{12*} \to \iota_{\mathrm{Ind}} \mathrm{Pro}(\overline{p_{12}}^* \overline{p_{12}}_*) i'_!.$$

Now we are ready to decompose the diagram $(*)'_2$. Let $\Omega_{i'*} : i'_* p_{12}^* p_{12*} \xrightarrow{\sim} \overline{p_{12}}^* \overline{p_{12}}_* i'_*$ and $\Omega^*_{i'} : i'^* \overline{p_{12}}^* \overline{p_{12}}_* \xrightarrow{\sim} p_{12}^* p_{12*} i'^*$ be natural functor isomorphisms. Set $i'^*_{Q \otimes Q^\vee}(-) := i'^*(-) \otimes^{\mathbf{L}} Q \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee}$, and let $\Omega^{i'^*}_{Q \otimes Q^\vee} : i'^*_{Q \otimes Q^\vee} \overline{p_{12}}^* \overline{p_{12}}_* \xrightarrow{\sim} {p_{12}}^* p_{12*} i'^*_{Q \otimes Q^\vee}$ be the functor isomorphism given by natural functor isomorphims $\Omega_2 : i'^*_Q \overline{p_{12}}^* \overline{p_{12}}_* \xrightarrow{\sim} p_{12}^* p_{12*} {i'_Q}^*$ and $\Omega_{Q^\vee} : p_{12}^* p_{12*}(-) \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee} \xrightarrow{\sim} p_{12}^* p_{12*}((-) \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee})$. Embedding the diagram $(*)'_2$ into the category $\mathrm{Ind}(\mathrm{Pro}(\mathrm{DQcoh}(G \times X_1 \times \overline{X_2}, \overline{\pi'_1}^* p_1^* W_1)))$ by the inclusion

$$\iota_{\mathrm{InPr}} : \mathrm{DQcoh}(G \times X_1 \times \overline{X_2}, \overline{\pi'_1}^* p_1^* W_1) \hookrightarrow \mathrm{Ind}(\mathrm{Pro}(\mathrm{DQcoh}(G \times X_1 \times \overline{X_2}, \overline{\pi'_1}^* p_1^* W_1))),$$

the diagram $(*)'_2$ is decomposed into the following diagram

$$\begin{tikzcd}
 & & i'_{Q*} {p_{12}}^* p_{12*} {i'_Q}^* \overline{p_{12}}^* \arrow[dr, "\Omega_{L2}"] \arrow[d, phantom, "(a)"] & & \\
i'_{Q*} i'^*_Q \overline{p_{12}}^* \arrow[r, "\eta_{\overline{p_{12}}}"] \arrow[d, "\gamma_{Q!}^{-1}"'] \arrow[dr, phantom, "(b)"] & i'_{Q*} i'^*_Q \overline{p_{12}}^* \overline{p_{12}}_* \overline{p_{12}}^* \arrow[ur, "\Omega_2"] \arrow[r, "\Omega^{i'^*}_{Q \otimes Q^\vee}"] \arrow[dr, phantom, "(c)"] & i'_* {p_{12}}^* p_{12*} i'^*_{Q \otimes Q^\vee} \overline{p_{12}}^* \arrow[r, "\Omega_{i'*}"] \arrow[dr, phantom, "(d)"] & \overline{p_{12}}^* \overline{p_{12}}_* i'_{Q*} i'^*_Q \overline{p_{12}}^* \arrow[d, "\gamma_{Q!}^{-1}"] \\
i'_{Q!} i'^*_Q \overline{p_{12}}^* \arrow[r, "\eta_{\overline{p_{12}}}"] \arrow[d, "\omega_Q"'] \arrow[dr, phantom, "(e)"] & i'_{Q\#} i'^*_Q \overline{p_{12}}^* \overline{p_{12}}_* \overline{p_{12}}^* \arrow[u, "\gamma_{Q\#}"'] \arrow[r, "\Omega^{i'^*}_{Q \otimes Q^\vee}"] \arrow[d, "\omega_Q"'] & i'_{\#} {p_{12}}^* p_{12*} i'^*_{Q \otimes Q^\vee} \overline{p_{12}}^* \arrow[u, "\gamma_{Q\#}"] \arrow[r, "\lambda"] \arrow[d, "\omega_Q"] \arrow[dr, phantom, "(f)"] & \overline{p_{12}}^* \overline{p_{12}}_* i'_{Q!} i'^*_Q \overline{p_{12}}^* \arrow[d, "\omega_Q"] \\
i'_! i'_* \overline{p_{12}}^* \arrow[r, "\eta_{\overline{p_{12}}}"] \arrow[d, "\omega_{i'_!}"'] & i'_{\#} i'^* \overline{p_{12}}^* \overline{p_{12}}_* \overline{p_{12}}^* \arrow[r, "\Omega^*_{i'}"] & i'_{\#} {p_{12}}^* p_{12*} i'^* \overline{p_{12}}^* \arrow[r, "\lambda"] & \overline{p_{12}}^* \overline{p_{12}}_* i'_! i'^* \overline{p_{12}}^* \arrow[d, "\omega_{i'_!}"] \\
\overline{p_{12}}^* \arrow[rrr, "\eta_{\overline{p_{12}}}"] & & & \overline{p_{12}}^* \overline{p_{12}}_* \overline{p_{12}}^*,
\end{tikzcd}$$

where functor morphisms attached to arrows are the ones which induce the functor morphisms, and we omit embedding functors ι_{InPr} and ι_{Pro} from the above diagram. The diagram (a) is commutative, since Ω_{L2} is given by $\Omega_{Q^\vee}$ and $\Omega_{i'*}$. The commutativity of the diagrams (b), (c), (e) and (f) follows from the functoriality of functor morphisms, and the diagram (d) is commutative by Lemma 4.44 (2). Hence, it is enough to verify the commutativity of the following diagrams

$$(*)'_{2a}: \quad \begin{tikzcd} i'^*\overline{p_{12}}^*\overline{p_{12}}_*(-) \otimes^{\mathbf{L}} Q \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee} \arrow[r, "\Omega^{i'*}_{Q\otimes Q^\vee}"] \arrow[d, "\omega_Q i'^*\overline{p_{12}}^*\overline{p_{12}}_*"'] & p_{12}^*p_{12*}(i'^*(-) \otimes^{\mathbf{L}} Q \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee}) \arrow[d, "p_{12}^*p_{12*}\omega_Q i'^*"] \\ i'\overline{p_{12}}^*\overline{p_{12}}_*(-) \arrow[r, "\Omega^*_{i'}"] & p_{12}^*p_{12*}i'^* \end{tikzcd}$$

and

$$(*)'_{2b}: \quad \begin{tikzcd} \iota_{\mathrm{Ind}}i'_!i'_* \arrow[r, "i'_\# i'^*\eta_{\overline{p_{12}}}"] \arrow[d, "\iota_{\mathrm{Ind}}\omega_{i'_!}"'] & i'_\# i'^*\overline{p_{12}}^*\overline{p_{12}}_* \arrow[r, "i'_\#\Omega^*_{i'}"] & i'_\# p_{12}{}^*p_{12}{}_*i'^* \arrow[r, "\lambda i'^*"] & \iota_{\mathrm{Ind}}\mathrm{Pro}(\overline{p_{12}}^*\overline{p_{12}}_*)i'_!i'^* \arrow[d, "\iota_{\mathrm{Ind}}\mathrm{Pro}(\overline{p_{12}}^*\overline{p_{12}}_*)\omega_{i'_!}"] \\ \iota_{\mathrm{InPr}} \arrow[rrr, "\iota_{\mathrm{InPr}}\eta_{\overline{p_{12}}}"] & & & \iota_{\mathrm{InPr}}\overline{p_{12}}^*\overline{p_{12}}_* \end{tikzcd}$$

We show that the diagram $(*)'_{2a}$ is commutative. Let $\Omega_Q : p_{12}^*p_{12*}(-) \otimes^{\mathbf{L}} Q \xrightarrow{\sim} p_{12}^*p_{12*}((-) \otimes^{\mathbf{L}} Q)$ be the natural functor isomorphism. Then, the functor morphism $\Omega_2 : {i'_Q}^*\overline{p_{12}}^*\overline{p_{12}}_* \xrightarrow{\sim} p_{12}{}^*p_{12*}{i'_Q}^*$ is the following compositions of functor morphisms

$$i'^*\overline{p_{12}}^*\overline{p_{12}}_*(-) \otimes^{\mathbf{L}} Q \xrightarrow{((-)\otimes^{\mathbf{L}}Q)\Omega^*_{i'}} p_{12}^*p_{12*}i'^*(-) \otimes^{\mathbf{L}} Q \xrightarrow{\Omega_Q i'^*} p_{12}^*p_{12*}(i'^*(-) \otimes^{\mathbf{L}} Q).$$

Moreover, the following diagram

$$\begin{tikzcd} i'^*\overline{p_{12}}^*\overline{p_{12}}_*(-) \otimes^{\mathbf{L}} Q \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee} \arrow[r, "((-)\otimes^{\mathbf{L}}Q\otimes^{\mathbf{L}}Q^{\mathbf{L}\vee})\Omega^*_{i'}"] \arrow[d, "\omega_Q i'^*\overline{p_{12}}^*\overline{p_{12}}_*"'] & p_{12}^*p_{12*}i'^*(-) \otimes^{\mathbf{L}} Q \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee} \arrow[d, "\omega_Q p_{12}^*p_{12*}i'^*"] \\ i'\overline{p_{12}}^*\overline{p_{12}}_*(-) \arrow[r, "\Omega^*_{i'}"] & p_{12}^*p_{12*}i'^* \end{tikzcd}$$

is commutative by the functoriality of the functor morphism ω_Q. Hence, to show that the diagram $(*)'_{2a}$ is commutative, we have only to show the commutativity of the following diagram

$$\begin{tikzcd} p_{12}^*p_{12*}i'^*(-) \otimes^{\mathbf{L}} Q \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee} \arrow[r, "((-)\otimes^{\mathbf{L}}Q^{\mathbf{L}\vee})\Omega_Q i'^*"] \arrow[dr, "\omega_Q p_{12}^*p_{12*}i'^*"'] & p_{12}^*p_{12*}(i'^*_Q(-)) \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee} \arrow[r, "\Omega_{Q^\vee}(i'^*(-)\otimes^{\mathbf{L}}Q)"] & p_{12}^*p_{12*}(i'^*_Q(-) \otimes^{\mathbf{L}} Q^{\mathbf{L}\vee}) \arrow[dl, "p_{12}^*p_{12*}\omega_Q i'^*"] \\ & p_{12}^*p_{12*}i'^* & \end{tikzcd}$$

Replacing the object $P \in \mathrm{Dcoh}(X_1 \times X_2, \pi_2^*W_2 - \pi_1^*W_1)$ with an object in $\mathrm{Dlfr}(X_1 \times X_2, \pi_2^*W_2 - \pi_1^*W_1)$, we may assume that the object $Q = p_{12}^*P$ is an object whose components are locally free sheaves. Then, the functors in the above diagram are underived functors. Hence, the commutativity of the diagram is verified by easy diagram chasing of morphisms between quasi-coherent sheaves, which is left to the reader.

Since all of the functors in $(*)'_{2b}$ are underived functors, the diagram $(*)'_{2b}$ is also verified by diagram chasing of map of systems, which is also left to the reader.

- **Proof of the commutativity of** $(*)'_3$

By the functoriality of ω'_3, the following diagram is commutative:

$$\begin{tikzcd}
{\pi'_2}^*\mathbf{R}\pi'_{2*}{p_{12}}^* \arrow[r, "{\pi'_2}^*\mathbf{R}\pi'_{2*}{p_{12}}^*\eta_{p_{12}}"] \arrow[d, "\omega'_3{p_{12}}^*"'] & {\pi'_2}^*\mathbf{R}\pi'_{2*}{p_{12}}^*p_{12*}{p_{12}}^* \arrow[d, "\omega'_3p_{12}^*p_{12*}p_{12}^*"] \\
{p_{12}}^* \arrow[r, "{p_{12}}^*\eta_{p_{12}}"] & {p_{12}}^*p_{12*}{p_{12}}^*
\end{tikzcd}$$

Hence, to prove that the diagram $(*)'_3$ is commutative, it is enough to prove the following diagram is commutative:

$$\begin{tikzcd}
{\pi'_2}^*\mathbf{R}\pi'_{2*}{p_{12}}^*p_{12*} \arrow[r, "{\pi'_2}^*\Omega_3"] \arrow[dr, "\omega'_3p_{12}^*p_{12*}"'] & {\pi'_2}^*p_2^*p_{2*}\mathbf{R}\pi'_{2*} \arrow[r, "\Omega_{L3}\mathbf{R}\pi'_{2*}"] & p_{12}^*p_{12*}{\pi'_2}^*\mathbf{R}\pi'_{2*} \arrow[dl, "p_{12}^*p_{12*}\omega'_3"] \\
& p_{12}^*p_{12*} &
\end{tikzcd}$$

Since we may replace any object in $\mathrm{Dcoh}(G \times X_1 \times X_2, {\pi'_2}^*p_2^*W_2)$ with an object whose components are injective sheaves, the commutativity of the above diagram can be checked by easy diagram chasing of morphisms between quasi-coherent sheaves, which is left to the reader.

5.3. **Main Theorem.** At first, to state the main theorem, we give the definition of G-linearizable objects.

Definition 5.4. Let X be a variety with G-action. An object F of $\mathrm{D^b}(\mathrm{coh}X)$ is called **G-linearizable**, if F is in the essential image of the forgetful functor $\mathrm{D^b}(\mathrm{coh}_G X) \to \mathrm{D^b}(\mathrm{coh}X)$.

We need the following lemma, which is an opposite version of [Joh, Lemma 1.1.1].[1] We give a proof for the reader's convenience.

Lemma 5.5 (cf. [Joh] Lemma 1.1.1)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between (arbitrary) categories, and suppose that F admits a right adjoint functor $G : \mathcal{B} \to \mathcal{A}$. Then, if there exists an isomorphism of functors $\alpha : \mathrm{id}_{\mathcal{A}} \xrightarrow{\sim} GF$ (α is not necessarily the adjunction morphism), then F is fully faithful.*

Proof. The isomorphism α implies that the following composition of maps is bijective;

$$\mathrm{Hom}(A, A') \xrightarrow{F} \mathrm{Hom}(F(A), F(A')) \xrightarrow{G} \mathrm{Hom}(GF(A), GF(A')).$$

Hence it is enough to show that G is fully faithful on the image of F. Since the above composition is bijective, G is full on the image of F. Let $\varepsilon : \mathrm{id}_{\mathcal{A}} \to GF$ and $\delta : FG \to \mathrm{id}_{\mathcal{B}}$ be the adjunction morphisms. For any $f \in \mathrm{Hom}(F(A), F(A'))$ we have

$$\delta_{F(A')} \circ FG(f) \circ F(\varepsilon_A) = f \circ \delta_{F(A)} \circ F(\varepsilon_A) = f,$$

where the first equation follows from the functoriality of δ and the the second equation follows from the property of the adjunction morphisms. Hence the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Hom}(F(A), F(A')) \arrow[r, equal] \arrow[d, "G"'] & \mathrm{Hom}(F(A), F(A')) \\
\mathrm{Hom}(GF(A), GF(A')) \arrow[r, "F"] & \mathrm{Hom}(FGF(A), FGF(A')), \arrow[u, "\delta_{F(A')}\circ(-)\circ F(\varepsilon_A)"']
\end{tikzcd}$$

and hence G is faithful on the image of F. □

[1] I would like to thank Timothy Logvinenko for informing me about this lemma.

Now we are ready to state and prove the main theorem.

Theorem 5.6. *Let $P \in \mathrm{D^b}(\mathrm{coh}X_1 \times_{\mathbb{A}^1} X_2)$ be a G-linearizable object whose support is proper over X_1 and X_2. If the integral functor $\Phi_{j_*(P)} : \mathrm{D^b}(\mathrm{coh}X_1) \to \mathrm{D^b}(\mathrm{coh}X_2)$ is an equivalence (resp. fully faithful), then there is an integral functor*

$$\Phi_{\widetilde{P}_G} : \mathrm{Dcoh}_G(X_1, W_1) \to \mathrm{Dcoh}_G(X_2, W_2)$$

which is also an equivalence (resp. fully faithful) for some $\widetilde{P}_G \in \mathrm{Dcoh}_G(X_1 \times X_2, \pi_2^* W_2 - \pi_1^* W_1)$.

Proof. Since P is G-linearizable, we may assume that there is an object $P_G \in \mathrm{D^b}(\mathrm{coh}_G X_1 \times_{\mathbb{A}^1} X_2)$ such that $\Pi(P_G) = P$, where $\Pi : \mathrm{D^b}(\mathrm{coh}_G X_1 \times_{\mathbb{A}^1} X_2) \to \mathrm{D^b}(\mathrm{coh}X_1 \times_{\mathbb{A}^1} X_2)$ is the forgetful functor. Set

$$\widetilde{P}_G := j_*(\Upsilon(P_G)) \in \mathrm{Dcoh}_G(X_1 \times X_2, \pi_2^* W_2 - \pi_1^* W_1),$$

where $\Upsilon : \mathrm{D^b}(\mathrm{coh}_G X_1 \times_{\mathbb{A}^1} X_2) \to \mathrm{Dcoh}_G(X_1 \times_{\mathbb{A}^1} X_2, 0)$ is the exact functor defined in Definition 3.14, and $j_* : \mathrm{Dcoh}_G(X_1 \times_{\mathbb{A}^1} X_2, 0) \to \mathrm{Dcoh}_G(X_1 \times X_2, \pi_2^* W_2 - \pi_1^* W_1)$ is the direct image of embedding $j : X_1 \times_{\mathbb{A}^1} X_2 \to X_1 \times X_2$. Let $\widetilde{P} := \mathrm{Res}_G(\widetilde{P}_G) \in \mathrm{Dcoh}(X_1 \times X_2, \pi_2^* W_2 - \pi_1^* W_1)$. Then we have

$$\widetilde{P} = j_*(\Upsilon(P)) = j_*(\mathrm{Tot}(\tau(P))) \cong \mathrm{Tot}(j_*(\tau(P))),$$

where $\tau : \mathrm{D^b}(\mathrm{coh}X_1 \times_{\mathbb{A}^1} X_2) \to \mathrm{D^b}(Z^0(\mathrm{coh}(X_1 \times_{\mathbb{A}^1} X_2, 0)))$ is the functor given by the same manner as in just before Definition 3.14, and j_* in the last one is the direct image

$$j_* : \mathrm{D^b}(Z^0(\mathrm{coh}(X_1 \times_{\mathbb{A}^1} X_2, 0))) \to \mathrm{D^b}(Z^0(\mathrm{coh}(X_1 \times X_2, 0)))$$

induced by an exact functor $j_* : Z^0(\mathrm{coh}(X_1 \times_{\mathbb{A}^1} X_2, 0)) \to Z^0(\mathrm{coh}(X_1 \times X_2, 0))$ between abelian categories. Since $\mathrm{Supp}(j_*(\tau(P))) = \mathrm{Supp}(P)$, $\widetilde{P}$ has a π_i-proper support $(i = 1, 2)$. By Theorem 5.1, it is enough to show that if the integral functor $\Phi_{j_*(P)} : \mathrm{D^b}(\mathrm{coh}X_1) \to \mathrm{D^b}(\mathrm{coh}X_2)$ is an equivalence (resp. fully faithful), then the integral functor

$$\Phi_{\widetilde{P}} : \mathrm{Dcoh}(X_1, W_1) \to \mathrm{Dcoh}(X_2, W_2)$$

is an equivalence (resp. fully faithful).

Assume that the integral functor $\Phi_{j_*(P)} : \mathrm{D^b}(\mathrm{coh}X_1) \to \mathrm{D^b}(\mathrm{coh}X_2)$ is fully faithful. The integral functor $\Phi_{j_*(P)}$ induces the extended functor $\Phi'_{j_*(P)} : \mathrm{D}(\mathrm{Qcoh}X_1) \to \mathrm{D}(\mathrm{Qcoh}X_2)$. Then the functor $\Phi'_{j_*(P)}$ is also fully faithful since it preserves any direct limit and any object in the unbounded derived category $\mathrm{D}(\mathrm{Qcoh}X_1)$ is isomorphic to the direct limit of a direct system of objects in $\mathrm{D^b}(\mathrm{coh}X_1)$ by [TT, Proposition 2.3.2]. Hence by the argument in the proof of [BDFIK1, Theorem 5.15], we obtain an isomorphism of functors

$$\Phi'_{\widetilde{P}_R} \circ \Phi'_{\widetilde{P}} \cong \mathrm{id}_{\mathrm{DQcoh}(X_1, W_1)},$$

where $\Phi'_{\widetilde{P}} : \mathrm{DQcoh}(X_1, W_1) \to \mathrm{DQcoh}(X_2, W_2)$ and $\Phi'_{\widetilde{P}_R} : \mathrm{DQcoh}(X_2, W_2) \to \mathrm{DQcoh}(X_1, W_1)$ are the extended functors from $\Phi_{\widetilde{P}}$ and its right adjoint $\Phi_{\widetilde{P}_R}$ respectively. This isomorphism of functors induces the restricted isomorphism of functors

$$\Phi_{\widetilde{P}_R} \circ \Phi_{\widetilde{P}} \cong \mathrm{id}_{\mathrm{Dcoh}(X_1, W_1)}.$$

Since $\Phi_{\widetilde{P}} \dashv \Phi_{\widetilde{P}_R}$ by Proposition 4.47, this isomorphism implies that the functor $\Phi_{\widetilde{P}} : \mathrm{Dcoh}(X_1, W_1) \to \mathrm{Dcoh}(X_2, W_2)$ is fully faithful by Lemma 5.5.

If the integral functor $\Phi_{j_*(P)} : \mathrm{D^b}(\mathrm{coh}X_1) \to \mathrm{D^b}(\mathrm{coh}X_2)$ is an equivalence, its left adjoint functor $\Phi_{j_*(P)_L}$ is fully faithful. Hence, by the above argument, we see that a left adjoint functor $\Phi_{\widetilde{P}_L} : \mathrm{Dcoh}(X_2, W_2) \to \mathrm{Dcoh}(X_1, W_1)$ of the fully faithful functor $\Phi_{\widetilde{P}}$ is also fully faithful. Hence $\Phi_{\widetilde{P}}$ is an equivalence. $\square$

5.4. **Applications.** In this last subsection, we give two applications of the main theorem.

5.4.1. *Flops of three folds.* Let X and X^+ be smooth quasi-projective threefolds, and let the diagram

$$X \xrightarrow{f} Y \xleftarrow{f^+} X^+$$

be a flop. Set $Z := X \times_Y X^+$ and let $\iota : Z \to X \times X^+$ be the embedding.

In [Bri], Bridgeland shows the following theorem:

Theorem 5.7 ([Bri]). *The integral functor*

$$\Phi_{\iota_*(\mathcal{O}_Z)} : \mathrm{D^b}(\mathrm{coh}X) \to \mathrm{D^b}(\mathrm{coh}X^+)$$

is an equivalence.

Let G be a reductive affine algebraic group acting on X, X^+ and Y with the morphisms f and f^+ equivariant. Take a semi invariant regular function $W_Y : Y \to \mathbb{A}^1$, and set $W := f^*W_Y$ and $W^+ := f^{+*}W_Y$. Consider the following cartesian square;

$$\begin{tikzcd} & X \times_{\mathbb{A}^1} X^+ \arrow[dl] \arrow[dr] & \\ X \arrow[dr, "W"'] & & X^+ \arrow[dl, "W^+"] \\ & \mathbb{A}^1 & \end{tikzcd}$$

The embedding $\iota : Z \to X \times X^+$ factors through $X \times_{\mathbb{A}^1} X^+$, i.e. ι is the composition of embeddings $i : Z \to X \times_{\mathbb{A}^1} X^+$ and $j : X \times_{\mathbb{A}^1} X^+ \to X \times X^+$. Set

$$P := i_*(\mathcal{O}_Z) \in \mathrm{D^b}(\mathrm{coh}X \times_{\mathbb{A}^1} X^+).$$

Since flopping contractions f and f^+ are proper morphisms, the support of P is proper over X and X^+. Furthermore, the object $\mathcal{O}_Z \in \mathrm{D^b}(\mathrm{coh}Z)$ has a tautological G-equivariant structure. Hence, P is a G-linearizable object. Consequently, we obtain the following corollary of Theorem 5.6:

Corollary 5.8. *We have an equivalence of derived factorization categories;*

$$\Phi_{\widetilde{P}_G} : \mathrm{Dcoh}_G(X, W) \xrightarrow{\sim} \mathrm{Dcoh}_G(X^+, W^+).$$

We define K-equivalence of gauged LG models. The above gauged LG models $(X, W)^G$ and $(X^+, W^+)^G$ are K-equivalent.

Definition 5.9. Let X_1 and X_2 be smooth varieties with group G-actions, and let $W_1 : X_1 \to \mathbb{A}^1$ and $W_2 : X_2 \to \mathbb{A}^1$ be χ-semi invariant regular functions for some character $\chi : G \to \mathbb{G}_m$. The gauged LG models $(X_1, \mathcal{O}(\chi), W_1)^G$ and $(X_2, \mathcal{O}(\chi), W_2)^G$ are called K**-equivalent**, if there exists a common G-equivariant resolution of X_1 and X_2

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X_1 & & X_2 \end{tikzcd}$$

such that $p^*W_1 = q^*W_2$ and $p^*\omega_{X_1} \cong q^*\omega_{X_2}$.

By Corollary 5.8 or [Seg, Conjecture 2,15], it is natural to expect the following conjecture:

Conjecture 5.10. *If two gauged LG models $(X, \mathcal{O}(\chi), W_X)^G$ and $(Y, \mathcal{O}(\chi), W_Y)^G$ are K-equivalent, then their derived factorization categories are equivalent;*

$$\mathrm{Dcoh}_G(X, W_X) \cong \mathrm{Dcoh}_G(Y, W_Y).$$

The above conjecture for K-equivalent gauged LG models of trivial σ-type is proposed by Kawamata [Kaw]. The converse of the above conjecture is not true in general. A counterexample to the converse of the Kawamata's conjecture is given by Uehara [Ueh].

5.4.2. *Equivariantizations of derived equivalences.* Let G be a reductive affine algebraic group, and let X_1 and X_2 be smooth quasi-projective varieties with G-actions.

Corollary 5.11. *Let $P \in \mathrm{D^b}(\mathrm{coh}X_1 \times X_2)$ be an object. Assume that P is G-linearizable object and the support of P is proper over X_1 and X_2. Choose an object $P_G \in \mathrm{D^b}(\mathrm{coh}_G X_1 \times X_2)$ such that $\Pi(P_G) \cong P$, where $\Pi : \mathrm{D^b}(\mathrm{coh}_G X_1 \times X_2) \to \mathrm{D^b}(\mathrm{coh}X_1 \times X_2)$ is the forgetful functor. If the integral functor $\Phi_P : \mathrm{D^b}(\mathrm{coh}X_1) \to \mathrm{D^b}(\mathrm{coh}X_1)$ is an equivalence (resp. fully faithful), then the integral functor*

$$\Phi_{P_G} : \mathrm{D^b}(\mathrm{coh}_G X_1) \to \mathrm{D^b}(\mathrm{coh}_G X_2)$$

is also an equivalence (resp. fully faithful).

Proof. Extend the G-action to $G \times \mathbb{G}_m$-action by $\mathbb{G}_m$ acting trivially. Then P is $G \times \mathbb{G}_m$-linearizable. By Theorem 5.6, there is an object $\widetilde{P}_{G\times\mathbb{G}_m} \in \mathrm{Dcoh}_{G\times\mathbb{G}_m}(X_1 \times X_2, 0)$ which induces an equivalence (resp. fully faithful)

$$\Phi_{\widetilde{P}_{G\times\mathbb{G}_m}} : \mathrm{Dcoh}_{G\times\mathbb{G}_m}(X_1, 0) \to \mathrm{Dcoh}_{G\times\mathbb{G}_m}(X_2, 0).$$

By Proposition 4.6 and equivalences $\mathrm{coh}_G X_i \cong \mathrm{coh}[X_i/G]$ for each $i = 1, 2$, we have equivalences

$$\Omega_i : \mathrm{Dcoh}_{G\times\mathbb{G}_m}(X_i, 0) \cong \mathrm{D^b}(\mathrm{coh}_G X_i).$$

Since the following diagram

$$\begin{tikzcd}
\mathrm{Dcoh}_{G\times\mathbb{G}_m}(X_1, 0) \arrow[r, "\Phi_{\widetilde{P}_{G\times\mathbb{G}_m}}"] \arrow[d, "\Omega_1"'] & \mathrm{Dcoh}_{G\times\mathbb{G}_m}(X_2, 0) \arrow[d, "\Omega_2"] \\
\mathrm{D^b}(\mathrm{coh}_G X_1) \arrow[r, "\Phi_{P_G}"] & \mathrm{D^b}(\mathrm{coh}_G X_2)
\end{tikzcd}$$

is commutative, the integral functor Φ_{P_G} is also an equivalence (resp. fully faithful). □

Corollary 5.11 is shown if the group G is finite by Ploog [Plo, Lemma 5]. We can also prove Corollary 5.11 for finite group actions by the result of [Ela2].

6. Relative singularity categories

Relative singularity categories are introduced in [EP], and it is shown that derived factorization categories (with some conditions on regular functions) are equivalent to relative singularity categories. In this section, we recall the definition and properties of relative singularity categories.

6.1. **Triangulated categories of relative singularities.** Let X be a quasi-projective scheme, and let G be an affine algebraic group acting on X. Throughout this section, we assume that *X has a G-equivariant ample line bundle.* If X is normal, this condition is satisfied by [Tho, Lemma 2.10]. The equivariant triangulated category of singularities $\mathrm{D}^{\mathrm{sg}}_G(X)$ of X is defined as the Verdier quotient of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G X)$ by the thick subcategory $\mathit{Perf}_G(X)$ of equivariant perfect complexes. Following [Orl1], we consider a larger category $\mathrm{D}^{\mathrm{cosg}}_G(X)$ defined as the Verdier quotient of $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X)$ by the thick subcategory $\mathit{Lfr}_G(X)$ of complexes which is quasi-isomorphic to bounded complexes of equivariant locally free sheaves (not necessarily of finite ranks). If G is trivial, we denote the singularity categories by $\mathrm{D}^{\mathrm{cosg}}(X)$ or $\mathrm{D}^{\mathrm{sg}}(X)$.

We recall relative singularity categories following [EP]. Let $Z \subset X$ be a G-invariant closed subscheme of X such that *$\mathcal{O}_Z$ has finite G-flat dimension as an $\mathcal{O}_X$-module* i.e., the G-equivariant sheaf $\mathcal{O}_Z \in \mathrm{coh}_G(X)$ has a finite resolution $F^\bullet \to \mathcal{O}_Z$ of G-equivariant flat sheaves on X. Under the assumption, we have the derived inverse image $\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z)$ between bounded derived categories for the closed immersion $i : Z \to X$. This functor preserves complexes of coherent sheaves; $\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{coh} X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh} Z)$.

Definition 6.1 ([EP] Section 2.1)**.** We define the following Verdier quotients

$$\mathrm{D}^{\mathrm{cosg}}_G(Z/X) := \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z)/\langle \mathrm{Im}(\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z))\rangle^{\oplus}$$

$$\mathrm{D}^{\mathrm{sg}}_G(Z/X) := \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G Z)/\langle \mathrm{Im}(\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G Z))\rangle,$$

where $\langle - \rangle$ (resp. $\langle - \rangle^{\oplus}$) denotes the smallest thick subcategory containing objects in $(-)$ (resp. and closed under infinite direct sums which exist in $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z)$). The quotient category $\mathrm{D}^{\mathrm{sg}}_G(Z/X)$ is called the **equivariant triangulated category of singularities of Z relative to X**. If G is trivial, we denote the categories defined above by $\mathrm{D}^{\mathrm{cosg}}(Z/X)$ or $\mathrm{D}^{\mathrm{sg}}(Z/X)$.

Proposition 6.2. *Assume that G is reductive. We have natural Verdier localizations by thick subcategories*

$$\pi^{\mathrm{co}} : \mathrm{D}^{\mathrm{cosg}}_G(Z) \to \mathrm{D}^{\mathrm{cosg}}_G(Z/X)$$

$$\pi : \mathrm{D}^{\mathrm{sg}}_G(Z) \to \mathrm{D}^{\mathrm{sg}}_G(Z/X).$$

Proof. It is enough to show that $\mathit{Lfr}_G(Z) \subset \langle \mathrm{Im}(\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z))\rangle^{\oplus}$ and $\mathit{Perf}_G(Z) \subset \langle \mathrm{Im}(\mathbf{L}i^* : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G Z))\rangle$. These inclusions follow from the assumption that X has a G-equivariant ample line bundle L. The proofs of the inclusions are similar, and we prove the only former inclusion. It is enough to show that any G-equivariant locally free sheaf E on Z is a direct summand of a bounded complex whose terms are direct sums of invertible sheaves of the form $i^* L^{\otimes n}$. By [Tho, Lemma 1.4], there is a bounded above locally free resolution $E^\bullet \xrightarrow{\sim} E$ whose terms are as above. For any $n > 0$, we have the following triangle in $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z)$

$$\sigma^{\geq -n} E^\bullet \to E \to H^{-n}(\sigma^{\geq -n} E^\bullet)[n+1] \to \sigma^{\geq -n} E^\bullet[1],$$

where $\sigma^{\geq -n}$ denotes the brutal truncation. If we choose a sufficiently large $n \gg 0$, we have

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z)}(E, H^{-n}(\sigma^{\geq -n} E^\bullet)[n+1]) = 0$$

by [Orl1, Lemma 1.12], since the restriction functor $\mathrm{Res}_G : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh} Z)$ is faithful by Lemma 4.62 (2). Hence the above triangle splits, and E is a direct summand of the complex $\sigma^{\geq -n} E^\bullet$. $\square$

Remark 6.3. Note that, if X is regular, then the thick subcategory $\langle \mathrm{Im}(\mathbf{L}i^*)\rangle \subset \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G Z)$ coincides with its thick subcategory $\mathit{Perf}_G(Z)$ of equivariant perfect complexes of Z. Hence the quotient category $\mathrm{D}^{\mathrm{sg}}_G(Z/X)$ is same as $\mathrm{D}^{\mathrm{sg}}_G(Z)$. Similarly, $\mathrm{D}^{\mathrm{cosg}}_G(Z/X)$ is also same as $\mathrm{D}^{\mathrm{cosg}}_G(Z)$ when X is regular.

The exact functors $\mathrm{Res}_G : \mathrm{Qcoh}_G Z \to \mathrm{Qcoh} Z$ and $\mathrm{Ind}_G : \mathrm{Qcoh} Z \to \mathrm{Qcoh}_G Z$ induce functors between relative singularity categories

$$\mathrm{Res}_G : \mathrm{D}^{\mathrm{cosg}}_{\mathrm{G}}(Z/X) \to \mathrm{D}^{\mathrm{cosg}}(Z/X)$$

$$\mathrm{Ind}_G : \mathrm{D}^{\mathrm{cosg}}(Z/X) \to \mathrm{D}^{\mathrm{cosg}}_{\mathrm{G}}(Z/X).$$

We need the following lemma in the proof of the main result.

Lemma 6.4. *Assume that G is reductive. Then the restriction functor*

$$\mathrm{Res}_G : \mathrm{D}^{\mathrm{cosg}}_{\mathrm{G}}(Z/X) \to \mathrm{D}^{\mathrm{cosg}}(Z/X)$$

is faithful.

Proof. This follows from a similar argument as in the proof of Lemma 4.62 (2). □

6.2. **Direct images and inverse images in relative singularity categories.** Let X_1 and X_2 be quasi-projective schemes with actions of an affine algebraic group G. Assume that X_1 and X_2 have G-equivariant ample line bundles. Let $\tilde{f} : X_2 \to X_1$ be a G-equivariant morphism. Let Z_1 be a G-invariant closed subscheme of X_1 such that $\mathcal{O}_{Z_1}$ has finite G-flat dimension as a $\mathcal{O}_{X_1}$-module, and let Z_2 be the fiber product $Z_1 \times_{X_1} X_2$. Denote by f the restriction $\tilde{f}|_{Z_2} : Z_2 \to Z_1$ of $\tilde{f}$ to Z_2. We assume that the cartesian square

$$\begin{tikzcd}
Z_2 \arrow[r, "f"] \arrow[d] & Z_1 \arrow[d] \\
X_2 \arrow[r, "\tilde{f}"] & X_1
\end{tikzcd}$$

is *exact* in the sense of [Kuz]. Then, $\mathcal{O}_{Z_2}$ also has finite G-flat dimension as a $\mathcal{O}_{X_2}$-module. Furthermore, we assume that $\tilde{f}$ *has finite G-flat dimension*, i.e. the derived inverse image $\mathbf{L}\tilde{f}^* : \mathrm{D}^-(\mathrm{Qcoh}_G X_1) \to \mathrm{D}^-(\mathrm{Qcoh}_G X_2)$ maps $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X_1)$ to $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G X_2)$. Then f also has finite G-flat dimension.

In the above setting, the derived inverse image $\mathbf{L}f^* : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z_1) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z_2)$ induces exact functors

$$f^\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z_1/X_1) \to \mathrm{D}^{\mathrm{cosg}}_G(Z_2/X_2)$$

$$f^\circ : \mathrm{D}^{\mathrm{sg}}_G(Z_1/X_1) \to \mathrm{D}^{\mathrm{sg}}_G(Z_2/X_2),$$

and the derived direct image $\mathbf{R}f_* : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z_2) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_G Z_1)$ induces a right adjoint functor of $f^\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z_1/X_1) \to \mathrm{D}^{\mathrm{cosg}}_G(Z_2/X_2)$

$$f_\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z_2/X_2) \to \mathrm{D}^{\mathrm{cosg}}_G(Z_1/X_1).$$

If f is a proper morphism, the direct image $\mathbf{R}f_* : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G Z_2) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_G Z_1)$ between bounded complexes of coherent sheaves induces a right adjoint functor

$$f_\circ : \mathrm{D}^{\mathrm{sg}}_G(Z_2/X_2) \to \mathrm{D}^{\mathrm{sg}}_G(Z_1/X_1)$$

of $f^\circ : \mathrm{D}^{\mathrm{sg}}_G(Z_1/X_1) \to \mathrm{D}^{\mathrm{sg}}_G(Z_2/X_2)$.

Let X be a quasi-projective scheme with an action of an affine algebraic group G, and let $U \subset X$ be a G-invariant open subscheme. Let $Z \subset X$ be a G-invariant closed subscheme such that $\mathcal{O}_Z$ has finite G-flat dimension, and consider the fiber product $U_Z := Z \times_X U$.

Denote by $\tilde{l} : U \to X$ and $l : U_Z \to Z$ the open immersions. Then we have the following exact cartesian square:

$$\begin{tikzcd} U_Z \arrow[r, "l"] \arrow[d] & Z \arrow[d] \\ U \arrow[r, "\tilde{l}"] & X. \end{tikzcd}$$

Lemma 6.5. *The inverse image*

$$l^\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z/X) \to \mathrm{D}^{\mathrm{cosg}}_G(U_Z/U)$$

is a Verdier localization by the kernel of l°.

Proof. The direct image $\mathbf{R}l_* : \mathrm{D^b}(\mathrm{Qcoh}U_Z) \to \mathrm{D^b}(\mathrm{Qcoh}Z)$ is fully faithful and right adjoint to the inverse image $l^* : \mathrm{D^b}(\mathrm{Qcoh}Z) \to \mathrm{D^b}(\mathrm{Qcoh}U_Z)$. By [Orl2, Lemma 1.1], the direct image functor $l_\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z/X) \to \mathrm{D}^{\mathrm{cosg}}_G(U_Z/U)$ is fully faithful. Hence, l° admits a right adjoint functor which is fully faithful, and this implies the result. ☐

6.3. **Relative singularity categories and derived factorization categories.** In this section, X and G are the same as in section 6.1, and we assume that G is *reductive.* Let $\chi : G \to \mathbb{G}_m$ be a character of G, and let $W : X \to \mathbb{A}^1$ be a χ-semi invariant regular function. In this section, we assume that the corresponding G-invariant section $W : \mathcal{O}_X \to \mathcal{O}(\chi)$ is injective. For example, if W is flat, this condition is satisfied. Denote by X_0 the fiber of W over $0 \in \mathbb{A}^1$, and let $i : X_0 \to X$ be the closed immersion. We have an exact functor $\tau : \mathrm{Qcoh}_G X_0 \to Z^0(\mathrm{Qcoh}_G(X, \chi, W))$ defined by

$$\tau(F) := \Big(0 \to i_*(F) \to 0\Big).$$

We define a natural functor

$$\Upsilon : \mathrm{D^b}(\mathrm{Qcoh}_G X_0) \to \mathrm{D^{co}Qcoh}_G(X, \chi, W)$$

as the composition of functors

$$\mathrm{D^b}(\mathrm{Qcoh}_G X_0) \xrightarrow{\tau} \mathrm{D^b}(Z^0(\mathrm{Qcoh}_G(X, \chi, W))) \xrightarrow{\mathrm{Tot}} \mathrm{D^{co}Qcoh}_G(X, \chi, W).$$

The functor Υ annihilates the thick category $\langle \mathrm{Im}(\mathbf{L}i^*)\rangle^{\oplus} \subset \mathrm{D^b}(\mathrm{Qcoh}_G X_0)$, since its nonequivariant functor $\Upsilon : \mathrm{D^b}(\mathrm{Qcoh}X_0) \to \mathrm{D^{co}Qcoh}(X, W)$ annihilates $\mathrm{Res}_G(\langle \mathrm{Im}(\mathbf{L}i^*)\rangle^{\oplus})$ (see the proof of [EP, Theorem 2.7 and Theorem 2.8]) and the restriction functor $\mathrm{Res}_G : \mathrm{D^{co}Qcoh}_G(X, \chi, W) \to \mathrm{D^{co}Qcoh}(X, W)$ is faithful. Hence it induces an exact functor

$$\Upsilon : \mathrm{D}^{\mathrm{cosg}}_G(X_0/X) \to \mathrm{D^{co}Qcoh}_G(X, \chi, W).$$

Similarly, we have the following exact functor

$$\Upsilon : \mathrm{D}^{\mathrm{sg}}_G(X_0/X) \to \mathrm{Dcoh}_G(X, \chi, W),$$

and the following diagram is commutative;

$$\begin{tikzcd} \mathrm{D}^{\mathrm{cosg}}_G(X_0/X) \arrow[r, "\Upsilon"] & \mathrm{D^{co}Qcoh}_G(X, \chi, W) \\ \mathrm{D}^{\mathrm{sg}}_G(X_0/X) \arrow[r, "\Upsilon"] \arrow[u] & \mathrm{Dcoh}_G(X, \chi, W), \arrow[u] \end{tikzcd}$$

where the vertical arrows are natural inclusion functors (which are fully faithful).

Theorem 6.6 (cf. [EP] Theorem 2.7, Theorem 2.8.)**.** *The functors*

$$\Upsilon : \mathrm{D}^{\mathrm{cosg}}_G(X_0/X) \to \mathrm{D^{co}Qcoh}_G(X, \chi, W)$$
$$\Upsilon : \mathrm{D}^{\mathrm{sg}}_G(X_0/X) \to \mathrm{Dcoh}_G(X, \chi, W)$$

are equivalences.

In order to prove the above theorem, we need to construct the quasi-inverse of Υ. We say that a G-equivariant quasi-coherent sheaf $F \in \mathrm{Qcoh}_G X$ is W-**flat**, if the morphism of sheaves $W : F \to F \otimes L$ is injective. Denote by $\mathrm{Flat}_G^W(X, \chi, W)$ the dg full subcategory of $\mathrm{Qcoh}_G(X, \chi, W)$ consisting of factorizations whose components are W-flat. Then $H^0(\mathrm{Flat}_G^W(X, \chi, W))$ is a full triangulated subcategory of $H^0(\mathrm{Qcoh}_G(X, \chi, W))$. Denote by $\mathrm{Acycl}^{\mathrm{co}}(\mathrm{Flat}_G^W(X, \chi, W))$ the smallest thick subcategory of $H^0(\mathrm{Flat}_G^W(X, \chi, W))$ containing all totalizations of short exact sequences in the exact category $Z^0(\mathrm{Flat}_G^W(X, \chi, W))$. Consider the corresponding Verdier quotients

$$\mathrm{D}^{\mathrm{co}}\mathrm{Flat}_G^W(X, \chi, W) := H^0(\mathrm{Flat}_G^W(X, \chi, W))/\mathrm{Acycl}^{\mathrm{co}}(\mathrm{Flat}_G^W(X, \chi, W)).$$

The restriction functor $\mathrm{Res}_G : \mathrm{Qcoh}_G(X, \chi, W) \to \mathrm{Qcoh}(X, W)$ and the induction functor $\mathrm{Ind}_G : \mathrm{Qcoh}(X, W) \to \mathrm{Qcoh}_G(X, \chi, W)$ preserve factorizations whose components are W-flat sheaves since $\mathrm{Res}_G : \mathrm{Qcoh}_G X \to \mathrm{Qcoh} X$ and $\mathrm{Ind}_G : \mathrm{Qcoh} X \to \mathrm{Qcoh}_G X$ are exact functors. Hence the restriction and the induction functors induce the following functors

$$\mathrm{Res}_G : \mathrm{D}^{\mathrm{co}}\mathrm{Flat}_G^W(X, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Flat}^W(X, W)$$
$$\mathrm{Ind}_G : \mathrm{D}^{\mathrm{co}}\mathrm{Flat}^W(X, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Flat}_G^W(X, \chi, W),$$

and these functors are adjoint to each other;

$$\mathrm{Res}_G \dashv \mathrm{Ind}_G.$$

Lemma 6.7. *The natural functor*

$$\mathrm{D}^{\mathrm{co}}\mathrm{Flat}_G^W(X, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W)$$

is an equivalence.

Proof. At first, we prove that the functor is essentially surjective. Let $F \in \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W)$ be an object. Since X has a G-equivariant ample line bundle, there are G-equivariant locally free sheaf E_i and a surjective morphism $p_i : E_i \to F_i$ in $\mathrm{Qcoh}_G X$ for each $i = 0, 1$. Let $E \in \mathrm{Qcoh}_G(X, \chi, W)$ be the factorization of the following form

$$E := \Big(E_1 \oplus E_0 \xrightarrow{W \oplus \mathrm{id}_{E_0}} E_1(\chi) \oplus E_0 \xrightarrow{\mathrm{id}_{E_1(\chi)} \oplus W} E_1(\chi) \oplus E_0(\chi)\Big).$$

Then p_1 and p_0 define a natural surjective morphism $p : E \to F$ in $Z^0(\mathrm{Qcoh}_G(X, \chi, W))$. The kernel $K := \mathrm{Ker}(p)$ of p is in $Z^0(\mathrm{Flat}_G^W(X, \chi, W))$ since the components of K are subsheaves of W-flat sheaves. Hence the totalization $\mathrm{Tot}(C^\bullet)$ of the complex

$$C^\bullet : \cdots \to 0 \to K \hookrightarrow E \to 0 \to \cdots$$

with the cohomological degree of E zero is in $\mathrm{D}^{\mathrm{co}}\mathrm{Flat}_G^W(X, \chi, W)$, and we see that the natural morphism $\mathrm{Tot}(C^\bullet) \to F$ induced by p is an isomorphism in $\mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W)$.

To show the functor $\mathrm{D}^{\mathrm{co}}\mathrm{Flat}_G^W(X, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X, \chi, W)$ is fully faithful, it suffices to prove that for any morphism $f : E \to F$ in $H^0(\mathrm{Qcoh}_G(X, \chi, W))$ with $F \in H^0(\mathrm{Flat}_G^W(X, \chi, W))$ and the cone of f in $\mathrm{Acycl}^{\mathrm{co}}(\mathrm{Qcoh}_G(X, \chi, W))$, there exists a morphism $g : F' \to E$ with $F' \in H^0(\mathrm{Flat}_G^W(X, \chi, W))$ such that the cone of $f \circ g$ is in $\mathrm{Acycl}^{\mathrm{co}}(\mathrm{Flat}_G^W(X, \chi, W))$ (see [LS, Proposition B.2. (*ff1*)$^{\mathrm{op}}$]). By the above argument in the previous paragraph, we can find a morphism $g : F' \to E$ with $F' \in H^0(\mathrm{Flat}_G^W(X, \chi, W))$ such that the cone of g is in $\mathrm{Acycl}^{\mathrm{co}}(\mathrm{Qcoh}_G(X, \chi, W))$, and then the cone of $f \circ g$ is in $H^0(\mathrm{Flat}_G^W(X, \chi, W)) \cap \mathrm{Acycl}^{\mathrm{co}}(\mathrm{Qcoh}_G(X, \chi, W))$. Hence it is enough to show that

$$H^0(\mathrm{Flat}_G^W(X, \chi, W)) \cap \mathrm{Acycl}^{\mathrm{co}}(\mathrm{Qcoh}_G(X, \chi, W)) \subseteq \mathrm{Acycl}^{\mathrm{co}}(\mathrm{Flat}_G^W(X, \chi, W)).$$

For this, let $A \in H^0(\mathrm{Flat}_G^W(X, \chi, W)) \cap \mathrm{Acycl}^{\mathrm{co}}(\mathrm{Qcoh}_G(X, \chi, W))$ be an object. We already know that $\mathrm{Res}_G(A) \in \mathrm{Acycl}^{\mathrm{co}}(\mathrm{Flat}^W(X, W))$ by [EP, Corollary 2.6 (a)]. Note that the restriction functor $\mathrm{Res}_G : \mathrm{D}^{\mathrm{co}}\mathrm{Flat}_G^W(X, \chi, W) \to \mathrm{D}^{\mathrm{co}}\mathrm{Flat}^W(X, W)$ is faithful by a similar argument as in the proof of Lemma 4.62 (2). Hence the fact that $\mathrm{Res}_G(A) \in \mathrm{Acycl}^{\mathrm{co}}(\mathrm{Flat}^W(X,W))$ implies that $A \in \mathrm{Acycl}^{\mathrm{co}}(\mathrm{Flat}^W_G(X,\chi,W))$. $\square$

For an object $F = \Big(F_1 \xrightarrow{\varphi_1^F} F_0 \xrightarrow{\varphi_0^F} F_1(\chi)\Big) \in Z^0(\mathrm{Flat}^W_G(X,\chi,W))$, define an object $\Xi(F) \in \mathrm{D}^{\mathrm{cosg}}_G(X_0/X)$ by $\Xi(F) := \mathrm{Cok}(\varphi_1^F)$. It is easy to see that this defines the following exact functor

$$\Xi : H^0(\mathrm{Flat}^W_G(X,\chi,W)) \to \mathrm{D}^{\mathrm{cosg}}_G(X_0/X).$$

If G is trivial, this exact functor annihilates $\mathrm{Acycl}^{\mathrm{co}}(\mathrm{Flat}^W(X,W))$ by [EP, Theorem 2.7, 2.8]. Hence, since $\mathrm{Res}_G : \mathrm{D}^{\mathrm{cosg}}_G(X_0/X) \to \mathrm{D}^{\mathrm{cosg}}(X_0/X)$ is faithful, we obtain the exact functor $\Xi : \mathrm{D}^{\mathrm{co}}\mathrm{Flat}^W_G(X,\chi,W) \to \mathrm{D}^{\mathrm{cosg}}_G(X_0/X)$. By Lemma 6.7, we have the left derived functor of Ξ;

$$\mathbf{L}\Xi : \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W) \to \mathrm{D}^{\mathrm{cosg}}_G(X_0/X).$$

Proof of Theorem 6.6: We will show that the functors Υ and $\mathbf{L}\Xi$ are mutually inverse. Let $E \in \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)$ be an object. By Lemma 6.7 we may assume that $E \in \mathrm{D}^{\mathrm{co}}\mathrm{Flat}^W_G(X,\chi,W)$. Then

$$\Upsilon\mathbf{L}\Xi(E) \cong \Upsilon\Xi(E) = \Big(0 \to \mathrm{Cok}(\varphi_1^E) \to 0\Big),$$

and the surjective morphism $E_0 \twoheadrightarrow \mathrm{Cok}(\varphi_1^E)$ induces the natural surjective morphism $\phi_E : E \to \Upsilon\Xi(E)$ in $Z^0(\mathrm{Qcoh}_G(X,\chi,W))$. Since the kernel of ϕ_E is the factorization $\Big(E_1 = E_1 \xrightarrow{W} E_1(\chi)\Big)$ and it is isomorphic to the zero object in $H^0(\mathrm{Flat}^W_G(X,\chi,W))$, the morphism $\phi_E : E \to \Upsilon\Xi(E)$ is an isomorphism in $\mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)$. It is easy to see that the isomorphisms $\phi_{(-)}$ define an isomorphism of functors

$$\phi : \mathrm{id}_{\mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)} \xrightarrow{\sim} \Upsilon\mathbf{L}\Xi.$$

Let $F \in \mathrm{D}^{\mathrm{cosg}}_G(X_0/X)$ be an object. Then we may assume that $F \in \mathrm{Qcoh}_G X_0$. Take a surjective morphism $p : P \twoheadrightarrow i_*F$ with P locally free. Set $K := \mathrm{Ker}(p) \in \mathrm{Qcoh}_G X$ and $Q := \big(K \xrightarrow{i} P \xrightarrow{W} K(\chi)\big) \in \mathrm{Qcoh}_G(X\chi,W)$, where $i : K \to P$ is the natural inclusion. Consider the natural surjective morphism $\pi : Q \to \big(0 \to i_*F \to 0\big)$ in $Z^0(\mathrm{Qcoh}_G(X,\chi,W))$. Then the kernel of π is the factorization $\big(K = K \xrightarrow{W} K(\chi)\big)$, and it is isomorphic to the zero object in $H^0(\mathrm{Qcoh}_G(X,\chi,W))$. Hence π is an isomorphism in $\mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}_G(X,\chi,W)$, and so we have a natural isomorphism $\psi_F : \mathbf{L}\Xi\Upsilon(F) \xrightarrow{\sim} F$ in $\mathrm{D}^{\mathrm{cosg}}_G(X_0/X)$ defined as the composition $\mathbf{L}\Xi\Upsilon(F) \xrightarrow{\sim} \Xi\Upsilon(Q) = \mathrm{Cok}(i) = F$. We need to show that the isomorphisms $\psi_{(-)}$ are functorial in $(-)$. Since the restriction functor Res_G is isomorphic to the forgetful functor Forg_G, we have a natural isomorphism of functors $\sigma : \mathrm{Res}_G\mathbf{L}\Xi\Upsilon \xrightarrow{\sim} \mathbf{L}\Xi\Upsilon\mathrm{Res}_G$ defined by the composition

$$\mathrm{Res}_G\mathbf{L}\Xi\Upsilon \xrightarrow{\sim} \mathrm{Forg}_G\mathbf{L}\Xi\Upsilon = \mathbf{L}\Xi\Upsilon\mathrm{Forg}_G \xrightarrow{\sim} \mathbf{L}\Xi\Upsilon\mathrm{Res}_G,$$

and the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Res}_G\mathbf{L}\Xi\Upsilon(F) \arrow[rr, "\mathrm{Res}_G(\psi_F)"] \arrow[dr, "\sigma_F"'] & & \mathrm{Res}_G(F) \\
& \mathbf{L}\Xi\Upsilon\mathrm{Res}_G(F) \arrow[ur, "\psi_{\mathrm{Res}_G(F)}"'] &
\end{tikzcd}$$

Hence we see that the isomorphisms $\psi_{(-)}$ are functorial by the fact that the isomorphisms $\psi_{(-)}$ are functorial if G is trivial and that the functor Res_G is faithful. This completes the proof of the former equivalence.

The latter equivalence follows from [EP, Remark 2.7], which is a generalized result of [EP, Theorem 2.7]. □

7. Main results (Part II)

Let X be a smooth quasi-projective variety, and let G be a reductive affine algebraic group acting on X. Let $\mathcal{E}$ be a G-equivariant locally free sheaf of rank r, and let $s \in \Gamma(X, \mathcal{E}^\vee)^G$ be a G-invariant section of $\mathcal{E}^\vee$. Denote by $Z \subset X$ the zero scheme of s. We assume that s is *regular*, i.e. the codimension of Z in X is r. Let

$$\mathrm{V}(\mathcal{E}(\chi)) := \underline{\mathrm{Spec}}(\mathrm{Sym}(\mathcal{E}(\chi)^\vee))$$

be a vector bundle over X with the G-action induced by the equivariant structure of the locally free sheaf $\mathcal{E}(\chi)$. Denote by $\mathrm{V}(\mathcal{E}(\chi))|_Z$ the restriction of the vector bundle $\mathrm{V}(\mathcal{E}(\chi))$ to Z. Let $j : Z \hookrightarrow X$ and $i : \mathrm{V}(\mathcal{E}(\chi))|_Z \hookrightarrow \mathrm{V}(\mathcal{E}(\chi))$ be the closed immersions, and let $q : \mathrm{V}(\mathcal{E}(\chi)) \to \mathrm{X}$ and $p : \mathrm{V}(\mathcal{E}(\chi))|_Z \to Z$ be the projections. Now we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{V}(\mathcal{E}(\chi))|_Z \arrow[r, "i"] \arrow[d, "p"'] & \mathrm{V}(\mathcal{E}(\chi)) \arrow[d, "q"] \\ Z \arrow[r, "j"] & X. \end{tikzcd}$$

The invariant section s induces a χ-semi invariant regular function

$$Q_s : \mathrm{V}(\mathcal{E}(\chi)) \to \mathbb{A}^1.$$

Let $W : X \to \mathbb{A}^1$ be a χ-semi invariant regular function on X. The function W induces χ-semi invariant functions on Z, $\mathrm{V}(\mathcal{E}(\chi))$ and $\mathrm{V}(\mathcal{E}(\chi))|_Z$, which we denote by the same notation W (by abuse of notation). Since the inverse image p^* and the direct image i_* are exact and commutative with arbitrary direct sums as functors between categories of quasi-coherent sheaves, these induce (underived) functors

$$p^* : \mathrm{D^{co}Qcoh}_G(Z, \chi, W) \to \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi))|_Z, \chi, W)$$

$$i_* : \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi))|_Z, \chi, W) \to \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, W + Q_s).$$

Restricting the composition $i_*p^* : \mathrm{D^{co}Qcoh}_G(Z, \chi, W) \to \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, W + Q_s)$ to $\mathrm{Dcoh}_G(Z, \chi, W)$, we obtain an exact functor

$$i_*p^* : \mathrm{Dcoh}_G(Z, \chi, W) \to \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, W + Q_s).$$

Shipman proved that the above functor i_*p^* is an equivalence when $G = \mathbb{G}_m$ trivially acts on X and $W = 0$ (see also [Isi]):

Theorem 7.1 ([Shi] Theorem 3.4)**.** *The composition*

$$i_*p^* : \mathrm{Dcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \xrightarrow{\sim} \mathrm{Dcoh}_{\mathbb{G}_m}(\mathrm{V}(\mathcal{E}(\chi_1)), \chi_1, Q_s)$$

is an equivalence.

The goal of this section is to show the following main result which is an analogy of the above theorem.

Theorem 7.2. *Assume that $W|_Z : Z \to \mathbb{A}^1$ is flat. The functor*

$$i_*p^* : \mathrm{Dcoh}_G(Z, \chi, W) \to \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), \chi, W + Q_s)$$

is an equivalence.

Remark 7.3. Let S be a smooth quasi-projective variety, and let G be an affine reductive group acting on S. Let $W : S \to \mathbb{A}^1$ be a $\chi := \chi_1 + \chi_2$-semi invariant non-constant regular function for some characters $\chi_i : G \to \mathbb{G}_m$. Let $X := \mathrm{V}(\mathcal{O}(\chi_1)) \cong S \times \mathbb{A}^1_{x_1}$ be the G-vector bundle over S, and let $s \in \Gamma(X, \mathcal{O}(\chi_1))^G$ be the section corresponding to the χ_1-semi invariant function $S \times \mathbb{A}^1_{x_1} \to \mathbb{A}^1$ which is defined as the projection $(s, x_1) \mapsto x_1$. Then, S is isomorphic to the zero scheme of s, and the G-vector bundle $\mathrm{V}(\mathcal{O}(-\chi_1)(\chi))$ over X is isomorphic to the G-variety $S \times \mathbb{A}^2_{x_1,x_2}$, where the G-weights of x_i is given by χ_i. By Theorem 7.2, we have the following equivalence

$$\mathrm{Dcoh}_G(S, \chi, W) \simeq \mathrm{Dcoh}_G(S \times \mathbb{A}^2_{x_1,x_2}, \chi, W + x_1x_2).$$

This kind of equivalence is know as *Knörrer periodicity*, so the above theorem is considered as a generalization of the original Knörrer periodicity [Knö, Theorem 3.1].

7.1. **Koszul factorizations.** Let $(X, \chi, W)^G$ be a gauged LG model such that X is a smooth variety. Let $\mathcal{E}$ be a G-equivariant locally free sheaf on X of rank r, and let

$$s : \mathcal{E} \to \mathcal{O}_X \quad \text{and} \quad t : \mathcal{O}_X \to \mathcal{E}(\chi)$$

be morphisms in $\mathrm{coh}_G X$ such that $t \circ s = W \cdot \mathrm{id}_{\mathcal{E}}$ and $s(\chi) \circ t = W$. Let $Z_s \subset X$ be the zero scheme of the section $s \in \Gamma(X, \mathcal{E}^\vee)^G$. We say that s is **regular** if the codimension of Z_s in X equals to the rank r.

Definition 7.4. We define an object $K(s, t) \in \mathrm{lfr}_G(X, \chi, W)$ as

$$K(s,t) := \Big(K_1 \xrightarrow{k_1} K_0 \xrightarrow{k_0} K_1(\chi)\Big)$$

where

$$K_1 := \bigoplus_{n=0}^{\lceil r/2 \rceil - 1} \big(\bigwedge^{2n+1} \mathcal{E}\big)(\chi^n), \qquad K_0 := \bigoplus_{n=0}^{\lfloor r/2 \rfloor} \big(\bigwedge^{2n} \mathcal{E}\big)(\chi^n)$$

and

$$k_i := t \wedge (-) \oplus s \vee (-).$$

The following property will be necessary in section 7.2.

Lemma 7.5 ([BFK1] Lemma 3.21 and Proposition 3.20)**.**
(1) *We have a natural isomorphism*

$$K(s,t)^\vee \cong K(t^\vee, s^\vee).$$

(2) *If s is regular, we have a natural isomorphisms in* $\mathrm{Dcoh}_G(X, \chi, W)$

$$\mathcal{O}_{Z_s} \cong K(s,t) \quad \text{and} \quad \mathcal{O}_{Z_s} \otimes \bigwedge^r \mathcal{E}^\vee(\chi^{-1})[-r] \cong K(s,t)^\vee,$$

where $\mathcal{O}_{Z_s} := \Big(0 \to \mathcal{O}_{Z_s} \to 0\Big)$ *and* $\bigwedge^r \mathcal{E}^\vee(\chi^{-1})[-r]$ *is a complex in* $\mathrm{coh}_G X$*.*

7.2. **Integral functors in Gorenstein cases.** We define integral functors between derived factorization categories. For simplicity, we consider the case when G is trivial. Let X_1 and X_2 be Gorenstein quasi-projective schemes, and let $W_i : X_i \to \mathbb{A}^1$ be a regular function. We denote the projection by $\pi_i : X_1 \times X_2 \to X_i$ for each $i = 1, 2$.

In order to define integral functors in Gorenstein cases, we need the following lemmas:

Lemma 7.6. *Assume that the scheme X is Noetherian. The natural functor*

$$H^0(\mathrm{Inj}_G(X, \chi, W)) \to \mathrm{D^{co}Qcoh}_G(X, \chi, W)$$

is an equivalence.

Proof. Since the abelian category $\mathrm{Qcoh}_G X$ of G-equivariant quasi-coherent sheaves is a locally Noetherian Grothendieck category, it has enough injective objects, and coproducts of injective objects are injective. Hence the result follows from [BDFIK1, Cororally 2.25]. $\square$

Lemma 7.7 ([EP] Corollary 2.3.e and 2.4.a)**.** *Let (X, W) be a LG model. Assume that the scheme X is a Gorenstein separated scheme of finite Krull dimension with an ample line bundle. Then the functor*

$$\mathrm{D^{co}LFr}(X, W) \to \mathrm{D^{co}Qcoh}(X, W)$$

induced by the embedding of dg functor $\mathrm{LFr}(X, W) \to \mathrm{Qcoh}(X, W)$ *is an equivalence.*

Note that, since X_1 and X_2 are Gorenstein, so is $X_1 \times X_2$ (cf. [TY]). By the above lemmas, for $P \in \mathrm{D^{co}Qcoh}(X_1 \times X_2, \pi_2^* W - \pi_1^* W)$, we can define the integral functor with respect to P, denoted by Φ_P, as the following functor

$$\mathbf{R}\pi_{2*}(\pi_1^*(-) \otimes^{\mathbf{L}} P) : \mathrm{D^{co}Qcoh}(X_1, W_1) \to \mathrm{D^{co}Qcoh}(X_2, W_2).$$

Similar result to Proposition 4.48 holds for integral functors in Gorenstein cases.

7.3. **Lemmas for the main theorem.** In this section, we provide some lemmas for the main result. Throughout this section, we consider the case when G is trivial.

Set

$$\omega_j := \bigwedge^r (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee \quad \text{and} \quad \omega_i := p^* \omega_j,$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z in X. These are invertible sheaves on Z and $\mathrm{V}(\mathcal{E})|_Z$ respectively. We define an exact functor

$$i^! : \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E}), W + Q_s) \to \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E})|_Z, W)$$

as $i^!(-) := \mathbf{L}i^*(-) \otimes \omega_i[-r]$. By [EP, Theorem 3.8], the above functor $i^!$ is right adjoint to $i_* : \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E})|_Z, W) \to \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E}), W + Q_s)$. Let

$$K := K(q^* s, t) \in \mathrm{lfr}(\mathrm{V}(\mathcal{E}), Q_s)$$

be the Koszul factorization of $q^* s \in \Gamma(\mathrm{V}(\mathcal{E}), q^* \mathcal{E}^\vee)$ and $t \in \Gamma(\mathrm{V}(\mathcal{E}), q^* \mathcal{E})$, where t is the tautological section. By abuse of notation, we denote by $\mathcal{O}_Z$ the object in $\mathrm{coh}(Z, 0)$ of the following form

$$\Big(0 \to \mathcal{O}_Z \to 0\Big).$$

Lemma 7.8. *Consider the case when $W = 0$. We have isomorphisms*

$$i_* p^*(\mathcal{O}_Z) \cong K \quad \text{and} \quad p_* i^!(K) \cong \mathcal{O}_Z$$

in $\mathrm{Dcoh}(\mathrm{V}(\mathcal{E}), Q_s)$ *and in* $\mathrm{D^{co}Qcoh}(Z, 0)$ *respectively.*

Proof. These isomorphisms follow from Lemma 7.5. In particular, the former isomorphism is an immediate consequence. Note that $\omega_i \cong i^* \bigwedge^r q^* \mathcal{E}^\vee$. We obtain the latter isomorphism as follows;

$$p_* i^!(K) \cong p_* \mathbf{L}i^*(\mathcal{O}_Z \otimes \bigwedge^r q^* \mathcal{E}^\vee[-r]) \cong p_* \mathbf{L}i^*(K^\vee) \cong p_* \mathbf{L}i^*(\mathcal{O}_{Z_{t^\vee}}) \cong \mathcal{O}_Z,$$

where the last isomorphism follows from the fact that the zero section $Z \subset \mathrm{V}(\mathcal{E})$ is isomorphic to the fiber product of closed subschemes $\mathrm{V}(\mathcal{E})|_Z \hookrightarrow \mathrm{V}(\mathcal{E})$ and $Z_{t^\vee} \hookrightarrow \mathrm{V}(\mathcal{E})$. $\square$

Lemma 7.9. *The functor*

$$i_* p^* : \mathrm{D^{co}Qcoh}(Z, W) \to \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E}), W + Q_s)$$

is fully faithful.

Proof. The functors i_*p^* and $p_*i^!$ can be represented as integral functors

$$i_*p^* \cong \Phi_{k_*\mathcal{O}_{\mathrm{V}(\mathcal{E})|_Z}} \quad \text{and} \quad p_*i^! \cong \Phi_{k_*\omega_i[-r]},$$

where $k := p \times i : \mathrm{V}(\mathcal{E})|_Z \to Z \times \mathrm{V}(\mathcal{E})$ and kernels $\mathcal{O}_{\mathrm{V}(\mathcal{E})|_Z}$ and $\omega_i[-r]$ are objects in $\mathrm{Dcoh}(\mathrm{V}(\mathcal{E})|_Z, 0)$. By easy computation, we see that there exists an object $P \in \mathrm{D^{co}Qcoh}(Z, 0)$ such that $p_*i^! \circ i_*p^* \cong \Phi_{\Delta_*P} \cong (-) \otimes P$, where $\Delta : Z \to Z \times Z$ is the diagonal embedding. Substituting $W = 0$, by Lemma 7.8, we have an isomorphism $P \cong \mathcal{O}_Z$. But P doesn't depend on the function W. Hence, for any W, we have an isomorphism of functors $p_*i^! \circ i_*p^* \cong \Phi_{\Delta_*P} \cong \mathrm{id}_{\mathrm{D^{co}Qcoh}(Z,W)}$. By Lemma 5.5, this implies that the functor $i_*p^* : \mathrm{D^{co}Qcoh}(Z, W) \to \mathrm{D^{co}Qcoh}(\mathrm{V}(\mathcal{E}), W + Q_s)$ is fully faithful. □

7.4. **Proof of the main theorem.** In this section, we prove the main theorem. Recall that G is a reductive affine algebraic group acting on a smooth quasi-projective variety X. Since X is smooth, there is a G-equivariant ample line bundle on X. In what follows, we assume that $W|_Z : Z \to \mathbb{A}^1$ is flat.

At first, we consider relative singularity categories. Let Z_0, $V|_{Z_0}$ and V_0 be the fibers of $W : Z \to \mathbb{A}^1$, $W : \mathrm{V}(\mathcal{E}(\chi))|_Z \to \mathbb{A}^1$ and $W + Q_s : \mathrm{V}(\mathcal{E}(\chi)) \to \mathbb{A}^1$ over $0 \in \mathbb{A}^1$ respectively. Denote by $p_0 : V|_{Z_0} \to Z_0$ and $i_0 : V|_{Z_0} \to V_0$ the restrictions of p and i respectively. By [Kuz, Corollary 2.27], the following cartesian squares are exact

$$\begin{tikzcd} V|_{Z_0} \arrow[r, "p_0"] \arrow[d] & Z_0 \arrow[d] \\ \mathrm{V}(\mathcal{E})|_Z \arrow[r, "p"] & Z \end{tikzcd} \qquad \begin{tikzcd} V|_{Z_0} \arrow[r, "i_0"] \arrow[d] & V_0 \arrow[d] \\ \mathrm{V}(\mathcal{E})|_Z \arrow[r, "i"] & \mathrm{V}(\mathcal{E}). \end{tikzcd}$$

Since p and i have finite flat dimensions, we have exact functors of relative singularity categories

$$p_0{}^\circ : \mathrm{D}_G^{\mathrm{cosg}}(Z_0/Z) \to \mathrm{D}_G^{\mathrm{cosg}}(V|_{Z_0}/\mathrm{V}(\mathcal{E})|_Z)$$
$$i_{0\circ} : \mathrm{D}_G^{\mathrm{cosg}}(V|_{Z_0}/\mathrm{V}(\mathcal{E})|_Z) \to \mathrm{D}^{\mathrm{cosg}}(V_0/\mathrm{V}(\mathcal{E})) = \mathrm{D}_G^{\mathrm{cosg}}(V_0).$$

Then the following diagram is commutative

$$\begin{tikzcd} \mathrm{D}_G^{\mathrm{cosg}}(Z_0/Z) \arrow[r, "i_{0\circ}p_0{}^\circ"] \arrow[d, "\Upsilon"'] & \mathrm{D}_G^{\mathrm{cosg}}(V_0) \arrow[d, "\Upsilon"] \\ \mathrm{D^{co}Qcoh}_G(Z, W) \arrow[r, "i_*p^*"] & \mathrm{D^{co}Qcoh}_G(\mathrm{V}(\mathcal{E}(\chi)), W + Q_s). \end{tikzcd}$$

Furthermore, we compactify V_0 and $V|_{Z_0}$. The compactifying technique appeared in [Orl2]. Let

$$P := \mathbb{P}(\mathcal{E}(\chi) \oplus \mathcal{O}_X) = \underline{\mathrm{Proj}}(\mathrm{Sym}(\mathcal{E}(\chi) \oplus \mathcal{O}_{\mathrm{X}})^\vee)$$

be the projective space bundle over X with a G-action induced by the equivariant structure of $\mathcal{E}(\chi) \oplus \mathcal{O}_X$. Then we have a natural equivariant open immersion

$$l : \mathrm{V}(\mathcal{E}(\chi)) \to P.$$

Denote by $l|_Z : \mathrm{V}(\mathcal{E}(\chi))|_Z \to P|_Z$ the pull-back of l by the closed immersion $\bar{i} : P|_Z \to P$. Now we have the following cartesian square

$$\begin{tikzcd} \mathrm{V}(\mathcal{E}(\chi))|_Z \arrow[r, "l|_Z"] \arrow[d, "i"'] & P|_Z \arrow[d, "\bar{i}"] \\ \mathrm{V}(\mathcal{E}(\chi)) \arrow[r, "l"] & P. \end{tikzcd}$$

Denote by $\overline{q} : P \to X$ the natural projection, and let $\overline{p} := \overline{q}|_Z : P|_Z \to Z$ be the pull-back of $\overline{q}$ by the closed immersion $j : Z \to X$. Let P_0 be the G-invariant subscheme of P

defined by the G-invariant section $\overline{s \oplus W} \in \Gamma(P, \mathcal{O}(1)(\chi))^G$ which is corresponding to the composition

$$\mathcal{O}_P \xrightarrow{\overline{q}^*(s \oplus W)} \overline{q}^*(\mathcal{E} \oplus \mathcal{O}(\chi^{-1}))^\vee \xrightarrow{\sigma} \mathcal{O}_P(1)(\chi),$$

where σ is the canonical surjection, and let $P|_{Z_0}$ be the zero scheme defined by the invariant section $\overline{i}^*(\overline{s \oplus W}) \in \Gamma(P|_Z, \mathcal{O}(1)(\chi))^G$. Since the pull-back of $\overline{s \oplus W}$ (resp. $\overline{i}^*(\overline{s \oplus W})$) by the open immersion l (resp. $l|_Z$) is equal to $W + Q_s$ (resp. W), we have the following exact cartesian square

$$\begin{tikzcd} V|_{Z_0} \arrow[r, "l|_{Z_0}"] \arrow[d, "i_0"'] & P|_{Z_0} \arrow[d, "\overline{i_0}"] \\ V_0 \arrow[r, "l_0"] & P_0. \end{tikzcd}$$

Denote by $\overline{p_0} : P|_{Z_0} \to Z_0$ be the pull-back of $\overline{p} : P|_Z \to Z$ by the closed immersion $Z_0 \to Z$. Since the morphisms $\overline{i_0} : P|_{Z_0} \to P_0$ and $\overline{p_0} : P|_{Z_0} \to Z_0$ have finite Tor dimensions, the direct images $\mathbf{R}\overline{i_0}_* : \mathrm{D^b}(\mathrm{coh}P|_{Z_0}) \to \mathrm{D^b}(\mathrm{coh}P_0)$ and $\mathbf{R}\overline{p_0}_* : \mathrm{D^b}(\mathrm{coh}P|_{Z_0}) \to \mathrm{D^b}(\mathrm{coh}Z_0)$ induce the following exact functors (cf. [TT, Proposition 2.7]),

$$\overline{i_0}_\circ : \mathrm{D}^{\mathrm{sg}}_G(P|_{Z_0}) \to \mathrm{D}^{\mathrm{sg}}_G(P_0)$$
$$\overline{p_0}_\circ : \mathrm{D}^{\mathrm{sg}}_G(P|_{Z_0}) \to \mathrm{D}^{\mathrm{sg}}_G(Z_0).$$

Now we have the following commutative diagram

$$\begin{tikzcd} \mathrm{D}^{\mathrm{sg}}_G(Z_0) \arrow[r, "\overline{i_0}_\circ \overline{p_0}^\circ"] \arrow[d, "\pi"'] & \mathrm{D}^{\mathrm{sg}}_G(P_0) \arrow[d, "l_0{}^\circ"] \\ \mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \arrow[r, "i_{0\circ} p_0{}^\circ"] & \mathrm{D}^{\mathrm{sg}}_G(V_0), \end{tikzcd}$$

where the vertical arrow on the left side is a Verdier localization by Proposition 6.2.

Remark 7.10. If Z is smooth, the above vertical arrows are equivalences. Indeed, in that case, the singular locus $\mathrm{Sing}(P_0)$ is contained in V_0, whence $l_0{}^\circ$ is an equivalence by a similar argument in the proof of [Orl1, Proposition 1.14]. The equivalence of π follows from Remark 6.3.

Let $\overline{i_0}^! : \mathrm{D^b}(\mathrm{coh}_G P_0) \to \mathrm{D^b}(\mathrm{coh}_G P|_{Z_0})$ be the functor defined by

$$\overline{i_0}^! := \mathbf{L}\overline{i_0}^*(-) \otimes \bigwedge^r (\mathcal{I}/\mathcal{I}^2)^\vee[-r],$$

where $\mathcal{I}$ is the ideal sheaf of $\overline{i_0} : P|_{Z_0} \hookrightarrow P_0$. The functor $\overline{i_0}^!$ is a right adjoint functor of $\overline{i_0}_* : \mathrm{D^b}(\mathrm{coh}_G P|_{Z_0}) \to \mathrm{D^b}(\mathrm{coh}_G P_0)$. Indeed, these functors are adjoint when G is trivial by [Har, III Theorem 6.7, Corollary 7.3], and the isomorphism

$$\mathrm{Hom}(\overline{i_0}_*(A), B) \cong \mathrm{Hom}(A, \overline{i_0}^!(B)),$$

where $A \in \mathrm{D^b}(\mathrm{coh}P|_{Z_0})$ and $B \in \mathrm{D^b}(\mathrm{coh}P_0)$, commutes with G-actions on each vector space of morphisms by the property in [Har, III Proposition 6.9.c]. Hence we see that $\overline{i_0}^!$ is right adjoint to $\overline{i_0}_*$ by [BFK2, Lemma 2.2.8]. Denote by

$$\overline{i_0}^\flat : \mathrm{D}^{\mathrm{sg}}_G(P_0) \to \mathrm{D}^{\mathrm{sg}}_G(P|_{Z_0})$$

the functor induced by $\overline{i_0}^!$. By the above argument, we have the following adjoint pair

$$\overline{i_0}_\circ \dashv \overline{i_0}^\flat.$$

Similarly, we have a right adjoint functor

$$i_0{}^\flat : \mathrm{D}^{\mathrm{cosg}}_G(V_0) \to \mathrm{D}^{\mathrm{cosg}}_G(V|_{Z_0}/\mathrm{V}(\mathcal{E}(\chi))|_Z)$$

of $i_{0\circ} : \mathrm{D}^{\mathrm{cosg}}_G(V|_{Z_0}/\mathrm{V}(\mathcal{E}(\chi))|_Z) \to \mathrm{D}^{\mathrm{cosg}}_G(V_0)$.

Proof of Theorem 7.2: We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \arrow[r, "i_{0\circ}p_0{}^\circ"] \arrow[d, "\Upsilon"'] & \mathrm{D}^{\mathrm{sg}}_G(V_0) \arrow[d, "\Upsilon"] \\
\mathrm{Dcoh}_G(Z,\chi,W) \arrow[r, "i_*p^*"] & \mathrm{Dcoh}_G(\mathrm{V}(\mathcal{E}(\chi)),\chi,W+Q_s),
\end{tikzcd}$$

where the vertical arrows are equivalences by Theorem 6.6. Hence it suffices to show that the functor $i_{0\circ}p_0{}^\circ : \mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{sg}}_G(V_0)$ is an equivalence.

At first, we prove that the functor $i_{0\circ}p_0{}^\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{cosg}}_G(V_0)$ is fully faithful. Let

$$\varepsilon_G : \mathrm{id}_{\mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z)} \to p_{0\circ}i_0{}^\flat \circ i_{0\circ}p_0{}^\circ$$

be the adjunction morphism of the adjoint pair $i_{0\circ}p_0{}^\circ \dashv p_{0\circ}i_0{}^\flat$. It is enough to show that for any object $A \in \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z)$, the cone $C_G(A)$ of the morphism $\varepsilon_G(A) : A \to p_{0\circ}i_0{}^\flat \circ i_{0\circ}p_0{}^\circ(A)$ is the zero object. But the object $\mathrm{Res}_G(C_G(A))$ is isomorphic to the cone $C(A)$ of the adjunction morphism of $\varepsilon(\mathrm{Res}_G(A)) : \mathrm{Res}_G(A) \to p_{0\circ}i_0{}^\flat \circ i_{0\circ}p_0{}^\circ(\mathrm{Res}_G(A))$ of the adjoint pair of functors between $\mathrm{D}^{\mathrm{cosg}}(Z_0/Z)$ and $\mathrm{D}^{\mathrm{cosg}}(V_0)$. Since we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{cosg}}(Z_0/Z) \arrow[r, "i_{0\circ}p_0{}^\circ"] \arrow[d, "\Upsilon"'] & \mathrm{D}^{\mathrm{cosg}}(V_0) \arrow[d, "\Upsilon"] \\
\mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}(Z,W) \arrow[r, "i_*p^*"] & \mathrm{D}^{\mathrm{co}}\mathrm{Qcoh}(\mathrm{V}(\mathcal{E}),W+Q_s),
\end{tikzcd}$$

where the vertical arrows are equivalences by Theorem 6.6, the functor $i_{0\circ}p_0{}^\circ$ is fully faithful by Lemma 7.9. This implies that the object $C(A)$ is the zero object. Hence $C_G(A)$ is also the zero object since the restriction functor Res_G is faithful by Lemma 6.4. Hence $i_{0\circ}p_0{}^\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{cosg}}_G(V_0)$ is fully faithful. This implies that $i_{0\circ}p_0{}^\circ : \mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{sg}}_G(V_0)$ is also fully faithful, since the natural inclusions $\mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z)$ and $\mathrm{D}^{\mathrm{sg}}_G(V_0) \to \mathrm{D}^{\mathrm{cosg}}_G(V_0)$ are fully faithful by Theorem 6.6 and Proposition 4.53 (1).

It only remains to show that the functor $i_{0\circ}p_0{}^\circ : \mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{sg}}_G(V_0)$ is essentially surjective. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{sg}}_G(Z_0) \arrow[r, "\overline{i_0}_\circ\overline{p_0}^\circ"] \arrow[d, "\pi"'] & \mathrm{D}^{\mathrm{sg}}_G(P_0) \arrow[d, "l_0{}^\circ"] \\
\mathrm{D}^{\mathrm{sg}}_G(Z_0/Z) \arrow[r, "i_{0\circ}p_0{}^\circ"] & \mathrm{D}^{\mathrm{sg}}_G(V_0)
\end{tikzcd}$$

By a similar argument as in the proof of [Orl1, Lemma 1.11], we see that every object in $\mathrm{D}^{\mathrm{sg}}_G(V_0)$ is isomorphic to an object $F[k]$ for some G-equivariant coherent sheaf F and for some integer $k \in \mathbb{Z}$. Hence the vertical arrow on the right hand side in the above diagram is essentially surjective, since for every object E in $\mathrm{coh}_G V_0$ there exists an object $\overline{E}$ in $\mathrm{coh}_G P_0$ such that $l_0{}^*(\overline{E}) \cong E$. Thus, we only need to prove that $\overline{i_0}_\circ\overline{p_0}^\circ : \mathrm{D}^{\mathrm{sg}}_G(Z_0) \to \mathrm{D}^{\mathrm{sg}}_G(P_0)$ is essentially surjective. To prove that, it is enough to show that the right adjoint functor $\overline{p_0}_\circ\overline{i_0}^\flat : \mathrm{D}^{\mathrm{sg}}_G(P_0) \to \mathrm{D}^{\mathrm{sg}}_G(Z_0)$ is fully faithful. Since the restriction functor $\mathrm{Res}_G : \mathrm{D}^{\mathrm{sg}}_G(P_0) \to \mathrm{D}^{\mathrm{sg}}(P_0)$ is faithful by Lemma 6.4 and [PV, Proposition 3.8], it follows from [Orl2, Theorem 2.1] that the adjunction $\overline{i_0}_\circ\overline{p_0}^\circ \circ \overline{p_0}_\circ\overline{i_0}^\flat \to \mathrm{id}_{\mathrm{D}^{\mathrm{sg}}_G(P_0)}$ is an isomorphism of functors by a similar argument as in the proof of the fully faithfulness of $i_{0\circ}p_0{}^\circ : \mathrm{D}^{\mathrm{cosg}}_G(Z_0/Z) \to \mathrm{D}^{\mathrm{cosg}}_G(V_0)$ in the previous paragraph. $\square$

7.5. **Cases when $W = 0$.** In the previous section, we prove the main result assuming that $W|_Z : Z \to \mathbb{A}^1$ is flat. In this section, we consider the cases when $W = 0$. In this cases, using results in [Shi], we can show the following:

With notation as above, consider $\mathbb{G}_m \times G$-action on X induced by the projection $\mathbb{G}_m \times G \to G$. Let $\theta : \mathbb{G}_m \times G \to \mathbb{G}_m$ be the character defined as the projection. Since the first factor of $\mathbb{G}_m \times G$ trivially acts on X, the G-equivariant locally free sheaf $\mathcal{E}$ has a natural $\mathbb{G}_m \times G$-equivariant structure.

Proposition 7.11. *We have an equivalence*

$$\mathrm{D^b}(\mathrm{coh}_G Z) \xrightarrow{\sim} \mathrm{Dcoh}_{\mathbb{G}_m \times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s).$$

Proof. By Proposition 4.6, we obtain an equivalence

$$\mathrm{D^b}(\mathrm{coh}_G Z) \xrightarrow{\sim} \mathrm{Dcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0).$$

Hence it is enough to show the functor

$$i_*p^* : \mathrm{Dcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0) \to \mathrm{Dcoh}_{\mathbb{G}_m \times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s)$$

is an equivalence.

By Lemma 7.9, it follows that

$$i_*p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(\mathrm{V}(\mathcal{E}(\chi_1)), \chi_1, Q_s)$$

is fully faithful since the forgetful functor $\mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \to \mathrm{D^{co}Qcoh}(Z, 0)$ is faithful. Furthermore, the above functor i_*p^* is an equivalence since the right orthogonal of the image of the restricted functor $i_*p^* : \mathrm{Dcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(\mathrm{V}(\mathcal{E}(\chi_1)), \chi_1, Q_s)$ vanishes by the argument in [Shi, Theorem 3.4]. In particular, the right adjoint functor

$$p_*i^! : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(\mathrm{V}(\mathcal{E}(\chi_1)), \chi_1, Q_s) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0)$$

of i_*p^* is also fully faithful.

Next we will show that the functor

$$i_*p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s)$$

is an equivalence. Let

$$\varepsilon_{\mathbb{G}_m \times G} : \mathrm{id}_{\mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z,\theta,0)} \to p_*i^! \circ i_*p^*$$

be the adjunction morphism. To show that the functor $i_*p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(\mathrm{V}(\mathcal{E}(\theta)), \theta, Q_s)$ is fully faithful, we will prove that the adjunction morphism $\varepsilon_{\mathbb{G}_m \times G}$ is an isomorphism of functors. For this, it suffices to show that for any object $F \in \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0)$ the cone $C_{\mathbb{G}_m \times G}(F)$ of the morphism $\varepsilon_{\mathbb{G}_m \times G}(F) : F \to p_*i^! \circ i_*p^*(F)$ is the zero object. Recall that the categories $\mathrm{Qcoh}_{\mathbb{G}_m} Z$ and $\mathrm{Qcoh}_{\mathbb{G}_m \times G} Z$ are equivalent to the categories $\mathrm{Qcoh}[Z/\mathbb{G}_m]$ and $\mathrm{Qcoh}_G[Z/\mathbb{G}_m]$ respectively, where $[Z/\mathbb{G}_m]$ denotes the quotient stack, and we can consider the restriction and the induction functors for algebraic stacks as in section 4.4. Let $\pi_G : \mathrm{Qcoh}_{\mathbb{G}_m \times G} Z \to \mathrm{Qcoh}_{\mathbb{G}_m} Z$ be the functor corresponding to the restriction functor $\mathrm{Res}_G : \mathrm{Qcoh}_G[Z/\mathbb{G}_m] \to \mathrm{Qcoh}[Z/\mathbb{G}_m]$ via the equivalences $\mathrm{Qcoh}_{\mathbb{G}_m} Z \cong \mathrm{Qcoh}[Z/\mathbb{G}_m]$ and $\mathrm{Qcoh}_{\mathbb{G}_m \times G} Z \cong \mathrm{Qcoh}_G[Z/\mathbb{G}_m]$. Then π_G naturally induces the following exact functor

$$\pi_G : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0),$$

and π_G has the right adjoint functor $\sigma_G : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m \times G}(Z, \theta, 0)$ induced by the induction functor. Since the argument in the proof of Lemma 4.62 works for algebraic stacks, the adjunction morphism $\mathrm{id} \to \sigma_G \circ \pi_G$ is a split mono. Hence π_G is faithful. The object $\pi_G(C_{\mathbb{G}_m \times G}(F))$ is isomorphic to the cone $C_{\mathbb{G}_m}(F)$ of the adjunction morphism $\varepsilon_{\mathbb{G}_m}(\pi_G(F)) : \pi_G(F) \to p_*i^! \circ i_*p^*(\pi_G(F))$, and $C_{\mathbb{G}_m}(F)$ is the zero object since the functor $i_*p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(Z, \chi_1, 0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m}(\mathrm{V}(\mathcal{E}(\chi_1)), \chi_1, Q_s)$ is fully faithful.

Hence we see that the object $C_{\mathbb{G}_m\times G}(F)$ is also the zero object since π_G is faithful. By an identical argument, we see that the right adjoint functor

$$p_*i^! : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(\mathrm{V}(\mathcal{E}(\theta)),\theta,Q_s) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(Z,\theta,0)$$

is also fully faithful. Hence the functor

$$i_*p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(Z,\theta,0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(\mathrm{V}(\mathcal{E}(\theta)),\theta,Q_s)$$

is an equivalence.

By Proposition 4.53 (1), we see that the equivalence $i_*p^* : \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(Z,\theta,0) \to \mathrm{D^{co}Qcoh}_{\mathbb{G}_m\times G}(\mathrm{V}(\mathcal{E}(\theta)),\theta,Q_s)$ induces an equivalence of the compact objects

$$i_*p^* : \overline{\mathrm{Dcoh}_{\mathbb{G}_m\times G}(Z,\theta,0)} \to \overline{\mathrm{Dcoh}_{\mathbb{G}_m\times G}(\mathrm{V}(\mathcal{E}(\theta)),\theta,Q_s)},$$

where $\overline{(-)}$ denotes the idempotent completion of $(-)$. But $\mathrm{Dcoh}_{\mathbb{G}_m\times G}(Z,\theta,0)$ on the left hand side is already idempotent complete since it is equivalent to $\mathrm{D^b}(\mathrm{coh}_G Z)$. Hence the functor

$$i_*p^* : \mathrm{Dcoh}_{\mathbb{G}_m\times G}(Z,\theta,0) \to \mathrm{Dcoh}_{\mathbb{G}_m\times G}(\mathrm{V}(\mathcal{E}(\theta)),\theta,Q_s)$$

is an equivalence. □

7.6. **Orlov's theorem for gauged LG models.** In this section, we obtain a gauged LG version of the following theorem of Orlov.

Theorem 7.12 ([Orl3] Theorem 40)**.** *Let $X \subset \mathbb{P}_k^{N-1}$ be the hypersurface defined by a section $f \in \Gamma(\mathbb{P}_k^{N-1},\mathcal{O}(d))$. Denote by F the corresponding homogeneous polynomial.*
(1) *If $d < N$, there is a semi-orthogonal decomposition*

$$\mathrm{D^b}(\mathrm{coh}X) = \langle \mathcal{O}_X(d-N+1),...,\mathcal{O}_X, \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N,\chi_d,F)\rangle.$$

(2) *If $d = N$, there is an equivalence*

$$\mathrm{D^b}(\mathrm{coh}X) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N,\chi_d,F).$$

(3) *If $d > N$, there is a semi-orthogonal decomposition*

$$\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}_k^N,\chi_d,F) = \langle k,...,k(N-d+1), \mathrm{D^b}(\mathrm{coh}X)\rangle.$$

We combine the main result with the theory of variations of GIT quotients to obtain a gauged LG version of the above theorem. For the theory of variations of GIT quotients, see [BFK2] or [BDFIK3, Section 2]. This kind of approach to Orlov's theorem appeared in [Shi], [BFK2], and [BDFIK3], and our argument is similar to the one in [BDFIK3, Section 3].

Let S be a smooth quasi-projective variety with $\mathbb{G}_m$-action, and set

$$Q := S \times \mathbb{A}^N \times \mathbb{A}^1.$$

For $i = 1,2$, set $G_i := \mathbb{G}_m$, and let $G := G_1 \times G_2$. For a positive integer $d > 1$, we define a G-action on Q as follows;

$$G \times Q \ni (g_1,g_2)\times(s,v_1,...v_N,u) \mapsto (g_2\cdot s, g_1v_1,...,g_1v_N, g_1^{-d}g_2u) \in Q,$$

where the action $\cdot$ is the original $\mathbb{G}_m$-action on S. Let $\lambda : \mathbb{G}_m \to G$ be the character defined by $\lambda(a) := (a,1)$. Denote by Z_λ the fixed locus of λ-action on Q. Then Z_λ coincides with the zero section $S\times 0\times 0 \subset Q$. Furthermore, set $S_+ := \{q \in Q \mid \lim_{a\to 0}\lambda(a)q \in Z_\lambda\}$ and $S_- := \{q \in Q \mid \lim_{a\to 0}\lambda(a)^{-1}q \in Z_\lambda\}$. Then

$$S_+ = S\times\mathbb{A}^N\times 0 \quad \text{and} \quad S_- = S\times 0\times\mathbb{A}^1.$$

Denote by Q_+ (resp. Q_-) be the complement of S_+ (resp. S_-) in Q. Then the stratifications

$$Q = Q_+ \sqcup S_+ \quad \text{and} \quad Q = Q_- \sqcup S_-$$

are elementary wall crossings in the sense of [BFK2].

Let $W : S \to \mathbb{A}^1$ be a χ_1-semi invariant function which is flat. Let $f \in \Gamma(\mathbb{P}_S^{N-1}, \mathcal{O}(d))^{\mathbb{G}_m}$ be a non-zero $\mathbb{G}_m$-invariant section, and denote by $F : \mathbb{A}_S^N \to \mathbb{A}^1$ the corresponding regular function. Since Q is the trivial line bundle over $\mathbb{A}_S^N$, the function F induces a regular function $\widetilde{F} : Q \to \mathbb{A}^1$. Then the function

$$W + \widetilde{F} : Q \to \mathbb{A}^1$$

is a $\chi_{0,1}$-semi invariant regular function, where W is the pull-back of $W : S \to \mathbb{A}^1$ by the projection $Q \to S$, and $\chi_{0,1} : G \to \mathbb{G}_m$ is the character defined by $\chi_{0,1}(g_1, g_2) := g_2$. By [BFK2, Lemma 3.4.4] and [BFK2, Theorem 3.5.2], we have the following:

Proposition 7.13. *Let $t_\pm$ be the λ-weight of the restriction of relative canonical bundle $\omega_{S_\pm/Q}$ to Z_λ, and set $\mu := -t_+ + t_-$. Let $\chi : G \to \mathbb{G}_m$ be the character defined by $\chi(g_1, g_2) := g_1 g_2$.*
(1) If $\mu < 0$, there exist fully faithful functors

$$\Upsilon_- : \mathrm{Dcoh}_{G/\lambda}(Z_\lambda, \chi_1, W + \widetilde{F}) \to \mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F})$$

$$\Phi_- : \mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}) \to \mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}),$$

and we have the following semi-orthogonal decomposition

$$\mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}) = \langle \Upsilon_-(\mu + 1), ..., \Upsilon_-, \Phi_-(\mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F})) \rangle,$$

where we denote by $\Upsilon_-(n)$ the the essential image of the composition $(-) \otimes \mathcal{O}(\chi^n) \circ \Upsilon_-$.
(2) If $\mu = 0$, we have an equivalence

$$\mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}) \cong \mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}).$$

(3) If $\mu > 0$, there exist fully faithful functors

$$\Upsilon_+ : \mathrm{Dcoh}_{G/\lambda}(Z_\lambda, \chi_1, W + \widetilde{F}) \to \mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F})$$

$$\Phi_+ : \mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}) \to \mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}),$$

and we have the following semi-orthogonal decomposition

$$\mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}) = \langle \Upsilon_+, ..., \Upsilon_+(-\mu + 1), \Phi_+(\mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F})) \rangle,$$

where we denote by $\Upsilon_+(n)$ the the essential image of the composition $(-) \otimes \mathcal{O}(\chi^n) \circ \Upsilon_+$.

Since $Z_\lambda = S \times 0 \times 0$, the function $\widetilde{F}$ vanishes on $Z_\lambda \subset Q$. Hence we have

$$\mathrm{Dcoh}_{G/\lambda}(Z_\lambda, \chi_1, W + \widetilde{F}) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(S, \chi_1, W).$$

Next, we have

$$Q_- = S \times \mathbb{A}^N \setminus 0 \times \mathbb{A}^1.$$

Since $F|_{S \times \mathbb{A}^N \setminus 0} \in \Gamma(S \times \mathbb{A}^N \setminus 0, \mathcal{O}(\chi_{-d,0})^\vee)^G$ and $Q_- = \mathrm{V}(\mathcal{O}(\chi_{-d,1}))$, Theorem 7.2 implies the following equivalence;

$$\mathrm{Dcoh}_G(Q_-, \chi_{0,1}, W + \widetilde{F}) \cong \mathrm{Dcoh}_G(Z, \chi_{0,1}, W),$$

where $Z \subset S \times \mathbb{A}^N \setminus 0$ is the zero scheme of F. Moreover, the quotient stack $[Z/G_1]$ is isomorphic to the hypersurface X in the projective space bundle $\mathbb{P}_S^{N-1}$ over S defined by the invariant section $f \in \Gamma(\mathbb{P}_S^{N-1}, \mathcal{O}(d))^{G_2}$. Hence we have an equivalence

$$\mathrm{Dcoh}_G(Z, \chi_{0,1}, W) \cong \mathrm{Dcoh}_{G_2}(X, \chi_1, W).$$

On the other hand, we have

$$Q_+ = S \times \mathbb{A}^N \times \mathbb{A}^1 \setminus 0.$$

We consider another action of G on Q_+ as follows;

$$G \times Q_+ \ni (g_1, g_2) \times (s, v, u) \mapsto (g_1^d \cdot s, g_1 v, g_1^{-d} g_2 u) \in Q_+.$$

We denote by $\widetilde{Q_+}$ the new G-variety. Then we have a G-equivariant isomorphism

$$\varphi : \widetilde{Q_+} \xrightarrow{\sim} Q_+,$$

given by $\varphi(s, v, u) := (u \cdot s, v, u)$, where $u \in \mathbb{A}^1 \setminus 0$ is considered as a point in $\mathbb{G}_m$. Since G_2 trivially acts on the first two components $S \times \mathbb{A}^N$ of $\widetilde{Q_+}$, we have

$$[\widetilde{Q_+}/G_2] \cong S \times \mathbb{A}^N \times [\mathbb{A}^1 \setminus 0/G_2] \cong \mathbb{A}^N_S.$$

Hence we have the following equivalence

$$\mathrm{Dcoh}_G(Q_+, \chi_{0,1}, W + \widetilde{F}) \cong \mathrm{Dcoh}_{G_1}(\mathbb{A}^N_S, \chi_d, W + F),$$

where, on the right hand side, G_1-action is given by the following

$$G_1 \times S \times \mathbb{A}^N \ni g_1 \times (s, v) \mapsto (g_1^d \cdot s, g_1 v).$$

Finally, note that $\mu = d - N$ and that the twisting by the G-equivariant invertible sheaf $\mathcal{O}(\chi)$ corresponds to the twisting, in $\mathrm{Dcoh}_{G_2}(X, \chi_1, W)$, by the G_2-equivariant invertible sheaf $\mathcal{O}(1)$ on X which is the pull-back of the tautological G_2-equivariant invertible sheaf on $\mathbb{P}^{N-1}_S$. Combining Proposition 7.13 and the above argument, we obtain the following gauged LG version of the Orlov's theorem:

Let S be a smooth quasi-projective variety with a $\mathbb{G}_m$-action, and let $W : S \to \mathbb{A}^1$ be a χ_1-semi invariant regular function which is flat. Consider $\mathbb{G}_m$-actions on $\mathbb{A}^N_S$ and on $\mathbb{P}^{N-1}_S$ given by

$$\mathbb{G}_m \times \mathbb{A}^N_S \ni t \times (s, v_1, ..., v_N) \mapsto (t^d \cdot s, tv_1, ...tv_N) \in \mathbb{A}^N_S$$
$$\mathbb{G}_m \times \mathbb{P}^{N-1}_S \ni t \times (s, v_1 : ... : v_N) \mapsto (t \cdot s, v_1 : ... : v_N) \in \mathbb{P}^{N-1}_S.$$

Theorem 7.14. *For $d > 1$, let $f \in \Gamma(\mathbb{P}^{N-1}_S, \mathcal{O}(d))^{\mathbb{G}_m}$ be a non-zero invariant section, and let $F : \mathbb{A}^N_S \to \mathbb{A}^1$ be the corresponding χ_d-semi invariant regular function. Let $X \subset \mathbb{P}^{N-1}_S$ be the hypersurface defined by f, and assume that the morphism $W|_X$ is flat.*
(1) If $d < N$, there are fully faithful functors

$$\Phi : \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F) \to \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W)$$
$$\Upsilon : \mathrm{Dcoh}_{\mathbb{G}_m}(S, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W),$$

and there is a semi-orthogonal decomposition

$$\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) = \langle \Upsilon_{d-N+1}, ..., \Upsilon_0, \Phi(\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F)) \rangle,$$

where Υ_i denotes the essential image of the composition $(-) \otimes \mathcal{O}(i) \circ \Upsilon$.

(2) If $d = N$, we have an equivalence

$$\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) \cong \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F).$$

(3) If $d > N$, there are fully faithful functors

$$\Psi : \mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F)$$
$$\Upsilon : \mathrm{Dcoh}_{\mathbb{G}_m}(S, \chi_1, W) \to \mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F),$$

and there is a semi-orthogonal decomposition

$$\mathrm{Dcoh}_{\mathbb{G}_m}(\mathbb{A}^N_S, \chi_d, W + F) = \langle \Upsilon_0, ..., \Upsilon_{N-d+1}, \Psi(\mathrm{Dcoh}_{\mathbb{G}_m}(X, \chi_1, W)) \rangle,$$

where Υ_i denotes the essential image of the composition $(-) \otimes \mathcal{O}(\chi_i) \circ \Upsilon$.

Remark 7.15. (1) We can view Orlov's Theorem 7.12 as the case when $S = \operatorname{Spec} k$ and $W = 0$ in the above theorem.
(2) If $N > 1$, the assumption that $W|_X$ is flat is satisfied whenever $W : S \to \mathbb{A}^1$ is flat.
(3) For positive integers $a_1, ..., a_N$, applying the similar argument to the G-action on Q defined by

$$G \times Q \ni (g_1, g_2) \times (s, v_1, ...v_N, u) \mapsto (g_2 \cdot s, g_1^{a_1} v_1, ..., g_1^{a_N} v_N, g_1^{-d} g_2 u) \in Q,$$

we can obtain the similar result for the hypersurface X in weighted projective stack bundle $\mathbb{P}_S^{N-1}(a_1, ..., a_N) := [S \times \mathbb{A}^N \setminus 0/G_1]$ over S defined by the section corresponding to a G_1-invariant section $F \in \Gamma(\mathbb{A}_S^N, \mathcal{O}(\chi_d))^{G_1}$.
(4) Of course, Orlov's theorem in [Orl3] is much more general. It covers noncommutative situations unlike our setting.

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Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1 Minamiohsawa, Hachioji-shi, Tokyo, 192-0397, Japan
E-mail address: yuki-hirano@ed.tmu.ac.jp