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GEOMETRIC PROPERTIES OF ORBITS OF COMMUTATIVE HERMANN ACTIONS

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Abstract. In this thesis, we study geometric properties of orbits of commutative Hermann actions. A Hermann action is a generalization of isotropy actions of compact symmetric spaces.

Contents

1. Introduction

In Riemannian geometry, often submanifolds appear with special properties. For example, minimal submanifolds have been studied by many mathematicians. In spacial cases of minimal submanifolds, there are austere submanifolds and totally geodesic submanifolds. Austere submanifolds are associated with special Lagrangian submanifolds in the cotangent bundle of the hypersphere. In addition, harmonic maps and biharmonic maps are interesting submanifolds. Geometric properties listed above are described by the local structure of submanifolds. A reflectivity and a weakly reflectivity are geometric properties which require a global structure of submanifolds. Reflective submanifolds and weakly reflective submanifolds are totally geodesic and austere, respectively.

To understand these geometric properties, it is an important problem which constructs an example. One method for constructing examples is a method using

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Lie group actions. W. Hsiang and H. B. Lawson constructed many examples of minimal hypersurfaces in the hypersphere using cohomogeneity one action on the hypersphere. This method can be applied to other geometric properties.

The author have studied Lie group actions on Riemannian symmetric spaces, such as isotropy representations and isotropy actions of compact symmetric spaces. The second fundamental form of orbits of such actions are expressed by root system. O. Ikawa ([I]) introduced the notion of symmetric triad as a generalization of the notion of irreducible root system to study orbits of commutative Hermann actions. O. Ikawa expressed orbit spaces of Hermann actions by using symmetric triads, and gave a characterization of the minimal, austere and totally geodesic orbits of Hermann actions in terms of symmetric triads.

In this thesis, we consider commutative Hermann actions and associated actions on compact Lie groups, and express the minimal, austere, weakly reflective, biharmonic properties of orbits of these actions in terms of symmetric triads.

In Section 2, we review the notion of root systems and symmetric triads. In particular, a minimal point, an austere point and a totally geodesic point are discussed.

In Section 3, we recall the definition of weakly reflective submanifolds, and their fundamental properties, and we gave sufficient conditions for orbits of these actions to be weakly reflective. Using the sufficient conditions, we obtain many examples of weakly reflective submanifolds in compact symmetric spaces.

In Section 4, we give a characterization of biharmonic orbits of commutative Hermann actions and associated actions on Lie groups in terms of symmetric triads. Using the characterization, we give examples of biharmonic submanifolds in compact symmetric spaces which is not necessarily hypersurfaces. The contents of this section is based on joint work with T. Sakai and H. Urakawa.

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2. Hermann actions and symmetric triads

2.1. **Symmetric triads.** O. Ikawa ([I]) introduced the notion of symmetric triad as a generalization of the notion of irreducible root system to study orbits of Hermann actions. Ikawa expressed orbit spaces of Hermann actions by using symmetric triads, and gave a characterization of the minimal, austere and totally geodesic orbits of Hermann actions in terms of symmetric triads. We recall the notions of root system and symmetric triad. See [I] for details.

Let $(\mathfrak{a}, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space over $\mathbb{R}$. For each $\alpha \in \mathfrak{a}$, we define an orthogonal transformation $s_\alpha : \mathfrak{a} \to \mathfrak{a}$ by

$$s_\alpha(H) = H - \frac{2\langle\alpha, H\rangle}{\langle\alpha,\alpha\rangle}\alpha \quad (H \in \mathfrak{a}),$$

namely s_α is the reflection with respect to the hyperplane $\{H \in \mathfrak{a} \mid \langle\alpha, H\rangle = 0\}$.

Definition 2.1. A finite subset Σ of $\mathfrak{a} \setminus \{0\}$ is a *root system* of $\mathfrak{a}$, if it satisfies the following three conditions:

(1) $\mathrm{Span}(\Sigma) = \mathfrak{a}$.
(2) If $\alpha, \beta \in \Sigma$, then $s_\alpha(\beta) \in \Sigma$.
(3) $2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z} \quad (\alpha, \beta \in \Sigma)$.

A root system of $\mathfrak{a}$ is said to be *irreducible* if it cannot be decomposed into two disjoint nonempty orthogonal subsets.

Let Σ be a root system of $\mathfrak{a}$. The Weyl group $W(\Sigma)$ of Σ is the finite subgroup of the orthogonal group $\mathrm{O}(\mathfrak{a})$ of $\mathfrak{a}$ generated by $\{s_\alpha \mid \alpha \in \Sigma\}$.

Definition 2.2 ([I] Definition 2.2)**.** A triple $(\tilde{\Sigma}, \Sigma, W)$ of finite subsets of $\mathfrak{a} \setminus \{0\}$ is a *symmetric triad* of $\mathfrak{a}$, if it satisfies the following six conditions:

(1) $\tilde{\Sigma}$ is an irreducible root system of $\mathfrak{a}$.
(2) Σ is a root system of $\mathfrak{a}$.
(3) $(-1)W = W$, $\tilde{\Sigma} = \Sigma \cup W$.
(4) $\Sigma \cap W$ is a nonempty subset. If we put $l := \max\{\|\alpha\| \mid \alpha \in \Sigma \cap W\}$, then $\Sigma \cap W = \{\alpha \in \tilde{\Sigma} \mid \|\alpha\| \leq l\}$.
(5) For $\alpha \in W$ and $\lambda \in \Sigma \setminus W$,
$$2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \text{ is odd if and only if } s_\alpha(\lambda) \in W \setminus \Sigma.$$
(6) For $\alpha \in W$ and $\lambda \in W \setminus \Sigma$,
$$2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \text{ is odd if and only if } \ s_\alpha(\lambda) \in \Sigma \setminus W.$$

Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$. We set
$$\begin{aligned} \Gamma =& \{H \in \mathfrak{a} \mid \langle \lambda, H \rangle \in (\pi/2)\mathbb{Z} \quad (\lambda \in \tilde{\Sigma})\}, \\ \Gamma_{\Sigma \cap W} =& \{H \in \mathfrak{a} \mid \langle \lambda, H \rangle \in (\pi/2)\mathbb{Z} \quad (\lambda \in \Sigma \cap W)\}. \end{aligned}$$
A point in Γ is called a *totally geodesic point*. It is known that $\Gamma = \Gamma_{\Sigma \cap W}$. We define an open subset $\mathfrak{a}_r$ of $\mathfrak{a}$ by
$$\mathfrak{a}_r = \bigcap_{\lambda \in \Sigma, \alpha \in W} \left\{ H \in \mathfrak{a} \;\middle|\; \langle \lambda, H \rangle \notin \pi\mathbb{Z},\ \langle \alpha, H \rangle \notin \frac{\pi}{2} + \pi\mathbb{Z} \right\}.$$
A point in $\mathfrak{a}_r$ is called a *regular point*, and a point in the complement of $\mathfrak{a}_r$ in $\mathfrak{a}$ is called a *singular point*. A connected component of $\mathfrak{a}_r$ is called a *cell*. The *affine Weyl group* $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ of $(\tilde{\Sigma}, \Sigma, W)$ is a subgroup of the affine group of $\mathfrak{a}$, which defined by the semidirect product $\mathrm{O}(\mathfrak{a}) \ltimes \mathfrak{a}$, generated by
$$\left\{ \left(s_\lambda, \frac{2n\pi}{\langle \lambda, \lambda \rangle}\lambda \right) \;\middle|\; \lambda \in \Sigma, n \in \mathbb{Z} \right\} \cup \left\{ \left(s_\alpha, \frac{(2n+1)\pi}{\langle \alpha, \alpha \rangle}\alpha \right) \;\middle|\; \alpha \in W, n \in \mathbb{Z} \right\}.$$
The action of $(s_\lambda, (2n\pi/\langle \lambda, \lambda \rangle)\lambda)$ on $\mathfrak{a}$ is the reflection with respect to the hyperplane $\{H \in \mathfrak{a} \mid \langle \lambda, H \rangle = n\pi\}$, and the action of $(s_\alpha, ((2n+1)\pi/\langle \alpha, \alpha \rangle)\alpha)$ on $\mathfrak{a}$ is the reflection with respect to the hyperplane $\{H \in \mathfrak{a} \mid \langle \alpha, H \rangle = (n+1/2)\pi\}$. The affine Weyl group $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ acts transitively on the set of all cells. More precisely, for each cell P, it holds that
$$\mathfrak{a} = \bigcup_{s \in \tilde{W}(\tilde{\Sigma}, \Sigma, W)} s\overline{P}.$$

We take a fundamental system $\tilde{\Pi}$ of $\tilde{\Sigma}$. We denote by $\tilde{\Sigma}^+$ the set of positive roots in $\tilde{\Sigma}$. Set $\Sigma^+ = \tilde{\Sigma}^+ \cap \Sigma$ and $W^+ = \tilde{\Sigma}^+ \cap W$. Denote by Π the set of simple roots of Σ. We set

$$W_0 = \{\alpha \in W^+ \mid \alpha + \lambda \notin W \ (\lambda \in \Pi)\}.$$

From the classification of symmetric triads, we have that W_0 consists of the only one element, denoted by $\tilde{\alpha}$. We define an open subset P_0 of $\mathfrak{a}$ by

$$P_0 = \left\{ H \in \mathfrak{a} \;\middle|\; \langle \tilde{\alpha}, H \rangle < \frac{\pi}{2},\ \langle \lambda, H \rangle > 0 \ (\lambda \in \Pi) \right\}. \tag{2.1}$$

Then P_0 is a cell. For an nonempty subset $\Delta \subset \Pi \cup \{\tilde{\alpha}\}$, set

$$P_0^{\Delta} = \left\{ \begin{array}{c|c} H \in \overline{P}_0 & \begin{array}{c} \langle \lambda, H \rangle > 0 \ (\lambda \in \Delta \cap \Pi) \\ \langle \mu, H \rangle = 0 \ (\mu \in \Pi \setminus \Delta) \\ \langle \tilde{\alpha}, H \rangle \begin{cases} < (\pi/2) \ (\text{if } \tilde{\alpha} \in \Delta) \\ = (\pi/2) \ (\text{if } \tilde{\alpha} \notin \Delta) \end{cases} \end{array} \end{array} \right\},$$

then

$$\overline{P}_0 = \bigcup_{\Delta \subset \Pi \cup \{\tilde{\alpha}\}} P_0^{\Delta} \ (\text{disjoint union}).$$

Definition 2.3 ([I] Definition 2.13)**.** Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$. Consider two mappings m and n from $\tilde{\Sigma}$ to $\mathbb{R}_{\geq 0} := \{a \in \mathbb{R} \mid a \geq 0\}$ which satisfy the following four conditions:

(1) For any $\lambda \in \tilde{\Sigma}$,
  (1-1) $m(\lambda) = m(-\lambda)$, $n(\lambda) = n(-\lambda)$,
  (1-2) $m(\lambda) > 0$ if and only if $\lambda \in \Sigma$,
  (1-3) $n(\lambda) > 0$ if and only if $\lambda \in W$.
(2) When $\lambda \in \Sigma,\ \alpha \in W,\ s \in W(\Sigma)$, then $m(\lambda) = m(s(\lambda)),\ n(\alpha) = n(s(\alpha))$.
(3) When $\lambda \in \tilde{\Sigma},\ \sigma \in W(\tilde{\Sigma})$, then $m(\lambda) + n(\lambda) = m(\sigma(\lambda)) + n(\sigma(\lambda))$.
(4) Let $\lambda \in \Sigma \cap W,\ \alpha \in W$. If $2\langle \alpha, \lambda \rangle / \langle \alpha, \alpha \rangle$ is even, then $m(\lambda) = m(s_\alpha(\lambda))$. If $2\langle \alpha, \lambda \rangle / \langle \alpha, \alpha \rangle$ is odd, then $m(\lambda) = n(s_\alpha(\lambda))$.

We call $m(\lambda)$ and $n(\alpha)$ the *multiplicities* of λ and α, respectively.

Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$ with multiplicities m and n. For $H \in \mathfrak{a}$, we set

$$m_H = -\sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H \rangle \notin \pi\mathbb{Z}}} m(\lambda) \cot\langle \lambda, H \rangle \lambda + \sum_{\substack{\alpha \in W^+ \\ \langle \alpha, H \rangle \notin (\pi/2) + \pi\mathbb{Z}}} n(\alpha) \tan\langle \alpha, H \rangle \alpha.$$

The vector m_H is called the mean curvature vector at H. A vector $H \in \mathfrak{a}$ is a *minimal point* if $m_H = 0$.

Proposition 2.4 ([I] Theorem 2.14)**.** *Let $(\tilde{\Sigma}, \Sigma, W)$ be a symmetric triad of $\mathfrak{a}$ with multiplicities. For $H \in \mathfrak{a}$ and $\sigma = (s, X) \in \tilde{W}(\tilde{\Sigma}, \Sigma, W)$, set $H' = \sigma H \in \mathfrak{a}$, then*

$$m_{H'} = s(m_H).$$

Theorem 2.5 ([I] Theorem 2.24)**.** *For any nonempty subset $\Delta \subset \Pi \cup \{\tilde{\alpha}\}$, there exists a unique minimal point $H \in P_0^{\Delta}$.*

A vector $H \in \mathfrak{a}$ is an *austere point* if the subset of $\mathfrak{a}$ with multiplicities defined by

$$\begin{aligned} & \{-\cot\langle\lambda, H\rangle\lambda \text{ (multiplicity} = m(\lambda)) \mid \lambda \in \Sigma^+, \langle\lambda, H\rangle \notin \pi\mathbb{Z}\} \\ \cup \quad & \{\tan\langle\alpha, H\rangle\alpha \text{ (multiplicity} = n(\alpha)) \mid \alpha \in W^+, \langle\alpha, H\rangle \notin (\pi/2) + \pi\mathbb{Z}\} \end{aligned}$$

is invariant with multiplicities under the multiplication by -1. An austere point is a minimal point.

Proposition 2.6 ([I] Theorem 2.18)**.** *A point $H \in \mathfrak{a}$ is austere if and only if the following three conditions holds:*

(1) $\langle\lambda, H\rangle \in (\pi/2)\mathbb{Z}$ *for any* $\lambda \in (\Sigma \setminus W) \cup (W \setminus \Sigma)$.
(2) $2H \in \Gamma_{\Sigma\cap W}$.
(3) $m(\lambda) = n(\lambda)$ *for any* $\lambda \in \Sigma \cap W$ *with* $\langle\lambda, H\rangle \in (\pi/4) + (\pi/2)\mathbb{Z}$.

Ikawa gave the classification of symmetric triad and determined austere points for symmetric triads with multiplicities.

2.2. **Minimal orbits and austere orbits.** In this section, we consider Hermann actions and associated actions on Lie groups which are hyperpolar actions on compact symmetric spaces. A. Kollross ([Kol]) classified the hyperpolar actions on compact irreducible symmetric spaces. By the classification, we can see that a hyperpolar action on a compact symmetric space whose cohomogeneity is two or greater, is orbit-equivalent to some Hermann action.

Let G be a compact, connected, semisimple Lie group, and K_1, K_2 be closed subgroups of G. For each $i = 1, 2$, assume that there exists an involutive automorphism θ_i of G which satisfies $(G_{\theta_i})_0 \subset K_i \subset G_{\theta_i}$, where G_{θ_i} is the set of fixed points of θ_i and $(G_i)_0$ is the identity component of G_{θ_i}. Then the triple (G, K_1, K_2) is called a *compact symmetric triad*. The pair (G, K_i) is a compact symmetric pair for $i = 1, 2$. We denote the Lie algebras of G, K_1 and K_2 by $\mathfrak{g}, \mathfrak{k}_1$ and $\mathfrak{k}_2$, respectively. The involutive automorphism of $\mathfrak{g}$ induced from θ_i will be also denoted by θ_i. Take an $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Then the inner product $\langle\cdot,\cdot\rangle$ induces a bi-invariant Riemannian metric on G and G-invariant Riemannian metrics on the coset manifolds $M_1 := G/K_1$ and $M_2 := K_2\backslash G$. We denote these Riemannian metrics on G, M_1 and M_2 by the same symbol $\langle\cdot,\cdot\rangle$. These Riemannian manifolds G, M_1 and M_2 are Riemannian symmetric spaces with respect to $\langle\cdot,\cdot\rangle$. We denote by π_i the natural projection from G to M_i ($i = 1, 2$), and consider the following three Lie group actions:

- $(K_2 \times K_1) \curvearrowright G \ : (k_2, k_1)g = k_2 g k_1^{-1} \ \ ((k_2, k_1) \in K_2 \times K_1)$,
- $K_2 \curvearrowright M_1 \ : k_2\pi_1(g) = \pi_1(k_2 g) \ \ (k_2 \in K_2)$,
- $K_1 \curvearrowright M_2 \ : k_1\pi_2(g) = \pi_2(g k_1^{-1}) \ \ (k_1 \in K_1)$,

for $g \in G$. The three actions have the same orbit space, and in fact, the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\pi_2} & M_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_1} \\ M_1 & \xrightarrow[\tilde{\pi}_2]{} & K_2\backslash G/K_1, \end{array}$$

where $\tilde{\pi}_i$ is the natural projection from M_i to the orbit space $K_2\backslash G/K_1$. Ikawa computed the second fundamental form of orbits of Hermann actions in the case

$\theta_1\theta_2 = \theta_2\theta_1$. We can apply Ikawa's method to the geometry of orbits of the $(K_2 \times K_1)$-action. For $g \in G$, we denote the left (resp. right) transformation of G by L_g (resp. R_g). The isometry on M_1 (resp. M_2) induced by L_g (resp. R_g) will be also denoted by the same symbol L_g (resp. R_g).

For $i = 1, 2$, we set

$$\mathfrak{m}_i = \{X \in \mathfrak{g} \mid \theta_i(X) = -X\}.$$

Then we have an orthogonal direct sum decomposition of $\mathfrak{g}$ that is the canonical decomposition:

$$\mathfrak{g} = \mathfrak{k}_i \oplus \mathfrak{m}_i.$$

The tangent space $T_{\pi_i(e)}M_i$ of M_i at the origin $\pi_i(e)$ is identified with $\mathfrak{m}_i$ in a natural way. We define a closed subgroup G_{12} of G by

$$G_{12} = \{g \in G \mid \theta_1(g) = \theta_2(g)\}.$$

Hence $((G_{12})_0, K_{12})$ is a compact symmetric pair, where K_{12} is a closed subgroup of $(G_{12})_0$ defined by

$$K_{12} = \{k \in (G_{12})_0 \mid \theta_1(k) = k\}.$$

The canonical decomposition of $((G_{12})_0, K_{12})$ is given by

$$\mathfrak{g}_{12} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2).$$

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}_1 \cap \mathfrak{m}_2$. Then $\exp(\mathfrak{a})$ is a torus subgroup in $(G_{12})_0$. Then $\exp(\mathfrak{a})$, $\pi_1(\exp(\mathfrak{a}))$ and $\pi_2(\exp(\mathfrak{a}))$ are sections of the $(K_2 \times K_1)$-action, the K_2-action and the K_1-action, respectively. To investigate the orbit spaces of the three actions, we consider a equivalent relation $\sim$ on $\mathfrak{a}$ defined as follows: For $H_1, H_2 \in \mathfrak{a}$, $H_1 \sim H_2$ if $K_2 \exp(H_1)K_1 = K_2 \exp(H_2)K_1$. Clearly, we have $H_1 \sim H_2$ if and only if $K_2\pi_1(\exp(H_1)) = K_2\pi_1(\exp(H_2))$, and similarly, $H_1 \sim H_2$ if and only if $K_1\pi_2(\exp(H_1)) = K_1\pi_2(\exp(H_2))$. Then we have $\mathfrak{a}/\!\sim = K_2\backslash G/K_1$. For each subgroup L of G, we define

$$N_L(\mathfrak{a}) = \{k \in L \mid \mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}\},$$
$$Z_L(\mathfrak{a}) = \{k \in L \mid \mathrm{Ad}(k)H = H \ (H \in \mathfrak{a})\}.$$

Then $Z_L(\mathfrak{a})$ is a normal subgroup of $N_L(\mathfrak{a})$. We define a group $\tilde{J}$ by

$$\tilde{J} = \{([s], Y) \in N_{K_2}(\mathfrak{a})/Z_{K_1\cap K_2}(\mathfrak{a}) \ltimes \mathfrak{a} \mid \exp(-Y)s \in K_1\}.$$

The group $\tilde{J}$ naturally acts on $\mathfrak{a}$ by the following:

$$([s], Y)H = \mathrm{Ad}(s)H + Y \quad (([s], Y) \in \tilde{J},\ H \in \mathfrak{a}).$$

Matsuki ([M]) proved that

$$K_2\backslash G/K_1 \cong \mathfrak{a}/\tilde{J}.$$

Hereafter, we suppose $\theta_1\theta_2 = \theta_2\theta_1$. Then we have an orthogonal direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{k}_2).$$

We define subspaces of $\mathfrak{g}$ as follows:

$$\mathfrak{k}_0 = \{X \in \mathfrak{k}_1 \cap \mathfrak{k}_2 \mid [\mathfrak{a}, X] = \{0\}\},$$
$$V(\mathfrak{k}_1 \cap \mathfrak{m}_2) = \{X \in \mathfrak{k}_1 \cap \mathfrak{m}_2 \mid [\mathfrak{a}, X] = \{0\}\},$$
$$V(\mathfrak{m}_1 \cap \mathfrak{k}_2) = \{X \in \mathfrak{m}_1 \cap \mathfrak{k}_2 \mid [\mathfrak{a}, X] = \{0\}\}.$$

For $\lambda \in \mathfrak{a}$,

$$\begin{aligned}
\mathfrak{k}_\lambda &= \{X \in \mathfrak{k}_1 \cap \mathfrak{k}_2 \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X \ (H \in \mathfrak{a})\},\\
\mathfrak{m}_\lambda &= \{X \in \mathfrak{m}_1 \cap \mathfrak{m}_2 \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X \ (H \in \mathfrak{a})\},\\
V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) &= \{X \in \mathfrak{k}_1 \cap \mathfrak{m}_2 \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X \ (H \in \mathfrak{a})\},\\
V_\lambda^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2) &= \{X \in \mathfrak{m}_1 \cap \mathfrak{k}_2 \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X \ (H \in \mathfrak{a})\}.
\end{aligned}$$

We set

$$\begin{aligned}
\Sigma &= \{\lambda \in \mathfrak{a} \setminus \{0\} \mid \mathfrak{k}_\lambda \neq \{0\}\},\\
W &= \{\alpha \in \mathfrak{a} \setminus \{0\} \mid V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) \neq \{0\}\},\\
\tilde{\Sigma} &= \Sigma \cup W.
\end{aligned}$$

It is known that $\dim \mathfrak{k}_\lambda = \dim \mathfrak{m}_\lambda$ and $\dim V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) = \dim V_\lambda^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)$ for each $\lambda \in \tilde{\Sigma}$. Thus we set $m(\lambda) := \dim \mathfrak{k}_\lambda,\ n(\lambda) := \dim V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$. Notice that Σ is the root system of the pair $((G_{12})_0, K_{12})$, and $\tilde{\Sigma}$ is a root system of $\mathfrak{a}$ (see [I]). We take a basis of $\mathfrak{a}$ and the lexicographic ordering $>$ on $\mathfrak{a}$ with respect to the basis. We set

$$\tilde{\Sigma}^+ = \{\lambda \in \tilde{\Sigma} \mid \lambda > 0\},\ \Sigma^+ = \Sigma \cap \tilde{\Sigma}^+,\ W^+ = W \cap \tilde{\Sigma}^+.$$

Then we have an orthogonal direct sum decomposition of $\mathfrak{g}$:

$$\begin{aligned}
\mathfrak{g} = \mathfrak{k}_0 \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda \oplus \mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{m}_\lambda \oplus V(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus \sum_{\alpha \in W^+} V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)\\
\oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus \sum_{\alpha \in W^+} V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2).
\end{aligned}$$

Furthermore, we have the following lemma.

Lemma 2.7 ([I] Lemmas 4.3 and 4.16)**.** (1) *For each $\lambda \in \Sigma^+$, there exist orthonormal bases $\{S_{\lambda,i}\}_{i=1}^{m(\lambda)}$ and $\{T_{\lambda,i}\}_{i=1}^{m(\lambda)}$ of $\mathfrak{k}_\lambda$ and $\mathfrak{m}_\lambda$ respectively such that for any $H \in \mathfrak{a}$,*

$$[H, S_{\lambda,i}] = \langle \lambda, H\rangle T_{\lambda,i}, \quad [H, T_{\lambda,i}] = -\langle \lambda, H\rangle S_{\lambda,i}, \quad [S_{\lambda,i}, T_{\lambda,i}] = \lambda,$$

$$\begin{aligned}
\mathrm{Ad}(\exp H) S_{\lambda,i} &= \cos\langle \lambda, H\rangle S_{\lambda,i} + \sin\langle \lambda, H\rangle T_{\lambda,i},\\
\mathrm{Ad}(\exp H) T_{\lambda,i} &= -\sin\langle \lambda, H\rangle S_{\lambda,i} + \cos\langle \lambda, H\rangle T_{\lambda,i}.
\end{aligned}$$

(2) *For each $\alpha \in W^+$, there exist orthonormal bases $\{X_{\alpha,j}\}_{j=1}^{n(\alpha)}$ and $\{Y_{\alpha,j}\}_{j=1}^{n(\alpha)}$ of $V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$ and $V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)$ respectively such that for any $H \in \mathfrak{a}$*

$$[H, X_{\alpha,j}] = \langle \alpha, H\rangle Y_{\alpha,j}, \quad [H, Y_{\alpha,j}] = -\langle \alpha, H\rangle X_{\alpha,j}, \quad [X_{\alpha,j}, Y_{\alpha,j}] = \alpha,$$

$$\begin{aligned}
\mathrm{Ad}(\exp H) X_{\alpha,j} &= \cos\langle \alpha, H\rangle X_{\alpha,j} + \sin\langle \alpha, H\rangle Y_{\alpha,j},\\
\mathrm{Ad}(\exp H) Y_{\alpha,j} &= -\sin\langle \alpha, H\rangle X_{\alpha,j} + \cos\langle \alpha, H\rangle Y_{\alpha,j}.
\end{aligned}$$

Using Lemma 2.7, Ikawa proved the following theorems.

Theorem 2.8 ([I] Lemma 4.22)**.** *Let $x = \exp H$ for $H \in \mathfrak{a}$. Then we have:*

(1) $dL_x^{-1} B_H(dL_x(T_{\lambda,i}), dL_x(T_{\mu,j})) = \cot(\langle \mu, H\rangle)[T_{\lambda,i}, S_{\mu,j}]^\perp$,
(2) $dL_x^{-1} B_H(dL_x(Y_{\alpha,i}), dL_x(Y_{\beta,j})) = -\tan(\langle \beta, H\rangle)[Y_{\alpha,i}, X_{\beta,j}]^\perp$,
(3) $B_H(dL_x(Y_1), dL_x(Y_2)) = 0$,
(4) $B_H(dL_x(T_{\lambda,i}), dL_x(Y_2)) = 0$,

(5) $B_H(dL_x(Y_{\alpha,i}), dL_x(Y_2)) = 0,$
(6) $dL_x^{-1} B_H(dL_x(T_{\lambda,i}), dL_x(Y_{\beta,j})) = -\tan(\langle \beta, H\rangle)[T_{\lambda,i}, X_{\beta,j}]^{\perp},$

for

$$\lambda, \mu \in \Sigma^+ \text{ with } \langle \lambda, H\rangle, \langle \mu, H\rangle \notin \pi\mathbb{Z},\ 1 \le i \le m(\lambda),\ 1 \le j \le m(\mu),$$
$$\alpha, \beta \in W^+ \text{ with } \langle \alpha, H\rangle, \langle \beta, H\rangle \notin \frac{\pi}{2} + \pi\mathbb{Z},\ 1 \le i \le n(\alpha),\ 1 \le j \le n(\beta),$$
$$Y_1, Y_2 \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2).$$

Here $X^{\perp}$ *is the normal component, i.e.* $(\mathrm{Ad}(x^{-1})\mathfrak{m}_2) \cap \mathfrak{m}_1$*-component, of a tangent vector* $X \in \mathfrak{m}_1$.

Theorem 2.9 ([I] Corollaries 4.23, 4.29, 4.24, and [GT] Theorem 5.3)**.** *Let* $g = \exp(H)$ $(H \in \mathfrak{a})$*. Denote the mean curvature vector of* $K_2\pi_1(g) \subset M_1$ *at* $\pi_1(g)$ *by* m_H^1*. Then we have:*

(1)
$$dL_g^{-1} m_H^1 = -\sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H\rangle \notin \pi\mathbb{Z}}} m(\lambda) \cot\langle \lambda, H\rangle \lambda + \sum_{\substack{\alpha \in W^+ \\ \langle \alpha, H\rangle \notin (\pi/2)+\pi\mathbb{Z}}} n(\alpha) \tan\langle \alpha, H\rangle \alpha.$$

(2) *The orbit* $K_2\pi_1(g) \subset M_1$ *is austere if and only if the finite subset of* $\mathfrak{a}$ *defined by*
$$\{-\lambda \cot\langle \lambda, H\rangle\ (\textit{multiplicity} = m(\lambda)) \mid \lambda \in \Sigma^+, \langle \lambda, H\rangle \notin \pi\mathbb{Z}\}$$
$$\cup\{\alpha \tan\langle \alpha, H\rangle\ (\textit{multiplicity} = n(\alpha)) \mid \alpha \in W^+, \langle \alpha, H\rangle \notin (\pi/2) + \pi\mathbb{Z}\}$$
is invariant under the multiplication by -1 *with multiplicities.*

(3) *The orbit* $K_2\pi_1(g) \subset M_1$ *is totally geodesic if and only if* $\langle \lambda, H\rangle \in (\pi/2)\mathbb{Z}$ *for each* $\lambda \in \tilde{\Sigma}^+$.

We can apply Theorem 2.9 for orbits $K_1\pi_2(g) \subset M_2$. Thus, we have the following corollary.

Corollary 2.10 ([I] Corollary 4.30)**.** *The orbit* $K_2\pi_1(g)$ *is minimal (resp. austere, totally geodesic) if and only if* $K_1\pi_2(g)$ *is minimal (resp. austere, totally geodesic).*

Now we consider the second fundamental form of orbits of the $(K_2 \times K_1)$-action on G. For $H \in \mathfrak{a}$, we set
$$\Sigma_H = \{\lambda \in \Sigma \mid \langle \lambda, H\rangle \in \pi\mathbb{Z}\},\ W_H = \{\alpha \in W \mid \langle \alpha, H\rangle \in (\pi/2) + \pi\mathbb{Z}\},$$
$$\tilde{\Sigma}_H = \Sigma_H \cup W_H,\ \Sigma_H^+ = \Sigma^+ \cap \Sigma_H,\ W_H^+ = W^+ \cap W_H,\ \tilde{\Sigma}_H^+ = \Sigma_H^+ \cup W_H^+.$$

Let $g = \exp(H)$ $(H \in \mathfrak{a})$. Then we have

$$T_g(K_2 g K_1) = \left\{ \frac{d}{dt} \exp(tX_2) g \exp(-tX_1)\Big|_{t=0} \;\middle|\; X_1 \in \mathfrak{k}_1,\ X_2 \in \mathfrak{k}_2 \right\}$$
$$= dL_g((\mathrm{Ad}(g)^{-1}\mathfrak{k}_2) + \mathfrak{k}_1) \tag{2.2}$$
$$= dL_g\Bigg(\mathfrak{k}_0 \oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus \sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} \mathfrak{m}_\lambda \oplus \sum_{\alpha \in W^+ \setminus W_H} V_\alpha^{\perp}(\mathfrak{m}_1 \cap \mathfrak{k}_2)$$
$$\oplus V(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda \oplus \sum_{\alpha \in W^+} V_\alpha^{\perp}(\mathfrak{k}_1 \cap \mathfrak{m}_2)\Bigg), \tag{2.3}$$

$$T_g^{\perp}(K_2gK_1) = dL_g((\mathrm{Ad}(g)^{-1}\mathfrak{m}_2)\cap\mathfrak{m}_1) \tag{2.4}$$

$$= dL_g\left(\mathfrak{a}\oplus\sum_{\lambda\in\Sigma_H^+}\mathfrak{m}_\lambda\oplus\sum_{\alpha\in W_H^+}V_\alpha^{\perp}(\mathfrak{m}_1\cap\mathfrak{k}_2)\right). \tag{2.5}$$

For $X=(X_2,X_1)\in\mathfrak{g}\times\mathfrak{g}$, we define a Killing vector field X^* on G by

$$(X^*)_p = \frac{d}{dt}\exp(tX_2)p\exp(-tX_1)\bigg|_{t=0} \quad (p\in G).$$

Then

$$(X^*)_p = (dL_p)(\mathrm{Ad}(p)^{-1}X_2 - X_1)$$

holds. If $X_2=0$, then X^* is a left invariant vector field. Denote by ∇ the Levi-Civita connection on G. By Koszul's formula, we have the following.

Lemma 2.11 ([O] Lemma 3)**.** *Let $g\in G$, $X=(X_2,X_1)$, $Y=(Y_2,Y_1)\in\mathfrak{g}\times\mathfrak{g}$. Then we have*

$$(\nabla_{X^*}Y^*)_g = -\frac{1}{2}dL_g[\mathrm{Ad}(g)^{-1}X_2 - X_1, \mathrm{Ad}(g)^{-1}Y_2 + Y_1].$$

Proof. By Koszul's formula, we have

$$\begin{aligned}2\langle\nabla_{X^*}Y^*, Z\rangle =& X^*\langle Y^*,Z\rangle + Y^*\langle Z,X^*\rangle - Z\langle X^*,Y^*\rangle\\ &+\langle[X^*,Y^*],\ Z\rangle - \langle[Y^*,Z],\ X^*\rangle + \langle[Z,X^*],\ Y^*\rangle\end{aligned}$$

for any $X=(X_2,X_1)$, $Y=(Y_2,Y_1)\in\mathfrak{g}\times\mathfrak{g}$, $Z\in\mathfrak{g}$. We compute the right side of the above equation at e. Since $\langle Y^*,Z\rangle_h = \langle\mathrm{Ad}(h^{-1})Y_2 - Y_1, Z\rangle$ $(h\in G)$, we have

$$\begin{aligned}(X^*\langle Y^*,Z\rangle)_e &= \frac{d}{dt}\langle\mathrm{Ad}(\exp(-t(X^*)_e))Y_2 - Y_1, Z\rangle|_{t=0}\\ &= \langle -[(X^*)_e, Y_2], Z\rangle = \langle -[X_2 - X_1, Y_2], Z\rangle.\end{aligned}$$

Similarly, we have

$$(Y^*\langle Z,X^*\rangle)_e = \langle -[Y_2 - Y_1, X_2], Z\rangle.$$

Since $\langle X^*,Y^*\rangle_h = \langle\mathrm{Ad}(h^{-1})X_2 - X_1, \mathrm{Ad}(h^{-1})Y_2 - Y_1\rangle$ $(h\in G)$, we have

$$\begin{aligned}Z(X^*,Y^*)_e &= \frac{d}{dt}\langle\mathrm{Ad}(\exp(-tZ))X_2 - X_1, \mathrm{Ad}(\exp(-tZ))Y_2 - Y_1\rangle|_{t=0}\\ &= \frac{d}{dt}\langle\mathrm{Ad}(\exp(-tZ))X_2, -Y_1\rangle + \langle -X_1, \mathrm{Ad}(\exp(-tZ))Y_2\rangle|_{t=0}\\ &= \langle[Z,X_2],Y_1\rangle + \langle X_1,[Z,Y_2]\rangle = \langle Z,[X_2,Y_1]\rangle + \langle Z,[Y_2,X_1]\rangle\\ &= \langle Z,[X_2,Y_1]+[Y_2,X_1]\rangle.\end{aligned}$$

Note the sign of the commutator product of $\mathfrak{X}(G)$ and $\mathfrak{g}\times\mathfrak{g}$. Then we have

$$[X^*,Y^*] = -(\mathrm{ad}_{\mathfrak{g}\times\mathfrak{g}}(X)Y)^*.$$

Thus,

$$\langle[X^*,Y^*],Z\rangle_e = \langle -\mathrm{ad}(X_2)Y_2 + \mathrm{ad}(X_1)Y_1, Z\rangle.$$

Since $Z=(0,-Z)^*$ we have

$$\begin{aligned}\langle[Y^*,Z],X^*\rangle_e &= \langle -\mathrm{ad}(Y_1)Z, X_2 - X_1\rangle = \langle Z, \mathrm{ad}(Y_1)(X_2 - X_1)\rangle,\\ \langle[Z,X^*],Y^*\rangle_e &= -\langle Z, \mathrm{ad}(X_1)(Y_2 - Y_1)\rangle.\end{aligned}$$

Therefore, we have

$$\begin{aligned}2(\nabla_{X^*}Y^*)_e =&(-[X_2-X_1,Y_2])+(-[Y_2-Y_1,X_2])-([X_2,Y_1]+[Y_2,X_1])\\&+(-[X_2,Y_2]+[X_1,Y_1])-([Y_1,X_2-X_1])+(-[X_1,Y_2-Y_1])\\=&[X_2-X_1,-Y_2+Y_1]+[X_1,Y_2+Y_1-Y_2+Y_1]\\&+[X_2,Y_2-Y_1-Y_1-Y_2]\\=&[X_2-X_1,-Y_2+Y_1]+2[X_1-X_2,Y_1]\\=&-[X_2-X_1,Y_2+Y_1].\end{aligned}$$

Hence we obtain

$$(\nabla_{X^*}Y^*)_e = -\frac{1}{2}[X_2-X_1,Y_2+Y_1]. \tag{2.6}$$

Since dL_g is an isometry, we have

$$(\nabla_{X^*}Y^*)_g = dL_g(\nabla_{dL_g^{-1}X^*}dL_g^{-1}Y^*)_e.$$

Further, we have

$$\begin{aligned}(dL_g^{-1}X^*)_h = dL_g^{-1}(X^*)_{gh} &= dL_g^{-1}dL_{gh}(\mathrm{Ad}(gh)^{-1}X_2-X_1)\\&= dL_h(\mathrm{Ad}(h)^{-1}\mathrm{Ad}(g)^{-1}X_2-X_1)\\&=(\mathrm{Ad}(g)^{-1}X_2,X_1)^*_h \qquad (h\in G).\end{aligned}$$

Thus,

$$dL_g^{-1}X^* = (\mathrm{Ad}(g)^{-1}X_2,X_1)^*$$

holds. Summarizing the above, we obtain

$$(\nabla_{X^*}Y^*)_g = -\frac{1}{2}dL_g[\mathrm{Ad}(g)^{-1}X_2-X_1,\mathrm{Ad}(g)^{-1}Y_2+Y_1].$$

□

For $H\in\mathfrak{a}$, we denote the second fundamental form of the orbit $K_2gK_1\subset G$ by B_H. By Lemma 2.11, we can express B_H for $H\in\mathfrak{a}$.

Theorem 2.12 ([O] Theorem 3). *For $H\in\mathfrak{a}$, we set $g=\exp(H)$ and*

$$\begin{aligned}V_1 &= \sum_{\lambda\in\Sigma^+\setminus\Sigma_H}\mathfrak{m}_\lambda\oplus\sum_{\alpha\in W^+\setminus W_H}V_\alpha^\perp(\mathfrak{m}_1\cap\mathfrak{k}_2),\\V_2 &= \sum_{\lambda\in\Sigma^+}\mathfrak{k}_\lambda\oplus\sum_{\alpha\in W^+}V_\alpha^\perp(\mathfrak{k}_1\cap\mathfrak{m}_2).\end{aligned}$$

Then we have the following:

1. *For $X\in\mathfrak{k}_0$, $B_H(dL_g(X),Y)=0$ where $Y\in T_g(K_2gK_1)$.*
2. *For $X\in V(\mathfrak{k}_1\cap\mathfrak{m}_2)$,*
$$dL_g^{-1}B_H(dL_g(X),dL_g(Y))=\begin{cases}0 & (Y\in\mathfrak{k}_1\oplus V(\mathfrak{m}_1\cap\mathfrak{k}_2))\\ -\frac{1}{2}[X,Y]^\perp & (Y\in V_1).\end{cases}$$
3. *For $X\in V(\mathfrak{m}_1\cap\mathfrak{k}_2)$,*
$$dL_g^{-1}B_H(dL_g(X),dL_g(Y))=\begin{cases}0 & (Y\in V(\mathfrak{m}_1\cap\mathfrak{k}_2)\oplus V_1)\\ \frac{1}{2}[X,Y]^\perp & (Y\in V_2).\end{cases}$$

(4) *For* $S_{\lambda,i}$ $(\lambda \in \Sigma^+,\ 1 \leq i \leq m(\lambda))$,

$$dL_g^{-1} B_H(dL_g(S_{\lambda,i}), dL_g(Y)) = \begin{cases} 0 & (Y \in V_2) \\ -\dfrac{1}{2}[S_{\lambda,i}, Y]^{\perp} & (Y \in V_1)\,. \end{cases}$$

(5) *For* $X_{\alpha,i}$ $(\alpha \in W^+,\ 1 \leq i \leq n(\alpha))$,

$$dL_g^{-1} B_H(dL_g(X_{\alpha,i}), dL_g(Y)) = \begin{cases} 0 & (Y \in V_2) \\ -\dfrac{1}{2}[X_{\alpha,i}, Y]^{\perp} & (Y \in V_1)\,. \end{cases}$$

(6) *For* $T_{\lambda,i}$ $(\lambda \in \Sigma^+ \setminus \Sigma_H,\ 1 \leq i \leq m(\lambda))$,

- $dL_g^{-1} B_H(dL_g(T_{\lambda,i}), dL_g(T_{\mu,j})) = \cot\langle\mu, H\rangle [T_{\lambda,i}, S_{\mu,j}]^{\perp}$ *where* $\mu \in \Sigma^+ \setminus \Sigma_H,\ 1 \leq j \leq m(\mu)$.
- $dL_g^{-1} B_H(dL_g(T_{\lambda,i}), dL_g(Y_{\beta,j})) = -\tan\langle\beta, H\rangle [T_{\lambda,i}, X_{\beta,j}]^{\perp}$ *where* $\beta \in W^+ \setminus W_H,\ 1 \leq j \leq n(\beta)$.

(7) *For* $Y_{\alpha,i}$ $(\alpha \in W^+ \setminus W_H,\ 1 \leq i \leq n(\alpha))$,

$$dL_g^{-1} B_H(dL_g(Y_{\alpha,i}), dL_g(Y_{\beta,j})) = -\tan\langle\beta, H\rangle [Y_{\alpha,i}, X_{\beta,j}]^{\perp}$$

where $\beta \in W^+ \setminus W_H,\ 1 \leq j \leq n(\beta))$.

Here, $X^{\perp}$ *is the normal component, i.e. the* $((\mathrm{Ad}(g)^{-1}\mathfrak{m}_2) \cap \mathfrak{m}_1)$*-component, of a tangent vector* $X \in \mathfrak{g}$.

Proof. By a simple calculation, we have the following:

- For $X \in \mathfrak{k}_0$, $dL_g(X) = (X, 0)^*_g$.
- For $X \in V(\mathfrak{k}_1 \cap \mathfrak{m}_2)$, $dL_g(X) = (0, -X)^*_g$.
- For $X \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$, $dL_g(X) = (X, 0)^*_g$.
- For $S_{\lambda,i}$ $(\lambda \in \Sigma^+, 1 \leq i \leq m(\lambda))$, $dL_g(S_{\lambda,i}) = (0, -S_{\lambda,i})^*_g$.
- For $T_{\lambda,i}$ $(\lambda \in \Sigma^+ \setminus \Sigma_H, 1 \leq i \leq m(\lambda))$,
$$dL_g(T_{\lambda,i}) = \left(-\frac{S_{\lambda,i}}{\sin\langle\lambda, H\rangle}, -\cot\langle\lambda, H\rangle S_{\lambda,i}\right)^*_g.$$
- For $X_{\alpha,i}$ $(\alpha \in W^+, 1 \leq i \leq n(\alpha))$, $dL_g(X_{\alpha,i}) = (0, -X_{\alpha,i})^*_g$.
- For $Y_{\alpha,i}$ $(\alpha \in W^+ \setminus W_H, 1 \leq i \leq n(\alpha))$,
$$dL_g(Y_{\alpha,i}) = \left(\frac{Y_{\alpha,i}}{\cos\langle\alpha, H\rangle}, \tan\langle\alpha, H\rangle X_{\alpha,i}\right)^*_g.$$

Then, applying Lemma 2.11, we have follows.

For (1), let $X \in \mathfrak{k}_0$. Then we can calculate as follows:

- For $Y \in \mathfrak{k}_0$,
$$\begin{aligned} B_H(dL_g(X), dL_g(Y)) &= \left(\nabla_{(\mathrm{Ad}(g)^{-1}X,0)^*}(\mathrm{Ad}(g)^{-1}Y, 0)^*\right)^{\perp}_g \\ &= -\frac{1}{2} dL_g([X, Y])^{\perp} = 0 \end{aligned}$$
since $[X, Y] \in \mathfrak{k}_0$ is a tangent vector.
- For $Y \in V(\mathfrak{k}_1 \cap \mathfrak{m}_2)$,
$$\begin{aligned} B_H(dL_g(X), dL_g(Y)) &= \left(\nabla_{(\mathrm{Ad}(g)^{-1}X,0)^*}(0, -Y)^*\right)^{\perp}_g \\ &= -\frac{1}{2} dL_g([X, -Y])^{\perp} = 0 \end{aligned}$$

since $[X,Y] \in \mathfrak{k}_1$ is a tangent vector.

- For $Y \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$,
$$B_H(dL_g(X), dL_g(Y)) = \left(\nabla_{(\mathrm{Ad}(g)^{-1}X,0)^*}(\mathrm{Ad}(g)^{-1}Y,0)^*\right)_g^{\perp} = -\frac{1}{2}dL_g([X,Y])^{\perp} = 0$$
since $[X,Y] \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$ is a tangent vector.
- For $S_{\lambda,i}$ $(\lambda \in \Sigma^+,\ 1 \le i \le m(\lambda))$,
$$B_H(dL_g(X), dL_g(S_{\lambda,i})) = \left(\nabla_{(\mathrm{Ad}(g)^{-1}X,0)^*}(0,-S_{\lambda,i})^*\right)_g^{\perp} = -\frac{1}{2}dL_g([X,-S_{\lambda,i}])^{\perp} = 0$$
since $[X,-S_{\lambda,i}] \in \mathfrak{k}_1$ is a tangent vector.
- For $X_{\alpha,j}$ $(\alpha \in W^+,\ 1 \le j \le n(\alpha))$,
$$B_H(dL_g(X), dL_g(X_{\alpha,j})) = \left(\nabla_{(\mathrm{Ad}(g)^{-1}X,0)^*}(0,-X_{\alpha,j})^*\right)_g^{\perp} = -\frac{1}{2}dL_g([X,-X_{\alpha,j}])^{\perp} = 0$$
since $[X,X_{\alpha,j}] \in \mathfrak{k}_1$ is a tangent vector.
- For $T_{\lambda,i}$ $(\lambda \in \Sigma^+ \setminus \Sigma_H,\ 1 \le i \le m(\lambda))$,
$$B_H(dL_g(X), dL_g(T_{\lambda,i})) = \left(\nabla_{(\mathrm{Ad}(g)^{-1}X,0)^*}(\mathrm{Ad}(g)^{-1}\frac{-S_{\lambda,i}}{\sin\langle\lambda,H\rangle}, -\cot\langle\lambda,H\rangle S_{\lambda,i})^*\right)_g^{\perp} = -\frac{1}{2}dL_g([X,-2\cot\langle\lambda,H\rangle S_{\lambda,i} - T_{\lambda,i}])^{\perp} = 0$$
since $[X,S_{\lambda,i}] \in \mathfrak{k}_1$ and $[X,T_{\lambda,i}] \in \mathfrak{m}_\lambda$ are tangent vectors.
- For $Y_{\alpha,j}$ $(\alpha \in W^+ \setminus W_H,\ 1 \le j \le n(\alpha))$,
$$B_H(dL_g(X), dL_g(Y_{\alpha,j})) = \left(\nabla_{(\mathrm{Ad}(g)^{-1}X,0)^*}(\mathrm{Ad}(g)^{-1}\frac{Y_{\alpha,j}}{\cos\langle\alpha,H\rangle}, \tan\langle\alpha,H\rangle X_{\alpha,j})^*\right)_g^{\perp} = -\frac{1}{2}dL_g([X,2\tan\langle\alpha,H\rangle X_{\alpha,j} - Y_{\alpha,j}])^{\perp} = 0$$
since $[X,X_{\alpha,j}] \in \mathfrak{k}_1$ and $[X,Y_{\alpha,j}] \in V_\alpha^{\perp}(\mathfrak{m}_1 \cap \mathfrak{k}_2)$ are tangent vectors.

For (2), let $X \in V(\mathfrak{k}_1 \cap \mathfrak{m}_2)$. Then we can calculate as follows:

- For $Y \in V(\mathfrak{k}_1 \cap \mathfrak{m}_2)$,
$$B_H(dL_g(X), dL_g(Y)) = \left(\nabla_{0,-X)^*}(0,-Y)^*\right)_g^{\perp} = -\frac{1}{2}dL_g([X,-Y])^{\perp} = 0$$
since $[X,Y] \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$ is a tangent vector.
- For $Y \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$,
$$B_H(dL_g(X), dL_g(Y)) = \left(\nabla_{(0,-X)^*}(\mathrm{Ad}(g)^{-1}Y,0)^*\right)_g^{\perp} = -\frac{1}{2}dL_g([X,Y])^{\perp}.$$
Then, $[X,Y] \in \mathfrak{a}$ and $\langle [X,Y], H'\rangle = \langle X, [Y,H']\rangle$ for all $H' \in \mathfrak{a}$, thus $[X,Y] = 0$. Hence $B_H(dL_g(X), dL_g(Y)) = 0$.

- For $S_{\lambda,i}$ $(\lambda \in \Sigma^+,\ 1 \leq i \leq m(\lambda))$,

$$\begin{aligned} B_H(dL_g(X), dL_g(S_{\lambda,i})) &= \left(\nabla_{(0,-X)^*}(0, -S_{\lambda,i})^*\right)_g^\perp \\ &= -\frac{1}{2} dL_g([X, -S_{\lambda,i}])^\perp = 0 \end{aligned}$$

since $[X, S_{\lambda,i}] \in \mathfrak{k}_1$ is a tangent vector.
- For $X_{\alpha,j}$ $(\alpha \in W^+,\ 1 \leq j \leq n(\alpha))$,

$$\begin{aligned} B_H(dL_g(X), dL_g(X_{\alpha,j})) &= \left(\nabla_{(0,-X)^*}(0, -X_{\alpha,j})^*\right)_g^\perp \\ &= -\frac{1}{2} dL_g([X, -X_{\alpha,j}])^\perp = 0 \end{aligned}$$

since $[X, X_{\alpha,j}] \in \mathfrak{k}_1$ is a tangent vector.
- For $T_{\lambda,i}$ $(\lambda \in \Sigma^+ \setminus \Sigma_H,\ 1 \leq i \leq m(\lambda))$,

$$\begin{aligned} B_H(dL_g(X), dL_g(T_{\lambda,i})) &= \left(\nabla_{(0,-X)^*}(\mathrm{Ad}(g)^{-1}\frac{-S_{\lambda,i}}{\sin\langle\lambda, H\rangle}, -\cot\langle\lambda, H\rangle S_{\lambda,i})^*\right)_g^\perp \\ &= -\frac{1}{2} dL_g([X, -2\cot\langle\lambda, H\rangle S_{\lambda,i} + T_{\lambda,i}])^\perp \\ &= -\frac{1}{2} dL_g([X, +T_{\lambda,i}])^\perp \end{aligned}$$

since $[X, S_{\lambda,i}] \in \mathfrak{k}_1$ is a tangent vector.
- For $Y_{\alpha,j}$ $(\alpha \in W^+ \setminus W_H,\ 1 \leq j \leq n(\alpha))$,

$$\begin{aligned} B_H(dL_g(X), dL_g(Y_{\alpha,j})) &= \left(\nabla_{(0,-X)^*}(\mathrm{Ad}(g)^{-1}\frac{Y_{\alpha,j}}{\cos\langle\alpha, H\rangle}, \tan\langle\alpha, H\rangle X_{\alpha,j})^*\right)_g^\perp \\ &= -\frac{1}{2} dL_g([X, 2\tan\langle\alpha, H\rangle X_{\alpha,j} + Y_{\alpha,j}])^\perp \\ &= -\frac{1}{2} dL_g([X, Y_{\alpha,j}])^\perp \end{aligned}$$

since $[X, X_{\alpha,j}] \in \mathfrak{k}_1$ is a tangent vector.

For (3), let $X \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$. Then we can calculate as follows:

- For $Y \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$,

$$\begin{aligned} B_H(dL_g(X), dL_g(Y)) &= \left(\nabla_{(X,0)^*}(\mathrm{Ad}(g)^{-1}Y, 0)^*\right)_g^\perp \\ &= -\frac{1}{2} dL_g([X, Y])^\perp = 0. \end{aligned}$$

since $[X, Y] \in \mathfrak{k}_0$ is a tangent vector.
- For $S_{\lambda,i}$ $(\lambda \in \Sigma^+,\ 1 \leq i \leq m(\lambda))$,

$$\begin{aligned} B_H(dL_g(X), dL_g(S_{\lambda,i})) &= \left(\nabla_{(X,0)^*}(0, -S_{\lambda,i})^*\right)_g^\perp \\ &= -\frac{1}{2} dL_g([X, -S_{\lambda,i}])^\perp = \frac{1}{2} dL_g([X, S_{\lambda,i}])^\perp. \end{aligned}$$

- For $X_{\alpha,j}$ $(\alpha \in W^+,\ 1 \le j \le n(\alpha))$,

$$\begin{aligned} B_H(dL_g(X), dL_g(X_{\alpha,j})) &= \left(\nabla_{(X,0)^*}(0, -X_{\alpha,j})^*\right)_g^{\perp} \\ &= -\frac{1}{2}dL_g([X, -X_{\alpha,j}])^{\perp} = \frac{1}{2}dL_g([X, X_{\alpha,j}])^{\perp}. \end{aligned}$$

- For $T_{\lambda,i}$ $(\lambda \in \Sigma^+ \setminus \Sigma_H,\ 1 \le i \le m(\lambda))$,

$$\begin{aligned} &B_H(dL_g(X), dL_g(T_{\lambda,i})) = B_H(dL_g(T_{\lambda,i}), dL_g(X)) \\ &= \left(\nabla_{(\mathrm{Ad}(g)^{-1}\frac{-S_{\lambda,i}}{\sin\langle\lambda,H\rangle}, -\cot\langle\lambda,H\rangle S_{\lambda,i})^*}(X,0)^*\right)_g^{\perp} \\ &= -\frac{1}{2}dL_g([T_{\lambda,i}, X])^{\perp} = 0 \end{aligned}$$

  since $[X, T_{\lambda,i}] \in \mathfrak{k}_1$ is a tangent vector.
- For $Y_{\alpha,j}$ $(\alpha \in W^+ \setminus W_H,\ 1 \le j \le n(\alpha))$,

$$\begin{aligned} &B_H(dL_g(X), dL_g(Y_{\alpha,j})) = B_H(dL_g(Y_{\alpha,j}), dL_g(X)) \\ &= \left(\nabla_{(\mathrm{Ad}(g)^{-1}\frac{Y_{\alpha,j}}{\cos\langle\alpha,H\rangle}, \tan\langle\alpha,H\rangle X_{\alpha,j})^*}(X,0)^*\right)_g^{\perp} \\ &= -\frac{1}{2}dL_g([Y_{\alpha,j}, X])^{\perp} = 0 \end{aligned}$$

  since $[X, Y_{\alpha,j}] \in \mathfrak{k}_1$ is a tangent vector.

For (4), let $\lambda \in \Sigma^+$ and $1 \le i \le m(\lambda)$. Then we can calculate as follows:

- For $S_{\mu,j}$ $(\mu \in \Sigma^+,\ 1 \le j \le m(\mu))$,

$$\begin{aligned} B_H(dL_g(S_{\lambda,i}), dL_g(S_{\mu,j})) &= \left(\nabla_{(0,-S_{\mu,j})^*}(0, -S_{\lambda,i})^*\right)_g^{\perp} \\ &= -\frac{1}{2}dL_g([S_{\lambda,i}, -S_{\mu,j}])^{\perp} = 0 \end{aligned}$$

  since $[S_{\lambda,i}, S_{\mu,j}] \in \mathfrak{k}_1$ is a tangent vector.
- For $X_{\alpha,j}$ $(\alpha \in W^+,\ 1 \le j \le n(\alpha))$,

$$\begin{aligned} B_H(dL_g(S_{\lambda,i}), dL_g(X_{\alpha,j})) &= \left(\nabla_{(0,-S_{\lambda,i})^*}(0, -X_{\alpha,j})^*\right)_g^{\perp} \\ &= -\frac{1}{2}dL_g([S_{\lambda,i}, -X_{\alpha,j}])^{\perp} = 0 \end{aligned}$$

  since $[S_{\lambda,i}, X_{\alpha,j}] \in \mathfrak{k}_1$ is a tangent vector.
- For $T_{\mu,j}$ $(\mu \in \Sigma^+ \setminus \Sigma_H,\ 1 \le j \le m(\mu))$,

$$\begin{aligned} &B_H(dL_g(S_{\lambda,i}), dL_g(T_{\mu,j})) = B_H(dL_g(T_{\mu,j}), dL_g(S_{\lambda,i})) \\ &= \left(\nabla_{(\mathrm{Ad}(g)^{-1}\frac{-S_{\mu,j}}{\sin\langle\mu,H\rangle}, -\cot\langle\mu,H\rangle S_{\mu,j})^*}(0, -S_{\lambda,i})^*\right)_g^{\perp} \\ &= -\frac{1}{2}dL_g([T_{\mu,j}, -S_{\lambda,i}])^{\perp} = -\frac{1}{2}dL_g([S_{\lambda,i}, T_{\mu,j}])^{\perp}. \end{aligned}$$

- For $Y_{\alpha,j}$ $(\alpha \in W^+ \setminus W_H,\ 1 \leq j \leq n(\alpha))$,

$$\begin{aligned} &B_H(dL_g(S_{\lambda,i}), dL_g(Y_{\alpha,j})) = B_H(dL_g(Y_{\alpha,j}), dL_g(S_{\lambda,i})) \\ &= \left(\nabla_{(\mathrm{Ad}(g)^{-1}\frac{Y_{\alpha,j}}{\cos\langle\alpha,H\rangle}, \tan\langle\alpha,H\rangle X_{\alpha,j})^*}(0, -S_{\lambda,i})^*\right)_g^\perp \\ &= -\frac{1}{2}dL_g([Y_{\alpha,j}, -S_{\lambda,i}])^\perp = -\frac{1}{2}dL_g([S_{\lambda,i}, Y_{\alpha,j}])^\perp. \end{aligned}$$

For (5), let $\alpha \in W^+$ and $1 \leq j \leq n(\alpha)$. Then we can calculate as follows:

- For $X_{\beta,i}$ $(\beta \in W^+,\ 1 \leq i \leq n(\beta))$,

$$\begin{aligned} B_H(dL_g(X_{\alpha,j}), dL_g(X_{\beta,i}),) &= \left(\nabla_{(0,-X_{\alpha,j})^*}(0, -X_{\beta,i})^*\right)_g^\perp \\ &= -\frac{1}{2}dL_g([X_{\alpha,j}, -X_{\beta,i}])^\perp = 0 \end{aligned}$$

  since $[X_{\alpha,j}, X_{\beta,i}] \in \mathfrak{k}_1$ is a tangent vector.

- For $T_{\lambda,i}$ $(\lambda \in \Sigma^+ \setminus \Sigma_H,\ 1 \leq i \leq m(\lambda))$,

$$\begin{aligned} &B_H(dL_g(X_{\alpha,j}), dL_g(T_{\lambda,i})) = B_H(dL_g(T_{\lambda,i}), dL_g(X_{\alpha,j})) \\ &= \left(\nabla_{(\mathrm{Ad}(g)^{-1}\frac{-S_{\lambda,i}}{\sin\langle\lambda,H\rangle}, -\cot\langle\lambda,H\rangle S_{\lambda,i})^*}(0, -X_{\alpha,j})^*\right)_g^\perp \\ &= -\frac{1}{2}dL_g([T_{\lambda,i}, -X_{\alpha,j}])^\perp = -\frac{1}{2}dL_g([X_{\alpha,j}, T_{\mu,j}])^\perp. \end{aligned}$$

- For $Y_{\beta,i}$ $(\beta \in W^+ \setminus W_H,\ 1 \leq i \leq n(\beta))$,

$$\begin{aligned} &B_H(dL_g(X_{\alpha,j}), dL_g(Y_{\beta,i})) = B_H(dL_g(Y_{\beta,i}), dL_g(X_{\alpha,j})) \\ &= \left(\nabla_{(\mathrm{Ad}(g)^{-1}\frac{Y_{\beta,i}}{\cos\langle\beta,H\rangle}, \tan\langle\beta,H\rangle X_{\beta,i})^*}(0, -X_{\alpha,j})^*\right)_g^\perp \\ &= -\frac{1}{2}dL_g([Y_{\beta,i}, -X_{\alpha,j}])^\perp = -\frac{1}{2}dL_g([X_{\alpha,j}, Y_{\beta,i}])^\perp. \end{aligned}$$

For (6), let $\lambda \in \Sigma^+ \setminus \Sigma_H$ and $1 \leq i \leq m(\lambda)$. Then we can calculate as follows:

- For $T_{\mu,j}$ $(\mu \in \Sigma^+ \setminus \Sigma_H,\ 1 \leq j \leq m(\mu))$,

$$\begin{aligned} &B_H(dL_g(T_{\lambda,i}), dL_g(T_{\mu,j})) \\ &= \left(\nabla_{(\mathrm{Ad}(g)^{-1}\frac{-S_{\lambda,i}}{\sin\langle\lambda,H\rangle}, -\cot\langle\lambda,H\rangle S_{\lambda,i})^*}(\mathrm{Ad}(g)^{-1}\frac{-S_{\mu,j}}{\sin\langle\mu,H\rangle}, -\cot\langle\mu,H\rangle S_{\mu,j})^*\right)_g^\perp \\ &= -\frac{1}{2}dL_g([T_{\lambda,i}, -2\cot\langle\mu,H\rangle S_{\mu,j} + T_{\mu,j}])^\perp = \cot\langle\mu,H\rangle dL_g[T_{\lambda,i}, S_{\mu,j}]^\perp. \end{aligned}$$

- For $Y_{\alpha,j}$ $(\alpha \in W^+ \setminus W_H,\ 1 \leq j \leq n(\alpha))$,

$$\begin{aligned} &B_H(dL_g(T_{\lambda,i}), dL_g(Y_{\alpha,j})) \\ &= \left(\nabla_{(\mathrm{Ad}(g)^{-1}\frac{-S_{\lambda,i}}{\sin\langle\lambda,H\rangle}, -\cot\langle\lambda,H\rangle S_{\lambda,i})^*}(\mathrm{Ad}(g)^{-1}\frac{Y_{\alpha,j}}{\cos\langle\alpha,H\rangle}, \tan\langle\alpha,H\rangle X_{\alpha,j})^*\right)_g^\perp \\ &= -\frac{1}{2}dL_g([T_{\lambda,i}, 2\tan\langle\alpha,H\rangle X_{\alpha,j} + Y_{\alpha,j}])^\perp = -\tan\langle\alpha,H\rangle dL_g([T_{\lambda,i}, X_{\alpha,j}])^\perp. \end{aligned}$$

For (7), let $\alpha, \beta \in W^+ \setminus W_H$, $1 \leq j \leq n(\alpha)$ and $1 \leq i \leq n(\beta)$. Then we have

$$\begin{aligned}
&B_H(dL_g(Y_{\alpha,j}), dL_g(Y_{\beta,i})) \\
&= \left(\nabla_{(\mathrm{Ad}(g)^{-1}\frac{Y_{\alpha,j}}{\cos\langle\alpha,H\rangle}, \tan\langle\alpha,H\rangle X_{\alpha,j})^*}(\mathrm{Ad}(g)^{-1}\frac{Y_{\beta,i}}{\cos\langle\beta,H\rangle}, \tan\langle\beta,H\rangle X_{\beta,i})^*\right)_g^\perp \\
&= -\frac{1}{2}dL_g([Y_{\alpha,j}, 2\tan\langle\beta,H\rangle X_{\beta,i} + Y_{\beta,i}])^\perp = \tan\langle\beta,H\rangle dL_g([Y_{\alpha,j}, X_{\beta,i}])^\perp.
\end{aligned}$$

Then, we have the consequence. □

We denote the mean curvature vector of the orbit K_2gK_1 at g by m_H. By Theorem 2.12, we can see that the following corollary.

Corollary 2.13 ([O] Corollary 2). *For $H \in \mathfrak{a}$,*

$$dL_g^{-1}m_H = -\sum_{\lambda\in\Sigma^+\setminus\Sigma_H} m(\lambda)\cot\langle\lambda,H\rangle\lambda + \sum_{\alpha\in W^+\setminus W_H} n(\alpha)\tan\langle\alpha,H\rangle\alpha.$$

Moreover, $dL_g^{-1}m_H = dL_g^{-1}m_H^1$ holds. Hence, an orbit $K_2gK_1 \subset G$ is minimal if and only if $K_2\pi_1(g) \subset M_1$ is minimal.

Proof. By Theorem 2.12, we have

$$\begin{aligned}
dL_g^{-1}B_H(dL_g(X), dL_g(X)) &= 0 \ (X \in \mathfrak{k}_1), \\
dL_g^{-1}B_H(dL_g(T_{\lambda,i}), dL_g(T_{\lambda,i})) &= -\cot\langle\lambda,H\rangle\lambda \ \ (\lambda \in \Sigma^+ \setminus \Sigma_H,\ 1 \leq i \leq m(\lambda)), \\
dL_g^{-1}B_H(dL_g(Y_{\alpha,j}), dL_g(Y_{\alpha,j})) &= \tan\langle\alpha,H\rangle\alpha \ \ (\alpha \in W^+ \setminus W_H,\ 1 \leq j \leq n(\alpha)).
\end{aligned}$$

Thus we have

$$dL_g^{-1}m_H = -\sum_{\lambda\in\Sigma^+\setminus\Sigma_H} m(\lambda)\cot\langle\lambda,H\rangle\lambda + \sum_{\alpha\in W^+\setminus W_H} n(\alpha)\tan\langle\alpha,H\rangle\alpha.$$

Moreover, by (1) of Theorem 2.9, we obtain $dL_g^{-1}m_H = dL_g^{-1}m_H^1$. □

Next, we consider austere orbits of the $(K_2 \times K_1)$-action on G. By using $(\tilde{\Sigma}, \Sigma, W)$, Ikawa gave a necessary and sufficient condition for an orbit of the K_2-action to be an austere submanifold. Similarly, in the $(K_2 \times K_1)$-action, we also have a necessary and sufficient condition for an orbit to be an austere submanifold. We investigate the set of eigenvalues of the shape operator $A^{dL_g\xi}$ of $K_1gK_2 \subset G$ for each normal vector $dL_g\xi \in T_g^\perp K_2gK_1 \cong dL_g((\mathrm{Ad}(g)^{-1}\mathfrak{m}_2) \cap \mathfrak{m}_1)$. For each $g \in G$, we denote the isotropy subgroup of the $(K_2 \times K_1)$-action on G at g by $(K_2 \times K_1)_g$. Notice that $(K_2 \times K_1)_g$ is isomorphic to the isotropy subgroup $(K_1)_{\pi_2(g)}$ of the K_1-action at $\pi_2(g)$. The isotropy subgroup $(K_2 \times K_1)_g$ acts on the normal space $T_g^\perp(K_2gK_1)$ by the differential of the $(K_2 \times K_1)$-action. Then we have

$$d(k_2, k_1)_g(dL_g(\xi)) = \frac{d}{dt}k_2g\exp(t\xi)k_1^{-1}\bigg|_{t=0} = dL_g(\mathrm{Ad}(k_1)\xi).$$

Therefore, the representation of $(K_2 \times K_1)_g$ is equivalent to the adjoint representation of $(K_1)_{\pi_2(g)}$ on $(\mathrm{Ad}(g)^{-1}\mathfrak{m}_2) \cap \mathfrak{m}_1$. Since $\mathrm{Lie}((K_1)_{\pi_2(g)}) = \mathfrak{k}_1 \cap (\mathrm{Ad}(g)^{-1}\mathfrak{k}_2)$, the Lie algebra $\mathrm{Lie}((K_1)_{\pi_2(g)}) \oplus ((\mathrm{Ad}(g)^{-1}\mathfrak{m}_2) \cap \mathfrak{m}_1)$ is an orthogonal symmetric Lie algebra with respect to θ_1. Moreover, when $g \in \exp(\mathfrak{a})$, $\mathfrak{a}$ is a maximal abelian subspace of $((\mathrm{Ad}(g)^{-1}\mathfrak{m}_2) \cap \mathfrak{m}_1)$. Thus, $\mathfrak{a}$ is a section of the representation of $(K_1)_{\pi_2(g)}$

on $(\mathrm{Ad}(g)^{-1}\mathfrak{m}_2)\cap\mathfrak{m}_1$. Therefore, we have

$$\bigcup_{(k_2,k_1)\in(K_2\times K_1)_g} d(k_2,k_1)_g dL_g(\mathfrak{a}) = T_g^{\perp}K_2gK_1. \tag{2.7}$$

Thus, without loss of generality we can assume $\xi\in\mathfrak{a}$. Hence, by Theorem 2.12 we have

$$\begin{aligned} &A^{dL_g\xi}(dL_g(S_{\lambda,i}), dL_g(T_{\lambda,i})) \\ &\quad = (dL_g(S_{\lambda,i}), dL_g(T_{\lambda,i}))\begin{bmatrix} 0 & -(1/2)\langle\lambda,\xi\rangle \\ -(1/2)\langle\lambda,\xi\rangle & -\cot\langle\lambda,H\rangle\langle\lambda,\xi\rangle \end{bmatrix} \\ &\qquad (\lambda\in\Sigma^+\setminus\Sigma_H,\ 1\le i\le m(\lambda)), \end{aligned} \tag{2.8}$$

$$\begin{aligned} &A^{dL_g\xi}(dL_g(X_{\alpha,j}), dL_g(Y_{\alpha,j})) \\ &\quad = (dL_g(X_{\alpha,j}), dL_g(Y_{\alpha,j}))\begin{bmatrix} 0 & -(1/2)\langle\alpha,\xi\rangle \\ -(1/2)\langle\alpha,\xi\rangle & \tan\langle\alpha,H\rangle\langle\alpha,\xi\rangle \end{bmatrix} \\ &\qquad (\alpha\in W^+\setminus W_H,\ 1\le j\le n(\alpha)), \end{aligned} \tag{2.9}$$

for $X\in\mathfrak{k}_0\oplus V(\mathfrak{k}_1\cap\mathfrak{m}_2)\oplus V(\mathfrak{m}_1\cap\mathfrak{k}_2)\oplus\sum_{\lambda\in\Sigma_H^+}\mathfrak{k}_\lambda\oplus\sum_{\alpha\in W_H^+}V_\alpha^{\perp}(\mathfrak{k}_1\cap\mathfrak{m}_2)$,

$$A^{dL_g\xi}dL_g(X)=0. \tag{2.10}$$

Therefore, the set of eigenvalues of $A^{dL_g\xi}$ is given by

$$\begin{aligned} &\left\{-\frac{\cos\langle\lambda,H\rangle\pm1}{2\sin\langle\lambda,H\rangle}\langle\lambda,\xi\rangle\ (\text{multiplicity}=m(\lambda))\ \middle|\ \lambda\in\Sigma^+\setminus\Sigma_H\right\} \\ &\cup\left\{\frac{\sin\langle\alpha,H\rangle\pm1}{2\cos\langle\alpha,H\rangle}\langle\alpha,\xi\rangle\ (\text{multiplicity}=n(\alpha))\ \middle|\ \alpha\in W^+\setminus W_H\right\} \\ &\cup\{0\ (\text{multiplicity}=l)\} \end{aligned} \tag{2.11}$$

where $l=\dim(\mathfrak{k}_0\oplus\sum_{\lambda\in\Sigma_H^+}\mathfrak{k}_\lambda\oplus V(\mathfrak{k}_1\cap\mathfrak{m}_2)\oplus\sum_{\alpha\in W_H^+}V_\alpha^{\perp}(\mathfrak{k}_1\cap\mathfrak{m}_2)\oplus V(\mathfrak{m}_1\cap\mathfrak{k}_2))$.

Proposition 2.14 ([IST2] p.459)**.** *Let E be a finite subset of a finite dimensional vector space $\mathfrak{a}$ with an inner product $\langle,\rangle$. Then,* (i) *and* (ii) *are equivalent.*

(i) *For any $\xi\in\mathfrak{a}$, the set $\{\langle a,\xi\rangle \mid a\in E\}$ with multiplicity is invariant under the multiplication by -1.*
(ii) *The set E is invariant under the multiplication by -1.*

Thus, we have the following corollary.

Corollary 2.15 ([O] Corollary 3)**.** *Let $g=\exp(H)$ $(H\in\mathfrak{a})$. Then the orbit $K_2gK_1\subset G$ is austere if and only if the finite subset of $\mathfrak{a}$ defined by*

$$\begin{aligned} &\left\{-\frac{\cos\langle\lambda,H\rangle\pm1}{2\sin\langle\lambda,H\rangle}\lambda\ (\textit{multiplicity}=m(\lambda))\ \middle|\ \lambda\in\Sigma^+\setminus\Sigma_H\right\} \\ &\cup\left\{\frac{\sin\langle\alpha,H\rangle\pm1}{2\cos\langle\alpha,H\rangle}\alpha\ (\textit{multiplicity}=n(\alpha))\ \middle|\ \alpha\in W^+\setminus W_H\right\} \end{aligned}$$

is invariant under the multiplication by -1.

It is easy to prove that the following proposition.

Proposition 2.16 ([O] Proposition 5)**.** *For each $H \in \mathfrak{a}$,*

$$\begin{aligned} E =& \{-\lambda \cot\langle\lambda, H\rangle \ (\textit{multiplicity} = m(\lambda)) \mid \lambda \in \Sigma^+ \setminus \Sigma_H\} \\ & \cup\{\alpha \tan\langle\alpha, H\rangle \ (\textit{multiplicity} = n(\alpha)) \mid \alpha \in W^+ \setminus W_H\} \end{aligned}$$

is invariant under the multiplication by -1 with multiplicities if and only if

$$\begin{aligned} E' =& \left\{ -\frac{\cos\langle\lambda, H\rangle \pm 1}{2\sin\langle\lambda, H\rangle}\lambda \ (\textit{multiplicity} = m(\lambda)) \ \middle| \ \lambda \in \Sigma^+ \setminus \Sigma_H \right\} \\ & \cup \left\{ \frac{\sin\langle\alpha, H\rangle \pm 1}{2\cos\langle\alpha, H\rangle}\alpha \ (\textit{multiplicity} = n(\alpha)) \ \middle| \ \alpha \in W^+ \setminus W_H \right\} \end{aligned}$$

is invariant under the multiplication by -1 with multiplicities.

Proof. The equation $E = -E$ holds if and only if (i) and (ii) hold, where

(i) $\langle\lambda, H\rangle \in (\pi/4)\mathbb{Z} \quad (\lambda \in \tilde{\Sigma}^+ \setminus \tilde{\Sigma}_H)$,
(ii) if $\langle\lambda, H\rangle \in (\pi/4) + (\pi/2)\mathbb{Z}$, then $m(\lambda) = n(\lambda)$.

When $E = -E$ holds, for each $\lambda \in \tilde{\Sigma}^+ \setminus \tilde{\Sigma}_H$, if $\langle\lambda, H\rangle \in (\pi/2)\mathbb{Z}$, then it holds either one of the following:

- $\lambda \in \Sigma_H$ and
$$\frac{\sin\langle\lambda, H\rangle + 1}{2\cos\langle\lambda, H\rangle} = -\frac{\sin\langle\lambda, H\rangle - 1}{2\cos\langle\lambda, H\rangle}.$$
- $\lambda \in W_H$ and
$$-\frac{\cos\langle\lambda, H\rangle + 1}{2\sin\langle\lambda, H\rangle} = \frac{\cos\langle\lambda, H\rangle - 1}{2\sin\langle\lambda, H\rangle}.$$

Further, if $\langle\lambda, H\rangle \in (\pi/4) + (\pi/2)\mathbb{Z}$, then it holds either one of the following:

- $m(\lambda) = n(\lambda)$ and
$$\frac{\cos\langle\lambda, H\rangle + 1}{2\sin\langle\lambda, H\rangle} = \frac{\sin\langle\lambda, H\rangle + 1}{2\cos\langle\lambda, H\rangle}, \quad \text{and} \quad \frac{\cos\langle\lambda, H\rangle - 1}{2\sin\langle\lambda, H\rangle} = \frac{\sin\langle\lambda, H\rangle - 1}{2\cos\langle\lambda, H\rangle}.$$
- $m(\lambda) = n(\lambda)$ and
$$\frac{\cos\langle\lambda, H\rangle + 1}{2\sin\langle\lambda, H\rangle} = \frac{\sin\langle\lambda, H\rangle - 1}{2\cos\langle\lambda, H\rangle}, \quad \text{and} \quad \frac{\cos\langle\lambda, H\rangle - 1}{2\sin\langle\lambda, H\rangle} = \frac{\sin\langle\lambda, H\rangle + 1}{2\cos\langle\lambda, H\rangle}.$$

This implies that $E' = -E'$. The converse is shown by the same way. □

Corollary 2.17 ([O] Corollary 4)**.** *Let $g = \exp(H)$ $(H \in \mathfrak{a})$. The orbit $K_2 g K_1 \subset G$ is austere if and only if $K_2\pi_1(g) \subset M_1$ is austere.*

Remark 2.18. There is no correspondence in totally geodesic orbits. For example, when θ_1 and θ_2 cannot be transformed each other by an inner automorphism of $\mathfrak{g}$, $K_2 e K_1 \subset G$ is not totally geodesic, but $K_2\pi_1(e) \subset M_1$ is totally geodesic (see (4) and (5) in Theorem 2.12).

3. Weakly reflective submanifolds in compact symmetric spaces

Ikawa, Sakai, and Tasaki ([IST2]) proposed the notion of weakly reflective submanifold as a generalization of the notion of reflective submanifold ([Le]). In [IST2], they detected a certain global symmetry of several austere submanifolds in a hypersphere, and classified austere orbits and weakly reflective orbits of the linear isotropy representation of irreducible symmetric spaces. They gave a necessary and sufficient condition for orbits of the linear isotropy representation of irreducible symmetric spaces to be an austere submanifold (further, weakly reflective submanifold) in the hypersphere in terms of root systems. We would like to generalize this fact to compact Riemannian symmetric spaces. However, it is known that austere orbits of the isotropy action of compact symmetric spaces are reflective submanifolds. Therefore, we consider Hermann actions, which are a generalization of isotropy actions of compact symmetric spaces. Ikawa ([I]) classified austere orbits of commutative Hermann actions. However, weakly reflective orbits have not been classified yet. In this section, we give sufficient conditions for orbits of Hermann actions to be weakly reflective in terms of symmetric triads.

3.1. Weakly reflective submanifolds.

We recall the definitions of reflective submanifold and weakly reflective submanifold. Let $(\tilde{M}, \langle , \rangle)$ be a complete Riemannian manifold.

Definition 3.1 ([Le]). Let M be a submanifold of $\tilde{M}$. Then M is a reflective submanifold of $\tilde{M}$ if there exists an involutive isometry σ_M of $\tilde{M}$ such that M is a connected component of the fixed point set of σ_M. Then, we call σ_M the reflection of M.

Definition 3.2 ([IST2]). Let M be a submanifold of $\tilde{M}$. For each normal vector $\xi \in T^\perp_x M$ at each point $x \in M$, if there exists an isometry σ_ξ on $\tilde{M}$ which satisfies $\sigma_\xi(x) = x$, $\sigma_\xi(M) = M$ and $(d\sigma_\xi)_x(\xi) = -\xi$, then we call M a weakly reflective submanifold and σ_ξ a reflection of M with respect to ξ.

If M is a reflective submanifold of $\tilde{M}$, then σ_M is a reflection of M with respect to each normal vector $\xi \in T^\perp_x M$ at each point $x \in M$. Thus, a reflective submanifold of $\tilde{M}$ is a weakly reflective submanifold of $\tilde{M}$. Notice that a reflective submanifold is totally geodesic, but a weakly reflective submanifold is not necessarily totally geodesic.

Definition 3.3 ([HL]). Let M be a submanifold of $\tilde{M}$. We denote the shape operator of M by A. M is called an austere submanifold if for each normal vector $\xi \in T^\perp_x M$, the set of eigenvalues with their multiplicities of A^ξ is invariant under the multiplication by -1.

It is clear that an austere submanifold is a minimal submanifold. Ikawa, Sakai and Tasaki proved that a weakly reflective submanifold is an austere submanifold.

Lemma 3.4 ([IST2] p. 439). *Let G be a Lie group acting isometrically on a Riemannian manifold $\tilde{M}$. For $x \in \tilde{M}$, we consider the orbit Gx. If for each $\xi \in T^\perp_x Gx$, there exists a reflection of Gx at x with respect to ξ, then Gx is a weakly reflective submanifold of $\tilde{M}$.*

Proposition 3.5 ([IST2] Proposition 2.7). *Any singular orbit of a cohomogeneity one action on a Riemannian manifold is a weakly reflective submanifold.*

Proposition 3.6 ([IST2] Proposition 2.9)**.** *Let G be a connected Lie group acting isometrically on a complete, connected Riemannian manifold $\tilde{M}$. Suppose that the action of G on $\tilde{M}$ is cohomogeneity one with two singular orbits. If there exists a principal orbit which is a weakly reflective submanifold of $\tilde{M}$, then it has a same distance from two singular orbits and two singular orbits are isometric.*

3.2. **Sufficient conditions for orbits to be weakly reflective.** In the previous section, we saw a correspondence of austereness of orbits of the $(K_2 \times K_1)$-action and the K_2-action. In this section, we consider weakly reflective orbits of the $(K_2 \times K_1)$-action, the K_2-action and the K_1-action, and give two sufficient conditions for an orbit to be weakly reflective. The first sufficient condition is the following:

Theorem 3.7 ([O] Theorem 4)**.** *Assume K_1 and K_2 are connected. Let $g = \exp(H)$ $(H \in \mathfrak{a})$. If $\langle \lambda, H \rangle \in (\pi/2)\mathbb{Z}$ for any $\lambda \in \tilde{\Sigma}$, then the orbit $K_2 g K_1 \subset G$ is weakly reflective.*

Proof. We set $\sigma = L_g \theta_1 L_g^{-1}$. Then σ satisfies the following conditions:

(1) $\sigma(g) = g$,
(2) $\sigma(K_2 g K_1) = K_2 g K_1$,
(3) $d\sigma(\xi) = -\xi \quad (\xi \in T_g^{\perp}(K_2 g K_1))$.

Clearly, $\sigma(g) = g$ holds. By Lemma 2.7, we have

$$\begin{aligned} \mathrm{Ad}(g^2)X &= X \quad (X \in \mathfrak{k}_0), \\ \mathrm{Ad}(g^2)S_{\lambda,i} &= -S_{\lambda,i} \quad (\lambda \in \Sigma^+, 1 \le i \le m(\lambda)), \\ \mathrm{Ad}(g^2)Y_{\alpha,j} &= -Y_{\alpha,j} \quad (\alpha \in W^+, 1 \le j \le n(\alpha)). \end{aligned}$$

Thus, we have $\mathrm{Ad}(g^2)\mathfrak{k}_2 = \mathfrak{k}_2$. Since K_2 is connected, we have $g^2 K_2 g^{-2} = K_2$. In addition, since $\theta_1\theta_2 = \theta_2\theta_1$, we have $\theta_1 \mathfrak{k}_2 = \mathfrak{k}_2$. Thus, we also have $\theta_1(K_2) = K_2$. Therefore, for $(k_2, k_1) \in K_2 \times K_1$,

$$\sigma(k_2 g k_1^{-1}) = (g^2 \theta_1(k_2) g^{-2}) g k_1^{-1} \in K_2 g K_1.$$

Hence, $\sigma(K_2 g K_1) = K_2 g K_1$. Since $T_g^{\perp}(K_2 g K_1) = dL_g(\mathrm{Ad}(g)^{-1}(\mathfrak{m}_2) \cap \mathfrak{m}_1)$, we have

$$d\sigma(\xi) = dL_g \theta_1(dL_g^{-1}(\xi)) = -dL_g dL_g^{-1}(\xi) = -\xi$$

Therefore, σ is a reflection of $K_2 g K_1$ at g with respect to each normal vector $dL_g \xi \in T_g^{\perp}(K_2 g K_1)$. □

Corollary 3.8 ([O] Corollary 5)**.** *The orbit $K_2 e K_1 \subset G$ is weakly reflective.*

Remark 3.9. Under the same condition as Theorem 3.7, we can prove that the orbits $K_2 \pi_1(g) \subset M_1$ and $K_1 \pi_2(g) \subset M_2$ are weakly reflective. However, Ikawa proved $K_2 \pi_1(g) \subset M_1$ and $K_1 \pi_2(g) \subset M_2$ are reflective. Hence $K_2 \pi_1(g) \subset M_1$ and $K_1 \pi_2(g) \subset M_2$ are totally geodesic, but $K_2 g K_1$ is not necessarily totally geodesic.

Let $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ be a subgroup of the affine group $\mathrm{O}(\mathfrak{a}) \ltimes \mathfrak{a}$ which is generated by

$$\left\{ \left(s_\lambda, \frac{2n\pi}{\|\lambda\|^2}\lambda \right) \;\middle|\; \lambda \in \Sigma,\ n \in \mathbb{Z} \right\} \cup \left\{ \left(s_\alpha, \frac{(2n+1)\pi}{\|\alpha\|^2}\alpha \right) \;\middle|\; \alpha \in W,\ n \in \mathbb{Z} \right\}.$$

Then, we have the following lemma.

Lemma 3.10 ([I] Lemmas 4.4 and 4.21)**.**

$$\tilde{W}(\tilde{\Sigma}, \Sigma, W) \subset \tilde{J}$$

Using the above lemma, we have the following lemma.

Lemma 3.11 ([O] Lemma 5)**.** *Let* $g = \exp(H)$ $(H \in \mathfrak{a})$. *Then, for each* $\lambda \in \tilde{\Sigma}_H$*, there exists* $k_\lambda \in N_{K_2}(\mathfrak{a})$*, such that*

(1)

$$\left(k_\lambda,\ \exp\left(-\frac{2\langle\lambda, H\rangle}{\langle\lambda,\lambda\rangle}\lambda\right)k_\lambda\right) \in (K_2 \times K_1)_g,$$

(2)

$$d\left(k_\lambda,\ \exp\left(-\frac{2\langle\lambda, H\rangle}{\langle\lambda,\lambda\rangle}\lambda\right)k_\lambda\right)_g (dL_g\xi) = dL_g(s_\lambda\xi) \quad (\xi \in \mathfrak{a}).$$

Proof. By the definition of $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$, for each $\lambda \in \tilde{\Sigma}_H$,

$$\left(s_\lambda, 2\frac{\langle\lambda, H\rangle}{\langle\lambda,\lambda\rangle}\lambda\right) \in \tilde{W}(\tilde{\Sigma}, \Sigma, W).$$

Since $\tilde{W}(\tilde{\Sigma}, \Sigma, W) \subset \tilde{J}$, there exists $k_\lambda \in N_{K_2}(\mathfrak{a})$, such that

$$\left([k_\lambda], 2\frac{\langle\lambda, H\rangle}{\langle\lambda,\lambda\rangle}\lambda\right) = \left(s_\lambda, 2\frac{\langle\lambda, H\rangle}{\langle\lambda,\lambda\rangle}\lambda\right).$$

By the definition of $\tilde{J}$, we have

$$\exp\left(-2\frac{\langle\lambda, H\rangle}{\langle\lambda,\lambda\rangle}\lambda\right)k_\lambda \in K_1.$$

For (1),

$$\begin{aligned}&\left(k_\lambda,\ \exp\left(-\frac{2\langle\lambda, H\rangle}{\langle\lambda,\lambda\rangle}\lambda\right)k_\lambda\right)g = k_\lambda \exp(H) k_\lambda^{-1} \exp\left(\frac{2\langle\lambda, H\rangle}{\langle\lambda,\lambda\rangle}\lambda\right)\\ &= \exp\left(\mathrm{Ad}(k_\lambda)H\right)\exp\left(\frac{2\langle\lambda, H\rangle}{\langle\lambda,\lambda\rangle}\lambda\right) = \exp\left(s_\lambda H + \frac{2\langle\lambda, H\rangle}{\langle\lambda,\lambda\rangle}\right) = \exp(H) = g.\end{aligned}$$

For (2),

$$d\left(k_\lambda,\ \exp\left(-\frac{2\langle\lambda, H\rangle}{\langle\lambda,\lambda\rangle}\lambda\right)k_\lambda\right)_g (dL_g\xi) = \frac{d}{dt}\exp\left(H + ts_\lambda(\xi)\right)\Big|_{t=0} = dL_g s_\lambda(\xi).$$

□

Proposition 3.12 ([O] Proposition 6)**.** *For any* $H \in \mathfrak{a}$*, if* $\tilde{\Sigma}_H$ *is nonempty, then* $\tilde{\Sigma}_H$ *is a root system of* $\mathrm{Span}(\tilde{\Sigma}_H)$.

Proof. We set $g = \exp(H)$. We consider the orthogonal symmetric Lie algebra

$$((\mathrm{Ad}(g)^{-1}\mathfrak{k}_2) \cap \mathfrak{k}_1) \oplus ((\mathrm{Ad}(g)^{-1}\mathfrak{m}_2) \cap \mathfrak{m}_1).$$

By Lemma 2.7, we can decompose the Lie algebra as the following:

$$\left(\mathfrak{k}_0 \oplus \sum_{\lambda\in\Sigma_H^+}\mathfrak{k}_\lambda \oplus \sum_{\alpha\in W_H^+} V_\alpha^\perp(\mathfrak{k}_1\cap\mathfrak{m}_2)\right) \oplus \left(\mathfrak{a} \oplus \sum_{\lambda\in\Sigma_H^+}\mathfrak{m}_\lambda \oplus \sum_{\alpha\in W_H^+} V_\alpha^\perp(\mathfrak{m}_1\cap\mathfrak{k}_2)\right).$$

It is the root space decomposition of the orthogonal symmetric Lie algebra with respect to $\mathfrak{a}$. □

For each $H \in \mathfrak{a}$, denote by $W(\tilde{\Sigma}_H)$ the Weyl group of $\tilde{\Sigma}_H$. The second sufficient condition is the following:

Theorem 3.13 ([O] Theorem 5)**.** *Let $g = \exp(H)$ $(H \in \mathfrak{a})$. If* $\mathrm{span}(\tilde{\Sigma}_H)$ *and* $-\mathrm{id}_{\mathfrak{a}} \in W(\tilde{\Sigma}_H)$, *then $K_2gK_1 \subset G$, $K_2\pi_1(g) \subset M_1$ and $K_1\pi_2(g) \subset M_2$ are weakly reflective.*

Proof. By the equation (2.7), it is sufficient to prove the existence of a reflection with respect to $dL_g\xi$ for each $\xi \in \mathfrak{a}$. Since $-\mathrm{id}_{\mathfrak{a}} \in W(\tilde{\Sigma}_H)$, there exist $\mu_1, \dots, \mu_l \in \tilde{\Sigma}_H$ such that $s_{\mu_1} \cdots s_{\mu_l} = -\mathrm{id}_{\mathfrak{a}}$. By Lemma 3.11, there exists $k_{\mu_i} \in N_{K_2}(\mathfrak{a})$ for each μ_i $(1 \leq i \leq l)$. We set

$$k'_{\mu_i} = \exp\left(-2\frac{\langle \mu_i, H\rangle}{\langle \mu_i, \mu_i\rangle}\mu_i\right) k_{\mu_i} \in K_1,$$

and

$$\sigma = (k_{\mu_1},\ k'_{\mu_1}) \cdots (k_{\mu_l},\ k'_{\mu_l}) \in (K_2 \times K_1).$$

Then, σ is a reflection of K_2gK_1 with respect to $dL_g\xi$ for each $\xi \in \mathfrak{a}$. Indeed,

$$\sigma(g) = g, \quad \sigma(K_2gK_1) = K_2gK_1, \quad d\sigma(dL_g(\xi)) = dL_g s_{\mu_1} \cdots s_{\mu_l}(\xi) = -dL_g\xi$$

hold. Similarly, $\sigma_1 = k_{\mu_1} \cdots k_{\mu_l}$ is a reflection of $K_2\pi_1(g)$ at $\pi_1(g)$ with respect to $dL_g(\xi)$. The isometry $\sigma_2 = k'_{\mu_1} \cdots k'_{\mu_l}$ is a reflection of $K_1\pi_2(g)$ at $\pi_2(g)$ with respect to $dR_g(\xi)$. □

Applying Theorems 3.13 and 3.7, we have new examples of weakly reflective submanifolds in compact symmetric spaces. We assume that (G, K_1, K_2) satisfies one of the following conditions (A), (B) or (C).

- **(A):** G is simple and θ_1 and θ_2 can not transform each other by an inner automorphism of $\mathfrak{g}$.
- **(B):** There exist a compact connected simple Lie group U and a symmetric subgroup $\overline{K}$ of U such that

$$G = U \times U, \quad K_1 = \Delta G = \{(u, u) \mid u \in U\}, \quad K_2 = \overline{K} \times \overline{K}.$$

- **(C):** There exist a compact connected simple Lie group U and an involutive outer automorphism σ such that

$$G = U \times U, \quad K_1 = \Delta G = \{(u, u) \mid u \in U\},$$
$$K_2 = \{(u_1, u_2) \mid (\sigma(u_2),\ \sigma(u_1)) = (u_1, u_2)\}.$$

Ikawa proved the following theorem.

Theorem 3.14 ([I2] Theorem 3.1)**.** *Let (G, K_1, K_2) be a compact symmetric triad which satisfies one of the conditions (A), (B) or (C). Then the triple $(\tilde{\Sigma}, \Sigma, W)$ defined as above is a symmetric triad with multiplicities. Conversely every symmetric triad is obtained in this way.*

It is known the following proposition.

Proposition 3.15 ([Ti])**.** *Let Σ be a irreducible root system of $\mathfrak{a}$. Then $-\mathrm{id}_{\mathfrak{a}} \notin W(\Sigma)$ if and only if $\Sigma \cong \mathrm{A}_r,\ \mathrm{D}_{2r+1},\ \mathrm{E}_6$ $(r \geq 2)$.*

Let $\Pi = \{\lambda_1, \dots, \lambda_r\}$ be a fundamental system of Σ, and set $W_0 = \{\tilde{\alpha}\}$. We define $H_i \in \mathfrak{a}$ by the following equations:

$$\langle H_i, \lambda_j \rangle = 0 \ (i \neq j), \ \langle H_i, \ \tilde{\alpha} \rangle = \pi/2.$$

Then, $\{H_1, \dots, H_r\}$ is a basis of $\mathfrak{a}$. We have the following lemma.

Lemma 3.16 ([O] Lemma 6)**.** Span$(\tilde{\Sigma}_H) = \mathfrak{a}$ *if and only if* $H = 0, H_1, \dots, H_r$ *for* $H \in \overline{P}_0$.

Proof. By definition of $\tilde{\Sigma}_H$, we have

$$\left(s_{\mu_i}, \frac{2\langle \lambda, H \rangle}{\langle \lambda, \lambda \rangle} \lambda \right) \in \tilde{W}(\tilde{\Sigma}, \Sigma, W), \ \left(s_{\mu_i}, \frac{2\langle \lambda, H \rangle}{\langle \lambda, \lambda \rangle} \lambda \right) H = H$$

for each $\lambda \in \tilde{\Sigma}_H$. By Proposition 2.4, we have $s_\lambda m_H = m_H$ for $\lambda \in \tilde{\Sigma}_H$. Thus, if Span$(\tilde{\Sigma}_H) = \mathfrak{a}$, then $m_H = 0$. On the other hand, for $H \in \overline{P}_0$, there exists the nonempty subset $\Delta \subset \Pi \cup \{\tilde{\alpha}\}$ such that $H \in P_0^\Delta$. By Lemma 2.25 in [I], Σ_H and W_H does not depend on H, but only Δ. Thus, when Span$(\tilde{\Sigma}_H) = \mathfrak{a}$, each point in P_0^Δ is a minimal point. Therefore, by Theorem 2.5, if when Span$(\tilde{\Sigma}_H) = \mathfrak{a}$, then $P_0^\Delta = \{H\}$. This implies that H is a vertex of $\overline{P}_0$. Therefore, $H = 0, H_1, \dots, H_r$. Conversely, when $H = 0, H_1, \dots, H_r$, we have Span$(\tilde{\Sigma}_H) = \mathfrak{a}$. □

For each symmetric triad of $\mathfrak{a}$, austere points are classified in [I]. Using the classification, we investigate $\tilde{\Sigma}_{H_i}$ $(1 \leq i \leq r)$ for each type of symmetric triads.

In order to state our results below, we shall follow the notations of irreducible root systems and the set of positive roots in [Bo]. For instance,

$$\begin{aligned}
\mathrm{B}_r^+ &= \{e_i \pm e_j \mid 1 \leq i < j \leq r\} \cup \{e_i \mid 1 \leq i \leq r\}, \\
\mathrm{C}_r^+ &= \{e_i \pm e_j \mid 1 \leq i < j \leq r\} \cup \{2e_i \mid 1 \leq i \leq r\}, \\
\mathrm{D}_r^+ &= \{e_i \pm e_j \mid 1 \leq i < j \leq r\}, \\
\mathrm{BC}_r^+ &= \{e_i \pm e_j \mid 1 \leq i < j \leq r\} \cup \{e_i \mid 1 \leq i \leq r\} \cup \{2e_i \mid 1 \leq i \leq r\}.
\end{aligned}$$

For the set of positive roots above, the sets of simple roots are given as follows:

$$\begin{aligned}
&\Pi(\mathrm{B}_r^+) = \Pi(\mathrm{BC}_r^+) = \{\lambda_1 = e_1 - e_2, \dots, \lambda_{r-1} = e_{r-1} - e_r, \lambda_r = e_r\}, \\
&\Pi(\mathrm{C}_r^+) = \{\lambda_1 = e_1 - e_2, \dots, \lambda_{r-1} = e_{r-1} - e_r, \lambda_r = 2e_r\}, \\
&\Pi(\mathrm{D}_r^+) = \{\lambda_1 = e_1 - e_2, \dots, \lambda_{r-1} = e_{r-1} - e_r, \lambda_r = e_{r-1} + e_r\}.
\end{aligned}$$

3.2.1. *Type* I-B_r. $\Sigma^+ = \mathrm{B}_r^+$, $W^+ = \{e_i \mid 1 \leq i \leq r\}$,

$$\tilde{\alpha} = e_1 = \lambda_1 + \cdots + \lambda_r.$$

(1) When $m(\pm e_i) = n(\pm e_i)$. A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = (1/2)H_r$. Since span$(\tilde{\Sigma}_H) \neq \mathfrak{a}$, the point $(1/2)H_r$ does not satisfies the sufficient condition in Theorem 3.13.
(2) When $m(\pm e_i) \neq n(\pm e_i)$. If $H \in \overline{P_0}$ is austere then it is totally geodesic. In this case, H_i is a totally geodesic point for each $1 \leq i \leq r$.

A compact symmetric triad whose symmetric triad is type I-B_r is one of the following:

(1) $(\mathrm{SO}(r+s+t), \ \mathrm{SO}(r+s) \times \mathrm{SO}(t), \ \mathrm{SO}(r) \times \mathrm{SO}(s+t)) \quad (r < t, \ 1 \leq s)$,

(2) (G, K_1, K_2) which satisfies condition (C) where

$$(U, \mathrm{Fix}(\sigma)) = (\mathrm{SO}(2m+2n+2), \mathrm{SO}(2m+1) \times \mathrm{SO}(2n+1))$$

for $r = m+n,\ m \geq 2$.

3.2.2. *Type* I-C_r. $\Sigma^+ = \mathrm{C}_r^+$, $W^+ = \mathrm{D}_r^+$,

$$\tilde{\alpha} = e_1 + e_2 = \lambda_1 + 2\lambda_2 + \cdots 2\lambda_{r-1} + \lambda_r.$$

Then a point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_i\ (2 \leq i \leq r-1),\ (1/2)H_1$. For each $H_i = (\pi/4)(e_1 + \cdots + e_i) \quad (2 \leq i \leq r-1)$, we have

$$\begin{aligned}\Sigma_{H_i}^+ =& \{e_s - e_t \mid 1 \leq s < t \leq i\} \cup \{e_s \pm e_t \mid i+1 \leq s < t \leq r\} \\ & \cup \{2e_s \mid i+1 \leq s \leq r\}, \\ W_{H_i}^+ =& \{e_s + e_t \mid 1 \leq s < t \leq i\}.\end{aligned}$$

Hence, $\tilde{\Sigma}_{H_i} \cong \mathrm{D}_i \oplus \mathrm{C}_{r-i}$. Therefore, by Proposition 3.15 and Theorem 3.13, if i is even, then $K_2 \exp(H_i) K_1 \subset G,\ K_2\pi_1(\exp(H_i)) \subset M_1,\ K_1\pi_2(\exp(H_i)) \subset M_2$ are weakly reflective. When i is odd, since $-\mathrm{id}_{\mathfrak{a}} \notin W(\Sigma)$, H_i does not satisfies the sufficient condition in Theorem 3.13.

A compact symmetric triad whose symmetric triad is type I-C_r is one of the following:

(1) $(\mathrm{SO}(4r),\ \mathrm{SO}(2r) \times \mathrm{SO}(2r),\ \mathrm{U}(2r))$,
(2) $(\mathrm{SU}(2r),\ \mathrm{SO}(2r),\ \mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(r)))$,
(3) $(E_7,\ \mathrm{SU}(8),\ E_6 \cdot \mathrm{U}(1)) \quad (r = 3)$,
(4) (G, K_1, K_2) which satisfies condition (C) where

$$\begin{aligned}(U, \mathrm{Fix}(\sigma)) =& (\mathrm{SU}(2r), \mathrm{SO}(2r)) \quad (r \geq 2) \text{ or} \\ & (\mathrm{SU}(2r), \mathrm{Sp}(r)) \quad (r \geq 2).\end{aligned}$$

3.2.3. *Type* I-BC_r-A_1^r. $\Sigma^+ = \mathrm{BC}_r^+$, $W^+ = \{e_i \mid 1 \leq i \leq r\}$,

$$\tilde{\alpha} = e_1 = \lambda_1 + \cdots + \lambda_r.$$

(1) When $m(\pm e_i) = n(\pm e_i)$. A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = (1/2)H_r$. Since $\mathrm{span}(\tilde{\Sigma}_H) \neq \mathfrak{a}$, H does not satisfies the sufficient condition in Theorem 3.13
(2) When $m(\pm e_i) \neq n(\pm e_i)$. If $H \in \overline{P_0}$ is austere then it is totally geodesic. In this case, H_i is a totally geodesic point for each $1 \leq i \leq r$.

A compact symmetric triad whose symmetric triad is type I-BC_r-A_1^r is one of the following:

(1) $(\mathrm{SU}(r+s+t),\ \mathrm{S}(\mathrm{U}(r+s) \times \mathrm{U}(t)),\ \mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(s+t))) \quad (r < t,\ 1 \leq s)$,
(2) $(\mathrm{Sp}(r+s+t),\ \mathrm{Sp}(r+s) \times \mathrm{Sp}(t),\ \mathrm{Sp}(r) \times \mathrm{Sp}(s+t)) \quad (r < t,\ 1 \leq s))$,
(3) $(\mathrm{SO}(4r+4),\ \mathrm{U}(2r+2),\ \mathrm{U}'(2r+2))$.

Where, we set

$$J = \left[\begin{array}{cc|cc} & & I_{n-1} & \\ & & & -1 \\ \hline -I_{n-1} & & & \\ & 1 & & \end{array}\right],$$

and define $\mathrm{U}(n)' := \{g \in \mathrm{SO}(2n) \mid JgJ^{-1} = g\}$.

3.2.4. *Type* I-BC$_r$-B$_r$. $\Sigma^+ = \mathrm{BC}_r^+$, $W^+ = \mathrm{B}_r^+$,

$$\tilde{\alpha} = e_1 + e_2 = \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_r.$$

When $r = 2$, if $m(\pm e_1 \pm e_2) = n(\pm e_1 \pm e_2)$, then $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = (1/2)H_1,\ H_2$. If $m(\pm e_1 \pm e_2) \neq n(\pm e_1 \pm e_2)$, then $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_2$. Since $H_2 = (\pi/4)(e_1 + e_2)$, we have $\Sigma^+_{H_2} = \{e_1 - e_2\}$, $W^+_{H_2} = \{e_1 + e_2\}$. Thus $\tilde{\Sigma}_{H_2} \cong \mathrm{A}_1^2$.

When $r \geq 3$, $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = (1/2)H_1,\ H_i \quad (2 \leq i \leq r)$. For each $H_i = (\pi/4)(e_1 + \cdots + e_i) \quad (2 \leq i \leq r)$, we have $\tilde{\Sigma}_{H_i} \cong \mathrm{D}_i \oplus \mathrm{BC}_1^{r-i}$. Therefore, by Proposition 3.15 and Theorem 3.13, if i is even, then $K_2 \exp(H_i)K_1 \subset G,\ K_2\pi_1(\exp(H_i)) \subset M_1,\ K_1\pi_2(\exp(H_i)) \subset M_2$ are weakly reflective for each $2 \leq i \leq r$. When i is odd, since $-\mathrm{id}_{\mathfrak{a}} \notin W(\Sigma)$, H_i does not satisfies the sufficient condition in Theorem 3.13 for $3 \leq i \leq r$. Since $\mathrm{span}(\tilde{\Sigma}_{(1/2)H_1}) \neq \mathfrak{a}$, the point $(1/2)H_1$ does not satisfies the sufficient condition in Theorem 3.13.

A compact symmetric triad whose symmetric triad is type I-BC$_r$-B$_r$ is one of the following:

(1) $(\mathrm{SO}(2r + 2s),\ \mathrm{S}(\mathrm{O}(2r) \times \mathrm{O}(2s)),\ \mathrm{U}(r + s)) \quad (r < s)$,
(2) $(E_6,\ \mathrm{SU}(6) \cdot \mathrm{SU}(2),\ \mathrm{SO}(10) \cdot \mathrm{U}(1)) \quad (r = 2)$,
(3) $(E_7,\ \mathrm{SO}(12) \cdot \mathrm{SU}(2),\ E_6 \cdot \mathrm{U}(1)) \quad (r = 2)$.

3.2.5. *Type* I-F$_4$. $\Sigma^+ = \mathrm{F}_4^+$, $W^+ = \{\text{short roots in } \mathrm{F}_4\} \cong \mathrm{D}_4$, $\Pi = \{\lambda_1 = e_2 - e_3, \lambda_2 = e_3 - e_4, \lambda_3 = e_4, \lambda_4 = (1/2)(e_1 - e_2 - e_3 - e_4)\}$, $\tilde{\alpha} = e_1 = \lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_4$. A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_4 = (\pi/2)e_1$. Then we have

$$\begin{aligned} \Sigma_{H_4} &= \{\pm e_2, \pm e_3, \pm e_4, \pm(e_2 \pm e_3), \pm(e_2 \pm e_4), \pm(e_3 \pm e_4)\}, \\ W_{H_4} &= \{\pm e_1, \pm(e_1 \pm e_2), \pm(e_1 \pm e_3), \pm(e_1 \pm e_4)\}. \end{aligned}$$

Hence

$$\tilde{\Sigma}^+_{H_4} \cong \mathrm{B}_4^+.$$

Therefore, by Proposition 3.15 and Theorem 3.13, the orbits $K_2 \exp(H_4)K_1 \subset G$, $K_2\pi_1(\exp(H_4)) \subset M_1$ and $K_1\pi_2(\exp(H_4)) \subset M_2$ are weakly reflective.

A compact symmetric triad whose symmetric triad is type I-F$_4$ is one of the following:

(1) $(E_6,\ \mathrm{Sp}(4),\ \mathrm{SU}(6) \cdot \mathrm{SU}(2))$,
(2) $(E_7,\ \mathrm{SU}(8),\ \mathrm{SO}(12) \cdot \mathrm{SU}(2))$,
(3) $(E_8,\ \mathrm{SO}(16),\ E_7 \cdot \mathrm{SU}(2))$,
(4) (G, K_1, K_2) which satisfies condition (C) where

$$(U, \mathrm{Fix}(\sigma)) = (E_6, \mathrm{Sp}(4)) \text{ or } (E_6, F_4).$$

3.2.6. *Type* II-BC$_r$. $\Sigma^+ = \mathrm{B}_r^+$, $W^+ = \mathrm{BC}_r^+$,

$$\tilde{\alpha} = 2e_1 = 2\lambda_1 + \cdots + 2\lambda_r.$$

A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_i$ $(1 \leq i \leq r)$. For $H_i = (\pi/4)(e_1 + \cdots + e_i)$, we have $\tilde{\Sigma}^+_{H_i} \cong \mathrm{C}_i \oplus \mathrm{B}_{r-i}$. Therefore, by Proposition 3.15 and Theorem 3.13, $K_2 \exp(H_i)K_1 \subset G,\ K_2\pi_1(\exp(H_i)) \subset M_1,\ K_1\pi_2(\exp(H_i)) \subset M_2$ are weakly reflective for each $1 \leq i \leq r$.

A compact symmetric triad whose symmetric triad is type II-BC_r is one of the following:

(1) $(\mathrm{SU}(r+s),\ \mathrm{SO}(r+s),\ \mathrm{S}(\mathrm{U}(r)\times \mathrm{U}(s)))\quad (r<s)$,
(2) $(\mathrm{SO}(4r+2),\ \mathrm{SO}(2r+1)\times \mathrm{SO}(2r+1),\ \mathrm{U}(2r+1))$,
(3) $(E_6,\ \mathrm{Sp}(4),\ \mathrm{SO}(10)\cdot \mathrm{U}(1))\quad (r=2)$.

3.2.7. *Type* III-A_r. By Proposition 3.15, $-\mathrm{id}_{\mathfrak{a}} \notin W(\tilde{\Sigma})$. Moreover, for each $H \in \mathfrak{a}$, $W(\tilde{\Sigma}_H) \subset W(\tilde{\Sigma})$ since $\tilde{\Sigma}_H \subset \tilde{\Sigma}$. Hence $-\mathrm{id}_{\mathfrak{a}} \notin W(\tilde{\Sigma}_H)$. Thus, any austere point does not satisfies the sufficient condition in Theorem 3.13.

A compact symmetric triad whose symmetric triad is type III-A_r is one of the following:

(1) $(\mathrm{SU}(2r+2),\ \mathrm{Sp}(r+1),\ \mathrm{SO}(2r+2))$,
(2) $(E_6,\ \mathrm{Sp}(4),\ F_4)\quad (r=2)$,
(3) $(U\times U, \Delta(U\times U), \overline{K}\times\overline{K})$ where $(U,\overline{K})$ is a compact symmetric pair whose root system is type A_r (condition (B)).

3.2.8. *Type* III-B_r. $\Sigma^+ = W^+ = \mathrm{B}_r^+$,

$$\tilde{\alpha} = e_1 + e_2 = \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_r.$$

A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = (1/2)H_1,\ H_i\ (2 \le i \le r)$.

For each $H_i = (\pi/4)(e_1 + \cdots + e_i)$, we have $\tilde{\Sigma}_{H_i} \cong \mathrm{D}_i \oplus \mathrm{B}_{r-i}$. Therefore, by Proposition 3.15 and Theorem 3.13, if i is even, then orbits $K_2\exp(H_i)K_1 \subset G,\ K_2\pi_1(\exp(H_i)) \subset M_1,\ K_1\pi_2(\exp(H_i)) \subset M_2$ are weakly reflective for each $2 \le i \le r$. When i is odd, since $-\mathrm{id}_{\mathfrak{a}} \notin W(\Sigma)$, H_i does not satisfies the sufficient condition in Theorem 3.13. Since $\mathrm{span}(\tilde{\Sigma}_{H_1}) \neq \mathfrak{a}$, the point $(1/2)H_1$ does not satisfies the sufficient condition in Theorem 3.13.

A compact symmetric triad whose symmetric triad is type III-B_r is one of the following:

(1) $(U\times U, \Delta(U\times U), \overline{K}\times\overline{K})$ where $(U,\overline{K})$ is a compact symmetric pair whose root system is type B_r (condition (B)).

3.2.9. *Type* III-C_r. $\Sigma^+ = W^+ = \mathrm{C}_r^+$,

$$\tilde{\alpha} = 2e_1 = 2\lambda_1 + \cdots + 2\lambda_{r-1} + \lambda_r.$$

If $m(\pm 2e_i) \neq n(\pm 2e_i)$, then a point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_i\ (1 \le i \le r-1)$. If $m(\pm 2e_i) = n(\pm 2e_i)$, then $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = (1/2)H_r, H_i\ (1 \le i \le r-1)$. For each $H_i = (\pi/4)(e_1 + \cdots + e_i)\ (1 \le i \le r-1)$, we have $\tilde{\Sigma}_{H_i} \cong \mathrm{C}_i \oplus \mathrm{C}_{r-i}$. Therefore, by Proposition 3.15 and Theorem 3.13, $K_2\exp(H_i)K_1 \subset G,\ K_2\pi_1(\exp(H_i)) \subset M_1,\ K_1\pi_2(\exp(H_i)) \subset M_2$ are weakly reflective for each $1 \le i \le r-1$. Since $\mathrm{span}(\tilde{\Sigma}_{(1/2)H_r}) \neq \mathfrak{a}$, the point $(1/2)H_r$ does not satisfies the sufficient condition in Theorem 3.13. A compact symmetric triad whose symmetric triad is type III-C_r is one of the following:

(1) $(\mathrm{SU}(4r),\ \mathrm{S}(\mathrm{U}(2r)\times \mathrm{U}(2r)),\ \mathrm{Sp}(2r))$,
(2) $(\mathrm{Sp}(2r),\ \mathrm{U}(2r),\ \mathrm{Sp}(r)\times \mathrm{Sp}(r))$,
(3) $(U\times U, \Delta(U\times U), \overline{K}\times\overline{K})$ where $(U,\overline{K})$ is a compact symmetric pair whose root system is type C_r (condition (B)).

3.2.10. *Type* III-BC_r. $\Sigma^+ = W^+ = \mathrm{BC}_r^+$,

$$\tilde{\alpha} = 2e_1 = 2\lambda_1 + \cdots + 2\lambda_r.$$

A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_i$ $(1 \leq i \leq r)$. For each $H_i = (\pi/4)(e_1 + \cdots + e_i)$ $(1 \leq i \leq r)$, we have $\tilde{\Sigma}_{H_i} \cong \mathrm{C}_i \oplus \mathrm{BC}_{r-i}$. Therefore, by Proposition 3.15 and Theorem 3.13, $K_2 \exp(H_i) K_1 \subset G$, $K_2\pi_1(\exp(H_i)) \subset M_1$, $K_1\pi_2(\exp(H_i)) \subset M_2$ are weakly reflective for each $1 \leq i \leq r$.

A compact symmetric triad whose symmetric triad is type III-BC_r is one of the following:

(1) $(\mathrm{SU}(2r+2s),\ \mathrm{S}(\mathrm{U}(2r) \times \mathrm{U}(2s)),\ \mathrm{Sp}(r+s))$ $(r < s)$,
(2) $(\mathrm{SU}(2(2r+1)),\ \mathrm{S}(\mathrm{U}(2r+1) \times \mathrm{U}(2r+1)),\ \mathrm{Sp}(2r+1))$ $(1 \leq r)$,
(3) $(\mathrm{Sp}(r+s),\ \mathrm{U}(r+s),\ \mathrm{Sp}(r) \times \mathrm{Sp}(s))$ $(r < s)$,
(4) $(E_6,\ \mathrm{SU}(6) \cdot \mathrm{SU}(2),\ F_4)$ $(r = 1)$,
(5) $(E_6,\ \mathrm{SO}(10) \cdot \mathrm{U}(1), F_4)$ $(r = 1)$,
(6) $(F_4,\ \mathrm{Sp}(3) \cdot \mathrm{Sp}(1), \mathrm{SO}(9))$ $(r = 1)$,
(7) $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$ where $(U, \overline{K})$ is a compact symmetric pair whose root system is type BC_r (condition (B)).

3.2.11. *Type* III-D_r. $\Sigma^+ = W^+ = \mathrm{D}_r^+$,

$$\tilde{\alpha} = e_1 + e_2 = \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r.$$

A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if H_i $(2 \leq i \leq r-1)$, $(1/2)H_1$, $(1/2)H_{r-1}$, $(1/2)H_r$, $(1/2)(H_1+H_{r-1})$, $(1/2)(H_1+H_r)$, $(1/2)(H_{r-1}+H_r)$. For each $H_i = (\pi/4)(e_1 + \cdots + e_i)$ $(2 \leq i \leq r-2)$, we have $\tilde{\Sigma}_{H_i} \cong \mathrm{D}_i \oplus \mathrm{D}_{r-i}$. Therefore, by Proposition 3.15 and Theorem 3.13, if r and i are even, then $K_2 \exp(H_i) K_1 \subset G$, $K_2\pi_1(\exp(H_i)) \subset M_1$, $K_1\pi_2(\exp(H_i)) \subset M_2$ are weakly reflective for each $1 \leq i \leq r$. When $H = H_i$ (i or r is odd), $(1/2)H_1$, $(1/2)H_{r-1}$, $(1/2)H_r$, $(1/2)(H_1+H_{r-1})$, $(1/2)(H_1+H_r)$, $(1/2)(H_{r-1}+H_r)$, H does not satisfies the sufficient condition in Theorem 3.13.

A compact symmetric triad whose symmetric triad is type III-D_r is one of the following:

(1) $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$ where $(U, \overline{K})$ is a compact symmetric pair whose root system is type D_r (condition (B)).

3.2.12. *Type* III-E_6. By Proposition 3.15, $-\mathrm{id}_{\mathfrak{a}} \notin W(\tilde{\Sigma})$. Moreover, for each $H \in \mathfrak{a}$, $W(\tilde{\Sigma}_H) \subset W(\tilde{\Sigma})$ since $\tilde{\Sigma}_H \subset \tilde{\Sigma}$. Hence $-\mathrm{id}_{\mathfrak{a}} \notin W(\tilde{\Sigma}_H)$. Thus, each austere point does not satisfies the sufficient condition in Theorem 3.13.

A compact symmetric triad whose symmetric triad is type III-E_6 is one of the following:

(1) $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$ where $(U, \overline{K})$ is a compact symmetric pair whose root system is type E_6 (condition (B)).

3.2.13. *Type* III-E_7. $\Sigma^+ = W^+ = \mathrm{E}_7^+$, $\Pi = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$,

$$\tilde{\alpha} = 2\lambda_1 + 2\lambda_2 + 4\lambda_3 + 4\lambda_4 + 3\lambda_5 + 2\lambda_6 + \lambda_7.$$

A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_1, H_2, H_6, (1/2)H_7$. Since $\mathrm{span}(\tilde{\Sigma}_{(1/2)H_7}) \neq \mathfrak{a}$, the point $(1/2)H_7$ does not satisfies the sufficient condition in Theorem 3.13.

(1) When $H = H_1$. We have $\Sigma^+_{H_1} = \Sigma^+ \cap \mathrm{span}_{\mathbb{Z}}\{\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$, $W^+_{H_1} = \{\tilde{\alpha}\}$. Since $\langle \tilde{\alpha}, \lambda_i \rangle = 0$ $(2 \leq i \leq 7)$, $\Sigma_{H_1} \perp W_{H_1}$. Hence, $\tilde{\Sigma}_{H_1}$ is isomorphic to $\Sigma_{H_1} \oplus W_{H_1}$ as a root system. Since $\{\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$ is a fundamental system of Σ_{H_1}, we can see $\Sigma_{H_1} \cong \mathrm{D}_6$. Hence, we have $\tilde{\Sigma}_{H_1} \cong \mathrm{D}_6 \oplus \mathrm{A}_1$. Therefore, by Proposition 3.15 and Theorem 3.13, $K_2 \exp(H_1) K_1 \subset G$, $K_2 \pi_1(\exp(H_1)) \subset M_1$, $K_1 \pi_2(\exp(H_1)) \subset M_2$ are weakly reflective.
(2) When $H = H_2$. We have

$$\begin{aligned}\Sigma^+_{H_2} =& \Sigma^+ \cap \mathrm{span}_{\mathbb{Z}}\{\lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\},\\ W_{H_2} =& \{\lambda_0,\ \lambda_0 + \lambda_7,\ \lambda_0 + \lambda_6 + \lambda_7,\ \lambda_0 + \lambda_5 + \lambda_6 + \lambda_7,\ \lambda_0 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7,\\ & \lambda_0 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7,\ \lambda_0 + \lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7\},\end{aligned}$$

where $\lambda_0 = \lambda_1 + 2\lambda_2 + 2\lambda_3 + 3\lambda_4 + 2\lambda_5 + \lambda_6$. Hence,

$$\Pi_{H_2} := \{\lambda_0, \lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$$

is a fundamental system of $\tilde{\Sigma}_{H_2}$. For $i = 1, 3 \leq i \leq 6$, we have $\langle \lambda_0,\ \lambda_i \rangle = 0, \langle \lambda_0,\ \lambda_7 \rangle = \langle \lambda_6,\ \lambda_7 \rangle$. Thus, Π_{H_2} corresponds to the Dynkin diagram of type A_7. Therefore, we obtain $\tilde{\Sigma}_{H_2} \cong \mathrm{A}_7$. By Proposition 3.15, we have $-\mathrm{id}_{\mathfrak{a}} \notin W(\tilde{\Sigma}_{H_2})$. Thus, H_2 does not satisfies the sufficient condition in Theorem 3.13.
(3) When $H = H_6$. Similarly, we set $\lambda_0 = \lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + 2\lambda_6 + \lambda_7$. Then, the set

$$\Pi_{H_6} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_7\}$$

is a fundamental system of $\tilde{\Sigma}_{H_6}$. For $2 \leq i \leq 5, i = 7$, we have $\langle \lambda_0,\ \lambda_i \rangle = 0, \langle \lambda_0,\ \lambda_1 \rangle = \langle \lambda_1,\ \lambda_3 \rangle$. The set Π_{H_6} corresponds to the Dynkin diagram of type $\mathrm{D}_6 \oplus \mathrm{A}_1$. Thus, we have $\tilde{\Sigma}_{H_6} \cong \mathrm{D}_6 \oplus \mathrm{A}_1$. Therefore, by Proposition 3.15 and Theorem 3.13, $K_2 \exp(H_6) K_1 \subset G$, $K_2 \pi_1(\exp(H_6)) \subset M_1$ and $K_1 \pi_2(\exp(H_6)) \subset M_2$ are weakly reflective.

A compact symmetric triad whose symmetric triad is type III-E_7 is one of the following:

(1) $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$ where $(U, \overline{K})$ is a compact symmetric pair whose root system is type E_7 (condition (B)).

3.2.14. *Type* III-E_8. $\Sigma^+ = W^+ = \mathrm{E}_8^+$, $\Pi = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8\}$, $\tilde{\alpha} = 2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 6\lambda_4 + 5\lambda_5 + 4\lambda_6 + 3\lambda_7 + 2\lambda_8$. A point $H \in \overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_1, H_8$.
(1) When $H = H_1$. We set $\lambda_0 = 2\lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_4 + 3\lambda_5 + 2\lambda_6 + \lambda_7$. Then, the set $\Pi_{H_1} = \{\lambda_0, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8\}$ is a fundamental system of $\tilde{\Sigma}_{H_1}$. For each $2 \leq i \leq 7$, we have $\langle \lambda_0,\ \lambda_i \rangle = 0, \langle \lambda_0,\ \lambda_8 \rangle = \langle \lambda_7,\ \lambda_8 \rangle$. Thus Π_{H_1} corresponds to the Dynkin diagram of type D_8. Hence, $\tilde{\Sigma}_{H_1} \cong \mathrm{D}_8$. Therefore, by Proposition 3.15 and Theorem 3.13, we have $K_2 \exp(H_1) K_1 \subset G$, $K_2 \pi_1(\exp(H_1)) \subset M_1$, $K_1 \pi_2(\exp(H_1)) \subset M_2$ are weakly reflective.
(2) When $H = H_8$. We have $\Sigma^+_{H_8} = \Sigma^+ \cap \mathrm{span}_{\mathbb{Z}}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$, $W_{H_8} = \{\tilde{\alpha}\}$. For each $1 \leq i \leq 7$, we have $\langle \tilde{\alpha},\ \lambda_i \rangle = 0$. Thus, $\Sigma_{H_8} \perp W_{H_8}$. Hence $\tilde{\Sigma}_{H_8}$ is isomorphic to $\tilde{\Sigma}_{H_8} \cong \Sigma_{H_8} \oplus W_{H_8}$ as a root system. Since the set of simple roots $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$ is a fundamental system of Σ_{H_8}, we can see that $\Sigma_{H_8} \cong \mathrm{E}_7$. Thus, $\tilde{\Sigma}_{H_8} \cong \mathrm{E}_7 \oplus \mathrm{A}_1$. Therefore, by Proposition 3.15 and Theorem 3.13, $K_2 \exp(H_8) K_1 \subset G$, $K_2 \pi_1(\exp(H_8)) \subset M_1$, $K_1 \pi_2(\exp(H_8)) \subset M_2$ are weakly reflective.

A compact symmetric triad whose symmetric triad is type III-E_8 is one of the following:

(1) $(U\times U, \Delta(U\times U), \overline{K}\times\overline{K})$ where $(U,\overline{K})$ is a compact symmetric pair whose root system is type E_8 (condition (B)).

3.2.15. *Type* III-F_4. $\Sigma^+ = W^+ = F_4^+$, $\Pi = \{\lambda_1 = e_2-e_3, \lambda_2 = e_3-e_4, \lambda_3 = e_4, \lambda_4 = (1/2)(e_1-e_2-e_3-e_4)\}$, $\tilde{\alpha} = e_1+e_2 = 2\lambda_1+3\lambda_2+4\lambda_3+2\lambda_4$. A point $H\in\overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_1 = (\pi/4)(e_1+e_2), H_4 = (\pi/2)e_1$.
(1) When $H = H_1$. We have $\tilde{\Sigma}_{H_1}\cong C_4$. Therefore, by Proposition 3.15 and Theorem 3.13, $K_2\exp(H_1)K_1\subset G,\ K_2\pi_1(\exp(H_1))\subset M_1,\ K_1\pi_2(\exp(H_1))\subset M_2$ are weakly reflective.
(2) When $H = H_4$. We have $\tilde{\Sigma}_{H_4}\cong B_4$. Therefore, by Proposition 3.15 and Theorem 3.13, $K_2\exp(H_4)K_1\subset G,\ K_2\pi_1(\exp(H_4))\subset M_1,\ K_1\pi_2(\exp(H_4))\subset M_2$ are weakly reflective.

A compact symmetric triad whose symmetric triad is type III-F_4 is one of the following:

(1) $(U\times U, \Delta(U\times U), \overline{K}\times\overline{K})$ where $(U,\overline{K})$ is a compact symmetric pair whose root system is type F_4 (condition (B)).

3.2.16. *Type* III-G_2. $\Sigma^+ = W^+ = G_2^+$, $\Pi = \{\lambda_1 = e_1-e_2, \lambda_2 = -2e_1-e_2+e_3\}$, $\tilde{\alpha} = -e_1-e_2+2e_3 = 3\lambda_1+2\lambda_2$.

A point $H\in\overline{P_0}$ is austere which is not totally geodesic if and only if $H = H_2 = (\pi/12)(-e_1-e_2+2e_3) = (\pi/12)(3\lambda_1+2\lambda_2)$. We have $\Sigma^+_{H_2} = \{\lambda_1\}$, $W^+_{H_2} = \{3\lambda_1+2\lambda_2\}$. Thus, $\tilde{\Sigma}^+_{H_2} = \{\lambda_1,\ 3\lambda_1+2\lambda_2\}$ Therefore, by Proposition 3.15 and Theorem 3.13, $K_2\exp(H_2)K_1\subset G,\ K_2\pi_1(\exp(H_2))\subset M_1,\ K_1\pi_2(\exp(H_2))\subset M_2$ are weakly reflective.

A compact symmetric triad whose symmetric triad is type III-G_2 is one of the following:

(1) $(U\times U, \Delta(U\times U), \overline{K}\times\overline{K})$ where $(U,\overline{K})$ is a compact symmetric pair whose root system is type G_2 (condition (B)).

4. Biharmonic submanifolds in compact symmetric spaces

Harmonic maps play a central role in geometry; they are critical points of the energy functional $E(\varphi) = (1/2)\int_M |d\varphi|^2\, v_g$ for smooth maps φ of (M,g) into (N,h). The Euler-Lagrange equations are given by the vanishing of the tension filed $\tau(\varphi)$. In 1983, J. Eells and L. Lemaire [EL1] extended the notion of harmonic map to biharmonic map, which are, by definition, critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2\, v_g.$$

After G.Y. Jiang [J] studied the first and second variation formulas of E_2, extensive studies in this area have been done (for instance, see [CMO], [IIU2], [IIU], [II], [LO2], [MO1], [OT2], [S1], etc.). Notice that harmonic maps are always biharmonic by definition. One of the important main problems is to ask whether the converse is true. B.Y. Chen raised ([C]) so called B.Y. Chen's conjecture and later, R. Caddeo, S. Montaldo, P. Piu and C. Oniciuc raised ([CMO]) the generalized B.Y. Chen's conjecture:

Every biharmonic submanifold of the Euclidean space $\mathbb{R}^n$ *must be harmonic (minimal).*

Every biharmonic submanifold of a Riemannian manifold of non-positive curvature must be harmonic (minimal).

For the generalized Chen's conjecture, Ou and Tang gave ([OT], [OT2]) a counter example in a Riemannian manifold of negative curvature. The Chen's conjecture was solved affirmatively in the case of surfaces in the three dimensional Euclidean space ([C]), and the case of hypersurfaces of the four dimensional Euclidean space ([D], [HV]), and the case of generic hypersurfaces in the Euclidean space ([KU]).

Furthermore, Akutagawa and Maeta gave ([AM]) a final supporting evidence to the Chen's conjecture: Every complete properly immersed biharmonic submanifold of the Euclidean space $\mathbb{R}^n$ is minimal.

It is also known (cf. [NU1], [NU2], [NUG]): every biharmonic map $\varphi : (M, g) \to (N, h)$ of a complete Riemannian manifold (M, g) into another Riemannian manifold (N, h) with non-positive sectional curvature with finite energy and finite bienergy is harmonic.

On the contrary to the above, the case that the target space (N, h) whose sectional curvature is non-negative, theory of biharmonic maps and/or biharmonic immersions is quite different. In 1986, Jiang [J] and in 2002, Oniciuc [On] constructed independently different examples of proper biharmonic immersions into the spheres. Here, *proper biharmonic* means that biharmonic, but not harmonic.

In this section, we study biharmonic submanifolds in compact symmetric spaces, and then we characterize the biharmonic property of orbits of commutative Hermann actions and associated actions in terms of symmetric triad with multiplicities (see Theorems 4.4 and 4.6). Moreover, we determine all the biharmonic hypersurfaces in the irreducible symmetric spaces of compact type which are regular orbits of commutative Hermann actions of cohomogeneity one (cf. Theorem 4.9). When cohomogeneity of the actions are two or greater, we obtain many examples of proper biharmonic submanifolds in compact symmetric spaces (see Subsection 4.6).

4.1. **Preliminaries.** We first prepare the materials for the first and second variational formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map $\varphi : (M, g) \to (N, h)$, of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h), which is an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi)\, v_g,$$

where $e(\varphi) := (1/2)|d\varphi|^2$ is called the energy density of φ. That is, for any variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\frac{d}{dt}\bigg|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0, \tag{4.1}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ which is given by $V(x) = \frac{d}{dt}\big|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$, $(x \in M)$, and the *tension field* of φ is given by $\tau(\varphi) = \sum_{i=1}^m B_\varphi(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, g), and B_φ is the second fundamental form of φ defined by

$$\begin{aligned} B_\varphi(X, Y) &= (\widetilde{\nabla} d\varphi)(X, Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y), \end{aligned}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, ∇, and ∇^h are Levi-Civita connections on TM, TN of (M, g), (N, h), respectively, and $\overline{\nabla}$, and $\widetilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (4.1), φ is *harmonic* if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that φ is harmonic. Then,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\varphi_t) = \int_M h(J(V), V) v_g,$$

where J is an elliptic differential operator, called the *Jacobi operator* acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta} V - \mathcal{R}(V), \tag{4.2}$$

where $\overline{\Delta} V = \overline{\nabla}^* \overline{\nabla} V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} V - \overline{\nabla}_{\nabla_{e_i} e_i} V\}$ is the *rough Laplacian* and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}(V) = \sum_{i=1}^m R^h(V, d\varphi(e_i)) d\varphi(e_i)$, and R^h is the curvature tensor of (N, h) given by $R^h(U, V)W = \nabla^h_U(\nabla^h_V W) - \nabla^h_V(\nabla^h_U W) - \nabla^h_{[U,V]} W$ for $U, V, W \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire [EL1] proposed polyharmonic (k-harmonic) maps and Jiang [J] studied the first and second variation formulas of biharmonic maps. Let us consider the *bienergy functional* defined by

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g,$$

where $|V|^2 = h(V, V)$, $V \in \Gamma(\varphi^{-1}TN)$.

The first variation formula of the bienergy functional is given by

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V) v_g.$$

Here,

$$\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}(\tau(\varphi)) - \mathcal{R}(\tau(\varphi)),$$

which is called the *bitension field* of φ, and J is given in (4.2).

A smooth map φ of (M, g) into (N, h) is said to be *biharmonic* if $\tau_2(\varphi) = 0$. By definition, every harmonic map is biharmonic. We say, for an immersion $\varphi : (M, g) \to (N, h)$ to be *proper biharmonic* if it is biharmonic but not harmonic.

4.2. **Biharmonic isometric immersions.** In the first part of this section, we first show a characterization theorem for an isometric immersion φ of an m dimensional Riemannian manifold (M, g) into an n dimensional Riemannian manifold (N, h) whose tension field $\tau(\varphi)$ satisfies that $\overline{\nabla}^\perp_X \tau(\varphi) = 0$ ($X \in \mathfrak{X}(M)$) to be biharmonic, where $\overline{\nabla}^\perp$ is the normal connection on the normal bundle $T^\perp M$. Let us recall the following theorem due to [J]:

Theorem 4.1 ([OSU] Theorem 3.1)**.** *Let $\varphi : (M^m, g) \to (N^n, h)$ be an isometric immersion. Assume that $\overline{\nabla}^{\perp}_X \tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$. Then, φ is biharmonic if and only if the following holds:*

$$
\begin{aligned}
&- \sum_{j,k=1}^{m} h\big(\tau(\varphi), R^h(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k)\big)\, d\varphi(e_j) \\
&\qquad + \sum_{j,k=1}^{m} h(\tau(\varphi), B_\varphi(e_j, e_k))\, B_\varphi(e_j, e_k) \\
&\qquad - \sum_{j=1}^{m} R^h(\tau(\varphi), d\varphi(e_j))\, d\varphi(e_j) = 0,
\end{aligned}
\tag{4.3}
$$

where $\{e_j\}_{j=1}^m$ is a locally defined orthonormal frame field on (M, g).

Here, let us apply the following general curvature tensorial properties ([KN], Vol. I, Pages 198, and 201) to the first term of the left hand side of (4.3):

$$
h\big(W_1, R^h(W_3, W_4)W_2\big) = h\big(W_3, R^h(W_1, W_2)W_4\big), \qquad (W_i \in \mathfrak{X}(N), i = 1, 2, 3, 4).
$$

Then, we have

$$
\begin{aligned}
&h\big(\tau(\varphi), R^h(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k)\big) \\
&\qquad = h\big(d\varphi(e_j), R^h(\tau(\varphi), d\varphi(e_k))d\varphi(e_k)\big).
\end{aligned}
$$

Therefore, for the first term of (4.3), we have that

$$
\sum_{j=1}^{m} h\Big(d\varphi(e_j), \sum_{k=1}^{m} R^h(\tau(\varphi), d\varphi(e_k))d\varphi(e_k)\Big)\, d\varphi(e_j)
$$

is equal to the tangential part of $\sum_{k=1}^m R^h(\tau(\varphi), d\varphi(e_k))\, d\varphi(e_k)$. Thus, the equation (4.3) is equivalent to

$$
\begin{aligned}
&-\Big(\sum_{k=1}^{m} R^h(\tau(\varphi), d\varphi(e_k))d\varphi(e_k)\Big)^{\top} \\
&\qquad + \sum_{j,k=1}^{m} h(\tau(\varphi), B_\varphi(e_j, e_k))\, B_\varphi(e_j, e_k) \\
&\qquad - \sum_{k=1}^{m} R^h(\tau(\varphi), d\varphi(e_k))\, d\varphi(e_k) = 0,
\end{aligned}
\tag{4.4}
$$

where $W^{\top}$ and $W^{\perp}$ mean the tangential part and the normal part of $W \in \mathfrak{X}(N)$, respectively. We have, by comparing the tangential part and the normal part of the equation (4.4), it is equivalent to that

$$
\begin{aligned}
&\Big(\sum_{k=1}^{m} R^h(\tau(\varphi), d\varphi(e_k))d\varphi(e_k)\Big)^{\top} = 0, \quad \text{and} \\
&\Big(\sum_{k=1}^{m} R^h(\tau(\varphi), d\varphi(e_k))\, d\varphi(e_k)\Big)^{\perp} = \sum_{j,k=1}^{m} h(\tau(\varphi), B_\varphi(e_j, e_k))\, B_\varphi(e_j, e_k).
\end{aligned}
$$

These two equations are equivalent to the following single equation:

$$\sum_{k=1}^{m} R^h(\tau(\varphi), d\varphi(e_k))\, d\varphi(e_k) = \sum_{j,k=1}^{m} h(\tau(\varphi), B_\varphi(e_j, e_k))\, B_\varphi(e_j, e_k). \tag{4.5}$$

Summarizing the above, we obtain:

Theorem 4.2 ([OSU] Theorem 3.2)**.** *Let $\varphi : (M^m, g) \to (N^n, h)$ be an isometric immersion. Assume that $\overline{\nabla}^\perp_X \tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$. Then, φ is biharmonic if and only if (4.5) holds.*

By Theorem 4.2, we can see that the following theorem.

Theorem 4.3. *Let (M^m, g) and (N^n, h) be Riemannian manifolds. Let $\varphi : M \to N$ be a isometric immersion which satisfies $\overline{\nabla}^\perp_X \tau(\varphi)$ $(X \in \mathfrak{X}(M))$. Then φ is biharmonic if and only if*

$$\sum_{i=1}^{m} \left(R^h(\tau(\varphi), d\varphi(e_i))d\varphi(e_i)\right)^\perp = \sum_{i=1}^{m} B_\phi(A_{\tau(\varphi)} d\varphi(e_i), d\varphi(e_i)) \tag{4.6}$$

holds.

4.3. **Characterization theorem.** In the previous section, we saw the second fundamental forms of orbits of the commutative associated actions and the Hermann actions. In this section, we obtain a necessary and sufficient condition for orbits to be biharmonic submanifolds.

First, we consider orbits of the $(K_2 \times K_1)$-action.

Theorem 4.4. *Let (G, K_1, K_2) be a commutative compact symmetric triad. For $H \in \mathfrak{a}$, we set $x = \exp(H)$. Then the orbit $K_2 x K_1$ is biharmonic if and only if*

$$\begin{aligned}
&\sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} m(\lambda) \langle dL_x^{-1}(\tau_H), \lambda\rangle \left(\frac{3}{2} - (\cot\langle\lambda, H\rangle)^2\right) \lambda \\
&+ \sum_{\alpha \in W^+ \setminus W_H} n(\alpha) \langle dL_x^{-1}(\tau_H), \alpha\rangle \left(\frac{3}{2} - (\tan\langle\alpha, H\rangle)^2\right) \alpha \\
&+ \sum_{\mu \in \Sigma_H^+} m(\mu) \langle dL_x^{-1}(\tau_H), \mu\rangle \mu + \sum_{\beta \in W_H^+} n(\beta) \langle dL_x^{-1}(\tau_H), \beta\rangle \beta = 0
\end{aligned}$$

holds.

Proof. Let $R^{\langle,\rangle}$ be the curvature tensor of $(G, \langle,\rangle)$. Since G is a symmetric space, we have

$$R^{\langle,\rangle}(dL_x(X), dL_x(Y))dL_x(Z) = -dL_x([[X, Y], Z]) \quad (X, Y, Z \in \mathfrak{g}).$$

Hence, we have

- $R^{\langle,\rangle}(\tau_H, dL_x(T_{\lambda,i}))dL_x(T_{\lambda,i}) = \langle\lambda, dL_x^{-1}(\tau_H)\rangle dL_x(\lambda)$ $(\lambda \in \Sigma^+ \setminus \Sigma_H,\ 1 \le i \le m(\lambda))$,
- $R^{\langle,\rangle}(\tau_H, dL_x(S_{\lambda,i}))dL_x(S_{\lambda,i}) = \langle\lambda, dL_x^{-1}(\tau_H)\rangle dL_x(\lambda)$ $(\lambda \in \Sigma^+,\ 1 \le i \le m(\lambda))$,
- $R^{\langle,\rangle}(\tau_H, dL_x(Y_{\alpha,j}))dL_x(Y_{\alpha,j}) = \langle\lambda, dL_x^{-1}(\tau_H)\rangle dL_x(\alpha)$ $(\alpha \in W^+ \setminus W_H,\ 1 \le j \le n(\alpha))$,

- $R^{\langle,\rangle}(\tau_H, dL_x(X_{\alpha,j}))dL_x(X_{\alpha,j}) = \langle \lambda, dL_x^{-1}(\tau_H)\rangle dL_x(\alpha)$ $(\alpha \in W^+,\ 1 \le j \le n(\alpha))$,
- $R^{\langle,\rangle}(\tau_H, dL_x(X))dL_x(X) = 0 \quad (X \in \mathfrak{k}_0 \oplus V(\mathfrak{k}_1 \cap \mathfrak{m}_2)\oplus) V(\mathfrak{m}_1 \cap \mathfrak{k}_2))$.

On the other hand, by Theorem 2.12, we have

- $$\begin{aligned} &B_H(A_{\tau_H} dL_x(T_{\lambda,i}), dL_x(T_{\lambda,i})) \\ =&-\frac{1}{2}\langle dL_x(\lambda), \tau_H\rangle B_H(dL_x(S_{\lambda,i}), dL_x(T_{\lambda,i})) \\ &-\langle dL_x(\lambda), \tau_H\rangle \cot(\langle \lambda, H\rangle) B_H(dL_x(T_{\lambda,i}), dL_x(T_{\lambda,i})) \\ =&\langle dL_x(\lambda), \tau_H\rangle \left(\frac{1}{4} + (\cot\langle \lambda, H\rangle)^2\right) dL_x(\lambda) \end{aligned}$$
  for $\lambda \in \Sigma^+ \setminus \Sigma_H, 1 \le i \le m(\lambda)$,
- $$\begin{aligned} &B_H(A_{\tau_H} dL_x(S_{\lambda,i}), dL_x(S_{\lambda,i})) \\ =&-\frac{1}{2}\langle dL_x(\lambda), \tau_H\rangle B_H(dL_x(T_{\lambda,i}), dL_x(S_{\lambda,i})) \\ =&\frac{1}{4}\langle dL_x(\lambda), \tau_H\rangle dL_x(\lambda) \end{aligned}$$
  for $\lambda \in \Sigma^+ \setminus \Sigma_H, 1 \le i \le m(\lambda)$,
- $$\begin{aligned} &B_H(A_{\tau_H} dL_x(Y_{\alpha,j}), dL_x(Y_{\alpha,j})) \\ =&-\frac{1}{2}\langle dL_x(\alpha), \tau_H\rangle B_H(dL_x(X_{\alpha,j}), dL_x(Y_{\alpha,j})) \\ &-\langle dL_x(\alpha), \tau_H\rangle \tan(\langle \alpha, H\rangle) B_H(dL_x(Y_{\alpha,j}), dL_x(Y_{\alpha,j})) \\ =&\langle dL_x(\alpha), \tau_H\rangle \left(\frac{1}{4} + (\tan\langle \alpha, H\rangle)^2\right) dL_x(\alpha) \end{aligned}$$
  for $\alpha \in W^+ \setminus W_H, 1 \le j \le n(\alpha)$,
- $$\begin{aligned} &B_H(A_{\tau_H} dL_x(X_{\alpha,j}), dL_x(X_{\alpha,j})) \\ =&-\frac{1}{2}\langle dL_x(\alpha), \tau_H\rangle B_H(dL_x(Y_{\alpha,j}), dL_x(X_{\alpha,j})) \\ =&\frac{1}{4}\langle dL_x(\alpha), \tau_H\rangle dL_x(\alpha) \end{aligned}$$
  for $\alpha \in W^+ \setminus W_H, 1 \le j \le n(\alpha)$,
- $$B_H(A_{\tau_H} dL_x(X), dL_x(X)) = 0$$
  for $X \in \mathfrak{k}_0 \oplus V(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus \sum_{\lambda \in \Sigma_H^+} \mathfrak{k}_\lambda \oplus \sum_{\alpha \in W_H^+} V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$.

Therefore, by Theorem 4.3, we have the consequence. □

When $\dim \mathfrak{a} = 1$, we have the following corollary.

Corollary 4.5. *Let (G, K_1, K_2) be a commutative compact symmetric triad. Suppose the condition $\dim \mathfrak{a} = 1$. Then, for $H \in \mathfrak{a}$, if $K_2 \exp(H) K_1$ is regular, then the orbit $K_2 \exp(H) K_1 \subset G$ is biharmonic if and only if*

$$\langle \tau_H, \alpha \rangle \left(m(\alpha) \left\{ \frac{3}{2} - (\cot\langle \alpha, H \rangle)^2 \right\} + 4m(2\alpha) \left\{ \frac{3}{2} - (\cot\langle 2\alpha, H \rangle)^2 \right\} \right.$$
$$\left. + n(\alpha) \left\{ \frac{3}{2} - (\tan\langle \alpha, H \rangle)^2 \right\} + 4n(2\alpha) \left\{ \frac{3}{2} - (\tan\langle 2\alpha, H \rangle)^2 \right\} \right) = 0$$

holds. Where $\alpha \in \Sigma$ and if $\lambda \notin \Sigma$ (resp. $\lambda \notin W$), then $m(\lambda) = 0$ (resp. $n(\lambda) = 0$) for $\lambda \in \mathfrak{a}$.

Next we consider commutative Hermann actions.

Theorem 4.6. *Let (G, K_1, K_2) be a commutative compact symmetric triad. For $H \in \mathfrak{a}$, we set $x = \exp(H)$. Then $K_2 \pi_1(x)$ is biharmonic if and only if*

$$\sum_{\lambda \in \Sigma^+ \setminus \Sigma_H} m(\lambda) \langle dL_x^{-1}(\tau_H), \lambda \rangle \left(1 - (\cot\langle \lambda, H \rangle)^2\right) \lambda$$
$$+ \sum_{\alpha \in W^+ \setminus W_H} n(\alpha) \langle dL_x^{-1}(\tau_H), \alpha \rangle \left(1 - (\tan\langle \alpha, H \rangle)^2\right) \alpha = 0$$

holds.

Proof. Let $R^{\langle,\rangle}$ be the curvature tensor of $(M_1, \langle,\rangle)$. Since G is a symmetric space, we have

$$R^{\langle,\rangle}(dL_x(X), dL_x(Y)) dL_x(Z) = -dL_x([[X, Y], Z]) \quad (X, Y, Z \in \mathfrak{m}_1).$$

Hence, we have

- for $\lambda \in \Sigma^+ \setminus \Sigma_H, 1 \leq i \leq m(\lambda)$,
$$\begin{aligned} R^{\langle,\rangle}(\tau_H, dL_x(T_{\lambda,i})) dL_x(T_{\lambda,i}) &= \langle \tau_H, dL_x(\lambda) \rangle dL_x[S_{\lambda,i}, T_{\lambda,i}] \\ &= \langle \tau_H, dL_x(\lambda) \rangle dL_x(\lambda), \end{aligned}$$
- for $\alpha \in W^+ \setminus W_H, 1 \leq j \leq m(\alpha)$,
$$\begin{aligned} R^{\langle,\rangle}(\tau_H, dL_x(Y_{\alpha,j})) dL_x(Y_{\alpha,j}) &= \langle \tau_H, dL_x(\alpha) \rangle dL_x[X_{\alpha,j}, Y_{\alpha,j}] \\ &= \langle \tau_H, dL_x(\alpha) \rangle dL_x(\alpha), \end{aligned}$$
- $R^{\langle,\rangle}(\tau_H, dL_x(X)) dL_x(X) = 0$ for $X \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$.

On the other hand, by Lemma 2.8, we have

- $$\begin{aligned} & B_H(A_{\tau_H} dL_x(T_{\lambda,i}), dL_x(T_{\lambda,i})) \\ &= -\langle \tau_H, dL_x(\lambda) \rangle (\cot\langle \lambda, H \rangle) B_H(dL_x(T_{\lambda,i}), dL_x(T_{\lambda,i})) \\ &= \langle \tau_H, dL_x(\lambda) \rangle (\cot\langle \lambda, H \rangle)^2 dL_x(\lambda) \end{aligned}$$

  for $\lambda \in \Sigma^+ \setminus \Sigma_H, 1 \leq i \leq m(\lambda)$,

- $$\begin{aligned} &B_H(A_{\tau_H} dL_x(Y_{\alpha,j}), dL_x(Y_{\alpha,j})) \\ &= -\langle \tau_H, dL_x(\alpha)\rangle (\tan\langle \alpha, H\rangle) B_H(dL_x(Y_{\alpha,j}), dL_x(Y_{\alpha,j})) \\ &= \langle \tau_H, dL_x(\alpha)\rangle (\tan\langle \alpha, H\rangle)^2 dL_x(\alpha) \end{aligned}$$
  for $\alpha \in W^+ \setminus W_H, 1 \leq j \leq n(\alpha)$,
- $B_H(A_{\tau_H} dL_x(X), dL_x(X)) = 0$ for $X \in V(\mathfrak{m}_1 \cap \mathfrak{k}_2)$.

By Theorem 4.3, we have consequence. □

Corollary 4.7. *Let (G, K_1, K_2) be a commutative compact symmetric triad. Suppose the condition $\dim \mathfrak{a} = 1$. Then, for $H \in \mathfrak{a}$, if $K_2\pi_1(\exp(H))$ is regular, then the orbit $K_2\pi_1(\exp(H)) \subset M_1$ is biharmonic if and only if*

$$\begin{aligned} \langle \tau_H, \alpha\rangle \big(&m(\alpha)\left\{1 - (\cot\langle \alpha, H\rangle)^2\right\} + 4m(2\alpha)\left\{1 - (\cot\langle 2\alpha, H\rangle)^2\right\} \\ &+ n(\alpha)\left\{1 - (\tan\langle \alpha, H\rangle)^2\right\} + 4n(2\alpha)\left\{1 - (\tan\langle 2\alpha, H\rangle)^2\right\}\big) = 0 \end{aligned}$$

holds. Where $\alpha \in \Sigma$ and if $\lambda \notin \Sigma$ (resp. $\lambda \notin W$), then $m(\lambda) = 0$ (resp. $n(\lambda) = 0$) for $\lambda \in \mathfrak{a}$.

4.4. **Biharmonic orbits of cohomogeneity one Hermann actions.** In this section, applying Corollary 4.7 we will study biharmonic regular orbits of cohomogeneity one Hermann actions.

Let (G, K_1, K_2) be a commutative compact symmetric triad where G is semisimple. It is known that the tension field of an orbit of a Hermann action is parallel in the normal bundle (see [IST1]), i.e. $\overline{\nabla}^{\perp}_X \tau_H = 0$ for every vector field X on the orbit $K_2\pi_1(x)$.

Hereafter we assume that $\dim \mathfrak{a} = 1$. Since the cohomogeneity of K_2-action on M_1 and that of K_1-action on M_2 are equal to $\dim \mathfrak{a}$, regular orbits of K_2-actions (resp. K_1-action) are homogeneous hypersurfaces in M_1 (resp. M_2). Hence we can apply Corollary 4.7 for regular orbits of these actions. Clearly, $K_2\pi_1(x)$ is a regular orbit if and only if $K_1\pi_2(x)$ is also a regular orbit. Therefore, we have the following proposition.

Proposition 4.8. *Let $x = \exp H$ for $H \in \mathfrak{a}$. Suppose that $K_2\pi_1(x)$ is a regular orbit of K_2-action on M_1, so $K_1\pi_2(x)$ is also a regular orbit of K_1-action on M_2. Then,*

1. *An orbit $K_2\pi_1(x)$ is harmonic if and only if $K_1\pi_2(x)$ is harmonic.*
2. *An orbit $K_2\pi_1(x)$ is proper biharmonic if and only if $K_1\pi_2(x)$ is proper biharmonic.*

Proof. The triad $(\tilde{\Sigma}, \Sigma, W)$ does not depend on the order of K_1 and K_2. Thus, by Corollary 4.7, we have the consequence. □

If G is simple and $\theta_1 \not\sim \theta_2$, then for a commutative compact symmetric triad (G, K_1, K_2) the triple $(\tilde{\Sigma}, \Sigma, W)$ is a symmetric triad with multiplicities $m(\lambda)$ and $n(\alpha)$ (cf. Theorem 3.14). In this case, for $x = \exp H$ $(H \in \mathfrak{a})$, the orbit $K_2\pi_1(x)$ is regular if and only if H is a regular point with respect to $(\tilde{\Sigma}, \Sigma, W)$.

All the symmetric triads with $\dim \mathfrak{a} = 1$ are classified into the following four types ([I]):

	Σ^+	W^+	$\tilde{\alpha}$
III-B$_1$	$\{\alpha\}$	$\{\alpha\}$	α
I-BC$_1$	$\{\alpha, 2\alpha\}$	$\{\alpha\}$	α
II-BC$_1$	$\{\alpha\}$	$\{\alpha, 2\alpha\}$	2α
III-BC$_1$	$\{\alpha, 2\alpha\}$	$\{\alpha, 2\alpha\}$	2α

Let $\vartheta := \langle \tilde{\alpha}, H \rangle$ for $H \in \mathfrak{a}$. Then, by (2.1), $P_0 = \{H \in \mathfrak{a} \mid 0 < \vartheta < \pi/2\}$ is a cell in these types. If M_1 is simply connected, then the orbit space of K_2-action on M_1 is identified with $\overline{P_0} = \{H \in \mathfrak{a} \mid 0 \leq \vartheta \leq \pi/2\}$, more precisely, each orbit meets $\pi_1(\exp \overline{P_0})$ at one point. A point in the interior of the orbit space corresponds to a regular orbit, and there exists a unique minimal (harmonic) orbit among regular orbits. On the other hand, two endpoints of the orbit space correspond to singular orbits. These singular orbits are minimal (harmonic), moreover these are weakly reflective ([IST2]).

4.4.1. *Type* III-B$_1$. By Corollary 4.7, the biharmonic condition is equivalent to

$$m(\alpha) + n(\alpha) = m(\alpha)(\cot \vartheta)^2 + n(\alpha)(\tan \vartheta)^2$$

for $H \in P_0$. Thus we have

$$\tan \vartheta = 1, \text{ or } \sqrt{\frac{m(\alpha)}{n(\alpha)}}.$$

On the other hand, by (1) of Theorem 2.9, the harmonic condition $\tau_H = 0$ is equivalent to

$$-m(\alpha) \cot \vartheta + n(\alpha) \tan \vartheta = 0.$$

Thus we have

$$\tan \vartheta = \sqrt{\frac{m(\alpha)}{n(\alpha)}}.$$

Therefore, the situation is divided into the following two cases:

(1) When $m(\alpha) = n(\alpha)$, if an orbit $K_2\pi_1(x)$ is biharmonic, then it is harmonic.
(2) When $m(\alpha) \neq n(\alpha)$, an orbit $K_2\pi_1(x)$ is proper biharmonic if and only if $(\tan \vartheta)^2 = 1$ for $H \in P_0$. In this case, a unique proper biharmonic orbit exists at the center of P_0, namely $\vartheta = \pi/4$.

4.4.2. *Type* I-BC$_1$. We denote $m_1 := m(\alpha)$, $m_2 := m(2\alpha)$ and $n_1 := n(\alpha)$ for short. Then, by Corollary 4.7, the biharmonic condition is equivalent to

$$m_1 + n_1 + 4m_2 = m_1(\cot \vartheta)^2 + n_1(\tan \vartheta)^2 + 4m_2(\cot 2\vartheta)^2.$$

Thus, we have

$$(\tan \vartheta)^2 = \frac{m_1 + n_1 + 6m_2 \pm \sqrt{(m_1 + n_1 + 6m_2)^2 - 4(n_1 + m_2)(m_1 + m_2)}}{2(n_1 + m_2)}.$$

By (1) of Theorem 2.9, the harmonic condition $\tau_H = 0$ is equivalent to

$$-m_1 \cot \vartheta + n_1 \tan \vartheta - 4m_2 \cot 2\vartheta = 0.$$

Thus, we have

$$(\tan \vartheta)^2 = \frac{m_1 + m_2}{n_1 + m_2}.$$

Since

$$\begin{aligned}0 &< \frac{m_1+n_1+6m_2-\sqrt{(m_1+n_1+6m_2)^2-4(n_1+m_2)(m_1+m_2)}}{2(n_1+m_2)}\\ &< \frac{m_1+m_2}{n_1+m_2}\\ &< \frac{m_1+n_1+6m_2+\sqrt{(m_1+n_1+6m_2)^2-4(n_1+m_2)(m_1+m_2)}}{2(n_1+m_2)},\end{aligned}$$

an orbit $K_2\pi_1(x)$ is proper biharmonic if and only if

$$(\tan\vartheta)^2 = \frac{m_1+n_1+6m_2 \pm \sqrt{(m_1+n_1+6m_2)^2-4(n_1+m_2)(m_1+m_2)}}{2(n_1+m_2)}$$

holds for $H \in P_0$. Furthermore, a unique harmonic regular orbit exists between two proper biharmonic orbits in P_0.

4.4.3. *Type* II-BC$_1$. By the definition of multiplicities, if $2\alpha \in W^+$, then $m(\alpha) = n(\alpha)$. Hence we denote $m_1 := m(\alpha) = n(\alpha)$ and $n_2 := n(2\alpha)$. Then, by Corollary 4.7, the biharmonic condition is equivalent to

$$2m_1 + 4n_2 = m_1\big((\cot(\vartheta/2))^2 + (\tan(\vartheta/2))^2\big) + 4n_2(\tan\vartheta)^2.$$

Thus, we have

$$(\tan\vartheta)^2 = \frac{n_2 \pm \sqrt{n_2^2 - 4n_2m_1}}{2n_2} = \frac{1}{2} \pm \sqrt{\frac{n_2-4m_1}{4n_2}}.$$

By (1) of Theorem 2.9, the harmonic condition $\tau_H = 0$ is equivalent to

$$m_1\big(-\cot(\vartheta/2) + \tan(\vartheta/2)\big) + 2n_2\tan\vartheta = 0.$$

Thus, we have

$$(\tan\vartheta)^2 = \frac{m_1}{n_2}.$$

Therefore, the situation is divided into the following three cases:

(1) When $n_2 < 4m_1$, if $K_2\pi_1(x)$ is biharmonic, then it is harmonic.
(2) When $n_2 = 4m_1$, an orbit $K_2\pi_1(x)$ is proper biharmonic if and only if $(\tan\vartheta)^2 = 1/2$ for $H \in P_0$.
(3) When $n_2 > 4m_1$, an orbit $K_2\pi_1(x)$ is proper biharmonic if and only if

$$(\tan\vartheta)^2 = \frac{n_2 \pm \sqrt{n_2^2 - 4n_2m_1}}{2n_2}$$

holds for $H \in P_0$, since

$$0 < \frac{m_1}{n_2} < \frac{n_2 - \sqrt{n_2^2-4n_2m_1}}{2n_2} < \frac{n_2 + \sqrt{n_2^2-4n_2m_1}}{2n_2}.$$

4.4.4. *Type* III-BC$_1$. By the definition of multiplicities, if $2\alpha \in W^+$, then $m(\alpha) = n(\alpha)$. Hence we denote $m_1 := m(\alpha) = n(\alpha), m_2 := m(2\alpha)$ and $n_2 := n(2\alpha)$. Then, by Corollary 4.7, the biharmonic condition is equivalent to

$$2m_1 + 4m_2 + 4n_2 = m_1\big((\cot(\vartheta/2))^2 + (\tan(\vartheta/2))^2\big) + 4m_2(\cot\vartheta)^2 + 4n_2(\tan\vartheta)^2.$$

Thus, we have

$$\begin{aligned}(\tan\vartheta)^2 &= \frac{m_2 + n_2 \pm \sqrt{(m_2+n_2)^2 - 4n_2(m_1+m_2)}}{2n_2}\\ &= \frac{m_2 + n_2 \pm \sqrt{(m_2-n_2)^2 - 4n_2 m_1}}{2n_2}.\end{aligned}$$

By (1) of Theorem 2.9, the harmonic condition $\tau_H = 0$ is equivalent to

$$m_1\big(\tan(\vartheta/2) - \cot(\vartheta/2)\big) - 2m_2\cot\vartheta + 2n_2\tan\vartheta = 0.$$

Thus, we have

$$(\tan\vartheta)^2 = \frac{m_1+m_2}{n_2}.$$

Therefore, we obtain the following results:

(1) When $(m_2-n_2)^2 - 4n_2m_1 < 0$, if $K_2\pi_1(x)$ is biharmonic, then it is harmonic.
(2) When $(m_2-n_2)^2 - 4n_2m_1 = 0$, an orbit $K_2\pi_1(x)$ is proper biharmonic if and only if $(\tan\vartheta)^2 = (m_2+n_2)/2n_2$ for $H \in P_0$.
(3) When $(m_2-n_2)^2 - 4n_2m_1 > 0$, an orbit $K_2\pi_1(x)$ is proper biharmonic if and only if

$$(\tan\vartheta)^2 = \frac{m_2 + n_2 \pm \sqrt{(m_2-n_2)^2 - 4n_2 m_1}}{2n_2}$$

for $H \in P_0$.

For the proof of (2), we will show that

$$\frac{m_1+m_2}{n_2} \neq \frac{m_2+n_2}{2n_2}.$$

If $(m_1+m_2)/n_2 = (m_2+n_2)/(2n_2)$, then $2m_1+m_2-n_2 = 0$. Hence $(m_2-n_2)^2 - 4n_2m_1 = -4m_1(m_1+m_2) < 0$, which is a contradiction.

For the proof of (3), we will show that

$$\frac{m_1+m_2}{n_2} \neq \frac{m_2 + n_2 \pm \sqrt{(m_2-n_2)^2 - 4n_2 m_1}}{2n_2}.$$

If the equality holds, then we have $(2m_1+m_2-n_2)^2 = (m_2-n_2)^2 - 4n_2m_1$. Hence $4m_1(m_1+m_2) = 0$, which is a contradiction.

In fact, in the cases of type III-BC$_1$, a compact symmetric triad which is not (1) is only $(E_6, \mathrm{SO}(10)\cdot\mathrm{U}(1), F_4)$ in the list below. In this case,

$$\begin{aligned}\frac{m_1+m_2}{n_2} &< \frac{m_2 + n_2 - \sqrt{(m_2-n_2)^2 - 4n_2 m_1}}{2n_2}\\ &< \frac{m_2 + n_2 + \sqrt{(m_2-n_2)^2 - 4n_2 m_1}}{2n_2}\end{aligned}$$

holds.

Let $b > 0$, $c > 1$ and $q > 1$. Each commutative compact symmetric triad (G, K_1, K_2) where G is simple, $\theta_1 \not\sim \theta_2$ and $\dim \mathfrak{a} = 1$ is one of the following (see [I2]):

Type III-B_1

(G, K_1, K_2)	$(m(\alpha), n(\alpha))$
$(SO(1+b+c), SO(1+b) \times SO(c), SO(b+c))$	$(c-1, b)$
$(SU(4), Sp(2), SO(4))$	$(2, 2)$
$(SU(4), S(U(2) \times U(2)), Sp(2))$	$(3, 1)$
$(Sp(2), U(2), Sp(1) \times Sp(1))$	$(1, 2)$

Type I-BC_1

(G, K_1, K_2)	$(m(\alpha), m(2\alpha), n(\alpha))$
$(SO(2+2q), SO(2) \times SO(2q), U(1+q))$	$(2(q-1), 1, 2(q-1))$
$(SU(1+b+c), S(U(1+b) \times U(c)), S(U(1) \times U(b+c))$	$(2(c-1), 1, 2b)$
$(Sp(1+b+c), Sp(1+b) \times Sp(c), Sp(1) \times Sp(b+c))$	$(4(c-1), 3, 4b)$
$(SO(8), U(4), U(4)')$	$(4, 1, 1)$

Type II-BC_1

(G, K_1, K_2)	$(m(\alpha), n(\alpha), n(2\alpha))$
$(SO(6), U(3), SO(3) \times SO(3))$	$(2, 2, 1)$
$(SU(1+q), SO(1+q), S(U(1) \times U(q)))$	$(q-1, q-1, 1)$

Type III-BC_1

(G, K_1, K_2)	$(m(\alpha), m(2\alpha), n(\alpha), n(2\alpha))$
$(SU(2+2q), S(U(2) \times U(2q)), Sp(1+q))$	$(4(q-1), 3, 4(q-1), 1)$
$(Sp(1+q), U(1+q), Sp(1) \times Sp(q))$	$(2(q-1), 1, 2(q-1), 2)$
$(E_6, SU(6) \cdot SU(2), F_4)$	$(8, 3, 8, 5)$
$(E_6, SO(10) \cdot U(1), F_4)$	$(8, 7, 8, 1)$
$(F_4, Sp(3) \cdot Sp(1), Spin(9))$	$(4, 3, 4, 4)$

Here, we define $U(4)' = \{g \in SO(8) \mid JgJ^{-1} = g\}$ where

$$J = \left[\begin{array}{cc|cc} & & I_3 & \\ & & & -1 \\ \hline -I_3 & & & \\ & 1 & & \end{array}\right]$$

and I_l denotes the identity matrix of $l \times l$.

4.5. **Classification theorem.** Summing up the previous sections, we classify all the biharmonic hypersurfaces in irreducible compact symmetric spaces which are orbits of commutative Hermann actions, namely we obtain the following theorem.

Theorem 4.9 ([OSU] Theorem 6.1)**.** *Let (G, K_1, K_2) be a commutative compact symmetric triad where G is simple, and suppose that K_2-action on $M_1 = G/K_1$ is cohomogeneity one (hence K_1-action on $M_2 = K_2\backslash G$ is also cohomogeneity one). Then all the proper biharmonic hypersurfaces which are regular orbits of K_2-action (resp. K_1-action) in the compact symmetric space M_1 (resp. M_2) are classified into the following lists:*

(1) *When (G, K_1, K_2) is one of the following cases, there exists a unique proper biharmonic hypersurface which is a regular orbit of K_2-action on M_1 (resp. K_1-action on M_2).*

(1-1) (SO$(1+b+c)$, SO$(1+b)\times$SO(c), SO$(b+c)$) $(b>0,\ c>1,\ c-1\neq b)$
(1-2) (SU(4), S(U(2) $\times$ U(2)), Sp(2))
(1-3) (Sp(2), U(2), Sp(1) $\times$ Sp(1))

(2) *When (G, K_1, K_2) is one of the following cases, there exist exactly two distinct proper biharmonic hypersurfaces which are regular orbits of of K_2-action on M_1 (resp. K_1-action on M_2).*

(2-1) (SO$(2+2q)$, SO(2) $\times$ SO$(2q)$, U$(1+q)$) $(q>1)$
(2-2) (SU$(1+b+c)$, S(U$(1+b)\times$U(c)), S(U(1)$\times$U$(b+c)$)) $(b\geq 0,\ c>1)$
(2-3) (Sp$(1+b+c)$, Sp$(1+b)\times$Sp(c), Sp(1)$\times$Sp$(b+c)$) $(b\geq 0,\ c>1)$
(2-4) (SO(8), U(4), U(4)$'$)
(2-5) (E_6, SO(10) $\cdot$ U(1), F_4)
(2-6) (SO$(1+q)$, SO(q), SO(q)) $(q>1)$
(2-7) (F_4, Spin(9), Spin(9))

(3) *When (G, K_1, K_2) is one of the following cases, any biharmonic regular orbit of K_2-action on M_1 (resp. K_1-action on M_2) is harmonic.*

(3-1) (SO$(2c)$, SO$(c)\times$SO(c), SO$(2c-1)$) $(c>1)$
(3-2) (SU(4), Sp(2), SO(4))
(3-3) (SO(6), U(3), SO(3) $\times$ SO(3))
(3-4) (SU$(1+q)$, SO$(1+q)$, S(U(1) $\times$ U(q))) $(q>1)$
(3-5) (SU$(2+2q)$, S(U(2) $\times$ U$(2q)$), Sp$(1+q)$) $(q>1)$
(3-6) (Sp$(1+q)$, U$(1+q)$, Sp(1) $\times$ Sp(q)) $(q>1)$
(3-7) (E_6, SU(6) $\cdot$ SU(2), F_4)
(3-8) (F_4, Sp(3) $\cdot$ Sp(1), Spin(9))

Remark 4.10. In Theorem 4.9, we determined all the biharmonic hypersurfaces in irreducible compact symmetric spaces which are orbits of commutative Hermann actions.

(1) In the previous section we assumed $\theta_1 \not\sim \theta_2$. If $\theta_1 \sim \theta_2$, then the action of K_2 on M_1 is orbit equivalent to the isotropy action of K_1 on M_1. We will discuss these cases in Section 6.3.
(2) The commutative condition $\theta_1\theta_2 = \theta_2\theta_1$ is essential for our discussion. Indeed, there exist some Hermann actions where $\theta_1\theta_2 \neq \theta_2\theta_1$. Moreover there exist some hyperpolar actions of cohomogeneity one on irreducible compact symmetric spaces which are not Hermann actions (cf. [Kol]).

We shall explain details of the cases (1-1), (2-2) and (3-1) in Theorem 4.9, and give new examples of proper biharmonic orbits. By Proposition 4.8, a proper biharmonic orbit $K_2\pi_1(x)$ in M_1 corresponds to a proper biharmonic orbit $K_1\pi_2(x)$ in M_2. In particular, we can obtain new examples of proper biharmonic orbits corresponding to some known examples.

We consider the isotropy subgroups of orbits of Hermann actions. For $x = \exp H$ $(H \in \mathfrak{a})$, we define the isotropy subgroups

$$(K_2)_{\pi_1(x)} = \{k \in K_2 \mid k\pi_1(x) = \pi_1(x)\},$$

$$(K_1)_{\pi_2(x)} = \{k \in K_1 \mid k\pi_2(x) = \pi_2(x)\}.$$

Then we can show that $(K_2)_{\pi_1(x)} \cong (K_1)_{\pi_2(x)}$ by an inner automorphism of G. The orbit $K_2\pi_1(x)$ (resp. $K_1\pi_2(x)$) is diffeomorphic to the homogeneous space

$K_2/((K_2)_{\pi_1(x)})$ (resp. $K_1/((K_1)_{\pi_2(x)})$). If $K_2\pi_1(x)$ is a regular orbit, then $K_1\pi_2(x)$ is also a regular orbit, and we have $\mathrm{Lie}((K_2)_{\pi_1(x)}) = \mathrm{Lie}((K_1)_{\pi_2(x)}) = \mathfrak{k}_0$.

Example 1. Let $(G, K_1, K_2) = (\mathrm{SO}(1+b+c), \mathrm{SO}(1+b)\times\mathrm{SO}(c), \mathrm{SO}(b+c))$ $(b > 0,\ c > 1)$. This is the case of (3-1) when $c-1 = b$, otherwise the case of (1-1) in Theorem 4.9. In this case, the involutions θ_1 and θ_2 are given by

$$\theta_1(k) = I'_{1+b} k I'_{1+b}, \quad \theta_2(k) = I'_1 k I'_1 \quad (k \in G),$$

where

$$I'_l = \begin{bmatrix} -I_l & 0 \\ 0 & I_{1+b+c-l} \end{bmatrix} \quad (1 \le l \le b+c).$$

Then, we have the canonical decompositions $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{m}_1 = \mathfrak{k}_2 \oplus \mathfrak{m}_2$ as

$$\mathfrak{k}_1 = \left\{ \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \;\middle|\; \begin{matrix} X \in \mathfrak{so}(1+b) \\ Y \in \mathfrak{so}(c) \end{matrix} \right\}, \quad \mathfrak{m}_1 = \left\{ \begin{bmatrix} 0 & X \\ -{}^tX & 0 \end{bmatrix} \;\middle|\; X \in \mathrm{M}_{1+b,c}(\mathbb{R}) \right\},$$

$$\mathfrak{k}_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix} \;\middle|\; X \in \mathfrak{so}(b+c) \right\}, \quad \mathfrak{m}_2 = \left\{ \begin{bmatrix} 0 & X \\ -{}^tX & 0 \end{bmatrix} \;\middle|\; X \in \mathrm{M}_{1,b+c}(\mathbb{R}) \right\}.$$

Thus, we have

$$\mathfrak{k}_1 \cap \mathfrak{k}_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & Y \end{bmatrix} \;\middle|\; \begin{matrix} X \in \mathfrak{so}(b) \\ Y \in \mathfrak{so}(c) \end{matrix} \right\},$$

$$\mathfrak{m}_1 \cap \mathfrak{m}_2 = \left\{ \begin{bmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ -{}^tX & 0 & 0 \end{bmatrix} \;\middle|\; X \in \mathrm{M}_{1,c}(\mathbb{R}) \right\},$$

$$\mathfrak{k}_1 \cap \mathfrak{m}_2 = \left\{ \begin{bmatrix} 0 & X & 0 \\ -{}^tX & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; X \in \mathrm{M}_{1,b}(\mathbb{R}) \right\},$$

$$\mathfrak{m}_1 \cap \mathfrak{k}_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & X \\ 0 & -{}^tX & 0 \end{bmatrix} \;\middle|\; X \in \mathrm{M}_{b,c}(\mathbb{R}) \right\}.$$

We take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}_1 \cap \mathfrak{m}_2$ as

$$\mathfrak{a} = \left\{ H(\vartheta) = \begin{bmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ -{}^tX & 0 & 0 \end{bmatrix} \;\middle|\; \begin{matrix} X = [0, \ldots, 0, \vartheta] \\ \vartheta \in \mathbb{R} \end{matrix} \right\}.$$

Then we have

$$\mathfrak{k}_0 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & Y & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \;\middle|\; \begin{matrix} X \in \mathfrak{so}(b) \\ Y \in \mathfrak{so}(c-1) \end{matrix} \right\},$$

$$V(\mathfrak{k}_1 \cap \mathfrak{m}_2) = \{0\},$$

$$V(\mathfrak{m}_1 \cap \mathfrak{k}_2) = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & -{}^tX & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \;\middle|\; X \in \mathrm{M}_{b,c-1}(\mathbb{R}) \right\}.$$

Let E_i^j be a matrix whose (i,j)-entry is one and all the other entries are zero. We define $A_i^j := E_i^j - E_j^i$. Then, we can see

$$[H(\vartheta), A_1^j] = -\vartheta A_{1+b+c}^j \qquad (2 \leq j \leq b+c),$$
$$[H(\vartheta), A_{1+b+c}^j] = \vartheta A_1^j \qquad (2 \leq j \leq b+c).$$

We define a vector $\alpha \in \mathfrak{a}$ by $\langle H(\vartheta), \alpha\rangle = \vartheta$ $(\vartheta \in \mathbb{R})$. Then

$$\mathfrak{k}_\alpha = \mathrm{Span}\{A_{1+b+c}^{2+b}, \dots, A_{1+b+c}^{b+c}\},$$
$$\mathfrak{m}_\alpha = \mathrm{Span}\{A_1^{2+b}, \dots, A_1^{b+c}\},$$
$$V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) = \mathrm{Span}\{A_1^2, \dots, A_1^{1+b}\},$$
$$V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2) = \mathrm{Span}\{A_{1+b+c}^2, \dots, A_{1+b+c}^{1+b}\}.$$

Hence, in this case, we have

$$\Sigma^+ = \{\alpha\}, \quad W^+ = \{\alpha\}, \quad m(\alpha) = c-1, \quad n(\alpha) = b.$$

Let $x_0 = \exp(H(\pi/4))$. By the computation in Section 4.4.1, we can see that $K_2\pi_1(x_0)$ and $K_1\pi_2(x_0)$ are biharmonic hypersurfaces in M_1 and M_2, respectively. These orbits exist at the center of the orbit space $\overline{P_0} = \{H(\vartheta) \mid 0 \leq \vartheta \leq \pi/2\}$. When $c-1 = b$, these orbits are harmonic. When $c-1 \neq b$, these are not harmonic, hence proper biharmonic. The orbit $K_2\pi_1(x_0)$ is the Clifford hypersurface $S^b(1/\sqrt{2}) \times S^{c-1}(1/\sqrt{2}) \cong (\mathrm{SO}(1+b) \times \mathrm{SO}(c))/(\mathrm{SO}(b) \times \mathrm{SO}(c-1))$ embedded in the sphere $S^{b+c}(1) \cong \mathrm{SO}(1+b+c)/\mathrm{SO}(b+c) \cong M_2$ ([J]). On the other hand, the orbit $K_2\pi_1(x_0)$ is diffeomorphic to $\mathrm{SO}(b+c)/(\mathrm{SO}(b)\times\mathrm{SO}(c-1))$, i.e. the universal covering of a real flag manifold, and embedded in the oriented real Grassmannian manifold $\widetilde{G_{1+b}}(\mathbb{R}^{1+b+c}) \cong \mathrm{SO}(1+b+c)/(\mathrm{SO}(1+b) \times \mathrm{SO}(c)) \cong M_1$ as the tube of radius $\pi/4$ over the totally geodesic sub-Grassmannian $\widetilde{G_b}(\mathbb{R}^{b+c})$. The orbit $K_2\pi_1(x_0)$ in M_1 gives a new example of a proper biharmonic hypersurface in the oriented real Grassmannian manifold.

Example 2. Let $(G, K_1, K_2) = (\mathrm{SU}(1+b+c), \mathrm{S}(\mathrm{U}(1+b) \times \mathrm{U}(c)), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(b+c)))$ $(b > 0,\ c > 1)$. This is the case of (2-2) except for $b = 0$ in Theorem 4.9. In this case, the involutions θ_1 and θ_2 are given by

$$\theta_1(k) = I'_{1+b} k I'_{1+b}, \quad \theta_2(k) = I'_1 k I'_1 \quad (k \in G).$$

Analogous to the previous example, in this case, we have

$$\Sigma^+ = \{\alpha, 2\alpha\},\ W^+ = \{\alpha\},\ m(\alpha) = 2(c-1),\ m(2\alpha) = 1,\ n(\alpha) = 2b.$$

Therefore, the symmetric triad $(\tilde{\Sigma}, \Sigma, W)$ is of type I-BC$_1$. By the computation in Section 4.4.2, we have two distinct proper biharmonic hypersurfaces in M_1, and also in M_2. More precisely, let $x_\pm = \exp(H(\vartheta_\pm))$ where $\vartheta_\pm$ is a solution of the equation

$$(\tan\vartheta)^2 = \frac{m_1 + n_1 + 6m_2 \pm \sqrt{(m_1+n_1+6m_2)^2 - 4(n_1+m_2)(m_1+m_2)}}{2(n_1+m_2)}$$
$$= \frac{(c-1)+b+3 \pm \sqrt{((c-1)+b+3)^2 - (2b+1)(2(c-1)+1)}}{2b+1}.$$

Then $K_2\pi_1(x_\pm)$ and $K_1\pi_2(x_\pm)$ are proper biharmonic hypersurfaces in M_1 and M_2, respectively. The orbit $K_1\pi_2(x_\pm) \cong \mathrm{S}(\mathrm{U}(1+b)\times\mathrm{U}(c))/\mathrm{S}(\mathrm{U}(b)\times\mathrm{U}(c-1)\times\mathrm{U}(1))$ is the tube of radius $\vartheta_\pm$ over the totally geodesic $\mathbb{C}P^b$ in the complex projective space $\mathbb{C}P^{b+c} \cong \mathrm{SU}(1+b+c)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(b+c)) \cong M_2$ (see Theorem 5 in [IIU]). On the other hand, the orbit $K_2\pi_1(x_\pm) \cong \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(b+c))/\mathrm{S}(\mathrm{U}(b) \times \mathrm{U}(c-1) \times \mathrm{U}(1))$ is the tube of radius $\vartheta_\pm$ over the totally geodesic sub-Grassmannian $G_b(\mathbb{C}^{b+c})$ in the complex Grassmannian manifold $G_{1+b}(\mathbb{C}^{1+b+c}) \cong \mathrm{SU}(1+b+c)/\mathrm{S}(\mathrm{U}(1+b)\times\mathrm{U}(c)) = M_1$. The orbit $K_2\pi_1(x_\pm)$ in M_1 gives a new example of a proper biharmonic hypersurface in the complex Grassmannian manifold.

In the above argument, we supposed that $\theta_1 \not\sim \theta_2$ in order to use the classification of commutative compact symmetric triads. However, we can apply our method to the cases of $\theta_1 \sim \theta_2$. When $\theta_1 \sim \theta_2$, a Hermann action is orbit equivalent to the isotropy action of a compact symmetric space (see [I]). Hence, it is sufficient to discuss the cases of isotropy actions, that is, $\theta_1 = \theta_2$. When $\theta_1 = \theta_2$, we have $W = \emptyset$, since $\mathfrak{k}_1 \cap \mathfrak{m}_2 = \mathfrak{m}_1 \cap \mathfrak{k}_2 = \{0\}$. Thus we have $\tilde{\Sigma} = \Sigma$. Moreover, Σ is the root system of the compact symmetric space M_1 with respect to $\mathfrak{a}$. Since we consider the cases of $\dim \mathfrak{a} = 1$, the rank of M_1 equals to one. All the simply connected, rank one symmetric spaces of compact type are classified as follows:

$$S^q,\ \mathbb{C}P^q,\ \mathbb{H}P^q,\ \mathbb{O}P^2 \quad (q \geq 2).$$

The isotropy actions of these symmetric spaces correspond to the cases (2-6), (2-2) with $b = 0$, (2-3) with $b = 0$, and (2-9) in Theorem 4.9, respectively. Except for the case of $\mathbb{O}P^2$, homogeneous biharmonic hypersurfaces in compact, rank one symmetric spaces were classified ([IIU2], [IIU]). Therefore, we consider the octonionic projective plane $\mathbb{O}P^2 \cong F_4/\mathrm{Spin}(9)$.

Let $(G, K_1, K_2) = (F_4, \mathrm{Spin}(9), \mathrm{Spin}(9))$ with $\theta_1 = \theta_2$. This is the case of (2-9) in Theorem 4.9. Since $K_1 = K_2$, we denote

$$\mathfrak{k} := \mathfrak{k}_1 = \mathfrak{k}_2, \quad \mathfrak{m} := \mathfrak{m}_1 = \mathfrak{m}_2.$$

We define an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ by $\langle\cdot,\cdot\rangle = -\mathrm{Killing}(\cdot,\cdot)$. Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$. Then we have $\Sigma^+ = \{\alpha, 2\alpha\}$ and $m(\alpha) = 8$, $m(2\alpha) = 7$ ([He], Page 534). By letting $n(\alpha) = n(2\alpha) = 0$ in Corollary 4.7 since $W^+ = \emptyset$, we can see that the biharmonic condition is equivalent to

$$9 = 2(\cot\langle\alpha, H\rangle)^2 + 7(\cot\langle 2\alpha, H\rangle)^2.$$

Thus we have

$$(\cot\langle\alpha, H\rangle)^2 = \frac{25 \pm 2\sqrt{130}}{15}.$$

The harmonic condition $\tau_H = 0$ is equivalent to

$$4\cot\langle\alpha, H\rangle + 7\cot\langle 2\alpha, H\rangle = 0.$$

Thus we have

$$(\cot\langle\alpha, H\rangle)^2 = \frac{7}{15}.$$

Since

$$0 < \frac{25 - 2\sqrt{130}}{15} < \frac{7}{15} < \frac{25 + 2\sqrt{130}}{15},$$

an orbit $K_2\pi_1(x)$ is proper biharmonic if and only if

$$(\cot\langle\alpha, H\rangle)^2 = \frac{25 \pm 2\sqrt{130}}{15}$$

holds for $H \in \mathfrak{a}$ with $0 < \langle \alpha, H \rangle < \pi/2$. Furthermore, a unique harmonic regular orbit exists between two proper biharmonic orbits in $\{H \in \mathfrak{a} \mid 0 < \langle \alpha, H \rangle < \pi/2\}$. These regular orbits are diffeomorphic to S^{15} embedded in $\mathbb{O}P^2$.

4.6. **Cases of cohomogeneity two or greater.** When $\dim \mathfrak{a} = 1$, proper biharmonic orbits are classified in [OSU]. Hence we consider cases of $\dim \mathfrak{a} \geq 2$. In particular, we consider cases of $\dim \mathfrak{a} = 2$. Then cohomogeneity two commutative Hermann action classified into the following cases:

- isotropy actions ($K_1 = K_2$)
  - Type A_2
    - $(\mathrm{SU}(3), \mathrm{SO}(3))$,
    - $(\mathrm{SU}(3) \times \mathrm{SU}(3), \mathrm{SU}(3))$,
    - $(\mathrm{SU}(6), \mathrm{Sp}(3))$,
    - (E_6, F_4),
  - Type B_2
    - $(\mathrm{SO}(3) \times \mathrm{SO}(3), \mathrm{SO}(3))$,
    - $(\mathrm{SO}(4+n), \mathrm{SO}(2) \times \mathrm{SO}(2+n))$,
  - Type C_2
    - $(\mathrm{Sp}(2), \mathrm{U}(2))$,
    - $(\mathrm{Sp}(2) \times \mathrm{Sp}(2), \mathrm{Sp}(2))$,
    - $(\mathrm{Sp}(4), \mathrm{Sp}(2) \times \mathrm{Sp}(2))$,
    - $(\mathrm{SU}(4), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)))$,
    - $(\mathrm{SO}(8), \mathrm{U}(4))$,
  - Type BC_2
    - $(\mathrm{SU}(4+n), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2+n)))$,
    - $(\mathrm{SO}(10), \mathrm{U}(5))$,
    - $(\mathrm{Sp}(4+n), \mathrm{Sp}(2) \times \mathrm{Sp}(2+n))$,
    - $(E_6, \mathrm{T}^1 \cdot \mathrm{Spin}(10))$,
  - Type G_2
    - $(G_2, \mathrm{SO}(4))$,
    - $(G_2 \times G_2, G_2)$,
- When $(\theta_1 \not\sim \theta_2)$
  - Type I-B_2
    - $(\mathrm{SO}(2+s+t), \mathrm{SO}(2+s) \times \mathrm{SO}(t), \mathrm{SO}(2) \times \mathrm{SO}(s+t))$ $(2 < t, 1 \leq s)$, $(m(e_1), m(e_1 - e_2), n(e_1)) = (t-2, 1, s)$,
    - $(\mathrm{SO}(6), \mathrm{SO}(3) \times \mathrm{SO}(3))$ (σ-action),
  - Type I-C_2
    - $(\mathrm{SO}(8), \mathrm{SO}(4) \times \mathrm{SO}(4), \mathrm{U}(4))$, $(m(e_1 - e_2), m(2e_1), n(e_1 - e_2)) = (2, 1, 2)$,
    - $(\mathrm{SU}(4), \mathrm{SO}(4), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)))$, $(m(e_1 - e_2), m(2e_1), n(e_1 - e_2)) = (1, 1, 1)$,
    - $(\mathrm{SU}(4), \mathrm{SO}(4))$ (σ-action),
    - $(\mathrm{SU}(4), \mathrm{Sp}(2))$ (σ-action),
  - Type I-BC_2–A_1^2
    - $(\mathrm{SU}(2+s+t), \mathrm{S}(\mathrm{U}(2+s) \times \mathrm{U}(t)), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(s+t)))$ $(2 < t, 1 \leq s)$, $(m(e_1), m(e_1 - e_2), m(2e_1), n(e_1)) = (2(t-2), 2, 1, 2s)$,
    - $(\mathrm{Sp}(2+s+t), \mathrm{Sp}(2+s) \times \mathrm{Sp}(t), \mathrm{Sp}(2) \times \mathrm{Sp}(s+t))$ $(2 < t, 1 \leq s)$, $(m(e_1), m(e_1 - e_2), m(2e_1), n(e_1)) = (4(t-2), 4, 3, 4s)$,

  * $(\mathrm{SO}(12), \mathrm{U}(6), \mathrm{U}(6)')$, $(m(e_1), m(e_1 - e_2), m(2e_1), n(e_1)) = (4, 4, 1, 4)$,
- Type I-BC_2–B_2
  * $(\mathrm{SO}(4+2s), \mathrm{SO}(4) \times \mathrm{SO}(2s), \mathrm{U}(2+s))$ $(2 < s)$, $(m(e_1), m(e_1 - e_2), m(2e_1), n(e_1 - e_2)) = (2(s-2), 2, 1, 2)$,
  * $(E_6, \mathrm{SU}(6) \cdot \mathrm{SU}(2), \mathrm{SO}(10) \cdot \mathrm{U}(1))$, $(m(e_1), m(e_1 - e_2), m(2e_1), n(e_1 - e_2)) = (4, 4, 1, 2)$,
  * $(E_7, \mathrm{SO}(12) \cdot \mathrm{SU}(2), E_6 \cdot \mathrm{U}(1))$ $(m(e_1), m(e_1 - e_2), m(2e_1), n(e_1 - e_2)) = (8, 6, 1, 2)$,
- Type II-BC_2
  * $(\mathrm{SU}(2+s), \mathrm{SO}(2+s), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(s)))$ $(2 < s)$, $(m(e_1), m(e_1 - e_2), n(2e_1)) = (s-2, 1, 1)$,
  * $(\mathrm{SO}(10), \mathrm{SO}(5) \times \mathrm{SO}(5), \mathrm{U}(5))$, $(m(e_1), m(e_1 - e_2), n(2e_1)) = (2, 2, 1)$,
  * $(E_6, \mathrm{Sp}(4), \mathrm{SO}(10) \cdot \mathrm{U}(1))$ $(m(e_1), m(e_1 - e_2), n(2e_1)) = (4, 3, 1)$,
- Type III-A_2
  * $(\mathrm{SU}(6), \mathrm{Sp}(3), \mathrm{SO}(6))$, $(m(e_1 - e_2), n(e_1 - e_2)) = (2, 2)$,
  * $(E_6, \mathrm{Sp}(4), F_4)$, $(m(e_1 - e_2), n(e_1 - e_2)) = (4, 4)$,
  * $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$, (condition B)
- Type III-B_2
  * $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$, (condition B)
- Type III-C_2
  * $(\mathrm{SU}(8), \mathrm{S}(\mathrm{U}(4) \times \mathrm{U}(4)), \mathrm{Sp}(4))$, $(m(e_1 - e_2), m(2e_1), n(e_1 - e_2), n(2e_1)) = (4, 3, 4, 1)$,
  * $(\mathrm{Sp}(4), \mathrm{U}(4), \mathrm{Sp}(2) \times \mathrm{Sp}(2))$, $(m(e_1 - e_2), m(2e_1), n(e_1 - e_2), n(2e_1)) = (2, 1, 2, 2)$,
  * $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$, (condition B)
- Type III-BC_2
  * $(\mathrm{SU}(4+2s), \mathrm{S}(\mathrm{U}(4) \times \mathrm{U}(2s)), \mathrm{Sp}(2+s))$ $(2 < s)$, $m(e_1), m(e_1 - e_2), m(2e_1), n(2e_1) = (4(s-2), 4, 3, 1)$,
  * $(\mathrm{SU}(10), \mathrm{S}(\mathrm{U}(5) \times \mathrm{U}(5)), \mathrm{Sp}(5))$, $m(e_1), m(e_1 - e_2), m(2e_1), n(2e_1) = (4, 4, 1, 3)$,
  * $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$, (condition B)
- Type III-G_2
  * $(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$, (condition B)

• reducible cases

Cases of $K_1 = K_2$

When $\Sigma \cap W = \emptyset$, Hermann actions are orbit equivalent to isotropy actions. Hence, we consider isotropy actions of compact symmetric spaces. We set a basis $\{H_\alpha\}_{\alpha \in \Pi}$ of $\mathfrak{a}$ as follows;

$$\langle H_\alpha, \beta \rangle = 0 \ (\alpha \neq \beta, \ \alpha, \beta \in \Pi), \quad \langle H_\alpha, \tilde{\alpha} \rangle = \pi,$$

where $\tilde{\alpha}$ is the highest root of Σ. We set a subset P_0 of $\mathfrak{a}$ by

$$P_0 = \{H \in \mathfrak{a} \mid \langle H, \alpha \rangle > 0 \ (\alpha \in \Pi), \ \langle H, \tilde{\alpha} \rangle < \pi\}.$$

then we have

$$P_0 = \left\{ \sum_{\alpha \in Pi} t_\alpha H_\alpha \mid t_\alpha > 0 \ (\alpha \in \Pi), \ \sum_{\alpha \in \Pi} t_\alpha < 1 \right\}.$$

4.6.1. *Type* A_2. We set

$$\mathfrak{a} = \{\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 \mid \xi_1 + \xi_3 + \xi_3 = 0\}.$$

Then, we have

$$\begin{aligned}&\Sigma^+ = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_1 + \alpha_2\},\ W^+ = \emptyset,\\ &m = m(\alpha) \qquad (\alpha \in \Sigma).\end{aligned}$$

When $H = tH_{\alpha_1}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_2\}$. Hence we have

$$\tau_H = m\cot\langle\alpha_1, H\rangle\alpha_1 + m\cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2) = m\cot\langle\alpha_1, H\rangle(2\alpha_1+\alpha_2).$$

Thus the orbit $K_2\pi_1(\exp H)$ is harmonic if and only if $\langle\alpha_1, H\rangle = \pi/2$.

By Theorem 4.6, the orbit $K_2\pi_1(\exp H)$ is biharmonic if and only if

$$\begin{aligned}0 =& m\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1\\ &+ m\langle\tau_H, \alpha_1+\alpha_2\rangle(1 - (\cot\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\ =& m\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)(2\alpha_1+\alpha_2)\end{aligned}$$

Thus we have $\tau_H = 0, \langle\alpha_1, H\rangle = (1/4)\pi,\ (3/4)\pi$. Therefor, the orbit $K_2\pi_1(\exp H)$ is proper biharmonic if and only if $\langle\alpha_1, H\rangle = (1/4)\pi,\ (3/4)\pi$.

By the same argument, we have the followings:

- The orbit $K_2\pi_1(\exp H)$ is proper biharmonic if and only if $\langle\alpha_2, H\rangle = (1/4)\pi,\ (3/4)\pi$ for $H = tH_{\alpha_2}$ $(0 < t < 1)$.
- The orbit $K_2\pi_1(\exp H)$ is proper biharmonic if and only if $\langle\alpha_1, H\rangle = (1/4)\pi,\ (3/4)\pi$ for $H = tH_{\alpha_1} + (1-t)H_{\alpha_2}$ $(0 < t < 1)$.

4.6.2. *Type* B_2 *and* C_2. We set

$$\begin{aligned}\Sigma^+ = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\},\ W^+ = \emptyset,\\ \tilde{\alpha} = \alpha_1 + 2\alpha_2 = e_1 + e_2,\end{aligned}$$

and

$$m_1 = m(e_1),\ m_2 = m(e_1 - e_2).$$

(1) When $H = tH_{\alpha_1}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{e_2\}$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& -m_2\cot\langle\alpha_1, H\rangle\alpha_1 - m_1\cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\ &- m_2\cot\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ =& -(2m_2+m_1)\cot\langle\alpha_1, H\rangle(\alpha_1+\alpha_2).\end{aligned}$$

Hence, $\tau_H = 0$ if and only if $\langle\alpha_1, H\rangle = \pi/2$. By Theorem 4.6, $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}0 =& m_2\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1\\ &+ m_1\langle\tau_H, \alpha_1+\alpha_2\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)(\alpha_1+\alpha_2)\\ &+ m_2\langle\tau_H, \alpha_1+2\alpha_2\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)(\alpha_1+2\alpha_2)\\ =& \langle\tau_H, \alpha_1\rangle(2m_2+m_1)(1 - (\cot\langle\alpha_1, H\rangle)^2)(\alpha_1+\alpha_2).\end{aligned}$$

Therefore, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or $\langle\alpha_1, H\rangle = \pi/4,\ (3/4)\pi$. In particular, $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if $\langle\alpha_1,\ H\rangle = \pi/4, (3/4)\pi$.

(2) When $H = tH_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{e_1 - e_2\}$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& - m_1 \cot\langle\alpha_2, H\rangle\alpha_2 - m_1 \cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\ &- m_2 \cot\langle\alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2)\\ =& - m_1 \cot\langle\alpha_2, H\rangle(\alpha_1 + 2\alpha_2)\\ &- (1/2)m_2(\cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle)(\alpha_1 + 2\alpha_2)\\ =& - (1/2)\{(2m_1 + m_2)\cot\langle\alpha_2, H\rangle - m_2 \tan\langle\alpha_2, H\rangle\}(\alpha_1 + 2\alpha_2).\end{aligned}$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{m_2}{2m_1 + m_2}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}0 =& m_1\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)\alpha_2\\ &+ m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2)\\ &+ m_2\langle\tau_H, \alpha_1 + 2\alpha_2\rangle(1 - (\cot\langle 2\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2)\\ =& + m_1\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2)\\ &+ m_2\langle\tau_H, 2\alpha_2\rangle(1 - (\cot\langle 2\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2)\\ =&\langle\tau_H, \alpha_2\rangle\{m_1(1 - (\cot\langle\alpha_2, H\rangle)^2)\\ &+ 2m_2(1 - (1/4)(\cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle)^2)\}(\alpha_1 + 2\alpha_2)\\ =&(1/2)\langle\tau_H, \alpha_2\rangle\{(2m_1 + m_2)(1 - (\cot\langle\alpha_2, H\rangle)^2)\\ &+ m_2(1 - (\tan\langle\alpha_2, H\rangle)^2) + 4m_2\}(\alpha_1 + 2\alpha_2).\end{aligned}$$

Therefore, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$(2m_1 + m_2)(1 - (\cot\langle\alpha_2, H\rangle)^2) + m_2(1 - (\tan\langle\alpha_2, H\rangle)^2) + 4m_2 = 0$$

holds. This equation is equivalent to

$$\big((2m_1 + m_2)(\cot\langle\alpha_2, H\rangle)^2 - m_2\big)((\cot\langle\alpha_2, H\rangle)^2 - 1) = 4m_2(\cot\langle\alpha_2, H\rangle)^2.$$

Since $m_2 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{m_1 + 3m_2 \pm \sqrt{m_1^2 + 4m_1m_2 + 8m_2^2}}{2m_1 + m_2}.$$

(3) When $H = tH_{\alpha_1} + (1 - t)H_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{e_1 + e_2\}$ and

$\langle\alpha_2, H\rangle = (\pi/2) - \langle\alpha_1, H\rangle$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m_2 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1 \cot\langle\alpha_2, H\rangle\alpha_2 - m_1 \cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\
=& - m_2 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1 \cot\big((\pi/2) - (\langle\alpha_1, H\rangle/2)\big)\alpha_2\\
& - m_1 \cot\big((\pi/2) + (\langle\alpha_1, H\rangle/2)\big)(\alpha_1+\alpha_2)\\
=& - m_2 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1 \tan(\langle\alpha_1, H\rangle/2)\alpha_2 + m_1 \tan(\langle\alpha_1, H\rangle/2)(\alpha_1+\alpha_2)\\
=& - m_2 \cot\langle\alpha_1, H\rangle\alpha_1 + m_1 \tan(\langle\alpha_1, H\rangle/2)\alpha_1\\
=& - (1/2)m_2(\cot(\langle\alpha_1, H\rangle/2) - \tan(\langle\alpha_1, H\rangle/2))\alpha_1 + m_1 \tan(\langle\alpha_1, H\rangle/2)\alpha_1\\
=&(1/2)\{-m_2 \cot(\langle\alpha_1, H\rangle/2) + (2m_1+m_2)\tan(\langle\alpha_1, H\rangle/2)\}\alpha_1.
\end{aligned}$$

Hence, $\tau_H = 0$ if and only if

$$\left(\cot\left(\frac{\langle\alpha_1, H\rangle}{2}\right)\right)^2 = \frac{2m_1+m_2}{m_2}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_2\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1 + m_1\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)\alpha_2\\
&+ m_1\langle\tau_H, \alpha_1+\alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\
=& m_2\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1 - (1/2)m_1\langle\tau_H, \alpha_1\rangle(1 - (\tan(\langle\alpha_2, H\rangle/2))^2)\alpha_2\\
&+ (1/2)m_1\langle\tau_H, \alpha_1\rangle(1 - (\tan(\langle\alpha_2, H\rangle/2))^2)(\alpha_1+\alpha_2)\\
=& m_2\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1 + (1/2)m_1\langle\tau_H, \alpha_1\rangle(1 - (\tan(\langle\alpha_2, H\rangle/2))^2)\alpha_1\\
=& (1/4)\langle\tau_H, \alpha_1\rangle\{4m_2 + m_2(1 - (\cot(\langle\alpha_1, H\rangle/2))^2)\\
&\quad + (2m_1+m_2)(1 - (\tan(\langle\alpha_2, H\rangle/2))^2)\}\alpha_1.
\end{aligned}$$

Therefore, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$4m_2 + m_2(1 - (\cot(\langle\alpha_1, H\rangle/2))^2) + (2m_1+m_2)(1 - (\tan(\langle\alpha_2, H\rangle/2))^2) = 0$$

holds. This equation is equivalent to

$$(m_2(\cot(\langle\alpha_1, H\rangle/2))^2 - (2m_1+m_2))((\cot(\langle\alpha_1, H\rangle/2))^2 - 1) = 4m_2(\cot(\langle\alpha_1, H\rangle/2))^2.$$

Since $m_2 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$(\cot(\langle\alpha_1, H\rangle/2))^2 = \frac{m_1 + 3m_2 \pm \sqrt{m_1^2 + 4m_1m_2 + 8m_2^2}}{m_2}$$

holds.

4.6.3. *Type* BC_2. We set

$$\begin{aligned}
\Sigma^+ =& \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, 2\alpha_2, 2\alpha_1+2\alpha_2\},\\
W^+ =& \emptyset,\ \tilde{\alpha} = 2\alpha_1 + 2\alpha_2,
\end{aligned}$$

and

$$m_1 = m(e_1),\ m_2 = m(e_1 - e_2),\ m_3 = m(2e_1).$$

(1) When $H = tH_{\alpha_1}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{e_2, 2e_2\}$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& - m_2 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1 \cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\ &- m_2 \cot\langle\alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2) - m_3 \cot\langle 2\alpha_1 + 2\alpha_2, H\rangle(2\alpha_1 + 2\alpha_2)\\ =& - (2m_2 + m_1) \cot\langle\alpha_1, H\rangle(\alpha_1 + \alpha_2) - m_3(\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle)(\alpha_1 + \alpha_2)\\ =& \{-(m_1 + 2m_2 + m_3) \cot\langle\alpha_1, H\rangle + m_3 \tan\langle\alpha_1, H\rangle\}(\alpha_1 + \alpha_2).\end{aligned}$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_1, H\rangle)^2 = \frac{m_3}{m_1 + 2m_2 + m_3}$$

holds. By Theorem 4.6, $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}0 =& m_2\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1\\ &+ m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)(\alpha_1 + \alpha_2)\\ &+ m_2\langle\tau_H, \alpha_1 + 2\alpha_2\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)(\alpha_1 + 2\alpha_2)\\ &+ m_3\langle\tau_H, 2\alpha_1 + 2\alpha_2\rangle(1 - (\cot\langle 2\alpha_1, H\rangle)^2)2(\alpha_1 + \alpha_2)\\ =& \langle\tau_H, \alpha_1\rangle(2m_2 + m_1)(1 - (\cot\langle\alpha_1, H\rangle)^2)(\alpha_1 + \alpha_2)\\ &+ m_3\langle\tau_H, \alpha_1\rangle(4 + (1 - (\cot\langle\alpha_1, H\rangle)^2) + (1 - (\tan\langle\alpha_1, H\rangle)^2))(\alpha_1 + \alpha_2)\\ =& \langle\tau_H, \alpha_1\rangle\{(2m_2 + m_1 + m_3)(1 - (\cot\langle\alpha_1, H\rangle)^2)\\ &\quad + m_3(1 - (\tan\langle\alpha_1, H\rangle)^2) + 4m_3\}(\alpha_1 + \alpha_2).\end{aligned}$$

Therefore, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$(2m_2 + m_1 + m_3)(1 - (\cot\langle\alpha_1, H\rangle)^2) + m_3(1 - (\tan\langle\alpha_1, H\rangle)^2) + 4m_3 = 0$$

holds. This equation is equivalent to

$$\big((2m_2 + m_1 + m_3)(\cot\langle\alpha_1, H\rangle)^2 - m_3\big)\big((\cot\langle\alpha_1, H\rangle)^2 - 1\big) = 4m_3(\cot\langle\alpha_1, H\rangle)^2.$$

Since $m_3 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$\begin{aligned}&(\cot\langle\alpha_1, H\rangle)^2\\ =& \frac{m_1 + 2m_2 + 6m_3 \pm \sqrt{(m_1 + 2m_2 + 6m_3) - 4(m_1 + 2m_2 + m_3)m_3}}{m_1 + 2m_2 + m_3}.\end{aligned}$$

(2) When $H = tH_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{e_1 - e_2\}$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& - m_1 \cot\langle\alpha_2, H\rangle\alpha_2 - m_1 \cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\ &- m_2 \cot\langle\alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2)\\ &- m_3 \cot\langle 2\alpha_2, H\rangle 2\alpha_2 - m_3 \cot\langle 2\alpha_1 + 2\alpha_2, H\rangle 2(\alpha_1 + \alpha_2)\\ =& - m_1 \cot\langle\alpha_2, H\rangle(\alpha_1 + 2\alpha_2)\\ &- (1/2)m_2(\cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle)(\alpha_1 + 2\alpha_2)\\ &- m_3(\cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle)(\alpha_1 + 2\alpha_2)\\ =& - (1/2)\{(2m_1 + m_2 + 2m_3) \cot\langle\alpha_2, H\rangle - (m_2 + 2m_3) \tan\langle\alpha_2, H\rangle\}(\alpha_1 + 2\alpha_2).\end{aligned}$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{m_2 + 2m_3}{2m_1 + m_2 + 2m_3}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_1\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)\alpha_2 \\
&+ m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2) \\
&+ m_2\langle\tau_H, \alpha_1 + 2\alpha_2\rangle(1 - (\cot\langle 2\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2) \\
&+ m_3\langle\tau_H, 2\alpha_2\rangle(1 - (\cot\langle 2\alpha_2, H\rangle)^2)2\alpha_2 \\
&+ m_3\langle\tau_H, 2\alpha_1 + 2\alpha_2\rangle(1 - (\cot\langle 2\alpha_2, H\rangle)^2)2(\alpha_1 + \alpha_2) \\
=& + m_1\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2) \\
&+ 2m_2\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle 2\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2) \\
&+ 4m_3\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle 2\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2) \\
=&\langle\tau_H, \alpha_2\rangle\{m_1(1 - (\cot\langle\alpha_2, H\rangle)^2) \\
&+ (2m_2 + 4m_3)(1 - (1/4)(\cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle)^2)\}(\alpha_1 + 2\alpha_2) \\
=&(1/2)\langle\tau_H, \alpha_2\rangle\{(2m_1 + m_2 + 2m_3)(1 - (\cot\langle\alpha_2, H\rangle)^2) \\
&+ (m_2 + 2m_3)(1 - (\tan\langle\alpha_2, H\rangle)^2) + 4(m_2 + 2m_3)\}(\alpha_1 + 2\alpha_2).
\end{aligned}$$

Therefore, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned}
&(2m_1 + m_2 + 2m_3)(1 - (\cot\langle\alpha_2, H\rangle)^2) \\
&\qquad + (m_2 + 2m_3)(1 - (\tan\langle\alpha_2, H\rangle)^2) + 4(m_2 + 2m_3) = 0
\end{aligned}$$

holds. This equation is equivalent to

$$\begin{aligned}
&\big((2m_1 + m_2 + 2m_3)(\cot\langle\alpha_2, H\rangle)^2 - (m_2 + 2m_3)\big)((\cot\langle\alpha_2, H\rangle)^2 - 1) \\
&\qquad = 4(m_2 + 2m_3)(\cot\langle\alpha_2, H\rangle)^2.
\end{aligned}$$

Since $m_2 + 2m_3 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{m_1 + 3(m_2 + 2m_3) \pm \sqrt{m_1^2 + 4m_1(m_2 + 2m_3) + 8(m_2 + 2m_3)^2}}{2m_1 + m_2 + 2m_3}.$$

(3) When $H = tH_{\alpha_1} + (1-t)H_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\tilde{\alpha} = 2e_1\}$ and $\langle \alpha_2, H\rangle = (\pi/2) - \langle \alpha_1, H\rangle$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m_2 \cot\langle \alpha_1, H\rangle \alpha_1 - m_1 \cot\langle \alpha_2, H\rangle \alpha_2 \\
&- m_1 \cot\langle \alpha_1 + \alpha_2, H\rangle (\alpha_1 + \alpha_2) - m_2 \cot\langle \alpha_1 + 2\alpha_2, H\rangle (\alpha_1 + 2\alpha_2) \\
&- m_3 \cot\langle 2\alpha_2, H\rangle 2\alpha_2 \\
=& - m_2 \cot\langle \alpha_1, H\rangle \alpha_1 - m_1 \cot\big((\pi/2) - \langle \alpha_1, H\rangle\big)\alpha_2 \\
&- m_1 \cot(\pi/2)(\alpha_1 + \alpha_2) - m_2 \cot\big(\pi - \langle \alpha_1, H\rangle\big)(\alpha_1 + 2\alpha_2) \\
&- m_3 \cot\big(\pi - \langle 2\alpha_1, H\rangle\big) 2\alpha_2 \\
=& - m_2 \cot\langle \alpha_1, H\rangle \alpha_1 - m_1 \tan\langle \alpha_1, H\rangle \alpha_2 \\
&+ m_2 \cot\langle \alpha_1, H\rangle (\alpha_1 + 2\alpha_2) \\
&+ m_3 (\cot\langle \alpha_1, H\rangle - \tan\langle \alpha_1, H\rangle)\alpha_2 \\
=& - (m_1 + m_3) \tan\langle \alpha_1, H\rangle \alpha_2 + (2m_2 + m_3) \cot\langle \alpha_1, H\rangle \alpha_2.
\end{aligned}$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle \alpha_1, H\rangle)^2 = \frac{m_1 + m_3}{2m_2 + m_3}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_2 \langle \tau_H, \alpha_1\rangle (1 - (\cot\langle \alpha_1, H\rangle)^2)\alpha_1 \\
&+ m_1 \langle \tau_H, \alpha_2\rangle (1 - (\cot\langle \alpha_2, H\rangle)^2)\alpha_2 \\
&+ m_1 \langle \tau_H, \alpha_1 + \alpha_2\rangle (1 - (\cot\langle \alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2) \\
&+ m_2 \langle \tau_H, \alpha_1 + 2\alpha_2\rangle (1 - (\cot\langle \alpha_1 + 2\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2) \\
&+ m_3 \langle \tau_H, 2\alpha_2\rangle (1 - (\cot\langle 2\alpha_2, H\rangle)^2) 2\alpha_2 \\
=& m_2 \langle \tau_H, \alpha_1\rangle (1 - (\cot\langle \alpha_1, H\rangle)^2)\alpha_1 \\
&+ m_1 \langle \tau_H, \alpha_2\rangle (1 - (\tan\langle \alpha_1, H\rangle)^2)\alpha_2 \\
&+ m_2 \langle \tau_H, \alpha_1 + 2\alpha_2\rangle (1 - (\cot\langle \alpha_1, H\rangle)^2)(\alpha_1 + 2\alpha_2) \\
&+ m_3 \langle \tau_H, 2\alpha_2\rangle (1 - (\cot\langle 2\alpha_1, H\rangle)^2) 2\alpha_2 \\
=& m_2 \langle \tau_H, \alpha_1\rangle (1 - (\cot\langle \alpha_1, H\rangle)^2)\alpha_1 \\
&- m_1 \langle \tau_H, \alpha_1\rangle (1 - (\tan\langle \alpha_1, H\rangle)^2)\alpha_2 \\
&- m_2 \langle \tau_H, \alpha_1\rangle (1 - (\cot\langle \alpha_1, H\rangle)^2)(\alpha_1 + 2\alpha_2) \\
&- m_3 \langle \tau_H, \alpha_1\rangle (4 + (1 - (\cot\langle \alpha_1, H\rangle)^2) + (1 - (\tan\langle \alpha_1, H\rangle)^2))\alpha_2 \\
=& - \langle \tau_H, \alpha_1\rangle \{(m_3 + 2m_2)(1 - (\cot\langle \alpha_1, H\rangle)^2) \\
&\quad + (m_1 + m_3)(1 - (\tan\langle \alpha_1, H\rangle)^2) + 4m_3\}\alpha_2.
\end{aligned}$$

Therefore, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$(m_3 + 2m_2)(1 - (\cot\langle \alpha_1, H\rangle)^2) + (m_1 + m_3)(1 - (\tan\langle \alpha_1, H\rangle)^2) + 4m_3 = 0$$

holds. This equation is equivalent to

$$\big((m_3 + 2m_2)(\cot\langle \alpha_1, H\rangle)^2 - (m_1 + m_3)\big)((\cot\langle \alpha_1, H\rangle)^2 - 1) = 4m_3(\cot\langle \alpha_1, H\rangle)^2$$

Since $m_3 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot(\langle\alpha_1, H\rangle))^2 = \frac{m_1 + 2m_2 + 6m_3 \pm \sqrt{(m_1 - 2m_2)^2 + 8m_3(m_1 + 2m_2 + 4m_3)}}{2(2m_2 + m_3)}$$

holds.

4.6.4. *Type* G_2. We set

$$\begin{aligned}
\Sigma^+ &= \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\},\ W^+ = \emptyset,\\
&\langle\alpha_1, \alpha_1\rangle = 1,\ \langle\alpha_1, \alpha_2\rangle = -\frac{3}{2},\ \langle\alpha_2, \alpha_2\rangle = 3,\\
\tilde{\alpha} &= 3\alpha_1 + 2\alpha_2,
\end{aligned}$$

and

$$m = m(\alpha_1) = m(\alpha_2).$$

(1) When $H = tH_{\alpha_1}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_2\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m\cot\langle\alpha_1, H\rangle\alpha_1 - m\cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\
&- m\cot\langle 2\alpha_1 + \alpha_2, H\rangle(2\alpha_1 + \alpha_2) - m\cot\langle 3\alpha_1 + \alpha_2, H\rangle(3\alpha_1 + \alpha_2)\\
&- m\cot\langle 3\alpha_1 + 2\alpha_2, H\rangle(3\alpha_1 + 2\alpha_2)\\
=& - m\{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle + 3\cot\langle 3\alpha_1, H\rangle\}(2\alpha_1 + \alpha_2)\\
=& - m\left\{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle + 3\frac{\cot\langle\alpha_1, H\rangle\cot\langle 2\alpha_1, H\rangle - 1}{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle}\right\}(2\alpha_1 + \alpha_2).
\end{aligned}$$

Thus, $\tau_H = 0$ if and only if

$$\left\{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle + 3\frac{\cot\langle\alpha_1, H\rangle\cot\langle 2\alpha_1, H\rangle - 1}{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle}\right\} = 0.$$

Since

$$\begin{aligned}
&\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle + 3\frac{\cot\langle\alpha_1, H\rangle\cot\langle 2\alpha_1, H\rangle - 1}{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle}\\
=&(\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle)^2 + 3\{\cot\langle\alpha_1, H\rangle\cot\langle 2\alpha_1, H\rangle - 1\}\\
=&\frac{1}{4}(3\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle)^2 + \frac{3}{2}\{\cot\langle\alpha_1, H\rangle(\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle) - 2\}\\
=&\frac{1}{4}(3\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle)^2 + \frac{3}{2}\{(\cot\langle\alpha_1, H\rangle)^2 - 3\}\\
=&\frac{1}{4}[(3\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle)^2 + 6\{(\cot\langle\alpha_1, H\rangle)^2 - 3\}]\\
=&\frac{1}{4}[9(\cot\langle\alpha_1, H\rangle)^2 - 6 + (\tan\langle\alpha_1, H\rangle)^2 + 6(\cot\langle\alpha_1, H\rangle)^2 - 18]\\
=&\frac{1}{4}[15(\cot\langle\alpha_1, H\rangle)^2 - 24 + (\tan\langle\alpha_1, H\rangle)^2]
\end{aligned}$$

The equation is equivalent to

$$15(\cot\langle\alpha_1, H\rangle)^4 - 24(\cot\langle\alpha_1, H\rangle) + 1 = 0.$$

Since $0 < \langle\alpha_1, H\rangle < (\pi/3)$, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_1, H\rangle)^2 = \frac{12+\sqrt{129}}{15}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m\{\langle\tau_H, \alpha_1\rangle(1-(\cot\langle\alpha_1, H\rangle)^2)\alpha_1 \\
&+ \langle\tau_H, \alpha_1+\alpha_2\rangle(1-(\cot\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2) \\
&+ \langle\tau_H, 2\alpha_1+\alpha_2\rangle(1-(\cot\langle 2\alpha_1+\alpha_2, H\rangle)^2)(2\alpha_1+\alpha_2) \\
&+ \langle\tau_H, 3\alpha_1+\alpha_2\rangle(1-(\cot\langle 3\alpha_1+\alpha_2, H\rangle)^2)(3\alpha_1+\alpha_2) \\
&+ \langle\tau_H, 3\alpha_1+2\alpha_2\rangle(1-(\cot\langle 3\alpha_1+2\alpha_2, H\rangle^2)(3\alpha_1+2\alpha_2)\} \\
=& m\langle\tau_H, \alpha_1\rangle\{(1-(\cot\langle\alpha_1, H\rangle)^2) \\
&\quad + 2(1-(\cot\langle 2\alpha_1, H\rangle)^2) + 9(1-(\cot\langle 3\alpha_1, H\rangle)^2)\}(2\alpha_1+\alpha_2).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&(1-(\cot\langle\alpha_1, H\rangle)^2) + 2(1-(\cot\langle 2\alpha_1, H\rangle)^2) + 9(1-(\cot\langle 3\alpha_1, H\rangle)^2) \\
=& 12 - \left[(\cot\langle\alpha_1, H\rangle)^2 + 2(\cot\langle 2\alpha_1, H\rangle)^2 + 9(\cot\langle 3\alpha_1, H\rangle)^2\right] \\
=& 12 - \left[(\cot\langle\alpha_1, H\rangle)^2 + 2(\cot\langle 2\alpha_1, H\rangle)^2 + 9\left(\frac{\cot\langle\alpha_1, H\rangle\cot\langle 2\alpha_1, H\rangle - 1}{\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle}\right)^2\right].
\end{aligned}$$

Thus, the orbit $K_2\pi_1(\exp H)$ is biharmonic if and only if

$$\begin{aligned}
0 =& \{(\cot\langle\alpha_1, H\rangle)^2 + 2(\cot\langle 2\alpha_1, H\rangle)^2\}(\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle)^2 \\
&+ 9(\cot\langle\alpha_1, H\rangle\cot\langle 2\alpha_1, H\rangle - 1)^2 - 12(\cot\langle\alpha_1, H\rangle + \cot\langle 2\alpha_1, H\rangle)^2 \\
=& \frac{1}{8}\{(3(\cot\langle\alpha_1, H\rangle)^2 - 2 - (\tan\langle\alpha_1, H\rangle)^2)(3\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle)^2\} \\
&+ \frac{9}{4}\{(\cot\langle\alpha_1, H\rangle)^2 - 3\}^2 - 3(3\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle)^2 \\
=& \frac{1}{8}\{(3(\cot\langle\alpha_1, H\rangle)^2 - 26 - (\tan\langle\alpha_1, H\rangle)^2)(3\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle)^2\} \\
&+ \frac{18}{8}\{(\cot\langle\alpha_1, H\rangle)^4 - 6(\cot\langle\alpha_1, H\rangle)^2 + 9\}^2 \\
=& \frac{1}{8}\{45(\cot\langle\alpha_1, H\rangle)^4 - 378(\cot\langle\alpha_1, H\rangle)^2 + 318 \\
&- 30(\tan\langle\alpha_1, H\rangle)^2 + (\tan\langle\alpha_1, H\rangle)^4\} \\
=& \frac{(\tan\langle\alpha_1, H\rangle)^4}{8}\{45(\cot\langle\alpha_1, H\rangle)^8 - 378(\cot\langle\alpha_1, H\rangle)^6 \\
&+ 318(\cot\langle\alpha_1, H\rangle)^4 - 30(\cot\langle\alpha_1, H\rangle)^2 + 1\}.
\end{aligned}$$

We set $x = (\cot\langle\alpha_1, H\rangle)^2$ and

$$f(x) = 45x^4 - 378x^3 + 318x^2 - 30x + 1.$$

Then,

$$\begin{aligned}&\frac{df}{dx}(x) = 180x^3 - 1026x^2 + 636x - 30 = 6(x-5)(30x^2 - 21x + 1)\\ =&180(x-5)\left(x - \frac{21+\sqrt{321}}{60}\right)\left(x - \frac{21-\sqrt{321}}{60}\right).\end{aligned}$$

Since

$$f(1/3) = (128/9) > 0, \frac{df}{dx}(1/3) = \frac{224}{3} > 0, f(5) = -6824 < 0 \text{ and } f(7) = 6112 > 0,$$

the equation $f(x) = 0$ has distinct two solutions for $(1/3) < x$. Therefore, there exist $0 < t_-, t_+ < 1$ such that the orbits $K_2\pi_1(\exp(t_\pm H_{\alpha_1}))$ are biharmonic. Since

$$f\left(\frac{12+\sqrt{129}}{15}\right) \neq 0$$

the orbits $K_2\pi_1(\exp(t_\pm H_{\alpha_1}))$ are proper biharmonic.

(2) When $H = tH_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_1\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& - m\cot\langle\alpha_2, H\rangle\alpha_2 - m\cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\ &- m\cot\langle 2\alpha_1+\alpha_2, H\rangle(2\alpha_1+\alpha_2) - m\cot\langle 3\alpha_1+\alpha_2, H\rangle(3\alpha_1+\alpha_2)\\ &- m\cot\langle 3\alpha_1+2\alpha_2, H\rangle(3\alpha_1+2\alpha_2)\\ =& - m\{2\cot\langle\alpha_2, H\rangle + \cot\langle 2\alpha_2, H\rangle\}(3\alpha_1+2\alpha_2)\\ =& -\frac{1}{2}m\{5\cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle\}(3\alpha_1+2\alpha_2).\end{aligned}$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{1}{5}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}0 =& m\{\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)\alpha_2\\ &+ \langle\tau_H, \alpha_1+\alpha_2\rangle(1 - (\cot\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\ &+ \langle\tau_H, 2\alpha_1+\alpha_2\rangle(1 - (\cot\langle 2\alpha_1+\alpha_2, H\rangle)^2)(2\alpha_1+\alpha_2)\\ &+ \langle\tau_H, 3\alpha_1+\alpha_2\rangle(1 - (\cot\langle 3\alpha_1+\alpha_2, H\rangle)^2)(3\alpha_1+\alpha_2)\\ &+ \langle\tau_H, 3\alpha_1+2\alpha_2\rangle(1 - (\cot\langle 3\alpha_1+2\alpha_2, H\rangle)^2)(3\alpha_1+2\alpha_2)\}\\ =& 2m\langle\tau_H, \alpha_2\rangle\{(1 - (\cot\langle\alpha_2, H\rangle)^2) + (1 - (\cot\langle 2\alpha_2, H\rangle^2)\}(3\alpha_1+2\alpha_2)\\ =& \frac{1}{2}m\langle\tau_H, \alpha_2\rangle\{5(1 - (\cot\langle\alpha_2, H\rangle)^2) + (1 - (\tan\langle\alpha_2, H\rangle)^2) + 4\}(3\alpha_1+2\alpha_2).\end{aligned}$$

Therefore, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$5(1 - (\cot\langle\alpha_2, H\rangle)^2) + (1 - (\tan\langle\alpha_2, H\rangle)^2) + 4 = 0$$

holds. This equation is equivalent to

$$\big(5(\cot\langle\alpha_2, H\rangle)^2 - 1\big)\big((\cot\langle\alpha_2, H\rangle)^2 - 1\big) = 4(\cot\langle\alpha_2, H\rangle)^2.$$

Thus, the solutions of the equation are not harmonic. Hence the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot(\langle\alpha_2, H\rangle))^2 = \frac{5 \pm 2\sqrt{5}}{5}$$

holds.

(3) When $H = tH_{\alpha_1} + (1-t)H_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{3\alpha_1 + \alpha_2\}, W_H^+ = \emptyset$. We set $\vartheta = (\pi/6)t$. Then,

$$\langle\alpha_1, H\rangle = 2\vartheta,\ \langle\alpha_2, H\rangle = \frac{\pi}{2} - 3\vartheta.$$

By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& - m\{\cot(2\vartheta)\alpha_1 + \cot((\pi/2) - 3\vartheta)\alpha_2 + \cot((\pi/2) - \vartheta)(\alpha_1 + \alpha_2)\\ &+ \cot((\pi/2) + \vartheta)(2\alpha_1 + \alpha_2) + \cot((\pi/2) + 3\vartheta)(3\alpha_1 + \alpha_2)\}\\ =& - m\{\cot(2\vartheta) + \tan\vartheta + \tan(3\vartheta)\}\alpha_1.\end{aligned}$$

Since

$$\tan(3\vartheta) = \frac{\cot\vartheta + \cot(2\vartheta)}{\cot\vartheta\cot(2\vartheta) - 1},$$

$\tau_H = 0$ if and only if,

$$\begin{aligned}0 =&(\cot(2\vartheta) - \tan\vartheta)(\cot\vartheta\cot(2\vartheta) - 1) - 3(\cot\vartheta + \cot(2\vartheta))\\ =&\{(\cot\vartheta)^4 - 24(\cot\vartheta)^2 + 15\}/(\cot\vartheta)\end{aligned}$$

Since $0 < \vartheta < (\pi/6)$, $\cot\vartheta > \sqrt{3}$. Hence $\tau_H = 0$ if and only if,

$$(\cot\vartheta)^2 = 12 + \sqrt{129}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}0 =&m\{\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1\\ &+ \langle\tau_H, \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)\alpha_2\\ &+ \langle\tau_H, (\alpha_1 + \alpha_2)\rangle(1 - (\cot\langle\alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2)\\ &+ \langle\tau_H, (2\alpha_1 + \alpha_2)\rangle(1 - (\cot\langle 2\alpha_1 + \alpha_2, H\rangle)^2)(2\alpha_1 + \alpha_2)\\ &+ \langle\tau_H, (3\alpha_1 + \alpha_2)\rangle(1 - (\cot\langle 3\alpha_1 + \alpha_2, H\rangle)^2)(3\alpha_1 + \alpha_2)\}\\ =&(m/2)\langle\tau_H, \alpha_1\rangle\{2(1 - (\cot(2\vartheta))^2)\alpha_1 - 3(1 - (\cot((\pi/2) - 3\vartheta))^2)\alpha_2\\ &- (1 - (\cot((\pi/2) - \vartheta))^2)(\alpha_1 + \alpha_2) + (1 - (\cot((\pi/2) + \vartheta))^2)(2\alpha_1 + \alpha_2)\\ &+ 3(1 - (\cot((\pi/2) + 3\vartheta)^2)(3\alpha_1 + \alpha_2)\}\\ =&(m/2)\langle\tau_H, \alpha_1\rangle\{2(1 - (\cot(2\vartheta))^2)\alpha_1 - 3(1 - (\tan(3\vartheta))^2)\alpha_2\\ &- (1 - (\tan(\vartheta))^2)(\alpha_1 + \alpha_2) + (1 - (\tan(\vartheta))^2)(2\alpha_1 + \alpha_2)\\ &+ 3(1 - (\tan(3\vartheta)^2)(3\alpha_1 + \alpha_2)\}\\ =&(m/2)\langle\tau_H, \alpha_1\rangle\{2(1 - (\cot(2\vartheta))^2) + (1 - (\tan(\vartheta))^2) + 9(1 - (\tan(3\vartheta))^2)\}\alpha_1.\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$2(1 - (\cot(2\vartheta))^2) + (1 - (\tan(\vartheta))^2) + 9(1 - (\tan(3\vartheta))^2) = 0$$

holds. Then we have

$$\begin{aligned}&2(1-(\cot(2\vartheta))^2)+(1-(\tan(\vartheta))^2)+9(1-(\tan(3\vartheta))^2)\\=&(12-2(\cot(2\vartheta))^2-(\tan(\vartheta))^2)-9\frac{(\cot(2\vartheta)+\cot(\vartheta))^2}{((\cot(2\vartheta))(\cot(\vartheta))-1)^2}\end{aligned}$$

Thus $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H=0$ or

$$\{12-2(\cot(2\vartheta))^2-(\tan(\vartheta))^2\}((\cot(2\vartheta))(\cot(\vartheta))-1)^2-9(\cot(2\vartheta)+\cot(\vartheta))^2=0$$

Then,

$$\begin{aligned}&\{12-2(\cot(2\vartheta))^2-(\tan(\vartheta))^2\}((\cot(2\vartheta))(\cot(\vartheta))-1)^2-9(\cot(2\vartheta)+\cot(\vartheta))^2\\=&\{12-(1/2)(\cot(\vartheta)-\tan(\vartheta))^2-(\tan(\vartheta))^2\}\times(1/4)\{(\cot(\vartheta))^2-3\}^2\\&+(9/4)\{3\cot(\vartheta)-\tan(\vartheta)\}^2\\=&-(1/8)[\{(\cot(\vartheta))^2-26+3(\tan(\vartheta))^2\}\{(\cot(\vartheta))^4-6(\cot(\vartheta))^2+9\}\\&+18\{9(\cot(\vartheta))^2-6+(\tan(\vartheta))^2\}]\\=&-(1/8)[(\cot(\vartheta))^6-32(\cot(\vartheta))^4+330(\cot(\vartheta))^2-360+45(\tan(\vartheta))^2]\\=&-(1/8)(\tan(\vartheta))^2[(\cot(\vartheta))^8-32(\cot(\vartheta))^6+330(\cot(\vartheta))^4-360(\cot(\vartheta))^2+45]\end{aligned}$$

We set $x=(\cot(\vartheta))^2$ and

$$f(x)=x^4-32x^3+330x^2-360x+45.$$

Then,

$$\begin{aligned}\frac{df}{dx}(x)&=4(3x^3-24x^2+165x-90)\\\frac{d^2f}{dx^2}(x)&=12(x-5)(x-11)\end{aligned}$$

Since

$$f(3)=1152>0,\frac{df}{dx}(3)=864>0,\frac{df}{dx}(11)=608>0,$$

$(df/dx)(x)>0 and f(x)>0$ for $3<x$. Thus the equation $f(x)=0$ has no solution for $3<x$. Therefore, if the orbits $K_2\pi_1(\exp(t_\pm H_{\alpha_1}))$ is harmonic, then it is harmonic.

Cases of $\theta_1\not\sim\theta_2$
Let $\tilde{\alpha}\in\{\alpha\in W^+\mid\alpha+\lambda\notin W\ (\lambda\in\Pi)\}$.

4.6.5. *Type* I-B$_2$ *and* I-BC$_2$-A$_1^2$. We set

$$\begin{aligned}\Sigma^+=\{e_1\pm e_2,e_1,e_2,2e_1,2e_2\},\ W^+=\{e_1,e_2\},\\\Pi=\{\alpha_1=e_1-e_2,\alpha_2=e_2\},\ \tilde{\alpha}=\alpha_1+\alpha_2=e_1\end{aligned}$$

and

$$m_1=m(e_1),\ m_2=m(e_1+e_2),\ m_3=m(2e_1),n_1=n(e_1),$$

where, if $(\tilde{\Sigma}, \Sigma, W)$ is type I-B$_2$, then $m_3 = 0$.
(1) When $H = tH_{\alpha_1}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_2, 2\alpha_2\}$ and $W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m_2 \cot\langle \alpha_1, H\rangle \alpha_1 - m_1 \cot\langle \alpha_1 + \alpha_2, H\rangle (\alpha_1 + \alpha_2) \\
& - m_2 \cot\langle \alpha_1 + 2\alpha_2, H\rangle (\alpha_1 + 2\alpha_2) - m_3 \cot\langle 2(\alpha_1 + \alpha_2), H\rangle 2(\alpha_1 + \alpha_2) \\
& + n_1 \tan\langle \alpha_1 + \alpha_2, H\rangle (\alpha_1 + \alpha_2) + n_1 \tan\langle \alpha_2, H\rangle \alpha_2 \\
=& - \{(2m_2 + m_1) \cot\langle \alpha_1, H\rangle e_1 + m_3 \cot\langle 2\alpha_1, H\rangle 2e_1\} + n_1 \tan\langle \alpha_1, H\rangle (\alpha_1 + \alpha_2) \\
=& - \{(2m_2 + m_1) \cot\langle \alpha_1, H\rangle e_1 + m_3 (\cot\langle \alpha_1, H\rangle - \tan\langle \alpha_1, H\rangle) e_1\} \\
& + n_1 \tan\langle \alpha_1, H\rangle (\alpha_1 + \alpha_2) \\
=& \{-(2m_2 + m_1 + m_3) \cot\langle \alpha_1, H\rangle + (n_1 + m_3) \tan\langle \alpha_1, H\rangle\} e_1.
\end{aligned}$$

Hence we have that $\tau_H = 0$ if and only if

$$(\cot\langle \alpha_1, H\rangle)^2 = \frac{n_1 + m_3}{m_1 + 2m_2 + m_3}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_2 \langle \tau_H, \alpha_1\rangle (1 - (\cot\langle \alpha_1, H\rangle)^2) \alpha_1 \\
& + m_1 \langle \tau_H, \alpha_1 + \alpha_2\rangle (1 - (\cot\langle \alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2) \\
& + m_2 \langle \tau_H, \alpha_1 + 2\alpha_2\rangle (1 - (\cot\langle \alpha_1 + 2\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2) \\
& + m_3 \langle \tau_H, 2(\alpha_1 + \alpha_2)\rangle (1 - (\cot\langle 2(\alpha_1 + \alpha_2), H\rangle)^2) 2(\alpha_1 + \alpha_2) \\
& + n_1 \langle \tau_H, (\alpha_1 + \alpha_2)\rangle (1 - (\tan\langle \alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2) \\
& + n_1 \langle \tau_H, \alpha_2\rangle (1 - (\tan\langle \alpha_2, H\rangle)^2) \alpha_2 \\
=& \langle \tau_H, \alpha_1\rangle \{(2m_2 + m_1)(1 - (\cot\langle \alpha_1, H\rangle)^2) + 4m_3 (1 - (\cot\langle 2\alpha_1, H\rangle)^2) \\
& + n_1 (1 - (\tan\langle \alpha_1, H\rangle)^2)\} e_1 \\
=& \langle \tau_H, \alpha_1\rangle \{(m_1 + 2m_2 + m_3)(1 - (\cot\langle \alpha_1, H\rangle)^2) + 4m_3 \\
& + (n_1 + m_3)(1 - (\tan\langle \alpha_1, H\rangle)^2)\} e_1.
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned}
(4.7) \qquad & (m_1 + 2m_2 + m_3)(\cot\langle \alpha_1, H\rangle)^4 \\
& - \{(m_1 + 2m_2 + m_3) + (n_1 + m_3) + 4m_3\}(\cot\langle \alpha_1, H\rangle)^2 + n_1 + m_3 = 0
\end{aligned}$$

holds. Since $\tau_H = 0$ if and only if

$$(\cot\langle \alpha_1, H\rangle)^2 = \frac{n_1 + m_3}{m_1 + 2m_2 + m_3},$$

$K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$\begin{aligned}
&(\cot\langle \alpha_1, H\rangle)^2 \\
=& \begin{cases} \frac{-(m_1+2m_2+6m_3+n_1) \pm \sqrt{(m_1+2m_2+6m_3+n_1)^2 - 4(m_1+2m_2+m_3)(n_1+m_3)}}{2(m_1+2m_2+m_3)} & (m_3 > 0) \\ 1 & (m_3 = 0). \end{cases}
\end{aligned}$$

Let H_+ and H_- denote the solutions of the biharmonic equation (4.7) such that $(\cot\langle \alpha_1, H_-\rangle)^2 \leq (\cot\langle \alpha_1, H_+\rangle)^2$. Let H_0 denotes the harmonic point such that

$0 < \langle \alpha_1, H_0 \rangle < \pi/2$. Since

$$\begin{aligned}&(m_1 + 2m_2 + m_3)(\cot\langle \alpha_1, H\rangle)^4\\&+\{(m_1 + 2m_2 + m_3) + (n_1 + m_3) + 4m_3\}(\cot\langle \alpha_1, H\rangle)^4 + n_1 + m_3 = 0\end{aligned}$$

if and only if

$$\begin{aligned}&\{(m_1 + 2m_2 + m_3)(\cot\langle \alpha_1, H\rangle)^2 - (n_1 + m_3)\}((\cot\langle \alpha_1, H\rangle)^2 - 1)\\=&4m_3(\cot\langle \alpha_1, H\rangle)^2,\end{aligned}$$

if $m_3 > 0$, then

$$\langle \alpha_1, H_-\rangle < \langle \alpha_1, H_0\rangle < \langle \alpha_1, H_+\rangle.$$

(2) When $H = tH_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_1\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& - m_1 \cot\langle \alpha_2, H\rangle\alpha_2 - m_1 \cot\langle \alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\&- m_2 \cot\langle \alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2) - m_3 \cot\langle 2(\alpha_1 + \alpha_2), H\rangle 2(\alpha_1 + \alpha_2)\\&- m_3 \cot\langle 2\alpha_2, H\rangle 2\alpha_2\\&+ n_1 \tan\langle \alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2) + n_1 \tan\langle \alpha_2, H\rangle\alpha_2\\=& - m_1 \cot\langle \alpha_2, H\rangle\alpha_2 - m_1 \cot\langle \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\&- m_2 \cot\langle 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2) - m_3 \cot\langle 2\alpha_2, H\rangle 2(\alpha_1 + \alpha_2)\\&- m_3 \cot\langle 2\alpha_2, H\rangle 2\alpha_2\\&+ n_1 \tan\langle \alpha_2, H\rangle(\alpha_1 + \alpha_2) + n_1 \tan\langle \alpha_2, H\rangle\alpha_2\\=& - m_1 \cot\langle \alpha_2, H\rangle(\alpha_1 + 2\alpha_2)\\&- (m_2 + 2m_3) \cot\langle 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2)\\&+ n_1 \tan\langle \alpha_2, H\rangle(\alpha_1 + 2\alpha_2)\\=&\frac{1}{2}\{-(2m_1 + m_2 + 2m_3) \cot\langle \alpha_2, H\rangle\\&\quad + (2n_1 + m_2 + 2m_3) \tan\langle \alpha_2, H\rangle\}(\alpha_1 + 2\alpha_2).\end{aligned}$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle \alpha_2, H\rangle)^2 = \frac{2n_1 + m_2 + 2m_3}{2m_1 + m_2 + 2m_3}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_1\langle\tau_H,\alpha_2\rangle(1-(\cot\langle\alpha_2,H\rangle)^2)\alpha_2\\
&+ m_1\langle\tau_H,\alpha_1+\alpha_2\rangle(1-(\cot\langle\alpha_1+\alpha_2,H\rangle)^2)(\alpha_1+\alpha_2)\\
&+ m_2\langle\tau_H,\alpha_1+2\alpha_2\rangle(1-(\cot\langle\alpha_1+2\alpha_2,H\rangle)^2)(\alpha_1+2\alpha_2)\\
&+ m_3\langle\tau_H,2(\alpha_1+\alpha_2)\rangle(1-(\cot\langle 2(\alpha_1+\alpha_2),H\rangle)^2)2(\alpha_1+\alpha_2)\\
&+ m_3\langle\tau_H,2\alpha_2\rangle(1-(\cot\langle 2\alpha_2,H\rangle)^2)2\alpha_2\\
&+ n_1\langle\tau_H,(\alpha_1+\alpha_2)\rangle(1-(\tan\langle\alpha_1+\alpha_2,H\rangle)^2)(\alpha_1+\alpha_2)\\
&+ n_2\langle\tau_H,\alpha_2\rangle(1-(\tan\langle\alpha_2,H\rangle)^2)\alpha_2\\
=& + m_1\langle\tau_H,\alpha_2\rangle(1-(\cot\langle\alpha_2,H\rangle)^2)(\alpha_1+2\alpha_2)\\
&+ 2m_2\langle\tau_H,\alpha_2\rangle(1-(\cot\langle 2\alpha_2,H\rangle)^2)(\alpha_1+2\alpha_2)\\
&+ 4m_3\langle\tau_H,\alpha_2\rangle(1-(\cot\langle 2\alpha_2,H\rangle)^2)(\alpha_1+2\alpha_2)\\
&+ n_1\langle\tau_H,\alpha_2\rangle(1-(\tan\langle\alpha_2,H\rangle)^2)(\alpha_1+2\alpha_2)\\
=&\langle\tau_H,\alpha_2\rangle\{m_1(1-(\cot\langle\alpha_2,H\rangle)^2)+(2m_2+4m_3)(1-(\cot\langle 2\alpha_2,H\rangle)^2)\\
&+ n_1(1-(\tan\langle\alpha_2,H\rangle)^2)\}(\alpha_1+2\alpha_2).
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned}
0 =& m_1(1-(\cot\langle\alpha_2,H\rangle)^2)+(2m_2+4m_3)(1-(\cot\langle 2\alpha_2,H\rangle)^2)\\
&+ n_1(1-(\tan\langle\alpha_2,H\rangle)^2)
\end{aligned}$$

holds. The equation is equivalent to

$$\begin{aligned}
&\big((2m_1+m_2+2m_3)(\cot\langle\alpha_2,H\rangle)^2-(2n_1+m_2+2m_3)\big)\big((\cot\langle\alpha_2,H\rangle)^2-1\big)\\
=&\qquad (2m_2+4m_3)(\cot\langle\alpha_2,H\rangle)^2.
\end{aligned}$$

Since $2m_2+4m_3>0$, the solutions of the equation are not harmonic. Hence the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$\begin{aligned}
&(\cot(\langle\alpha_2,H\rangle))^2\\
=&\frac{m_1+n_1+l\pm\sqrt{(m_1+n_1+l)^2-(2n_1+m_2+2m_3)(2m_1+m_2+2m_3)}}{2n_1+m_2+2m_3}
\end{aligned}$$

holds, where $l = 2m_2+2m_3$

(3) When $H = tH_{\alpha_1}+(1-t)H_{\alpha_2}$ $(0<t<1)$, we have $\Sigma_H^+ = \emptyset, W_H^+ = \{\alpha_1+\alpha_2\}$. We set $\vartheta = \langle\alpha_1,H\rangle$. Then, $\langle\alpha_2,H\rangle = (\pi/2)-\vartheta$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m_2\cot\langle\alpha_1,H\rangle\alpha_1 - m_1\cot\langle\alpha_2,H\rangle\alpha_2\\
&- m_1\cot\langle\alpha_1+\alpha_2,H\rangle(\alpha_1+\alpha_2) - m_2\cot\langle\alpha_1+2\alpha_2,H\rangle(\alpha_1+2\alpha_2)\\
&- m_3\cot\langle 2\alpha_2,H\rangle 2\alpha_2\\
&+ n_1\tan\langle\alpha_2,H\rangle\alpha_2\\
=& - m_2\cot(\vartheta)\alpha_1 - m_1\cot((\pi/2)-\vartheta)\alpha_2 - m_1\cot(\pi/2)(\alpha_1+\alpha_2)\\
&- m_2\cot(\pi-\vartheta)(\alpha_1+2\alpha_2) - m_3\cot(\pi-2\vartheta)(2\alpha_2) + n_1\tan((\pi/2)-\vartheta)\alpha_2\\
=&\{(2m_2+m_3+n_1)\cot(\vartheta)-(m_1+m_3)\tan(\vartheta)\}\alpha_2
\end{aligned}$$

Hence, $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_2, H\rangle)^2 = \frac{m_1 + m_3}{2m_2 + m_3 + n_1}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_2\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1 \\
&+ m_1\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle)^2)\alpha_2 \\
&+ m_2\langle\tau_H, \alpha_1 + 2\alpha_2\rangle(1 - (\cot\langle\alpha_1 + 2\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2) \\
&+ m_3\langle\tau_H, 2\alpha_2\rangle(1 - (\cot\langle 2\alpha_2, H\rangle)^2)2\alpha_2 \\
&+ n_1\langle\tau_H, (\alpha_1 + \alpha_2)\rangle(1 - (\tan\langle\alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2) \\
=& \langle\tau_H, \alpha_1\rangle\{m_2(1 - (\cot\vartheta)^2)\alpha_1 - m_1(1 - (\tan\vartheta)^2)\alpha_2 \\
&- m_2(1 - (\cot\vartheta)^2)(\alpha_1 + 2\alpha_2) - 4m_3(1 - (\cot 2\vartheta)^2)\alpha_2 + n_1(1 - (\cot\vartheta)^2)\alpha_2\} \\
=& -\langle\tau_H, \alpha_1\rangle\{(2m_2 + n_1)(1 - (\cot\vartheta)^2) + m_1(1 - (\tan\vartheta)^2) \\
&\quad + 4m_3(1 - (\cot 2\vartheta)^2)\}\alpha_2 \\
=& -\langle\tau_H, \alpha_1\rangle\{(2m_2 + n_1 + m_3)(1 - (\cot\vartheta)^2) \\
&\quad + (m_1 + m_3)(1 - (\tan\vartheta)^2) + 4m_3\}\alpha_2.
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = \{(2m_2 + n_1 + m_3)(1 - (\cot\vartheta)^2) + (m_1 + m_3)(1 - (\tan\vartheta)^2) + 4m_3\}$$

holds. The equation is equivalent to

$$\big((2m_2 + m_3 + n_1)(\cot\vartheta)^2 - (m_1 + m_3)\big)((\cot\vartheta)^2 - 1) = 4m_3(\cot\vartheta)^2.$$

Since $m_3 > 0$, the solutions of the equation are not harmonic. Hence the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$\begin{aligned}
&(\cot\vartheta)^2 \\
=&\frac{(m_1 + 2m_2 + 6m_3 + n_1) \pm \sqrt{(m_1 + 2m_2 + n_1)^2 + 8m_3(m_1 + 2m_2 + 4m_3 + n_1)}}{2(2m_2 + m_3 + n_1)}
\end{aligned}$$

holds.

4.6.6. *Type* I-C$_2$. We set

$$\Sigma^+ = \{e_1 \pm e_2, 2e_1, 2e_2\}, W^+ = \{e_1 - e_2, e_1 + e_2\},$$
$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = 2e_2\},\ \tilde{\alpha} = \alpha_1 + \alpha_2 = e_1 + e_2.$$

When we set $m_1 = m(e_1 + e_2), m_2 = m(2e_1), n_1 = n(e_1 + e_2)$, then we have same result as cases of Type I-B$_2$.

4.6.7. *Type* I-BC$_2$−B$_2$. We set

$$\Sigma^+ = \{e_1 \pm e_2, e_1, e_2, 2e_1, 2e_2\}, W^+ = \{e_1 \pm e_2, e_1, e_2\},$$
$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\},\ \tilde{\alpha} = \alpha_1 + 2\alpha_2 = e_1 + e_2$$

and

$$m_1 = m(e_1),\ m_2 = m(e_1 + e_2),\ m_3 = m(2e_1),\ n_1 = n(e_1),\ n_2 = n(e_1 + e_2).$$

Since $e_1 \in \Sigma \cap W$, $e_1 - e_2 \in W$ and $(2\langle e_1, e_1 - e_2\rangle)/(\langle e_1 - e_2, e_1 - e_2\rangle)$ is odd, by definition of multiplicities, we have $m_1 = m(e_1) = n(e_1) = n_1$.
(1) When $H = tH_{\alpha_1}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_2, 2\alpha_2\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m_2 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1 \cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2) \\
& - m_2 \cot\langle\alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2) - m_3 \cot\langle 2(\alpha_1 + \alpha_2), H\rangle 2(\alpha_1 + \alpha_2) \\
& + n_2 \tan\langle\alpha_1, H\rangle\alpha_1 + n_2 \tan\langle\alpha_2, H\rangle\alpha_2 \\
& + n_1 \tan\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2) + n_2 \tan\langle\alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2) \\
=& - (2m_2 + m_1) \cot\langle\alpha_1, H\rangle e_1 - 2m_3 \cot\langle 2\alpha_1, H\rangle e_1 \\
& (2n_2 + n_1) \tan\langle\alpha_1, H\rangle e_1 \\
=& \{-(m_1 + 2m_2 + m_3) \cot\langle\alpha_1, H\rangle + (m_1 + 2n_2 + m_3) \tan\langle\alpha_1, H\rangle\} e_1.
\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle\alpha_1, H\rangle)^2 = \frac{m_1 + 2n_2 + m_3}{m_1 + 2m_2 + m_3}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_2\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1 \\
& + m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle(1 - (\cot\langle\alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2) \\
& + m_2\langle\tau_H, \alpha_1 + 2\alpha_2\rangle(1 - (\cot\langle\alpha_1 + 2\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2) \\
& + m_3\langle\tau_H, 2(\alpha_1 + \alpha_2)\rangle(1 - (\cot\langle 2(\alpha_1 + \alpha_2), H\rangle)^2) 2(\alpha_1 + \alpha_2) \\
& + n_2\langle\tau_H, \alpha_1\rangle(1 - (\tan\langle\alpha_1, H\rangle)^2)\alpha_2 \\
& + n_1\langle\tau_H, (\alpha_1 + \alpha_2)\rangle(1 - (\tan\langle\alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2) \\
& + n_2\langle\tau_H, (\alpha_1 + 2\alpha_2)\rangle(1 - (\tan\langle(\alpha_1 + 2\alpha_2), H\rangle)^2)(\alpha + 2\alpha_2) \\
=& \langle\tau_H, \alpha_1\rangle\{m_2(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1 + m_1(1 - (\cot\langle\alpha_1, H\rangle)^2)(\alpha_1 + \alpha_2) \\
& + 4m_3(1 - (\cot\langle 2\alpha_1, H\rangle)^2)(\alpha_1 + \alpha_2) \\
& + (n_1 + 2n_2)(1 - (\tan\langle\alpha_1, H\rangle)^2)(\alpha_1 + 2\alpha_2)\} \\
=& \langle\tau_H, \alpha_1\rangle\{(m_1 + 2m_2 + m_3)(1 - (\cot\langle\alpha_1, H\rangle)^2) \\
& + (m_1 + 2n_2 + m_3)(1 - (\tan\langle\alpha_1, H\rangle)^2) + 4m_3\}(\alpha_1 + \alpha_2).
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned}
0 =& (m_1 + 2m_2 + m_3)(1 - (\cot\langle\alpha_1, H\rangle)^2) \\
& + (m_1 + 2n_2 + m_3)(1 - (\tan\langle\alpha_1, H\rangle)^2) + 4m_3
\end{aligned}$$

holds. The equation is equivalent to

$$\begin{aligned}
& \big((m_1 + 2m_2 + m_3)(\cot\langle\alpha_2, H\rangle)^2 - (m_1 + 2n_2 + m_3)\big)\big((\cot\langle\alpha_2, H\rangle)^2 - 1\big) \\
= \quad & 4m_3(\cot\langle\alpha_2, H\rangle)^2.
\end{aligned}$$

Since $m_3 > 0$, the solutions of the equation are not harmonic. Set

$$a = m_1 + 2m_2 + m_3,\ b = n_1 + 2n_2 + m_3,\ c = 4m_3.$$

Hence the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$\begin{aligned}(\cot(\langle\alpha_1, H\rangle))^2 =&\frac{a+b+c\pm\sqrt{(a+b+c)^2-4ab}}{2a}\\ =&\frac{a+b+c\pm\sqrt{(a+b)^2+c(2(a+b)+c)}}{2a}\end{aligned}$$

holds.

(2) When $H = tH_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_1\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& -m_1\cot\langle\alpha_2, H\rangle\alpha_2 - m_1\cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\ &- m_2\cot\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2) - m_3\cot\langle 2(\alpha_1+\alpha_2), H\rangle 2(\alpha_1+\alpha_2)\\ &- m_3\cot\langle 2\alpha_2, H\rangle 2\alpha_2\\ &+ n_2\tan\langle\alpha_1, H\rangle\alpha_1\\ &+ n_1\tan\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2) + n_2\tan\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ =& -m_1\cot\langle\alpha_2, H\rangle(\alpha_1+2\alpha_2) - (m_2+2m_3)\cot\langle 2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ &+ n_1\tan\langle\alpha_2, H\rangle(\alpha_1+2\alpha_2) + n_2\tan\langle 2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ =&\{-(2m_1+m_2+2m_3)\cot\langle 2\alpha_2, H\rangle + n_2\tan\langle 2\alpha_2, H\rangle\}(\alpha_1+2\alpha_2).\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_2, H\rangle)^2 = \frac{n_2}{2m_1+m_2+2m_3}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}0 =& m_1\langle\tau_H, \alpha_2\rangle(1-(\cot\langle\alpha_2, H\rangle)^2)\alpha_2\\ &+ m_1\langle\tau_H, \alpha_1+\alpha_2\rangle(1-(\cot\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\ &+ m_2\langle\tau_H, \alpha_1+2\alpha_2\rangle(1-(\cot\langle\alpha_1+2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2)\\ &+ m_3\langle\tau_H, 2(\alpha_1+\alpha_2)\rangle(1-(\cot\langle 2(\alpha_1+\alpha_2), H\rangle)^2)2(\alpha_1+\alpha_2)\\ &+ m_3\langle\tau_H, 2\alpha_2\rangle(1-(\cot 2\langle\alpha_2, H\rangle)^2)2\alpha_2\\ &+ n_2\langle\tau_H, \alpha_1\rangle(1-(\tan\langle\alpha_1, H\rangle)^2)\alpha_2\\ &+ n_1\langle\tau_H, (\alpha_1+\alpha_2)\rangle(1-(\tan\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\ &+ n_2\langle\tau_H, (\alpha+2\alpha_2)\rangle(1-(\tan(\alpha+2\alpha_2), H\rangle)^2)(\alpha+2\alpha_2)\\ =&\langle\tau_H, \alpha_2\rangle\{m_1(1-(\cot\langle\alpha_2, H\rangle)^2) + n_1(1-(\tan\langle\alpha_2, H\rangle)^2)\\ &+ (2m_2+4m_3)(1-(\cot\langle 2\alpha_2, H\rangle)^2) + 2n_2(1-(\tan\langle 2\alpha_2, H\rangle)^2)\}(\alpha_1+2\alpha_2)\\ =& 2\langle\tau_H, \alpha_2\rangle\{(2m_1+m_2+2m_3)(1-(\cot\langle 2\alpha_2, H\rangle)^2)\\ &+ n_2(1-(\tan\langle 2\alpha_2, H\rangle)^2) - 4m_3\}(\alpha_1+2\alpha_2).\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned}0 =& (2m_1+m_2+2m_3)(1-(\cot\langle 2\alpha_2, H\rangle)^2)\\ &+ n_2(1-(\tan\langle 2\alpha_2, H\rangle)^2) - 4m_3\end{aligned}$$

holds. The equation is equivalent to

$$\begin{aligned}&\big((2m_1+m_2+2m_3)(\cot\langle 2\alpha_2,H\rangle)^2-2n_2\big)((\cot\langle\alpha_2,H\rangle)^2-1)\\=&-4m_3(\cot\langle\alpha_2,H\rangle)^2.\end{aligned}$$

Since $m_3>0$, the solutions of the equation are not harmonic. When

$$(-2m_1+m_2+2m_3+n_2)^2-4(2m_1+m_2+2m_3)n_2>0$$

the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot(\langle 2\alpha_2,H\rangle))^2=\frac{l\pm\sqrt{l^2-4(2m_1+m_2+2m_3)n_2}}{2(2m_1+m_2+2m_3)}$$

holds, where $l=-2m_1+m_2+2m_3+n_2$.

(3) When $H=tH_{\alpha_1}+(1-t)H_{\alpha_2}$ $(0<t<1)$, we have $\Sigma_H^+=\emptyset, W_H^+=\{\alpha_1+2\alpha_2\}$. We set $2\vartheta=\langle\alpha_1,H\rangle$. Then $\langle\alpha_2,H\rangle=(\pi/4)-\vartheta$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H=&-m_2\cot(2\vartheta)\alpha_1-m_1\cot((\pi/4)-\vartheta)\alpha_2-m_1\cot((\pi/4)+\vartheta)(\alpha_1+\alpha_2)\\&-m_2\cot(\pi/2)(\alpha_1+2\alpha_2)-m_3\cot((\pi/2)+2\vartheta)2(\alpha_1+\alpha_2)\\&-m_3\cot((\pi/2)-2\vartheta)2\alpha_2\\&+n_2\tan(2\vartheta)\alpha_1+n_1\tan((\pi/4)-\vartheta)\alpha_2+n_1\tan((\pi/4)-\vartheta)(\alpha_1+\alpha_2)\\=&-m_2\cot(2\vartheta)\alpha_1-2m_1\tan(2\vartheta)\alpha_2+2m_1\tan(2\vartheta)(\alpha_1+\alpha_2)\\&+n_2\tan(2\vartheta)\alpha_1\\=&\{-m_2\cot(2\vartheta)+(2m_1+m_3+n_2)\tan(2\vartheta)\}\alpha_1.\end{aligned}$$

Hence we have $\tau_H=0$ if and only if

$$(\cot(2\vartheta))^2=\frac{2m_1+2m_3+n_2}{m_2}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_2\langle\tau_H,\alpha_1\rangle(1-(\cot(2\vartheta))^2)\alpha_1\\
&+ m_1\langle\tau_H,\alpha_2\rangle(1-(\cot((\pi/4)-\vartheta))^2)\alpha_2\\
&+ m_1\langle\tau_H,\alpha_1+\alpha_2\rangle(1-(\cot((\pi/4)+\vartheta))^2)(\alpha_1+\alpha_2)\\
&+ m_2\langle\tau_H,\alpha_1+2\alpha_2\rangle(1-(\cot(\pi/2))^2)(\alpha_1+2\alpha_2)\\
&+ m_3\langle\tau_H,2(\alpha_1+\alpha_2)\rangle(1-(\cot((\pi/2)+2\vartheta))^2)2(\alpha_1+\alpha_2)\\
&+ m_3\langle\tau_H,2\alpha_2\rangle(1-(\cot((\pi/2)-2\vartheta))^2)2\alpha_2\\
&+ n_2\langle\tau_H,\alpha_1\rangle(1-(\tan(2\vartheta))^2)\alpha_1\\
&+ n_1\langle\tau_H,\alpha_2\rangle(1-(\tan((\pi/4)-\vartheta))^2)\alpha_2\\
&+ n_1\langle\tau_H,(\alpha_1+\alpha_2)(1-(\tan((\pi/4)+\vartheta))^2)(\alpha_1+\alpha_2)\\
=&\langle\tau_H,\alpha_1+\alpha_2\rangle\{2m_2(1-(\cot(2\vartheta))^2)\alpha_1\\
&+ m_1\{(1-(\cot((\pi/4)-\vartheta))^2)+(1-(\tan((\pi/4)-\vartheta))^2)\}\alpha_2\\
&+ m_1\{(1-(\cot((\pi/4)+\vartheta))^2)+(1-(\tan((\pi/4)+\vartheta))^2)\}(\alpha_1+\alpha_2)\\
&+ 2m_3(1-(\tan(2\vartheta))^2)(2\alpha_1)+2n_2(1-(\tan(2\vartheta))^2)(\alpha_1)\}\\
=&\langle\tau_H,\alpha_1+\alpha_2\rangle\{2m_2(1-(\cot(2\vartheta))^2)\alpha_1+4m_1\tan(2\vartheta)^2\alpha_2\\
&- 4m_1\tan(2\vartheta)^2(\alpha_1+\alpha_2)+(4m_3+2n_2)(1-(\tan(2\vartheta))^2)\alpha_1\}\\
=&2\langle\tau_H,\alpha_1+\alpha_2\rangle\{m_2(1-(\cot(2\vartheta))^2)\\
&+ (2m_1+2m_3+n_2)(1-(\tan(2\vartheta))^2)-2m_1\}\alpha_1.
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H=0$ or

$$0 =(m_2(1-(\cot(2\vartheta))^2)+(2m_1+2m_3+n_2)(1-(\tan(2\vartheta))^2)-2m_1$$

holds. The equation is equivalent to

$$\{m_2(\cot(2\vartheta))^2-(2m_1+2m_3+n_2)\}((\cot(2\vartheta))^2-1)=-2m_1(\cot(2\vartheta))^2.$$

Since $m_1>0$, the solutions of the equation are not harmonic. When

$$(2m_3+m_2+m_2)^2-4m_2(2m_1+2m_3+n_2)>0,$$

the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot(2\vartheta))^2=\frac{2m_3+m_2+m_2\pm\sqrt{(2m_3+m_2+m_2)^2-4m_2(2m_1+2m_3+n_2)}}{2m_2}$$

holds.

4.6.8. *Type* II-BC$_2$. We set

$$\Sigma^+=\{e_1\pm e_2,e_1,e_2\},W^+=\{e_1\pm e_2,e_1,e_2,2e_1,2e_2\},$$
$$\Pi=\{\alpha_1=e_1-e_2,\alpha_2=e_2\},\ \tilde{\alpha}=2\alpha_1+2\alpha_2=2e_1$$

and

$$m_1=m(e_1),\ m_2=m(e_1+e_2),n_1=n(e_1),\ n_2=n(e_1+e_2),\ n_3=n(2e_1).$$

Since $e_1, e_1+e_2 \in \Sigma \cap W,\ 2e_1 \in W$ and $(2\langle e_1, 2e_1\rangle)/(\langle 2e_1, 2e_1\rangle) = 1$ and $(2\langle e_1 + e_2, 2e_1\rangle)/(\langle 2e_1, 2e_1\rangle) = 1$ are odd, by definition of multiplicities, we have $m_1 = m(e_1) = n(e_1) = n_1$, $m_2 = m(e_1+e_2) = n(e_1+e_2) = n_2$.
(1) When $H = tH_{\alpha_1}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_2\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& - m_2 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1 \cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\ &- m_2 \cot\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ &+ m_2 \tan\langle\alpha_1, H\rangle\alpha_1 + m_2 \tan\langle\alpha_2, H\rangle\alpha_2\\ &+ m_1 \tan\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2) + m_2 \tan\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ &+ n_3 \tan\langle 2\alpha_1+2\alpha_2, H\rangle(2\alpha_1+2\alpha_2)\\ =& - 2m_2\{\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle\}(\alpha_1+\alpha_2)\\ &- m_1\{\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle\}(\alpha_1+\alpha_2)\\ &+ 2n_3 \tan\langle 2\alpha_1, H\rangle(\alpha_1+\alpha_2)\\ =&2\{-(m_1+2m_2)\cot\langle 2\alpha_1, H\rangle + n_3 \tan\langle 2\alpha_1, H\rangle\}e_1.\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_1, H\rangle)^2 = \frac{n_3}{m_1+2m_2}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}0 =&m_2\langle\tau_H, \alpha_1\rangle(1-(\cot\langle\alpha_1, H\rangle)^2)\alpha_1\\ &+ m_1\langle\tau_H, \alpha_1+\alpha_2\rangle(1-(\cot\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\ &+ m_2\langle\tau_H, \alpha_1+2\alpha_2\rangle(1-(\cot\langle\alpha_1+2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2)\\ &+ m_2\langle\tau_H, \alpha_1\rangle(1-(\tan\langle\alpha_1, H\rangle)^2)\alpha_1\\ &+ m_1\langle\tau_H, (\alpha_1+\alpha_2)\rangle(1-(\tan\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\ &+ m_2\langle\tau_H, (\alpha_1+2\alpha_2)\rangle(1-(\tan(\alpha_1+2\alpha_2), H\rangle)^2)(\alpha_1+2\alpha_2)\\ &+ n_3\langle\tau_H, 2(\alpha_1+\alpha_2)\rangle(1-(\tan\langle 2(\alpha_1+\alpha_2), H\rangle)^2)2(\alpha_1+\alpha_2)\\ =&2\langle\tau_H, \alpha_1\rangle\{(m_1+2m_2)\big((1-(\cot\langle\alpha_1, H\rangle)^2) + (1-(\tan\langle\alpha_1, H\rangle)^2)\big)\\ &+ 4n_3(1-(\tan\langle 2\alpha_1, H\rangle)^2)\}(\alpha_1+\alpha_2)\\ =&2\langle\tau_H, \alpha_1\rangle\{-4(m_1+2m_2)(\cot\langle 2\alpha_1, H\rangle)^2 + 4n_3(1-(\tan\langle 2\alpha_1, H\rangle)^2)\}(\alpha_1+\alpha_2)\\ =&8\langle\tau_H, \alpha_1\rangle\{-(m_1+2m_2)(\cot\langle 2\alpha_1, H\rangle)^2 + n_3(1-(\tan\langle 2\alpha_1, H\rangle)^2)\}(\alpha_1+\alpha_2).\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = -(m_1+2m_2)(\cot\langle 2\alpha_1, H\rangle)^2 + n_3(1-(\tan\langle 2\alpha_1, H\rangle)^2)$$

holds. The equation is equivalent to

$$\begin{aligned}&\{(m_1+2m_2)(\cot\langle 2\alpha_1, H\rangle)^2 - n_3\}((\cot\langle 2\alpha_1, H\rangle)^2 - 1)\\ =& -(m_1+2m_2)(\cot\langle 2\alpha_1, H\rangle)^2\end{aligned}$$

Since $m_1 + 2m_2 > 0$, the solutions of the equation are not harmonic. When $n_3^2 - 4(m_1+2m_2)n_3 > 0$, the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot(\langle 2\alpha_1, H\rangle))^2 = \frac{n_3 \pm \sqrt{n_3^2 - 4(m_1+2m_2)n_3}}{2(m_1+2m_2)}$$

holds.

(2) When $H = tH_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_1\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& - m_1 \cot\langle\alpha_2, H\rangle\alpha_2 - m_1 \cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\ &- m_2 \cot\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ &+ m_1 \tan\langle\alpha_2, H\rangle\alpha_2\\ &+ m_1 \tan\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2) + m_2 \tan\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ &+ n_3 \tan\langle 2(\alpha_1+\alpha_2), H\rangle 2(\alpha_1+\alpha_2)\\ &+ n_3 \tan\langle 2\alpha_2, H\rangle 2\alpha_2\\ =& - m_1 \cot\langle\alpha_2, H\rangle(\alpha_1+2\alpha_2) - m_2 \cot\langle 2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ &+ m_1 \tan\langle\alpha_2, H\rangle(\alpha_1+2\alpha_2) + m_2 \tan\langle 2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ &+ n_3 \tan\langle 2\alpha_2, H\rangle 2(\alpha_1+2\alpha_2)\\ =&\{-(2m_1+m_2)\cot\langle 2\alpha_2, H\rangle + (m_2+2n_3)\tan\langle 2\alpha_2, H\rangle\}(\alpha_1+2\alpha_2).\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_2, H\rangle)^2 = \frac{m_2+2n_3}{2m_1+m_2}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}0 =& m_1\langle\tau_H, \alpha_2\rangle(1-(\cot\langle\alpha_2, H\rangle)^2)\alpha_2\\ &+ m_1\langle\tau_H, \alpha_1+\alpha_2\rangle(1-(\cot\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\ &+ m_2\langle\tau_H, \alpha_1+2\alpha_2\rangle(1-(\cot\langle\alpha_1+2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2)\\ &+ m_1\langle\tau_H, \alpha_2\rangle(1-(\tan\langle\alpha_2, H\rangle)^2)\alpha_2\\ &+ m_1\langle\tau_H, (\alpha_1+\alpha_2)\rangle(1-(\tan\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\ &+ m_2\langle\tau_H, (\alpha_1+2\alpha_2)\rangle(1-(\tan(\alpha_1+2\alpha_2), H\rangle)^2)(\alpha_1+2\alpha_2)\\ &+ n_3\langle\tau_H, 2(\alpha_1+\alpha_2)\rangle(1-(\tan\langle 2(\alpha_1+\alpha_2), H\rangle)^2)2(\alpha_1+\alpha_2)\\ &+ n_3\langle\tau_H, 2\alpha_2\rangle(1-(\tan 2\langle\alpha_2, H\rangle)^2)2\alpha_2\\ =&\langle\tau_H, \alpha_2\rangle\{m_1\big((1-(\cot\langle\alpha_2, H\rangle)^2)+(1-(\tan\langle\alpha_2, H\rangle)^2)\big)\\ &+ m_2(1-(\cot\langle 2\alpha_2, H\rangle)^2) + 2m_2(1-(\tan\langle 2\alpha_2, H\rangle)^2)\\ &+ 4n_3(1-(\tan\langle 2\alpha_2, H\rangle)^2)\}(\alpha_1+2\alpha_2)\\ =&\langle\tau_H, \alpha_2\rangle\{-4m_1(\cot\langle 2\alpha_2, H\rangle)^2 + 2m_2(1-(\cot\langle 2\alpha_2, H\rangle)^2)\\ &+ 2m_2(1-(\tan\langle 2\alpha_2, H\rangle)^2)\\ &+ 4n_3(1-(\tan\langle 2\alpha_2, H\rangle)^2)\}(\alpha_1+2\alpha_2)\\ =&\langle\tau_H, \alpha_2\rangle\{-(4m_1+2m_2)(\cot\langle 2\alpha_2, H\rangle)^2\\ &+ (2m_2+4n_3)(1-(\tan\langle 2\alpha_2, H\rangle)^2) - 4m_1\}(\alpha_1+2\alpha_2).\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned}0 =&(2m_1+m_2)(1-(\cot\langle 2\alpha_2, H\rangle)^2)\\ &+ (m_2+2n_3)(1-(\tan\langle 2\alpha_2, H\rangle)^2) - 2m_1\end{aligned}$$

holds. The equation is equivalent to

$$\begin{aligned}&\big((2m_1+m_2)(\cot\langle 2\alpha_2,H\rangle)^2-(m_2+2n_3)\big)((\cot\langle 2\alpha_2,H\rangle)^2-1)\\=&-2m_1(\cot\langle 2\alpha_2,H\rangle)^2.\end{aligned}$$

Since $2m_1>0$, the solutions of the equation are not harmonic. When

$$(m_2+n_3)^2-(2m_1+m_2)(m_2+2n_3)>0$$

the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$\begin{aligned}&(\cot(\langle 2\alpha_2,H\rangle))^2\\=&\frac{m_2+n_3\pm\sqrt{(m_2+n_3)^2-(2m_1+m_2)(m_2+2n_3)}}{(2m_1+m_2)}\end{aligned}$$

holds.

(3) When $H=tH_{\alpha_1}+(1-t)H_{\alpha_2}$ $(0<t<1)$, we have $\Sigma_H^+=\emptyset, W_H^+=\{\tilde{\alpha}=2\alpha_1+2\alpha_2\}$. We set $\vartheta=\langle 2\alpha_1,H\rangle$. Then $\langle 2\alpha_2,H\rangle=(\pi/2)-\vartheta$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H=&-m_2\cot\langle\alpha_1,H\rangle\alpha_1-m_1\cot\langle\alpha_2,H\rangle\alpha_2-m_1\cot\langle\alpha_1+\alpha_2,H\rangle(\alpha_1+\alpha_2)\\&-m_2\cot\langle\alpha_1+2\alpha_2,H\rangle(\alpha_1+2\alpha_2)\\&+m_2\tan\langle\alpha_1,H\rangle\alpha_1+m_1\tan\langle\alpha_2,H\rangle\alpha_2+m_1\tan\langle\alpha_1+\alpha_2,H\rangle(\alpha_1+\alpha_2)\\&+m_2\tan\langle\alpha_1+2\alpha_2,H\rangle(\alpha_1+2\alpha_2)\\&+n_3\tan\langle 2\alpha_2,H\rangle 2\alpha_2\\=&-m_2(\cot\langle\alpha_1,H\rangle-\tan\langle\alpha_1,H\rangle)\alpha_1-m_1(\cot\langle\alpha_2,H\rangle-\tan\langle\alpha_2,H\rangle)\alpha_2\\&-m_2(\cot\langle\alpha_1+2\alpha_2,H\rangle-\tan\langle\alpha_1+2\alpha_2,H\rangle)(\alpha_1+2\alpha_2)\\&+n_3\tan((\pi/2)-\vartheta)(2\alpha_2)\\=&-2m_2\cot(\vartheta)\alpha_1-2m_1\cot((\pi/2)-\vartheta)\alpha_2\\&-2m_2\cot(\pi-\vartheta)(\alpha_1+2\alpha_2)+2n_3\tan((\pi/2)-\vartheta)\alpha_2\\=&2\{(2m_2+n_3)\cot(\vartheta)-m_1\tan(\vartheta)\}\alpha_2.\end{aligned}$$

Hence we have $\tau_H=0$ if and only if

$$(\cot\vartheta)^2=\frac{m_1}{2m_2+n_3}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =&m_2\langle\tau_H,\alpha_1\rangle(1-(\cot\langle\alpha_1,H\rangle)^2)\alpha_1\\
&+m_1\langle\tau_H,\alpha_2\rangle(1-(\cot\langle\alpha_2,H\rangle^2)\alpha_2\\
&+m_1\langle\tau_H,\alpha_1+\alpha_2\rangle(1-(\cot\langle\alpha_1+\alpha_2,H\rangle)^2)(\alpha_1+\alpha_2)\\
&+m_2\langle\tau_H,\alpha_1+2\alpha_2\rangle(1-(\cot\langle\alpha_1+2\alpha_2,H\rangle)^2)(\alpha_1+2\alpha_2)\\
&+m_2\langle\tau_H,\alpha_1\rangle(1-(\tan\langle\alpha_1,H\rangle)^2)\alpha_1\\
&+m_1\langle\tau_H,\alpha_2\rangle(1-(\tan\langle\alpha_2,H\rangle)^2)\alpha_2\\
&+m_1\langle\tau_H,(\alpha_1+\alpha_2)(1-(\tan\langle\alpha_1+\alpha_2,H\rangle)^2)(\alpha_1+\alpha_2)\\
&+m_2\langle\tau_H,\alpha_1+2\alpha_2\rangle(1-(\tan\langle\alpha_1+2\alpha_2,H\rangle)^2)(\alpha_1+2\alpha_2)\\
&+n_3\langle\tau_H,2\alpha_2\rangle(1-(\tan\langle 2\alpha_2,H\rangle)^2)2\alpha_2\\
=&-m_2\langle\tau_H,\alpha_1\rangle(\cot\langle\alpha_1,H\rangle-\tan\langle\alpha_1,H\rangle)^2\alpha_1\\
&-m_1\langle\tau_H,\alpha_2\rangle(\cot\langle\alpha_2,H\rangle-\tan\langle\alpha_2,H\rangle)^2\alpha_2\\
&-m_2\langle\tau_H,\alpha_1+2\alpha_2\rangle(\cot\langle\alpha_1+2\alpha_2,H\rangle-\tan\langle\alpha_1+2\alpha_2,H\rangle)^2(\alpha_1+2\alpha_2)\\
&+n_3\langle\tau_H,2\alpha_2\rangle(1-(\cot\vartheta)^2)2\alpha_2\\
=&-4m_2\langle\tau_H,\alpha_1\rangle(\cot\vartheta)^2\alpha_1+4m_1\langle\tau_H,\alpha_1\rangle(\cot((\pi/2)-\vartheta))^2\alpha_2\\
&+4m_2\langle\tau_H,\alpha_1\rangle(\cot(\pi-\vartheta))^2(\alpha_1+2\alpha_2)-4n_3\langle\tau_H,\alpha_1\rangle(1-(\cot\vartheta)^2)\alpha_2\\
=&-4\langle\tau_H,\alpha_1\rangle\{2m_2(\cot\vartheta)^2+m_1(\tan\vartheta)^2-n_3(1-(\cot\vartheta)^2)\}\alpha_2\\
=&4\langle\tau_H,\alpha_1\rangle\{(2m_2+n_3)(1-(\cot\vartheta)^2)+m_1(1-(\tan\vartheta)^2)-(2m_2+m_1)\}\alpha_2
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H=0$ or

$$0=(2m_2+n_3)(1-(\cot\vartheta)^2)+m_1(1-(\tan\vartheta)^2)-(2m_2+m_1)$$

holds. The equation is equivalent to

$$\{(2m_2+n_3)(\cot\vartheta)^2-m_1\}((\cot(\vartheta))^2-1)=-(2m_2+m_1)(\cot(2\vartheta))^2.$$

Since $2m_2+m_1>0$, the solutions of the equation are not harmonic. When

$$n_3^2-4(2m_2+n_3)m_1>0,$$

the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot\vartheta)^2=\frac{n_3\pm\sqrt{n_3^2-4(2m_2+n_3)m_1}}{2(2m_2+n_3)}$$

holds.

4.6.9. *Type* III-A$_2$. We set

$$\mathfrak{a}=\{x_1e_1+x_2e_2+x_3e_3 \mid x_i\in\mathbb{R},\ x_1+x_2+x_3=0\},$$

and

$$\begin{aligned}
&\Sigma^+=W^+=\{e_1-e_2,\ e_2-e_3,\ e_1-e_3\},\\
&\Pi=\{\alpha_1=e_1-e_2,\alpha_2=e_2-e_3\},\tilde{\alpha}=\alpha_1+\alpha_2,\\
&m:=m(\lambda)=n(\lambda)\quad(\lambda\in\tilde{\Sigma}).
\end{aligned}$$

(1) When $H = tH_{\alpha_1}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_2\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& m\{-\cot\langle\alpha_1, H\rangle\alpha_1 - \cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\ &\tan\langle\alpha_1, H\rangle\alpha_1 + \tan\langle\alpha_2, H\rangle\alpha_2 + \tan\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\}\\ =& m\{-\cot\langle\alpha_1, H\rangle + \tan\langle\alpha_1, H\rangle\}(2\alpha_1+\alpha_2).\end{aligned}$$

Hence we have that $\tau_H = 0$ if and only if $\langle\alpha_1, H\rangle = \pi/4$. By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}0 =& m\{\langle\tau_H, \alpha_1\rangle(1-(\cot\langle\alpha_1, H\rangle)^2)\alpha_1\\ &+ \langle\tau_H, \alpha_1+\alpha_2\rangle(1-(\cot\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\ &+ \langle\tau_H, \alpha_1\rangle(1-(\tan\langle\alpha_1, H\rangle)^2)\alpha_1\\ &+ \langle\tau_H, \alpha_2\rangle(1-(\tan\langle\alpha_2, H\rangle)^2)\alpha_2\\ &+ \langle\tau_H, \alpha_1+\alpha_2\rangle(1-(\tan\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\}\\ =& m\langle\tau_H, \alpha_1\rangle\{(1-(\cot\langle\alpha_1, H\rangle)^2) + (1-(\tan\langle\alpha_1, H\rangle)^2)\}(2\alpha_1+\alpha_2)\\ =& -m\langle\tau_H, \alpha_1\rangle(\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle)^2(2\alpha_1+\alpha_2)\end{aligned}$$

Hence, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\langle\alpha_1, H\rangle = \pi/4$. Therefore, if the orbit $K_2\pi_1(\exp(H))$ is biharmonic, then that is harmonic.
(2) When $H = tH_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_1\}, W_H^+ = \emptyset$. By the same calculation as (1), we have that the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\langle\alpha_2, H\rangle = \pi/4$ and if the orbit is biharmonic, then that is harmonic.
(3) When $H = tH_{\alpha_1} + (1-t)H_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \emptyset, W_H^+ = \{\alpha_1+\alpha_2\}$. By the same calculation as (1), we have that the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\langle\alpha_1, H\rangle = \pi/4$ and if the orbit is biharmonic, then that is harmonic.

4.6.10. *Type* III-B_2 *and* III-C_2. We set

$$\begin{aligned}\Sigma^+ = \{e_1 \pm e_2, e_1, e_2\}, W^+ = \{e_1 \pm e_2, e_1, e_2\},\\ \Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\},\ \tilde{\alpha} = \alpha_1 + 2\alpha_2 = e_1 + e_2\end{aligned}$$

and

$$m_1 = m(e_1),\ m_2 = m(e_1+e_2), n_1 = n(e_1),\ n_2 = n(e_1+e_2).$$

Since $e_1 \in \Sigma \cap W$, $e_1 + e_2 \in W$ and $(2\langle e_1, e_1+e_2\rangle)/(\langle e_1+e_2, e_1+e_2\rangle) = 1$ is odd, by definition of multiplicities, we have $m_1 = m(e_1) = n(e_1) = n_1$.
(1) When $H = tH_{\alpha_1}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_2\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}\tau_H =& -m_2\cot\langle\alpha_1, H\rangle\alpha_1 - m_1\cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\ &- m_2\cot\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ &+ m_2\tan\langle\alpha_1, H\rangle\alpha_1 + m_2\tan\langle\alpha_2, H\rangle\alpha_2\\ &+ m_1\tan\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2) + m_2\tan\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\ =& -(2m_2+m_1)\cot\langle\alpha_1, H\rangle(\alpha_1+\alpha_2)\\ &+ (2n_2+m_1)\tan\langle\alpha_1, H\rangle(\alpha_1+\alpha_2)\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_1, H\rangle)^2 = \frac{2m_2 + m_1}{2n_2 + m_1}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_2\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1 \\
&+ m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle(1 - (\cot\langle\alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2) \\
&+ m_2\langle\tau_H, \alpha_1 + 2\alpha_2\rangle(1 - (\cot\langle\alpha_1 + 2\alpha_2, H\rangle)^2)(\alpha_1 + 2\alpha_2) \\
&+ m_2\langle\tau_H, \alpha_1\rangle(1 - (\tan\langle\alpha_1, H\rangle)^2)\alpha_2 \\
&+ m_1\langle\tau_H, (\alpha_1 + \alpha_2)\rangle(1 - (\tan\langle\alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2) \\
&+ m_2\langle\tau_H, (\alpha + 2\alpha_2)\rangle(1 - (\tan(\alpha + 2\alpha_2), H\rangle)^2)(\alpha + 2\alpha_2) \\
=& \langle\tau_H, \alpha_1\rangle\{(2m_2 + m_1)(1 - (\cot\langle\alpha_1, H\rangle)^2) \\
&(2n_2 + m_1)(1 - (\tan\langle\alpha_1, H\rangle)^2)\}(\alpha_1 + 2\alpha_2).
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = (2m_2 + m_1)(1 - (\cot\langle\alpha_1, H\rangle)^2)(2n_2 + m_1)(1 - (\tan\langle\alpha_1, H\rangle)^2)$$

holds. The equation is equivalent to

$$\{(2m_2 + m_1)(\cot\langle\alpha_1, H\rangle)^2 - (2n_2 + m_1)\}((\cot\langle\alpha_1, H\rangle)^2 - 1) = 0$$

Therefore, when $m_2 \neq n_2$ the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot(\langle 2\alpha_1, H\rangle))^2 = 1 \qquad (\text{i.e.}\langle\alpha_1, H\rangle = (\pi/4))$$

holds.

(2) When $H = tH_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_1\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m_1\cot\langle\alpha_2, H\rangle\alpha_2 - m_1\cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2) \\
&- m_2\cot\langle\alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2) \\
&+ m_1\tan\langle\alpha_2, H\rangle\alpha_2 + m_1\tan\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2) \\
&+ n_2\tan\langle\alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2) \\
=& - m_1(\cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle)(\alpha_1 + 2\alpha_2) \\
&- m_2\cot\langle 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2) \\
&+ n_2\tan\langle 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2) \\
=& \{-(2m_1 + m_2)\cot\langle 2\alpha_2, H\rangle + n_2\tan\langle 2\alpha_2, H\rangle\}(\alpha_1 + 2\alpha_2).
\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_2, H\rangle)^2 = \frac{n_2}{2m_1 + m_2}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_1\langle \tau_H, \alpha_2\rangle(1-(\cot\langle\alpha_2, H\rangle)^2)\alpha_2 \\
&+ m_1\langle \tau_H, \alpha_1+\alpha_2\rangle(1-(\cot\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2) \\
&+ m_2\langle \tau_H, \alpha_1+2\alpha_2\rangle(1-(\cot\langle\alpha_1+2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2) \\
&+ m_1\langle \tau_H, \alpha_2\rangle(1-(\tan\langle\alpha_2, H\rangle)^2)\alpha_2 \\
&+ m_1\langle \tau_H, \alpha_1+\alpha_2\rangle(1-(\tan\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2) \\
&+ n_2\langle \tau_H, \alpha_1+2\alpha_2\rangle(1-(\tan\langle\alpha_1+2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2) \\
=& m_1\langle \tau_H, \alpha_2\rangle\{(1-(\cot\langle\alpha_2, H\rangle)^2)+(1-(\tan\langle\alpha_2, H\rangle)^2)\}(\alpha_1+2\alpha_2) \\
&+ m_1\langle \tau_H, \alpha_2\rangle(1-(\cot\langle 2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2) \\
&+ n_2\langle \tau_H, \alpha_2\rangle(1-(\tan\langle 2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2) \\
=& \langle \tau_H, \alpha_2\rangle\{-4m_1(\cot\langle 2\alpha_2, H\rangle)^2+2m_2(1-(\cot\langle 2\alpha_2, H\rangle)^2) \\
&+ 2n_2(1-(\tan\langle 2\alpha_2, H\rangle)^2)\}(\alpha_1+2\alpha_2) \\
=& \langle \tau_H, 2\alpha_2\rangle\{(2m_1+m_2)(1-(\cot\langle 2\alpha_2, H\rangle)^2) \\
&+ n_2(1-(\tan\langle 2\alpha_2, H\rangle)^2)-2m_1\}(\alpha_1+2\alpha_2)
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = (2m_1+m_2)(1-(\cot\langle 2\alpha_2, H\rangle)^2)+n_2(1-(\tan\langle 2\alpha_2, H\rangle)^2)-2m_1$$

holds. The equation is equivalent to

$$\big((2m_1+m_2)(\cot\langle 2\alpha_2, H\rangle)^2-n_2\big)((\cot\langle 2\alpha_2, H\rangle)^2-1) = -2m_1(\cot\langle 2\alpha_2, H\rangle)^2$$

Since $2m_1 > 0$, the solutions of the equation are not harmonic. When

$$(m_2+n_2)^2-4(2m_1+m_2)n_2 > 0$$

the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot(\langle 2\alpha_2, H\rangle))^2 = \frac{m_2+n_2 \pm \sqrt{(m_2+n_2)^2-4(2m_1+m_2)n_2}}{2(2m_1+m_2)}$$

holds.

(3) When $H = tH_{\alpha_1}+(1-t)H_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \emptyset, W_H^+ = \{\tilde{\alpha} = \alpha_1+2\alpha_2\}$. We set $\vartheta = \langle\alpha_1, H\rangle$. Then $\langle 2\alpha_2, H\rangle = (\pi/2)-\vartheta$. By Theorem 2.9, we

have

$$\begin{aligned}\tau_H =& - m_2 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1 \cot\langle\alpha_1, H\rangle\alpha_2 - m_1 \cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\ &+ n_2 \tan\langle\alpha_1, H\rangle\alpha_1 + m_1 \tan\langle\alpha_2, H\rangle\alpha_2 + m_1 \tan\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\ =& - m_2 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1(\cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle)\alpha_2\\ &- m_1(\cot\langle\alpha_1 + \alpha_2, H\rangle - \tan\langle\alpha_1 + \alpha_2, H\rangle)(\alpha_1 + \alpha_2)\\ &+ n_2 \tan\langle\alpha_1, H\rangle\alpha_1\\ =& - m_2 \cot(\vartheta)\alpha_1 + n_2 \tan(\vartheta)\alpha_1 - 2m_1 \cot((\pi/2) - \vartheta)\alpha_2\\ &- m_1 \cot((\pi/2) + \vartheta)(\alpha_1 + \alpha_2)\\ =& - m_2 \cot(\vartheta)\alpha_1 + n_2 \tan(\vartheta)\alpha_1 - 2m_1 \tan(\vartheta)\alpha_2 + m_1 \tan(\vartheta)(\alpha_1 + \alpha_2)\\ =&\{-m_2 \cot(\vartheta) + (n_2 + 2m_1) \tan(\vartheta)\}\alpha_1.\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\vartheta)^2 = \frac{n_2 + 2m_1}{m_2}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}0 =& m_2\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1 + m_1\langle\tau_H, \alpha_2\rangle(1 - (\cot\langle\alpha_2, H\rangle^2)\alpha_2\\ &+ m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle(1 - (\cot\langle\alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2)\\ &+ n_2\langle\tau_H, \alpha_1\rangle(1 - (\tan\langle\alpha_1, H\rangle)^2)\alpha_1 + m_1\langle\tau_H, \alpha_2\rangle(1 - (\tan\langle\alpha_2, H\rangle)^2)\alpha_2\\ &+ m_1\langle\tau_H, (\alpha_1 + \alpha_2)(1 - (\tan\langle\alpha_1 + \alpha_2, H\rangle)^2)(\alpha_1 + \alpha_2)\\ =&\langle\tau_H, \alpha_1\rangle\{m_2(1 - (\cot\langle\alpha_1, H\rangle)^2) + n_2(1 - (\tan\langle\alpha_1, H\rangle)^2)\}\alpha_1\\ &+ m_1\langle\tau_H, \alpha_2\rangle\{(1 - (\cot\langle\alpha_2, H\rangle^2) + (1 - (\tan\langle\alpha_2, H\rangle^2)\}\alpha_2\\ &+ m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle\{(1 - (\cot\langle\alpha_1 + \alpha_2, H\rangle^2)\\ &+ (1 - (\tan\langle\alpha_1 + \alpha_2, H\rangle^2)\}(\alpha_1 + \alpha_2)\\ =&\langle\tau_H, \alpha_1\rangle\{m_2(1 - (\cot\langle\alpha_1, H\rangle)^2) + n_2(1 - (\tan\langle\alpha_1, H\rangle)^2)\}\alpha_1\\ &- m_1\langle\tau_H, \alpha_2\rangle(\cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle)^2\alpha_2\\ &- m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle(\cot\langle\alpha_1 + \alpha_2, H\rangle - \tan\langle\alpha_1 + \alpha_2, H\rangle)^2(\alpha_1 + \alpha_2)\\ =&\langle\tau_H, \alpha_1\rangle\{m_2(1 - (\cot\langle\alpha_1, H\rangle)^2) + n_2(1 - (\tan\langle\alpha_1, H\rangle)^2)\}\alpha_1\\ &- 4m_1\langle\tau_H, \alpha_2\rangle(\cot((\pi/2) - \vartheta))^2\alpha_2\\ &- 4m_1\langle\tau_H, \alpha_1 + \alpha_2\rangle(\cot((\pi/2) + \vartheta))^2(\alpha_1 + \alpha_2)\\ =&\langle\tau_H, \alpha_1\rangle\{m_2(1 - (\cot\langle\alpha_1, H\rangle)^2) + n_2(1 - (\tan\langle\alpha_1, H\rangle)^2) - 2m_1(\tan\vartheta)^2\}\alpha_1\\ =&\langle\tau_H, \alpha_1\rangle\{m_2(1 - (\cot\langle\alpha_1, H\rangle)^2) + (n_2 + 2m_1)(1 - (\tan\langle\alpha_1, H\rangle)^2) - 2m_1\}\alpha_1.\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = m_2(1 - (\cot\langle\alpha_1, H\rangle)^2) + (n_2 + 2m_1)(1 - (\tan\langle\alpha_1, H\rangle)^2) - 2m_1$$

holds. The equation is equivalent to

$$\{m_2(\cot\vartheta)^2 - (n_2 + 2m_1)\}((\cot(\vartheta))^2 - 1) = -2m_1(\cot(\vartheta))^2.$$

Since $m_1 > 0$, the solutions of the equation are not harmonic. When

$$(m_2 + n_2)^2 - 4m_2(n_2 + 2m_1) > 0,$$

the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot\vartheta)^2 = \frac{(m_2+n_2) \pm \sqrt{(m_2+n_2)^2 - 4m_2(n_2+2m_1)}}{2m_2}$$

holds.

4.6.11. *Type* III-BC$_2$. We set

$$\Sigma^+ = W^+ = \{e_1 \pm e_2, e_1, e_2, 2e_1, 2e_2\},$$
$$\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\},\ \tilde{\alpha} = 2\alpha_1 + 2\alpha_2 = 2e_1,$$

$$m_1 = m(e_1),\ m_2 = m(e_1+e_2),\ m_3 = (2e_1),$$
$$n_1 = n(e_1),\ n_2 = n(e_1+e_2),\ n_3 = (2e_1).$$

Since $e_1, e_1+e_2 \in \Sigma \cap W$, $2e_1 \in W$ and $(2\langle e_1, 2e_1\rangle)/(\langle 2e_1, 2e_1\rangle) = 1$ and $(2\langle e_1 + e_2, 2e_1\rangle)/(\langle 2e_1, 2e_1\rangle) = 1$ are odd, by definition of multiplicities, we have $m_1 = m(e_1) = n(e_1) = n_1, m_2 = m(e_1+e_2) = n(e_1+e_2) = n_2$.
(1) When $H = tH_{\alpha_1}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_2, 2\alpha_2\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m_2\cot\langle\alpha_1, H\rangle\alpha_1 - m_1\cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\
&- m_2\cot\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2) - m_3\cot\langle 2\alpha_1+2\alpha_2, H\rangle(2\alpha_1+2\alpha_2)\\
&+ m_2\tan\langle\alpha_1, H\rangle\alpha_1 + m_1\tan\langle\alpha_2, H\rangle\alpha_2\\
&+ m_1\tan\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2) + m_2\tan\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\
&+ n_3\tan\langle 2\alpha_1+2\alpha_2, H\rangle(2\alpha_1+2\alpha_2)\\
=& - m_2\{\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle\}\alpha_1\\
&- m_1\{\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle\}(\alpha_1+\alpha_2)\\
&- m_2\{\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle\}(\alpha_1+2\alpha_2)\\
&- m_3\cot\langle 2\alpha_1, H\rangle(2\alpha_1+2\alpha_2) + n_3\tan\langle 2\alpha_1, H\rangle(2\alpha_1+2\alpha_2)\\
=& - 4m_2\cot\langle 2\alpha_1, H\rangle(\alpha_1+\alpha_2) - 2m_1\cot\langle 2\alpha_1, H\rangle(\alpha_1+\alpha_2)\\
&- 2m_3\cot\langle 2\alpha_1, H\rangle(\alpha_1+\alpha_2) + 2n_3\tan\langle 2\alpha_1, H\rangle(\alpha_1+\alpha_2)\\
=&2\{-(2m_2+m_1+m_3)\cot\langle 2\alpha_1, H\rangle + n_3\tan\langle 2\alpha_1, H\rangle\}(\alpha_1+\alpha_2).
\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_1, H\rangle)^2 = \frac{n_3}{m_1+2m_2+m_3}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_2\langle \tau_H, \alpha_1\rangle(1-(\cot\langle\alpha_1, H\rangle)^2)\alpha_1 \\
&+ m_1\langle \tau_H, \alpha_1+\alpha_2\rangle(1-(\cot\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2) \\
&+ m_2\langle \tau_H, \alpha_1+2\alpha_2\rangle(1-(\cot\langle\alpha_1+2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2) \\
&+ m_3\langle \tau_H, 2\alpha_1+2\alpha_2\rangle(1-(\cot\langle 2\alpha_1+2\alpha_2, H\rangle)^2)(2\alpha_1+2\alpha_2) \\
&+ m_2\langle \tau_H, \alpha_1\rangle(1-(\tan\langle\alpha_1, H\rangle)^2)\alpha_2 \\
&+ m_1\langle \tau_H, (\alpha_1+\alpha_2)\rangle(1-(\tan\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2) \\
&+ m_2\langle \tau_H, (\alpha+2\alpha_2)\rangle(1-(\tan(\alpha+2\alpha_2), H\rangle)^2)(\alpha+2\alpha_2) \\
&+ n_3\langle \tau_H, 2\alpha_1+2\alpha_2\rangle(1-(\tan\langle 2\alpha_1+2\alpha_2, H\rangle)^2)(2\alpha_1+2\alpha_2) \\
=&\langle \tau_H, \alpha_1\rangle\{-2m_2(\cot\langle\alpha_1, H\rangle-\tan\langle\alpha_1, H\rangle)^2-m_1(\cot\langle\alpha_1, H\rangle-\tan\langle\alpha_1, H\rangle)^2 \\
&+4m_3(1-(\cot\langle 2\alpha_1, H\rangle)^2)+4n_3(1-(\tan\langle 2\alpha_1, H\rangle)^2)\}(\alpha_1+\alpha_2) \\
=&\langle \tau_H, \alpha_1\rangle\{(8m_2+4m_1+4m_3)(1-(\cot\langle 2\alpha_1, H\rangle)^2) \\
&+4n_3(1-(\tan\langle 2\alpha_1, H\rangle)^2)-(8m_2+4m_1)\}(\alpha_1+\alpha_2) \\
=&4\langle \tau_H, \alpha_1\rangle\{(2m_2+m_1+m_3)(1-(\cot\langle 2\alpha_1, H\rangle)^2) \\
&+n_3(1-(\tan\langle 2\alpha_1, H\rangle)^2)-(2m_2+m_1)\}(\alpha_1+\alpha_2).
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = (2m_2+m_1+m_3)(1-(\cot\langle 2\alpha_1, H\rangle)^2)+n_3(1-(\tan\langle 2\alpha_1, H\rangle)^2)-(2m_2+m_1)$$

holds. The equation is equivalent to

$$\begin{aligned}
&\{(2m_2+m_1+m_3)(\cot\langle 2\alpha_1, H\rangle)^2-n_3\}((\cot\langle 2\alpha_1, H\rangle)^2-1) \\
&\quad = -(2m_2+m_1)(\cot\langle 2\alpha_1, H\rangle)^2.
\end{aligned}$$

Since $(2m_2+m_1) > 0$, the solutions of the equation are not harmonic. When $(m_3+n_3)^2-4(2m_2+m_1+m_3)n_3 > 0$, the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot(\langle 2\alpha_1, H\rangle))^2 = \frac{m_3+n_3 \pm \sqrt{(m_3+n_3)^2-4(2m_2+m_1+m_3)n_3}}{2(2m_2+m_1+m_3)}$$

holds.

(2) When $H = tH_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_1\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m_1 \cot\langle\alpha_2, H\rangle\alpha_2 - m_1 \cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\
&- m_2 \cot\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\
&- m_3 \cot\langle 2\alpha_2, H\rangle(2\alpha_2) - m_3 \cot\langle 2\alpha_1+2\alpha_2, H\rangle(2\alpha_1+2\alpha_2)\\
&+ m_1 \tan\langle\alpha_2, H\rangle\alpha_2 + m_1 \tan\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\
&+ m_2 \tan\langle\alpha_1+2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\
&+ n_3 \tan\langle 2\alpha_2, H\rangle(2\alpha_2) + n_3 \tan\langle 2\alpha_1+2\alpha_2, H\rangle(2\alpha_1+2\alpha_2)\\
=& - m_1(\cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle)(\alpha_1+2\alpha_2) - m_2 \cot\langle 2\alpha_2, H\rangle(\alpha_1+2\alpha_2)\\
&+ m_2 \tan\langle 2\alpha_2, H\rangle(\alpha_1+2\alpha_2) - 2m_3 \cot\langle 2\alpha_2, H\rangle(\alpha_2+2\alpha_2)\\
&+ 2n_3 \tan\langle 2\alpha_2, H\rangle(\alpha_2+2\alpha_2)\\
=&\{-(2m_1+m_2+2m_3)\cot\langle 2\alpha_2, H\rangle + (m_2+2n_2)\tan\langle 2\alpha_2, H\rangle\}(\alpha_1+2\alpha_2).
\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\langle 2\alpha_2, H\rangle)^2 = \frac{m_2+2n_2}{2m_1+m_2+2m_3}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_1\langle\tau_H, \alpha_2\rangle(1-(\cot\langle\alpha_2, H\rangle)^2)\alpha_2\\
&+ m_1\langle\tau_H, \alpha_1+\alpha_2\rangle(1-(\cot\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\
&+ m_2\langle\tau_H, \alpha_1+2\alpha_2\rangle(1-(\cot\langle\alpha_1+2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2)\\
&+ m_3\langle\tau_H, 2\alpha_2\rangle(1-(\cot\langle 2\alpha_2, H\rangle)^2)(2\alpha_2)\\
&+ m_3\langle\tau_H, 2\alpha_1+2\alpha_2\rangle(1-(\cot\langle 2\alpha_1+2\alpha_2, H\rangle)^2)(2\alpha_1+2\alpha_2)\\
&+ m_1\langle\tau_H, \alpha_2\rangle(1-(\tan\langle\alpha_2, H\rangle)^2)\alpha_2\\
&+ m_1\langle\tau_H, \alpha_1+\alpha_2\rangle(1-(\tan\langle\alpha_1+\alpha_2, H\rangle)^2)(\alpha_1+\alpha_2)\\
&+ m_2\langle\tau_H, \alpha_1+2\alpha_2\rangle(1-(\tan\langle\alpha_1+2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2)\\
&+ n_3\langle\tau_H, 2\alpha_2\rangle(1-(\tan\langle 2\alpha_2, H\rangle)^2)(2\alpha_2)\\
&+ n_3\langle\tau_H, 2\alpha_1+2\alpha_2\rangle(1-(\tan\langle 2\alpha_1+2\alpha_2, H\rangle)^2)(2\alpha_1+2\alpha_2)\\
=& m_1\langle\tau_H, \alpha_2\rangle\{(1-(\cot\langle\alpha_2, H\rangle)^2) + (1-(\tan\langle\alpha_2, H\rangle)^2)\}(\alpha_1+2\alpha_2)\\
&+ 2m_2\langle\tau_H, \alpha_2\rangle(1-(\cot\langle 2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2)\\
&+ 2m_2\langle\tau_H, \alpha_2\rangle(1-(\tan\langle 2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2)\\
&+ 4m_3\langle\tau_H, \alpha_2\rangle(1-(\cot\langle 2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2)\\
&+ 4n_3\langle\tau_H, \alpha_2\rangle(1-(\tan\langle 2\alpha_2, H\rangle)^2)(\alpha_1+2\alpha_2)\\
=& 2\langle\tau_H, \alpha_2\rangle\{-2m_1(\cot\langle 2\alpha_2, H\rangle)^2 + (m_2+2m_3)(1-(\cot\langle 2\alpha_2, H\rangle)^2)\\
&+ (n_2+2n_3)(1-(\tan\langle 2\alpha_2, H\rangle)^2)\}(\alpha_1+2\alpha_2)\\
=& \langle\tau_H, 2\alpha_2\rangle\{(2m_1+m_2+2m_3)(1-(\cot\langle 2\alpha_2, H\rangle)^2)\\
&+ (n_2+2n_3)(1-(\tan\langle 2\alpha_2, H\rangle)^2) - 2m_1\}(\alpha_1+2\alpha_2).
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$\begin{aligned} 0 =&(2m_1 + m_2 + 2m_3)(1 - (\cot\langle 2\alpha_2, H\rangle)^2) \\ &+ (n_2 + 2n_3)(1 - (\tan\langle 2\alpha_2, H\rangle)^2) - 2m_1 \end{aligned}$$

holds. The equation is equivalent to

$$\begin{aligned} &\big((2m_1 + m_2 + 2m_3)(\cot\langle 2\alpha_2, H\rangle)^2 - (m_2 + 2n_3)\big)\big((\cot\langle 2\alpha_2, H\rangle)^2 - 1\big) \\ =& - 2m_1(\cot\langle 2\alpha_2, H\rangle)^2 \end{aligned}$$

Since $2m_1 > 0$, the solutions of the equation are not harmonic. When

$$(m_2 + m_3 + n_3)^2 - (2m_1 + m_2 + 2m_3)(m_2 + 2n_2) > 0$$

the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$\begin{aligned} &(\cot(\langle 2\alpha_2, H\rangle))^2 \\ =&\frac{m_2 + m_3 + n_3 \pm \sqrt{(m_2 + m_3 + n_3)^2 - (2m_1 + m_2 + 2m_3)(m_2 + 2n_2)}}{2m_1 + m_2 + 2m_3} \end{aligned}$$

holds.

(3) When $H = tH_{\alpha_1} + (1 - t)H_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \emptyset, W_H^+ = \{\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 = 2e_1\}$. We set $\vartheta = \langle 2\alpha_1, H\rangle$. Then $\langle 2\alpha_2, H\rangle = (\pi/2) - \vartheta$. By Theorem 2.9, we have

$$\begin{aligned} \tau_H =& - m_2 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1 \cot\langle\alpha_2, H\rangle\alpha_2 - m_1 \cot\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2) \\ &- m_2 \cot\langle\alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2) - m_3 \cot\langle 2\alpha_2, H\rangle(2\alpha_2) \\ &+ m_2 \tan\langle\alpha_1, H\rangle\alpha_1 + m_1 \tan\langle\alpha_2, H\rangle\alpha_2 + m_1 \tan\langle\alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2) \\ &+ m_2 \tan\langle\alpha_1 + 2\alpha_2, H\rangle(\alpha_1 + 2\alpha_2) + n_3 \tan\langle 2\alpha_2, H\rangle(2\alpha_2) \\ =& - m_2(\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle)\alpha_1 - m_1(\cot\langle\alpha_2, H\rangle - \tan\langle\alpha_2, H\rangle)\alpha_2 \\ &- m_2(\cot\langle\alpha_1 + 2\alpha_2, H\rangle - \tan\langle\alpha_1 + 2\alpha_2, H\rangle)(\alpha_1 + 2\alpha_2) \\ &- m_3 \cot\langle 2\alpha_2, H\rangle(2\alpha_2) + n_3 \tan\langle 2\alpha_2, H\rangle(2\alpha_2) \\ =& - 2m_2 \cot(\vartheta)\alpha_1 - 2m_1 \cot((\pi/2) - \vartheta)\alpha_2 - 2m_2 \cot(\pi - \vartheta)(\alpha_1 + 2\alpha_2) \\ &- m_3 \cot((\pi/2) - \vartheta)2\alpha_2 + n_3 \tan((\pi/2) - \vartheta)2\alpha_2 \\ =&\{(4m_2 + 2n_3) \cot(\vartheta) - (2m_1 + 2m_3) \tan((\pi/2) - \vartheta)\}\alpha_2. \end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$(\cot\vartheta)^2 = \frac{m_1 + m_3}{2m_2 + n_3}.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_2\langle\tau_H,\alpha_1\rangle(1-(\cot\langle\alpha_1,H\rangle)^2)\alpha_1 + m_1\langle\tau_H,\alpha_2\rangle(1-(\cot\langle\alpha_2,H\rangle^2)\alpha_2 \\
&+ m_1\langle\tau_H,\alpha_1+\alpha_2\rangle(1-(\cot\langle\alpha_1+\alpha_2,H\rangle)^2)(\alpha_1+\alpha_2) \\
&+ m_2\langle\tau_H,\alpha_1+2\alpha_2\rangle(1-(\cot\langle\alpha_1+2\alpha_2,H\rangle)^2)(\alpha_1+2\alpha_2) \\
&+ m_3\langle\tau_H,2\alpha_2\rangle(1-(\cot\langle2\alpha_2,H\rangle)^2)(2\alpha_2) \\
&+ m_2\langle\tau_H,\alpha_1\rangle(1-(\tan\langle\alpha_1,H\rangle)^2)\alpha_1 + m_1\langle\tau_H,\alpha_2\rangle(1-(\tan\langle\alpha_2,H\rangle)^2)\alpha_2 \\
&+ m_1\langle\tau_H,(\alpha_1+\alpha_2)\rangle(1-(\tan\langle\alpha_1+\alpha_2,H\rangle)^2)(\alpha_1+\alpha_2) \\
&+ m_2\langle\tau_H,(\alpha_1+2\alpha_2)\rangle(1-(\tan\langle\alpha_1+2\alpha_2,H\rangle)^2)(\alpha_1+2\alpha_2) \\
&+ n_3\langle\tau_H,2\alpha_2\rangle(1-(\tan\langle2\alpha_2,H\rangle)^2)(2\alpha_2) \\
=& m_2\langle\tau_H,\alpha_1\rangle\{(1-(\cot\langle\alpha_1,H\rangle)^2)+(1-(\tan\langle\alpha_1,H\rangle)^2)\}\alpha_1 \\
&+ m_1\langle\tau_H,\alpha_2\rangle\{(1-(\cot\langle\alpha_2,H\rangle)^2)+(1-(\tan\langle\alpha_2,H\rangle)^2)\}\alpha_2 \\
&+ m_2\langle\tau_H,\alpha_1+2\alpha\rangle\{(1-(\cot\langle\alpha_1+2\alpha,H\rangle)^2) \\
&\qquad + (1-(\tan\langle\alpha_1+2\alpha,H\rangle)^2)\}(\alpha_1+2\alpha) \\
&+ m_3\langle\tau_H,2\alpha_2\rangle(1-(\cot((\pi/2)-\vartheta))^2)(2\alpha_2) \\
&+ n_3\langle\tau_H,2\alpha_2\rangle(1-(\tan((\pi/2)-\vartheta))^2)(2\alpha_2) \\
=& -4m_2\langle\tau_H,\alpha_1\rangle(\cot(\vartheta))^2\alpha_1 - 4m_1\langle\tau_H,\alpha_2\rangle(\cot((\pi/2)-\vartheta))^2\alpha_2 \\
&- 4m_2\langle\tau_H,\alpha_1+2\alpha_2\rangle(\cot(\pi-\vartheta))^2(\alpha+2\alpha_2) \\
&+ m_3\langle\tau_H,2\alpha_2\rangle(1-(\tan(\vartheta))^2)(2\alpha_2) + n_3\langle\tau_H,2\alpha_2\rangle(1-(\cot(\vartheta))^2)(2\alpha_2) \\
=& 4\langle\tau_H,\alpha_2\rangle\{-2m_2(\cot(\vartheta))^2 - m_1(\tan(\vartheta))^2 + m_3\rangle(1-(\tan(\vartheta))^2) \\
&+ n_3(1-(\cot(\vartheta))^2)\}\alpha_2 \\
=& 4\langle\tau_H,\alpha_2\rangle\{(2m_2+n_3)(1-(\cot(\vartheta))^2) \\
&+ (m_1+m_3)(1-(\tan(\vartheta))^2) - (2m_2+m_1)\}\alpha_2.
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = (2m_2+n_3)(1-(\cot(\vartheta))^2) + (m_1+m_3)(1-(\tan(\vartheta))^2) - (2m_2+m_1)$$

holds. The equation is equivalent to

$$\{(2m_2+n_3)(\cot\vartheta)^2 - (m_1+m_3)\}((\cot(\vartheta))^2-1) = -(m_1+2m_2)(\cot(\vartheta))^2.$$

Since $m_1+2m_2>0$, the solutions of the equation are not harmonic. When

$$(m_3+n_3)^2 - 4(2m_2+n_3)(m_1+m_3) > 0,$$

the orbit $K_2\pi_1(\exp(H))$ is proper biharmonic if and only if

$$(\cot\vartheta)^2 = \frac{(m_3+n_3) \pm \sqrt{(m_3+n_3)^2 - 4(2m_2+n_3)(m_1+m_3)}}{2(2m_2+n_3)}$$

holds.

4.6.12. *Type* III-G_2. We set

$$\Sigma^+ = W^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\},$$

$$\langle\alpha_1,\alpha_1\rangle = 1,\ \langle\alpha_1,\alpha_2\rangle = -\frac{3}{2},\ \langle\alpha_2,\alpha_2\rangle = 3,$$

$$\tilde{\alpha} = 3\alpha_1 + 2\alpha_2,$$

and

$$m_1 = m(\alpha_1), m_2 = m(\alpha_2).$$

(1) When $H = tH_{\alpha_1}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_2\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m_1 \cot\langle\alpha_1, H\rangle\alpha_1 - m_1\cot\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2) \\
&- m_1\cot\langle 2\alpha_1+\alpha_2, H\rangle(2\alpha_1+\alpha_2) - m_2\cot\langle 3\alpha_1+\alpha_2, H\rangle(3\alpha_1+\alpha_2) \\
&- m_2\cot\langle 3\alpha_1+2\alpha_2, H\rangle(3\alpha_1+2\alpha_2) \\
&- m_1\tan\langle\alpha_1, H\rangle\alpha_1 - m_1\tan\langle\alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2) \\
&- m_1\tan\langle 2\alpha_1+\alpha_2, H\rangle(2\alpha_1+\alpha_2) - m_2\tan\langle 3\alpha_1+\alpha_2, H\rangle(3\alpha_1+\alpha_2) \\
&- m_2\tan\langle 3\alpha_1+2\alpha_2, H\rangle(3\alpha_1+2\alpha_2) \\
&= \big[- m_1\{(\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle) + (\cot\langle 2\alpha_1, H\rangle - \tan\langle 2\alpha_1, H\rangle)\} \\
&- 3m_2(\cot\langle 3\alpha_1, H\rangle - \tan\langle 3\alpha_1, H\rangle)\big](2\alpha_1+\alpha_2) \\
=& 2\big[- m_1\{\cot\langle 2\alpha_1, H\rangle + \cot\langle 4\alpha_1, H\rangle\} - 3m_2\cot\langle 6\alpha_1, H\rangle\big](2\alpha_1+\alpha_2)
\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$-m_1\{\cot\langle 2\alpha_1, H\rangle + \cot\langle 4\alpha_1, H\rangle\} - 3m_2\cot\langle 6\alpha_1, H\rangle = 0.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_1\langle\tau_H, \alpha_1\rangle(1 - (\cot\langle\alpha_1, H\rangle)^2)\alpha_1 \\
&+ m_1\langle\tau_H, (\alpha_1+\alpha_2)\rangle(1 - (\cot\langle(\alpha_1+\alpha_2), H\rangle)^2)(\alpha_1+\alpha_2) \\
&+ m_1\langle\tau_H, (2\alpha_1+\alpha_2)\rangle(1 - (\cot\langle(2\alpha_1+\alpha_2), H\rangle)^2)(\alpha_1+\alpha_2) \\
&+ m_2\langle\tau_H, (3\alpha_1+\alpha_2)\rangle(1 - (\cot\langle(3\alpha_1+\alpha_2), H\rangle)^2)(3\alpha_1+\alpha_2) \\
&+ m_2\langle\tau_H, (3\alpha_1+2\alpha_2)\rangle(1 - (\cot\langle(3\alpha_1+2\alpha_2), H\rangle)^2)(3\alpha_1+2\alpha_2) \\
&+ m_1\langle\tau_H, \alpha_1\rangle(1 - (\tan\langle\alpha_1, H\rangle)^2)\alpha_1 \\
&+ m_1\langle\tau_H, (\alpha_1+\alpha_2)\rangle(1 - (\tan\langle(\alpha_1+\alpha_2), H\rangle)^2)(\alpha_1+\alpha_2) \\
&+ m_1\langle\tau_H, (2\alpha_1+\alpha_2)\rangle(1 - (\tan\langle(2\alpha_1+\alpha_2), H\rangle)^2)(2\alpha_1+\alpha_2) \\
&+ m_2\langle\tau_H, (3\alpha_1+\alpha_2)\rangle(1 - (\tan\langle(3\alpha_1+\alpha_2), H\rangle)^2)(3\alpha_1+\alpha_2) \\
&+ m_2\langle\tau_H, (3\alpha_1+2\alpha_2)\rangle(1 - (\tan\langle(3\alpha_1+2\alpha_2), H\rangle)^2)(3\alpha_1+2\alpha_2) \\
=& \langle\tau_H, \alpha_1\rangle\{-m_1(\cot\langle\alpha_1, H\rangle - \tan\langle\alpha_1, H\rangle)^2 - 2m_1(\cot\langle 2\alpha_1, H\rangle - \tan\langle 2\alpha_1, H\rangle)^2 \\
&- 9m_2(\cot\langle 3\alpha_1, H\rangle - \tan\langle 3\alpha_1, H\rangle)^2\}(2\alpha_1+\alpha_2) \\
=& - 4\langle\tau_H, \alpha_1\rangle\{m_1(\cot\langle 2\alpha_1, H\rangle)^2 + 2m_1(\cot\langle 4\alpha_1, H\rangle)^2 \\
&+ 9m_2(\cot\langle 6\alpha_1, H\rangle)^2\}(2\alpha_1+\alpha_2).
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = m_1(\cot\langle 2\alpha_1, H\rangle)^2 + 2m_1(\cot\langle 4\alpha_1, H\rangle)^2 + 9m_2(\cot\langle 6\alpha_1, H\rangle)^2$$

holds. Clearly,

$$m_1(\cot\langle 2\alpha_1, H\rangle)^2 + 2m_1(\cot\langle 4\alpha_1, H\rangle)^2 + 9m_2(\cot\langle 6\alpha_1, H\rangle)^2 > 0$$

for $0 < t < 1$. Therefore, if the orbit $K_2\pi_1(\exp(H))$ is biharmonic, then it is harmonic.

(2) When $H = tH_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \{\alpha_1\}, W_H^+ = \emptyset$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m_2 \cot\langle \alpha_2, H\rangle\alpha_2 - m_1 \cot\langle \alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\
&- m_1 \cot\langle 2\alpha_1 + \alpha_2, H\rangle(2\alpha_1 + \alpha_2) - m_2 \cot\langle 3\alpha_1 + \alpha_2, H\rangle(3\alpha_1 + \alpha_2)\\
&- m_2 \cot\langle 3\alpha_1 + 2\alpha_2, H\rangle(3\alpha_1 + 2\alpha_2)\\
&- m_2 \tan\langle \alpha_2, H\rangle\alpha_2 - m_1 \tan\langle \alpha_1 + \alpha_2, H\rangle(\alpha_1 + \alpha_2)\\
&- m_1 \tan\langle 2\alpha_1 + \alpha_2, H\rangle(2\alpha_1 + \alpha_2) - m_2 \tan\langle 3\alpha_1 + \alpha_2, H\rangle(3\alpha_1 + \alpha_2)\\
&- m_2 \tan\langle 3\alpha_1 + 2\alpha_2, H\rangle(3\alpha_1 + 2\alpha_2)\\
&= \big[- m_2\{(\cot\langle \alpha_2, H\rangle - \tan\langle \alpha_2, H\rangle) + (\cot\langle 2\alpha_2, H\rangle - \tan\langle 2\alpha_2, H\rangle)\}\\
&- m_1(\cot\langle \alpha_2, H\rangle - \tan\langle \alpha_2, H\rangle)\big](3\alpha_1 + 2\alpha_2)\\
=& - 2\big[(m_1 + m_2) \cot\langle 2\alpha_1, H\rangle + m_2 \cot\langle 4\alpha_1, H\rangle\big](3\alpha_1 + 2\alpha_2)
\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$(m_1 + m_2) \cot\langle 2\alpha_1, H\rangle + m_2 \cot\langle 4\alpha_1, H\rangle = 0.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =& m_2\langle \tau_H, \alpha_2\rangle(1 - (\cot\langle \alpha_2, H\rangle)^2)\alpha_2\\
&+ m_1\langle \tau_H, (\alpha_1 + \alpha_2)\rangle(1 - (\cot\langle(\alpha_1 + \alpha_2), H\rangle)^2)(\alpha_1 + \alpha_2)\\
&+ m_1\langle \tau_H, (2\alpha_1 + \alpha_2)\rangle(1 - (\cot\langle(2\alpha_1 + \alpha_2), H\rangle)^2)(\alpha_1 + \alpha_2)\\
&+ m_2\langle \tau_H, (3\alpha_1 + \alpha_2)\rangle(1 - (\cot\langle(3\alpha_1 + \alpha_2), H\rangle)^2)(3\alpha_1 + \alpha_2)\\
&+ m_2\langle \tau_H, (3\alpha_1 + 2\alpha_2)\rangle(1 - (\cot\langle(3\alpha_1 + 2\alpha_2), H\rangle)^2)(3\alpha_1 + 2\alpha_2)\\
&+ m_2\langle \tau_H, \alpha_2\rangle(1 - (\tan\langle \alpha_2, H\rangle)^2)\alpha_2\\
&+ m_1\langle \tau_H, (\alpha_1 + \alpha_2)\rangle(1 - (\tan\langle(\alpha_1 + \alpha_2), H\rangle)^2)(\alpha_1 + \alpha_2)\\
&+ m_1\langle \tau_H, (2\alpha_1 + \alpha_2)\rangle(1 - (\tan\langle(2\alpha_1 + \alpha_2), H\rangle)^2)(2\alpha_1 + \alpha_2)\\
&+ m_2\langle \tau_H, (3\alpha_1 + \alpha_2)\rangle(1 - (\tan\langle(3\alpha_1 + \alpha_2), H\rangle)^2)(3\alpha_1 + \alpha_2)\\
&+ m_2\langle \tau_H, (3\alpha_1 + 2\alpha_2)\rangle(1 - (\tan\langle(3\alpha_1 + 2\alpha_2), H\rangle)^2)(3\alpha_1 + 2\alpha_2)\\
=& - \langle \tau_H, \alpha_2\rangle[(m_1 + m_2)(\cot\langle \alpha_2, H\rangle - \tan\langle \alpha_2, H\rangle)^2\\
&+ 2m_2(\cot\langle 2\alpha_1, H\rangle - \tan\langle 2\alpha_1, H\rangle)^2](2\alpha_1 + \alpha_2).
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = (m_1 + m_2)(\cot\langle 2\alpha_2, H\rangle)^2 + 2m_2(\cot\langle 4\alpha_1, H\rangle)^2$$

holds. Clearly,

$$(m_1+m_2)(\cot\langle 2\alpha_2, H\rangle)^2 + 2m_2(\cot\langle 4\alpha_1, H\rangle)^2 > 0$$

for $0 < t < 1$. Therefore, if the orbit $K_2\pi_1(\exp(H))$ is biharmonic, then it is harmonic.

(3) When $H = tH_{\alpha_1} + (1-t)H_{\alpha_2}$ $(0 < t < 1)$, we have $\Sigma_H^+ = \emptyset, W_H^+ = \{3\alpha_1 + 2\alpha_2\}$. We set $\vartheta = \langle \alpha_1, H\rangle$. Then $\langle 2\alpha_2, H\rangle = (\pi/2) - 3\vartheta$ and $0, < \vartheta < (\pi/6)$. By Theorem 2.9, we have

$$\begin{aligned}
\tau_H =& - m_1 \cot\langle \alpha_1, H\rangle \alpha_1 - m_2 \cot\langle \alpha_2, H\rangle \alpha_2 - m_1 \cot\langle \alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\
& - m_1 \cot\langle 2\alpha_1+\alpha_2, H\rangle(2\alpha_1+\alpha_2) - m_2 \cot\langle 3\alpha_1+\alpha_2, H\rangle(3\alpha_1+\alpha_2)\\
& - m_1 \tan\langle \alpha_1, H\rangle \alpha_1 - m_2 \tan\langle \alpha_2, H\rangle \alpha_2 - m_1 \tan\langle \alpha_1+\alpha_2, H\rangle(\alpha_1+\alpha_2)\\
& - m_1 \tan\langle 2\alpha_1+\alpha_2, H\rangle(2\alpha_1+\alpha_2) - m_2 \tan\langle 3\alpha_1+\alpha_2, H\rangle(3\alpha_1+\alpha_2)\\
=&2[-m_1 \cot\langle 2\alpha_1, H\rangle \alpha_1 - m_2 \cot\langle 2\alpha_2, H\rangle \alpha_2 - m_1 \cot\langle 2(\alpha_1+\alpha_2), H\rangle(\alpha_1+\alpha_2)\\
& - m_1 \cot\langle 2(2\alpha_1+\alpha_2), H\rangle(2\alpha_1+\alpha_2) - m_2 \cot\langle 2(3\alpha_1+\alpha_2), H\rangle(3\alpha_1+\alpha_2)]\\
=& - 2[m_1 \cot(2\vartheta)\alpha_1 + m_2 \cot((\pi/2) - 3\vartheta)\alpha_2 + m_1 \cot((\pi/2) - \vartheta)(\alpha_1+\alpha_2)\\
& + m_1 \cot((\pi/2 + \vartheta))(2\alpha_1+\alpha_2) - m_2 \cot((\pi/2) + 3\vartheta)(3\alpha_1+\alpha_2)]\\
=& - 2\big[m_1 \cot(2\vartheta)\alpha_1 - m_2 \tan(3\vartheta)(3\alpha_1) - m_1 \tan(\vartheta)\alpha_1\big].
\end{aligned}$$

Hence we have $\tau_H = 0$ if and only if

$$m_1 \cot(2\vartheta) - 3m_2 \tan(3\vartheta) - m_1 \tan(\vartheta) = 0.$$

By Theorem 4.6, the orbit $K_2\pi_1(\exp(H))$ is biharmonic if and only if

$$\begin{aligned}
0 =&m_1\langle \tau_H, \alpha_1\rangle(1 - (\cot\langle \alpha_1, H\rangle)^2)\alpha_1 + m_2\langle \tau_H, \alpha_2\rangle(1 - (\cot\langle \alpha_2, H\rangle)^2)\alpha_2\\
& + m_1\langle \tau_H, (\alpha_1+\alpha_2)\rangle(1 - (\cot\langle(\alpha_1+\alpha_2), H\rangle)^2)(\alpha_1+\alpha_2)\\
& + m_1\langle \tau_H, (2\alpha_1+\alpha_2)\rangle(1 - (\cot\langle(2\alpha_1+\alpha_2), H\rangle)^2)(\alpha_1+\alpha_2)\\
& + m_2\langle \tau_H, (3\alpha_1+\alpha_2)\rangle(1 - (\cot\langle(3\alpha_1+\alpha_2), H\rangle)^2)(3\alpha_1+\alpha_2)\\
& + m_1\langle \tau_H, \alpha_1\rangle(1 - (\tan\langle \alpha_1, H\rangle)^2)\alpha_1 + m_2\langle \tau_H, \alpha_2\rangle(1 - (\tan\langle \alpha_2, H\rangle)^2)\alpha_2\\
& + m_1\langle \tau_H, (\alpha_1+\alpha_2)\rangle(1 - (\tan\langle(\alpha_1+\alpha_2), H\rangle)^2)(\alpha_1+\alpha_2)\\
& + m_1\langle \tau_H, (2\alpha_1+\alpha_2)\rangle(1 - (\tan\langle(2\alpha_1+\alpha_2), H\rangle)^2)(2\alpha_1+\alpha_2)\\
& + m_2\langle \tau_H, (3\alpha_1+\alpha_2)\rangle(1 - (\tan\langle(3\alpha_1+\alpha_2), H\rangle)^2)(3\alpha_1+\alpha_2)\\
=& - m_1\langle \tau_H, \alpha_1\rangle(\cot(2\vartheta))^2\alpha_1 - m_2\langle \tau_H, \alpha_2\rangle(\cot((\pi/2) - 3\vartheta))^2\alpha_2\\
& - m_1\langle \tau_H, (\alpha_1+\alpha_2)\rangle(\cot((\pi/2) - \vartheta))^2(\alpha_1+\alpha_2)\\
& - m_1\langle \tau_H, (2\alpha_1+\alpha_2)\rangle(\cot((\pi/2) + \vartheta))^2(\alpha_1+\alpha_2)\\
& - m_2\langle \tau_H, (3\alpha_1+\alpha_2)\rangle(\cot((\pi/2) - 3\vartheta))^2(3\alpha_1+\alpha_2)\\
=& - \langle \tau_H, \alpha_1\rangle[m_1(\cot(2\vartheta))^2\alpha_1 + (3/2)m_2(\tan(3\vartheta))^2(3\alpha_1) + (1/2)m_1(\tan(\vartheta))^2\alpha_1]\\
=& - \langle \tau_H, \alpha_1\rangle[m_1(\cot(2\vartheta))^2 + (9/2)m_2(\tan(3\vartheta))^2 + (1/2)m_1(\tan(\vartheta))^2]\alpha_1.
\end{aligned}$$

Therefore, $K_2\pi_1(\exp(H))$ is biharmonic if and only if $\tau_H = 0$ or

$$0 = m_1(\cot(2\vartheta))^2 + (9/2)m_2(\tan(3\vartheta))^2 + (1/2)m_1(\tan(\vartheta))^2$$

holds. Clearly,

$$m_1(\cot(2\vartheta))^2 + (9/2)m_2(\tan(3\vartheta))^2 + (1/2)m_1(\tan(\vartheta))^2 > 0$$

for $0 < t < 1$. Therefore, if the orbit $K_2\pi_1(\exp(H))$ is biharmonic, then it is harmonic.

4.6.13. *Tables of proper biharmonic orbits.* By the above arguments, we obtain many examples of proper biharmonic submanifolds in compact symmetric spaces as orbits of Hermann actions. The co-dimension of these submanifolds are greater than two, since we consider singular orbits of cohomogeneity two action.

Theorem 4.11. *Let (G, K_1, K_2) be a compact symmetric triad which satisfies the one of the following conditions (A), (B) or (C) in Theorem 3.14. Assume that the K_2-action on $M_1 = G/K_1$is cohomogeneity two. Then, for each orbit type which is an one parameter family in the orbit space, we can divide into the following three cases:*

(1) *There exists a unique proper biharmonic orbit.*
(2) *There exist exactly two distinct proper biharmonic orbit.*
(3) *Any biharmonic orbit is harmonic.*

We list results of the above computations below.

Isotropy actions ($K_1 = K_2$)

Type A_2

(G, K_1, K_2)	$m(\alpha)$	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{SU}(3), \mathrm{SO}(3))$	1	(2)	(2)	(2)
$(\mathrm{SU}(3) \times \mathrm{SU}(3), \mathrm{SU}(3))$	2	(2)	(2)	(2)
$(\mathrm{SU}(6), \mathrm{Sp}(3))$	4	(2)	(2)	(2)
(E_6, F_4)	8	(2)	(2)	(2)

Type B_2

(G, K_1, K_2)	(m_1, m_2)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{SO}(3) \times \mathrm{SO}(3), \mathrm{SO}(3))$	$(2, 2)$	(2)	(2)	(2)
$(\mathrm{SO}(4+n), \mathrm{SO}(2) \times \mathrm{SO}(2+n))$	$(n, 1)$	(2)	(2)	(2)

Type C_2

(G, K_1, K_2)	(m_1, m_2)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{Sp}(2), \mathrm{U}(2))$	$(1, 1)$	(2)	(2)	(2)
$(\mathrm{Sp}(2) \times \mathrm{Sp}(2), \mathrm{Sp}(2))$	$(2, 2)$	(2)	(2)	(2)
$(\mathrm{Sp}(4), \mathrm{Sp}(2) \times \mathrm{Sp}(2))$	$(4, 3)$	(2)	(2)	(2)
$(\mathrm{SU}(4), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)))$	$(2, 1)$	(2)	(2)	(2)
$(\mathrm{SO}(8), \mathrm{U}(4))$	$(4, 1)$	(2)	(2)	(2)

Type BC_2

(G, K_1, K_2)	(m_1, m_2, m_3)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{SU}(4+n), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2+n)))$	$(2n, 2, 1)$	(2)	(2)	(2)
$(\mathrm{SO}(10), \mathrm{U}(5))$	$(4, 4, 1)$	(2)	(2)	(2)
$(\mathrm{Sp}(4+n), \mathrm{Sp}(2) \times \mathrm{Sp}(2+n))$	$(4n, 4, 3)$	(2)	(2)	(2)
$(E_6, \mathrm{T}^1 \cdot \mathrm{Spin}(10))$	$(8, 6, 1)$	(2)	(2)	(2)

Type G_2

(G, K_1, K_2)	(m_1, m_2)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(G_2, \mathrm{SO}(4))$	$(1, 1)$	(2)	(2)	(3)
$(G_2 \times G_2, G_2)$	$(2, 2)$	(2)	(2)	(3)

When $(\theta_1 \not\sim \theta_2)$

Type I-B_2

(G, K_1, K_2)	(m_1, m_2, n_1)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{SO}(2+s+t), \mathrm{SO}(2+s) \times \mathrm{SO}(t), \mathrm{SO}(2) \times \mathrm{SO}(s+t))$	$(t-2, 1, s)$	(1)	(2)	(2)
$(\mathrm{SO}(6), \mathrm{SO}(3) \times \mathrm{SO}(3))$ (C)	$(2, 2, 2)$	(1)	(2)	(2)

Here $(2 < t, 1 \leq s)$.

Type I-C_2

(G, K_1, K_2)	(m_1, m_2, n_1)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{SO}(8), \mathrm{SO}(4) \times \mathrm{SO}(4), \mathrm{U}(4))$	$(2, 1, 2)$	(1)	(2)	(2)
$(\mathrm{SU}(4), \mathrm{SO}(4), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)))$	$(1, 1, 1)$	(1)	(2)	(2)
$(\mathrm{SU}(4), \mathrm{SO}(4))$ (C)	$(2, 2, 2)$	(1)	(2)	(2)
$(\mathrm{SU}(4), \mathrm{Sp}(2))$ (C)	$(2, 2, 2)$	(1)	(2)	(2)

Type I-BC_2–A_1^2

(G, K_1, K_2)	(m_1, m_2, m_3, n_1)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{SU}(2+s+t), \mathrm{S}(\mathrm{U}(2+s) \times \mathrm{U}(t)), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(s+t)))$	$(2(t-2), 2, 1, 2s)$	(2)	(2)	(2)
$(\mathrm{Sp}(2+s+t), \mathrm{Sp}(2+s) \times \mathrm{Sp}(t), \mathrm{Sp}(2) \times \mathrm{Sp}(s+t))$	$(4(t-1), 4, 3, 4s)$	(2)	(2)	(2)
$(\mathrm{SO}(12), \mathrm{U}(6), \mathrm{U}(6)')$	$(4, 4, 1, 4)$	(2)	(2)	(2)

Here $2 < t, 1 \leq s$.

Type I-BC_2–B_2

(G, K_1, K_2)	(m_1, m_2, m_3, n_2)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{SO}(4+2s), \mathrm{SO}(4) \times \mathrm{SO}(2s), \mathrm{U}(2+s))$	$(2(s-2), 2, 1, 2)$	(2)	(2)	(3)
$(E_6, \mathrm{SU}(6) \cdot \mathrm{SU}(2), \mathrm{SO}(10) \cdot \mathrm{U}(1))$	$(4, 4, 1, 2)$	(2)	(3)	(3)
$(E_7, \mathrm{SO}(12) \cdot \mathrm{SU}(2), E_6 \cdot \mathrm{U}(1))$	$(8, 6, 1, 2)$	(2)	(3)	(3)

Here $2 \leq s$.

Type II-BC_2

(G, K_1, K_2)	(m_1, m_2, n_3)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{SU}(2+s), \mathrm{SO}(2+s), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(s)))$	$(s-2, 1, 1)$	(3)	(3)	(3)
$(\mathrm{SO}(10), \mathrm{SO}(5) \times \mathrm{SO}(5), \mathrm{U}(5))$	$(2, 2, 1)$	(3)	(3)	(3)
$(E_6, \mathrm{Sp}(4), \mathrm{SO}(10) \cdot \mathrm{U}(1))$	$(4, 3, 1)$	(3)	(3)	(3)

Here $2 \leq s$.

Type III-A_2

(G, K_1, K_2)	(m_1, n_1)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{SU}(6), \mathrm{Sp}(3), \mathrm{SO}(6))$	$(2, 2)$	(3)	(3)	(3)
$(E_6, \mathrm{Sp}(4), F_4)$	$(4, 4)$	(3)	(3)	(3)
$(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$, (B)	(a, a)	(3)	(3)	(3)

Here $2 \leq s$, and a is the multiplicity of the root system of the symmetric pair $(U, \overline{K})$.

Type III-B_2

(G, K_1, K_2)	(m_1, m_2, n_2)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$, (B)	(a, b, b)	(3)	(3)	(3)

Here $2 \leq s$, and (a, b) is the multiplicity of the root system of the symmetric pair $(U, \overline{K})$.

Type III-C_2

(G, K_1, K_2)	(m_1, m_2, n_2)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{SU}(8), \mathrm{S}(\mathrm{U}(4) \times \mathrm{U}(4)), \mathrm{Sp}(4))$	$(4, 3, 1)$	(1)	(3)	(3)
$(\mathrm{Sp}(4), \mathrm{U}(4), \mathrm{Sp}(2) \times \mathrm{Sp}(2))$	$(2, 1, 2)$	(1)	(3)	(3)
$(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$ (B)	(a, b, b)	(3)	(3)	(3)

Here (a, b) is the multiplicity of the root system of the symmetric pair $(U, \overline{K})$.

Type III-BC_2

(G, K_1, K_2)	(m_1, m_2, m_3, n_3)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(\mathrm{SU}(4+2s), \mathrm{S}(\mathrm{U}(4) \times \mathrm{U}(2s)), \mathrm{Sp}(2+s))$	$(4(s-2), 4, 3, 1)$	(3)	(3)	(3)
$(\mathrm{SU}(10), \mathrm{S}(\mathrm{U}(5) \times \mathrm{U}(5)), \mathrm{Sp}(5))$	$(4, 4, 1, 3)$	(3)	(3)	(3)
$(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$ (B)	(a, b, c, c)	(3)	(3)	(3)

Here $2 \leq s$, and (a, b, c) is the multiplicity of the root system of the symmetric pair $(U, \overline{K})$.

Type III-G_2

(G, K_1, K_2)	(m_1, m_2, n_1, n_2)	tH_{α_1}	tH_{α_2}	$tH_{\alpha_1} + (1-t)H_{\alpha_2}$
$(U \times U, \Delta(U \times U), \overline{K} \times \overline{K})$ (B)	(a, b, a, b)	(3)	(3)	(3)

Here (a, b) is the multiplicity of the root system of the symmetric pair $(U, \overline{K})$.

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