SELF-AFFINITY OF LANDFORM AND ITS MEASUREMENT

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Abstract Fractal geometry is expected to provide a new quantitative way to express a certain property of landform. Transect profiles of landform are considered to be self-affine because vertical and horizontal coordinates should be scaled differently, while contour lines are isotropic and self-similar. The method developed to analyze the self-affinity of curves in two-dimensional space can well express these fractal characteristics of both transect profiles and contour lines. This method is extended to analyze the three-dimensional land surfaces. The variance of elevation change Z^2, surface area S and bottom area A are measured in a number of scaling unit areas of various sizes to see whether Z^2 and A are scaled as $Z^2 \sim S^{\nu_z}$ and $A \sim S^{\nu_A}$. The three dimensional land surface is appeared to be self-affine with $\nu_A \cong 1$ and $0 < \nu_z < 1$, and the value of ν_z is equivalent to the value of scaling factor H of fractional Brownian motion traces. This method of measurement well reproduces the initially introduced H value of the computer-generated fractional Brownian surfaces, and it is confirmed to work also on the real topography

Key words: fractal geometry, self-affinity, scaling parameter, random surfaces, landform

1. Introduction

Landform is considered to be a good example of fractal geometry (*e.g.* Mandelbrot, 1982), which is now well known as the concept that made mathematical description possible on the complex forms found in nature. Coastlines that show statistical self-similarity (Mandelbrot, 1967) are a classic example of fractals. Ahnert (1984) examined the relationship between local relief and diameter of the reference area on some topographic maps of Europe and Africa, and he pointed out the existence of the same rule formulated by Hurst (1951) for hydrological data. The Hurst phenomenon (Klemeš, 1974), which is one of the important findings promoting the development of the concept of fractal geometry (Mandelbrot and Wallis, 1968, 1969a, 1969b, and also Mandelbrot, 1982), is that the rescaled range of stream discharge R/σ is proportional to the Hth power of the period of record N ($R/\sigma \sim N^H$, and $0.5 < H < 1.0$, where R is the cumulative sample range, and σ is the standard deviation of the data) instead of $N^{0.5}$ as predicted for the

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Gaussian random processes. The symbol "$\sim$", which is often used by physicists, means "behaves like", and it is regarded as similar to "$\propto$". The Hurst parameter H is considered as equivalent to the scaling parameter H ($0<H<1$) of fractonal Brownian motion (Mandelbrot, 1965, as explained in Mandelbrot, 1982, and Mandelbrot and Van Ness, 1968), the trace of which is known to be an example of fractals. For the fractional Brownian motion the parameter H relates the variance of increments $var(x(t_2)-x(t_1))$ or X_t^2 to the time difference $|t_2-t_1|$ or T by the scaling law

$$X_t^2 \sim T^{2H}.$$

The Hurst parameter is also related to the Hausdorff dimension (Culling, 1986). Culling and Datko (1987) analyzed fractal characteristics of landform on some topographic maps of southern England in addition to the elaborated mathematical discussion, and they concluded that where a diffusion degradation regime seems to dominate the land surface can be represented by a fractional Brownian surface.

The concept of fractal geometry possibly provides a new quantitative parameter that expresses certain characteristics of landform. Matsushita and Ouchi (1989a, b) examined the characteristics of various curves as fractals, and recognized that vertical and horizontal coordinates of transect profiles should be scaled differently, while contour lines have the isotropic nature. In other words, the transect profiles are self-affine just like the trace of one-dimensional random walk, and the contour lines are self-similar (a special case of self-affinity) as the Koch curve, which is a well known example of the regular fractals. Matsushita and Ouchi (1989a, b) then proposed a method to analyze the self-affinity of various curves. This method, which scales each coordinate differently, is simple, and it can be applied to a variety of curves from the self-similar Koch curve and contour lines to self-affine noise curves and transect profiles of landform without any modification. In this paper the author explains this method and shows the possibility of its application to the three-dimensional landform.

2. Measurement of Self-affinity of Various Curves

In order to analyze the self-affinity of a curve in two-dimensional space ($y=f(x)$), Matsushita and Ouchi (1989a, b) developed a method that calculates the scaling parameters for x and y coordinates separately. Between two arbitrary points on a curve, x- and y- variances (X^2 and Y^2) of all measured points in the section are calculated as

$$X^2=1/n\sum_{i=1}^{n}(x_i-x_c)^2,$$

$$Y^2=1/n\sum_{i=1}^{n}(y_i-y_c)^2,$$

where

$$x_c=1/n\sum_{i=1}^{n}x_i\,;\ y_c=1/n\sum_{i=1}^{n}y_i.$$

And curve length N is also measured. This procedure is repeated for many sections of different sizes. If the log-log plots of X and Y (standard deviations) versus N can be expressed as

$$X \sim N^{\nu_x};\ Y \sim N^{\nu_y}, \qquad (1)$$

the curve can be characterized as a fractal and self-affine. X and Y are then scaled with each other as

$$Y \sim X^{H};\ H = \nu_y/\nu_x. \qquad (2)$$

Particularly, if $\nu_x = \nu_y$, the curve is self-similar with the fractal dimension $D = 1/\nu$. In practice, the curve is divided into sections by the yardstick method, and the values of X, Y, and N are obtained for each section. The respective average values of X, Y, and N are regarded as the representing values for this yardstick length.

The distance L between the end points of the self-similar triadic Koch curve (Fig. 1, fractal dimension D=1.26...) is scaled with its length N as $L \sim N^{\nu}$ ($\nu = 1/D$) by definition. This means that the Koch curve is expected to be scaled as $X \sim N^{\nu_x}$, $Y \sim N^{\nu_y}$ ($\nu_x = \nu_y = 1/D = 0.79...$) in two-dimensional space. In the case of the self-affine one-dimensional random walk (Brownian motion) trace (Fig. 2, a fractional Brownian motion trace with H=0.5) the variance of increments X_t^2 is known to be proportional to the corresponding time interval T as $X_t^2 \sim T$ or $X_t \sim T^{H}$ (H=0.5). In other words, the curve length N between two points on the curve is scaled with T as $T \sim N^{\nu_t}$ ($\nu_t = 1$), and with X_t as $X_t \sim N^{\nu_{xt}}$ ($\nu_{xt} = 0.5$) and $H = \nu_{xt}/\nu_t$ in the same form as equations (1) and (2).

The results of measurement by the method mentioned above on the triadic Koch curve

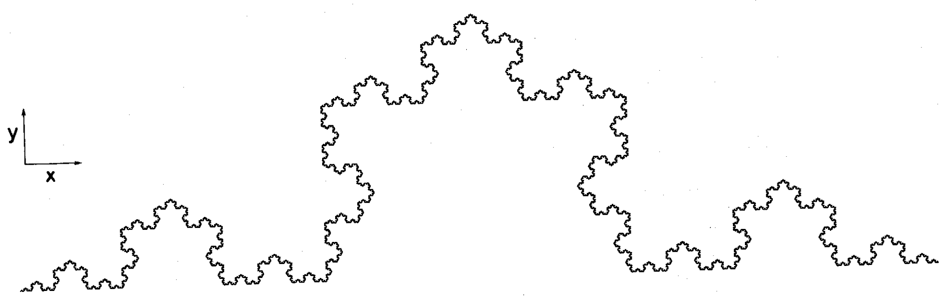

Fig. 1 The triadic Koch curve
A typical self-similar fractal curve generated by the recursive procedure as follows. A simple line is divided into thirds and the middle segment is replaced by two equal segment forming part of an equilateral triangle, and the same process is repeated for each segment at the next stage.
Its fractal dimension can be obtained as;
1) Taking the distance between two end points L (initial line length) as constant;
$a = (1/3)^n$ and $N = 4^n$, where a is the length of unit line and N is the number of the unit line. Then, the fractal dimension D is expressed as; $N \sim a^{-D}$; $D = \ln 4/\ln 3 = 1.26...$
2) Taking the length of unit line a as constant;
$L = 3^n a$ and $N = 4^n$. Then, $N \sim L^{D}$; $D = \ln 4/\ln 3 = 1.26...$
or $L \sim N^{\nu}$; $\nu = 1/D$.

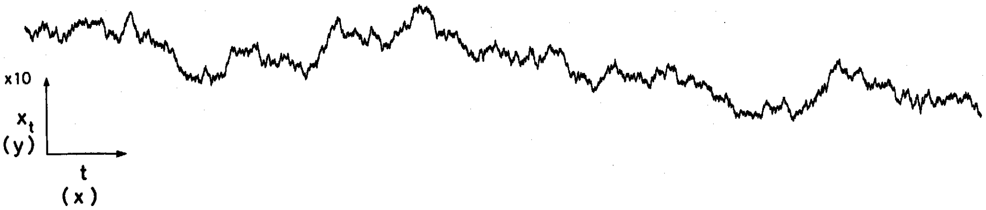

Fig. 2 An example of one-dimensional random walk (Brownian motion) trace $X_t \sim T^H$ ($H=0.5$). And $X_t \sim N^{\nu_{xt}}$; $T \sim N^{\nu_t}$, when the two coordinates are scaled differently ($H=\nu_{xt}/\nu_t$).

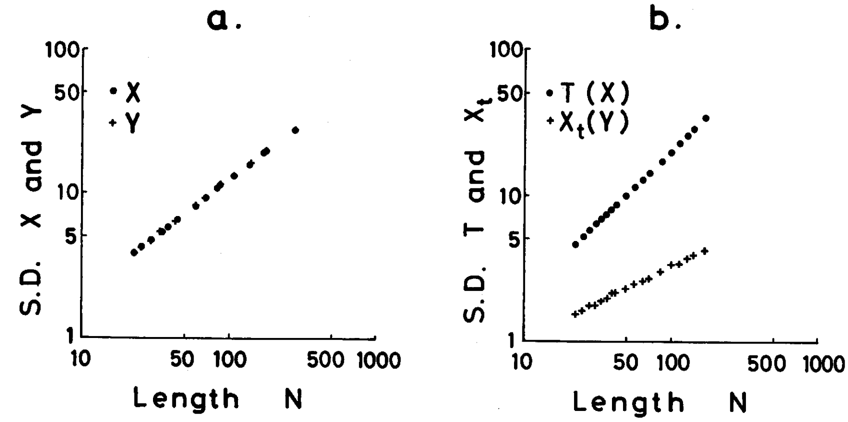

Fig. 3 Graphs showing the results of measurement on the curves in Figs. 1 and 2 Relation of the standard deviations of x and y coordinates (X and Y) to the curve length N in certain sections on the curves indicates the fractal characteristics of the curves. Both coordinates of the graph take arbitrary units.

a. The triadic Koch curve. The slopes of best fitting lines yield the self-similar exponents $\nu_x \cong \nu_y \cong 0.79$.... The equations for the best fitting lines and their values of square of correlation coefficients r^2 are;
$X=0.321N^{0.7956}$ ($r^2=0.9997$),
$Y=0.313N^{0.7997}$ ($r^2=0.9996$).

b. The one-dimensional random walk trace. The slopes of best fitting lines yield the self-affine exponents $\nu_t \cong 1$ and $\nu_{xt} \cong 0.5$. The best fitting lines are;
$T = 0.209N^{0.9953}$ ($r^2=1.0000$),
$X_t = 0.326N^{0.4667}$ ($r^2=0.9978$).

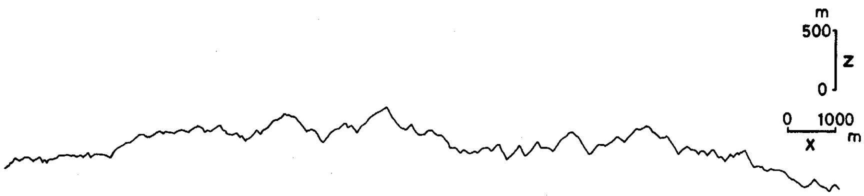

Fig. 4 Transect profile near Mt. Yamizo in Fukushima Prefecture The approximate position of the profile is shown in Fig. 6.

and on the one-dimensional random walk trace are shown in Figure 3. These results indicate that the method works well. It was also confirmed that the values of ν_y or H are very insensitive to vertical or horizontal exaggeration.

This method was then applied to transect profiles and contour lines of real landform. Figure 4 shows a transect profile taken from a 1/25,000 scale topographic map of the well-dissected Mt. Yamizo (1,022 m high) area in Fukushima Prefecture, Japan. The results of measurement by the method shown in Figure 5 ($\nu_x \cong 1$, $\nu_z < 1$) indicate that the transect profile is self-affine and has characteristics similar to the fractional Brownian motion trace. Figure 6 shows a 700m contour line in the same map. The results of measurement on this contour line (Fig. 7, $\nu_x \cong \nu_y \cong 0.73$) indicate that the contour line is self-similar and its fractal dimension $D \cong 1/\nu_x \cong 1/\nu_y \cong 1.37$.

For the self-affine fractional Brownian motion ($\nu_t = 1$ and $\nu_{xt} = H$) it is known that the distribution of the set of points which satisfy $x_t(t) = 0$ (zero set) is self-similar with the fractal dimension $D_0 = 1 - H$. In the same manner the set of level-crossing points with the same altitude on the transect profile of landform, which is self-affine with $\nu_x \cong 1$ and $\nu_z = H$, is considered to be self-similar with the fractal dimension $D_0 = 1 - H$. Contour lines of the landform, which can be considered as the horizontal integration of level-crossing points, are then expected to be self-similar with the fractal dimension

$$D = D_0 + 1 = 2 - H.$$

The results of measurements (Figs. 5 and 7) well explain this relationship.

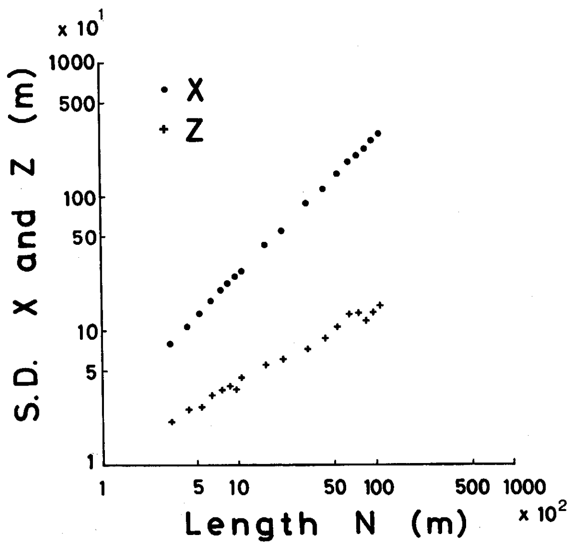

Fig. 5 Graph showing the result of measurement on the transect profile in Fig. 4
The slopes of the plots yield the self-affine exponents $\nu_x \cong 1.0$ and $\nu_z \cong 0.55$.
The best fitting lines are;
$X = 0.227N^{1.020}$ ($r^2 = 0.9997$),
$Z = 0.888N^{0.5537}$ ($r^2 = 0.9895$).

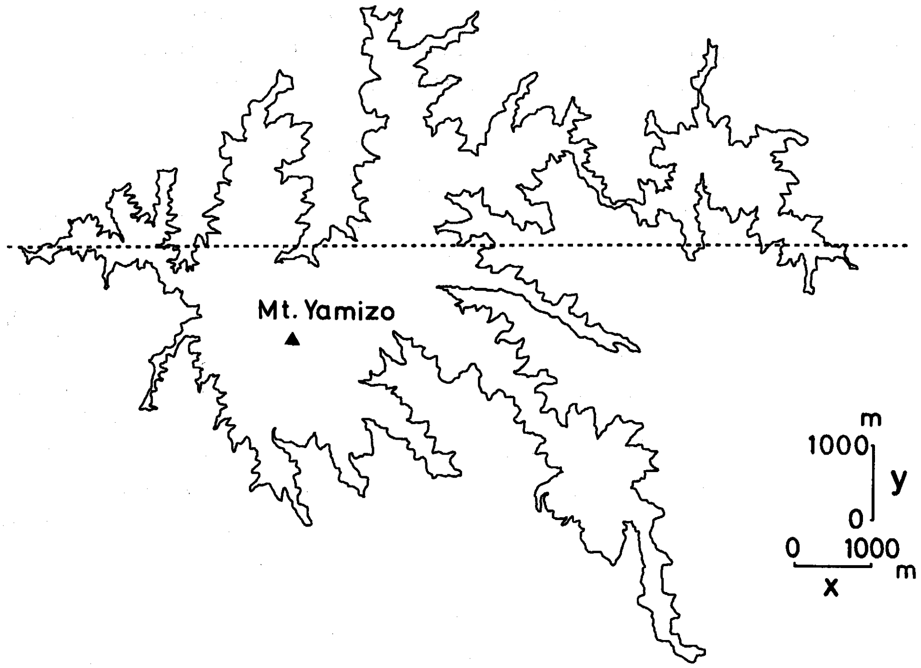

Fig. 6 Contour line of 700 m taken from the 1/25000 topographic map of the Mt. Yamizo area

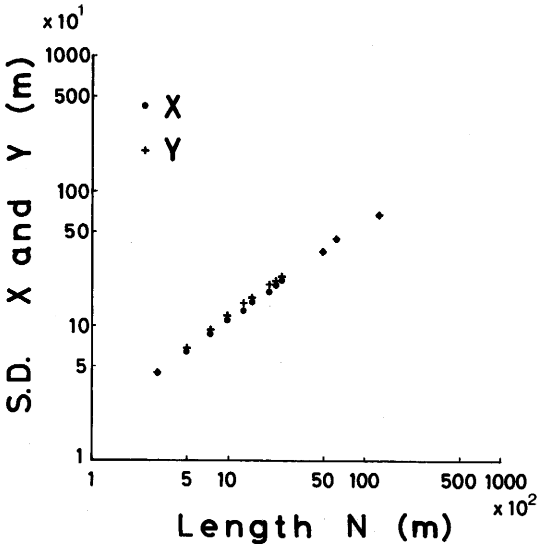

Fig. 7 Graph showing the result of measurement on the contour line in Fig. 6
The slopes yield the self-similar exponents $\nu_x \cong \nu_y \cong 0.73$ ($D \cong 1.37$). The best fitting lines are;
$X = 0.656N^{0.7415}$ ($r^2 = 0.9981$),
$Y = 0.822N^{0.7193}$ ($r^2 = 0.9952$).

3. Application to Three-dimensional Landform

The earth surface develops in three-dimensional space; and therefore, transect profiles and contour lines may not be sufficient to express the characteristics of landform as a whole. The extension of the concept of measuring ν_x and ν_z to the three-dimensional landform is straightforward. Assuming that a land surface has the same characteristics as indicated by its transect profiles in any directions, the variance of elevation change $var(z(x_2,y_2)-z(x_1,y_1))$ or Z^2 in a certain scaling unit area is considered to be related to the corresponding distance change as

$$Z^2 \sim [(x_2-x_1)^2+(y_2-y_1)^2]^H,$$

or

$$Z^2 \sim (x^2+y^2)^H,$$

just as the fractional Brownian motion trace. When the mesh data are used, this relation is considered to be equivalent to

$$Z^2 \sim A^H,$$

where A is bottom area of the scaling unit. Using the surface area S, the relation can be expressed as

$$Z^2 \sim S^{\nu_z};\ A \sim S^{\nu_A} \qquad (3)$$

and

$$H = \nu_z/\nu_A, \qquad (4)$$

ν_z and ν_A respectively correspond to ν_y and ν_x of the equations (1) and (2).

In practice, of course, the procedure of measurement requires certain expedients, because we have to deal with limited number of measurement points. The best worked method obtained after a long procedure of trial and error is explained as follows. The surface to be measured is divided into a number of scaling unit area (square) using the area of a surface formed by four corner points of the square, which can be calculated as the total area of two triangles, as a yardstick. The variance Z^2 of z, the bottom area A and the surface area S are calculated for each scaling unit area, and the respective average values are regarded as the representing values for this scaling unit area. This procedure is repeated for different sizes of scaling unit area, and the values of ν_z and ν_A are obtained from the log-log plots of Z^2 and A versus S.

The method was first applied to the computer-generated fractional Brownian surfaces in order to examine its effectiveness. The computer program to generate fractional Brownian surfaces with arbitrary values of H ($0<H<1$) was developed following the algorithm of the random midpoint displacement method (Saupe, 1988). The surface is generated by the program with a series of random numbers, and different seed values for the random number generator result in different series of random numbers and in different surfaces with the same value of H. The impression on the "roughness" or "texture" of these surfaces, however, is the same as far as the value of H is unchanged. The hundred different surfaces were generated with the seed values 1-100 for each value of H (0.8, 0.5 and 0.2). This procedure was repeated for the three different meshes, 64 × 64, 128 × 128 and 256 × 256. Figure 8 shows the examples of block diagrams of the surfaces (64 × 64 mesh) generated by this program with the values of H, 0.8, 0.5 and 0.

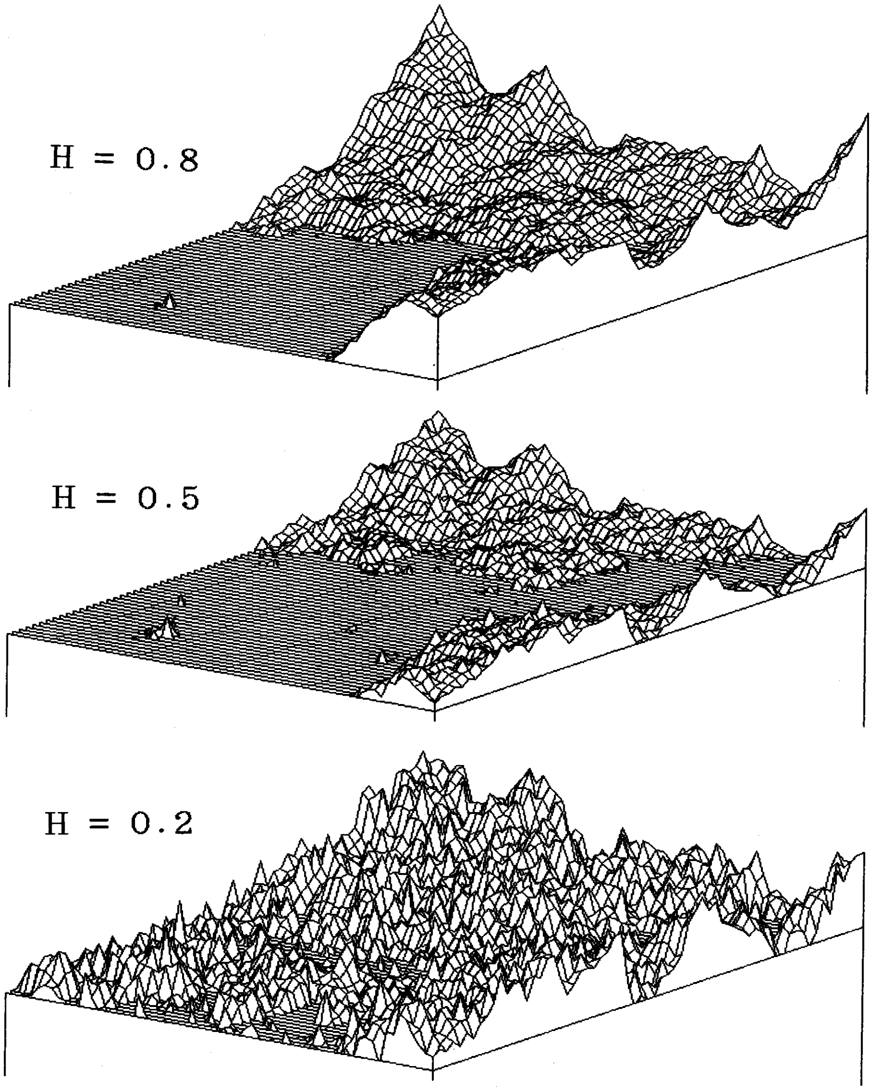

Fig. 8 Block diagrams showing the examples of fractional Brownian surfaces (64 × 64 mesh) generated by the random midpoint displacement method with different values of H. The seed value for the random number generator is 1 for all these three surfaces.

2. Although these surfaces are different from the real topography because they do not have the development of drainage basins at all, which is one of the most important processes of the evolution of the earth surface, the block diagrams give us an impression very similar to the real landform. As it is expected, the maximum relative height and smoothness increase with the value of H, and high local roughness appears with the low value of H. The measurement was performed on each surface generated. Figure 9 shows the example of measurement results on the surfaces. The value of ν_z, which is equivalent to H because the value of ν_A is always close to 1, well reproduces the H value introduced at the beginning. The value of measured ν_z becomes closer to the specified H value as the number of mesh increased. When the specified H value is 0.5, the average of hundred values of measured ν_z is 0.476 (standard deviation $\sigma=0.0538$) for the 64 × 64 mesh, 0.490 ($\sigma=0.0375$) for the 128 × 128 mesh, and 0.498 ($\sigma=0.0228$) for the 256 × 256 mesh. The same trend is clearly observed also in the cases of the specified H=0.2 and 0.8. For the 256 × 256 mesh the average of hundred values of measured ν_z is 0.254 ($\sigma=0.0126$) when the specified H=0.2, and 0.763 ($\sigma=0.0399$) when the specified H=0.8. The results indicate that the method of measurement works well on the three dimensional land surface.

The 80 × 80 mesh of 5 mm mesh size was drawn on the 1/25,000 topographic map of

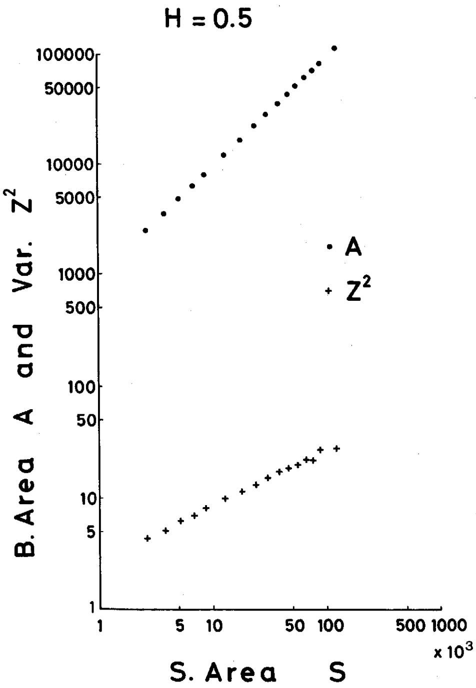

Fig. 9 Graph showing the result of measurement on the fractional Brownian surface of the specified H=0.5 (256 × 256 mesh)
The value of ν_z ($\cong 0.497$) well agrees with the initially introduced H=0.5. Both coordinates of the graph take arbitrary units. The best fitting lines are;
$A = 0.956S^{1.000}$ ($r^2=0.9981$),
$Z^2 = 0.885S^{0.4968}$ ($r^2= 0.9964$).

the Mt. Yamizo area (mesh size is 125 m), and the elevation of all 81 × 81 points was read to make the mesh data to be analyzed. Ablock diagram showing the landscape of the area and a contour map drawn from the mesh data are shown in Figure 10. The mesh data were then analyzed by the method mentioned above. The result (Fig. 11, $\nu_z \cong H \cong 0.53$) is reasonable for the block diagram in Figure 10 comparing with Figure 8, and it

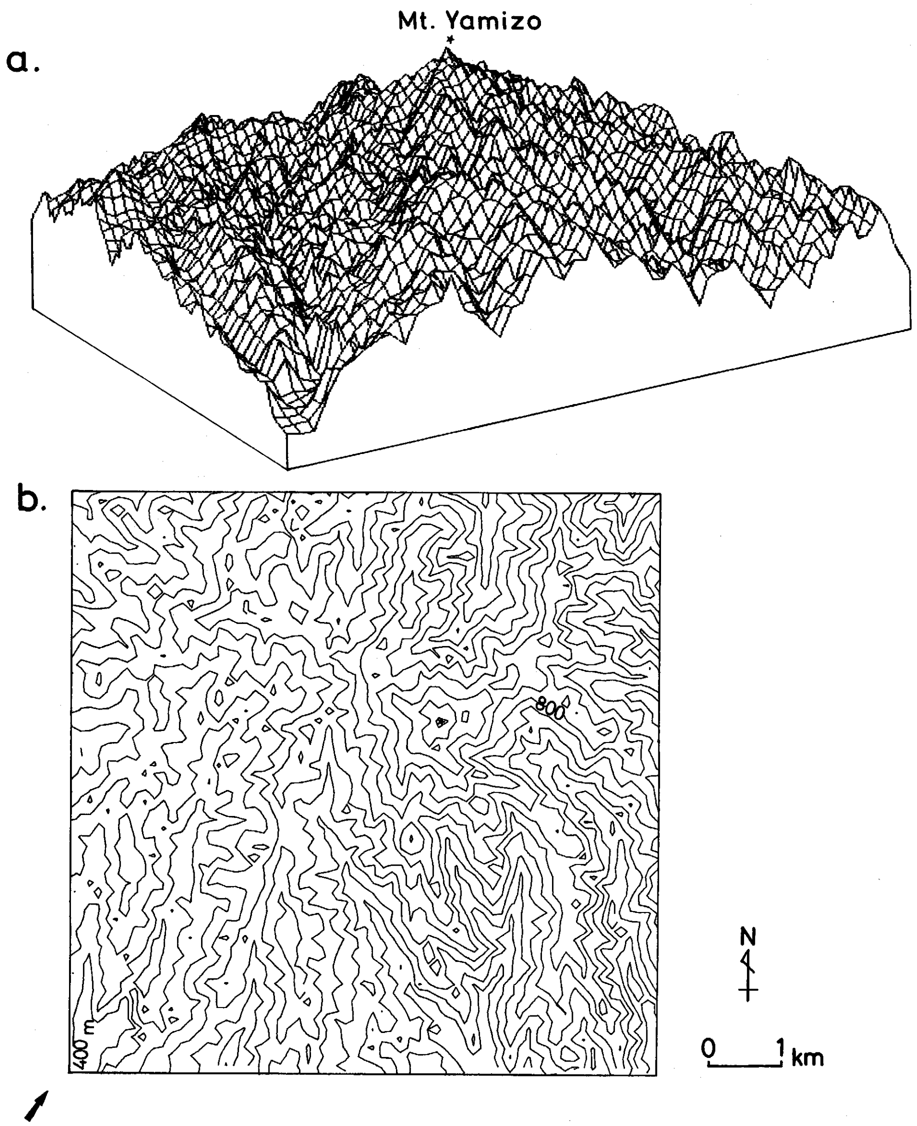

Fig. 10 a. Block diagram and b. Contour map of the area around Mt. Yamizo drawn from the mesh data
The block diagram shows a perspective view from the point of 1,000 m high in the direction indicated by an arrow. * indicates the location of Mt. Yamizo.

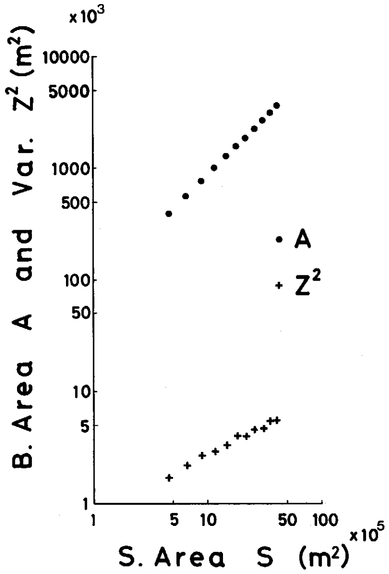

Fig. 11 Graph showing the result of measurement on the mesh data of the Mt. Yamizo area. $\nu_A \cong 1.0$ and $\nu_z \cong 0.53$. The best fitting lines are;
$A = 0.893S^{0.9974}$ ($r^2 = 1.0000$),
$Z^2 = 1.874S^{0.5265}$ ($r^2 = 0.9871$).

seems compatible with the results of measurement on a transect profile and a contour line of the area ($\nu_z \cong H \cong 0.55$ for the profile, and $D \cong 1.37$ for the contour line).

4. Discussion

The earth surface is composed of various forms of different scale and different origin, and the value of parameter ν_z or H would be different depending on the scale. When a huge area is measured, ν_z or H is expected to have a smaller value for the larger area section, because the height of mountains seems to have a certain limit. Matsushita and Ouchi (1989b) showed the transect profile whose log-log plots of Z versus N can be divided into two sections of different slopes, steeper in the section of smaller N and flatter in the section of larger N. Although their explanation that the smaller or local structure may be formed by erosion and the larger or global structure may be formed by the crustal movement needs further examination, this seems to be a interesting and necessary subject to be studied. Ohmori (1978) pointed out that the "dispersion of altitude" X, which is the standard deviation of altitude of all measurement points in a unit area, is related to the size of unit area S as $X \sim S^{0.16}$ for Japanese mountains. This relationship can be considered as equivalent to $Z^2 \sim A^H$ with $H = 0.32$ for the relationship examined in this study. The value seems to be small comparing with the value obtained in the Mt. Yamizo area. This apparent discrepancy can probably be explained by the complexity of real landform as mentioned above. Ohmori (1978) used the basic data of

250 m mesh size and dealt with larger areas, while 125 m mesh size is used in a 10 km × 10 km area around Mt. Yamizo in this study.

In conclusion, the method to analyze the self-affinity of various curves proposed by Matsushita and Ouchi (1989a, b) appeared to be basically applicable to the three-dimensional landform. The parameter ν_z or H in the equations (3) and (4) seems to be useful to express the self-affine "roughness" or "texture" of landform, although further examination and much more measurement on various types of landform are necessary to make clear its meaning in the quantitative expression of landform.

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