GENERALIZATION OF RATIONAL FORMULA METHOD

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Abstract The current rational formula was generalized based on the assumption that both a unit hydrograph and a normalized rainfall hyetograph were approximated to the normal distribution. It was concluded that there were two cases of relation between the peak discharge and rainfall characteristics. The one is that the peak discharge is proportional to the maximum rainfall intensity when the drainage area is small; the other is that it is proportional to the total amounts of rainfall when the area is so large. In addition, under the condition that not only the rainfall but also the runoff coefficient are random variables, the frequency distribution of peak discharge was derived.

1. Introduction

The rational formula method may be one of the most familiar model to estimate runoff from rainfall. Although this model was developed by Mulvaney more than a hundred years ago (Chow, 1964), many engineers have been still obliged to utilize for estimation of peak discharge of small rivers and/or sewers (ASCE, 1960). The main reasons of such continuation of old fashion may be: a) the simplicity of calculation, b) the possibility of obtaining the parameter's values without actual hydrological data, and c) that the frequency standard, which must be needed for engineering planning, is involved in the method.

On the other hand, there have been many hydrologists standing against the rational formula method (Chow, 1964) and a lot of models have been developed instead of it. However, hydrologists can never ignore the fact that the rational formula method has not falled into disuse as the above mention.

The study herein is firstly to clarify that the rational formula method is based on the very steady mathematical conception, which is often used in the field of hydrology but which might not be investigated in relation to the rational method. It must be, of course, clear that there are much difficulties of the runoff estimation by means of a simple model with only a couple of parameters because the natural phenomena is so complicated. The second purpose of this study is to improve on the rational formula method so as to concern the uncertainty beyond the model ability.

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2. Theoretical Approach

The basic formula of the rational method is

$$Q_p = \frac{1}{3.6} \cdot C \cdot r \cdot A \tag{1}$$

where Q_p is the peak discharge in m^3/sec, A is the drainage area in km^2, r is the average rainfall intensity during "the time of concentration" or t_c in mm/h and C is the runoff coefficient.

As far as the writer knows, the equation (1) is based on six assumptions as follows:

1) The relation of discharge to rainfall must depend on a linear response system.

2) The unit hydrograph (or the area-time diagram) is approximated by a rectangular distribution with respect to time.

3) The hyetograph normalized by total rainfall is also a rectangular distribution with time.

4) The actual peak discharge must be direct proportion to the calculated one. The parameter is named as "the runoff coefficient".

5) The runoff coefficient and the time of concentration must be invariable on the same conditions of rainstorm and of drainage.

6) The frequency of peak discharge is the same as that of the rainfall intensity.

In 1932 Sherman developed the unit hydrograph method to estimate runoff hydrograph from rainfall hyetograph. He assumed a drainage as a linear system, which is mathematically described by the convolution-integral form. The rational method can also explained by means of this type of mathematics. When the rectangular distributions are adopted as a unit hydrograph as well as a hyetograph and the operation of convolution integral is excuted, then the derived form is only a equilateral triangle or a equilateral trapezoid as a runoff hydrograph. Accordingly, we can reach a conclusion that "the maximum rate of flow or peak discharge must be produced by a rainfall which is maintained for a period equal to the time from beginning to and of unit hydrograph or the time of concentration." (Chow, 1964).

First three assumptions are needed the above explanations. The assumption 4) is necessary to make a calculated peak flow fit on a observed. If the assumption 5) is realized or if the runoff coefficient as well as the time of concentration are the same on a given drainage, we are able to obtain the value of peak discharge without hydrological data. In fact, many hydrologists desired to determine these values (for example, Kirpich, 1940; Yoshino and Yoneda, 1973). The assumption 6) may support strongly the planners or designers because they often want to determine a discharge on the probabilistic basis.

Now, although all of the above assumptions must be discussed, the assumptions 2) and 3) may be the most questionable because it must be very rare case that not only the shape of unit hydrograph but also of hyetograph are of rectangular. These do not seem to satisfy the theoretical generarity at all. It may be more reasonable assumption that the only nearby peaks of hyetograph, unit hydrograph as well as hydrograph must be resemble to normal distribution with respect to time. That is, the following equations will be assumed.

$$I(t) = \frac{1}{\sqrt{2\pi}\sigma_i} \cdot \exp\left\{-\frac{(t-m_i)^2}{2\sigma_i^2}\right\} \quad (2)$$

$$R(t) = \frac{R_A}{\sqrt{2\pi}\sigma_r} \cdot \exp\left\{-\frac{(t-m_r)^2}{2\sigma_r^2}\right\} \quad (3)$$

$$Q(t) = \frac{Q_A}{\sqrt{2\pi}\sigma_q} \cdot \exp\left\{-\frac{(t-m_q)^2}{2\sigma_q^2}\right\} \quad (4)$$

where I, R and Q mean the unit hydrograph, the hyetograph and the hydrograph respectively; t is time; m represents the time of peak and σ means the dispersion of each curves. R_A and Q_A are needed for normalization and may be equal to total rainfall and total runoff, respectively. Figure 1 shows schematically the relationship the above three equations. Under the assumption 1), the result of convolution-integral of equations (2) and (3) will be,

$$Q'(t) = \int_0^t I(\tau) \cdot R(t-\tau)\, d\tau = \frac{R_A}{\sqrt{2\pi}\sqrt{\sigma_r^2+\sigma_i^2}} \cdot \exp\left\{-\frac{(t-m_i-m_r)^2}{2(\sigma_i^2+\sigma_r^2)}\right\} \quad (5)$$

When $t = m_i + m_r$, then $Q'(t)$ becomes maximum. Therefore, the calculated peak discharge,

$$Q'_p = \frac{R_A}{\sqrt{2\pi}\sqrt{\sigma_i^2+\sigma_r^2}} \quad (6)$$

Taking the assumption 4), that is the relation of the actual peak, Q_p to the calculated peak, Q'_p is proportional, then

$$Q_p = C \cdot Q'_p = \frac{C \cdot R_A}{\sqrt{2\pi}\sqrt{\sigma_i^2+\sigma_r^2}} \quad (7)$$

where C is the runoff coefficient. Eq. (7) is just the desired relation. The total rainfall, R_A and the dispersion of hyetograph, σ_r will be obtained from the concerned hyetograph. At least several sets of storm data may be needed in order to determine the runoff coefficient, C and the dispersion of unit hydrograph, σ_i. In eq. (3), if t is equal to m_r, then R(t) is maximum, that is $R_p = R_A/(\sqrt{2\pi} \cdot \sigma_r)$. Accordingly, eq. (7) is rearranged as follows:

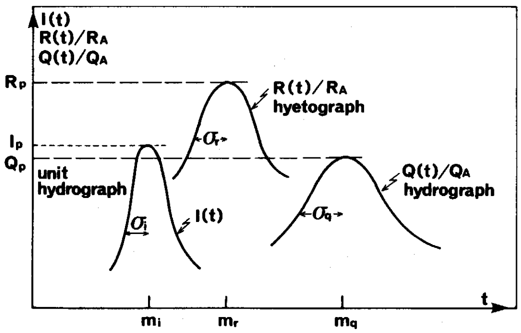

Fig. 1 Schematic diagrams of unit hydrograph, hyetograph and hydrograph

$$\left(\frac{R_A}{\sqrt{2\pi}\cdot R_p}\right)^2 = C^2\cdot\left(\frac{R_A}{\sqrt{2\pi}\cdot Q_p}\right)^2 - \sigma_i^2 \tag{8}$$

The runoff coefficient and the dispersion of unit hydrograph will be determined by means of eq. (8).

In eq. (7), if σ_i is negligible in comparison with σ_r, then

$$Q_p \doteqdot \frac{C\cdot R_A}{\sqrt{2\pi}\cdot\sigma_r} = \frac{C\cdot R_A}{\sqrt{2\pi}}\cdot\frac{\sqrt{2\pi}\cdot R_p}{R_A} = C_1\cdot R_p\ . \tag{9}$$

The eq. (9) means that the runoff peak is proportional to the rainfall peak. On the other hand, if σ_r is negligible and if σ_i seems to be the same on a given drainage, then

$$Q_p \doteqdot \frac{C\cdot R_A}{\sqrt{2\pi}\cdot\sigma_i} = C_2\cdot R_A. \tag{10}$$

This equation describes that the runoff peak is proportional to the total rainfall. In case of the practical application, it may be difficult to choose the more pertinent equation between the eqs. (9) and (10). But it is obvious from the derivation these relations that eq. (9) is valid for small drainage and that eq. (10) is for large. And also eq. (8) will be usefull to confirm the designer's perplexity.

If the assumption 6) is acceptable, in other words, if the frequency of peak discharge is the same as that of the maximum rainfall intensity or the total runoff and if the runoff coefficient C_1 or C_2 is unchanged, then the frequency density function of Q_p will be derived by the next equations.

$$f_{Q_p}(q_p) = \frac{1}{C_1}\cdot f_{R_p}\left(\frac{q_p}{C_1}\right)$$

or

$$f_{Q_p}(q_p) = \frac{1}{C_2}\cdot f_{R_p}\left(\frac{q_p}{C_2}\right) \tag{11}$$

where $f_{R_p}(r_p)$ and $f_{R_A}(r_A)$ are the probability density functions (PDF) of peak rainfall and of total rainfall, respectively.

However no one may accept that the runoff coefficient is the unchanged parameters because the runoff phenomenon is never explained by a simple system being involved only one parameter but so complicated.

Therefore it must be more reasonable that the runoff coefficient is also a random variable. Thus, the PDF of peak discharge will be derived by the form of

$$f_{Q_p}(q_p) = \int_{-\infty}^{\infty}\left|\frac{1}{r_p}\right|\cdot f_{C_1}\left(\frac{q_p}{r_p}\right)\cdot f_{R_p}(r_p)\,dr_p$$

or

$$f_{Q_p}(q_p) = \int_{-\infty}^{\infty}\left|\frac{1}{r_A}\right|\cdot f_{C_2}\left(\frac{q_p}{r_A}\right)\cdot f_{R_A}(r_A)\,dr_A \tag{12}$$

3. Application

In this section, the theoretical consideration described in the previous section will be inspected by means of actual storm data of small rivers in Tokyo.

Figure 2 shows the drainage shapes and the location of gauges. Table 1 shows the numerical summaries of three drainages. The values of design peak discharge were calculated in accordance with the design specifications prepared by the Ministry of Construction (1977). The runoff coefficient of these rivers was equal to 0.5, and the time of concentration was calculated by the Ruziha's formula. The form of intensity-duration curve is that $r = 5000/(40 + t)$, of which return period was five years. Figure 3 was prepared to show the relation of eq. (8). It may be easily understood that σ_i is negligible but that the square of C is varying from flood to flood. As shown in Figure 4, the distribution of the yearly maximum hourly rainfall in Tokyo is approximated to be the log-normal distribution. For convenience of calculation, the PDF of the runoff coefficient, C_1 is also able to make an approximation with the log-normal form. Thus the forms will be written as

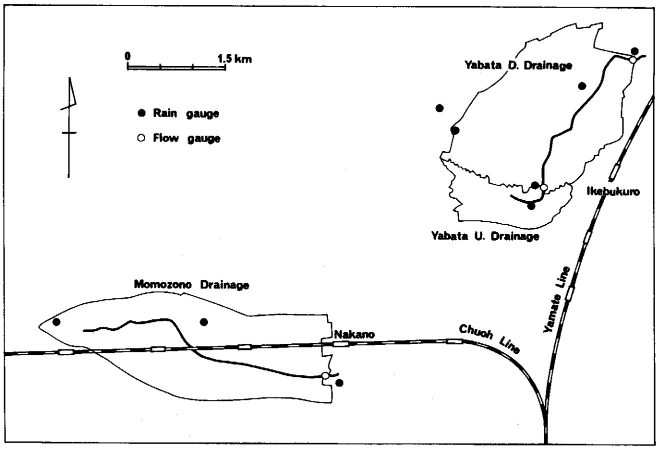

Fig. 2 Illustration of drainages

Table 1 Numerical summaries of the concerned drainage

Drainage		Momozono	Yabata D.	Yabata U.
Area	(km^2)	5.1	5.4	1.1
Channel Length	(km)	3.6	3.5	1.2
Average Slope	(%)	17	14	15
Impreviousty	(%)	49	53	40
Design Peak Discharge	(mm/h)	34	31	45

Table 2 Rainfall and runoff data of Momozono sewerage district, Tokyo

Flood event	(1) R_A	(2) Q_A	(3) R_p	(4) Q_p	$\left(\frac{R_A}{\sqrt{2\pi Q_p}}\right)^2$	$\left(\frac{R_A}{\sqrt{2\pi R_p}}\right)^2$	(5) $\frac{Q_p}{R_p}$
1973							
June 22	9	3	7	6.0	0.36	0.26	0.86
July 2	25	9	12	8.2	1.49	0.69	0.68
July 20	8	4	5	6.0	0.28	0.41	1.20
Aug. 1	21	7	6	7.4	1.29	1.96	1.23
Aug. 4	75	17	52	24.0	1.56	0.33	0.46
Sept. 5	34	17	10	6.4	4.52	1.84	0.64
Oct. 13	78	49	39	34.2	0.83	0.64	0.88
Oct. 21	36	16	8	6.4	5.06	3.24	0.80
Nov. 9	55	32	20	23.5	0.88	1.21	1.18
1974							
July 7	51	21	17	18.7	1.15	1.44	1.10
July 20	76	65	43	41.6	0.52	0.50	0.97
July 20	22	11	22	22.8	0.15	0.16	1.04
July 25	12	14	11	10.0	0.23	0.19	0.91
Aug. 1	32	10	18	9.8	1.71	0.51	0.54
Aug. 14	35	11	35	28.8	0.23	0.16	0.82
Aug. 25	39	14	10	6.3	6.13	2.43	0.63
Sept. 19	9	4	8	7.2	0.25	0.20	0.90
Nov. 17	27	11	8	7.3	2.19	1.82	0.91

(1) Total Rainfall (mm)
(2) Total Runoff (mm)
(3) Hourly Peak Rainfall (mm/h)
(4) Peak Runoff (mm/h)
(5) Mean of log $\left(\frac{Q_p}{R_p}\right) = -0.168$, Standard deviation of log $\left(\frac{Q_p}{R_p}\right) = 0.270$

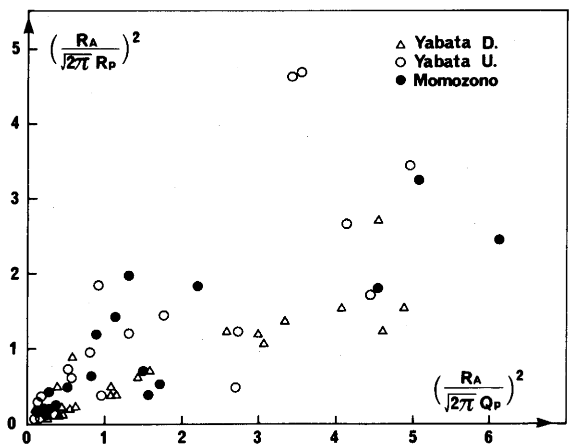

Fig. 3 Estimation of C and σ_i for Momozono, Yabata U. and Yabata D.

Table 3 Rainfall and runoff data of Yabata D sewerage district, Tokyo

Flood event	(1) R_A	(2) Q_A	(3) R_p	(4) Q_p	$(\frac{R_A}{\sqrt{2\pi}Q_p})^2$	$(\frac{R_A}{\sqrt{2\pi}R_p})^2$	(5) $\frac{Q_p}{R_p}$
1972							
July 11	143	79	20	18.5	9.56	8.18	0.93
July 14	125	63	20	11.3	19.58	6.25	0.57
Sept. 14	169	89	15	18.5	13.4	20.3	1.23
Dec. 8	25	12	9	6.2	2.60	1.23	0.69
Dec. 12	15	7	10	5.6	1.15	0.36	0.56
Dec. 23	91	48	11	5.9	38.1	10.9	0.54
1973							
Jan. 24	44	24	15	9.6	3.36	1.38	0.64
June 8	16	7	15	8.6	0.55	0.18	0.57
June 22	12	5	11	6.1	0.62	0.19	0.55
June 24	19	18	19	23.2	0.11	0.16	1.22
Aug. 4	18	10	7	4.1	3.08	1.06	0.59
Oct. 13	72	44	41	27.7	1.08	0.49	0.68
Oct. 21	34	21	11	6.7	4.12	1.53	0.61
Oct. 28	40	28	13	7.2	4.94	1.51	0.55
Nov. 9	78	48	34	38.3	0.66	0.84	1.13
1974							
Apr. 21	28	12	10	5.2	4.64	1.25	0.52
July 7	46	23	22	14.4	1.63	0.70	0.65
July 20	71	35	36	23.8	1.42	0.62	0.66
July 20	37	20	21	24.6	0.36	0.50	1.17
July 24	26	11	26	20.6	0.25	0.16	0.79
Aug. 1	59	26	38	22.7	1.08	0.39	0.60
Aug. 14	19	8	22	11.2	0.46	0.12	0.51
Aug. 31	107	49	26	19.9	4.63	2.71	0.77
Sept. 9	42	23	32	26.1	0.41	0.28	0.82
Sept. 9	20	10	20	13.0	0.38	0.16	0.65
Oct. 27	40	19	9	5.4	8.78	3.16	0.60
Nov. 17	27	12	10	6.2	3.03	1.71	0.62

(1) Total Rainfall (mm)
(2) Total Runoff (mm)
(3) Hourly Peak Rainfall (mm/h)
(4) Peak Runoff (mm/h)
(5) Mean of $\log(\frac{Q_p}{R_p}) = -0.368$, Standard deviation of $\log(\frac{Q_p}{R_p}) = 0.264$

and

$$\left.\begin{aligned} f_{R_p}(r_p) &= \frac{1}{\sqrt{2\pi}\,\sigma_{\ln r_p}\cdot r_p}\cdot\exp\left\{-\frac{(\ln r_p - m_{\ln r_p})^2}{2\sigma^2_{\ln r_p}}\right\} \\ f_{C_1}(C_1) &= \frac{1}{\sqrt{2\pi}\,\sigma_{\ln C_1}\cdot C_1}\cdot\exp\left\{-\frac{(\ln C_1 - m_{\ln C_1})^2}{2\sigma^2_{\ln C_1}}\right\} \end{aligned}\right\} \quad (13)$$

Under the condition of Eq. (12), the PDF of the peak discharge will be derived as follows:

Table 4 Rainfall and runoff data of Yabata U sewerage district, Tokyo

Flood event	(1) R_A	(2) Q_A	(3) R_p	(4) Q_p	$(\frac{R_A}{\sqrt{2\pi}\,Q_p})^2$	$(\frac{R_A}{\sqrt{2\pi}\,R_p})^2$	(5) $\frac{Q_p}{R_p}$
1972							
July 11	83	45	30	29	1.31	1.22	0.97
July 14	123	40	25	13	14.32	3.87	0.52
July 22	13	3	10	8	0.42	0.27	0.80
July 23	19	4	11	4	3.61	0.48	0.36
Dec. 8	25	6	9	6	2.78	1.23	0.67
1973							
Jan. 24	43	15	13	8.1	4.51	1.75	0.62
June 8	19	6	15	18.7	0.17	0.26	1.24
June 22	10	4	11	6.8	0.35	0.13	0.62
Aug. 1	17	4	5	7.1	0.92	1.85	1.42
Oct. 13	66	37	31	34.9	0.57	0.73	1.12
Oct. 21	24	9	8	7.2	1.78	1.44	0.90
Oct. 28	24	15	13	5.8	2.74	0.55	0.45
Nov. 9	78	33	32	34.9	0.80	0.95	1.09
1974							
July 20	65	34	33	34.3	0.57	0.62	1.03
July 21	37	16	26	36.5	0.16	0.32	1.40
July 24	25	12	25	32.7	0.09	0.16	1.31
Aug. 1	60	25	39	24.1	0.99	0.38	0.62
Aug. 14	24	11	24	22.4	0.18	0.16	0.93
Aug. 31	130	42	28	23.2	5.02	3.45	0.83
Sept. 4	27	7	5	5.8	3.47	4.67	1.16

(1) Total Rainfall (mm)
(2) Total Runoff (mm)
(3) Hourly Peak Rainfall (mm/h)
(4) Peak Runoff (mm/h)
(5) Mean of log $(\frac{Q_p}{R_p}) = -0.169$, Standard deviation of log $(\frac{Q_p}{R_p}) = 0.380$

$$f_{Q_p}(q_p) = \frac{1}{\sqrt{2\pi}\,\sigma_{\ln q_p} \cdot q_p} \cdot \exp -\frac{(q_p - m_{\ln q_p})^2}{2\sigma^2_{\ln q_p}} \tag{14}$$

where $m_{\ln q_p} = m_{\ln r_p} + m_{\ln C_1}$ and $\sigma_{\ln q_p} = \sqrt{\sigma^2_{\ln r_p} + \sigma^2_{\ln C_1}}$

The three lines showed in Figure 4 are the above derived PDF of each river. The values of discharge shown in Table 1 are marked on the lines of cumulative function by black circles in Figure 4. According to the exsisting rational method, the occurrence changes of those discharge may be explained as the same with each other. On the other hand, it must be understood from Figure 4 that the chance of those discharges is different with each river.

In any case, the latter method may be more reasonable than the former because the PDF of discharge is, though imperfectly, derived.

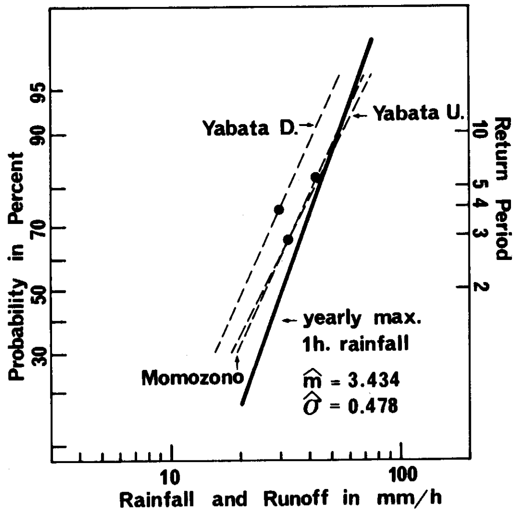

Fig. 4 Cumulated probability function of yearly maximum 1 hour rainfall and the corresponded runoff derived by eq. (14)

4. Problems in the Future

In order to estimate the runoff from the rainfall, lots of hydrological model have been developed during past about a hundred years and is still investigated by many hydrologists. And this tendency of investigation seems to be going to the direction of the construction the more complicated model by means of the highly specialized mathematics and of computer technics.

However this direction of the investigation may not worth for the actual application. The reasons are; 1) even if a model describing the phenomenon in detail would be developed, the estimation by the model must be usually accompanied by any probable errors because the model can never trace the nature itself but is just a model; 2) moreover, specialy in hydrology, the rainfall and runoff data utilized for determination of the numerical values of parameter have to have noticeable errors produced by gauging and conversing. Thus the determined values of parameters usually involve the obvious errors. Therefore the more any model has parameters, the larger the estimation error may be becoming. It may be gone so far as to say that the over complicated model loses its meaning for estimation.

This report was written under the recognition as mentioned above. In the future of runoff study, it may be more important to investigate the old but theoretically solid model like the rational method as well as the errors of estimation.

The author would like to dedicate this paper to Professor Takamasa Nakano in commemoration of his retirement from Tokyo Metropolitan University.

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(* in Japanese)