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On the blow-analytic equivalence of plane curve singularities

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Contents

Introduction

The aim of this paper is to provide a classification of real plane curve singularities.

In the study of singularities, a key role is played by the choice of a suitable equivalence relation. One possible approach comes from differential topology, and leads to a chain of increasingly stronger equivalence relations, starting from the topological equivalence and ending with the analytic equivalence. In this direction, the most notable result is the classification started by Arnold [1] of hypersurface singularities up to right equivalence (i.e., up to analytic coordinate transformation). Arnold introduced the concept of "modality", related to Riemann's idea of moduli and classified all singularities of modality less or equal to 2, providing explicit lists of their normal forms.

On the other hand, a classical problem in algebraic geometry is the study of algebraic varieties up to birational transformations. Hironaka proved [5] that, over a field of characteristic zero, every singular variety is birationally equivalent to a smooth variety. The idea, then, is to classify resolutions up to birational equivalence by finding a "simplest" variety in each birational equivalence class. For real algebraic surface singularities, the existence and uniqueness of minimal resolutions is shown, for example, in [9].

A third approach comes from interaction of the previous two methods. Blow-analytic equivalence was introduced by Kuo [11] in order to define a classification of real analytic singularities stronger than $\mathcal{C}^0$-equivalence and more flexible than $\mathcal{C}^1$-equivalence. Namely, we say that two real analytic plane curve germs are blow-analytically equivalent if they are homeomorphic and there exists an analytic isomorphism between a pair of respective resolutions. While analytic isomorphisms between germs rarely exist, blow-analytic homeomorphisms form a much wider class of functions, which can be employed for a less strict classification of singularities.

In the past decades many results have appeared on the classification of function germs up to blow-analytic equivalence, by Kuo [10] [11], Fukui [4], Paunescu [14], Koike and Parusiński [8], among the others. Yet, the classification of the zero sets of real singularities up to blow-analytic homeomorphism remained an open problem. The first work in this direction was published in 1998 by Kobayashi and Kuo, and contained the following statement:

Theorem ([6]). *All unibranched real plane curve germs are blow-analytically equivalent to a line.*

Shortly after, Kobayashi defined the blow-analytic invariant μ' and proved the following result for singular plane curve germs having two local analytic irreducible components:

Theorem ([7]). *Two bibranched germs of plane curves have isomorphic resolutions graph if and only if they have same μ', where μ' is a blow-analytic invariant which takes values in the set of natural numbers.*

As a direct consequence, we learned that the classification of bibranched plane curve germs is non-trivial: indeed, there are infinitely many equivalence classes, which can be neatly labelled by a discrete invariant μ'.

We investigate the general classification of embedded plane curve germs. Given two singularities one should, a priori, look for analytic isomorphisms between any of their respective embedded resolutions. Our method consists in translating blow-up and blow-down operations into graph-theoretic operations, and defining a minimal graph form (i.e., a standard form) up to blow-analytic homeomorphism. Namely, let $(C, 0)$ be an n-branched plane curve germ with an isolated singularity at the origin. A *standard form* is the dual graph of a good embedded resolution of $(C, 0)$ which is minimal under smooth contractions and up to the parity of some exceptional curves.

Moreover, we produce an algorithm to find a standard form given the dual graph of any good embedded resolution.

By studying the properties of standard forms, we prove our main result:

Main Theorem. *The number of blow-analytic equivalence classes of n-branched germs of plane curves with $\mu' = k$ is finite for any fixed natural numbers n and k.*

The blow-analytic classification of plane curves in the branched case is infinite, but the equivalence classes are partitioned in subsets of finite size by fixing the value of μ'.

Next, we look for an estimate of the number of equivalence classes as a function of μ'. By employing combinatorial techniques, we are able to prove the following explicit upper bound:

Proposition 3.1 *In the tribranched case, the number of graph standard forms with $\mu' = k$ is less than or equal to*

$$(k^3 - 2k^2 - k + 11)2^{k-2}.$$

Finally, we explicitly study the curve germs up to blow-analytic equivalence. As a first step, we generalize μ' by providing a family of blow-analytic invariants which include μ' as a particular case. Then, we produce explicit lists of graph standard forms and use the refined invariants to prove that each one represents a different blow-analytic equivalence class.

Theorems 4.1, 4.2 *Up to blow-analytic homeomorphism, there are exactly* 2 *and* 4 *tribranched plane curve singularities with* $\mu' = 0$ *and* 1 *respectively.*

Theorem 4.4 *Up to blow-analytic homeomorphism, there are exactly* 8 *blow-analytically distinct four-branched plane curve germs with* $\mu' = 0$.

Chapter 1

Preliminaries and previous results

We define blow-analytic equivalence in the context of the classification of singularities, and outline our approach to the subject. In Section 1.2 we recall a few facts in algebraic topology, and use them to prove an important result on intersection numbers. Section 1.3 reports the results on the classification of unibranched and bibranched plane curve singularities, which have been known previous to our study.

1.1 Blow-analytic equivalence

Definition. Let $f : (\mathbb{R}^2, 0) \to \mathbb{R}$, $g : (\mathbb{R}^2, 0) \to \mathbb{R}$ be two analytic function germs. We say that f and g are *$\mathcal{C}^r$-equivalent* with $r = 0, 1, \ldots, \infty, \omega$ if there exists a $\mathcal{C}^r$ isomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f = g \circ h$.

$$\begin{array}{ccc} (\mathbb{R}^2, 0) & \xrightarrow{h} & (\mathbb{R}^2, 0) \\ \downarrow f & & \downarrow g \\ \mathbb{R} & \xrightarrow{\text{id}} & \mathbb{R} \end{array}$$

When $r = 0$ we also say that f and g are *topologically equivalent*, and when $r = \omega$ we say that f and g are *analytically equivalent*.

By definition we have the following chain of implications:

$$\mathcal{C}^0\text{-equiv.} \Leftarrow \mathcal{C}^1\text{-equiv.} \Leftarrow \ \ldots \ \Leftarrow \mathcal{C}^\infty\text{-equiv.} \Leftrightarrow \mathcal{C}^\omega \text{ equiv.},$$

where the converse directions do not hold, except for the last one, which follows from Artin's Approximation Theorem [2].

Example 1.1 (Whitney's family [18])**.** Consider the family of function germs in $\mathbb{R}^2$ given by the equation:

$$f_t(x, y) = y(y - x)(y - 2x)(y - tx), \qquad 2 < t < \infty.$$

For any fixed value of t, $f_t(x, y)$ represents the intersection of 4 distinct lines.

Any two functions $f_{t_1}(x, y)$ and $f_{t_2}(x, y)$ in the family are topologically equivalent, however Whitney showed that no $\mathcal{C}^1$-isomorphism exists between them if $t_1 \neq t_2$.

One can then ask if it possible to find an equivalence relation which "kills" continuous moduli of curves such as the Whitney's family by putting them into the same equivalence class. This was the spirit which led Kuo and others to explore a different route of advance from the chain of differential relations described above.

Let us recall the definition of point blow-up, which plays a major role in our investigations. An extensive treatment with proofs can be found, for example, in [15].

Definition. The *blow-up of* $\mathbb{R}^2$ *at the origin* is the space

$$X = \{((x, y), [u, v]) \in \mathbb{R}^2 \times \mathbb{RP}^1 \,|\, xv - yu = 0\}$$

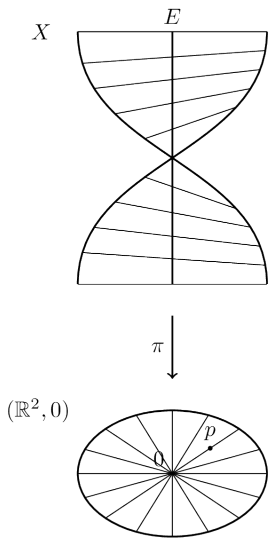
together with the natural projection on the first factor

$$\pi : X \to (\mathbb{R}^2, 0).$$

We say that π is a *blow-up* of the plane at the origin or, alternatively, a *contraction* or *blow-down* of X.

Let X be a real smooth algebraic surface and x a point in X. We define the *blow-up of* X *at* x by choosing suitable local coordinates in a neighbourhood of x and reducing to the case above.

From a set-theoretic point of view, the blow-up operation consists in associating to a point p in $\mathbb{R}^2$ the pair (p, l), where l is the line passing through the origin and p. This construction is a bijection (it is easy to check that π is in fact a diffeomorphism) for any p outside of the origin. When $p = 0$, we have $E = \pi^{-1}(0) \cong \mathbb{RP}^1$, which we call the *exceptional divisor* of the blow-up. A tubular neighbourhood of E in X is the non-trivial line bundle over $\mathbb{RP}^1$, which is diffeomorphic to a Möbius strip.



By allowing blow-up operations, we can define blow-analytic equivalence.

Definition. A function germ $f : (\mathbb{R}^2, 0) \to \mathbb{R}$ is *blow-analytic* if there exists a composition of blow-ups $\beta = \beta_1 \circ \cdots \circ \beta_m : X \to (\mathbb{R}^2, 0)$ such that the composition $f \circ \beta$ is analytic.

A homeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is *blow-analytic* if h and its inverse h^{-1} both have blow-analytic components.

Definition. Let $f : (\mathbb{R}^2, 0) \to \mathbb{R}$, $g : (\mathbb{R}^2, 0) \to \mathbb{R}$ be two analytic function germs. We say that f and g are *blow-analytically equivalent* if there exists a blow-analytic homeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with blow-analytic inverse such that $f = g \circ h$.

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \big\downarrow \beta & & \big\downarrow \beta' \\ (\mathbb{R}^2, 0) & \xrightarrow{h} & (\mathbb{R}^2, 0) \\ \big\downarrow f & & \big\downarrow g \\ \mathbb{R} & \xrightarrow{\text{id}} & \mathbb{R} \end{array}$$

It follows from the definition that two analytically equivalent curves are also blow-analytically equivalent. On the other hand, there are examples of blow-analytically equivalent function germs that are not $\mathcal{C}^1$-equivalent or bi-Lipschitz equivalent [8]. In the previous chain of differential relations, blow-analytic equivalence develops on a new branch:

$$\begin{array}{l} \mathcal{C}^0\text{-equiv.} \Leftarrow \mathcal{C}^1\text{-equiv.} \Leftarrow \ \dots \ \Leftarrow \mathcal{C}^\infty\text{-equiv.} \Leftrightarrow \mathcal{C}^\omega \text{ equiv.}, \\ \qquad\qquad\qquad \Leftarrow \text{blow-analytic equiv.} \end{array}$$

The blow-analytic equivalence relation solves the problem of the existence of continuous moduli: in fact, Kuo proved [10] that this phenomenon does not appear in the blow-analytic setting. All curves in the Whitney family, in particular, belong to the same equivalence class.

Several papers have appeared on the classification of function germs up to blow-analytic homeomorphism. For a survey of results in dimension two we refer to [8]. The same problems can be formulated in higher dimension, though the situation is more delicate and slightly different versions of the definition of blow-analytic equivalence have been proposed (see for example [4], [14]).

Our approach differs from these works in the sense that we choose to focus on the classification of the zero sets of analytic function germs instead of the function germs themselves. Namely, our goal is to investigate *germs of embedded real plane curves* up to blow-analytic homeomorphism.

Definition. Let $(C,0)$, $(D,0)$ be two curve germs in $\mathbb{R}^2$ with an isolated singularity at the origin. We say that $(C,0)$ and $(D,0)$ are *blow-analytically equivalent* if there exist a blow-analytic homeomorphism $h : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ that carries $(C,0)$ to $(D,0)$.

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \big\downarrow \beta & & \big\downarrow \beta' \\ (\mathbb{R}^2,0) & \xrightarrow{\ h\ } & (\mathbb{R}^2,0) \\ \big\uparrow & & \big\uparrow \\ (C,0) & \xrightarrow{\ h|_C\ } & (D,0) \end{array}$$

Let $\pi : X \to \mathbb{R}^2$ be a blow-up of $\mathbb{R}^2$ at the origin, and let $(C,0)$ be the germ a plane curve with an isolated singularity at the origin. The closure $\tilde{C} = \overline{\pi^{-1}(C \setminus \{0\})}$ is the *strict transform* of C. If C is locally defined by $f = 0$, the *total transform* $\pi^* C$ of C is defined by $f \circ \pi = 0$ in X.

Definition. Let $(C,0)$ be the germ of a curve in $\mathbb{R}^2$ with an isolated singularity at the origin. We say that $\pi : \tilde{X} \to \mathbb{R}^2$ is an *embedded resolution* of $(C,0)$ if $\tilde{X}$ is a smooth real algebraic surface and π is the composition of a finite sequence of blow-ups at a point such that the strict transform $\tilde{C}$ of C is smooth in $\tilde{X}$.

We call $\pi : \tilde{X} \to \mathbb{R}^2$ a *good embedded resolution* if the total transform of C is simple normal crossing, i.e., each irreducible component of $\pi^{-1}(C)$ is smooth and the support of the total transform has at most nodal singularities.

Hironaka's Main Theorem II [5] states that it is possible to find embedded good resolutions of varieties over fields of characteristic zero by repeatedly blowing up along non-singular subvarieties a finite number of times.

Concretely, given two plane curve germs $(C, 0)$, $(D, 0)$ with an isolated singularity at the origin, we are going to consider pairs of good embedded resolutions $\tilde{X}$, $\tilde{X}'$, and say that $(C, 0)$ and $(D, 0)$ are blow-analytically equivalent if we can find an analytic isomorphism between $\tilde{X}$ and $\tilde{X}'$.

1.2 A digression on intersection numbers

Through this section we will assume X to be a real algebraic surface, orientable or non-orientable. We consider the homology and cohomology groups of X with coefficients in $\mathbb{Z}/2\mathbb{Z}$. For convenience, we will omit the coefficients domain, and write $H_*(X)$, $H^*(X)$ and $H_c^*(X)$ for the homology, cohomology and cohomology with compact support of X, respectively.

Theorem 1.2 (Poincaré duality). *Let X be a smooth real algebraic surface, then $H_c^1(X)$ is isomorphic to $H_1(X)$ under the correspondence*

$$a \mapsto a \frown [X],$$

where $\frown$ is the cap product and $[X]$ denotes the fundamental homology class of X with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

This theorem implies, in particular, that there is a non-degenerate bilinear pairing

$$H_1(X) \times H_1(X) \longrightarrow H_0(X),$$

which we define by applying Poincaré duality, taking the cup product, and then applying Poincaré duality again:

$$(a, b) \longmapsto a \cdot b := (a^* \smile b^*) \frown [X],$$

where a^*, b^* are the Poincaré duals in cohomology of a, b respectively.

We call $a \cdot b$ the $\mathbb{Z}/2\mathbb{Z}$-*valued intersection number* of a, b. When $a = b$, $a \cdot a = a^2$ is the $\mathbb{Z}/2\mathbb{Z}$-*valued self-intersection number* of a.

Next, let us recall the definition of the Stiefel-Whitney classes of a vector bundle. The existence and uniqueness is of these classes is proven in [12].

Definition. Let $E \to M$ be a vector bundle over a smooth manifold M. There exists a sequence of cohomology classes

$$w_i(E) \in H^i(M), \quad i = 0, 1, 2, \ldots$$

associated to E, called *Stiefel-Whitney classes* of the vector bundle, which are characterised by the following properties:

(i) $w_0(E) = 1 \in H^0(M)$ and $w_i(E) = 0$ for any i greater than the rank of E.

(ii) (Naturality) If $f : M' \to M$ is a vector bundle map, then

$$w_i(f^*(E)) = f^*(w_i(E)),$$

where $f^*(E)$ is the pull-back vector bundle of E under f.

(iii) (Whitney product theorem) If E_1, E_2 are vector bundles over M, then

$$w_k(E_1 \oplus E_2) = \sum_{i=0}^{k} w_i(E_1) \smile w_{k-i}(E_2).$$

(iv) The class $w_1(\gamma_1)$ of the non-trivial line bundle γ_1 over $\mathbb{RP}^1 \cong S^1$ is non-zero.

Example 1.3 ([12])**.** The set of line bundles over the same smooth manifold M is denoted $\mathrm{Vect}_1(M)$ and has a group structure with the operation of tensor product. The first Stiefel-Whitney class gives an isomorphism between $\mathrm{Vect}_1(M)$ and $H^1(M)$ as groups, in fact:

$$w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2),$$

for all line bundles L_1, L_2 over M.

When $M = \mathbb{RP}^1 \cong S^1$, we have $H^1(S^1) = \mathbb{Z}/2\mathbb{Z}$. It follows that there are only two real line bundles over the circle up to $\mathcal{C}^\infty$-bundle isomorphism: the trivial one ($S^1 \times \mathbb{R}$) and the open Möbius strip (γ_1).

To any Cartier divisor D (i.e., a codimension one cycle defined locally by one algebraic equation) in a real smooth algebraic surface X corresponds a line bundle $\mathcal{L}(D)$ in X. Two Cartier divisors D_1, D_2 have isomorphic associated line bundles $\mathcal{L}(D_1) = \mathcal{L}(D_2)$ if and only if D_1, D_2 belong to the same linear equivalence class.

Conversely, given a line bundle $L \to M$ in X, the zero set of a smooth section is locally represented by $f = 0$ which is a divisor D in X, and the first Stiefel-Whitney class $w_1(L)$ can be identified with the Poincaré dual of the homology class $[D]$ in $H_1(X)$. This translates in terms of intersection numbers as follows:

$$D_1 \cdot D_2 = w_1(\mathcal{L}(D_1)) \cdot w_1(\mathcal{L}(D_2)), \qquad D^2 = w_1(\mathcal{L}(D)).$$

In particular, the $\mathbb{Z}/2\mathbb{Z}$-valued self-intersection of the exceptional divisor E in the blow-up of a real smooth algebraic surface at a point is $E^2 = w_1(\gamma_1) = 1$.

Let $p : X \to Y$ be an algebraic map between real smooth algebraic surfaces, and let D be a divisor in Y locally defined by $f = 0$. Then the total transform p^*D is locally defined by $f \circ p = 0$, thus it is the zero set of a section of the pull-back bundle $p^*\mathcal{L}(D)$, and the irreducible decomposition of $p^*\mathcal{L}(D)$ corresponds to the tensor product of line bundles.

In particular, let $\beta : X \to Y$ be the blow-up of a real smooth surface Y at a point p with exceptional divisor E. Let D be a divisor in Y passing through p, then the pull-back bundle $\beta^*\mathcal{L}(D)$ is isomorphic to $\mathcal{L}(\tilde{D}) \otimes \gamma_1$, where $\tilde{D}$ is the strict transform of D. We have:

$$\begin{aligned}(\beta^* D)^2 &= w_1(\beta^*\mathcal{L}(D))^2 = w_1(\mathcal{L}(\tilde{D}) \otimes \gamma_1) \cdot w_1(\mathcal{L}(\tilde{D}) \otimes \gamma_1) \\ &= w_1(\mathcal{L}(\tilde{D}))^2 + 1 = \tilde{D}^2 + 1.\end{aligned}$$

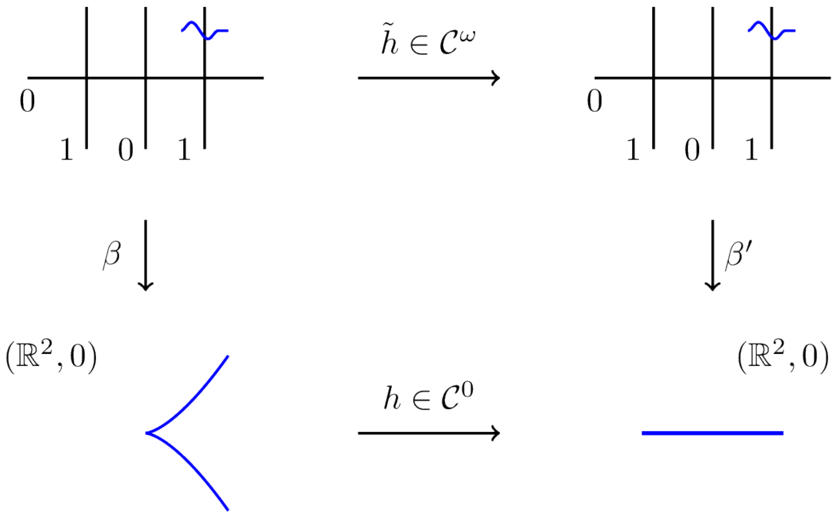
Therefore, when blowing up a point of a smooth curve, the $\mathbb{Z}/2\mathbb{Z}$-valued self-intersection number of this curve changes parity. In the context of blow-analytic equivalence, this allows us to compute the $\mathbb{Z}/2\mathbb{Z}$-valued self-intersection numbers of the exceptional divisors after successive blow-ups and blow-downs.

1.3 Unibranched and bibranched singularities

In order to explain the uniqueness of the blow-analytic classification of embedded curve germs, let us start with an example.

Example 1.4 (Kobayashi-Kuo example, [6])**.** A cusp $(\{y^2 - x^3 = 0\}, 0)$ and a line $(\{y = 0\}, 0)$ are blow-analytically equivalent.

An explicit blow-analytic homeomorphism h can be found by blowing up both the embedded cusp and the embedded line four times as shown in the diagram below:



where the number near each divisor is the corresponding $\mathbb{Z}/2\mathbb{Z}$-valued self-intersection number.

Observe that a cusp and a line are $\mathcal{C}^0$-equivalent, but not $\mathcal{C}^r$-equivalent for any $r \geq 1$.

The existence of a real analytic isomorphism between diffeomorphic resolution spaces as in the example above, is guaranteed by Nash's theorem:

Theorem 1.5 (Nash, [13])**.** *If two real analytic compact algebraic manifolds are diffeomorphic, then they are also equivalent as algebraic manifolds.*

In the unibranched case the classification up to blow-analytic homeomorphism of plane curve singularities has been completely studied by Kobayashi and Kuo, who proved the following result in 1998:

Theorem 1.6 (Kobayashi-Kuo, [6])**.** *All unibranched germs of plane curves are blow-analytically equivalent to a line.*

The first examples of non-equivalent embedded curve germs come from the bibranched case. In this setting Kobayashi proved the following:

Theorem 1.7 (Kobayashi, [7])**.** *Bibranched germs of plane curves have isomorphic resolution graphs if and only if they have the same μ'.*

A definition of μ' will be introduced in later chapters. For now it suffices to know that μ' is a blow-analytic invariant which takes values in $\mathbb{N}$.

Chapter 2

General n-branched case

In this chapter we study plane curve germs with n local analytical irreducible components up to blow-analytic homeomorphism. Firstly, we recall the construction of the dual graph associated to an embedded resolution, which will be our main tool in the treatment of singularities. We then develop a family of graph invariants under the blow-up and blow-down operations. In the second section, we introduce the concept of graph standard form associated to a resolution of singularity. Finally, we use the existence and the properties of standard forms to prove our main result regarding the "local" finiteness of the blow-analytic classification.

2.1 Blow-analytic invariants

Let X be a surface which is a tubular neighbourhood of the union of compact smooth curves $\{E_j\}_{j=1}^{m}$ intersecting transversally. We construct the weighted dual graph Γ associated to X by drawing a vertex v_i for each central curve E_i, and connecting two vertices by an edge if and only if the corresponding curves intersect. To each vertex we assign as weight the $\mathbb{Z}/2\mathbb{Z}$-valued self-intersection number of the corresponding curve. In figures of the graphs Γ, we represent odd curves as white vertices and even curves as black vertices.

Definition. We say that X is *smoothly contractible* if it is a surface obtained from $(\mathbb{R}^2, 0)$ by a finite sequence of blow-ups and blow-downs.

It follows easily from the definition that if X is smoothly contractible, then its dual graph Γ is a tree.

Let $A = (a_{ij})$ be the $\mathbb{Z}/2\mathbb{Z}$-valued intersection matrix associated to Γ, i.e., the matrix whose entries are the $\mathbb{Z}/2\mathbb{Z}$-valued intersection numbers $a_{ij} = E_i \cdot E_j$. It has been proven in [6] that X is smoothly contractible if and only if the determinant of A is 1. Since the information about the intersection matrix is captured by Γ, we also say that Γ is *smoothly contractible* if and only if the determinant of A is 1.

In the unibranched case, this the invariant which has been used to completely classify plane curve singularities up to blow-analytic homeomorphism.

Now assume that $(C, 0)$ has more than one branches and set $C = \bigcup_{i=1}^{n} C_i$ its irreducible decomposition. Let X be a good resolution of $\mathbb{R}^2$ at the origin, i.e., an embedded resolution which is a composition of successive blow-ups and blow-downs such that the support of the total transform of C is simple normal crossing. In order to keep track of the respective positions of each component of the strict transform, we define Γ^* as the extension of Γ obtained by adding a vertex for each component of the strict transform and an edge where a non-compact component (i.e., a component of the strict transform) intersects an exceptional curve. It follows from the goodness of the resolution that Γ^* is a tree.

Blow-analytic equivalence of curve germs determines an equivalence relation for triplets $(X, \cup_{i=1}^{n} \tilde{C}_i, \cup_j E_j)$ (where $\tilde{C}_i$ is the strict transform of C_i), which induces an equivalence relation for trees Γ and Γ^*. In case of ambiguity, we specify which curve germ corresponds to the dual graphs by writing $\Gamma(C)$ and $\Gamma^*(C)$.

Let $C = \bigcup_{i=1}^{n} C_i$ and $C' = \bigcup_{i=1}^{n} C'_i$ be two blow-analytically equivalent plane curve germs, then the blow-analytic homeomorphism $h : (\mathbb{R}^2, C, 0) \to (\mathbb{R}^2, C', 0)$ induces a bijection $\bar{h} : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $h(C_i) = C'_{\bar{h}(i)}$.

Let $(X, \cup_{i=1}^{n} \tilde{C}_i, \cup_j E_j)$ and $(X', \cup_{i=1}^{n} \tilde{C}'_i, \cup_{j'} E'_{j'})$ be good embedded resolutions of $(C, 0)$ and $(C', 0)$ respectively, and let $(\tilde{X}, \cup_{i=1}^{n} \tilde{\tilde{C}}_i, \cup_{\tilde{j}} \tilde{E}_{\tilde{j}})$ be a common good resolution which dominates $(X, \cup_{i=1}^{n} \tilde{C}_i, \cup_j E_j)$ and $(X', \cup_{i=1}^{n} \tilde{C}'_i, \cup_{j'} E'_{j'})$. Consider a path γ_{ij} in the exceptional set of X connecting the strict transforms of C_i and C_j, with $i \neq j$. We restrict ourselves to minimal paths, i.e., those γ_{ij} which, amongst all paths connecting the strict transforms of C_i and C_j, go through the minimum number of exceptional curves. Each path γ_{ij} has a lift in $\tilde{X}$, and thus an image $\gamma'_{\bar{h}(i)\bar{h}(j)}$ in X'.

In the dual graph, γ_{ij} (resp. $\gamma'_{\bar{h}(i)\bar{h}(j)}$) determines a path γ^*_{ij} (resp. $(\gamma'_{\bar{h}(i)\bar{h}(j)})^*$) in $\Gamma^*(C)$ (resp. $\Gamma^*(C')$) between the vertices corresponding to the strict transforms of C_i and C_j (resp. $C'_{\bar{h}(i)}$ and $C'_{\bar{h}(j)}$). Since Γ^* is a tree, for fixed i, j there is a unique path γ^*_{ij} in the dual graph corresponding to all minimal paths γ_{ij} in the resolution.

Lemma 2.1. *Let $\Gamma_{ij}(C)$ (resp. $\Gamma^*_{ij}(C)$) be the graph obtained by removing all vertices in γ^*_{ij} and the connecting edges from $\Gamma(C)$ (resp. $\Gamma^*(C)$), and let $\Delta_{ij}(C)$ be the set of connected components G in $\Gamma_{ij}(C)$ such that $\mu(G) \neq 0$, where $\mu(G)$ is the corank of the $\mathbb{Z}/2\mathbb{Z}$-valued intersection matrix associated to G. Let $\Delta^*_{ij}(C)$ denote the natural extension of $\Delta_{ij}(C)$ in $\Gamma^*(C)$.*

Then, a blow-analytic homeomorphism $h : (\mathbb{R}^2, C, 0) \to (\mathbb{R}^2, C', 0)$ induces a bijection $\hat{h} : \Delta^*_{ij}(C) \to \Delta^*_{\bar{h}(i)\bar{h}(j)}(C')$. In particular, $\mu(\Gamma_{ij}(C)) = \mu(\Gamma_{\bar{h}(i)\bar{h}(j)}(C'))$.

Proof. As before, take a good resolution $(\tilde{X}, \cup_{i=1}^n \tilde{\tilde{C}}_i, \cup_{\tilde{j}} \tilde{E}_{\tilde{j}})$ dominating $(X, \cup_{i=1}^n \tilde{C}_i, \cup_j E_j)$ and $(X', \cup_{i=1}^n \tilde{C}'_i, \cup_{j'} E'_{j'})$, and let β_k be a step in the sequence of blow-ups from X to $\tilde{X}$.

For any k, if the centre of β_k is a point in γ_{ij}, then the exceptional curve E_k intersects the lift of γ_{ij}, therefore it does not contribute to $\Gamma^*_{ij}(C)$.

If the centre of β_k is not in γ_{ij} but on a curve belonging to γ^*_{ij}, after the blow-up an isolated odd vertex is added to $\Gamma^*_{ij}(C)$, thus creating a new connected component which is smoothly contractible and does not contribute to $\Delta_{ij}(C)$.

Finally, if the centre of β_k is not on any curve in γ^*_{ij}, the vertex corresponding to the exceptional curve E_k extends one of the connected components G of $\Gamma_{ij}(C)$. Let us call G' the extended component. Slightly abusing the notation, let $A' = (E'_p \cdot E'_q)$ be the $\mathbb{Z}/2\mathbb{Z}$-valued intersection matrix associated to G'. By a change of basis, $A' \approx_{\mathbb{Z}} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$, where A is the $\mathbb{Z}/2\mathbb{Z}$-valued intersection matrix associated to G. Clearly, $\mu(G') = \mu(G)$, so β_k preserves the corank of the connected components of $\Gamma_{ij}(C)$.

Since this holds for each step β_k in the blow-up sequence, there exists a bijection between the elements of $\Delta^*_{ij}(C)$ and the elements of its lift in $\tilde{X}$. Furthermore, since $\gamma'_{\bar{h}(i)\bar{h}(j)}$ is the image of γ_{ij} in X', we have a bijection $\hat{h} : \Delta^*_{ij}(C) \to \Delta^*_{\bar{h}(i)\bar{h}(j)}(C')$. □

Let $I = \{I_k : k = 1, \dots, p\}$ denote a partition of $\{1, 2, \dots, n\}$ (i.e., $I_1 \cup \cdots \cup I_p = \{1, \dots, n\}$ and $I_k \cap I_l = \emptyset$ if $1 \le k < l \le p$). By considering the union of minimal paths γ_{ij} between two components $\tilde{C}_i$ and $\tilde{C}_j$ of the strict transform with $i, j \in I_k$ and $i \neq j$, the proof of the above lemma can be generalised to a partition on the set of strict transform components. Namely, we have that a blow-analytic homeomorphism $h : (\mathbb{R}^2, C, 0) \to (\mathbb{R}^2, C', 0)$ induces a bijection $\hat{h} : \Delta_I(C) \to \Delta_{\bar{h}(I)}(C')$ (where $\bar{h}(I) = \{\bar{h}(I_k) : k = 1, \dots, p\}$) and, in particular, the corank $\mu(\Gamma_I)$

is a blow-analytic invariant.

When $I = \{\{1\}, \{2\}, \ldots, \{n\}\}$, μ_I generalises the invariant μ defined in [7], although the author uses a different method to prove its invariance.

When $I = \{1, 2, \ldots, n\}$, we write μ_I as μ' for convenience. As we shall see, the value of μ' bounds from below the least number of components in the exceptional divisor of a good resolution of any curve germ in that equivalence class.

In the case of bibranched singularities, μ' has been used by Kobayashi to classify their resolution graphs up to blow-analytic homeomorphism (Theorem 1.7).

2.2 Standard forms

We approach the problem of the classification of embedded plane curve singularities by providing a classification of the dual graphs of their resolutions. Namely, given a smoothly contractible graph Γ, we perform blow-ups and blow-downs to simplify Γ and reduce it to a standard form, without changing the blow-analytic equivalence class of the corresponding embedded curve germ $(C, 0)$. This method allows us to make easy combinatorial computations and graphic representations.

Two blow-analytically equivalent germs have by definition a pair of isomorphic dual graphs. It should be noted that the converse is not true: in fact, we can explicitly construct examples of non-equivalent singularities with isomorphic dual graphs, as shown in the last section of this paper. However, to any dual graph correspond only a finite number of blow-analytically distinct embedded plane curve germs.

In what follows, we denote Q an even vertex with valency 1 in Γ^*, where the *valency* of a vertex is the number of edges incident to it. We call a vertex *extremal* if it has valency 1 in Γ, and we call *special vertex* a vertex with valency 3 or more in Γ^*. We remark that a configuration is not smoothly contractible if Γ contains two vertices of type Q attached to the same vertex. In fact, if the graph contains such a part, then the determinant of the $\mathbb{Z}/2\mathbb{Z}$-valued intersection matrix associated to Γ vanishes [6].

Let X be a good resolution and Γ^* its extended dual graph. The operations listed below are a composition of blow-ups and blow-downs of X, expressed for simplicity in the graph language.

C1 (*Contraction* 1): contract an odd vertex with valency 1 in Γ^*;

- C2 (*Contraction* 2): contract an odd vertex with valency 2 in Γ^*;
- C3 (*Contraction* 3): remove two adjacent even vertices, each having valency at most 2 in Γ^*, by first blowing up at the intersection of the two exceptional curves and then performing C2 three times (contracting the newly created exceptional curve last);
- M1 (*Modification* 1): if a vertex of type Q is attached to an odd vertex, change the parity of the latter as shown in [7]; namely, perform a blow-up at the point where the even curve in Q intersects the odd curve and then contract the extremal odd vertex.

Given a graph Γ^* as above, perform contractions C1, C2 and C3 repeatedly, until no more contractions can be made. Since the size of the graph is finite and each contraction decreases the number of vertices in Γ, after a finite number of steps Γ will be minimal under C1, C2 and C3. Next, apply M1 wherever it is possible. If $n = 0$ (i.e., the embedded curve germ is an isolated point), the minimal graph under the above operations is an odd vertex with valency 0, which we further contract, obtaining the empty graph.

The resulting surface X is blow-analytically equivalent to the original one, and its dual graph is reduced to a *standard form* of Γ.

Proposition 2.2. *A standard form of Γ satisfies the following properties:*

- P1 *All non-special vertices are even;*
- P2 *All special vertices adjacent a vertex of type Q are even;*
- P3 *The segment between two special vertices is at most one even vertex;*
- P4 *There are exactly μ' vertices of type Q.*

Proof. P1 follows from the fact that any odd non-special vertex has been contracted by C1 or C2. P2 is a consequence of M1. P3 follows from P1 and by C3. To prove P4, set $I = \{1, 2, \ldots, n\}$ and consider the extremal vertices of each connected component in Γ_I. By P1, they are all even. Moreover, by C3 they can only be part of a path of length 1 and two of them cannot be connected to the same vertex, since Γ is smoothly contractible. Thus Γ_I contains only even vertices, all disconnected, and exactly μ' of them, since each contributes to the corank by 1. □

An arbitrary Γ can always be reduced as shown above. Therefore, we shall restrict our attention to the easier task of classifying standard configurations.

Proposition 2.3. *For an n-branched embedded plane curve germ, a standard form of Γ has at most $\mu' + n$ extremal vertices.*

Proof. Since we assume Γ to be a standard form, its extremal vertices must be either even vertices (corresponding to vertices of type Q) or vertices adjacent to at least a non-compact component. There are exactly μ' vertices of the first kind and at most n of the second kind, thus there are at most $\mu' + n$ extremal vertices. □

Remark. The number of extremities could be strictly less than $\mu' + n$. In fact, more than one non-compact components could be adjacent to the same extremal vertex, or it could also happen that some non-compact components are attached to non-extremal vertices.

We can gain additional information about standard forms by looking at the weights of extremal vertices and at the vertices to which they are connected.

Consider the case where an extremal vertex is adjacent to a non-compact component. If the extremal vertex is odd, then it must be adjacent to at least two non-compact components (otherwise it has valency 2 in Γ^* and can be smoothly contracted).

Now, assume that the extremal vertex is even and adjacent to exactly one non-compact component. Then the preceding vertex v must be a special vertex (if it were a even vertex with valency 2, Γ could be further reduced by C3 without losing normal crossingness) and, to avoid configurations which are not smoothly contractible, there cannot be a vertex of type Q attached to v. Thus v must be either adjacent to a non-compact component or have valency at least 3 in Γ.

The above considerations prove the following lemma.

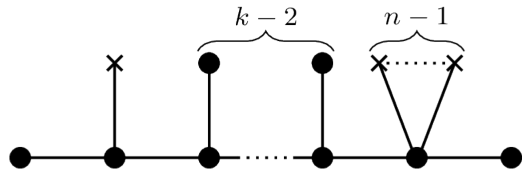
Lemma 2.4. *There are only four kinds of extremal vertices in a standard form:*

- *vertices of type Q;*
- *vertices adjacent to at least two non-compact components;*
- *even vertices adjacent to exactly one non-compact component and preceded by another vertex adjacent to a non-compact component;*
- *even vertices adjacent to exactly one non-compact component and preceded by a vertex with valency at least 3 in Γ.*

2.3 Main result

As we have seen in Section 1.3, the classification of unibranched plane curve germs is trivial: there is only one equivalence class. The classification is also obviously trivial when we consider an embedded plane curve with no branches (i.e., a point in $\mathbb{R}^2$). However, things change for embedded curves with two or more branches: using the standard forms and the invariants introduced in this chapter, we can in fact show that the number of blow-analytic equivalence classes is infinite for any fixed number of branches $n \geq 2$.

Example 2.5. For any $n \in \mathbb{N}$, $n \geq 2$ and for any $k \in \mathbb{N}$, $k \geq 2$, consider the following family of resolution graphs:



$\Gamma^{(k)}$

Each $\Gamma^{(k)}$ is smoothly contractible and it is a standard form.

Since we have $\mu'(\Gamma^{(k)}) = k$ for any k in $\mathbb{N}$, there exists no blow-analytic homeomorphism between $\Gamma^{(k)}$ and $\Gamma^{(l)}$ with $k \neq l$. Thus, we have infinitely many blow-analytic equivalence classes of n-branched curve germs, at least one for each standard form $\Gamma^{(k)}$.

The invariant μ' plays an important role in the blow-analytic classification of standard forms. The main reason is that, by fixing it, we reduce the infinite classification problem to a finite one.

Main Theorem. *The number of blow-analytic equivalence classes of n-branched germs of plane curves with $\mu' = k$ is finite for any k in $\mathbb{N}$.*

Proof. Given a germ of plane curve $(C, 0)$, take a good resolution of the embedded singularity and consider its dual graph Γ. By the process described above, the tree Γ can be reduced to its standard form, which, by Proposition 2.3, has at most $k + n$ branches. Furthermore, the length of each branch is limited by the properties of standard forms. Since the number of trees of a finite size is finite, it follows that there are only a finite number of standard forms, given n and k.

Observe that the number of smooth surfaces X corresponding to a given standard form, as well as the number of choices for the positions of the n non-compact components on X, is finite up to diffeomorphism. Thus, only finitely many blow-analytic equivalence classes of embedded plane curve singularities exist for fixed n and k in $\mathbb{N}$. □

After establishing the local finiteness of the blow-analytic classification, two possible study directions arise. The first one is the estimate of the number of equivalence classes as a function of μ'. For example, in the bibranched case, Theorem 1.7 tells us that there is exactly one standard form for each value of μ'. This holds for any μ' in $\mathbb{N}$, except for $\mu' = 1$, which is not achieved by any smoothly contractible configuration in the bibranched case. We will address the problem of finding a generating function in the next chapter for tribranched plane curves.

The second problem is to produce explicit lists of the standard forms. We do this for low values of μ', in the case of tribranched and 4-branched curve germs.

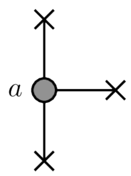
Chapter 3

An upper bound

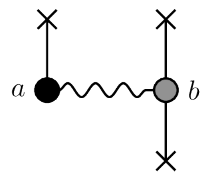
While it is difficult to recover a generating formula for the exact number of blow-analytic equivalence classes given the number of branches n and the value of μ', an upper bound to the number of standard forms can be estimated using combinatorial methods and some observations about the shape of Γ.

Definition. We call the *trunk* of Γ the topological structure of the minimal subtree connecting all non-compact components in Γ.

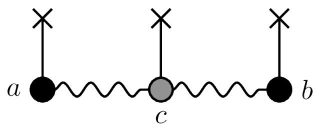
In the tribranched case there are four possible shapes for the trunk of a standard form:



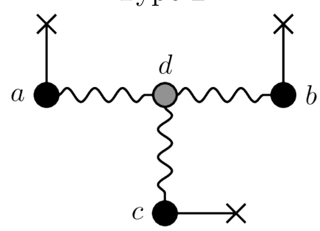
Type A



Type B



Type C



Type D

where $\times$ represents a non-compact component, the grey vertices can be either even or odd exceptional curves, and waved edges between two vertices represent finite chains of even curves connecting them.

Proposition 3.1. *In the tribranched case, the number of standard forms of Γ with $\mu' = k$ is less than or equal to*

$$(k^3 - 2k^2 - k + 11)2^{k-2}.$$

Proof. For $\mu' = k$, standard forms of Γ can be obtained by adding k vertices of type Q to the trunks above.

Observe that, in order to avoid not smoothly contractible configurations, two vertices of type Q cannot be attached to the same vertex. This implies that graphs of type A exist only for $k = 0$ and $k = 1$.

In what follows assume $k > 2$ for simplicity. The formula still holds for $k = 0, 1, 2$, as shown by the computations in the next section.

Type B. By Lemma 2.4, the vertex a cannot be extremal, so there must be a vertex of type Q attached to it.

If another vertex of type Q is attached to b, the remaining $k - 2$ vertices of type Q must be placed in the middle. The segment between each pair of special vertices (if it exists) is at most one even vertex, so 2^{k-1} different configurations are obtained this way.

Similarly, if b is extremal, then $k - 1$ vertices of type Q are attached to the edge of the trunk, which gives 2^k configurations.

Furthermore, b can be either odd or even, so there are

$$2(2^{k-1} + 2^k) = 3 \cdot 2^k$$

configurations of type B.

Type C. If vertices of type Q are attached to both a and b, there are

$$\sum_{\alpha+\beta=k-2} (2^{\alpha+1}2^{\beta+1}) + \sum_{\alpha+\beta=k-3} (2^{\alpha+1}2^{\beta+1}) =$$

$$= (k-2)(k-1)2^{k-1} + (k-2)(k-3)2^{k-2} = (3k^2 - 11k + 10)2^{k-2}$$

configurations, where the two terms in the sum count separately whether there is a vertex of type Q attached to c or not.

On the other hand, if b is an extremal vertex, then, by Lemma 2.4, the right edge is empty. Since $k > 2$, there must be one vertex of type Q attached to a and the other $k - 1$ to the left edge. This gives

$$2^{k-1} + 2^{k-2} = 3 \cdot 2^{k-2}$$

new configurations.

Since c can be either odd or even, the total number of configurations of type C is

$$2[(3k^2 - 11k + 10)2^{k-2} + 3 \cdot 2^{k-2}] = (3k^2 - 11k + 13)2^{k-1}.$$

Type D. First consider the case in which a, b and c each have a vertex of type Q attached to them. This gives

$$\sum_{\alpha+\beta+\gamma=k-3} (2^{\alpha+1}2^{\beta+1}2^{\gamma+1}) + \sum_{\alpha+\beta+\gamma=k-4} (2^{\alpha+1}2^{\beta+1}2^{\gamma+1}) =$$

$$= \frac{1}{3}(k-3)(k-2)(k-1)2^{k-1} + \frac{1}{3}(k-4)(k-3)(k-2)2^{k-2}$$

configurations.

Next, assume that c is extremal. By Lemma 2.4, this means that the downward edge is empty and there cannot be vertices of type Q attached to d, else not smoothly contractible configurations arise. So we have

$$\sum_{\alpha+\beta=k-2} (2^{\alpha+1}2^{\beta+1}) = (k-2)(k-1)2^{k-1}$$

configurations.

Observe that if two of the vertices in the trunk are extremal, Lemma 2.4 implies that the corresponding edges are empty, which leads to configurations that are not smoothly contractible. So the previous two cases cover all possible configurations.

Since there are two colour choices for the vertex d, in total there are

$$(k-2)(k^2 - 3k + 4)2^{k-1}$$

configurations of type D.

Adding the numbers obtained for each type, we get the upper bound

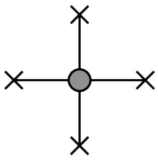
$$(k^3 - 2k^2 - k + 11)2^{k-2}.$$

□

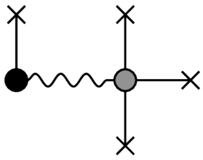
Remark. The above formula is merely an upper estimate of the number of standard forms for $\mu' = k$. In fact, the number includes some not smoothly contractible configurations as well as pairs of configurations which are blow-analytically equivalent (in the pair, one configuration is a standard form, to which the other can be reduced).

The same combinatorial techniques can be applied to any number of branches, although the complexity of the trunk of the standard forms rapidly increases, making it harder to do the computations by hand.

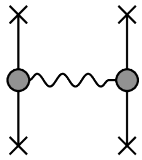
For example, in the 4-branched case there are ten possible shapes for the trunk of a standard form:



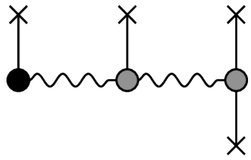
Type A



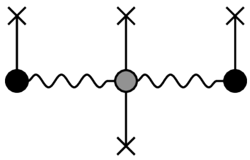
Type B



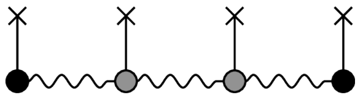
Type C



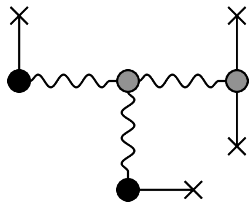
Type D



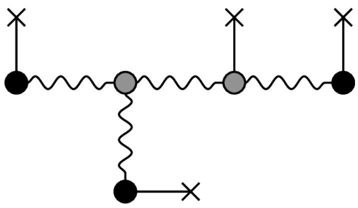
Type E



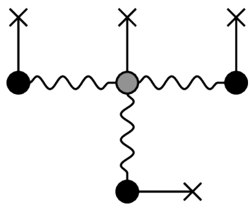
Type F



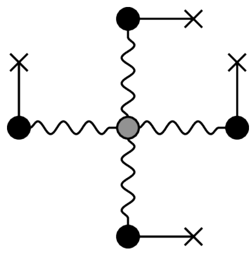
Type G



Type H



Type I



Type J

In general, it is reasonable to always expect the number of standard forms to grow exponentially, since this is strongly related to counting the number of trees of a given size, which is known to grow exponentially.

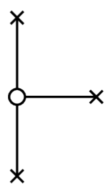
Chapter 4

Explicit classification

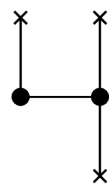
In this chapter, we restrict our attention to tribranched and 4-branched germs of plane curves and determine explicitly a standard form for each blow-analytic equivalence class, for low values of the invariant μ'.

4.1 Tribranched singularities

Theorem 4.1 (Kobayashi, [7]). *A germ of a tribranched plane curve with $\mu' = 0$ is blow-analytically equivalent to one of the following:*



$(\{xy(x-y)=0\},0)$



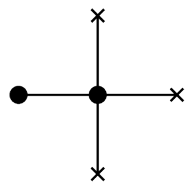
$(\{xy(x-y^2)=0\},0)$

The following results provide a classification of the dual graphs of good resolutions with $\mu' = 1, 2$.

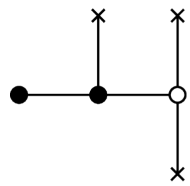
Up to this point, we are not able to prove in general the uniqueness of standard forms in a given blow-analytic equivalence class. In the proofs of Theorems 4.2, 4.4 and Proposition 4.3, we use the invariants μ_I to show that the standard forms listed in the statements are in fact blow-analytically distinct.

It may happen that two blow-analytically non-equivalent curve germs share the same graph standard form. However, to each standard form correspond at most a finite number of blow-analytic equivalence classes of plane curve germs, so we feel that a classification of the dual graphs is still a strong one from the blow-analytic point of view.

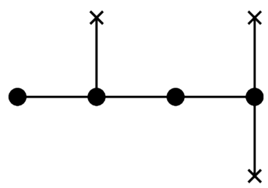
Theorem 4.2. *The dual graph of any good resolution of a tribranched plane curve germ with* $\mu' = 1$ *is blow-analytically equivalent to exactly one of the following standard forms:*



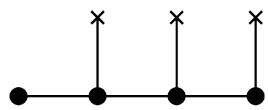
$A_2 : (\{y(y - x^2)(y + x^2) = 0\}, 0)$



$B_1 : (\{x(y - x)(y^2 - x^3) = 0\}, 0)$



$B_4 : (\{y(y - x^2)(y - x^4) = 0\}, 0)$

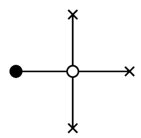


$C_2 : (\{y(y - x^2)(y^2 - x^5) = 0\}, 0)$

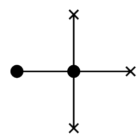
Proof. Consider a tribranched germ of plane curve $(C, 0)$ and assume $\mu' = 1$. Take a good embedded resolution of $(C, 0)$, construct its dual graph Γ and reduce it to a standard form as described in Section 2.2.

Since $(C, 0)$ is tribranched, the trunk of the reduced Γ must be of type A, B, C or D. Furthermore, the assumption $\mu' = 1$ implies that Γ contains exactly one vertex of type Q.

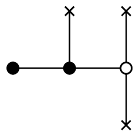
Draw all configurations with $\mu' = 1$ for each type, remembering that a segment between two special vertices is at most one even vertex and using Lemma 2.4 for the extremal vertices. Then, Γ must be blow-analytically equivalent to one of the following configurations:



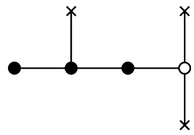
A_1



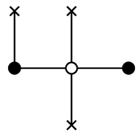
A_2



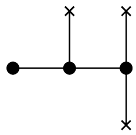
B_1



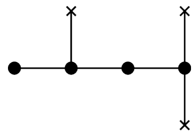
B_3



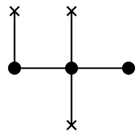
B_5 $(\mu \neq 0)$



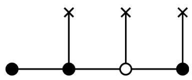
B_2 $(\mu \neq 0)$



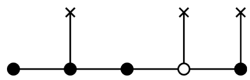
B_4



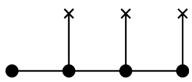
B_6 $(\mu \neq 0)$



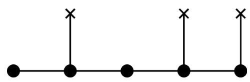
C_1



C_3 $(\mu \neq 0)$



C_2



C_4 $(\mu \neq 0)$

Observe that all configurations of type D with $\mu' = 1$ are not smoothly contractible, thus cannot be the dual graph of a resolution. For the same reason, we also cross out of the list all configurations with $\mu \neq 0$.

For the remaining configurations, A_1, B_3 and C_1 are blow-analytically equivalent to A_2, B_4 and C_2 respectively. Only 4 graphs are left and they are those of the statement.

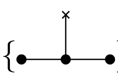
Finally, the equation of a representative for each configuration can be found by contracting all exceptional curves (possibly performing blow-ups if no odd curves are present).

To show that the four configurations are blow-analytically distinct, label $\{1, 2, 3\}$ the vertices corresponding to the three non-compact components and consider the triplets $\{\mu(\mathit{\Gamma}_{12}), \mu(\mathit{\Gamma}_{13}), \mu(\mathit{\Gamma}_{23})\}$, which are blow-analytic invariants by Lemma 2.1. We have:

$$A_2, B_4 : \{1, 1, 1\} \qquad B_1 : \{0, 1, 1\} \qquad C_2 : \{0, 1, 2\}.$$

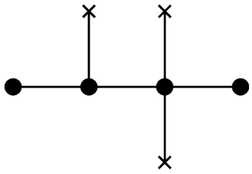
Since the coranks are not sufficient to distinguish between A_2 and B_4, we look explicitly at the sets Δ^*_{ij} for $1 \leq i < j \leq 3$:

$$\Delta^*_{12}(A_2) = \Delta^*_{13}(A_2) = \Delta^*_{23}(A_2) = \{\bullet, \times\};$$

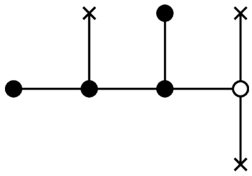
$$\Delta^*_{12}(B_4) = \Delta^*_{13}(B_4) = \{\bullet, \times\}, \qquad \Delta^*_{23}(B_4) = \{$$  $$\}.$$

There is no bijection between $\Delta^*_{ij}(A_2)$ and $\Delta^*_{23}(B_4)$ for any choice of ij, so we conclude that no blow-analytic homeomorphism exists between plane curve germs having good resolutions equivalent to A_2 and B_4 respectively. □

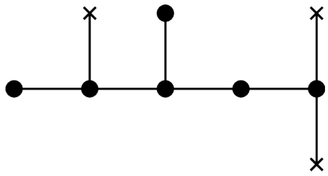
Proposition 4.3. *The dual graph of any good resolution of a tribranched plane curve germ with* $\mu' = 2$ *is blow-analytically equivalent to exactly one of the following standard forms:*



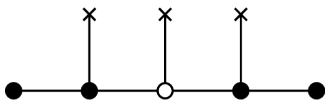
B_2



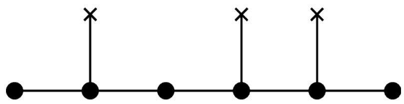
B_5



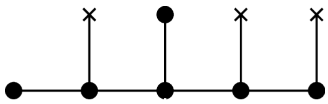
B_{12}



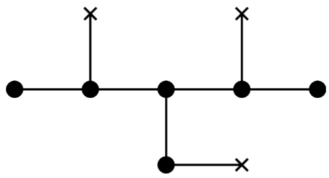
C_1



C_4

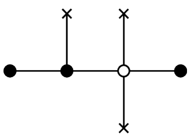


C_8

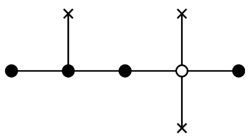


D_2

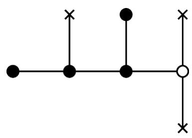
Proof. The proof is similar to that of the previous proposition. For each type, draw all reduced configurations in which exactly two vertices of type Q appear. The dual graph of any resolution of a tribranched singularity with $\mu' = 2$ is blow-analytically equivalent to one of the graphs in the list below. Again, notice that there are no smoothly contractible configurations of type A.



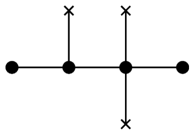
B_1



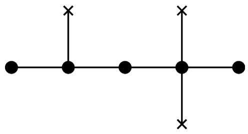
B_3 ($\mu \neq 0$)



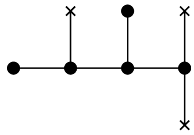
B_5



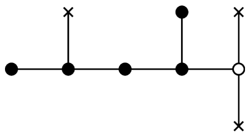
B_2



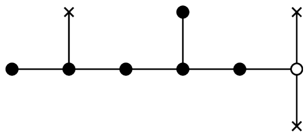
B_4 ($\mu \neq 0$)



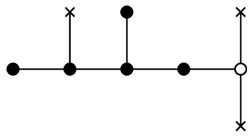
B_6 ($\mu \neq 0$)



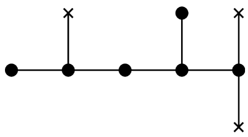
B_7 ($\mu \neq 0$)



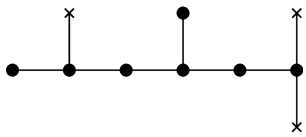
B_9 ($\mu \neq 0$)



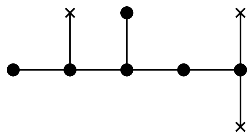
B_{11}



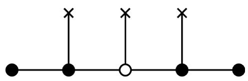
B_8 ($\mu \neq 0$)



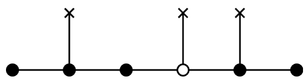
B_{10} ($\mu \neq 0$)



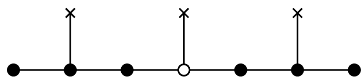
B_{12}



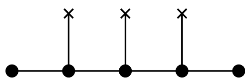
C_1



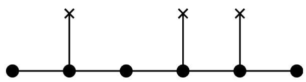
C_3



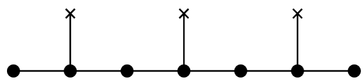
C_5 ($\mu \neq 0$)



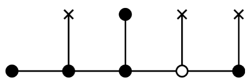
C_2 ($\mu \neq 0$)



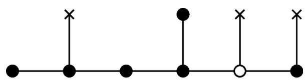
C_4



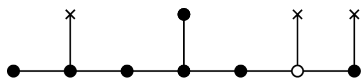
C_6 ($\mu \neq 0$)



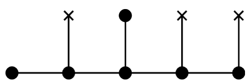
C_7



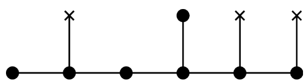
C_9 ($\mu \neq 0$)



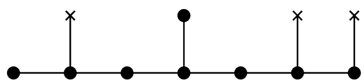
C_{11} ($\mu \neq 0$)



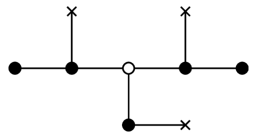
C_8



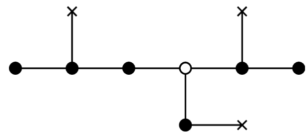
C_{10} ($\mu \neq 0$)



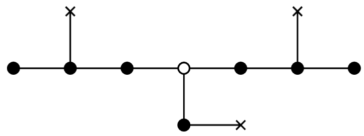
C_{12} ($\mu \neq 0$)



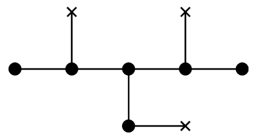
D_1



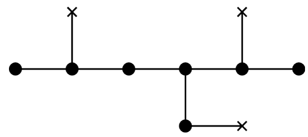
D_3 ($\mu \neq 0$)



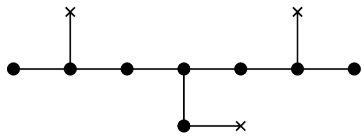
D_5 ($\mu \neq 0$)



D_2



D_4 ($\mu \neq 0$)



D_6 ($\mu \neq 0$)

Next, remove all configurations having $\mu \neq 0$, as they are not smoothly contractible.

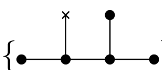
Finally, observe that some of the remaining configurations are pairwise blow-analytically equivalent (namely, B_1, B_{11}, C_3, C_7 and D_1 are equivalent to B_2, B_{12}, C_4, C_8 and D_2 respectively).

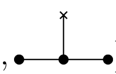
Again, we label $\{1, 2, 3\}$ the vertices corresponding to the three non-compact components and consider the values of the invariants $\{\mu(\Gamma_{12}),\ \mu(\Gamma_{13}),\ \mu(\Gamma_{23})\}$ to show that several of the configurations are non-equivalent. In fact, we have the following:

$$B_2, B_{12}, C_4 : \{1,2,2\} \qquad B_5 : \{0,2,2\} \qquad C_1 : \{1,1,2\}$$
$$C_8 : \{0,2,3\} \qquad D_2 : \{1,1,3\}.$$

To further distinguish between B_2, B_{12} and C_4, we look explicitly at the sets Δ^*_{ij} for $1 \leq i < j \leq 3$:

$$\Delta^*_{12}(B_2) = \Delta^*_{13}(B_2) = \{\bullet, \bullet, \times\}, \qquad \Delta^*_{23}(B_2) = \{\bullet, \times\};$$

$\Delta^*_{12}(B_{12}) = \Delta^*_{13}(B_{12}) = \{\bullet, \bullet, \times\}, \qquad \Delta^*_{23}(B_{12}) = \{$  $\}$;

$\Delta^*_{12}(C_4) = \{\bullet, \times\},\ \Delta^*_{12}(C_4) = \{\bullet,$  $\}$, $\Delta^*_{23}(C_4) = \{\bullet, \bullet, \times\}$.

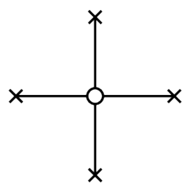
Since we do not have the bijections implied by Lemma 2.1, we can say that B_2, B_{12}, C_4 define different blow-analytic equivalence classes. □

4.2 4-branched singularities

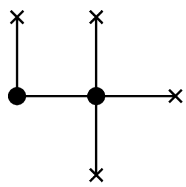
Due to the algorithmic nature of the process for finding all standard forms for a given number of branches and a given value of μ', it is possible to produce lists similar to the ones in the previous section. Here we present the list of standard form for 4-branched plane curve germs with $\mu' = 0$, as a further example.

Observe that the singular germ corresponding to the intersection of n lines at the origin has $\mu' = 0$ for any n (the dual graph of a good resolution is a single odd vertex). Hence the case $\mu' = 0$ is of particular interest in the sense that it contains the "easiest" singularities.

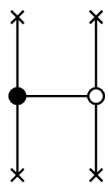
Theorem 4.4. *The dual graph of any good resolution of a* 4*-branched plane curve germ with* $\mu' = 0$ *is blow-analytically equivalent to exactly one of the following standard forms:*



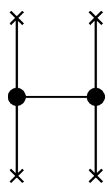
A_1



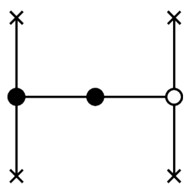
B_2



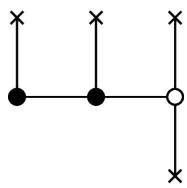
C_2



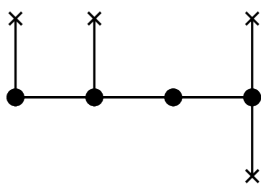
C_3



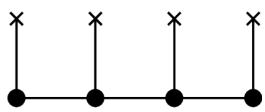
C_5



D_2



D_8

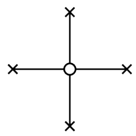


F_3

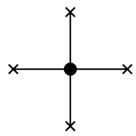
Proof. The proof is analogue to the tribranched case. Let $(C, 0)$ be a 4-branched plane curve germ with $\mu' = 0$, take a good embedded resolution of the singularity and reduce its dual graph to a standard form Γ. The assumption $\mu' = 0$ implies that no vertices type Q appear in Γ.

For each trunk type we list all candidates for Γ. The task is simplified by the application of Lemma 2.4, which dictates conditions on the extremal vertices of a standard form. Observe, for example, that the lemma implies that no smoothly contractible standard form with $\mu' = 0$ has trunk of type G, H, J or I.

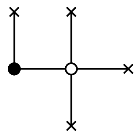
We get:



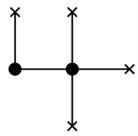
A_1



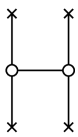
A_2 $(\mu \neq 0)$



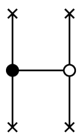
B_1



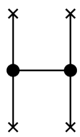
B_2



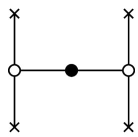
C_1 $(\mu \neq 0)$



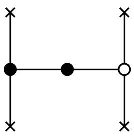
C_2



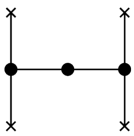
C_3



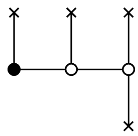
C_4 $(\mu \neq 0)$



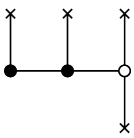
C_5



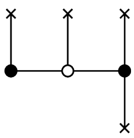
C_6 $(\mu \neq 0)$



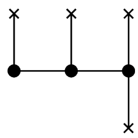
D_1



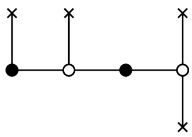
D_2



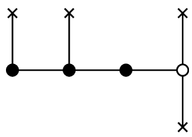
D_3



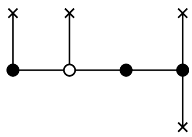
D_4 $(\mu \neq 0)$



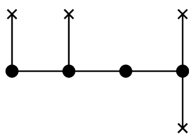
D_5



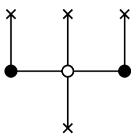
D_6



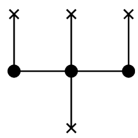
D_7



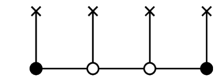
D_8



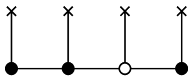
E_1 $(\mu \neq 0)$



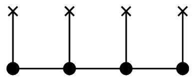
E_2 $(\mu \neq 0)$



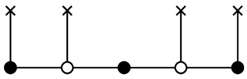
F_1



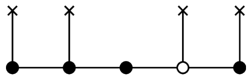
F_2



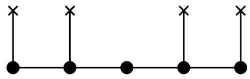
F_3



F_4 ($\mu \neq 0$)



F_5 ($\mu \neq 0$)



F_6 ($\mu \neq 0$)

We eliminate all non smoothly contractible configurations from the list (i.e., those with $\mu \neq 0$).

Next, observe that some of the remaining configurations can be transformed one into the other by simple blow-ups and blow-downs. Namely, B_1 is blow-analytically equivalent to B_2; D_1 is equivalent to D_2; D_5, D_6, D_7 are equivalent to D_8; F_1, F_2 are equivalent to F_3.

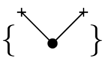
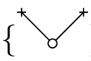
We are left with the 8 standard forms which appear in the statement: $A_1, B_2, C_2, C_3, C_5, D_2, D_8, F_3$. The most delicate step is proving that these are non-equivalent. We do it by computing the invariants μ_I.

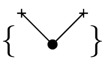
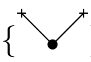
Label $\{1,2,3,4\}$ the vertices corresponding to the three non-compact components and consider the values of the invariants $\{\mu(\mathit{\Gamma}_{123}),\ \mu(\mathit{\Gamma}_{124}),\ \mu(\mathit{\Gamma}_{134}),\ \mu(\mathit{\Gamma}_{234})\}$ to show that several of the configurations are non-equivalent. We have the following:

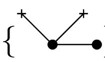
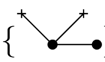
$$A_1, C_2, C_3, C_5 : \{0,0,0,0\} \qquad B_2, D_2, D_8 : \{0,0,0,1\} \qquad F_3 : \{0,0,1,1\}$$

To further distinguish between A_1, C_2, C_3, C_5, we look at the sets $\{\Delta^*_{ij} \mid 1 \leq i < j \leq 4\}$:

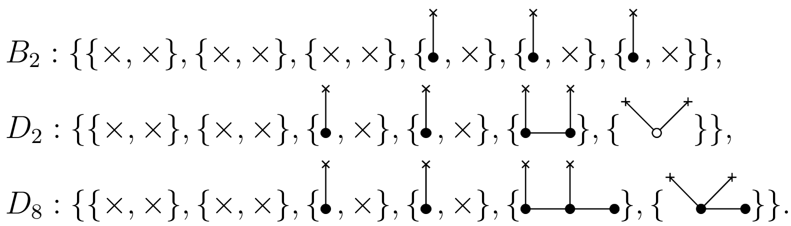
$A_1 : \{\{\times,\times\},\{\times,\times\},\{\times,\times\},\{\times,\times\},\{\times,\times\},\{\times,\times\}\}$,

$C_2 : \{\{\times,\times\},\{\times,\times\},\{\times,\times\},\{\times,\times\},\{$  $\},\{$  $\}\}$,

$C_3 : \{\{\times,\times\},\{\times,\times\},\{\times,\times\},\{\times,\times\},\{$  $\},\{$  $\}\}$,

$C_5 : \{\{\times,\times\},\{\times,\times\},\{\times,\times\},\{\times,\times\},\{$  $\},\{$  $\}\}$.

Similarly, to distinguish between B_2, D_2, D_8:



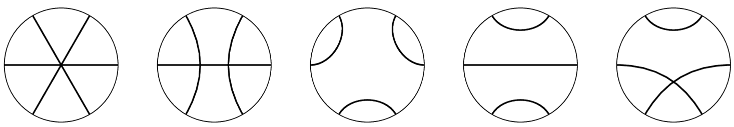
Since we do not have the bijections implied by Lemma 2.1, we can say that all standard forms belong to different blow-analytic equivalence classes. □

Chapter 5

Adding chord diagrams to the picture

Blow-ups and blow-downs are local transformations, so, in particular, they do not change the order in which the semi-branches intersect the boundary of a small of circle around the origin. We represent this piece of information in a *chord diagram* by drawing vertices on S^1 where the semi-branches intersect such boundary, and joining two vertices if they belong to the same local analytic component.

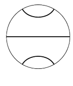
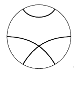
For example, in the tribranched case, there are five possible chord diagrams [3]:



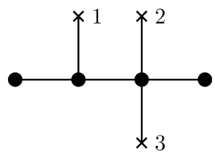
.

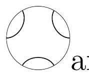
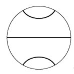
Since chord diagrams are blow-analytic invariants, we can prove that two configurations are non-equivalent by showing that they have different chord diagrams.

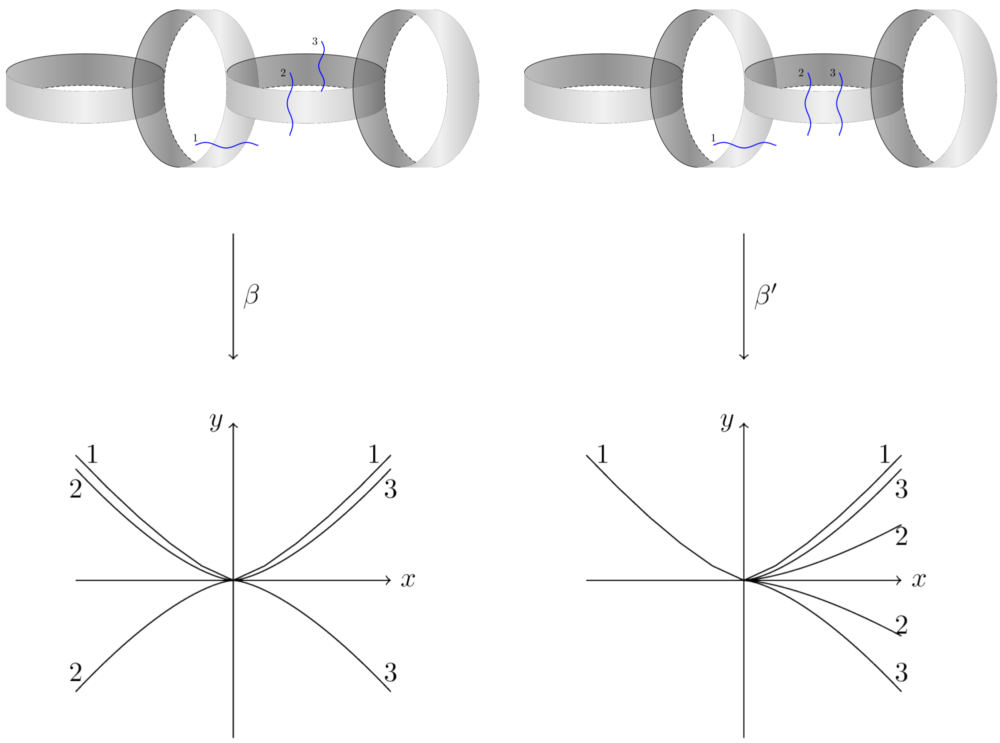
This invariant does not add new information to the classification of standard forms in Theorem 4.2. In fact, to each configuration corresponds exactly one chord diagram in the following way:

$A_2, B_4 :$  $\quad B_1, C_2 :$  .

As we consider configurations with a larger value of μ', however, a new phenomenon appears. For example, the standard form B_2 in Proposition 4.3



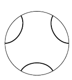
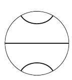
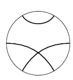
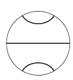
has two possible chords diagrams:  and . Resolutions corresponding to this dual graph are smooth surfaces diffeomorphic to a chain of four cylinders intersecting transversally, and different choices for the respective positions of the strict transform components give different chord diagrams.



$(\{(y^3 - x^4)(y^2 + x^3)(y^2 - x^3) = 0\}, 0)$ $(\{(y^3 - x^4)(2y^2 - x^3)(y^2 - x^3) = 0\}, 0)$

Thus we can have two blow-analytically distinct embedded plane curve germs with the same dual graph, as shown above.

The complete list of chord diagrams for the standard forms in Proposition 4.3 is as follows:

B_2, C_1, C_4, D_2 : ,  B_5, C_8 :  B_{12} : .

To each of the configurations B_2, C_1, C_4, D_2 correspond pairs of plane curve germs which are blow-analytically non-equivalent.

We remark that the induced equivalence of dual graphs is weaker than the blow-analytic equivalence of embedded plane curve germs. This follows from the fact that some topological information is lost in the passage from a resolution to its dual graph, namely, we lose track of the respective position of the strict transform components. One should pay attention to this kind of phenomena when passing from the equivalence of dual graphs Γ to the blow-analytic classification of germs.

However, for each standard form, there is only a finite possibility of equivalence classes of germs. In some cases, as above, we can distinguish the classes by using chord diagrams, which are determined solely by the order of the branches near the origin.

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