Comprehensive research of Integration Theory

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Contents

Chapter 1

Introduction

This thesis contains two topics. First, we shall investigate the Radon-Nikodym theorem for general measure spaces by the method of Daniell scheme and we shall discuss its applications. Second, we consider the Kakeya problem, especially we plan to prove some weighted estimates for the Kakeya maximal operator. Before stating our results we shall explain some background of the problems.

The Radon-Nikodym theorem dates back to the papers [53, 55]. Roughly speaking, the Radon-Nikodym theorem answers the following question: Is there a density function whenever ν is absolutely continuous with respect to μ (we denote $\nu \ll \mu$)? The answer is NO. There is a counter-example which fails to have a density function. We will describe it in Section 3.4. In view of this, some additional assumptions for two measures are required in order to obtain a density function. We will describe this situation more precisely. Let (Ω, Σ, μ) be a measure space and ν a measure satisfying $\nu \ll \mu$. In the standard textbooks, one discusses as follows. If μ is σ-finite, then we can find an increasing sequence of μ-finite sets $\{\Omega_n\}$ such that $\Omega = \cup_n \Omega_n$. We can construct a non-negative density function h_n on each Ω_n which has the uniqueness condition $h_{n+1} = h_n$ on Ω_n, and obtain the desired one as $h := \sup_n h_n$. A tacit understanding is that h_n vanishes a.e. outside Ω_n. However, what do we do if the underlying space Ω is not σ-finite ? The family of density functions h_{Ω_λ} on each Ω_λ having finite measure with $\Omega = \cup_\lambda \Omega_\lambda$ does not ensure us the measurability of $h = \sup_\lambda h_\lambda$. If Ω is measurable, it is natural to compel μ to ensure the measurability of $\sup_\lambda h_\lambda$ which is known as the *localizable* measure, but if Ω is not measurable, we must come up with other ideas. In this thesis, we will control the family of density functions $\{h_{\Omega_\lambda}\}$ appropriately, and obtain the Radon-Nikodym type equality:

$$\nu(E) = \int_E \langle h \rangle \, d\mu, \tag{1.1}$$

where the symbol $\langle h \rangle$ is an appropriate family of functions, which is called *folder*, (see Section 3.1). Our scheme covers the situation where Ω is *not* necessarily measurable. When the measure μ is not necessarily localizable, the Radon-Nikodym derivative fails to be a function, but forms a folder, as a consequence, we shall newly formulate the Radon-Nikodym density folder, which allows us to obtain the Radon-Nikodym theorem on non-σ-finite and/or σ-ring measure spaces, in other words, we do not have to work on the σ-finite measure spaces by our framework. Moreover, in [59] the author has established that the localizability is equivalent that all folders are represented by "one" measurable function.

As its applications, we shall characterize the dual space of integrable functions and the Lebesgue decomposition theorem. For a normed space $(E, \|\cdot\|)$, the dual space E^* is the set of all bounded linear functionals F. Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. For general measure space (Ω, Σ, μ), the dual space of L^p is identified with L^q, more precisely, each $F \in (L^p)^*$ is determined by

$$F(f) = \int_\Omega fg\, d\mu, \quad \text{for } g \in L^q,$$

and $\|F\| = \|g\|_{L^q}$. However, when $p = 1$, the above relation does not hold for arbitrary measure spaces. The essential reason why the dual of L^p with $1 < p < \infty$ can be identified without any condition on (Ω, Σ, μ) is that each function f in L^p has σ-finite carrier, but L^∞ functions does not necessarily have σ-finite one. That is because, if μ is σ-finite, we obtain the identification $(L^1)^* \cong L^\infty$. However, Segal [63] also proved that one can identify the dual of L^1 in the above manner if and only if the measure μ is localizable. This argument also can be seen in the textbooks, Rao [56] discussed this result from the view point of measure theory while Zaanen [78] discuss this result from the view point of Daniell integral. The Lebesgue decomposition thoerem asserts that given two measures μ and ν on Σ, we can decompose ν to its absoutely continuous part with respect to μ and its singular part with respect to μ. In the classical theory, the σ-finiteness plays a key role and there is a counter-example of a measure space which does not admit usual decomposition. We will reconsider carefully this theorem by using folders and reformulate a new version of the decomposition theorem. All of results require the general notion of density of measures. As a consequence, the folder works appropriately for characterization of the dual space and for decomposition of measures.

From Chapter 2 to Chapter 6, we discuss mainly the Daniell method as basic argument of integration. From the Daniell standpoint, not only measure theory appears as an almost self-evident consequence of the theory of the integral, but also non-σ-finite measure space appears naturally and this gives us an opportunity to study σ-ring measure spaces.

The second topic is the Kakeya problem. In 1917 Soichi Kakeya posed the Kakeya needle problem: what is the smallest area which is required to rotate a unit line segment (a "needle") by 180 degrees in the plane? A construction due to A. S. Besicovitch shows that such sets may have arbitrary small measure. At first glance, Kakeya needle problem and Bescovitch's solution appear to be little more than mathematical curiosities. However, in the last three decades it has gradually been realized that this type of problem is connected to many other seemingly unrelated problems in number theory, geometric combinatorics, arithmetic combinatorics, oscillatory integrals, and even the analysis of dispersive and wave equations. For a more quantitative approach the problem will be translated into bounds for Kakeya maximal functions.

Fix a sufficiently large natural number N. For a real number $a > 0$ let $\mathcal{B}_{a,N}$ be the family of all cylinders in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$, which are congruent to the cylinders with height Na and width a, but with arbitrary directions and centers. For a locally integrable function f on $\mathbb{R}^n$ the "small" Kakeya maximal operator $K_{a,N}$ is defined by

$$K_{a,N}f(x) := \sup_{x \in R \in \mathcal{B}_{a,N}} \frac{1}{|R|} \int_R |f(y)|\, dy$$

and the Kakeya maximal operator K_N is defined by

$$K_N f(x) := \sup_{a>0} K_{a,N} f(x),$$

where $|R|$ denotes the Lebesgue measure of R. It is conjectured that K_N is bounded on $L^n(\mathbb{R}^n)$ with the norm which grows no faster than $O((\log N)^{\alpha_n})$ for some $\alpha_n > 0$ as $N \to \infty$. In the case $n = 2$, this conjecture was solved affirmatively by Córdoba [14] with the exponent $\alpha_2 = 2$ and improved by Strömberg [68] with $\alpha_2 = 1$. About the $L^p(\mathbb{R}^2)$ estimates, we have the following result:

Theorem 1. *If* $n = 2$,

$$\|K_N f\|_{L^p(\mathbb{R}^2)} \leq C_{N,p} \|f\|_{L^p(\mathbb{R}^2)}$$

holds with

$$C_{N,p} := \begin{cases} O(N^{2/p-1}(\log N)^{2/p'}) & 1 < p < 2 \\ O((\log N)^{2/p}) & 2 \leq p < \infty, \end{cases}$$

where p' *is the conjugate exponent of* p (*see also* [33]).

In the higher dimensional case, $n > 2$, these estimates were proved so far only for some restricted class of functions. For the functions of product type $f(x) = f_1(x_1)f_2(x_2)\cdots f_n(x_n)$, Igari [36] proved the estimate for $K_{a,N}$ with the exponent $\alpha_n = 3/2$ and Tanaka [70] reproved with the exponent $\alpha_n = (n-1)/n$. When the functions are of radial type $f(x) = f_0(\|x\|_{l^2})$, Carbery, Hernández and Soria [11] proved the estimate for K_N with the exponent $\alpha_n = 1$. In [71], for the functions of radial type $f(x) = f_0(\|x\|_{l^1})$, Tanaka proved the estimate for $K_{a,N}$ with the exponent $\alpha_n = 1$. In [27], for the functions of radial type $f(x) = f_0(\|x\|_{l^q})$, $1 \leq q \leq n$, Duoandikoetxea and Naibo proved the estimate for K_N with the exponent $\alpha_n = 1$. We will describe its background in detail in Chapter 7.

Instead of the difficult operator K_N, a more powerful but slightly complicated maximal operator has been studied on the plane. Let Ω be a set of unit vectors in $\mathbb{R}^2$ with cardinality N. For a locally integrable function f on $\mathbb{R}^2$, the directional maximal operator M_Ω is defined by

$$M_\Omega f(x) := \sup_{r>0, \omega \in \Omega} \frac{1}{2r} \int_{-r}^{r} |f(x + t\omega)|\, dt.$$

In [39] and [40], Katz established

$$\|M_\Omega f\|_{L^2(\mathbb{R}^2)} \leq C \log N \|f\|_{L^2(\mathbb{R}^2)}.$$

holds for arbitrary Ω with $|\Omega| = N$. Since if Ω is an equidistributed set of directions then $K_N f(x) \le CM_\Omega f(x)$ holds, we obtain the bounds of the Kakeya maximal operator. We will investigate the weighted version of this operator,

$$M_{\Omega,w} f(x) := \sup_{x \in R \in \mathcal{B}_\Omega} \frac{1}{w(R)} \int_R |f(y)| w(y)\, dy,$$

and establish

$$\|M_{\Omega,w} f\|_{L^2(w)} \le C \log N \|f\|_{L^2(w)}.$$

for a certain weight w. The precise definitions will be described in the next chapter.

In the past fifteen years, the variable exponent Lebesgue spaces have been studied intensively and many people tried to extend the classical theory of function spaces. One of the most interesting problems on spaces with variable exponent is the boundedness of the Hardy-Littlewood maximal operator. The last study of this thesis is the bounds for the Kakeya maximal operator on the variable Lebesgue space. The important sufficient conditions, called log-Hölder continuity, have been obtained by [17] and [22]. Under the conditions, we shall establish the following estimate:

$$\|K_N f\|_{L^{p(\cdot)}} \le CN^{1-\frac{p_-}{p_+}} (\log N)^{2/p_-} \|f\|_{L^{p(\cdot)}}.$$

Since we can find easily the pointwise estimate

$$K_N f(x) \le CN \cdot Mf(x),$$

where $Mf(x)$ is the Hardy-Littlewood maximal operator, then we see immediately that $\|K_N f\|_{L^{p(\cdot)}} \le CN \|f\|_{L^{p(\cdot)}}$, but we can obtain a sharper estimate as above. Moreover, we discuss the lower bounds of the exponent of N, and show that we can not eliminate the power of N.

Below we describe the organization of this thesis.

In Chapter 2, we shall summarize the Daniell scheme and describe some essential properties of the Daniell integral. It should be noted repeatedly that there do exist some versions of Daniell's integration theories, so for the sake of the completeness, we give the proofs for some essential propositions.

One of the main themes of this thesis is the Radon-Nikodym theorem in Chapter 3. The folder is the most important concept to discuss on general measure spaces. This allows us to obtain the Radon-Nikdoym density (derivative) $\langle h \rangle$ for two measures satisfying $\nu \ll \mu$ and to formulate indefinite integral (1.1). We also characterize the localizable measure by using the completeness of the folder.

As its application, in Chapters 4, 5 and 6, we discuss the dual space of the integrable functions space L^1 and the Lebesgue decomposition theorem in general measure spaces.

We begin to consider the Kakeya problems in Chapter 7. In Chapter 7, we describe the background of the Kakeya problems and the relationship from the viewpoint of real analysis. In Chapter 8, we shall follow the paper [61], which concerns the boundedness of the weighted directional maximal operator and extend the Katz result.

In Chapter 9, we prove the boundedness of the Kakeya maximal operator on the variable Lebesgue spaces on the plane.

NOTATION

We shall use the following terminology and notational conventions:

- We write $N \gg 1$, when we are given a non-negative integer N which is large enough.
- The symbol $\overline{\mathbb{R}}$ denotes the extended reals, with the usual conventions concerning the arithmetic and order structure.
- If a sequence of functions $\{f_n\}$ converges to f increasingly or decreasingly, we write $f_n \nearrow f$ or $f_n \searrow f$, respectively.
- Let χ_A be the characteristic or indicator function of a set A, alternatively, if a set A is more complicative, we prefer to denote $\chi(A)$. In Section 3.2, we will use this notation enthusiastically.
- Given a sequence of measurable subsets $\{E_n\}_{n=1}^{\infty}$, we write $E_n \nearrow \bigcup_{n=1}^{\infty} E_n$ to express that $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$.

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Chapter 2

Daniell Integral

One of the basic concepts of analysis is that of the integral. It is well known that Lebesgue's integration theory is an essential tool for wide applications in which countability plays a key role. These include functional analysis, probability and statistics, harmonic analysis, many aspects of differential equations, and others. For some of the latter applications one can start with the concept of linear functionals, without mention of measure, and proceed to the theory of integration so-called Daniell integral. In this thesis, we discuss mainly the Daniell method as basic argument of integration. From the Daniell standpoint, not only measure theory appears as an almost self-evident consequence of the theory of the integral, but also non-σ-finite measure space arises naturally and this gives us an opportunity to study σ-ring measure spaces.

This chapter is devoted to a description of the Daniell scheme of extending certain linear functionals on a vector space and of the Stone condition. The extended function class $\mathcal{L}$ of integrable functions is defined via difference of monotone limits of elementary functions. The Stone condition (2.1) below ensures the measurability of the pointwise product fg for any measurable functions f, g. It should be noted that there are various schemes called Daniell integral [3, 8, 13, 18, 46, 48, 56, 66, 74, 78] and these schemes are not equivalent each other. Except for the extension procedure, one of the most essential differences between these schemes is that of measurability. For instance, Shilov and Gurevich [66] proposed the measurable functions are defined to be pointwise limit of elementary function functions, and it means the whole space is not necessarily measurable. However, they postulated on the set of all elementary function spaces Stone's condition, which serves as the condition of its measurability. Meanwhile, Weir [74] proposed another notion of measurablitiy, some authors called Stone's measurability, which ensures that of the whole space. In particular, we adopt the scheme that the whole space is not necessarily measurable throughout of this thesis.

2.1 Summary of Daniell scheme

This section gives a brief discussion of the Daniell extension procedure. We establish the Monotone Convergence Theorem and the Dominated Convergence Theorem.

A vector space $\mathcal{H}$ consisting of all $\mathbb{R}$-valued functions on a set $\Omega(\neq \emptyset)$ is said to be an *elementary function space* if $\mathcal{H}$ is closed under taking absolute value. The functions in $\mathcal{H}$ are called elementary. The set $\mathcal{H}$ is also called a *vector lattice* or a *Riesz space*, if it is a partially ordered vector space closed under taking pointwise maxima, minima of functions h, k, denoted by $h \vee k$, $h \wedge k$, respectively.

A $\mathbb{R}$-valued linear functional $\int$ on $\mathcal{H}$ satisfying
(1) non-negativity: $\mathcal{H} \ni h \geq 0 \Rightarrow \int h \geq 0$,
(2) continuity: $h_n \searrow 0 \Rightarrow \int h_n \to 0$
is said to be an *elementary integral* or a *Daniell integral* [18, 66, 69, 74]. The triplet $(\Omega, \mathcal{H}, \int)$ is called a *Daniell system.*

We denote by $\mathcal{H}^+$ the class of all pointwise limit functions f which can be expressed as the limit of a sequence of the monotone increasing elementary functions [66, 69, 74]. Here, we understand that any function in $\mathcal{H}^+$ assumes its value in $\overline{\mathbb{R}}$. We define the integral of $f \in \mathcal{H}^+$ by $\int f = \lim \int h_n$, where $\{h_n\}_{n=1}^{\infty}$ is a sequence of the monotone increasing elementary functions. This definition is independent of the choice of an approximating functions h_n. The integral on $\mathcal{H}^+$, for which we still write $\int$, is an $\overline{\mathbb{R}}$-valued functional.

Remark 2.1.1. Obviously, $\mathcal{H} \subset \mathcal{H}^+$ and $\int$ on $\mathcal{H}^+$ extends the elementary integral. The extended integral $\int$ is closed under addition, and it has non-negative homogeneity, and continuity of increasing sequence of $\mathcal{H}^+$.

A function $f \in \mathcal{H}^+$ is said to be *integrable* if $\int f < \infty$ and we denote the set of all such f by $\mathcal{H}^+_{\text{int}}$. A subset $Z \subset \Omega$ is said to be a *null set*, if it is realized as a subset of $\{f = +\infty\}$ for some $f \in \mathcal{H}^+_{\text{int}}$ (see [66, 74]). A subset of a null set, and a countable union of null sets are still null sets. When a given property holds on Ω except on a null set, we say that the property holds *almost everywhere* on Ω, or "a.e." for short. For example, it is immediate that $f \in \mathcal{H}^+_{\text{int}}$ takes in $\mathbb{R}$ almost everywhere.

Proposition 2.1.2. *If $f, g \in \mathcal{H}^+$ and $f = g$ a.e., then $\int f = \int g$.*

Proof. There exist null set $Z \subset \Omega$ and $h \in \mathcal{H}^+_{\text{int}}$ such that $Z \subset \{h = +\infty\}$ and $f(x) = g(x)$ holds for $x \notin Z$. We see that $f(x) + h(x) = g(x) + h(x)$ for all $x \in \Omega$. Then it follows $\int f + \int h = \int g + \int h$. Since $h \in \mathcal{H}^+_{\text{int}}$, we can subtract $\int h$ from both side. □

Proposition 2.1.3. *Let $\varphi \in \mathcal{L}^+$. For any $\varepsilon > 0$, there exist $f \in \mathcal{H}^+$ and $g \in \mathcal{H}^+_{\text{int}}$ such that $\varphi = f - g$ a.e., where $\int g < \varepsilon$ and $g \geq 0$.*

Proof. We can write that $\varphi = f' - g'$, for some $f' \in \mathcal{H}^+$ and $g' \in \mathcal{H}^+_{\text{int}}$. By the definition of $\mathcal{H}^+_{\text{int}}$, there exists $\{h_n\} \subset \mathcal{H}$ such that $h_n \nearrow g'$. We have $\varphi = (f' - h_n) - (g' - h_n)$ and

$g' - h_n \geq 0$ for all $n \in \mathbb{N}$. Since $\int(g' - h_n) \searrow 0$, we have $\int(g' - h_{n_0}) < \varepsilon$ for sufficiently large $n_0 \in \mathbb{N}$. Then $f := f' - h_{n_0}$ and $g := g' - h_{n_0}$ are desired functions. $\square$

An $\overline{\mathbb{R}}$-valued φ, defined a.e. on Ω, is said to be *measurable* if it is an a.e. limit of a sequence of elementary functions [66]. The set of all measurable functions is denoted by $\mathcal{M}$. (Here, $f \in \mathcal{M}$ takes values in $\overline{\mathbb{R}}$, and $\mathcal{H}^+ \subset \mathcal{M}$.) We note that this definition is essentially different from any other definition in [66, 74] and so on.

Remark 2.1.4. We should point out the following example: Let $\Omega = [0, 1]$. An $\mathcal{H}$ is the set of all $\mathbb{R}$-valued functions whose carrier is a finite subset of Ω, i.e., $h \in \mathcal{H}$ can be written as

$$h(x) = \sum_{k \in A} a_k \chi_{\{k\}}(x), \quad a_k \in \mathbb{R}, A \subset \Omega : \text{finite}.$$

We define $\int : \mathcal{H} \to \mathbb{R}$ as $\int h := \sum_{k \in A} a_k$ so that all Daniell measurable functions $\mathcal{M}$ is the set of all $\overline{\mathbb{R}}$-valued functions whose carriers are countable in Ω. This implies the constant function c is not measurable.

The following proposition is obvious:

Proposition 2.1.5. *We have the following assertions:*

(1) *If $\varphi, \psi \in \mathcal{M}$ then $\varphi \vee \psi, \varphi \wedge \psi \in \mathcal{M}$.*

(2) *Let $\varphi, \psi \in \mathcal{M}$. If $\varphi(x) + \psi(x)$ can be defined for a.e. x, i.e., $\varphi(x) = +\infty$ and $\psi(x) = +\infty$ (or $\varphi(x) = -\infty$ and $\psi(x) = -\infty$) occur only on a null set, then $\varphi + \psi$ is Daniell measurable.*

(3) *If $f_n \in \mathcal{M}$, then $\inf_n f_n$ and $\sup_n f_n$ are in $\mathcal{M}$.*

(4) *Let ψ be a function satisfying $\psi = \varphi$ a.e. for some $\varphi \in \mathcal{M}$. Then $\psi \in \mathcal{M}$.*

A subset $D \subset \Omega$ is said to be *measurable*, or more precisely *Daniell measurable* if $\chi_D \in \mathcal{M}$ and we denote the set of all such D by $\mathcal{D}$.

Proposition 2.1.6. (1) *The family of Daniell measurable sets $\mathcal{D}$ forms a σ-ring, i.e.,* (i) $\emptyset \in \mathcal{D}$. (ii) *If $A, B \in \mathcal{D}$, then $A \setminus B \in \mathcal{D}$.* (iii) *If $A_n \in \mathcal{D}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$ (in general it is not necessarily Ω is in $\mathcal{D}$ by Remark 2.1.4).*

(2) *Let D be a Daniell null set. If $Z \subset D$, then Z is a Daniell measurable set (completeness). In particular, $\chi_D = 0$ a.e.*

Proof. (1): (i) It is immediate from $0 = \chi_\emptyset \in \mathcal{M}$. (ii) Let $A, B \in \mathcal{D}$. Then $\chi_{A \setminus B} = (\chi_A - \chi_B) \vee 0 \in \mathcal{M}$. (iii) Let $A_n \in \mathcal{D}$. Then $\chi_{\cup_n A_n} = \sup_n \chi_{A_n} \in \mathcal{M}$.

(2) There exists $f \in \mathcal{H}^+_{\text{int}}$ such that $D \subset \{f = +\infty\}$. Then $\chi_Z(x) \leq \frac{1}{n} f(x) \in \mathcal{H}^+_{\text{int}}$, and this implies $\chi_Z = 0$ a.e. because f is finite a.e. By Proposition 2.1.5 (4), $\chi_Z \in \mathcal{M}$. Taking $Z = D$, we see that $\chi_D = 0$ a.e. $\square$

A function $\varphi \in \mathcal{M}$ is said to be in $\mathcal{L}^+$ if it can be represented as $\varphi = f - g$ a.e. for some $f \in \mathcal{H}^+$ and $g \in \mathcal{H}^+_{\text{int}}$, and we define $\int \varphi := \int f - \int g \in (-\infty, \infty]$. We can verify that the definition is independent of the choice of functions f and g.

Remark 2.1.7. Obviously, $\mathcal{H}^+ \subset \mathcal{L}^+ \subset \mathcal{M}$. The integral $\int$ on $\mathcal{L}^+$ is an extension of $\int$ on $\mathcal{H}^+$. The space $\mathcal{L}^+$ is not a vector space and the extended integral $\int$ on $\mathcal{L}^+$ is not linear. But as far as we ignore the difference on a null set, $\mathcal{L}^+$ is closed under addition, multiplication by non-negative constants. The extended integral $\int$ is closed under addition, and it has non-negative homogeneity, and continuity of increasing sequence of $\mathcal{L}^+$. If the integral of $\varphi \in \mathcal{L}^+$ is finite, φ is said to be an *integrable* function [66, 74], and the set of all such functions is denoted by $\mathcal{L}$. We deduce $\mathcal{H} \subset \mathcal{H}^+_{\text{int}} \subset \mathcal{L} \subset \mathcal{L}^+$. As far as we ignore the difference on a null set, $\mathcal{L}$ has a linear structure and the integral $\int$ on $\mathcal{L}$ is a real-valued linear functional. We will use the fact that any $\varphi \in \mathcal{L}$ is finite almost everywhere.

Remark 2.1.8. The above procedure is called a *Daniell scheme*. Several types of the Daniell scheme are described in [8, 48, 66, 69, 74], with different contents and constructions, and are not equivalent to one another. Our scheme is almost the same as that adopted in [66].

In what follows, we will prove the Dominated Convergence Theorem and the Monotone Convergence Theorem. We need study the properties of $\mathcal{L}, \mathcal{L}^+$ and $\mathcal{M}$.

Proposition 2.1.9. *For any $\varphi \in \mathcal{L}^+$, there exists $\varphi_n \in \mathcal{H}^+$ such that $\varphi_n \searrow \varphi$ a.e.*

Proof. For $\varphi \in \mathcal{L}^+$, there exist $f \in \mathcal{H}^+$ and $g \in \mathcal{H}^+_{\text{int}}$ such that $\varphi = f - g$ a.e. Since we find $g_n \in \mathcal{H}$ with $g_n \nearrow g$, it follows that $\varphi_n := f - g_n \in \mathcal{H}^+$ and that $\varphi_n \searrow \varphi$ a.e. $\square$

Corollary 2.1.10. *For any $\varphi \in \mathcal{L}$, there exists $\varphi_n \in \mathcal{H}^+_{\text{int}}$ such that $\varphi_n \searrow \varphi$ a.e.*

Lemma 2.1.11. (1) *If $\varphi \in \mathcal{L}^+$ then $\varphi^\pm \in \mathcal{L}^+$. In particular, If $\varphi \in \mathcal{L}$ then $\varphi^\pm \in \mathcal{L}$.*

(2) *If $\varphi, \psi \in \mathcal{L}^+$ satisfy $\varphi \le \psi$ then $\int \varphi \le \int \psi$. In particular, If $\varphi = \psi$ then $\int \varphi = \int \psi$.*

Proof. (1) For $\varphi, \psi \in \mathcal{L}^+$,

$$\varphi \vee \psi = (f + g_2) \vee (f_2 + g) - (g + g_2)$$

with $(f + g_2) \vee (f_2 + g) \in \mathcal{H}^+$ and $g + g_2 \in \mathcal{H}^+_{\text{int}}$. Put $\psi = 0$. Then we see $\varphi^+ \in \mathcal{L}^+$. similarly, $\varphi^- \in \mathcal{L}^+$, $|\varphi| = \varphi^+ + \varphi^- \in \mathcal{L}^+$.

(2) Let $f_1, f_2 \in \mathcal{H}^+$, and $g_1, g_2 \in \mathcal{H}^+_{\text{int}}$ with $\varphi = f_1 - g_1$ a.e. and $\psi = f_2 - g_2$ a.e. Since g_1, g_2 is finite almost everywhere, we can write $f_1 + g_2 \le f_2 + g_1$ a.e. and functions in both sides are in $\mathcal{H}^+$. By Proposition 2.1.2, we have $\int f_1 + \int g_2 \le \int f_2 + \int g_2$. Hence, it follows from $g_1, g_2 \in \mathcal{H}^+_{\text{int}}$ that

$$\int \varphi = \int f_1 - \int g_1 \le \int f_2 - \int g_2 = \int \psi. \quad \square$$

Proposition 2.1.12. (1) *If* $\mathcal{L} \ni \varphi_n \nearrow \varphi$ *a.e., then* $\varphi \in \mathcal{L}^+$*, and* $\int \varphi_n \nearrow \int \varphi$.

(2) *For any* $\varphi \in \mathcal{L}^+$*, there exists* $\varphi_n \in \mathcal{L}$ *such that* $\varphi_n \nearrow \varphi$ *a.e.*

(3) *If* $\mathcal{L}^+ \ni \varphi_n \nearrow \varphi$ *a.e., then* $\varphi \in \mathcal{L}^+$*, and* $\int \varphi_n \nearrow \int \varphi$.

Proof. (1) Taking $\psi_2 := \varphi_2 - \varphi_1$ a.e., $\psi_3 := \varphi_3 - \varphi_2$ a.e.,..., we have $\psi_n \in \mathcal{L}^+$,

$$0 \leq \psi_n \text{ a.e.,} \quad \text{and} \quad \varphi_n = \varphi_1 + \psi_2 + \cdots + \psi_n \text{ a.e. for } n \geq 2.$$

We write $\psi_n = f_n - g_n$ a.e. for some $f_n \in \mathcal{H}^+$, $g_n \in \mathcal{H}^+_{\mathrm{int}}$. By Proposition 2.1.3, we can assume $g_n \geq 0$ and $0 \leq \int g_n \leq 2^{-n}$, and this implies $f_n = \psi_n + g_n \geq 0$ almost everywhere. Since $f_n^+ \in \mathcal{H}^+$, we may assume $f_n \geq 0$ "everywhere". We also write $\varphi_1 = f_1 - g_1$ a.e. for some $f_1 \in \mathcal{H}^+$ and $g_1 \in \mathcal{H}^+_{\mathrm{int}}$, so that

$$\varphi_n = (f_1 + \cdots + f_n) - (g_1 + \cdots + g_n) \text{ a.e.}$$

Since $\mathcal{H}^+ \ni f_1 + \cdots + f_n \nearrow f \in \mathcal{H}^+$, $\mathcal{H}^+_{\mathrm{int}} \ni g_1 + \cdots + g_n \nearrow g \in \mathcal{H}^+_{\mathrm{int}}$, and $\varphi = f - g$ a.e., we obtain $\varphi \in \mathcal{L}^+$ and

$$\int \varphi_n = \int (f_1 + \cdots + f_n) - \int (g_1 + \cdots + g_n) \to \int f - \int g = \int \varphi.$$

(2) It is immediate from definition.

(3) By (2), there exists $h_{m,n} \in \mathcal{L}$ such that $h_{m,n} \nearrow \varphi_n$ a.e. as $m \to \infty$. Then

$$\varphi = \sup_n \varphi_n = \sup_n \sup_m h_{m,n} = \sup_N \sup_{n \leq N} \sup_{m \leq N} h_{m,n} \text{ a.e.}$$

By Lemma 2.1.11 (1),

$$\mathcal{L} \ni \sup_{n \leq N} \sup_{m \leq N} h_{m,n} \nearrow_{n \to \infty} \varphi \text{ a.e.}$$

Then, by (1), we have $\varphi \in \mathcal{L}^+$ and

$$\int \sup_{n \leq N} \sup_{m \leq N} h_{m,n} \nearrow_{n \to \infty} \int \varphi.$$

Since $\sup_{n \leq N} \sup_{m \leq N} h_{m,n} \leq \varphi_N \leq \varphi$ a.e., it follows that $\int \varphi_N \nearrow_{N \to \infty} \int \varphi$. □

Proposition 2.1.13. *If* $\varphi \in \mathcal{M}$*, then* $\varphi^{\pm}, |\varphi| \in \mathcal{L}^+$*. In particular, every non-negative Daniell measurable function belongs to* $\mathcal{L}^+$*.*

Proof. We prove only $\varphi \in \mathcal{M}$ implies $|\varphi| \in \mathcal{L}^+$. The rest of proof is identical. By Proposition 2.1.5 (1), there exists $h_n \in \mathcal{H}$ such that $0 \leq h_n \to |\varphi|$. Then

$$\mathcal{H} \ni \inf_{N \leq n \leq M} h_n \searrow_{M \to \infty} \inf_{N \leq n} h_n \in \mathcal{L}.$$

Indeed,

$$\mathcal{H} \ni -\inf_{N \leq n \leq M} h_n \nearrow_{M \to \infty} \inf_{N \leq n} h_n \leq 0,$$

then $\infty < -\int \inf_{N\leq n\leq M} h_n \leq -\int \inf_{N\leq n} h_n \leq 0$ so that $-\inf_{N\leq n} h_n \in \mathcal{H}^+_{\rm int}$. We find $\inf_{N\leq n} h_n \in \mathcal{L}$.

Now,

$$\inf_{N\leq n} h_n \nearrow \liminf_{n\to\infty} h_n = |\varphi| \ a.e.$$

By Proposition 2.1.12, $|\varphi| \in \mathcal{L}^+$, the proof is compete. □

Proposition 2.1.14. *$\varphi \in \mathcal{L}$ if and only if $\varphi \in \mathcal{M}$ and $\int |\varphi| < \infty$.*

Proof. If $\varphi \in \mathcal{L}$, then $\varphi \in \mathcal{M}$ and $|\varphi| = \varphi^+ + \varphi^- \in \mathcal{L}$ by Lemma 2.1.11 (1).

Conversely, if $\varphi \in \mathcal{M}$, then $\varphi^\pm, |\varphi| \in \mathcal{L}^+$ by Proposition 2.1.13 to follow, and $\int |\varphi| < \infty$ implies $\int |\varphi^\pm| < \infty$. Therefore, $\varphi = \varphi^+ - \varphi^- \in \mathcal{L}$. □

Finally, we can prove the following convergence theorems:

Theorem 2.1.15 (Monotone Convergence Theorem)**.** *Let $\{\varphi_n\}_n$ be an increasing sequence of non-negative Daniell measurable functions such that $\varphi := \lim_{n\to\infty} \varphi_n$ exists. Then $\varphi \in \mathcal{L}^+$ and*

$$\int \varphi = \lim_{n\to\infty} \int \varphi_n.$$

Proof. The result follows from Propositions 2.1.13 and 2.1.12(3). □

Theorem 2.1.16 (Dominated Convergence Theorem)**.** *Let $\{\varphi_n\}_n$ be a sequence of Daniell measurable functions. Suppose that $\varphi := \lim_{n\to\infty} \varphi_n$ exists almost everywhere. If there exists an integrable function g such that $|\varphi_n| \leq g$ a.e. then φ is in $\mathcal{L}$ and*

$$\int \varphi = \lim_{n\to\infty} \int \varphi_n.$$

Proof. By Lemma 2.1.11 and Propositions 2.1.13 and 2.1.14, we see that $|\varphi_n|$, $|\inf_{N\leq n} \varphi_n|$, and $|\varphi|$ are in $\mathcal{L}$. By Proposition 2.1.12(1), we see

$$\inf_{N\leq n} \varphi_n \nearrow_{N\to\infty} \varphi \quad \text{and} \quad \lim_{N\to\infty} \int \inf_{N\leq n} \varphi_n = \int \varphi.$$

We apply the same argument to $-\varphi_n$;

$$\sup_{N\leq n} \varphi_n \searrow_{N\to\infty} \varphi \quad \text{and} \quad \lim_{N\to\infty} \int \sup_{N\leq n} \varphi_n = \int \varphi.$$

Combining these results and $\inf_{N\leq n} \varphi_n \leq \varphi_N \leq \sup_{N\leq n} \varphi_n$, we have $\int \varphi_n \to \int \varphi$. □

2.2 Stone condition

In general, Ω has no topological structure and continuity of functions defined on Ω is not even defined. We cannot even be sure that the constant functions are measurable. It was Stone who saw clearly how important it is to satisfy the condition:

Definition 2.2.1 (Stone condition)**.** We say that $\mathcal{H}$ satisfies the Stone condition if

$$h \in \mathcal{H} \Rightarrow h \wedge 1 \in \mathcal{H}. \tag{2.1}$$

Remark 2.2.2. This condition guarantees the measurability of the pointwise product of measurable functions. It should be noted that the Stone condition in [66] includes the assumption of σ-finiteness of the whole space Ω.

Let $\mathcal{D}_0$ be the set of all measurable sets with finite integral, i.e., $\int \chi_A < \infty$ for $A \in \mathcal{D}$ and we call them integrable sets. The space $\mathcal{H}(\mathcal{D}_0)$ denotes the set of all $\mathcal{D}_0$-simple functions. We see that $\mathcal{H}(\mathcal{D}_0)$ is an elementary function space satisfying the Stone condition. Moreover, we define $\mu(A) := \int \chi_A$ for any $A \in \mathcal{D}$, and we define

$$\int h\,d\mu := \sum_{k=1}^{N} a_k \mu(A_k), \quad \text{for} \quad h(x) = \sum_{k=1}^{N} a_k \chi_{A_k}(x), \quad a_k \in \mathbb{R}, A_k \in \mathcal{D}_0.$$

Then $(\Omega, \mathcal{H}(\mathcal{D}_0), \int d\mu)$ is a Daniell system satisfying the Stone condition. Invoking the Daniell scheme, we obtain the Daniell integrable function space and the Daniell measurable functions from this simple function space. We write these extended spaces by $\mathcal{L}(\mathcal{D}_0), \mathcal{L}^+(\mathcal{D}_0)$, and $\mathcal{M}(\mathcal{D}_0)$. It is easy to see $\mathcal{H}(\mathcal{D}_0) \subset \mathcal{L}$, $\mathcal{H}^+(\mathcal{D}_0) \subset \mathcal{L}^+$, and $\mathcal{H}^+_{\text{int}}(\mathcal{D}_0) \subset \mathcal{L}$ by the Dominated Convergence Theorem.

Proposition 2.2.3. *The null set induced by $\mathcal{H}(\mathcal{D}_0)$ is the same as the null set induced by the original elemetary space $\mathcal{H}$.*

Proof. Suppose that $Z \subset \{f = +\infty\}$ for some $f \in \mathcal{H}^+_{\text{int}}(\mathcal{D}_0)$. Since we have $f \in \mathcal{L}$ and f is finite almost everywhere in the sense $\int$, we see that Z is a null set in the sense $\mathcal{H}$.

Conversely, let Z is a null set induced by $\mathcal{H}$. Then by Proposition 2.1.6(2), we have $\chi_Z = 0$ a.e. and this implies $\chi_Z \in \mathcal{L}$, that is Z is an integrable set. Since $Z \in \mathcal{D}_0$, we see $\chi_Z \in \mathcal{H}(\mathcal{D}_0)$. By the Monotone Convergence Theorem, we have $\infty\chi_Z \in \mathcal{H}^+_{\text{int}}(\mathcal{D}_0)$. We observe that $Z = \{\infty\chi_Z = +\infty\}$ and this implies Z is a null set induced by $\mathcal{H}(\mathcal{D}_0)$. $\square$

Theorem 2.2.4. *If $\mathcal{H}$ satisfies the Stone condition, then the following assertions hold:*

(1) *For each $\varphi \in \mathcal{L}$ and $\alpha > 0$, $\{\varphi > \alpha\} \in \mathcal{D}_0$ holds,*

(2) $\mathcal{L}(\mathcal{D}_0) = \mathcal{L}$,

(3) $\mathcal{L}^+(\mathcal{D}_0) = \mathcal{L}^+$,

(4) $\mathcal{M}(\mathcal{D}_0) = \mathcal{M}$.

Proof. (1) It is clear that $f \in \mathcal{M} \Rightarrow f \wedge 1 \in \mathcal{M}$ by the Stone condition, and hence we see $\varphi \wedge 1 \in \mathcal{L}$ for all $\varphi \in \mathcal{L}$. Defining $\psi := \varphi - \varphi \wedge 1$ a.e., we have $\mathcal{L} \ni \psi \geq 0$ a.e. Let $A := \{\varphi > 1\} = \{\psi > 0\}$. Then $0 \leq (n\psi) \wedge 1 \nearrow_{n\to\infty} \chi_A$ a.e. and this implies $\chi_A \in \mathcal{M}$. Since $\int \chi_A \leq \int \varphi < \infty$, it follows $A \in \mathcal{D}_0$. Applying the same argument, we have $\{\varphi > \alpha\} \in \mathcal{D}_0$.

(2) We have only to prove $\mathcal{L}(\mathcal{D}_0) \supset \mathcal{L}$. Let $\varphi \in \mathcal{L}$. By (1), we have $A_{n,k} := \{k/2^n < \varphi\} \in \mathcal{D}_0$ for $n, k = 0, 1, \ldots$. Since $\frac{k}{2^n}\chi_{A_{n,k}} \leq \varphi^+$, we verify $f_n := 2^{-n}\sum_k \chi_{A_{n,k}} \in \mathcal{L}^+$. Since $0 \leq f_n \leq \varphi^+$ and $f_n \nearrow \varphi^+$ a.e., it follows $\varphi^+ \in \mathcal{L}(\mathcal{D}_0)$. By the same way, we see $\varphi^- \in \mathcal{L}(\mathcal{D}_0)$, and hence $\varphi \in \mathcal{L}(\mathcal{D}_0)$.

(3) and (4) follow from an easy limiting argument. □

Suppose that $\mathcal{H}$ satisfies the Stone condition. For any $\varphi, \psi \in \mathcal{M}$, by Theorem 2.2.4, there exist $h_n, k_n \in \mathcal{H}(\mathcal{D}_0)$ such that $h_n \to \varphi$ a.e. and that $k_n \to \psi$ a.e. It is clear that $h_n \cdot k_n \in \mathcal{H}(\mathcal{D}_0)$, and this implies $h_n \cdot k_n \to \varphi \cdot \psi \in \mathcal{M}$.

Theorem 2.2.5 (Riesz-Fisher [66])**.** *The Daniell integrable function space $\mathcal{L}$ is a complete normed space over $\mathbb{R}$ with the norm,*

$$\|f\| := \int |f|.$$

That is to say, if $\{f_n\} \subset \mathcal{L}$ is a Cauchy sequence $\|f_n - f_m\| \to 0$, then $\|f_n - f\| \to 0$ for some $f \in \mathcal{L}$. Suppose in addition that $\mathcal{H}$ satisfies the Stone condition. Then the square integrable function space can be defined

$$\mathcal{L}^2 := \{f \in \mathcal{M} : \int |f|^2 < \infty\},$$

and $\mathcal{L}^2$ is a complete inner-product space over $\mathbb{R}$ with the inner-product,

$$(f, g) := \int fg.$$

The proof can be found in [66, 74].

Chapter 3

Radon-Nikodym Theorem

In one of the most excellent textbooks, Halmos [34] considered mainly the σ-ring measure spaces, but when one considers the Radon-Nikodym theorem he assumes the σ-finiteness for underlying spaces and this automatically implies its measurability. Many authors believe that the measurability of the whole space is natural when we consider the Radon-Nikodym theorem and related results. Many people have already studied what guarantees the existence of density and extended the results to the framework of Daniell scheme to some extent; see [8, 56, 66, 74, 78] among others. In this chapter, we give a more comprehensive discussion on the Radon-Nikodym theorem based on a type of Daniell scheme slightly different from the above literatures. In [8, 66, 74, 78], they studied different types of Daniell schemes assuming the underlying space Ω is *measurable.* In most textbooks of analysis, it was essential to assume that Ω is σ-finite. Further, in [56, 63, 78], the authors considered non-σ-finite cases and found necessary and sufficient conditions, which is so-called *localizability* of measure μ, under which the Radon-Nikodym theorem holds, where they assumed that the whole space Ω is measurable. However, if Ω is not measurable, it does not seems to be essential to compel to ensure the localizability of μ. We shall investigate that when the measure μ is not necessarily localizable, the Radon-Nikodym derivative fails to be a function, but forms a particular family of functions, which is called a *folder*, as a consequence, we shall newly formulate the Radon-Nikodym density folder, which allows us to obtain the Radon-Nikodym theorem on non-σ-finite and/or σ-ring measure spaces, in other words, we do not have to work on the σ-finite measure spaces by the framework of the Daniell integral.

3.1 Folders

In this section, we introduce the notion of folders so that we can describe the density of the Radon-Nikodym theorem.

Definition 3.1.1. (1) A subset $E \subset \Omega$ is said to be an *elementary measurable set* if $\chi_E \in \mathcal{H}^+$ and the totality of all elementary measurable sets is denoted by $\mathcal{E}$. (2) A subset E is said to be an *elementary integrable set* if there exists $\varphi \in \mathcal{H}$ such that $E = \{x \in \Omega \ : \varphi(x) > 1\}$, and the totality of all elementary integrable sets is denoted by $\mathcal{E}_0$.

Remark 3.1.2. Since $\mathcal{H}$ and $\mathcal{H}^+$ are closed under $\vee$, $\wedge$, we deduce $\mathcal{E}_0$ and $\mathcal{E}$ are closed under $\cup$, $\cap$. Further, all elementary measurable(integrable) sets are measurable(integrable) with respect to all integral on $\mathcal{H}$.

Proposition 3.1.3. *Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of measurable subsets.*

(1) *If $\mathcal{E} \ni E_n \nearrow E$ then $E \in \mathcal{E}$.*

(2) *The following assertions are equivalent:*
  (a) $E \in \mathcal{E}$,
  (b) *there exists $\varphi \in \mathcal{H}^+$ such that $E = \{\varphi > 1\}$,*
  (c) *there exists $\varphi \in \mathcal{H}^+$ such that $E = \{\varphi > 0\}$.*

(3) $\mathcal{E}_0 \subset \mathcal{E}$.

(4) *For any $E \in \mathcal{E}$, there exists $E_n \in \mathcal{E}_0$ $(n = 1, 2, \cdots)$ such that $E_n \nearrow E$.*

Proof. (1) It follows easily from $\mathcal{H}^+ \ni \chi_{E_n} \nearrow \chi_E \in \mathcal{H}^+$.

(2) We may assume $\varphi \in \mathcal{H}^+$ to be non-negative in (b) and (c), since $\varphi^+ = \varphi \vee 0$ is in $\mathcal{H}^+$ for all $\varphi \in \mathcal{H}^+$. (a) $\Rightarrow$ (b): If $E \in \mathcal{E}$, then $\chi_E \in \mathcal{H}^+$, and hence it suffices to set $\varphi = 2\chi_E$. (b) $\Rightarrow$ (c): For $\varphi \in \mathcal{H}^+$ of (b), there exists $h_n \in \mathcal{H}; h_n \nearrow \varphi$. Then $h_n - h_n \wedge 1$ is in $\mathcal{H}$ and converges to $\varphi - \varphi \wedge 1$. Since $h_n - h_n \wedge 1 = (h_n - 1) \vee 0$, we deduce that $h_n - h_n \wedge 1$ converges increasingly to $(\varphi - 1) \wedge 0 = \varphi - \varphi \wedge 1$, and this implies $\varphi - \varphi \wedge 1 \in \mathcal{H}^+$. Then we obtain $E = \{\varphi - \varphi \wedge 1 > 0\}$. (c) $\Rightarrow$ (a): Since $(n\varphi) \wedge 1 \in \mathcal{H}^+$, it follows that $(n\varphi) \wedge 1 \nearrow \chi_E$ and that $\chi_E \in \mathcal{H}^+$.

(3) It follows from (2) and $\mathcal{H}^+ \supset \mathcal{H}$.

(4) For any $E \in \mathcal{E}$, there exists $\varphi \in \mathcal{H}^+$ such that $E = \{\varphi > 1\}$ by (2). Choose $\varphi_n \in \mathcal{H}$ so that $\varphi_n \nearrow \varphi$ and put $E_n := \{\varphi_n > 1\}$, then $E_n \in \mathcal{E}_0$ and $E_n \nearrow E$. □

Proposition 3.1.4. (1) *If $\varphi \in \mathcal{M}$, then $\{\varphi \neq 0\} \in \mathcal{D}$.*

(2) *For any $D \in \mathcal{D}$, there exists $E \in \mathcal{E}$ such that $D \subset E$.*

Remark 3.1.5. Recall that $\mathcal{M}$ is the set of all $\overline{\mathbb{R}}$-valued functions φ, defined a.e. on Ω such that φ is an a.e. limit of a sequence of elementary functions [66]. Since $\varphi \in \mathcal{M}$ is defined almost everywhere, we have

$$\{\varphi \neq 0\} = \{|\varphi| > 0\} \cup \{x \in \Omega \ ; \varphi(x) \text{ is undefined}\},$$

$\{\varphi \neq 0\} = \{|\varphi| > 0\}$ is not the case.

Proof of Proposition 3.1.4. (1) If $\varphi \in \mathcal{M}$, then $\infty\chi_{\{\varphi\neq 0\}} = \infty|\varphi|$ a.e. and $\infty|\varphi| = \lim_{n\to\infty} n\chi_{\{\varphi\neq 0\}} \in \mathcal{L}^+$. Noting $\chi_{\{\varphi\neq 0\}} = (\infty\chi_{\{\varphi\neq 0\}}) \wedge 1$, we deduce $(\infty\chi_{\{\varphi\neq 0\}}) \wedge 1 \in \mathcal{L}^+$

by the Stone condition. This implies $\chi_{\{\varphi\neq 0\}} \in \mathcal{L}^+$, and hence $\{\varphi \neq 0\}$ is a measurable set.

(2) Since D is measurable, χ_D is in $\mathcal{L}^+$ by Proposition 2.1.13, and hence there exists $f_n \in \mathcal{H}^+$ such that $0 \leq f_n \searrow \chi_D$ holds outside some null set Z. There exists $0 \leq f_0 \in \mathcal{H}^+_{\mathrm{int}}$ such that $Z \subset \{f_0 = +\infty\}$, and hence $\chi_D \leq f_1 + f_0$ holds everywhere, because if $x \in Z$ then $\chi_D(x) \leq f_1(x) + f_0(x) = f_1(x) + \infty$. Since $f_1 + f_0 \in \mathcal{H}^+$, $E := \{f_1 + f_0 > 0\}$ is a desired elementary measurable set. □

Definition 3.1.6. (1) Let $(f_E)_{E\in\mathcal{E}}$ be a family of functions defined a.e. We call it a *folder*, if

$$f_F\chi_E = f_{E\cap F} \text{ a.e.} \tag{3.1}$$

for any $E, F \in \mathcal{E}$, and write $\langle f\rangle := (f_E)_{E\in\mathcal{E}}$. Each f_E is called a *file*.

(2) If $(f_E)_{E\in\mathcal{E}_0}$ satisfies the condition (3.1), for any $E, F \in \mathcal{E}_0$, then we also denote this system by $\langle f\rangle$ and call it a *prefolder*. Each f_E is called a file, too.

(3) Let $\langle f\rangle, \langle g\rangle$ be folders. Then, we say that $\langle f\rangle = (\text{or } \leq)\langle g\rangle$ a.e. if $f_E = (\text{or } \leq) g_E$ a.e. for all $E \in \mathcal{E}$. Similarly, for prefolders $\langle f\rangle$ and $\langle g\rangle$, we define $\langle f\rangle = (\text{or } \leq)\langle g\rangle$ a.e. analogously.

Let $(f_E)_{E\in\mathcal{E}}$ be a folder. Obviously, for $E \in \mathcal{E}$, by putting $E = F$ in (3.1), we have $f_E\chi_E = f_E$ a.e. and for $E, F \in \mathcal{E}$,

$$f_E\chi_F = f_F\chi_E = f_{E\cap F} \tag{3.2}$$

a.e. holds. In addition, for a folder $(f_E)_{E\in\mathcal{E}}$, the restriction $\langle f\rangle|_{\mathcal{E}_0} = (f_E)_{E\in\mathcal{E}_0}$ is a prefolder.

Example 1. A mapping from $E \in \mathcal{E}$ to the indicator function $\chi_E \in \mathcal{M}$ is a folder. We denote this folder by $\langle I\rangle$ and call it the *indicator folder*.

Given any prefolder $\langle h\rangle$, it can be extended uniquely to the folder $\langle f\rangle$ as in the following sense:

Proposition 3.1.7. *For any prefolder $\langle h\rangle$, there exists a folder $\langle f\rangle$ such that:*

(1) $\langle f\rangle|_{\mathcal{E}_0} = \langle h\rangle$ *a.e.*

(2) *and if there exists a folder $\langle g\rangle$ such that $\langle h\rangle = \langle g\rangle|_{\mathcal{E}_0}$ a.e., then $\langle f\rangle = \langle g\rangle$ a.e.*

Proof. (1) For any $E \in \mathcal{E}$, there exists $E_n \in \mathcal{E}_0$ such that $E_n \nearrow E$ by Proposition 3.1.3 (4). Let $E_0 := \emptyset$, and we set a function defined a.e. by:

$$f_E := \sum_{n=1}^{\infty} h_{E_n}\chi_{(E_n\setminus E_{n-1})}. \tag{3.3}$$

For any sequence $\mathcal{E}_0 \ni F_m \nearrow E$, we have

$$\begin{aligned} f_E \chi_{F_m} &= \sum_n h_{E_n} \chi_{F_m} \chi_{(E_n \setminus E_{n-1})} \\ &= \sum_n h_{F_m} \chi_{E_n} \chi_{(E_n \setminus E_{n-1})} \\ &= h_{F_m} \chi_E = h_{F_m} \chi_{F_m} \chi_E = h_{F_m} \chi_{F_m} = h_{F_m}, \end{aligned}$$

where the above equalities hold a.e., and we obtain $f_E = \lim_m h_{F_m}$ a.e. This implies $f_E = \lim_n h_{E_n}$ a.e. holds independently of the choice of a sequence E_n.

Now, we shall prove that $(f_E)_{E \in \mathcal{E}}$ forms a folder. Let $E, F \in \mathcal{E}$, and let $E_n, F_m \in \mathcal{E}_0$ be approximating sequences of E, F, respectively. Then $E_n \cap F_m \nearrow E \cap F$ as $m, n \to \infty$. Since $h_{F_m} \chi_{E_n} = h_{E_n \cap F_m}$ a.e., we have $f_F \chi_E = f_{E \cap F}$ a.e. as $m, n \to \infty$, this implies $E \mapsto h_E$ is a folder, which we denote it by $\langle f \rangle$. Obviously $\langle f \rangle|_{\mathcal{E}_0} = \langle h \rangle$ a.e. follows from (3.3).

(2) For any $E \in \mathcal{E}$, choose $E_n \in \mathcal{E}_0$ so that $E_n \nearrow E$. Then $h_{E_n} = g_{E_n}$ a.e., and hence $f_E = g_E$ a.e. as $n \to \infty$. □

Remark 3.1.8. Let φ be a function defined a.e. and $\langle h \rangle$ be a folder. Then $\mathcal{E} \ni E \mapsto \varphi h_E$ is also a folder. We denote this folder by $\varphi \langle h \rangle$. In particular, if we put $\varphi = \chi_F$ $(F \in \mathcal{E})$, then we have $\chi_F \langle h \rangle = h_F \langle I \rangle$ a.e.

Definition 3.1.9. We say $\langle f \rangle$ is a *complete* folder if there exists $E_0 \in \mathcal{E}$ such that $f_F = f_{E_0 \cap F}$ a.e. holds for any $F \in \mathcal{E}$. The file f_{E_0} is called a complete file of the folder $\langle f \rangle$.

Remark 3.1.10. (1) We say that $\mathcal{H}$ is σ-finite if $1 \in \mathcal{H}^+$ (cf. [74]). This condition is equivalent to $\Omega \in \mathcal{E}$ so that if $\mathcal{H}$ is σ-finite then all folders are complete because we can choose the complete file as h_Ω whenever we are given a folder $\langle h \rangle$.

(2) By definition, $f_{E_0 \cap F} = f_{E_0} \chi_F$ a.e. holds, and this implies the complete folder satisfies $\langle f \rangle = f_{E_0} \langle I \rangle = \chi_{E_0} \langle f \rangle$ a.e. The set $E_0 \in \mathcal{E}$, corresponding to the complete file f_{E_0}, is not unique but the complete file is unique as a function as follows:

Proposition 3.1.11. *Let $\langle f \rangle, \langle g \rangle$ be folders. Suppose that $\langle f \rangle = \langle g \rangle$ a.e., and that $\langle f \rangle$ is complete. Then*

(1) *$\langle g \rangle$ is also complete.*

(2) *Let f_{E_0}, g_{E_1} be complete files of $\langle f \rangle$, $\langle g \rangle$, respectively. Then it follows $f_{E_0} = g_{E_1}$ a.e.*

Proof. (1) We can choose f_{E_0} as a complete file of $\langle g \rangle$.

(2) Since $f_{E_0} \chi_F = g_{E_1} \chi_F$ a.e. for any $F \in \mathcal{E}$, taking $F = E_0 \cup E_1$, we obtain $f_{E_0} = g_{E_1}$ a.e. □

Proposition 3.1.11 says a complete folder $\langle f \rangle$ can be naturally identified with its complete file f_{E_0}. Hereafter, unless stated otherwise, $\langle f \rangle$ is abbreviated to f_{E_0}. For example, $\langle f \rangle \in \mathcal{L}$ means $f_{E_0} \in \mathcal{L}$. In general, the indicator folder $\langle I \rangle$ is not necessarily complete.

We say a folder(or prefolder) $\langle h \rangle$ is *measurable* if all its files are measurable. Note that if $\langle f \rangle$ and $\langle g \rangle$ are folders (or prefolders) with $\langle f \rangle = \langle g \rangle$ a.e. and $\langle f \rangle$ is measurable, then $\langle g \rangle$ is also measurable.

Proposition 3.1.12. (1) *A folder $\langle h \rangle$ is measurable if and only if the prefolder $\langle h \rangle|_{\mathcal{E}_0}$ is measurable.*

(2) *Let $\langle h \rangle$ be a measurable folder and $\varphi \in \mathcal{M}$. Then $\varphi\langle h \rangle$ is measurable and complete.*

Proof. (1) If $\langle h \rangle$ is measurable, then $\langle h \rangle|_{\mathcal{E}_0}$ is obviously measurable. Conversely, suppose that $\langle h \rangle|_{\mathcal{E}_0}$ is measurable. Since $\mathcal{M}$ is closed under taking the a.e. limit, the measurability of $\langle h \rangle$ follows from the proof of Proposition 3.1.7.

(2) By the Stone condition, $\mathcal{M}$ is closed under multiplication, and this implies each file φh_E of $\varphi\langle h \rangle$ is measurable. We choose $E_0 \in \mathcal{E}; \{\varphi \neq 0\} \subset E_0$ by Proposition 3.1.4, then, $\varphi h_F = \varphi \chi_{E_0} h_F = \varphi h_{E_0 \cap F}$ a.e. for any $F \in \mathcal{E}$. □

3.2 Density

Now we describe how to define the linear functional when we are given a folder $\langle f \rangle$. In this section, we denote the characteristic function of the set A by $\chi(A)$ instead of χ_A, when confusion occurs.

Definition 3.2.1. We say the measurable folder $\langle h \rangle$ is a *density folder*, if for every $f \in \mathcal{H}$, $f\langle h \rangle$ is integrable.

Given a density folder $\langle h \rangle$ and $f \in \mathcal{H}$, the folder $f\langle h \rangle$ is complete by Proposition 3.1.12, where its complete file is fh_{E_0}; there exists $E_0 \in \mathcal{E}$ such that $\{f \neq 0\} \subset E_0$. Note that E_0 depends on f. Now we define the integral $\int f\langle h \rangle := \int fh_{E_0}$. We can show that it does not depend the choice of $E_0 \in \mathcal{E}$ containing the carrier of f.

Proposition 3.2.2. *Let $\langle h \rangle$ be a folder. The mapping $P : \mathcal{H} \to \mathbb{R}$ with $P(f) = \int f\langle h \rangle$ is linear.*

Proof. Let $\varphi, \psi \in \mathcal{H}$. For any $E \in \mathcal{E}$, both φh_E and ψh_E are finite almost everywhere. It follows that $(\varphi + \psi)h_E = \varphi h_E + \psi h_E$ a.e., and hence this implies $(\varphi + \psi)\langle h \rangle = \varphi\langle h \rangle + \psi\langle h \rangle$ a.e. The additivity of P follows from that of $\int$. Homogeneity is obvious. □

Proposition 3.2.3. *Let $\langle h \rangle$ be a density folder. Then every file of the prefolder $\langle h \rangle_{\mathcal{E}_0}$ is integrable, that is, function h_E is integrable for any $E \in \mathcal{E}_0$.*

Proof. For $E \in \mathcal{E}_0$, there exists $\varphi \in \mathcal{H}$ such that $E = \{\varphi > 1\}$, where φ may be assumed non-negative. By the Stone condition, we have $\varphi \wedge 1 \in \mathcal{H}$ and hence, from the definition of E, $(\varphi \wedge 1)\chi_E = \chi_E$. Then, $(\varphi \wedge 1)h_E = h_E$ a.e. Since the left-hand-side is integrable, h_E is integrable. Indeed, the Stone condition implies $(\varphi \wedge 1) \in \mathcal{H}$. Since $\langle h \rangle$ is a density folder, it follows that $(\varphi \wedge 1)h_E$ is integrable. Thus, by Lemma 2.1.11 (2) h_E is integrable. □

Corollary 3.2.4. (1) *Every file of a density folder $\langle h \rangle$ is finite almost everywhere.*

(2) *If $\varphi_n \in \mathcal{H}$; $\varphi_n \searrow 0$ then $P(\varphi_n) \to 0$, where P is a linear mapping from Proposition 3.2.2.*

Proof. (1) Let $E \in \mathcal{E}$. We aim to prove that h_E is finite a.e. By Proposition 3.1.3(4), we can choose $\mathcal{E}_0 \ni E_n \nearrow E$. Since $h_{E_n} = h_E \chi_{E_n}$ a.e., we have $\chi\{|h_{E_n}| = +\infty\} = \chi\{|h_E| = +\infty)\}\chi_{E_n}$ a.e. and the left-hand-side $= 0$ a.e. by Proposition 3.2.3. Letting $n \to \infty$, we obtain $\chi\{|h_E| = +\infty\}\chi_E = \chi\{|h_E| = +\infty\} = 0$ a.e.

(2) Let $\varphi_n \in \mathcal{H}; \varphi_n \searrow 0$ and $E \in \mathcal{E}$. By (1), it follows that $|\varphi_n h_E| \to 0$ $(n \to \infty)$ a.e. Since we have $|\varphi_n h_E| \le |\varphi_1 h_E|$ a.e. and $\varphi_1 h_E \in \mathcal{L}$, the Dominated Convergence Theorem gives $P(\varphi_n) = \int \varphi_n \langle h \rangle \to 0$. □

Theorem 3.2.5. (1) *Let $\langle h \rangle$ be a non-negative density folder. If another folder $\langle g \rangle$ satisfies $\langle h \rangle = \langle g \rangle$ a.e., then $\langle g \rangle$ is a density folder. Furthermore, for any $f \in \mathcal{H}$ it follows that $\int f \langle h \rangle = \int f \langle g \rangle$.*

(2) *Conversely, if non-negative density folders $\langle h \rangle, \langle g \rangle$ satisfy $\int f \langle h \rangle = \int f \langle g \rangle$ for any $f \in \mathcal{H}$, then it follows $\langle h \rangle = \langle g \rangle$ a.e.*

Proof. (1) is clear. We will prove (2). Note that

$$\int f \langle h \rangle = \int f \langle g \rangle \tag{3.4}$$

remains valid for $f \in \mathcal{H}^+$. By Proposition 3.2.3, for any $E \in \mathcal{E}_0$, h_E and g_E are integrable. Also h_E and g_E vanish almost everywhere outside E, and this implies $\{h_E - g_E > 0\} \subset E$ a.e. Hence, $\chi\{h_E - g_E > 0\} \in \mathcal{L}$. By Corollary 2.1.10, there exists $0 \le f_n \in \mathcal{H}^+_{\text{int}}$ such that $f_n \searrow \chi\{h_E - g_E > 0\}$ a.e. and hence,

$$f_n \wedge \chi_E \searrow \chi\{h_E - g_E > 0\}\chi_E = \chi\{h_E - g_E > 0\} \ \ (a.e.).$$

We write $|\langle h \rangle| = (|h_E|)_{E \in \mathcal{E}}$. Since

$$|(f_n \wedge \chi_E)\langle h \rangle| = (f_n \wedge \chi_E)|\langle h \rangle| \le \chi_E |\langle h \rangle| = |h_E \langle I \rangle| (\in \mathcal{L})$$

almost everywhere, we see $(f_n \wedge \chi_E)\langle h \rangle$ is integrable for all $n \in \mathbb{N}$. Since $f_n \wedge \chi_E \in \mathcal{H}^+_{\text{int}}$,

$$\int f_n \wedge \chi_E \langle h \rangle = \int f_n \wedge \chi_E \langle g \rangle$$

holds by (3.4). Thus, the Dominated Convergence Theorem gives

$$\int \chi\{h_E - g_E > 0\} h_E = \int \chi\{h_E - g_E > 0\} g_E < \infty.$$

This implies $\int \chi\{h_E - g_E > 0\}(h_E - g_E) = 0$, and hence we obtain $h_E \leq g_E$ a.e. By a similar argument, we have the opposite inequality. Hence it follows $h_E = g_E$ a.e. for any $E \in \mathcal{E}_0$. Therefore, we obtain $\langle h \rangle = \langle g \rangle$ a.e. by Proposition 3.1.7. □

Combining Proposition 3.2.3 and Corollary 3.2.4, we can easily see the following lemma:

Lemma 3.2.6. *If the density $\langle h \rangle$ is non-negative, that is, for all $E \in \mathcal{E}$, $h_E \geq 0$ (a.e.), then $P: \mathcal{H} \to \mathbb{R}$ is a Daniell integral on $\mathcal{H}$.*

In the following we shall assume that the density folder $\langle h \rangle$ is non-negative. In particular, the indicator folder $\langle I \rangle$ is a non-negative density, and the Daniell integral P induced by $\langle I \rangle$ is nothing else but $\int$. We shall say that $\langle h \rangle$ *is a non-negative density of* P, if the non-negative density $\langle h \rangle$ defines a Daniell integral in the sense just described in Lemma 3.2.6.

Hereafter, we consider several integrals at the same time. The null sets and the integrabilities depend on each integral, and thereby we will use $\mathcal{H}^+_{\mathrm{int}}(P)$, P-null set, and P-*a.e.* and so on. For simplicity, we may use "a.e." for "$\int$-a.e.".

Proposition 3.2.7. *Let $\langle h \rangle$ be a non-negative density folder. We set $P(f) = \int f\langle h \rangle$ for $f \in \mathcal{H}$.*

(1) *Let $f \in \mathcal{H}^+$. Then $f\langle h \rangle$ belongs to $\mathcal{L}^+$, and $P(f) = \int f\langle h \rangle$ remains valid for $f \in \mathcal{H}^+$.*

(2) *If $f \in \mathcal{H}^+_{\mathrm{int}}(P)$ if and only if $f\langle h \rangle \in \mathcal{L}$.*

(3) *If $Z \subset \Omega$ is an $\int$-null set, then Z is P-null.*

Proof. (1) and (2) follow from definition and convergence theorems.

(3) Let Z be a null set. There exists $f \in \mathcal{H}^+_{\mathrm{int}}$ such that $Z \subset \{f = +\infty\}$.

We claim $\{f = +\infty\}$ is P-null. To do this, fix $E \in \mathcal{E}$ arbitrarily and choose $\mathcal{E}_0 \ni E_n \nearrow E$. We observe that $\mathcal{E} \ni \{f > m\} \searrow \{f = +\infty\} \in \mathcal{D}$. Then we have

$$\mathcal{E} \ni E_n \cap \{f > m\} \searrow E_n \cap \{f = +\infty\} \nearrow E \cap \{f = +\infty\},$$

and $\chi(E_n \cap \{f > m\})$ is integrable. We apply the Dominated Convergence Theorem to $P(\chi(E_n \cap \{f > m\})) = \int \chi(E_n \cap \{f > m\})\langle h \rangle$, and we obtain

$$P(\chi(E \cap \{f > m\})) = \int \chi(E \cap \{f > m\})\langle h \rangle.$$

We choose E containing $\{f = +\infty\}$. Then the Monotone Convergence Theorem gives

$$P(\chi_{\{f=+\infty\}}) = \int \chi_{\{f=+\infty\}}\langle h \rangle = 0.$$

It follows that $\{f = +\infty\}$ is P-null. □

3.3 Radon-Nikodym Theorem

Now we formulate the Radon-Nikodym theorem. Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system satisfying the Stone condition. We consider another Daniell integral Q on $\mathcal{H}$.

Definition 3.3.1. A Daniell integral Q on $\mathcal{H}$ is said to be absolutely continuous (with respect to $\int$) if any null set is a Q-null set, and we denote $Q \ll \int$.

Proposition 3.2.7(3) implies that the Daniell integral having non-negative density $\langle h \rangle$ is absolutely continuous. The Radon-Nikodym theorem asserts its converse as follows:

Theorem 3.3.2. *Suppose that $\mathcal{H}$ satisfies the Stone condition and that Q is a Daniell integral on $\mathcal{H}$.*

(1) *If Q is absolutely continuous, then Q has a non-negative density $\langle h \rangle$.*

(2) *This density is unique in the a.e. sense.*

To prove this, we need some lemmas. We will first show the following proposition:

Proposition 3.3.3. *Let Q be a Daniell integral on $\mathcal{H}$ and suppose that $Q \ll \int$.*

(1) *If we define $\left(\int +Q\right)(f) := \int f + Q(f)$ for $f \in \mathcal{H}$, then $(\int +Q)$ is a Daniell integral on $\mathcal{H}$.*

(2) *$(\int +Q)(f) = \int f + Q(f)$ holds for $f \in \mathcal{H}^+$.*

(3) *Z is $(\int +Q)$-null set if and only if Z is null.*

(4) *$\mathcal{M}(\int +Q) = \mathcal{M}$.*

(5) *If $\varphi \in \mathcal{L}^+(\int +Q)$, then $\varphi \in \mathcal{L}^+ \cap \mathcal{L}^+(Q)$ and $(\int +Q)(\varphi) = \int \varphi + Q(\varphi)$.*

Proof. (1) and (2) are evident from the definition of $(\int +Q)$.

(3) $(\Rightarrow)$ is clear. $(\Leftarrow)$: if Z is null, then Z is Q-null, and hence there exist $f \in \mathcal{H}^+_{\text{int}}$ and $g \in \mathcal{H}^+_{\text{int}}(Q)$ such that $Z \subset \{f, g = +\infty\} = \{f \wedge g = +\infty\}$. Since $f \wedge g \in \mathcal{H}^+$ and

$$\left(\int +Q\right) f \wedge g \leq \int f + Q(g) < \infty,$$

which proves Z is $(\int +Q)$-null.

(4) is clear by (3).

(5) If $\varphi \in \mathcal{L}^+(\int +Q)$, then $\varphi = f - g \quad ((\int +Q)\text{-}a.e.)$ for some $f \in \mathcal{H}^+$, $g \in \mathcal{H}^+_{\text{int}}(\int +Q)$. By (2), g is in $\mathcal{H}^+_{\text{int}} \cap \mathcal{H}^+_{\text{int}}(Q)$. This implies $\varphi = f - g$ a.e. $(g \in \mathcal{H}^+_{\text{int}})$ and $\varphi = f - g \quad (Q\text{-}a.e.)$ $(g \in \mathcal{H}^+_{\text{int}}(Q))$ by (3). Thus, we see $\varphi \in \mathcal{L}^+$ and $\varphi \in \mathcal{L}^+(Q)$. The last equation easily follows from the definition of $(\int +Q)$. ☐

We recall that $\mathcal{L}^2$ is the set of all measurable functions φ for which $|\varphi|^2 \in \mathcal{L}$, see Section 2.2. The set $\mathcal{L}^2$ is a Hilbert space with respect to $(f,g) = \int fg$.

Lemma 3.3.4. *Suppose that $\mathcal{H}$ satisfies the Stone condition and Q is an absolutely continuous Daniell integral on $\mathcal{H}$. Let $E \in \mathcal{E}_0$.*

(1) *There exists a non-negative measurable function h_E such that $h_E = h_E\chi_E$ a.e.*

(2) *For any $f \in \mathcal{L}^+(\int +Q)$, it follows that $fh_E \in \mathcal{L}^+$ and that $Q(f\chi_E) = \int fh_E$. Furthermore, this h_E is unique in the a.e. sense.*

Proof. Let us fix $E \in \mathcal{E}_0$. For any $f \in \mathcal{L}^2(\int +Q)$, $f\chi_E$ is measurable and $f^2\chi_E \leq f^2$, and hence $f\chi_E \in \mathcal{L}^2(\int +Q)$. By Proposition 3.3.3 (5), we see $f\chi_E \in \mathcal{L}^2(Q)$. In general, we can show that $f \in \mathcal{L}^2(Q)$ if and only if $f \in \mathcal{M}(Q)$ and $Q(|f|^2) < \infty$. From this, f is Q-measurable, and hence by Schwarz's inequality,

$$\begin{aligned} |Q(|f|\chi_E)|^2 &\leq Q(f^2) \cdot Q(\chi_E^2) \\ &\leq M \cdot \left(\int +Q\right) f^2 \quad (M := Q(\chi_E)) \\ &= M \cdot \|f\|^2_{\mathcal{L}^2(\int +Q)} < \infty. \end{aligned}$$

This implies that $F(f) := Q(f\chi_E)$ is a bounded linear functional on $\mathcal{L}^2(Q)$, and on $\mathcal{L}^2(\int +Q)$.

By Riesz's Representation Theorem there exists a unique $g_E \in \mathcal{L}^2(\int +Q)$ for which $fg_E \in \mathcal{L}(\int +Q)$ and

$$Q(f\chi_E) = \left(\int +Q\right) fg_E \tag{3.5}$$

for all f in $\mathcal{L}^2(\int +Q)$. Replacing f with $f\chi_E \in \mathcal{L}^2(\int +Q)$, we have

$$\left(\int +Q\right) fg_E = \left(\int +Q\right) fg_E\chi_E,$$

and hence we obtain $g_E = g_E\chi_E$ a.e. by uniqueness.

We shall prove $0 \leq g_E < 1$ a.e. Since $\{g_E < 0\} \subset E$ a.e., $\chi_{\{g_E<0\}} \in \mathcal{L}^2(\int +Q)$. Replacing f with $\chi_{\{g_E<0\}}$ in (3.5), we have $Q(\chi_{\{g_E<0\}}) = \left(\int +Q\right) \chi_{\{g_E<0\}}g_E \leq 0$, and hence $\chi_{\{g_E<0\}} = 0$ Q-a.e. Substituting it for (3.5), we obtain $\int \chi_{\{g_E<0\}}g_E = 0$. This implies $\chi_{\{g_E<0\}}g_E = 0$ a.e., and hence it follows $\chi_{\{g_E<0\}} = 0$ a.e., that is, $g_E \geq 0$ a.e. Similarly, we obtain $\chi_{\{g_E\geq 1\}} = 0$ a.e.

Thus, it follows $|fg_E| \leq |f|$ a.e. for any $f \in \mathcal{L}^2(\int +Q)$, and this implies fg_E is in $\mathcal{L}^2(\int +Q)$. By equation (3.5),

$$\begin{aligned} Q(f\chi_E) &= \left(\int +Q\right) fg_E = \int fg_E + Q(fg_E) \\ &= \int fg_E + \left(\int +Q\right) fg_E^2 = \int f(g_E + g_E^2) + Q(fg_E^2). \end{aligned}$$

Repeating this procedure, we obtain

$$Q(f\chi_E) = \int f(g_E + g_E^2 + \cdots + g_E^n) + Q(fg_E^n). \tag{3.6}$$

Since $0 \le g_E < 1$ a.e.,

$$\lim_{n\to\infty} Q(fg_E^n) = 0$$

and $0 \le g_E + g_E^2 + \cdots + g_E^n$ a.e. converges increasingly to a function assuming its value in $\mathbb{R}$ almost everywhere as $n \to \infty$, let h_E denote the limit function. Observe that h_E is measurable and that $0 \le h_E < \infty$ a.e. By definition, $h_E = h_E\chi_E$ a.e.

To prove (2), we use the truncation argument. We first assume $f \in \mathcal{L}(\int +Q)$ is non-negative. Then $(f\wedge m)\chi_E$ is in $\mathcal{L}^2(\int +Q)$ for any $m \in \mathbb{N}$. Noting

$$0 \le (f\wedge m)(g_E + g_E^2 + \cdots + g_E^n) \nearrow_{n\to\infty} (f\wedge m)h_E \ \ (a.e.) \ \in \mathcal{L}^+$$
$$(f\wedge m)g_E^n \searrow_{n\to\infty} 0 \ \ (Q\text{-}a.e.), \quad \text{(absolute continuity)}$$

we replace f with $(f\wedge m)\chi_E$ in (3.6) and applying the convergence theorems, it follows $Q((f\wedge m)\chi_E) = \int (f\wedge m)h_E$. The Monotone Convergence Theorem gives, $fh_E \in \mathcal{L}^+$ and

$$Q(f\chi_E) = \int fh_E. \tag{3.7}$$

This implies fh_E is integrable. For general $f \in \mathcal{L}(\int +Q)$, we apply the same argument to f^+, f^- separately. Since $f^\pm h_E \in \mathcal{L}$ and $Q(f^\pm\chi_E) = \int f^\pm h_E$, it follows $Q(f\chi_E) = \int fh_E$. If $f \in \mathcal{L}^+(\int +Q)$, there exists $f_n \in \mathcal{L}(\int +Q)$ such that $f_n \nearrow f$ a.e., and hence we obtain the desired equation as $n \to \infty$.

The uniqueness is proved by the same way as the proof of Theorem 3.2.5 (2). □

Lemma 3.3.5. *The set $(h_E)_{E\in\mathcal{E}_0}$ is a prefolder, where h_E is defined in Lemma 3.3.4.*

Proof. Let $E, F \in \mathcal{E}_0$. Replacing χ_E with $\chi_{E\cap F}$ in (3.7), it follows $Q(f\chi_{E\cap F}) = \int fh_{E\cap F}$. On the other hand, replacing f with $f\chi_F$ in (3.7) we have $Q(f\chi_{E\cap F}) = \int f\chi_F h_E$. By the uniqueness of Lemma 3.3.4 (2), we obtain $h_{E\cap F} = h_E\chi_F$ a.e. □

By Proposition 3.1.7, we immediately obtain the following lemma:

Lemma 3.3.6. *Suppose that $\mathcal{H}$ satisfies the Stone condition and that Q is an absolutely continuous Daniell integral on $\mathcal{H}$.*

(1) *There exists a non-negative density $\langle h\rangle$ such that for any $f \in \mathcal{L}^+(\int +Q)$, it follows $f\langle h\rangle \in \mathcal{L}^+$ and*

$$Q(f) = \int f\langle h\rangle. \tag{3.8}$$

(2) *This $\langle h\rangle$ is unique in the a.e. sense.*

Finally, we may take $f \in \mathcal{H}$ in Lemma 3.3.6 (1) and obtain Theorem 3.3.2.

3.4 Localizable measures

We will apply Theorem 3.3.2 to the classical measure theory. We fix a complete measure space (Ω, Σ, μ). Put $\Sigma_0 := \{A \in \Sigma : \ \mu(A) < \infty\}$, and let $\mathcal{H}(\Sigma_0)$ be the set of all finite linear combinations of indicator functions of the sets of Σ_0. We define the functional $\int$ on $\mathcal{H}(\Sigma_0)$ by

$$\int h := \sum_{k=1}^{n} a_k \mu(A_k), \qquad \left(h = \sum_{k=1}^{n} a_k \chi_{A_k}\right).$$

Then $(\Omega, \mathcal{H}(\Sigma_0), \int)$ is a Daniell system satisfying the Stone condition. Since the measure space is complete, each null set obtained by the Daniell scheme is also a μ-null set and the converse is true. We see that $\mathcal{E}_0 = \Sigma_0$, and $\mathcal{E} = \{$all countable unions of elements of $\Sigma_0\}$, i.e., $\mathcal{E}$ is the set of all σ-finite sets in Σ. Further, all Daniell measurable functions are Σ-measurable, and all Σ-measurable functions having σ-finite carrier are Daniell measurable. The set $\mathcal{D}$ of all the Daniell measurable sets is a σ-ring generated by the union of the elements of $\mathcal{E}$ and the null sets.

Let ν be a finite measure on Σ and absolutely continuous with respect to μ. If we put $Q(h) := \sum_{k=1}^{n} a_k \nu(A_k)$ for $h = \sum_{k=1}^{n} a_k \chi_{A_k} \in \mathcal{H}(\Sigma_0)$, then Q is a Daniell integral on $\mathcal{H}(\Sigma_0)$ and $Q \ll \int$. By Theorem 3.3.2, there exists a density folder $\langle h \rangle$ such that

$$Q(f) = \int f \langle h \rangle, \quad (f \in \mathcal{H}(\Sigma_0)).$$

Let $f = \chi_F \in \mathcal{H}^+(\Sigma_0)$, we obtain

$$\nu(F) = \int_F h_E d\mu, \quad (F \in \mathcal{E} : \ F \subset E)$$

for $E \in \mathcal{E}$. This means that for any $E \in \mathcal{E}$, h_E plays the role of a density function with respect to ν on E. This is nothing but for the measure-theoretic Radon-Nikodym theorem. Conversely, for the measure-theoretic Radon-Nikodym density on each σ-finite set, we can verify that these functions form a folder by the uniqueness.

But in general, there is no single Daniell measurable function (namely, complete file of folder) which connects all these h_E, that is to say, it is impossible to construct a Daniell measurable function h_0 defined on a certain subset of Ω agreeing with h_E on each $E \in \mathcal{E}$. We will consider the condition under which such function h_0 exists. To do this, we introduce a more comprehensive notion:

Definition 3.4.1. Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system with the Stone condition.

(1) A function $f : \Omega \to \overline{\mathbb{R}}$ is said to be *locally* (*Daniell*) *measurable* if fh is Daniell measurable for all $h \in \mathcal{H}$.

(2) A folder $\langle h \rangle$ is said to be *weakly complete* if there exists a locally measurable function f_0 such that

$$\langle h \rangle = f_0 \langle I \rangle \ a.e.$$

By definition, all complete folders are weakly complete. We call f_0 a *weakly complete file.*

3.4.1 σ-finite measure space

As we mention in Remark 3.1.10, if the complete measure space (Ω, Σ, μ) is σ-finite, then Ω belongs to $\mathcal{E}$, and hence all the folders are complete. This implies the classical Radon-Nikodym theorem remains valid for the σ-finite measure space.

3.4.2 Characterization of the Localizability

We will characterize the localizable measure by means of the folders. Let (Ω, Σ, μ) is a complete localizable measure space (see Chapter 1 and also [31, 63, 56, 78]). We induce the Daniell system $(\Omega, \mathcal{H}(\Sigma_0), \int)$ in the same way of the above. For any non-negative folder $\langle h \rangle = (h_E)_{E \in \mathcal{E}}$, let $\mathcal{F} := \{h_E : E \in \mathcal{E}\} \subset \mathcal{M}$. Since $\mathcal{F}$ is the subset of Σ-measurable functions, there exists an essential supremum f_0 for $\mathcal{F}$ by the localizability of μ (cf. [56, 78]). It is not difficult to verify that

$$h_E = f_0 \chi_E \text{ a.e. for all } E \in \mathcal{E}.$$

The essential supremum f_0 is Σ-measurable but not Daniell measurable. However, we can obtain the following characterization:

Theorem 3.4.2. *Let (Ω, Σ, μ) be a complete measure space. Then the measure μ is localizable if and only if any non-negative folder $\langle h \rangle$ is weakly complete, and its weakly complete file f_0 is Σ-measurable*

The "if" part is shown as above. We will prove the "only if" part.

Lemma 3.4.3. *For any Σ-measurable non-negative subcollection $\{f_\lambda : \lambda \in \Lambda\}$, there exists a folder $\langle h \rangle$ such that*

$$f_\lambda \langle I \rangle \leq \langle h \rangle \text{ a.e. for all } \lambda \in \Lambda. \tag{3.9}$$

Moreover, we can choose $\langle h \rangle$ is minimal, i.e., if there exists another folder $\langle g \rangle$ satisfying (3.9), then $\langle h \rangle \leq \langle g \rangle$ holds a.e.

Proof. Fix $E \in \mathcal{E}$. Then E is a σ-finite measure, so that the family $\{f_\lambda \chi_E : \lambda \in \Lambda\}$ has an essential supremum h_E satisfying $f_\lambda \chi_E \leq h_E$ a.e., and h_E is Σ-measurable. Obviously, the carrier of h_E is contained in E of σ-finite measure, and this implies h_E is a Daniell measurable function.

We prove $(h_E)_{E \in \mathcal{E}}$ satisfies the folder condition. Indeed, let h_E and $h_{E \cap F}$ be supremums of $\{f_\lambda \chi_E : \lambda \in \Lambda\}$ and $\{f_\lambda \chi_{E \cap F} : \lambda \in \Lambda\}$, respectively. Since $h_E \chi_F \geq f_\lambda \chi_{E \cap F}$, $h_E \chi_F$ is an upper bound of $\{f_\lambda \chi_{E \cap F} : \lambda \in \Lambda\}$. This implies $h_E \chi_F \geq h_{E \cap F}$. We define

$$h'_E := h_{E \cap F} + h_E \chi_{(E \setminus F)},$$

then h'_E is Daniell measurable and

$$\begin{aligned} f_\lambda \chi_E &= f_\lambda \chi_{E \cap F} + f_\lambda \chi_{(E \setminus F)} \\ &\leq h_{E \cap F} + h_E \chi_{(E \setminus F)} = h'_E. \end{aligned}$$

This implies h'_E is an upper bound of $\{f_\lambda \chi_E : \lambda \in \Lambda\}$. Hence $h_E \leq h'_E$ and $h_E \chi_F \leq h'_E \chi_F = h_{E\cap F}$. This implies $h_E \chi_F = h_{E\cap F}$ a.e.

The minimality of $\langle h \rangle$ is immediately obtained by the minimality of each h_E. □

Proof of Theorem 3.4.2. We suffices to consider $\mathcal{A} \subset L^1(\mu)$ (cf. [56, 78]). By Lemma 3.4.3, there exists a folder $\langle h \rangle$ such that

$$f_\lambda \langle I \rangle \leq \langle h \rangle \text{ a.e. for all } \lambda \in \Lambda,$$

and we choose a minimal $\langle h \rangle$. By assumption, there exists an $\mathcal{F}$-measurable non-negative complete file f_0 of the folder $\langle h \rangle$ such that

$$f_\lambda \langle I \rangle \leq \langle h \rangle = f_0 \langle I \rangle \text{ a.e. for all } \lambda \in \Lambda.$$

Let $\lambda \in \Lambda$. The carrier of $f_\lambda \in L^1(\mu)$ is σ-finite, so that we can have

$$f_\lambda \leq f_0 \chi_E \leq f_0 \text{ a.e. for all } \lambda \in \Lambda,$$

where E is containing the carrier of f_λ. This implies that f_0 is an upper bound of $\mathcal{A}$

We will show the minimality of f_0. If there exists an $\mathcal{F}$-measurable g such that $f_\lambda \leq g$, then it follows that $f_\lambda \langle I \rangle \leq g \langle I \rangle = \langle g \rangle$, where $\langle g \rangle = (g\chi_E)_{E\in\mathcal{E}}$. By the minimality of $\langle h \rangle$, we obtain $\langle h \rangle \leq \langle g \rangle$. This implies $f_0 \leq g$. □

Corollary 3.4.4. *Let (Ω, Σ, μ) be a localizable measure space, and $\nu : \Sigma \to \mathbb{R}$ be a finite measure with $\nu \ll \mu$. Then there is an a.e.-unique Σ-measurable function f_0 such that*

$$\nu(E) = \int_E f_0 \, d\mu \text{ for all } E \in \Sigma_0.$$

Remark 3.4.5. At last, we are in the position of describing the classical counter-example which fails to hold the Radon-Nikodym theorem. Let $([0,1], \Sigma, \mu)$ be a measure space with countable-cocountable σ-algebra Σ on an interval $[0,1] \subset \mathbb{R}$, that is,

$$\Sigma := \{A \subset [0,1] : A \text{ is countable or } A^c \text{ is countable}\},$$

and μ the counting measure on Σ. We observe that μ is not σ-finite, say, $[0,1]$ cannot be covered with countably many subsets $A_n \subset [0,1], n = 1, 2, \ldots$ of finite μ-measure. Taking ν to be the Lebesgue measure on $[0,1]$, we have $\nu \ll \mu$ but we cannot find a density function. We suppose that there exists a non-negative density function h such that

$$\nu(E) = \int_E h \, d\mu,$$

for all $E \in \Sigma$. Since $\nu(\{h \neq 0\}) = 0$, we see $\nu(\{h > 0\}) = 1$. This implies $\{h > 0\}$ is an uncountable set. Observing $\{h > 0\} = \bigcup_{n=1}^\infty \{h > 1/n\}$, we can find n_0 such that $\{h > 1/n_0\}$ is a countable set and is not finite. Therefore,

$$1 \geq \nu(\{h > 1/n_0\}) = \int_{\{h>1/n_0\}} h \, d\mu \geq \frac{1}{n_0} \mu(h > 1/n_0) = \infty,$$

and it is contradiction, see also [31, 34].

Moreover, we will reconsider this situation by using folders and show that the Radon-Nikodym type equality holds on the above measure space in Chapter 6.

Chapter 4

Dual Space

The first discussion of the dual space is in 1930s. Nikodym, in [54], considered the duality L^1-L^∞ based upon the result of the Radon and Lebesgue. The key to the discussion is the density of measure. The density of measure is already obtained in the paper due to Lebesgue and Radon, which is described in page 131 in [54]. In the paper, in Theorem II (Théoréme II) Nikodym stated the result as follows:

Theorem 2. *For any linear functional $U(f)$ defined on the set of all μ-integrable functions, which assumes its value in $\mathbb{R}$ to be absolutely continuous with respect to μ, it is necessary and sufficient for μ to enjoy the following properties: If $\mu(E) > 0$, there exists a μ-measurable partition $E_1 + E_2 = E$ such that $\mu(E_1)\mu(E_2) > 0$,*

Spaces L^p on a σ-finite measure space appeared in 1939 in [26] by Dunford-Pettis. They proved that if measure μ is σ-finite and $1 \leq p < \infty$, then $(L^p)^* = L^q$ for the conjugate exponent q (Theorem 2.1.6 p.345 in [26]). We should point out that they considered the space $(L^p)^*$ as the set of all mappings the space L^p into a Banach space. Moreover, they discussed that the reperesentations are given in terms of abstract integrals and kernel integrals. Now we shall summarize the recent study of this fact. In [5, 29, 37, 38, 47, 62], they studied the scalar-valued function spaces. It should be noted that Fedorova [29] considered this theorem by using Daniell-type integration. Kakutani [37] considered the dual space of L^∞ and characterized the condition for $(L^\infty)^* = L^1$ when we do not admit the choice of axiom. In [6, 35, 46], they studied the dual space of the set of Banach space-valued functions. At last, the author [60], showed that $(L^1)^*$ can be identified with the space of essentially bounded folders when the measure space is not necessarily localizable. We will restate this result and describe the relationship with the measure theory in Chapter 6.

4.1 Preliminaries

We first recall the L^p-norm and the semi-finiteness of $\int$. A semi-finite measure μ can be found in [31, 56, 78], and some of the authors refer to it as "finite subset property".

For any measurable function φ,

$$\operatorname*{ess\,sup}_{x\in\Omega} |\varphi(x)| \quad \text{or} \quad \|\varphi\|_\infty$$

denotes the greatest lower bound of all numbers C such that $|\varphi| \le C$ almost everywhere. A function $\varphi \in \mathcal{M}$ is essentially bounded if $\|\varphi\|_\infty < \infty$ and all such functions is denoted by $\mathcal{L}^\infty$. And for $\varphi \in \mathcal{L}$ we write

$$\|\varphi\|_1 := \int |\varphi|$$

as usual.

The next proposition is often referred to as the semi-finiteness of $\int$; see [31, 56, 78].

Proposition 4.1.1 (semi-finiteness)**.** *For any $0 \le \varphi \in \mathcal{L}^+$ satisfying $\int \varphi > 0$, there exists $\psi \in \mathcal{L}$ such that $0 \le \psi \le \varphi$ and that $\int \psi > 0$.*

Here for the sake of convenience for readers we recall the proof.

Proof. By the definition of $\varphi \in \mathcal{L}^+$, there exist $f \in \mathcal{H}^+, g \in \mathcal{H}^+_{\text{int}}$ such that $\varphi = f - g$ a.e. Since $f \in \mathcal{H}^+$, we may find a sequence $h_n \in \mathcal{H}$, $n \in \mathbb{N}$ such that $h_n \nearrow f$. Now, defining $\psi_n := h_n - g$, we learn this is integrable and hence so is positive part ψ_n^+. Since φ is assumed non-negative, $0 \le \psi_n^+ \nearrow \varphi$ almost everywhere. The Monotone Convergence Theorem gives $\int \psi_n^+ \nearrow \int \varphi > 0$ and we can find a sufficient large integer n_0 such that $\int \psi_{n_0}^+ > 0$. This $\psi_{n_0}^+$ is the desired function. □

Furthermore, we will consider the sum and product of folders in this chapter. Let $\langle h \rangle$, $\langle k \rangle$ be two folders. Then the mappings $\mathcal{E} \ni E \mapsto h_E \pm k_E$ and $E \mapsto h_E k_E$ satisfy the axiom of folder. Therefore, we denote these folders as:

$$\begin{aligned} \langle h \pm k \rangle \quad &\text{or} \quad \langle h \rangle \pm \langle k \rangle, \\ \langle hk \rangle \quad &\text{or} \quad \langle h \rangle \langle k \rangle. \end{aligned}$$

The following is obvious:

$$f\langle h + k \rangle = f\langle h \rangle + f\langle k \rangle \ \textit{a.e.} \tag{4.1}$$

for any $f \in \mathcal{H}$.

Theorem 3.2.5(2) can be extended for general folders:

Theorem 4.1.2. *If two arbitary density folders $\langle h \rangle, \langle g \rangle$ satisfy $\int f\langle h \rangle = \int f\langle g \rangle$ for any $f \in \mathcal{H}$, then it follows that $\langle h \rangle = \langle g \rangle$ a.e.*

Proof. By (4.1), we have

$$\int f\langle h \rangle = \int f\langle h^+ \rangle - \int f\langle h^- \rangle < \infty, \quad \text{and} \quad \int f\langle g \rangle = \int f\langle g^+ \rangle - \int f\langle g^- \rangle < \infty,$$

for any $f \in \mathcal{H}$. Since all terms are finite, we observe

$$\int f\langle h^+ + g^-\rangle = \int f\langle g^+ + h^-\rangle < \infty.$$

This implies that $\langle h^+ + g^-\rangle$ and $\langle g^+ + h^-\rangle$ are non-negative density folder. Theorem 3.2.5(2) gives $\langle h^+ + g^-\rangle = \langle g^+ + h^-\rangle$ a.e. i.e.,

$$h_E^+ + g_E^- = g_E^+ + h_E^- \quad \textit{a.e.} \text{ for all } E \in \mathcal{E}.$$

Hence, we obtain $h_E = h_E^+ - h_E^- = g_E^+ - g_E^- = g_E$ a.e. for any $E \in \mathcal{E}$. This completes the proof. □

4.2 Signed integral

In this section, we describe the property of the signed Daniell integral, which is a functional having linearity and continuity. The proofs can be found in [66].

Let $\Phi : \mathcal{H} \to \mathbb{R}$ be a linear mapping. For positive elementary functions, we define the total variation $|\Phi|$, positive variation Φ^+, negative variation Φ^- as follows:

$$\begin{aligned} |\Phi|(h) &:= \sup\{\Phi(k) \ ; \ |k| \le h,\ k \in \mathcal{H}\} \\ \Phi^+(h) &:= \sup\{\Phi(k) \ ; \ 0 \le k \le h,\ k \in \mathcal{H}\} \\ \Phi^-(h) &:= \sup\{-\Phi(k) \ ; \ 0 \le k \le h,\ k \in \mathcal{H}\}. \end{aligned}$$

We say Φ has finite variation if $|\Phi|(h)$ is finite for any positive elementary functions h.

Theorem 4.2.1. *If Φ has finite variation then $|\Phi|, \Phi^+$ and Φ^- can be extended uniquely to the non-negative linear mapping on $\mathcal{H}$ and*

$$\Phi = \Phi^+ - \Phi^- \tag{4.2}$$

holds. This decomposition is essentially minimum, in the sense that if there exists any other decomposition $\Phi = \Psi_1 - \Psi_2$, then $\Phi^+ \le \Psi_1, \Phi^- \le \Psi_2$ hold for any non-negative elementary functions. We call this decomposition (Φ^+, Φ^-) Jordan Decomposition.

Proof. Suppose that Φ has finite variation. We decompose $\mathcal{H} \ni h = h^+ - h^-$, and define $|\Phi|h := |\Phi|h^+ - |\Phi|h^-$. Then the linearity and non-negativity are obviously valid for all $h \in \mathcal{H}$. Moreover,

$$\Phi^+ := \frac{|\Phi| + \Phi}{2}, \quad \text{and} \quad \Phi^- := \frac{|\Phi| - \Phi}{2} \tag{4.3}$$

have the desired properties.

We prove the minimality of the decomposition. Let $\Phi = \Psi_1 - \Psi_2$ be a general decomposition of non-negative linear functionals. For any $h \in \mathcal{H}$ and any $k \in \mathcal{H}$ with $|k| \le h$, we have

$$-\Psi_2 k^+ \le \Phi k^+ \le \Psi_1 k^+, \quad -\Psi_2 k^- \le \Phi k^- \le \Psi_1 k^-,$$

and hence

$$\Phi k \leq \Psi_1 k^+ + \Psi_2 k^- \leq (\Psi_1 + \Psi_2)h, \quad \text{for } |k| \leq h.$$

Taking supremum over all such k, we obtain $|\Phi|h \leq (\Psi_1 + \Psi_2)h$ for $0 \leq h \in \mathcal{H}$. Combining the definition $\Phi h = (\Psi_1 - \Psi_2)h$ and (4.3), we have $\Phi^+ \leq h \leq \Psi_1 h$ and $\Phi^- h \leq \Psi_2 h$. ☐

Definition 4.2.2. We say a linear map $\Phi : \mathcal{H} \to \mathbb{R}$ is *a signed Daniell integral* if $\Phi(h_n) \to 0$ for any sequence of elementary functions with $h_n \searrow 0$.

Theorem 4.2.3. *If Φ is a signed Daniell integral on $\mathcal{H}$, then Φ has finite variation and the Jordan decomposition (Φ^+, Φ^-) is non-negative and continuous, i.e., Φ^+, Φ^- are Daniell integrals.*

The proof can be found in [66] of Theorem 2.11.6.

Definition 4.2.4. Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system with the Stone condition and Q be a signed Daniell integral. We say Q is absolutely continuous with respect to $\int$ if any $\int$-null set is a $|Q|$-null set. This is written by $Q \ll \int$ analogously to the non-negative integral.

We extend the domain of signed integral Q. If $f \in \mathcal{H}^+_{\text{int}}(|Q|)$, then $Q^+(f)$ and $Q^-(f)$ are finite and hence we can define

$$Q(f) := Q^+(f) - Q^-(f).$$

For $f \in \mathcal{L}(|Q|)$, since $f = f_1 - f_2$ with $f_1, f_2 \in \mathcal{H}^+_{\text{int}}(|Q|)$, we define

$$Q(f) := Q(f_1) - Q(f_2).$$

The Radon-Nikodym theorem still holds for the signed Daniell integrals.

Theorem 4.2.5. *Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system with the Stone condition, and Q be a signed Daniell integral on $\mathcal{H}$ such that $Q \ll \int$. Then there exists a density folder $\langle h \rangle$, such that for any $f \in \mathcal{L}(\int + |Q|)$,*

$$Q(f) = \int f\langle h \rangle. \tag{4.4}$$

This $\langle h \rangle$ is determined a.e.-uniquely.

Proof. Since Q is a signed Daniell integral on $\mathcal{H}$, Q has finite variation and $Q^\pm$ is a Daniell integral by Theorem 4.2.3. Moreover, by Theorem 4.2.1 $Q = Q^+ - Q^-$ holds on $\mathcal{H}$.

We will show $Q^\pm \ll \int$. If Z is an $\int$-null set, then it is a $|Q|$-null set by the assumption. There exists $f \in \mathcal{H}^+_{\text{int}}(|Q|)$ such that $Z \subset \{f = +\infty\}$. Since $f \in \mathcal{H}^+$, $|Q|(f) = Q^+(f) + Q^-(f)$ holds. But the left-hand is finite, so $f \in \mathcal{H}^+_{\text{int}}(Q^\pm)$. This means Z is a $Q^\pm$-null set.

By Lemma 3.3.6, there exist unique non-negative density folders $\langle h^{\pm}\rangle$ such that

$$Q^{\pm}(f) = \int f\langle h^{\pm}\rangle, \quad \text{for all } f \in \mathcal{L}(\textstyle\int +Q^{\pm}).$$

We observe $\mathcal{L}(\int +Q^{+}) \cap \mathcal{L}(\int +Q^{-}) = \mathcal{L}(\int +|Q|)$. By Corollary 3.2.4 (1), each file h_E of $\langle h\rangle$ is finite $\int$-a.e., so that we can take difference of each side and obtain

$$\begin{aligned} Q^{+}(f) - Q^{-}(f) &= \int f\langle h^{+}\rangle - \int f\langle h^{-}\rangle \\ &= \int f(\langle h^{+}\rangle - \langle h^{-}\rangle). \end{aligned}$$

Therefore, $\langle h\rangle := \langle h^{+}\rangle - \langle h^{-}\rangle$ is obviously a density folder, and $Q(f) = \int f\langle h\rangle$ holds for any $f \in \mathcal{L}(\int +|Q|)$. The uniqueness of $\langle h\rangle$ follows from Theorem 4.1.2. □

4.3 The dual space of $\mathcal{L}$

In this section, we shall prove that the dual space of $\mathcal{L}$ is identified with the set of all essentially bounded folders.

Definition 4.3.1. We say that $\langle h\rangle$ is an *essentially bounded folder* if

$$\sup_{E\in\mathcal{E}} \|h_E\|_\infty = \sup_{E\in\mathcal{E}}(\operatorname{ess.sup}_{x\in E} |h_E(x)|) < \infty.$$

We denote this by $\|\langle h\rangle\|_\infty$, and the set of all such folders is denoted by $\mathcal{L}^\infty$.

We first consider the elementary estimation of norm inequalities.

Lemma 4.3.2. *Let $\langle h\rangle \in \mathcal{L}^\infty$. It follows*

$$\|\langle h\rangle\|_\infty = \sup\left\{\left|\int f\langle h\rangle\right| ; f \in \mathcal{L}, \|f\|_1 = 1\right\}.$$

Proof. Let $f \in \mathcal{L}$. We choose $E_0 \in \mathcal{E}$ such that $\{f \neq 0\} \subset E_0$, then

$$\begin{aligned} \left|\int f\langle h\rangle\right| &= \left|\int f h_{E_0}\right| \\ &\leq \int |f||h_{E_0}| \leq \|\langle h\rangle\|_\infty \int |f| = \|\langle h\rangle\|_\infty \|f\|_1. \end{aligned}$$

Now, taking supremum over all $f \in \mathcal{L}$ with $\|f\|_1 = 1$, we have $\displaystyle\sup_{\|f\|_1=1}\left|\int f\langle h\rangle\right| \leq \|\langle h\rangle\|_\infty$.

To show the converse, let $\alpha := \|\langle h\rangle\|_\infty > 0$. Since $\|\langle h\rangle\|_\infty = \sup_{E\in\mathcal{E}} \|h_E\|_\infty$, for any a with $0 < a < \alpha$, there exists $E_a \in \mathcal{E}$ such that $a < \|h_{E_a}\|_\infty$. We deduce

$\chi_{\{h_{E_a}>a\}} \in \mathcal{L}^+$ and $\int \chi_{\{h_{E_a}>a\}} > 0$. By Proposition 4.1.1 there exists $g_{E_a} \in \mathcal{L}$ such that

$$0 \leq g_{E_a} \leq \chi_{\{h_{E_a}>a\}} \text{ and } 0 < \int g_{E_a}. \tag{4.5}$$

We define $f_{E_a} := (\int g_{E_a})^{-1} g_{E_a} \cdot \mathrm{sgn} h_{E_a}$, then $f_{E_a} \in \mathcal{L}$ and $\|f_{E_a}\|_1 = 1$. By (4.5), we deduce

$$\{g_{E_a} \neq 0\} \subset \{h_{E_a} > a\} \subset \{h_{E_a} > 0\} \subset E_a.$$

Hence

$$\begin{aligned}\left|\int f_{E_a} h_{E_a}\right| &= \frac{1}{\int g_{E_a}} \int g_{E_a} |h_{E_a}| \\ &= \frac{1}{\int g_{E_a}} \int g_{E_a} \chi_{\{g_{E_a} \neq 0\}} \chi_{\{h_{E_a}>a\}} |h_{E_a}| > a.\end{aligned}$$

Now, since $\{f_{E_a} \neq 0\} = \{g_{E_a} \neq 0\} \subset E_a$, we see $\int f_{E_a} h_{E_a} = \int f_{E_a} \langle h \rangle$. Moreover, taking supremum over all elements such that $\|f\|_1 = 1$, we see that for any a with $0 < a < \alpha$, there exists $E_a \in \mathcal{E}$ such that

$$\sup_{\|f\|_1=1} \left|\int f\langle h\rangle\right| \geq \left|\int f_{E_a} h_{E_a}\right| > a,$$

which yields the inverse inequality. □

Lemma 4.3.3. *Let $\langle h \rangle$ be a density folder. If there exists $0 < C < \infty$ such that for any $f \in \mathcal{L}$*

$$\left|\int f\langle h\rangle\right| \leq C\|f\|_1,$$

then it follows that $\langle h \rangle \in \mathcal{L}^\infty$.

Proof. Let us fix $E \in \mathcal{E}$ and its corresponding file h_E. We shall prove $h_E(x) \leq C$ a.e. $x \in \Omega$ by the use of the reduction to absurdity. For any $\varepsilon > 0$, putting $F_{E,\varepsilon} := \{|h_E| > C + \varepsilon\}$, then we have $0 \leq \chi_{F_{E,\varepsilon}} \in \mathcal{L}^+$. Now we assume that $\int \chi_{F_{E,\varepsilon}} > 0$ (if not, we have nothing to prove). By Proposition 4.1.1, there exists $g_E \in \mathcal{L}$ such that $0 \leq g_E \leq \chi_{F_E}$ and $\int g_E > 0$. Defining $\varphi_E := g_E \cdot (\mathrm{sgn} h_E)$, we see $\varphi_E \in \mathcal{L}$. Since

$$\{\varphi_E \neq 0\} \subset \{g_E \neq 0\} \subset F_{E,\varepsilon} \subset \{h_E > 0\} \subset E$$

as observed in the proof of Lemma 4.3.2, we deduce that

$$\begin{aligned}(C+\varepsilon)\|g_E\|_1 &\leq \left|\int g_E |h_E|\right| = \left|\int \varphi_E h_E\right| \\ &= \left|\int \varphi_E \langle h\rangle\right| \leq C\|\varphi_E\|_1 = C\|g_E\|_1.\end{aligned}$$

It follows that $\|g_E\|_1 = 0$ and it contradicts $\int g_E > 0$. This means that $\|h_E\| \leq C$ for any $E \in \mathcal{E}$. The proof is complete. □

Theorem 4.3.4. *Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system satisfying the Stone condition. Then there exists a one-to-one linear and norm preserving mapping τ between essentially bounded folders space $\mathcal{L}^\infty$ and the dual space $\mathcal{L}^*$; the correspondence is given by*

$$\tau(\langle h \rangle)f = \int f\langle h \rangle \quad (f \in \mathcal{L}).$$

Proof. Let $\langle h \rangle \in \mathcal{L}^\infty$. Defining

$$T_{\langle h \rangle} f := \int f\langle h \rangle \quad (f \in \mathcal{L}), \tag{4.6}$$

then $T_{\langle h \rangle}$ is linear and $|T_{\langle h \rangle} f| = |\int f\langle h \rangle| \leq \|\langle h \rangle\|_\infty \|f\|_1$, hence $\|T_{\langle h \rangle}\| \leq \|\langle h \rangle\| < \infty$, so $T_{\langle h \rangle} \in \mathcal{L}^*$. Moreover, from equation (4.6) and Lemma 4.3.2 we have $\|T_{\langle h \rangle}\| = \|\langle h \rangle\|_\infty$. It is shown that τ is the isometry from $\mathcal{L}^\infty$ to $\mathcal{L}^*$, This immediately implies that τ is injective.

Therefore, it suffices to prove that this mapping is surjective. Let $T \in \mathcal{L}^*$. Defining

$$Q(g) := Tg \quad \text{for } g \in \mathcal{L},$$

we see that Q is a signed Daniell integral on $\mathcal{L}$. Indeed, the linearity is obvious. If $\mathcal{L} \ni g_n \searrow 0$, then $|Q(g_n)| = |Tg_n| \leq \|T\|\|g_n\|_1 \to 0$ by the Dominated Convergence Theorem, so that Q is a signed Daniell integral on $\mathcal{L}$. We next prove $Q \ll \int$. Let Z be an $\int$-null set. Then there exists $f \in \mathcal{H}^+_{\text{int}}$ such that $Z \subset \{f = +\infty\}$. Since $f \in \mathcal{H}^+_{\text{int}} \subset \mathcal{L}$, we have $Q(f) < \infty$ and $|Q|(f) < \infty$ because $Q(f) = Q^+(f) - Q^-(f) < \infty$ and $|Q|(f) = Q^+(f) + Q^-(f) < \infty$ hold for $f \in \mathcal{L}$ by the Jordan Decomposition. This means $f \in \mathcal{H}^+_{\text{int}}(|Q|)$, and hence Z is Q-null.

Using Theorem 4.2.5, we can uniquely construct a density folder $\langle h \rangle$ such that $Q(f) = \int f\langle h \rangle$ holds for $f \in \mathcal{L}(\int + |Q|)$. However, since $\mathcal{L} \subset \mathcal{L}(\int + |Q|)$ by the definition of Q, we have

$$Tf = Q(f) = \int f\langle h \rangle, \quad \text{for all } f \in \mathcal{L}.$$

Since $|\int f\langle h \rangle| \leq \|T\|\|f\|_1$, by Lemma 4.3.3 it follows $\langle h \rangle \in \mathcal{L}^\infty$. It means that τ is surjective. $\square$

Chapter 5

Lebesgue Decomposition

The general Lebesgue decomposition theorem has been studied various context. In [7, 19, 57, 73], the authors considered the decomposition of additive set functions defined on a certain group, or measures take valued in a certain group, but all measures are assumed bounded. On the other hand, we are interested in general σ-additive measures on an arbitrary set taking their values in positive real number but unbounded. In classical measure theory, the Lebesgue decomposition theorem asserts that for two σ-finite measures μ, ν on a measurable space (Ω, Σ), there exist two σ-finite measures ν_a and ν_s such that $\nu = \nu_a + \nu_s$ with $\nu_a \ll \mu$ and that $\nu_s \perp \mu$, where $\nu_s \perp \mu$ means that ν_s and μ are mutually singular, that is, there exists an $F \in \Sigma$ such that $\nu_s(F) = 0$ and $\mu(F^c) = 0$. In this chapter, we reconsider the σ-finiteness of μ, ν. It is not easy to consider the non-σ-finite case [56]: Let $\Omega = \mathbb{R}$ and Σ be the Borel sets. Let μ be a Lebesgue measure and

$$\nu(E) := \begin{cases} \#(E) & E: \text{ finite set} \\ \infty & E: \text{ infinite set.} \end{cases}$$

We observe that ν is not σ-finite. We suppose there exists a decomposition;

$$\nu = \nu_1 + \nu_2, \quad \nu_1 \ll \mu, \nu_2 \perp \mu.$$

Since one point set $\{x\}$ is a μ-null set, then a ν_1-null set and $\nu_2(\{x\}) = 1$. Hence, we find $\nu_2(E) = 0$ if and only if $E = \emptyset$. Since $\nu_2 \perp \mu$, there exists a measurable F such that

$$\nu_2(F) = 0, \quad \text{and} \quad \mu(F^c) = 0.$$

Then $\nu_2(F) = 0$ implies $F = \emptyset$ and $F^c = \mathbb{R}$. This contradicts $\mu(F^c) = \mu(\mathbb{R}) = +\infty$.

5.1 Lebesgue decomposition

In this section, we will consider the general measure μ and ν by Daniell integral. To do this, we first reformulate the singularity of measure by means of folders. Finally, we prove general Lebesgue decomposition theorem.

Definition 5.1.1. Let $\int, Q$ be Daniell integrals on $\mathcal{H}$. We say that $\int$ and Q are mutually singular if there exists a collection $(Z_E)_{E\in\mathcal{E}}$ of elements of $(\int +Q)$-measurable sets such that: (1) $\int \chi_{Z_E\cap E} = Q(\chi_{Z_E^c\cap E}) = 0$ for any $E \in \mathcal{E}$, (2) the family of the indicator functions $(\chi_{Z_E})_{E\in\mathcal{E}}$ forms a folder. We denote the two mutually singular Daniell integrals $\int$ and Q by $Q\perp \int$.

Theorem 5.1.2. (1) *The integral $\int : \mathcal{H} \to \mathbb{R}$ is zero if and only if $\int \chi_E = 0$ for any $E \in \mathcal{E}$.*

(2) *If $Q\perp \int$ and $Q \ll \int$ then $Q = 0$.*

Proof. We first note that, in general, an elementary integral $\int$ on $\mathcal{H}$ is zero if and only if $\int f = 0$ for any $f \in \mathcal{H}^+$. Indeed, the sufficiency is clear. To prove the necessity, choosing $h_n \in \mathcal{H}$, so that $h_n \nearrow f$, we obtain $\int f = \lim_{n\to\infty} \int h_n = 0$.

(1) The necessity is clear. To prove the sufficiency, let f be a positive function in $\mathcal{H}^+$. Defining

$$s_n := \frac{1}{2^n}\sum_{k=1}^{\infty} \chi\left\{f > \frac{k}{2^n}\right\},$$

we find $s_n \in \mathcal{H}^+$ and $0 \le s_n \nearrow f$. Since $\chi\{f > k/2^n\} \in \mathcal{H}^+$ for each $n, k \in \mathbb{N}$, we see $Q(\chi\{f > k/2^n\}) = 0$ by assumption. The Monotone Convergence Theorem gives us $Q(s_n) = 0$ and also gives $0 = Q(s_n) \nearrow Q(f) = 0$. For general $f \in \mathcal{H}$, we apply the same argument to f^+, f^- separately.

(2) Suppose that there exists an $(\int +Q)$-measurable folder $\langle Z\rangle$ such that

$$Q(\chi_{Z_E^c\cap E}) = \int \chi_{Z_E\cap E} = 0 \quad \text{for any } E \in \mathcal{E}.$$

By absolute continuity, we see $Q(\chi_{Z_E\cap E}) = 0$. Therefore,

$$Q(\chi_{Z_E^c\cap E}) + Q(\chi_{Z_E\cap E}) = 0$$

and hence $Q(\chi_E) = 0$ for any E. By (1), we obtain $Q = 0$. ☐

Keeping in mind that the folder plays a key role in our result, we formulate and prove the Lebesgue decomposition theorem in our setting.

Theorem 5.1.3. *Let $(\Omega, \mathcal{H})$ be an elementary space satisfying the Stone condition, and let $\int, Q$ be Daniell integrals. Then Q can be uniquely expressed as $Q = Q_a + Q_s$ where $Q_a \ll \int$ and $Q_s\perp \int$.*

Proof. Since we see $Q \ll (\int +Q)$, it follows from Lemma 3.3.6 that there exists a non-negative $(\int +Q)$-density $\langle g\rangle$ such that

$$Q(f) = \left(\int +Q\right) f\langle g\rangle \tag{5.1}$$

for any $f \in \mathcal{L}^+(\int +Q)$.

We first prove $\langle g\rangle \leq \langle I\rangle \quad ((\int +Q)\text{-}a.e.)$ and $\langle g\rangle < \langle I\rangle$ $\int$-a.e. For every $E \in \mathcal{E}$, we can choose $E_n \in \mathcal{E}_0$ so that $E_n \nearrow E$. Noting that $\{g_E > 1\} \subset E \quad ((\int +Q)\text{-}a.e.)$, we substitute $f := \chi_{E_n}\chi_{\{g_E>1\}} \in \mathcal{L}(\int +Q)$ for the equation (5.1). Then we have

$$\begin{aligned} Q(\chi_{E_n}\chi_{\{g_E>1\}}) &= \int \chi_{E_n}\chi_{\{g_E>1\}}g_E + Q(\chi_{E_n}\chi_{\{g_E>1\}}g_E) \\ &\geq \int \chi_{E_n}\chi_{\{g_E>1\}} + Q(\chi_{E_n}\chi_{\{g_E>1\}}). \end{aligned}$$

Since $Q(\chi_{E_n}\chi_{\{g_E>1\}}) < \infty$, we obtain $\int \chi_{E_n}\chi_{\{g_E>1\}} = 0$, and hence $\chi_{E_n}\chi_{\{g_E>1\}} = 0$ a.e. Letting $n \to \infty$, we find $\chi_{\{g_E>1\}} = 0$ a.e. Again, substituting it for the equation (5.1), we obtain $Q((1-g_E)\chi_{\{g_E>1\}}) = 0$, and hence $\chi_{\{g_E>1\}} = 0 \quad (Q\text{-}a.e.)$. To show that $\langle g\rangle < \langle I\rangle$ a.e., substituting $f = \chi_{\{g_E=1\}}$ in (5.1) and applying the same argument, we can deduce that $\{g_E = 1\}$ is null.

Next, the family $(\chi_{\{g_E=1\}})_{E\in\mathcal{E}}$ is obviously an $(\int +Q)$-density folder, so that we denote it by $\langle G\rangle$. Now, Since $f\langle g\rangle \in \mathcal{L}(\int +Q)$ for any $f \in \mathcal{L}(\int +Q)$, we have

$$\begin{aligned} Q(f) &= \left(\int +Q\right) f\langle g\rangle = \int f\langle g\rangle + Q(f\langle g\rangle) \\ &= \int f\langle g\rangle + \left(\int +Q\right) f\langle g^2\rangle \\ &= \int f(\langle g\rangle + \langle g^2\rangle) + Q(f\langle g^2\rangle) \\ &= \int f(\langle g\rangle + \langle g^2\rangle + \cdots + \langle g^n\rangle) + Q(f\langle g^n\rangle). \end{aligned}$$

Since $\langle g^n\rangle \searrow \langle G\rangle \quad ((\int +Q)\text{-}a.e.)$ and

$$\langle 0\rangle \leq \langle g\rangle + \langle g^2\rangle + \cdots + \langle g^n\rangle \nearrow \quad ((\textstyle\int +Q)\text{-}a.e.),$$

we can denote this limit folder by $\langle h\rangle$. It follows that $\langle h\rangle$ is $(\int +Q)$-measurable and takes value in $[0,\infty]$. In fact, $\langle h\rangle$ takes real values almost everywhere. For any non-negative function $f \in \mathcal{L}(\int +Q)$, note that $f\langle g^n\rangle \searrow f\langle G\rangle \quad ((\int +Q)\text{-}a.e.)$ and

$$\langle 0\rangle \leq f(\langle g\rangle + \langle g^2\rangle + \cdots + \langle g^n\rangle) \nearrow f\langle h\rangle \quad ((\textstyle\int +Q)\text{-}a.e.),$$

applying the Monotone Convergence Theorem and the Dominated Convergence Theorem to Q and $\int$, we obtain

$$Q(f) = \int f\langle h\rangle + Q(f\langle G\rangle). \tag{5.2}$$

For general $f \in \mathcal{L}(\int +Q)$, we apply the same argument to f^+, f^- separately. This equation is valid for $f \in \mathcal{L}^+(\int +Q)$ because $f^- \in \mathcal{L}(\int +Q)$. If we take $f \in \mathcal{H}$ in (5.2), we deduce $\langle h\rangle$ is an $\int$-density.

We define $Q_a(f) := \int f\langle h\rangle$, $Q_s(f) := Q(f\langle G\rangle)$ for any $f \in \mathcal{H}$. Since $\langle h\rangle$ is an $\int$-density and Q_a, Q_s are non-negative, we see $Q_a \ll \int$ and $Q_s \ll Q$ by Proposition

3.2.7 (3). To prove $Q_s \perp \int$, noting that $\langle G \rangle$ is an $(\int + Q_s)$-measurable folder because $Q_s \ll Q$, we can easily see that $\langle G \rangle$ is a $(\int + Q_s)$-measurable folder satisfying the definition of $Q_s \perp \int$.

Finally, we will show the uniqueness of the decomposition. Suppose that $Q = Q_1 + Q_2$ for some Q_1 and Q_2 with $Q_1 \ll \int, Q_2 \perp \int$. Then we have $Q_1 + Q_2 = Q_a + Q_s$. We define a signed Daniell integral $\lambda : \mathcal{H} \to \mathbb{R}$ to be

$$\lambda(f) := Q_1(f) - Q_a(f) = Q_s(f) - Q_2(f), \quad \text{for } f \in \mathcal{H}.$$

By Theorem 4.2.3, we obtain the Jordan Decomposition $\lambda = \lambda^+ - \lambda^-$. For non-negative $h \in \mathcal{H}$,

$$\begin{aligned} \lambda^+(h) &= \sup\{\lambda(k) : 0 \le k \le h,\ k \in \mathcal{H}\} \\ &= \sup\{Q_1(k) - Q_a(k) : 0 \le k \le h,\ k \in \mathcal{H}\} \le Q_1(h), \end{aligned}$$

by the non-negativity of Q_a. similarly, we have $\lambda^-(h) \le Q_a(h)$. Therefore, we obtain

$$|\lambda|(h) = \lambda^+(h) + \lambda^-(h) \le Q_1(h) + Q_a(h) = (Q_1 + Q_a)(h), \tag{5.3}$$

for all non-negative $h \in \mathcal{H}$. (5.3) remains valid for non-negative $f \in \mathcal{H}^+$, and similarly we have $|\lambda| \le Q_2 + Q_s$ for non-negative $f \in \mathcal{H}^+$. Hence, we have

$$|\lambda| \ll Q_a + Q_1,\ |\lambda| \ll Q_2 + Q_s. \tag{5.4}$$

By (5.4) and $Q_a + Q_1 \ll \int$, we obtain $|\lambda| \ll \int$.

We shall next show $|\lambda| \perp \int$. By the assumption of $Q_s \perp \int$ and $Q_2 \perp \int$, there exist a $(Q_s + \int)$-measurable folder $\langle Z_s \rangle$ and a $(Q_2 + \int)$-measurable folder $\langle Z_2 \rangle$ such that

$$\int \chi_{Z_{s,E} \cap E} = Q_s(\chi_{Z^c_{s,E} \cap E}) = 0, \text{ and } \int \chi_{Z_{2,E} \cap E} = Q_2(\chi_{Z^c_{2,E} \cap E}) = 0,$$

respectively. We note that $Z_{s,E}$ and $Z_{2,E}$ are both $\int$-measurable and $\int$-null sets. Defining $Z_E := Z_{s,E} \cup Z_{2,E}$, we see that $\langle Z \rangle := (\chi_{Z_E})_{E \in \mathcal{E}}$ is obviously a $\int$-measurable and $\int$-null folder. Moreover, we recall $|\lambda| \ll \int$, so that $\langle Z \rangle$ is $(|\lambda| + \int)$-measurable and $(|\lambda| + \int)$-null folder. Since

$$Z^c_E \cap E \subset Z^c_{s,E} \cap E, \quad Z^c_E \cap E \subset Z^c_{2,E} \cap E,$$

and the right-hand-sides are Q_s-null and Q_2-null, respectively. It follows that $Z^c_E \cap E$ is a $(Q_s + Q_2)$-null set. By the fact that $\lambda \ll Q_s + Q_2$, we verify that $Z^c_E \cap E$ is a $|\lambda|$-null set. It means that $|\lambda| \perp \int$, and hence by Theorem 5.1.2 (2), we have $\lambda = 0$. This completes the proof of $Q_1 = Q_a,\ Q_2 = Q_s$. $\square$

Chapter 6

Applications

In this chapter, we apply our results of Sections 3.3, 4.3 and 5.1 to general measure spaces and localizable measure spaces. Classically, the following results are decisive, which was proved by Segal in the middle of 20th century:

Theorem 3 (Segal [63]). *Let (Ω, Σ, μ) be a measure space. The following assertions are equivalent (the definition will be given in Definition 1 below):*

(1) *μ is localizable.*

(2) *For any absolutely continuous measure ν, there exists a density function h such that*

$$\nu(E) = \int_E h \, d\mu \quad \text{for all } E \in \Sigma.$$

For the proof we refer to the textbooks [31, 56, 78]. The localizable measure space was also introduced by Segal:

Definition 1 (Segal [63]). (1) *A measure space (Ω, Σ, μ) is said to be semi-finite if whenever $A \in \Sigma$ with $\mu(A) = +\infty$, there exists a subset $B \subset A$ such that $B \in \Sigma$ and $0 < \mu(B) < \infty$.*

(2) *A semi-finite measure space (Ω, Σ, μ) is said to be localizable if for every $\mathcal{A} \subset \Sigma$, there exists $H \in \Sigma$ such that* (i) *$\mu(A \setminus H) = 0$ for every $A \in \mathcal{A}$,* (ii) *if another $G \in \Sigma$ satisfies the condition* (i) *then $\mu(H \setminus G) = 0$.*

In this chapter, we will prove the new characterization of the localizability, and we will also describe the example of the measure which is not localizable but has the property of the Radon-Nikodym type equation.

6.1 Reconsideration of measure theory by Daniell scheme

We recall first some notions. Fix a complete measure space (Ω, Σ, μ) and let $(\Omega, \mathcal{H}(\Sigma_0), \int d\mu)$ be the Daniell system induced by (Ω, Σ, μ), where $\mathcal{H}(\Sigma_0)$ is the set of all Σ_0-simple functions and Σ_0 is the set of all μ-finite sets in Σ. A functional $\int d\mu$ is an elementary integral defined by $\int h\, d\mu := \sum_{k=1}^{n} a_k \chi_{A_k}$ for $h \in \mathcal{H}(\Sigma_0)$, $a_k \in \mathbb{R}, A_k \in \Sigma_0$. Since the measure space is complete, each null set obtained by Daniell scheme is also a μ-null set and the converse is true. We see that $\mathcal{E}_0 = \Sigma_0$, and $\mathcal{E} = \{$all countable unions of elements of $\Sigma_0\}$, i.e., $\mathcal{E}$ is the set of all σ-finite sets in Σ. Further, all Daniell measurable functions are Σ-measurable, and all Σ-measurable functions having σ-finite carrier are Daniell measurable. The set $\mathcal{D}$ of all the Daniell measurable sets is a σ-ring generated by the union of the elements of $\mathcal{E}$ and the null sets.

By Theorem 3.4.2, a Radon-Nikodym density folder can be determined by a Σ-measurable function if and only if μ is localizable. However, we can find that Theorem 3.3.2 covers more general situations.

(1) We first consider the counter-example which is described at the end of Section 3.4. Let $\Omega = [0,1] \subset \mathbb{R}$, $\Sigma = \{A \subset \Omega : A$ is countable or A^c is countable $\}$. Let μ be a counting measure on Σ. Then Σ_0 consists of all finite subsets in $[0,1]$, and the set of all Daniell measurable functions $\mathcal{M}$, induced by the Daniell system $(\Omega, \mathcal{H}(\Sigma_0), \int)$, is the set of all extended real-valued functions whose carriers are countable subsets of Ω. Therefore, an arbitrary function on Ω is locally Daniell measurable, and hence an arbitrary folder $\langle h \rangle$ can be determined by some f_0 with $\langle h \rangle = f_0 \langle I \rangle$. To the contrary, the measure space (Ω, Σ, μ) is known to be non-localizable [31]. This means that there exists a non-localizable measure space but the induced folder becomes weakly complete. Forever, let ν be a Lebesgue measure. Then $\nu \ll \mu$ holds and $\nu(E) = \int_E h\, d\mu$ does not hold for any non-negative Σ-measurable function. However, the general Radon-Nikodym theorem

$$\nu(E) = \int_E \langle h \rangle \, d\mu$$

holds for all $E \in \mathcal{E}$ and the weakly complete file f_0 is zero function.

(2) In view of this, we consider the following two measures:

$$\mu_0(E) := \sum_{a \in \mathbb{R}} \delta_a(E), \quad \mu^*(E) := \sum_{a \in \mathbb{R}} \varphi(a) \delta_a(E)$$

for arbitrary function $\varphi : \mathbb{R} \to (0, \infty)$, where δ_a is a Dirac measure and E is an element of the countable-cocountable σ-algebra Σ on Ω. Note that the only μ_0-null set is an empty set, then (Ω, Σ, μ_0) is a complete measure space. Moreover, it is non-localizable measure space and $\mu^* \ll \mu_0$.

We observe that

$$\Sigma_0 := \{B \in \Sigma : \mu_0(B), \mu^*(B) < \infty\} = \{A \subset \Omega : \text{ finite set}\},$$

and that $\mathcal{E}$ consists of all countable subset of Ω. Then, the set of all Daniell measurable functions $\mathcal{M}(\Sigma_0)$ consists of all extended-real-valued functions having countable carrier.

By Theorem 3.3.2, we can find unique density folder $\langle h \rangle$ such that

$$\mu^*(E) = \int_E \langle h \rangle \, d\mu_0 \text{ for all } E \in \mathcal{E}.$$

Furthermore, a simple observation shows

$$\varphi \chi_E = h_E \text{ for all } E \in \mathcal{E},$$

where φ is not Σ-measurable but locally Daniell measurable, so that we can obtain

$$\mu^*(E) = \int_E \varphi \, d\mu_0 \text{ for all } E \in \mathcal{E}.$$

This is an example showing that the Radon-Nikodym theorem remains valid for non-localizable measure and weakly complete file is a non-zero function.

(3) Let (Ω, Σ, μ) be a complete localizable measure space. We induce the Daniell system $(\Omega, \mathcal{H}(\Sigma_0), \int)$ in the same way as above. For any non-negative folder $\langle h \rangle = (h_E)_{E \in \mathcal{E}}$, let $\mathcal{A} := \{h_E : E \in \mathcal{E}\} \subset \mathcal{M}$. Since $\mathcal{A}$ is the subset of Σ-measurable functions, by the localizability of μ there exists an essential supremum f_0 for $\mathcal{A}$ such that

$$h_E = f_0 \chi_E \text{ a.e. for all } E \in \mathcal{E}.$$

The essential supremum f_0 is $\mathcal{A}$-measurable but not Daniell measurable. The non-negativity can be eliminated, because the usual argument is available to $\langle h \rangle = \langle h^+ \rangle - \langle h^- \rangle$, where $\langle h^{\pm} \rangle = (h_E^{\pm})_{E \in \mathcal{E}}$.

Now, we obtain the following results:

Corollary 6.1.1. *Let (Ω, Σ, μ) be a localizable measure space. Then there exists a one-to-one linear and norm preserving mapping τ between essentially bounded "function" space L^∞ and the dual space $(L^1)^*$; the correspondence is given by*

$$\tau(g)f = \int fg, \quad \text{for } f \in L^1.$$

Corollary 6.1.2. *Let (Ω, Σ, μ) be a localizable measurable space, and let ν be a signed measure on Σ. Then ν can be uniquely expressed as $\nu = \nu_a + \nu_s$ where $\nu_a \ll \mu$ and $\nu_s \perp \mu$. Moreover, each measure can be expressed as follows: there exists a unique Σ-measurable function h such that*

$$\nu_a(E) = \int_E h \, d\mu, \quad \text{for any } E \in \Sigma_0,$$

and there exists $Z \in \Sigma$ such that

$$\nu_s(Z) = \mu(Z^c) = 0.$$

Chapter 7

Introduction to the Kakeya Problem

From this chapter, we shall consider the Kakeya problem and related topics. The Kakeya problem is a representative member of a much larger family of problems of a similar one. As we mentioned in Chapter 1, the Kakeya needle problem is very geometrical, and it is natural to apply elementary incidence geometry facts to this problem. Although this approach has had some success, it does not seem sufficient to solve the problem. However, in the last three decades it has been realized that the problem is connected to many other mathematical fields.

As we described in Chapter 1, Besicovitch gave the answer; we can rotate a needle using arbitrary small area. The fact relied on two observations. The first observation is that one can translate a needle to any location using arbitrary small are. The second one is that one can construct open subset of $\mathbb{R}^2$ of arbitrary small area which contain a unit line segment in every direction. For $n \geq 2$, we define a Kakeya set to be a subset in $\mathbb{R}^2$ which contains a unit line segment in every direction. In applications we wish to have more quantitative understanding of the Kakeya set. For example, we could replace unit line segment by $1 \times \delta$ tubes for some $0 < \delta \ll 1$ and consider the optimal compression of these tubes. That is to say, we can ask for bounds of the area of the δ-neighborhood of a Kakeya set. The answer is that these bounds are logarithmic in two dimensions, and it is known that the δ-neighborhood of a Kakeya set in $\mathbb{R}^2$ must have area at least $\sim 1/\log(1/\delta)$. The Kakeya conjecture is stated as follows: let $\mathcal{N}_\delta(E)$ is a neighborhood of the Kakeya set in n-dimensional Euclidean space $\mathbb{R}^n$. Then does it hold

$$\frac{|\mathcal{N}_\delta(E)|}{\delta^a} \not\to 0 \quad (\delta \to 0),$$

for all $a > 0$? It is true for the case $n = 2$. The first applications of the Kakeya conjecture to analysis arose in the study of Fourier summation in the 1970s. For a function f on the n-dimensional Euclidean space $\mathbb{R}^n$ the partial sum operator T_S is defined by

$$\widehat{T_S f}(\xi) = \chi_S(\xi)\widehat{f}(\xi).$$

One asks ourselves whether for $f \in L^p$, $T_S f$ converges to f in L^p, which is equivalent to asking whether $\widehat{T_S f}$ is bounded on L^p. If S is rectangle, then we would have boundedness for all p, $1 < p < \infty$, since this operator may be built out of Hilbert transforms as described in [64]. If we regard the characteristic function of a ball as an infinite product of the characteristic functions of rotated cubes, then an infinite product of $C_p > 1$ would make T_{ball} unbounded on L^p, $p \neq 2$. As a matter of fact, if S is a ball and $p \neq 2$, C. Fefferman [30] gave a counter example showing that T_{ball} is unbounded on L^p. The counter example, which he constructed, involved the construction of a Kakeya-type set; the proof shows that if $\widehat{T_S f}$ is bounded for ball S, then Kakeya sets could never have measure 0.

Instead of the operator T_S, S is ball, Fefferman and Stein proposed to deal with the slightly less singular Riesz-Bochner operator S_δ, $\delta > 0$, defined by

$$\widehat{S_\delta f}(\xi) = (1 - |\xi|^2)_+^\delta \widehat{f}(\xi), \quad t_+^\delta = \max(0, t^\delta).$$

A fundamental problem for S_δ is that for which range of $\delta > 0$ the L^p-bound

$$\|S_\delta f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$$

is true. There also have been affirmative answers for $n = 2$. But this conjecture for $n \geq 3$ is still open [14, 65, 76]. Many of the problems related to the Kakeya conjecture are still unsolved. However, these works reveal that estimates for the Riesz-Bochner operators are closely related to estimates for the Kakeya maximal operator. And the difficulty of problems for $n \geq 3$ lies in the lack of an appropriate estimate for the Kakeya maximal operator and the lack of an appropriate covering theory. Thus, the study of the Kakeya maximal operator would be meaningful. Moreover, there are many problems which are related to the above problems, for instance, a geometrical dimension of the Kakeya set, and the Restriction problem. These details are can be found in many surveys, [41, 72, 76, 77].

In Chapter 8, we investigate the weighted version of Alfonseca, Soria and Vargas's method [1, 2] and we obtain a weighted version of the Katz result [39, 40].

In Chapter 9, we consider the Kakeya maximal operator on the variable Lebesgue spaces and prove its boundedness.

Chapter 8

Directional Maximal Operator and Radial Weights

Instead of the difficult operator K_N, a more powerful but slightly complicated maximal operator has been studied on the plane. Let Ω be a set of unit vectors in $\mathbb{R}^2$ with cardinality N. For a locally integrable function f on $\mathbb{R}^2$, the directional maximal operator M_Ω is defined by

$$M_\Omega f(x) := \sup_{r>0,\omega\in\Omega} \frac{1}{2r}\int_{-r}^{r} |f(x+t\omega)|\, dt.$$

Strömberg showed in [68] that if Ω is an equidistributed set of directions with cardinality N then

$$\|M_\Omega f\|_{L^2(\mathbb{R}^2)} \leq C \log N \|f\|_{L^2(\mathbb{R}^2)}. \tag{8.1}$$

Notice that (8.1) yields the sharp $L^2(\mathbb{R}^2)$ estimate of the Kakeya maximal operator K_N, since we have

$$K_N f(x) \leq C M_\Omega f(x).$$

In [39] and [40], Katz established that (8.1) holds without the condition that Ω is an equidistributed set of directions. In [11] and [27], for the functions of radial type $f(x) = f_0(\|x\|_{l^q})$, $1 \leq q \leq n$, it is essentially proved that

$$\|M_\Omega f\|_{L^n(\mathbb{R}^n)} \leq C \log N \|f\|_{L^n(\mathbb{R}^n)}.$$

In [1] and [2], Alfonseca, Soria and Vargas proposed a new method to study this operator and they got a simple proof of the Katz result. In this chapter we investigate the weighted version of their method and we obtain a weighted version of the Katz result.

8.1 Preliminaries and main theorems

In order to state our theorem, we first introduce some notations due to [1] and [2].

Let Ω be a subset of $[0, \pi/4)$ and w be a weight on $\mathbb{R}^2$. We define the weighted directional maximal operator $M_{\Omega,w}$, acting on locally integrable functions f on $\mathbb{R}^2$, by

$$M_{\Omega,w} f(x) := \sup_{x \in R \in \mathcal{B}_\Omega} \frac{1}{w(R)} \int_R |f(y)| w(y)\, dy,$$

where $\mathcal{B}_\Omega$ denotes the basis of all rectangles with longest side forming an angle θ with the x-axis for some $\theta \in \Omega$, and $w(R)$ denotes $\int_R w(x)\, dx$. Let $\Omega_0 = \{\theta_1 > \theta_2 > \cdots > \theta_j > \cdots\}$ be an ordered subset of Ω. We take $\theta_0 = \pi/4$ and consider, for each $j \geq 1$, sets $\Omega_j = [\theta_j, \theta_{j-1}) \cap \Omega$, such that $\theta_j \in \Omega_0$ for all j. Assume also that $\Omega = \bigcup \Omega_j$. To each set Ω_j, $j = 0, 1, 2, \ldots$, we associated the corresponding basis $\mathcal{B}_j$. We define the weighted maximal operators associated to each basis for Ω_j by

$$M_{\Omega_j,w} f(x) := \sup_{x \in R \in \mathcal{B}_j} \frac{1}{w(R)} \int_R |f(y)| w(y)\, dy, \quad j = 0, 1, 2, \ldots.$$

Throughout this thesis we always assume that the weight w is a radial weight: $w(x) = w_0(\|x\|_{l^2}) = w_0(|x|)$ for some non-negative function w_0 on $\mathbb{R}_+$. We assume further that w_0 satisfies the following two conditions:

Doubling condition: For all $0 \leq r_1 \leq r_1' \leq r_2' \leq r_2 < \infty$ with $r_2 - r_1 = 2(r_2' - r_1')$,

$$\int_{r_1}^{r_2} w_0(r)\, dr \leq C \int_{r_1'}^{r_2'} w_0(r)\, dr; \tag{8.2}$$

Supremum condition: For all $0 < r_1 < r_2 < \infty$,

$$\sup_{r_1 < r < r_2} w_0(r) \leq \frac{C}{r_2 - r_1} \int_{r_1}^{r_2} w_0(r)\, dr. \tag{8.3}$$

Notice that r^a with $a > 0$ satisfies these conditions. Indeed, the doubling condition is clear and, for all $0 < r_1 < r_2 < \infty$,

$$(r_2)^a = \frac{a+1}{r_2} \int_0^{r_2} r^a\, dr \leq \frac{a+1}{r_2 - r_1} \int_{r_1}^{r_2} r^a\, dr.$$

The main result of this chapter is the following:

Theorem 8.1.1. *Let w be a radial weight satisfying (8.2) and (8.3). Then there exists a constant C independent of Ω such that*

$$\|M_{\Omega,w}\|_{L^2(w) \to L^2(w)} \leq \sup_{j \geq 1} \|M_{\Omega_j,w}\|_{L^2(w) \to L^2(w)} + C \|M_{\Omega_0,w}\|_{L^2(w) \to L^2(w)},$$

where $\|T\|_{L^2(w) \to L^2(w)}$ denotes the operator norm $T : L^2(w) \to L^2(w)$.

It is known that the weight $|x|^a$, $a > 0$, is in $A_\infty^*(\mathbb{R}^2)$ (cf. [45, p236]), where $A_\infty^*(\mathbb{R}^2)$ is the Muckenhoupt weight classes replacing the cubes Q by the rectangles R with sides parallel to the coordinate axes. From this fact and rotation invariance of the radial weights we can apply the proof of Corollary 4 in [2], and it allows us to give a weighted estimate of the Katz result (cf. [32, Theorems 6.7 and 6.13]).

Corollary 8.1.2. *Let Ω be a set of unit vectors on $\mathbb{R}^2$ with cardinality $N \gg 1$ and $w(x) = |x|^a$, $a > 0$. Then there exists a constant C depending on only a such that*

$$\|M_{\Omega,w}\|_{L^2(w)\to L^2(w)} \leq C \log N.$$

To prove Theorem 8.1.1, we essentially adapted the arguments in [1, 2]. In particular, to prove the theorem, we observe some geometry for $\mathbb{R}^2$. The following is a weighted version of the key geometric observation used in [1].

Proposition 8.1.3. *Let $0 < \theta_1, \theta_2 < \pi/4$. Let*

$$\omega_0 = (1, 0), \quad \omega_1 = (\cos\theta_1, \sin\theta_1) \text{ and } \omega_2 = (\cos(-\theta_2), \sin(-\theta_2)).$$

Let B be a rectangle whose longest side is parallel to ω_1 and let R be a rectangle whose longest side is parallel to ω_2. Suppose that $B \cap R \neq \emptyset$ and that the long side length of B be bigger than that of R. Then there exists a rectangle $\widetilde{R} \supset R$ whose longest side is parallel to ω_0 such that

$$\frac{w(R \cap B)}{w(R)} \leq C \frac{w(\widetilde{R} \cap B)}{w(\widetilde{R})},$$

where the constant C is independent of θ_1, θ_2, B and R.

To prove the proposition we need several technical lemmas.

8.2 Geometry on the plane

The aim of this section is to prove Proposition 8.1.3. To do so we first introduce our notation. We write $X \lesssim Y$ or $Y \gtrsim X$ if there is a constant C such that $X \leq CY$. The constant C may vary from line to line but the constants with subscripts, such as C_1, C_2, do not change in different occurrences. We write further $X \approx Y$ if $X \lesssim Y$ and $X \gtrsim Y$.

Given rectangle $R \subset \mathbb{R}^2$, let cR be the rectangle with the same center as R, but with the c times sidelengths oriented to the same direction of R. Given measurable set $E \subset \mathbb{R}^2$, let $|E|$ denote the Lebesgue measure of E and $w(E)$ denote $\int_E w$.

Our first goal is to show two key lemmas.

8.2.1 First key lemma

Recall that we always suppose that the weight w fulfill $w(x) = w_0(|x|)$ and that w_0 satisfy the doubling condition (8.2) and the supremum condition (8.3). For an $A \subset \mathbb{R}^2$ we set $r_1(A) := \inf_{x\in A} |x|$, $r_2(A) := \sup_{x\in A} |x|$ and $\operatorname{rad}(A) := r_2(A) - r_1(A)$. By definition we can easily see that, if $A \subset B \subset \mathbb{R}^2$, then $\operatorname{rad}(A) \leq \operatorname{rad}(B)$. We also see that $\operatorname{rad}(2R) \lesssim \operatorname{rad}(R)$ for any rectangle $R \subset \mathbb{R}^2$. The following is our first key lemma.

Lemma 8.2.1. *Let $R \subset \mathbb{R}^2$ be a rectangle. Then*

$$\frac{w(R)}{|R|} \approx \frac{1}{\operatorname{rad}(R)} \int_{r_1(R)}^{r_2(R)} w_0(r)\, dr.$$

Proof. Notice that

$$w(R) = \int_{r_1(R)}^{r_2(R)} \operatorname{arc}(R \cap \mathcal{C}_r) w_0(r)\, dr, \tag{8.4}$$

where $\mathcal{C}_r$ is the circle of radius r and centered at the origin and $\operatorname{arc}(R \cap \mathcal{C}_r)$ is the arc length of the arc $R \cap \mathcal{C}_r$. It follows from (8.4) and the supremum condition (8.3) that

$$\begin{aligned}
\frac{w(R)}{|R|} &= \frac{1}{|R|} \int_{r_1(R)}^{r_2(R)} \operatorname{arc}(R \cap \mathcal{C}_r) w_0(r)\, dr \\
&\leq \sup_{r_1(R)<r<r_2(R)} w_0(r) \cdot \frac{1}{|R|} \int_{r_1(R)}^{r_2(R)} \operatorname{arc}(R \cap \mathcal{C}_r)\, dr \\
&= \sup_{r_1(R)<r<r_2(R)} w_0(r) \lesssim \frac{1}{\operatorname{rad}(R)} \int_{r_1(R)}^{r_2(R)} w_0(r)\, dr.
\end{aligned}$$

Thus, we shall prove the converse:

$$\frac{w(R)}{|R|} \gtrsim \frac{1}{\operatorname{rad}(R)} \int_{r_1(R)}^{r_2(R)} w_0(r)\, dr.$$

Since w_0 satisfies the doubling condition (8.2), we need only verify the following claim:

There exists a set $A \subset R$ such that

$$\operatorname{rad}(R) \leq C_1 \operatorname{rad}(A), \tag{8.5}$$

and that

$$\operatorname{rad}(A) \inf_{r_1(A)<r<r_2(A)} \operatorname{arc}(A \cap \mathcal{C}_r) \geq C_2 |R|, \tag{8.6}$$

where the constants C_1 and C_2 are independent of R and A.

If this claim is true, then it follows from (8.4) and the doubling condition (8.2) that

$$\begin{aligned}
w(R) &\geq \int_{r_1(A)}^{r_2(A)} w_0(r)\, dr \cdot \inf_{r_1(A)<r<r_2(A)} \operatorname{arc}(A \cap \mathcal{C}_r) \\
&\gtrsim \frac{1}{\operatorname{rad}(R)} \int_{r_1(R)}^{r_2(R)} w_0(r)\, dr \cdot \operatorname{rad}(A) \inf_{r_1(A)<r<r_2(A)} \operatorname{arc}(A \cap \mathcal{C}_r) \\
&\gtrsim \frac{1}{\operatorname{rad}(R)} \int_{r_1(R)}^{r_2(R)} w_0(r)\, dr \cdot |R|.
\end{aligned}$$

We now prove the claim.

Because of the rotation invariance and the symmetry of the problem, we may assume that the rectangle R forms

$$R = (a_1, a_2) \times (b_1, b_2), \quad 0 < a_1 < a_2 < \infty,\, 0 < b_1 < b_2 < a_2.$$

Let

$$r_1 = \sqrt{a_1^2 + b_1^2}, \quad r_2 = \sqrt{a_2^2 + b_1^2}, \quad r_3 = \sqrt{a_1^2 + b_2^2} \text{ and } r_4 = \sqrt{a_2^2 + b_2^2}.$$

Then $r_1 = r_1(R)$ and $r_4 = r_2(R)$ and a simple calculation shows that

$$r_3 - r_1 \geq r_4 - r_2 \text{ and } r_2 - r_1 \geq r_4 - r_3. \tag{8.7}$$

We now consider two cases.

The case $r_2 \leq r_3$. For $t \geq -1$, we set

$$u(t) := \sqrt{a_2^2 + t(a_2^2 - a_1^2)}.$$

Let

$$t_1 = \frac{b_1^2}{a_2^2 - a_1^2} \text{ and } t_2 = \frac{b_2^2}{a_2^2 - a_1^2} - 1.$$

Then we observe that

$$r_1 = u(t_1 - 1), \quad r_2 = u(t_1), \quad r_3 = u(t_2), \quad r_4 = u(t_2 + 1) \text{ and, hence, } t_1 \leq t_2.$$

We choose an $A \subset R$ to be a set lying between the circles $\mathcal{C}_{u(t_1 - 1/2)}$ and $\mathcal{C}_{r_3}$.

We first show (8.5). It follows that

$$\begin{aligned}\frac{r_2 - r_1}{r_2 - u(t_1 - 1/2)} &= \frac{u(t_1) - u(t_1 - 1)}{u(t_1) - u(t_1 - 1/2)} = 2\frac{u(t_1) + u(t_1 - 1/2)}{u(t_1) + u(t_1 - 1)} \\ &\leq 2\frac{2u(t_1)}{u(t_1)} = 4.\end{aligned}$$

This and (8.7) imply

$$\begin{aligned}r_4 - r_1 &= (r_4 - r_3) + (r_3 - r_2) + (r_2 - r_1) \\ &\leq (r_3 - r_2) + 2(r_2 - r_1) \\ &\leq 8(r_3 - r_2) + 8(r_2 - u(t_1 - 1/2)) \\ &= 8(r_3 - u(t_1 - 1/2)),\end{aligned}$$

which means $\operatorname{rad}(R) \leq 8 \operatorname{rad}(A)$ and proves (8.5).

We next show (8.6). Observe that if $t \in [t_1, t_2]$ then the circle $\mathcal{C}_{u(t)}$ intersects with the both vertical sides of R. Furthermore, we observe that the circle $\mathcal{C}_{u(t)}$ intersects with the vertical line $x = a_2$ at the height $\sqrt{t}\sqrt{a_2^2 - a_1^2}$ and intersects with the vertical line $x = a_1$ at the height $\sqrt{t+1}\sqrt{a_2^2 - a_1^2}$ (see Figure 1). Hence, for all $t_1 \leq t \leq t_2$, we have

$$\operatorname{arc}(R \cap \mathcal{C}_{u(t)}) \geq \left(\sqrt{t+1} - \sqrt{t}\right)\sqrt{a_2^2 - a_1^2} \geq \frac{\sqrt{a_2^2 - a_1^2}}{2\sqrt{t+1}}.$$

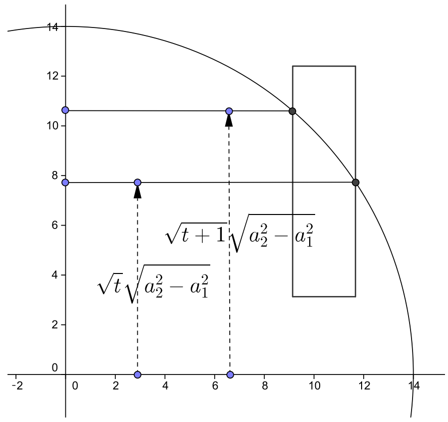

Figure 8.1: The circle $C_{u(t)}$ intersects with the both vertical sides of R.

This gives that

$$\inf_{r_2<r<r_3} \operatorname{arc}(A\cap\mathcal{C}_r) \geq \frac{\sqrt{a_2^2-a_1^2}}{2\sqrt{t_2+1}} \geq \frac{\sqrt{a_2^2-a_1^2}}{4\sqrt{t_2+1}}. \tag{8.8}$$

We also observe that the circle $\mathcal{C}_{u(t_1-1/2)}$ intersects with the vertical line $x=a_1$ at the height

$$\sqrt{t_1+1/2}\sqrt{a_2^2-a_1^2}.$$

This gives that

$$\begin{aligned}\inf_{u(t_1-1/2)<r<r_2} \operatorname{arc}(A\cap\mathcal{C}_r) = \operatorname{arc}(R\cap\mathcal{C}_{u(t_1-1/2)}) &\geq \left(\sqrt{t_1+1/2}-\sqrt{t_1}\right)\sqrt{a_2^2-a_1^2}\\ &\geq \frac{\sqrt{a_2^2-a_1^2}}{4\sqrt{t_1+1/2}}.\end{aligned}$$

Thus, by (8.8) and $t_1 \leq t_2$, we obtain

$$\begin{aligned}&(r_3-u(t_1-1/2))\inf_{u(t_1-1/2)<r<r_3}\operatorname{arc}(R\cap\mathcal{C}_r) \geq (r_3-u(t_1-1/2))\frac{\sqrt{a_2^2-a_1^2}}{4\sqrt{t_2+1}}\\ &= \frac{1}{r_3+u(t_1-1/2)}\frac{1/2+t_2-t_1}{4\sqrt{t_2+1}}\sqrt{a_2^2-a_1^2}(a_2-a_1)(a_2+a_1)\\ &\geq \frac{a_2+a_1}{8(r_3+u(t_1-1/2))}\left(\sqrt{t_2+1}-\sqrt{t_1}\right)\sqrt{a_2^2-a_1^2}(a_2-a_1)\\ &\geq \frac{a_2}{32a_2}|R| = \frac{|R|}{32},\end{aligned}$$

where we have used

$$\begin{aligned}\frac{1/2+t_2-t_1}{4\sqrt{t_2+1}} = \frac{1+2(t_2-t_1)}{8\sqrt{t_2+1}} &\geq \frac{t_2+1-t_1}{8\sqrt{t_2+1}}\\ &= \frac{1}{8}\left(\sqrt{t_2+1}-\frac{t_1}{\sqrt{t_2+1}}\right) \geq \frac{1}{8}\left(\sqrt{t_2+1}-\sqrt{t_1}\right)\end{aligned}$$

and

$$\left(\sqrt{t_2+1}-\sqrt{t_1}\right)\sqrt{a_2^2-a_1^2}(a_2-a_1)=(b_2-b_1)(a_2-a_1)=|R|.$$

These prove (8.6) in this case.

The case $r_2 > r_3$. For $t \geq -1$, we set

$$v(t) := \sqrt{b_2^2+t(b_2^2-b_1^2)}.$$

Let

$$t_3 = \frac{a_1^2}{b_2^2-b_1^2} \text{ and } t_4 = \frac{a_2^2}{b_2^2-b_1^2} - 1.$$

Then we see that

$$r_1 = v(t_3-1), \quad r_3 = v(t_3), \quad r_2 = v(t_4), \quad r_4 = v(t_4+1) \text{ and, hence, } t_3 \leq t_4.$$

We choose an $A \subset R$ to be a set lying between the circles $\mathcal{C}_{v(t_3-1/2)}$ and $\mathcal{C}_{r_2}$.

As in the previous case, we start with showing (8.5). It follows that

$$\frac{r_3-r_1}{r_3-v(t_3-1/2)} = \frac{v(t_3)-v(t_3-1)}{v(t_3)-v(t_3-1/2)} \leq 4.$$

This and (8.7) imply

$$\begin{aligned} r_4-r_1 &= (r_4-r_2)+(r_2-r_3)+(r_3-r_1) \\ &\leq (r_2-r_3)+2(r_3-r_1) \\ &\leq 8(r_2-r_3)+8(r_3-v(t_3-1/2)) \\ &= 8(r_2-v(t_3-1/2)), \end{aligned}$$

which means $\operatorname{rad}(R) \leq 8\operatorname{rad}(A)$ and proves (8.5).

We next show (8.6). Observe that

$$\inf_{r_3<r<r_2} \operatorname{arc}(A\cap\mathcal{C}_r) \geq b_2-b_1.$$

We also observe that the circle $\mathcal{C}_{v(t_3-1/2)}$ intersects with the vertical line $x=a_1$ at the height

$$\sqrt{(b_1^2+b_2^2)/2},$$

which gives that

$$\begin{aligned} &\inf_{v(t_3-1/2)<r<r_3} \operatorname{arc}(A\cap\mathcal{C}_r) \geq \sqrt{(b_1^2+b_2^2)/2}-b_1 \geq \frac{(b_2^2-b_1^2)/2}{\sqrt{(b_1^2+b_2^2)/2}+b_1} \\ &\geq \frac{b_2+b_1}{4b_2}(b_2-b_1) \geq \frac{b_2-b_1}{4}, \end{aligned}$$

where we have used $b_2 > b_1$. Notice that

$$\begin{aligned} r_4 - r_1 = \sqrt{a_2^2 + b_2^2} - \sqrt{a_1^2 + b_1^2} &= \frac{a_2^2 + b^2 - a_1^2 - b_1^2}{\sqrt{a_2^2 + b_2^2} + \sqrt{a_1^2 + b_1^2}} \\ &\geq \frac{(a_2 - a_1)(a_2 + a_1)}{2\sqrt{2}a_2} \geq \frac{a_2 - a_1}{2\sqrt{2}}, \end{aligned}$$

where we have used $a_2 > b_2 > b_1 > 0$ and $a_2 > a_1$. Thus, we obtain

$$\begin{aligned} &(r_3 - v(t_3 - 1/2)) \inf_{v(t_3-1/2)<r<r_3} \operatorname{arc}(R \cap \mathcal{C}_r) \\ &\geq \frac{1}{32}(r_4 - r_1)(b_2 - b_1) \geq \frac{\sqrt{2}}{128}(a_2 - a_1)(b_2 - b_1) = \frac{\sqrt{2}}{128}|R|, \end{aligned}$$

which proves (8.6) in this case, and, the proof of Lemma 8.2.1 is now complete. □

8.2.2 Second key lemma

We next show the second key lemma.

Lemma 8.2.2. *Let R be a rectangle which lies on the upper half plane and whose sides are parallel to the x and y axes with height $2n$ and width $2m$, $m > n > 0$. Let $\mathrm{C}_0 = (a, b)$ be the center of R. Set*

$$\begin{aligned} \mathrm{A}_0 &= (a, b) + (-m, n), & \mathrm{A}_1 &= (a, b) + (m, n), \\ \mathrm{B}_0 &= (a, b) + (-m, -n), & \mathrm{B}_1 &= (a, b) + (m, -n). \end{aligned}$$

Then there exists a constant $C > 0$ such that the following statements hold:

(a) *When $a \leq m$ and $b > n$,*

$$\frac{\operatorname{rad}(R)}{\operatorname{rad}(\mathrm{A}_0\mathrm{B}_1)} \leq C;$$

(b) *When $a > m$ and $b > n$,*

$$\min\left\{\frac{\operatorname{rad}(R)}{\operatorname{rad}(\mathrm{A}_0\mathrm{B}_1)}, \frac{\operatorname{rad}(R)}{\operatorname{rad}(\mathrm{B}_0\mathrm{B}_1)}\right\} \leq C \text{ and } \min\left\{\frac{\operatorname{rad}(R)}{\operatorname{rad}(\mathrm{A}_0\mathrm{B}_1)}, \frac{\operatorname{rad}(R)}{\operatorname{rad}(\mathrm{A}_0\mathrm{B}_0)}\right\} \leq C.$$

Proof. Let D be the point on the line joining A_0 and B_1 which is closest to the origin. Then D lies on the line $l : -mx + ny = 0$. We let $\mathrm{D}_0 \in \mathrm{A}_0\mathrm{B}_1$ be the closest point from the origin to the line segment $\mathrm{A}_0\mathrm{B}_1$ and let $\mathrm{D}_1 \in R$ be the closest point from the origin to the rectangle R. By the definition we have $r_1(\mathrm{A}_0\mathrm{B}_1) = \|\mathrm{D}_0\|$, $r_1(R) = \|\mathrm{D}_1\|$ and $\|\mathrm{D}\| \leq \|\mathrm{D}_0\|$.

Proof of (a). We start with showing part (a). It is clear that if R lies on the second quadrant, then $\operatorname{rad}(R) = \operatorname{rad}(\mathrm{A}_0\mathrm{B}_1)$. So, we prove the statement in three cases, namely,

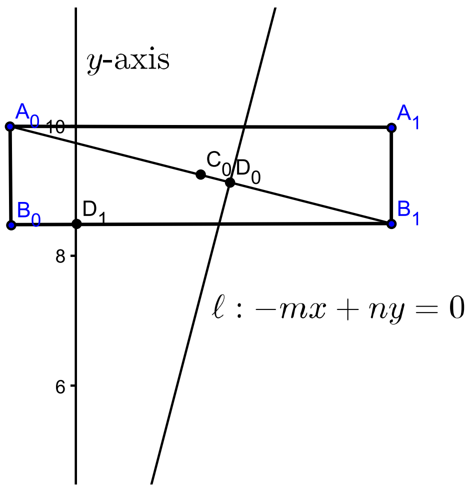

Figure 8.2: Proof of (a) case (ii)

Case (i): $-m \le a \le 0$ and $b > n$;
Case (ii): $m \ge a > 0$, $b > n$ and C_0 lies above the line l;
Case (iii): $m \ge a > 0$, $b > n$ and C_0 lies below the line l.

Case (i). If $-m \le a \le 0$ and $b > n$, then $r_2(R) = r_2(A_0B_1) = \|A_0\|$, $D_1 = (0, b-n)$ and C_0 lies above the line l. Thus, $-ma + nb > 0$ and

$$\mathrm{rad}\,(A_0B_1) = \|A_0\| - \|D_0\| \ge \|A_0\| - \|C_0\|. \tag{8.9}$$

Hence,

$$\frac{\mathrm{rad}\,(R)}{\mathrm{rad}\,(A_0B_1)} \le \frac{\|A_0\| - \|D_1\|}{\|A_0\| - \|C_0\|} = \frac{\|A_0\|^2 - \|D_1\|^2}{\|A_0\|^2 - \|C_0\|^2} \cdot \frac{\|A_0\| + \|C_0\|}{\|A_0\| + \|D_1\|} \le 2\frac{\|A_0\|^2 - \|D_1\|^2}{\|A_0\|^2 - \|C_0\|^2}$$
$$\lesssim \frac{a^2 + m^2 - 2ma + 4nb}{m^2 + n^2 + 2(-ma + nb)} \lesssim \frac{2m^2 + 4(-ma + nb)}{m^2 + n^2 + 2(-ma + nb)} \lesssim 1,$$

where we have used $-ma > 0$ and $a^2 \le m^2$.

Case (ii). If $m \ge a > 0$, $b > n$ and if C_0 lies above the line l, then $r_2(R) = \|A_1\|$, $D_1 = (0, b-n)$ and we have $-ma + nb > 0$ and (8.9). Therefore,

$$\frac{\mathrm{rad}\,(R)}{\mathrm{rad}\,(A_0B_1)} \lesssim \frac{a^2 + m^2 + 2ma + 4nb}{m^2 + n^2 + 2(-ma + nb)} \lesssim 1 + \frac{nb}{m^2 + n^2 - ma + nb},$$

where we have used $a \le m$. Since $a \le m$, we have $m^2 + n^2 - ma + nb \ge n^2 + nb \ge nb$ and, hence,

$$\frac{nb}{m^2 + n^2 - ma + nb} \le 1.$$

Case (iii). If $m \ge a > 0$, $b > n$ and if C_0 lies below the line l, then $r_2(R) = \|A_1\|$, $D_1 = (0, b-n)$ and we have $-ma + nb \le 0$ and

$$\mathrm{rad}\,(A_0B_1) = \|B_1\| - \|D_0\| \ge \|B_1\| - \|C_0\|. \tag{8.10}$$

Hence,

$$\frac{\mathrm{rad}\,(R)}{\mathrm{rad}\,(\mathrm{A}_0\mathrm{B}_1)} \lesssim \frac{a^2+m^2+2ma+4nb}{m^2+n^2+2(ma-nb)} \lesssim 1,$$

where we have used $m^2 \geq ma \geq nb$.

Proof of (b). Next,http://wmad.blog27.fc2.com/blog-entry-3026.html we show part (b). As for part (a), we consider the following two cases:

Case (i): $a > m$, $b > n$ and C_0 lies above the line l;
Case (ii): $a > m$, $b > n$ and C_0 lies below the line l.

Case (i). If $a > m$, $b > n$ and if C_0 lies above the line l, then $-ma+nb > 0$ and

$$\mathrm{rad}\,(R) = \|\mathrm{A}_1\| - \|\mathrm{B}_0\|. \tag{8.11}$$

It then follows that

$$\frac{\mathrm{rad}\,(R)}{\mathrm{rad}\,(\mathrm{A}_0\mathrm{B}_0)} \lesssim \frac{4(ma+nb)}{4nb} \lesssim 1,$$

where we have used $nb > ma$. This implies the second inequality of (b) holds.

We show the first inequality of (b). We recall that $-ma+nb > 0$ and that (8.9) and (8.11) hold. Thus,

$$\frac{\mathrm{rad}\,(R)}{\mathrm{rad}\,(\mathrm{A}_0\mathrm{B}_1)} \lesssim \frac{4(ma+nb)}{m^2+n^2+2(-ma+nb)} \lesssim \frac{ma+nb}{-ma+nb}$$

and

$$\frac{\mathrm{rad}\,(R)}{\mathrm{rad}\,(\mathrm{B}_0\mathrm{B}_1)} \lesssim \frac{ma+nb}{ma}.$$

Now, under the condition $-ma+nb > 0$, we shall estimate $\sup\min\{X, Y\}$, where

$$X := \frac{ma+nb}{-ma+nb} \text{ and } Y := \frac{ma+nb}{ma}.$$

Set

$$\begin{cases} \mathrm{C}_0 = (a,b), & \mathrm{C}_1 = (m,0), & \mathrm{C}_2 = (m,n), \\ \mathrm{C}_3 = (n,m), & \mathrm{C}_4 = (-m,n), & \mathrm{O} = (0,0). \end{cases}$$

Since

$$\begin{aligned} ma+nb &= \|\mathrm{C}_0\|\,\|\mathrm{C}_2\|\,\cos\angle\mathrm{C}_0\mathrm{O}\mathrm{C}_2, \\ -ma+nb &= \|\mathrm{C}_0\|\,\|\mathrm{C}_4\|\,\cos\angle\mathrm{C}_0\mathrm{O}\mathrm{C}_4, \\ ma &= \|\mathrm{C}_0\|\,\|\mathrm{C}_1\|\,\cos\angle\mathrm{C}_0\mathrm{O}\mathrm{C}_1, \end{aligned}$$

we have

$$X = \frac{ma+nb}{-ma+nb} = \frac{\cos\angle\mathrm{C}_0\mathrm{O}\mathrm{C}_2}{\cos\angle\mathrm{C}_0\mathrm{O}\mathrm{C}_4}, \qquad Y = \frac{ma+nb}{ma} \leq \sqrt{2}\frac{\cos\angle\mathrm{C}_0\mathrm{O}\mathrm{C}_2}{\cos\angle\mathrm{C}_0\mathrm{O}\mathrm{C}_1},$$

where the inequality $\sqrt{2}m = \sqrt{2m^2} \geq \sqrt{m^2+n^2}$ is used. Moreover, as C_0 is assumed to lie above the line l, we have

$$\cos\angle\mathrm{C}_0\mathrm{O}\mathrm{C}_2 \leq \cos\angle\mathrm{C}_3\mathrm{O}\mathrm{C}_2 = \frac{2mn}{m^2+n^2}.$$

As

$$\min\left\{\frac{1}{\cos\angle \mathrm{C_0OC_4}}, \frac{1}{\cos\angle \mathrm{C_0OC_1}}\right\}$$

is attained its maximum at $\angle \mathrm{C_0OC_4} = \angle \mathrm{C_0OC_1}$, it follows that

$$\cos\angle \mathrm{C_0OC_1} = \cos\left(\frac{\pi}{2} - \frac{\angle \mathrm{C_1OC_2}}{2}\right) = \sin\frac{\angle \mathrm{C_1OC_2}}{2}.$$

Thus,

$$\sup\min\left\{\frac{1}{\cos\angle \mathrm{C_0OC_4}}, \frac{1}{\cos\angle \mathrm{C_0OC_1}}\right\} = \frac{1}{\sin\frac{\angle \mathrm{C_1OC_2}}{2}} \approx \frac{\sqrt{m^2+n^2}}{n}.$$

In conclusion, we obtain

$$\min\{X, Y\} \lesssim \frac{mn/(m^2+n^2)}{n/\sqrt{m^2+n^2}} \approx \frac{m}{\sqrt{m^2+n^2}} \lesssim 1.$$

Case (ii). If $a > m$, $b > n$ and if $\mathrm{C_0}$ lies below the line l, then $-ma + nb \leq 0$ and (8.11) holds. Thus, as $ma \geq nb$, we have

$$\frac{\mathrm{rad}\,(R)}{\mathrm{rad}\,(\mathrm{B_0B_1})} \lesssim \frac{4(ma+nb)}{4ma} \lesssim 1.$$

The first inequality of (b) follows.

As in the previous case, we now show the second inequality of (b). The arguments are essentially the same as one for Case (i). First observe that since $-ma + nb \leq 0$, and since (8.10) and (8.11) hold, we have

$$\frac{\mathrm{rad}\,(R)}{\mathrm{rad}\,(\mathrm{A_0B_1})} \lesssim \frac{4(ma+nb)}{m^2+n^2+2(ma-nb)} \lesssim \frac{ma+nb}{ma-nb}$$

and

$$\frac{\mathrm{rad}\,(R)}{\mathrm{rad}\,(\mathrm{A_0B_0})} \lesssim \frac{ma+nb}{nb}.$$

Now, under the condition $-ma + nb \leq 0$, we shall estimate $\sup\min\{X', Y'\}$, where

$$X' := \frac{ma+nb}{ma-nb} \text{ and } Y' := \frac{ma+nb}{nb}.$$

As observed before, we have

$$\min\{X', Y'\} = \min\left\{\frac{\cos\angle \mathrm{C_0OC_2}}{\cos\angle \mathrm{C_0OC_4'}}, \frac{\sqrt{m^2+n^2}}{\sqrt{n^2}} \cdot \frac{\cos\angle \mathrm{C_0OC_2}}{\cos\angle \mathrm{C_0OC_1'}}\right\},$$

where $\mathrm{C_4'} = (m, -n)$ and $\mathrm{C_1'} = (0, n)$. Hence, $\sup\min\{X', Y'\}$ is attained when

$$\cos\angle \mathrm{C_0OC_4'} = \cos\angle \mathrm{C_0OC_1'} = \cos\left(\frac{\pi}{2} + \frac{\theta}{2}\right),$$

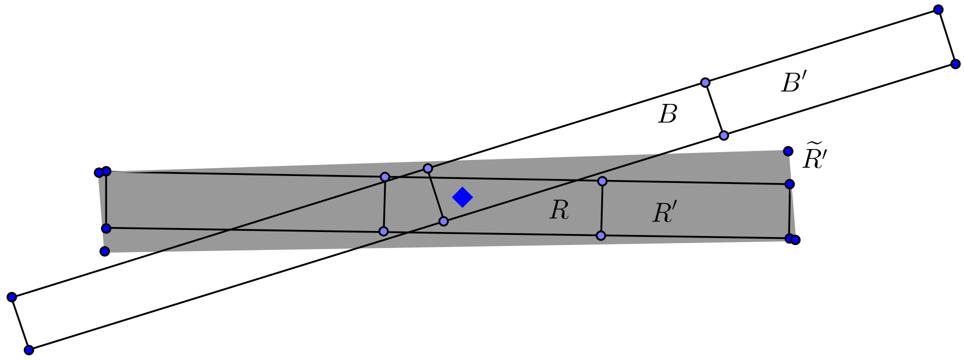

Figure 8.3: The star shape is the common center of R' and B'. The rectangle $\widetilde{R'}$ is the shadowed one.

where θ is the angle that the vector (m, n) forms with the x-axis. Since $\theta/2 \leq \pi/8$, $\cos\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$ is bounded from below and hence we obtain

$$\min\left\{\frac{\cos\angle \mathrm{C}_0\mathrm{O}\mathrm{C}_2}{\cos\angle \mathrm{C}_0\mathrm{O}\mathrm{C}_4'}, \frac{\sqrt{m^2+n^2}}{\sqrt{n^2}} \cdot \frac{\cos\angle \mathrm{C}_0\mathrm{O}\mathrm{C}_2}{\cos\angle \mathrm{C}_0\mathrm{O}\mathrm{C}_1'}\right\} \leq \min\left\{C, \frac{\sqrt{m^2+n^2}}{\sqrt{n^2}} \cdot C\right\} \lesssim 1.$$

The proof of Lemma 8.2.2 is now complete. □

8.2.3 Proof of the proposition

Now we are going to show Proposition 8.1.3.

Proof. We use the formula proved in Lemma 8.2.1. Notice that

$$w(R) \leq w(\hat{R}) \lesssim w(R) \text{ for any rectangle } R, \tag{8.12}$$

where $\hat{R}$ is a rectangle with the same center and the same short side length but twice bigger long side length of R, or a rectangle with the same center and the same long side length but twice bigger short side length of R.

Now we take rectangles R' and B' that satisfy the following conditions:

- R' and B' have the common center;
- R' (resp. B') is expanded from R (resp. B) toward the long sides;
- The long side of R' (reps. B') is three times bigger than that of R (reps. B);
- $R \cap B \subset R' \cap B'$.

Let $\widetilde{R'}$ be a smallest rectangle in the direction ω_0 containing R' (see Figure 3). Observe that if R' can be covered by N sets that are congruent to $R' \cap B'$ and that have disjoint interiors, then $\widetilde{R'}$ is covered by the corresponding sets that are congruent to $\widetilde{R'} \cap B'$. (This can be proved by the fact that the long side length of B is bigger than that of R.) Taking the smallest N, we obtain

$$\frac{|R' \cap B'|}{|\widetilde{R'} \cap B'|} = \frac{N|R' \cap B'|}{N|\widetilde{R'} \cap B'|} \lesssim \frac{|R'|}{|\widetilde{R'}|}. \tag{8.13}$$

We now verify

$$\frac{w(R' \cap B')}{w(\widetilde{R'} \cap B')} \lesssim \frac{w(R')}{w(\widetilde{R'})}. \tag{8.14}$$

Let P be a parallelogram and P' be a smallest rectangle containing P. Then there exists a rectangle $P'' \subset P$ such that P' is the dilation of P'' by a factor of eight. From this observation, the doubling property (8.12) and Lemma 8.2.1, we see that

$$w(R' \cap B') \approx \frac{|E|}{\mathrm{rad}\,(E)} \int_{r_1(E)}^{r_2(E)} w_0(r)\,dr,$$
$$w(\widetilde{R'} \cap B') \approx \frac{|F|}{\mathrm{rad}\,(F)} \int_{r_1(F)}^{r_2(F)} w_0(r)\,dr,$$

where E and F are the smallest rectangles containing $R' \cap B'$ and $\widetilde{R'} \cap B'$, respectively. By Lemma 8.2.1 and (8.13), to prove (8.14) we need only verify that

$$\frac{\frac{1}{\mathrm{rad}\,(E)} \int_{r_1(E)}^{r_2(E)} w_0(r)\,dr}{\frac{1}{\mathrm{rad}\,(R')} \int_{r_1(R')}^{r_2(R')} w_0(r)\,dr} \lesssim \frac{\frac{1}{\mathrm{rad}\,(F)} \int_{r_1(F)}^{r_2(F)} w_0(r)\,dr}{\frac{1}{\mathrm{rad}\,(\widetilde{R'})} \int_{r_1(\widetilde{R'})}^{r_2(\widetilde{R'})} w_0(r)\,dr}. \tag{8.15}$$

To verify (8.15), we show the following claim.

There exists a geometric constant $C_0 > 0$ *such that*

$$\min\left\{ \frac{\mathrm{rad}\,(\widetilde{R'})}{\mathrm{rad}\,(R')}, \frac{\mathrm{rad}\,(\widetilde{R'})}{\mathrm{rad}\,(F)} \right\} \le C_0.$$

This claim can be proved by use of Lemma 8.2.2. If $\widetilde{R'}$ contains the origin, then we can easily verify that $\dfrac{\mathrm{rad}\,(\widetilde{R'})}{\mathrm{rad}\,(R')} \le C_0$. By symmetry we have only to consider the cases for which $\widetilde{R'}$ lies on the upper half plane and B' crosses $\widetilde{R'}$ from left-side to right-side or from bottom to top. For each case we may regard $\widetilde{R'} \cap B'$ as the segments $\mathrm{B_0B_1}$ or $\mathrm{A_0B_0}$ in Lemma 8.7. Thus, the claim holds.

We return to the proof of Proposition 8.1.3.

If $\frac{\operatorname{rad}(\widetilde{R'})}{\operatorname{rad}(R')} \leq \frac{\operatorname{rad}(\widetilde{R'})}{\operatorname{rad}(F)}$ holds, then

$$r_2(\widetilde{R'}) - r_1(\widetilde{R'}) \leq C_0(r_2(R') - r_1(R')).$$

Hence, using the doubling property of w_0, we obtain

$$\int_{r_1(\widetilde{R'})}^{r_2(\widetilde{R'})} w_0(r)\,dr \lesssim \int_{r_1(R')}^{r_2(R')} w_0(r)\,dr.$$

By the supremum condition (8.3) and $E \subset F$, we have

$$\begin{aligned}\frac{1}{\operatorname{rad}(E)} \int_{r_1(E)}^{r_2(E)} w_0(r)\,dr &\leq \sup_{r_1(E)<r<r_2(E)} w_0(r) \leq \sup_{r_1(F)<r<r_2(F)} w_0(r) \\ &\lesssim \frac{1}{\operatorname{rad}(F)} \int_{r_1(F)}^{r_2(F)} w_0(r)\,dr.\end{aligned}$$

Since $\operatorname{rad}(R') \leq \operatorname{rad}(\widetilde{R'})$, combining the above estimation, we obtain (8.15).

Similarly, if $\frac{\operatorname{rad}(\widetilde{R'})}{\operatorname{rad}(R')} \geq \frac{\operatorname{rad}(\widetilde{R'})}{\operatorname{rad}(F)}$, then

$$\operatorname{rad}(F) \leq \operatorname{rad}(2\widetilde{R'}) \lesssim \operatorname{rad}(\widetilde{R'})$$

and so, by the similar arguments as above, (8.14) holds.

Lastly, let $\widetilde{R}$ be the rectangle with the same center and whose short side length is three times bigger than that of $\widetilde{R'}$. Observe that there exists a rectangle $U \subset \mathbb{R}^2$ such that $U \subset \widetilde{R} \cap B$ and $\widetilde{R} \cap B' \subset \hat{U}$, where $\hat{U}$ is the rectangle expanded from U toward the long sides with 5th bigger lengths, and, hence,

$$w(\widetilde{R} \cap B') \leq w(\hat{U}) \lesssim w(U) \leq w(\widetilde{R} \cap B).$$

Therefore, from $R' \subset 6R$, $\widetilde{R} \subset 3\widetilde{R'}$ and the doubling property of w, we obtain

$$\frac{w(R \cap B)}{w(R)} \lesssim \frac{w(R' \cap B')}{w(R')} \lesssim \frac{w(\widetilde{R'} \cap B')}{w(\widetilde{R'})} \lesssim \frac{w(\widetilde{R} \cap B')}{w(\widetilde{R})} \lesssim \frac{w(\widetilde{R} \cap B)}{w(\widetilde{R})},$$

where we have used (8.14) in the second inequality. The proof of Proposition 8.1.3 is now complete. □

8.3 The proof of Theorem 8.1.1

The following argument is due to [2]. We first linearize the operators $M_{\Omega,w}$ and $M_{\Omega_j,w}$. Given a set $\Lambda \subset [0, \pi/4)$, we observe that there exists a countable subset $\Lambda_0 \subset \Lambda$ such that $\overline{\Lambda_0} \supset \Lambda$. Let

$$\widetilde{\mathcal{B}}_{\Lambda_0} := \{R \in \mathcal{B}_{\Lambda_0} : \text{both length of short-side and long-side of } R \text{ are in } \mathbb{Q}\}.$$

Then, we can immediately verify

$$M_{\Lambda,w}f(x) = \sup_{x\in R\in\widetilde{\mathcal{B}}_{\Lambda_0}} \frac{1}{w(R)} \int_R |f(y)|w(y)\,dy.$$

Since $\widetilde{\mathcal{B}}_{\Lambda_0}$ is countable, we write $\widetilde{\mathcal{B}}_{\Lambda_0} = \{R_1, \dots, R_\alpha, \dots\}$. For any $x \in \mathbb{R}^2$, there exists $\alpha(x)$ such that

$$M_{\Lambda,w}f(x) < \frac{2}{w(R_{\alpha(x)})} \int_{R_{\alpha(x)}} |f(y)|w(y)\,dy, \quad x \in R_{\alpha(x)}, \tag{8.16}$$

and that each α, $1 \le \alpha \le \alpha(x)$, fails to hold (8.16). Putting $Q_\alpha := \{x \in R_\alpha : \alpha(x) = \alpha\}$, we obtain a pair (R_α, Q_α) satisfying

$$\sum_{\alpha=1}^{\infty} \chi_{Q_\alpha}(x) = 1 \text{ for all } x \in \mathbb{R}^2 \quad \text{and } Q_\alpha \subset R_\alpha. \tag{8.17}$$

For any $x \in \mathbb{R}^2$, choosing the pair (R_α, Q_α) satisfying (8.17), we define the operator $T_{\Lambda,w}$ by

$$T_{\Lambda,w}f(x) := \sum_\alpha \frac{1}{w(R_\alpha)} \left(\int_{R_\alpha} fw \right) \chi_{Q_\alpha}(x).$$

It follows from the definition of $T_{\Lambda,w}$ that

$$T_{\Lambda,w}f(x) \le M_{\Lambda,w}f(x). \tag{8.18}$$

By (8.16), we need only prove Theorem 8.1.1 with $M_{\Omega,w}$ replaced by $T_{\Lambda,w}$.

The following lemma is originally due to Carbery in [10].

Lemma 8.3.1. *Let $T_{\Lambda,w}$ be as above. Then $T_{\Lambda,w}$ is of strong type (p,p) with respect to the measure $w(x)\,dx$ if and only if there exists a constant C_q, such that for any sequence $\{\lambda_\alpha\} \subset \mathbb{R}_+$, we have*

$$\int \left(\sum_\alpha \lambda_\alpha \frac{w(Q_\alpha)}{w(R_\alpha)} \chi_{R_\alpha}(x) \right)^q w(x)\,dx \le C_q \sum_\alpha |\lambda_\alpha|^q w(Q_\alpha), \tag{8.19}$$

where q is the conjugate of p. Moreover, the infimum of the constants $(C_q)^{1/q}$ satisfying (8.19) is $\|T_\Lambda\|_{L^p(w)\to L^p(w)}$.

Proof. The proof is due to Carbery in [10]. If $T_{\Lambda,w}$ is of strong type (p,p) with respect to the measure $w(x)dx$, then it is easy to see that the adjoint $T^*_{\Lambda,w}$ is defined as

$$T^*_{\Lambda,w}g(x) = \sum_\alpha \left(\int_{Q_\alpha} gw \right) \frac{1}{w(R_\alpha)} \chi_{R_\alpha}(x),$$

and is of strong type (q,q) with respect to the same measure, i.e.,

$$\int |T^*_\Lambda g(x)|^q w(x)dx \le \|T^*_\Lambda\|^q_{L^q(w)\to L^q(w)} \int |g|^q w.$$

Taking $g = \sum_\alpha \lambda_\alpha \chi_{Q_\alpha}$, then $\int_{Q_\alpha} gw = \lambda_\alpha w(Q_\alpha)$. At the same time

$$\int |g|^q w = \int \left|\sum_\alpha \lambda_\alpha \chi_{Q_\alpha}\right|^q w = \int \sum_\alpha |\lambda_\alpha|^q \chi_{Q_\alpha} w = \sum_\alpha |\lambda_\alpha|^q w(Q_\alpha)$$

we obtain (8.19) with $C_q = \|T_\Lambda^*\|^q_{L^q(w)\to L^q(w)}$.

On the other hand, if we have (8.19), then, for all non-negative $h \in L^q$, taking $\lambda_\alpha = \frac{1}{w(Q_\alpha)} \int_{Q_\alpha} hw$, we have

$$|\lambda_\alpha|^q = \left|\frac{1}{w(Q_\alpha)} \int_{Q_\alpha} hw\right|^q \le \frac{1}{w(Q_\alpha)} \int_{Q_\alpha} |h|^q w.$$

Then,

$$\begin{aligned}
&\int |T_\Lambda^* h(x)|^q w(x)dx = \int \left(\sum_\alpha \int_{Q_\alpha} hw \frac{1}{w(R_\alpha)} \chi_{R_\alpha}(x)\right)^q w(x)dx \\
&= \int \left(\sum_\alpha \lambda_\alpha \frac{w(Q_\alpha)}{w(R_\alpha)} \chi_{R_\alpha}(x)\right)^q w(x)dx \\
&\le C_q \sum_\alpha \lambda_\alpha^q w(Q_\alpha) \le C_q \sum_\alpha \int_{Q_\alpha} |h|^q w \\
&= C_q \int |h|^q w.
\end{aligned}$$

Here for the last equality, we used the fact that $\{Q_\alpha\}_\alpha$ is disjoint. Hence, $\|T_\Lambda^* h\|_{L^q(w)} \le C_q^{1/q} \|h\|_{L^q(w)}$ holds, i.e., T_Λ is of strong type (p,p) with respect to the measure $w(x)dx$ and its norm is bounded by $(C_q)^{1/q}$. □

By Lemma 8.3.1 with $p = q = 2$ it is sufficient to show that the inequality (8.19) holds with $C_2^{1/2} = \sup_{j\ge 1} \|M_{\Omega_j,w}\|_{L^2(w)\to L^2(w)} + C\|M_{\Omega_0,w}\|_{L^2(w)\to L^2(w)}$.

We denote

$$\begin{aligned}
I^2 &= \int \left(\sum_\alpha \lambda_\alpha \frac{w(Q_\alpha)}{w(R_\alpha)} \chi_{R_\alpha}(x)\right)^2 w(x)\, dx \\
&= \int \left(\sum_l \sum_{\alpha: R_\alpha \in \Omega_l} \lambda_\alpha \frac{w(Q_\alpha)}{w(R_\alpha)} \chi_{R_\alpha}\right)^2 w(x)\, dx \\
&= \int \sum_l \left(\sum_{\alpha: R_\alpha \in \Omega_l} \lambda_\alpha \frac{w(Q_\alpha)}{w(R_\alpha)} \chi_{R_\alpha}\right)^2 w(x)\, dx \\
&\quad + 2 \sum_l \sum_{j<l} \int \sum_{R_\alpha \in \Omega_l} \sum_{R_\beta \in \Omega_j} \lambda_\alpha \lambda_\beta \frac{w(Q_\alpha) w(Q_\beta)}{w(R_\alpha) w(R_\beta)} \chi_{R_\alpha}(x) \chi_{R_\beta}(x) w(x)\, dx \\
&=: A + B.
\end{aligned}$$

For the first term we use (8.18) and Lemma 8.3.1 with $\Lambda = \Omega_l$. We obtain

$$
\begin{aligned}
A &\le \sum_l \|M_{\Omega_l,w}\|^2_{L^2(w)\to L^2(w)} \left(\sum_{\alpha: R_\alpha \in \Omega_l} |\lambda_\alpha|^2 w(Q_\alpha) \right) \\
&\le \left(\sup_l \|M_{\Omega_l,w}\|^2_{L^2(w)\to L^2(w)} \right) \left(\sum_l \sum_{\alpha: R_\alpha \in \Omega_l} |\lambda_\alpha|^2 w(Q_\alpha) \right) \\
&\le \left(\sup_l \|M_{\Omega_l,w}\|^2_{L^2(w)\to L^2(w)} \right) \left(\sum_\alpha |\lambda_\alpha|^2 w(Q_\alpha) \right). \qquad (8.20)
\end{aligned}
$$

By Proposition 8.1.3 there exists a constant C such that if $R_\alpha \in \Omega_l$ and $R_\beta \in \Omega_j$ with $j < l$, then we can find certain rectangles $\widetilde{R}_\alpha^-$ and $\widetilde{R}_\beta^+$, containing R_α and R_β, respectively, pointing in the direction of θ_j and so that

$$
\frac{w(R_\alpha \cap R_\beta)}{w(R_\alpha) w(R_\beta)} \lesssim \frac{w(\widetilde{R}_\alpha^- \cap R_\beta)}{w(\widetilde{R}_\alpha^-) w(R_\beta)} + \frac{w(R_\alpha \cap \widetilde{R}_\beta^+)}{w(R_\alpha) w(\widetilde{R}_\beta^+)}.
$$

Observe that both $\widetilde{R}_\alpha^-$ and $\widetilde{R}_\beta^+$ are rectangles of the basis $\mathcal{B}_0$. Then

$$
\begin{aligned}
B &\le 2C \sum_l \sum_{j<l} \int \sum_{R_\alpha \in \Omega_l} \sum_{R_\beta \in \Omega_j} \lambda_\alpha \lambda_\beta \frac{w(Q_\alpha) w(Q_\beta)}{w(\widetilde{R}_\alpha^-) w(R_\beta)} \chi_{\widetilde{R}_\alpha^-}(x) \chi_{R_\beta}(x) w(x)\, dx \\
&+ 2C \sum_l \sum_{j<l} \int \sum_{R_\alpha \in \Omega_l} \sum_{R_\beta \in \Omega_j} \lambda_\alpha \lambda_\beta \frac{w(Q_\alpha) w(Q_\beta)}{w(R_\alpha) w(\widetilde{R}_\beta^+)} \chi_{R_\alpha}(x) \chi_{\widetilde{R}_\beta^+}(x) w(x)\, dx \\
&= B^- + B^+.
\end{aligned}
$$

We shall only work with the B^- (the other term is analogous). So,

$$
\begin{aligned}
B^- &= 2C \sum_l \sum_{j<l} \int \sum_{R_\alpha \in \Omega_l} \sum_{R_\beta \in \Omega_j} \lambda_\alpha \lambda_\beta \frac{w(Q_\alpha) w(Q_\beta)}{w(\widetilde{R}_\alpha^-) w(R_\beta)} \chi_{\widetilde{R}_\alpha^-}(x) \chi_{R_\beta}(x) w(x)\, dx \\
&\le 2C \int \left(\sum_l \sum_{R_\alpha \in \Omega_l} \lambda_\alpha \frac{w(Q_\alpha)}{w(\widetilde{R}_\alpha^-)} \chi_{\widetilde{R}_\alpha^-}(x) w(x)^{1/2} \right) \\
&\qquad \times \left(\sum_j \sum_{R_\beta \in \Omega_j} \lambda_\beta \frac{w(Q_\beta)}{w(R_\beta)} \chi_{R_\beta}(x) w(x)^{1/2} \right) dx.
\end{aligned}
$$

By the Cauchy-Schwarz inequality

$$
\begin{aligned}
B^- &\le 2C \left(\int \left(\sum_l \sum_{R_\alpha \in \Omega_l} \lambda_\alpha \frac{w(Q_\alpha)}{w(\widetilde{R}_\alpha^-)} \chi_{\widetilde{R}_\alpha^-} \right)^2 w(x)\, dx \right)^{1/2} \\
&\qquad \times \left(\int \left(\sum_j \sum_{R_\beta \in \Omega_j} \lambda_\beta \frac{w(Q_\beta)}{w(R_\beta)} \chi_{R_\beta} \right)^2 w(x)\, dx \right)^{1/2}.
\end{aligned}
$$

Now, notice that $\widetilde{R}_\alpha^- \in \Omega_0$ for all α. Then by Lemma 8.3.1 and (8.18)

$$B^- \leq 2C\|M_{\Omega_0,w}\|_{L^2(w)\to L^2(w)} \left(\sum_\alpha |\lambda_\alpha|^2 w(Q_\alpha)\right)^{1/2} I. \tag{8.21}$$

Similarly, we can obtain the same bound for B^+. Combining the bounds (8.20) for A and (8.21) for $B^\pm$, we obtain

$$\begin{aligned} I^2 \leq &\left(\sup_l \|M_{\Omega_l,w}\|^2_{L^2(w)\to L^2(w)}\right)\left(\sum_\alpha |\lambda_\alpha|^2 w(Q_\alpha)\right) \\ &+ C\|M_{\Omega_0,w}\|_{L^2(w)\to L^2(w)} \left(\sum_\alpha |\lambda_\alpha|^2 w(Q_\alpha)\right)^{1/2} I. \end{aligned} \tag{8.22}$$

This implies

$$I \leq \left(\sup_l \|M_{\Omega_l,w}\|^2_{L^2(w)\to L^2(w)} + C\|M_{\Omega_0,w}\|_{L^2(w)\to L^2(w)}\right)\left(\sum_\alpha |\lambda_\alpha|^2 w(Q_\alpha)\right)^{1/2}.$$

By Lemma 8.3.1 this finishes the proof of Theorem 8.1.1.

Chapter 9

The Kakeya Maximal Operator on the Variable Lebesgue Spaces

In what follows, we also deal with variable L^p spaces on $\mathbb{R}^n$ The celebrated paper [44] by Kovácik and Rákosiík has greatly developed the theory of variable L^p spaces $L^{p(\cdot)}(\Omega)$ and established fundamental properties. After that conditions for the boundedness of the Hardy-Littlewood maximal operator M on variable L^p spaces $L^{p(\cdot)}(\Omega)$ Cruz-Uribe, Fiorenza and Neugebauer [17] and Nekvinda [52] gave the sufficient conditions on the exponent function $p(\cdot)$ independently. Diening [23] studied the necessary and sufficient conditions in terms of the conjugate exponent function $p'(\cdot)$. In the case of $\Omega = \mathbb{R}^n$, he has proved that the boundedness of M on $L^{p(\cdot)}(\mathbb{R}^n)$ is equivalent to that on $L^{p'(\cdot)}(\mathbb{R}^n)$. Recently Cruz-Uribe, Fiorenza, Martell and Pérez [16] have showed that many important operators are bounded on $L^{p(\cdot)}(\Omega)$ when M is bounded on $L^{p(\cdot)}(\Omega)$. For example, their result ensures the boundedness of singular integral operators, commutators and fractional integral operators on $L^{p(\cdot)}(\Omega)$ have been studied in [17, 22, 23, 24, 52]. E. Nakai and Y. Sawano [49] also investigated the variable Hardy spaces and generalized Campanato spaces by the grand maximal function, and then developed their several properties. In the following, we discuss the Kakeya maximal operator on the variable Lebesgue spaces.

9.1 Preliminaries and main result

Given a measurable function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, we define the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ to be the set of measurable functions such that for some $\lambda > 0$,

$$\rho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\}.$$

The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ generalizes the classical Lebesgue space $L^p(\mathbb{R}^n)$: if $p(x) \equiv p_0$, then $L^{p(\cdot)}(\mathbb{R}^n) = L^{p_0}(\mathbb{R}^n)$. We say that $p(\cdot)$ is locally log-Hölder continuous, and write $p(\cdot) \in \mathrm{LH}_0$, if there exists a constant C_0 such that

$$|p(x) - p(y)| \leq \frac{C_0}{-\log|x-y|}, \qquad x, y \in \mathbb{R}^n, \quad |x-y| < 1/2.$$

Also, we say that $p(\cdot)$ is log-Hölder continuous at infinity, and write $p(\cdot) \in \mathrm{LH}_\infty$, if there exist constants C_∞ and $p(\infty)$ such that

$$|p(x) - p(\infty)| \leq \frac{C_\infty}{\log(e + |x|)}, \qquad x \in \mathbb{R}^n.$$

We say $p(\cdot)$ is (globally) log-Hölder continuous if $p(\cdot) \in \mathrm{LH}_0 \cap \mathrm{LH}_\infty$ and we write $p(\cdot) \in \mathrm{LH}$. Finally, given a measurable set $E \subset \mathbb{R}^n$, let

$$p_-(E) := \operatorname*{ess\,inf}_{x \in E} p(x) \text{ and } p_+(E) := \operatorname*{ess\,sup}_{x \in E} p(x).$$

If $E = \mathbb{R}^n$, then we simply write p_- and p_+.

The main result of this paper is the following (see also [15]):

Theorem 9.1.1. *Let $N \gg 1$. Suppose that $p(\cdot) : \mathbb{R}^2 \to [2, \infty)$ belongs to* LH *and $p_+ < \infty$. Let*

$$c(p(\cdot), N) := p_- \cdot \sup_R \left(\frac{1}{p_-(R)} - \frac{1}{p_+(R)} \right),$$

where the supremum is taken over all rectangles $R \in \mathcal{B}_N$ with $|R| \leq 1$. Then there exists C independent of N such that

$$\|K_N f\|_{L^{p(\cdot)}(\mathbb{R}^2)} \leq C N^{c(p(\cdot),N)} (\log N)^{2/p_-} \|f\|_{L^{p(\cdot)}(\mathbb{R}^2)}. \tag{9.1}$$

Remark. (1) We remark that

$$c(p(\cdot), N) \leq p_- \left(\frac{1}{p_-} - \frac{1}{p_+} \right) = 1 - \frac{p_-}{p_+} \leq 1.$$

Also we see immediately that $c(p(\cdot), N) = 0$ if $p(\cdot)$ is constant.
(2) The technique of the proof is due to [15], which is used the machinery of Calderón-Zygmund cubes, and we apply this technique to the rectangles in $\mathcal{B}_N$. They also pointed out in [15], these theories will be applicable to other problems in variable Lebesgue spaces and the Calderón-Zygmund theory.
(3) One might naturally expect that

$$\|K_N f\|_{L^{p(\cdot)}(\mathbb{R}^2)} \leq C(\log N)^{2/p_-} \text{ when } 2 \leq p_- \leq p_+ < \infty.$$

However, we will show the following in the next section: Let $N \gg 1$ and $1 < p_- < p_+ < \infty$. Suppose that K_N is bounded from $L^{p(\cdot)}(\mathbb{R}^2)$ to $L^{p(\cdot)}(\mathbb{R}^2)$ and that $p(\cdot)$ is continuous. Then there exist a positive constant C, independent of N, and a small constant $\varepsilon > 0$ such that

$$\|K_N\|_{L^{p(\cdot)}(\mathbb{R}^2) \to L^{p(\cdot)}(\mathbb{R}^2)} \geq C N^{\varepsilon}.$$

9.2 Lower estimate of the boundedness constant

We first consider the lower estimate for $c(p(\cdot), N)$ in Theorem 9.1.1. If the exponent function $p(\cdot)$ is constant, then $c(p(\cdot), N) = 0$. However, we can show that if $p(\cdot)$ is not constant, then $c(p(\cdot), N)$ cannot be vanished. The following argument is due to T. Kopaliani [43] (see also [42]).

Recall that the conjugate function $p'(x)$ is defined by $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. The following generalized Hölder inequality and a duality relation can be found in [44]:

$$\int_{\mathbb{R}^2} |f(x)g(x)|\, dx \le 2\|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

$$\|f\|_{L^{p(\cdot)}} \le \sup_{\|g\|_{L^{p'(\cdot)}} \le 1} \int_{\mathbb{R}^2} |f(x)g(x)|\, dx.$$

Suppose that K_N is bounded from $L^{p(\cdot)}(\mathbb{R}^2)$ to $L^{p(\cdot)}(\mathbb{R}^2)$ and that $p(\cdot)$ belongs to LH. Then for every rectangle $R \in \mathcal{B}_N$, we have

$$\|K_N\|_{L^{p(\cdot)} \to L^{p(\cdot)}} \ge \|K_N f\|_{L^{p(\cdot)}} \ge \left\| \frac{1}{|R|} \int_R f(y)\, dy \cdot \chi_R \right\|_{L^{p(\cdot)}} = \frac{1}{|R|} \int_R f(y)\, dy \cdot \|\chi_R\|_{L^{p(\cdot)}}$$

for all nonnegative f with $\|f\|_{p(\cdot)} \le 1$. Taking supremum all such f, we have

$$\|K_N\|_{L^{p(\cdot)} \to L^{p(\cdot)}} \ge \frac{1}{|R|} \|\chi_R\|_{L^{p'(\cdot)}} \|\chi_R\|_{L^{p(\cdot)}} \tag{9.2}$$

for all $R \in \mathcal{B}_N$. Since $p(\cdot) \in$ LH implies $p(\cdot)$ is continuous, we can find two closed squares B_1 and B_2 in $\mathbb{R}^2$ with $|B_1|, |B_2| < 1$, such that

$$p_+(B_1) = \sup_{x \in B_1} p(x) < \inf_{x \in B_2} p(x) = p_-(B_2). \tag{9.3}$$

Without loss of generality, rotating B_1 and B_2 if necessary, we can assume

$$B_1 = [s - \varepsilon, s + \varepsilon] \times [t - \varepsilon, t + \varepsilon], \quad B_2 = [s - \varepsilon, s + \varepsilon] \times [t' - \varepsilon, t' + \varepsilon],$$

for some $\varepsilon > 0$, $0 < \varepsilon < 1/2$ and s, t, t'. Let $\widetilde{R}$ be the smallest rectangle containing B_1 and B_2 and $a := |t - t'| + 2\varepsilon$. We take N with $a/N < 2\varepsilon$, and choose $R \subset \widetilde{R}$ with sides parallel to $\widetilde{R}$ and its size is $a \times a/N$. We have

$$|R \cap B_1| = \frac{2a\varepsilon}{N} = |R \cap B_2|.$$

Observe now that the following embeddings hold:

$$L^{p(\cdot)}(B_2) \hookrightarrow L^{p_-(B_2)}(B_2)$$
$$L^{p'(\cdot)}(B_1) \hookrightarrow L^{(p_+(B_1))'}(B_1),$$

where $(p_+(B_1))' := \frac{p_+(B_1)}{p_+(B_1)-1}$. We have that

$$
\begin{aligned}
(9.2)\quad \frac{1}{|R|}\|\chi_R\|_{L^{p(\cdot)}}\|\chi_R\|_{L^{p'(\cdot)}} &\geq |R|^{-1}\|\chi_{R\cap B_2}\|_{L^{p(\cdot)}}\|\chi_{R\cap B_1}\|_{L^{p'(\cdot)}} \\
&\geq |R|^{-1}\|\chi_{R\cap B_2}\|_{L^{p_-(B_2)}(B_2)}\|\chi_{R\cap B_1}\|_{L^{(p_+(B_1))'}(B_1)} \\
&= |R|^{-1}\cdot|R\cap B_2|^{\frac{1}{p_-(B_2)}}\cdot|R\cap B_1|^{\frac{1}{(p_+(B_1))'}} \\
&= a^{-2}\cdot(2a\varepsilon)^{\frac{1}{p_-(B_2)}+\frac{1}{(p_+(B_1))'}}N^{1-\frac{1}{p_-(B_2)}-1+\frac{1}{p_+(B_1)}},
\end{aligned}
$$

where $\frac{1}{p_+(B_1)} - \frac{1}{p_-(B_2)} > 0$ by (9.3), and we have used the fact that $|B_1|, |B_2| < 1$ in the second inequality. This implies $\|K_N\|_{L^{p(\cdot)}\to L^{p(\cdot)}}$ has a lower bound $N^{\frac{1}{p_+(B_1)}-\frac{1}{p_-(B_2)}}$.

9.3 Proof of Theorem 9.1.1

In what follows we shall prove Theorem 9.1.1. Recall that we set

$$
c(p(\cdot), N) := p_- \cdot \sup_R \left(\frac{1}{p_-(R)} - \frac{1}{p_+(R)}\right),
$$

where the supremum is taken over all rectangles $R \in \mathcal{B}_N$ with $|R| \leq 1$. We need two lemmas.

Lemma 9.3.1. *Let $N \gg 1$. Suppose that $p(\cdot) : \mathbb{R}^2 \to [1,\infty)$ belongs to* LH *and $p_+ < \infty$. Then, for any rectangle $R \in \mathcal{B}_N$ and any $x \in R$,*

$$
|R|^{p(x)-p_+(R)} \leq C_p N^{p_+(R)-p(x)}, \quad |R|^{p_-(R)-p(x)} \leq C_p N^{p(x)-p_-(R)},
$$

where C_p is independent of N.

Proof. We prove the first inequality; the proof of the second is identical. When $|R| \geq 1$, there is nothing to prove. Suppose that $|R| < 1$ with the size $a/N \times a$. We observe $|R| = a^2/N$ and $a < \sqrt{N}$. In particular, since $p(\cdot)$ is continuous, there exists $y \in R$ such that $p(y) = p_+(R)$. If $1/4 \leq a$, then

$$
\begin{aligned}
|R|^{p(x)-p(y)} &= (a^2N^{-1})^{p(x)-p(y)} \\
&= \left(\frac{1}{a^2}\right)^{p(y)-p(x)} \cdot N^{p(y)-p(x)} \\
&\leq 16^{p_+-p_-}N^{p_+(R)-p(x)}.
\end{aligned}
$$

If $0 < a < 1/4$, then

$$
\begin{aligned}
N^{p(y)-p(x)}\exp\{2(p(x)-p(y))\log a\} &\leq N^{p_+(R)-p(x)}\exp\{2|p(x)-p(y)|\log(1/a)\} \\
&\leq N^{p_+(R)-p(x)}\exp\{2|p(x)-p(y)|\log(2/|x-y|)\} \\
&\leq N^{p_+(R)-p(x)}\exp\{2\frac{\log 2 - \log|x-y|}{-\log|x-y|}\} \\
&\leq e^4 N^{p_+(R)-p(x)},
\end{aligned}
$$

where we used $|x-y|<2a$ and the local log-Hölder continuity of $p(\cdot)$, because $|x-y|<1/2$ holds for all $x,y\in R$ from $a<1/4$. $\square$

The following lemma is due to [15]:

Lemma 9.3.2. *Let $p(\cdot):\mathbb{R}^2\to[1,\infty)$ be such that $p(\cdot)\in \mathrm{LH}_\infty$, and let $P(x):=(e+|x|)^{-M}$, $M\geq 2$. Then there exists a constant C depending on M and the LH_∞ constant of $p(\cdot)$ such that given any set E and any function F such that $0\leq F(y)\leq 1$, $y\in E$,*

$$\int_E F(y)^{p(y)}\,dy \leq C\int_E F(y)^{p(\infty)}\,dy + C\int_E P(y)^{p(\infty)}\,dy, \tag{9.4}$$

$$\int_E F(y)^{p(\infty)}\,dy \leq C\int_E F(y)^{p(y)}\,dy + C\int_E P(y)^{p(\infty)}\,dy. \tag{9.5}$$

Proof. We will prove (9.4); the proof of the second inequality is essentially the same. By the LH_∞ condition,

$$P(y)^{-|p(y)-p(\infty)|} = \exp(N\log(e+|y|)|p(y)-p(\infty)|)\leq \exp(NC_\infty).$$

Write the set E as $E_1\cup E_2$, where $E_1\{x\in E: F(y)\leq P(y)\}$ and $E_2=\{x\in E: P(y)<F(y)\}$. Then

$$\begin{aligned}\int_{E_1} F(y)^{p(y)}\,dy &\leq \int_{E_1} P(y)^{p(y)}\,dy\\ &\leq \int_{E_1} P(y)^{p(\infty)}P(y)^{-|p(y)-p(\infty)|}\,dy \leq \exp(NC_\infty)\int_{E_1} P(y)^{p(\infty)}\,dy.\end{aligned}$$

Similarly, since $F(y)\leq 1$,

$$\begin{aligned}\int_{E_2} F(y)^{p(y)}\,dy &\leq \int_{E_2} F(y)^{p(\infty)}F(y)^{-|p(y)-p(\infty)|}\,dy\\ &\leq \int_{E_2} F(y)^{p(\infty)}P(y)^{-|p(y)-p(\infty)|}\,dy \leq \exp(NC_\infty)\int_{E_2} F(y)^{p(\infty)}\,dy.\end{aligned}$$

$\square$

Proof of Theorem 9.1.1 We may assume that f is nonnegative. We first linearize the operator K_N. For $k\in\mathbb{N}$, we denote by $\mathcal{D}_k$ the family of all dyadic cubes $Q=2^{-k}(m+[0,1)^2)$, $m\in\mathbb{Z}^2$. For each $Q\in\mathcal{D}_k$ we choose a rectangle $R(Q)\in\mathcal{B}_N$, such that $R(Q)\supset Q$. We denote the operator T_k as

$$T_kf(x):=\sum_{Q\in\mathcal{D}_k}\frac{1}{|R(Q)|}\int_{R(Q)} f(y)\,dy\chi_Q(x).$$

By definition it is easy to see that

$$T_kf(x)\leq K_Nf(x) \tag{9.6}$$

for any choice of rectangles $\{R(Q)\}$. On the other hand, there is a sequence of linearized operators $\{T_k f\}$ which converge pointwise to $K_N f$ as k tends to infinity. Thus, we need only prove Theorem 9.1.1 with K_N replaced by T_k with a constant C not depending on k.

By homogeneity we may assume that $\|f\|_{p(\cdot)} = 1$. Then

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{R}^2} f(x)^{p(x)}\, dx \le 1.$$

Decompose f as $f_1 + f_2$, where $f_1 := f\chi_{\{x\colon f(x)>1\}}$ and $f_2 := f\chi_{\{x\colon f(x)\le 1\}}$. Then

$$\rho_{p(\cdot)}(f_i) \le \|f_i\|_{p(\cdot)} \le 1.$$

It will suffice to show that, for $i = 1, 2$, if $\lambda \ge CN^{c(p(\cdot),N)}(\log N)^{2/p_-}$, then

$$\rho_{p(\cdot)}\left(\frac{T_k f_i}{\lambda}\right) = \int_{\mathbb{R}^2} \left(\frac{T_k f_i(x)}{\lambda}\right)^{p(x)} dx \le 1$$

The estimate for f_1. It follows from Hölder's inequality that

$$\begin{aligned}\rho_{p(\cdot)}\left(\frac{T_k f_1}{\lambda}\right) &= \sum_{Q\in\mathcal{D}_k} \int_Q \left(\frac{1}{\lambda}\frac{1}{|R(Q)|}\int_{R(Q)} f_1(y)\, dy\right)^{p(x)} dx \\ &\le \sum_{Q\in\mathcal{D}_k} \int_Q \frac{N^{-c(p(\cdot),N)p(x)}}{C^{p_-}(\log N)^2}\left(\frac{1}{|R(Q)|}\int_{R(Q)} f_1(y)^{\frac{p_-(R(Q))}{p_-}}\, dy\right)^{\frac{p_- p(x)}{p_-(R(Q))}} dx.\end{aligned} \tag{9.7}$$

There holds by Lemma 9.3.1

$$|R(Q)|^{-p(x)} \le \begin{cases} C_p N^{p(x)-p_-(R(Q))}|R(Q)|^{-p_-(R(Q))} & \text{if } |R(Q)| \le 1 \\ |R(Q)|^{-p_-(R(Q))} & \text{if } |R(Q)| > 1 \end{cases}$$

which yields

$$(9.7) \le \frac{C_p}{C^{p_-}(\log N)^2}\sum_{Q\in\mathcal{D}_k}\int_Q N^{A_p}|R(Q)|^{-p_-}\left(\int_{R(Q)} f_1(y)^{\frac{p_-(R(Q))}{p_-}}\, dy\right)^{\frac{p_- p(x)}{p_-(R(Q))}} dx,$$

where we have used $p_-/p_-(R(Q)) \le 1$ and

$$A_p := \begin{cases} (p(x) - p_-(R(Q)))\frac{p_-}{p_-(R(Q))} - c(p(\cdot),N)p(x), & \text{if } |R(Q)| \le 1 \\ -c(p(\cdot),N)p(x), & \text{if } |R(Q)| > 1. \end{cases}$$

Then we find $A_p \le 0$. Indeed, if $|R(Q)| \le 1$, then

$$A_p \le p_-\left(\frac{p(x)}{p_-(R(Q))} - 1\right) - p_-\left(\frac{p(x)}{p_-(R(Q))} - \frac{p(x)}{p_+(R(Q))}\right) \le -p_- + p_-\frac{p_+(R(Q))}{p_+(R(Q))} \le 0.$$

If $|R(Q)| > 1$, there is nothing to prove:

$$\rho_{p(\cdot)}\left(\frac{T_k f_1}{\lambda}\right) \le \frac{C_p}{C^{p_-}(\log N)^2} \sum_{Q\in\mathcal{D}_k} |R(Q)|^{-p_-} \int_Q \left(\int_{R(Q)} f_1(y)^{\frac{p_-(R(Q))}{p_-}}\,dy\right)^{\frac{p_-p(x)}{p_-(R(Q))}} dx$$

$$\le \frac{C_p}{C^{p_-}(\log N)^2} \sum_{Q\in\mathcal{D}_k} |R(Q)|^{-p_-} \times$$

$$\int_Q \left(\int_{R(Q)} f_1(y)^{\frac{p(y)}{p_-}}\,dy\right)^{p_-} \left(\int_{R(Q)} f_1(y)^{p(y)}\,dy\right)^{p_-\{\frac{p(x)}{p_-(R(Q))}-1\}} dx$$

$$\le \frac{C_p}{C^{p_-}(\log N)^2} \sum_{Q\in\mathcal{D}_k} \int_Q \left(\frac{1}{|R(Q)|}\int_{R(Q)} f_1(y)^{\frac{p(y)}{p_-}}\,dy\right)^{p_-} dx,$$

where we have used

$$\left(\int_{R(Q)} f_1(y)^{p(y)}\,dy\right)^{p_-\{\frac{p(x)}{p_-(R(Q))}-1\}} \le \left(\int_{\mathbb{R}^2} f(y)^{p(y)}\,dy\right)^{p_-\{\frac{p(x)}{p_-(R(Q))}-1\}} \le 1.$$

Therefore, since for $Q \subset R(Q)$,

$$\rho_{p(\cdot)}\left(\frac{T_k f_1}{\lambda}\right) \le \frac{C_p}{C^{p_-}(\log N)^2} \int_{\mathbb{R}^2} K_N[f_1^{p(\cdot)/p_-}](x)^{p_-}\,dx$$

$$\le \frac{C_p}{C^{p_-}(\log N)^2} C_K(\log N)^2 \int_{\mathbb{R}^2} f_1(x)^{p(x)}\,dx \le \frac{C_pC_K}{C^{p_-}}.$$

Therefore, choosing C with $(C_pC_K)^{1/p_-} \le C$, we have $\rho_{p(\cdot)}(T_k f_1/\lambda) \le 1$.

The estimate for f_2. Since $f_2 \le 1$, we immediately see that

$$F := \frac{1}{|R(Q)|}\int_{R(Q)} f_2(y)\,dy \le 1.$$

Therefore, by Lemma 9.3.2, with $R(x) = (e+|x|)^{-2}$,

$$\rho_{p(\cdot)}\left(\frac{T_k f_2}{\lambda}\right) = \sum_{Q\in\mathcal{D}_k} \int_Q \left(\frac{1}{\lambda}\frac{1}{|R(Q)|}\int_{R(Q)} f_2(y)\,dy\right)^{p(x)} dx$$

$$\le C^{(1)} \sum_{Q\in\mathcal{D}_k} \int_Q \left(\frac{1}{\lambda}\frac{1}{|R(Q)|}\int_{R(Q)} f_2(y)\,dy\right)^{p(\infty)} dx$$

$$+ C^{(2)} \sum_{Q\in\mathcal{D}_k} \int_Q R(x)^{p(\infty)}\,dx.$$

We can immediately estimate the second term: since $p(\infty) ge 2$ and the sets $Q \in \mathcal{D}_k$ are disjoint for each k,

$$C^{(2)} \sum_{Q\in\mathcal{D}_k} \int_Q R(x)^{p(\infty)}\,dx = C^{(2)} \int_{\mathbb{R}^2} R(x)^{p(\infty)}\,dx \le C'.$$

We estimate the first term. Since $p(\infty) \geq 2$, K_N is bounded on $L^{p(\infty)}$. Therefore,

$$
\begin{aligned}
&C^{(1)} \sum_{Q \in \mathcal{D}_k} \int_Q \left(\frac{1}{\lambda} \frac{1}{|R(Q)|} \int_{R(Q)} f_2(y)\, dy \right)^{p(\infty)} dx \\
&\leq \frac{C^{(1)}}{C^{p(\infty)}(\log N)^2} \sum_{Q \in \mathcal{D}_k} \int_Q K_N f_2(x)^{p(\infty)}\, dx \\
&= \frac{C^{(1)}}{C^{p(\infty)}(\log N)^2} \int_{\mathbb{R}^2} K_N f_2(x)^{p(\infty)}\, dx \\
&\leq \frac{C^{(1)} C_K (\log N)^2}{C^{p(\infty)}(\log N)^2} \int_{\mathbb{R}^2} f_2(x)^{p(\infty)}\, dx,
\end{aligned}
$$

since $f_2 \leq 1$ we can apply Lemma 9.3.2 again,

$$
\begin{aligned}
\frac{C^{(1)} C_K}{C^{p(\infty)}} \int_{\mathbb{R}^2} f_2(x)^{p(\infty)}\, dx &\leq \frac{C^{(1)} C_K}{C^{p(\infty)}} \left(C^{(3)} \int_{\mathbb{R}^2} f_2(x)^{p(x)}\, dx + C^{(4)} \int_{\mathbb{R}^2} R(x)^{p(\infty)}\, dx \right) \\
&\leq \frac{C^{(1)} C_K}{C^{p(\infty)}} (C^{(3)} + C'').
\end{aligned}
$$

Combining the above constants, we can find sufficiently large constant C such that $\rho_{p(\cdot)}(T_k f_2 / \lambda) \leq 1$. □

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