Applications of torus actions to moment maps and Schubert calculus

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Contents

Chapter 1

Introduction

The relations between torus actions on manifolds and combinatorics has been studied by many mathematicians. Among those, hamiltonian torus actions on symplectic manifolds provides us a powerful tool to study their geometry and topology from combinatorics. On one hand, we can consider symplectic toric manifolds. The moment polytopes, the images of moment maps, are the best invariants for them in the sense that two symplectic toric manifolds are isomorphic if and only if their moment polytopes are the same ([12]). On the other hand, we can consider non-singular projective varieties with an algebraic torus action. In some nice situations, the GKM theory ([20]) describes the torus equivariant cohomology combinatorially in terms of the data of 0 and 1 dimensional orbits. In this thesis, we will study torus actions on symplectic manifolds (or orbifolds) in these two extremal situations.

In chapter 1, we will generalize the convexity theorem of moment polytopes mentioned above. Let (M, ω) be a compact symplectic manifold. Suppose that a torus T acts on M in a hamiltonian fashion with a moment map μ. Then the image $\mu(M)$ of the moment map is a convex polytope whose vertices are the image of the fixed points of the T-action. This theorem is proved by Atiyah [7] and Guillemin-Sternberg [23]. Our generalization is motivated by integrable systems; the integrable structure of the Toda lattice has a distinguished property that it admits two integrable structures in which their intersection gives the hamiltonian function of the Toda lattice [6] and that these functions together provides a super-integrable system. Namely, there are $2N-1$ independent functions

$$f_N, \cdots, f_2, H, g_2, \cdots, g_N$$

where both of $\{f_N, \cdots, f_2, H\}$ and $\{H, g_2, \cdots, g_N\}$ are pairwise Poisson commutative and $2N$ is the dimension of the Toda lattice. We will formulate this situation in terms of *tangled* hamiltonian torus actions on symplectic manifolds. We will include many explicit examples. We will also discuss a relation between our convexity theorem and super-integrable systems. This chapter is based on [1] which is the detailed version of an announcement [2].

In chapter 2, we will introduce and study the basic theory of *equivariant cohomology*. For a topoplogical group G and a topological space X with a G-action, the G-equivariant cohomolgy is $H^*_G(X)$ is defined by

$$H^*_G(X) = H^*(EG \times_G X)$$

where $EG \to BG$ is a contractible universal principal G-bundle, and the G action on $EG \times_G X$ is given by $g(\alpha, x) = (\alpha g^{-1}, gx)$ for $g \in G$, $\alpha \in EG$ and $x \in X$. In particular, torus equivariant cohomology $H^*_T(X)$ is interesting since, under some topological condition of X, the equivariant cohomology $H^*_T(X)$ can be presented combinatorially and we can also study the non-equivariant cohomology $H^*(X)$ from the equivariant cohomology. This chapter includes approximations of equivariant cohomology, constructions of equivariant fundamental classes of subvarieties, and the GKM theory ([20]). We will use all of these techniques in the next chapter.

In chapter 3, we will study Schubert calculus on weighted Grassmannians $\mathrm{wGr}(d, n)$. Let $d < n$ be positive integers and $\mathrm{aPl}(d, n) \subset \wedge^d \mathbb{C}^n$ the non-singular variety defined by the Plücker relations. We consider a linear $\mathbb{C}^\times$-action on $\mathbb{C}^n$ with weights $w_1, \cdots, w_n \in \mathbb{Z}_{\geq 0}$, and we then we get the induced linear $\mathbb{C}^\times$-action on $\wedge^d \mathbb{C}^n$ which preserves $\mathrm{aPl}(d, n)$. Although we will not give the precise description of these actions here, but it will follow that these actions have at most finite stabilizers, and the weighted Grassmannian $\mathrm{wGr}(d, n)$ is defined by

$$\mathrm{wGr}(d, n) = (\mathrm{aPl}(d, n) \backslash \{0\})/\mathbb{C}^\times.$$

This is a projective variety, with at worst orbifold singularities which was introduced by Corti and Reid [11]. Since there is a natural $\mathbb{C}^\times$-invariant symplectic form on $\mathrm{aPl}(d, n)$, we can also think of $\mathrm{wGr}(d, n)$ as a symplectic orbifold. Although Schubert calculus has been studied only on partial flag varieties which are smooth, recently some progress has been made for Schubert calculus on singular spaces ([25], [8], [13] etc). The weighted Grassmannian $\mathrm{wGr}(d, n)$ is particularly nice, because $\mathrm{wGr}(d, n)$ is presented by the quotient of a non-singular quasi-projective variety by a $\mathbb{C}^\times$-action with finite isotropies as mentioned above. This picture enables us to introduce the *weighted Schubert classes*. In fact, we consider a natural torus action on $\mathrm{wGr}(d, n)$ and introduced equivariant weighted Schubert class, and we compute the equivariant structure constants with respect to this basis. We will derive our structure constants by twisting the structure constants with respect to equivariant Schubert classes in $H^*_T(\mathrm{Gr}(d, n))$ studied by Knutson-Tao ([35]). An interesting corollary of this computation is that the non-equivariant structure constants are non-negative rational numbers when the weights of $\mathrm{wGr}(d, n)$ are increasing. This chapter is based on [3] in collaboration with Tomoo Matsumura.

In chapter 4, we will introduce polynomials which represent weighted Schubert classes in the cohomology ring of the weighted Grassmannian, and will study the cohomology ring of infinite dimensional weighted Grassmannians in terms of these polynomials. These polynomials are generalizations of Schur polynomials and factorial Schur polynomials representing the cohomology and equivariant cohomology of Grassmannians. This chapter is based on [4] in collaboration with Tomoo Matsumura.

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Chapter 2

A Convexity Theorem for Three Tangled Hamiltonian Torus actions, and super-integrable systems

2.1 Background and Results

In this chapter, we give a generalization of the convexity theorem in symplectic geometry as an approach to a special class of integrable systems, called super-integrable systems. We explain some background and our results in this section, and prove them in later sections. We also present some examples which illustrate our main theorem.

A *symplectic manifold* (M, ω) is a pair consisting of a smooth manifold M and a non-degenerate closed two-form ω on M. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. A G-action on M is *hamiltonian* if the action preserves the symplectic form ω and has a G-equivariant map $\mu : M \to \mathfrak{g}^*$ such that

$$\omega(X^\sharp, \cdot) = d\mu^X \quad (X \in \mathfrak{g}),$$

where $X^\sharp$ is the vector field on M induced by the infinitesimal action X, and μ^X is the function on M defined by $\mu^X = \langle \mu, X \rangle$. The map μ is called a *moment map*.

Atiyah and Guillemin-Sternberg proved that the image of a moment map of a hamiltonian torus action on a compact connected symplectic manifold is a convex polytope ([7], [23]).

Theorem 2.1.1 (Atiyah [7], Guillemin-Sternberg [23]). *Let (M, ω) be a compact connected symplectic manifold. Let T be a torus and $\mathfrak{t}$ its Lie algebra. Suppose that M has a hamiltonian T-action with a moment map $\mu : M \to \mathfrak{t}^*$. Then the image of μ is the convex hull of the image of the fixed point set of the action.*

Kirwan gave a generalization of Theorem 2.1.1 for hamiltonian group actions of compact connected Lie groups ([33]).

Theorem 2.1.2 (Kirwan [33]). *Let (M, ω) be a compact connected symplectic manifold. Let G be a compact connected Lie group and $\mathfrak{g}$ its Lie algebra. Suppose that M has a hamiltonian G-action with a moment map $\mu : M \to \mathfrak{g}^*$. Then the intersection of the image of μ with the positive Weyl chamber is a convex polytope.*

Recently, other various generalizations of Theorem 2.1.1 have been studied (see [22]).

Before stating our main theorem, we explain some physical background. Let (M, ω) be a symplectic manifold. By Darboux's theorem, ω looks locally like the standard linear symplectic form on an even dimensional Euclidean space. On this Euclidean space, the integral curves of the Hamiltonian flow of a function are the solutions of the Hamiltonian equation for this function, and this gives us the relations with classical mechanics. To analyze a physical system, it is important to find its symmetries since they give integrals of motion of the Hamiltonian equation. We can regard a moment map of a hamiltonian action on a symplectic manifold as a vector-valued map whose components are integrals of motion. If the manifold is compact and the group is a torus of dimension $\frac{1}{2}\dim M$ which acts effectively, we have a completely integrable system. In terms of integrals of motion, a *super-integrable system* is a completely integrable system with $\frac{1}{2}\dim M$ pairwise Poisson commutative integrals of motion (including the hamiltonian function) which also has k $(1 \leq k \leq \frac{1}{2}\dim M - 1)$ extra integrals of motion where the total $\frac{1}{2}\dim M + k$ functions are independent. Roughly speaking, a super-integrable system is a completely integrable system in which each trajectory is contained in a smaller torus than the Liouville tori. For an analogue of Arnold-Liouville theorem for super-integrable systems, see [16] and [44].

Several important completely integrable systems are super-integrable: the harmonic oscillators, the Kepler sysetem, the Euler top, the non-periodic Toda lattice, etc. In the case $k = \frac{1}{2}\dim M - 1$ (which is called maximally super-integrable), there are $\dim M - 1$ independent integrals of motion which implies that the generic trajectory has to be periodic if M is compact.

Let H be the hamiltonian function of the non-periodic Toda lattice of dimension $2N$. It is well known that this system is a completely integrable system (i.e. there exists N pairwise Poisson commutative independent functions including H). In [6], Agrotis-Damianou-Sophocleous showed that the non-periodic Toda lattice is maximally super-integrable. As they show, this system has an additional property: it has $2N - 1$ independent functions

$$f_N, \cdots, f_2, H, g_2, \cdots, g_N$$

where both of $\{f_N, \cdots, f_2, H\}$ and $\{H, g_2, \cdots, g_N\}$ are pairwise Poisson commutative. In this paper, we give a generalization of Theorem 2.1.1 motivated by this additional property. It takes a somewhat different form compared to

Theorem 2.1.2. In particular, we compose the moment map $M \to \mathfrak{g}^*$ with the restriction map to the dual of a sum of two commutative Lie subalgebra, and study its convexity. We will see that the convex property also holds for these maps.

Now, let us give the explicit statement of our main theorem.

Theorem 2.1.3. *Let (M,ω) be a compact connected symplectic manifold. Let G be a compact Lie group and $\mathfrak{g}$ its Lie algebra. Suppose that M has a hamiltonian G-action with a moment map $\mu : M \to \mathfrak{g}^*$. Assume that the Lie algebras $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3$ of maximal tori T_1, T_2, T_3 of G satisfy the condition*

$$\mathfrak{t}_i = \mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3 + [\mathfrak{t}_j, \mathfrak{t}_k] \qquad \textit{for } \{i,j,k\} = \{1,2,3\}. \tag{2.1.1}$$

Let $\mathcal{R}_{ij} : \mathfrak{g}^* \to (\mathfrak{t}_i + \mathfrak{t}_j)^*$ be the restriction map. Then for any i, j, k satisfying $\{i,j,k\} = \{1,2,3\}$, the image of $\mathcal{R}_{ij} \circ \mu : M \to (\mathfrak{t}_i + \mathfrak{t}_j)^*$ is the convex hull of the $\mathrm{Ad}^*(T_k)$-orbit of the image of the fixed point set of the T_i-action.

Remark 1. This is an extension of Theorem 2.1.1, which is the case $T_1 = T_2 = T_3$.
Remark 2. We need the third T_k to study the convexity of $\mathcal{R}_{ij} \circ \mu$. It will be shown that T_k acts on $(\mathfrak{t}_i + \mathfrak{t}_j)^*$ through the adjoint action of G.
Remark 3. The fibers of $\mathcal{R}_{ij} \circ \mu$ may not be connected, though the fibers of μ in Theorem 2.1.1 are connected. Also, the image $\mathcal{R}_{ij} \circ \mu(M)$ may not be a polytope.
Remark 4. The main technique we use in this paper is Lie theoretic, and we will use Theorem 2.1.1 itself to prove Theorem 2.1.3.

We also characterize the Lie subalgebras generated by $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{t}_3$ that satisfy equation (2.1.1) :

Proposition 2.1.4. *Let G be a compact Lie group and T_1, T_2, T_3 maximal tori of G. Then the Lie subalgebras $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3$ of T_1, T_2, T_3 satisfy the condition* (2.1.1) *if and only if the linear subspace $\mathfrak{h} = \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3$ is a Lie subalgebra of $\mathfrak{g}$, and*

$$\mathfrak{h} \cong \mathbb{R}^m \oplus \mathfrak{su}(2)^{\oplus n} \quad \textit{(as Lie algebras)}$$

where $\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3$ corresponds to $\mathbb{R}^m$ and $m + n = \mathrm{rank}\, G$, and for each summand in $\mathfrak{su}(2)^{\oplus n}$, there exists a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{su}(2)$ such that $e_i \in \mathfrak{su}(2) \cap \mathfrak{t}_i$ $(i = 1,2,3)$ which satisfy $[e_1, e_2] = e_3, [e_2, e_3] = e_1$ and $[e_3, e_1] = e_2$.

Finally, we obtain a super-integrable system by the following proposition. We denote by $(T_1 \cap T_2 \cap T_3)_0$ the identity component of $T_1 \cap T_2 \cap T_3$.

Proposition 2.1.5. *Under the assumptions of Theorem 2.1.3, if T_1 of dimension $\frac{1}{2}\dim M$ acts on M effectively and $T_3 = (\mathbb{R}/\mathbb{Z}) \times (T_1 \cap T_2 \cap T_3)_0$, then for any $X \in \mathfrak{t}_1 \cap \mathfrak{t}_2$, the triple (M, ω, μ^X) is a super-integrable system which has $\frac{1}{2}\dim M + 1$ independent integrals of motion.*

This chapter is based on [1] which is the detailed version of an announcement [2].

Organization: We will give a proof of Theorem 2.1.3 in section 2, and will study the equation (2.1.1) in section 3. In section 4, we will explain some examples illustrating Theorem 2.1.3. We will discuss a relation between Theorem 2.1.3 and super-integrable systems in section 5.

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2.2 A proof of the main theorem (Theorem 2.1.3)

In this section, we prove Theorem 2.1.3. Let us rewrite the theorem in a convenient fashion to give a proof. Let i, j, k satisfy $\{i, j, k\} = \{1, 2, 3\}$ below. Because G is a compact Lie group, we can take a G-invariant inner product $\phi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Let us define identifications $\widetilde{\phi} : \mathfrak{g} \to \mathfrak{g}^*$ and $\widetilde{\phi}' : \mathfrak{t}_i + \mathfrak{t}_j \to (\mathfrak{t}_i + \mathfrak{t}_j)^*$ by $\widetilde{\phi}(X) = \phi(X, \cdot\ )(X \in \mathfrak{g})$ and $\widetilde{\phi}'(Y) = \phi(Y, \cdot\ |_{\mathfrak{t}_i+\mathfrak{t}_j})(Y \in \mathfrak{t}_i + \mathfrak{t}_j)$. The orthogonal decompositions $\mathfrak{g} = (\mathfrak{t}_i + \mathfrak{t}_j) \oplus (\mathfrak{t}_i + \mathfrak{t}_j)^\perp$ with respect to ϕ induce the projections

$$\pi_{ij} : \mathfrak{g} = (\mathfrak{t}_i + \mathfrak{t}_j) \oplus (\mathfrak{t}_i + \mathfrak{t}_j)^\perp \to \mathfrak{t}_i + \mathfrak{t}_j.$$

In fact, we have $\mathrm{Ad}^*(\theta_k)(\mathfrak{t}_i + \mathfrak{t}_j) \subset \mathfrak{t}_i + \mathfrak{t}_j$ (see Lemma 2.2.2(a) below) and the following commutative diagrams for any $\theta_k \in T_k$:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\pi_{ij}} & \mathfrak{t}_i + \mathfrak{t}_j \\ {\scriptstyle\widetilde{\phi}}\downarrow & & \downarrow{\scriptstyle\widetilde{\phi}'} \\ \mathfrak{g}^* & \xrightarrow{R_{ij}} & (\mathfrak{t}_i + \mathfrak{t}_j)^* \end{array} \qquad \begin{array}{ccc} \mathfrak{t}_i + \mathfrak{t}_j & \xrightarrow{\mathrm{Ad}(\theta_k)} & \mathfrak{t}_i + \mathfrak{t}_j \\ {\scriptstyle\widetilde{\phi}'}\downarrow & & \downarrow{\scriptstyle\widetilde{\phi}'} \\ (\mathfrak{t}_i + \mathfrak{t}_j)^* & \xrightarrow{\mathrm{Ad}^*(\theta_k)} & (\mathfrak{t}_i + \mathfrak{t}_j)^*. \end{array}$$

Let us define $\widetilde{\mu} = \widetilde{\phi}^{-1} \circ \mu : M \to \mathfrak{g}$. Although the target space of $\widetilde{\mu}$ is $\mathfrak{g}$ rather than $\mathfrak{g}^*$, we regard both of μ and $\widetilde{\mu}$ moment maps. For any subset A of $\mathfrak{g}$, let us denote by

$$\mathrm{cvx}(A)$$

the convex hull of A in $\mathfrak{g}$. In this settings, we can rewrite theorem 2.1.3 as follows.

Theorem 2.2.1. *Let (M, ω) be a compact connected symplectic manifold. Let G be a compact Lie group and $\mathfrak{g}$ its Lie algebra. Suppose that M has a hamiltonian G-action with a moment a map $\mu : M \to \mathfrak{g}^*$. Assume that the Lie algebras $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3$ of maximal tori T_1, T_2, T_3 of G satisfy the condition* (2.1.1)*. Then the following holds for any $\{i, j, k\} = \{1, 2, 3\}$:*

$$\pi_{ij} \circ \widetilde{\mu}(M) = \mathrm{cvx}\left(\mathrm{Ad}(T_k) \cdot \pi_{ij} \circ \widetilde{\mu}\left(M^{T_i}\right)\right) = \mathrm{cvx}\left(\mathrm{Ad}(T_k) \cdot \pi_{ij} \circ \widetilde{\mu}\left(M^{T_j}\right)\right).$$

Without loss of generality we may assume $i = 1, j = 2$. We first prepare a few lemmas.

Lemma 2.2.2. *The following hold for any* $\{i, j, k\} = \{1, 2, 3\}$*:*

(a) T_k *acts on* $\mathfrak{t}_i + \mathfrak{t}_j$ *via the adjoint action of* G;

(b) $\mathfrak{t}_i + \mathfrak{t}_j = \mathrm{Ad}(T_k)\mathfrak{t}_i$.

Proof. First, let us prove (a). It is sufficient to show that T_3 acts on $\mathfrak{t}_1 + \mathfrak{t}_2$ via the adjoint action of G because of the symmetry of the assumption (2.1.1) with respect to the symbols i, j, k. By the two conditions $[\mathfrak{t}_3, \mathfrak{t}_1] \subset \mathfrak{t}_2$ and $[\mathfrak{t}_3, \mathfrak{t}_2] \subset \mathfrak{t}_1$ implied by the assumption (2.1.1), we have

$$[\mathfrak{t}_3, \mathfrak{t}_1 + \mathfrak{t}_2] \subset \mathfrak{t}_1 + \mathfrak{t}_2.$$

Hence, for every element X_3 of $\mathfrak{t}_3$, we have

$$\mathrm{Ad}(\exp X_3)(\mathfrak{t}_1 + \mathfrak{t}_2) = \exp(\mathrm{ad}(X_3))(\mathfrak{t}_1 + \mathfrak{t}_2) \subset \mathfrak{t}_1 + \mathfrak{t}_2.$$

Now we obtain (a) from the fact $T_3 = \exp \mathfrak{t}_3$. Next, let us prove (b). It suffices to prove $\mathfrak{t}_1 + \mathfrak{t}_2 = \mathrm{Ad}(T_3)\mathfrak{t}_1$. We have $\mathfrak{t}_1 + \mathfrak{t}_2 \supset \mathrm{Ad}(T_3)\mathfrak{t}_1$ by (a). Let us show the converse. That is, we show that for any element X of $\mathfrak{t}_1 + \mathfrak{t}_2$, there exists an element θ_3 of T_3 which satisfy

$$\mathrm{Ad}(\theta_3)X \in \mathfrak{t}_1. \tag{2.2.1}$$

To begin with, recall that there exists an element X' of $\mathfrak{t}_1$ which satisfies $Z_{\mathfrak{g}}(X') = Z_{\mathfrak{g}}(\mathfrak{t}_1)$ because $\mathfrak{t}_1$ is a Lie algebra of T_1 which is a maximal torus of the compact Lie group G. Here, for any subset $S \subset \mathfrak{g}$, $Z_{\mathfrak{g}}(S)$ is defined to be the centralizer of S, i.e.

$$Z_{\mathfrak{g}}(S) = \{W \in \mathfrak{g} \mid [W, A] = 0, A \in S\}.$$

Now, we have $Z_{\mathfrak{g}}(\mathfrak{t}_1) = \mathfrak{t}_1$ because $\mathfrak{t}_1$ is a maximal abelian subalgebra of $\mathfrak{g}$. Hence X' satisfies

$$Z_{\mathfrak{g}}(X') = \mathfrak{t}_1.$$

Let us define a function $F : T_3 \to \mathbb{R}$ by

$$F(\theta) = \phi(\mathrm{Ad}(\theta)X, X') \quad \text{for } \theta \in T_3.$$

Since T_3 is compact, there exists an element θ_3 of T_3 for which $F(\theta_3)$ is a maximum value of F. Then, for any element Y_3 of $\mathfrak{t}_3$, we have

$$\left.\frac{d}{dt}\phi(\mathrm{Ad}(e^{tY_3}\theta_3)X, X')\right|_{t=0} = 0,$$

and the left hand side can be calculated as

$$\phi(\mathrm{ad}(Y_3)(\mathrm{Ad}(\theta_3)X), X') = \phi([Y_3, \mathrm{Ad}(\theta_3)X], X') = \phi(Y_3, [\mathrm{Ad}(\theta_3)X, X']).$$

Thus we obtain

$$\phi(Y_3, [\mathrm{Ad}(\theta_3)X, X']) = 0 \quad \text{for } Y_3 \in \mathfrak{t}_3. \tag{2.2.2}$$

Since (a) implies that $\mathrm{Ad}(\theta_3)X$ is an element of $\mathfrak{t}_1 + \mathfrak{t}_2$, we have

$$[\mathrm{Ad}(\theta_3)X, X'] \in [\mathfrak{t}_1 + \mathfrak{t}_2, \mathfrak{t}_1] = [\mathfrak{t}_2, \mathfrak{t}_1] \subset \mathfrak{t}_3$$

by the assumption (2.1.1). This and (2.2.2) show that $[\mathrm{Ad}(\theta_3)X, X'] = 0$ by the non-degeneracy of the inner product ϕ on $\mathfrak{t}_3$. Hence, we obtain

$$\mathrm{Ad}(\theta_3)X \in Z_{\mathfrak{g}}(X') = \mathfrak{t}_1$$

by the choice of X'. This shows (2.2.1). □

We denote $\pi_{ij} \circ \widetilde{\mu}$ by $\pi_{ij}\widetilde{\mu}$ for brevity.

Lemma 2.2.3. *$\pi_{ij}\widetilde{\mu} : M \to \mathfrak{t}_i + \mathfrak{t}_j$ is T_k-equivariant for any $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. First, the statement makes sense by lemma 2.2.2. Since the moment map $\widetilde{\mu}$ is G-equivariant, it is enough to notice the T_3-equivariance of the projection $\pi_{12} : \mathfrak{g} \to \mathfrak{t}_1 + \mathfrak{t}_2$. This equivariance follows because the inner product ϕ is G-invariant and hence T_3 preserves the orthogonal decomposition $\mathfrak{g} = (\mathfrak{t}_1 + \mathfrak{t}_2) \oplus (\mathfrak{t}_1 + \mathfrak{t}_2)^{\perp}$. □

Let us define

$$\mathfrak{t}'_i = \{X \in \mathfrak{t}_i \mid \phi(X, Z) = 0,\ Z \in \mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3\} \quad (i = 1, 2, 3). \tag{2.2.3}$$

Then the following holds.

Lemma 2.2.4. *$\mathfrak{t}'_i \perp \mathfrak{t}'_j \quad (i \neq j)$*

Proof. It is sufficient to prove that $\mathfrak{t}'_1 \perp \mathfrak{t}'_2$. Take an element X'_1 of $\mathfrak{t}'_1$. By the assumption (2.1.1), we have

$$\mathfrak{t}'_2 \subset \mathfrak{t}_2 = \mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3 + [\mathfrak{t}_3, \mathfrak{t}_1]. \tag{2.2.4}$$

Recall that $[\mathfrak{t}_3, \mathfrak{t}_1]$ is the linear subspace of $\mathfrak{g}$ generated by the subset $\{[Y_3, Y_1] \in \mathfrak{g} \mid Y_3 \in \mathfrak{t}_3, Y_1 \in \mathfrak{t}_1\}$. For any element Z of $\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3$, we have

$$\phi(X'_1, Z) = 0,$$

by the definition of $\mathfrak{t}'_1$. For any element Y_3 of $\mathfrak{t}_3$ and any element Y_1 of $\mathfrak{t}_1$, we have

$$\phi(X'_1, [Y_3, Y_1]) = \phi([Y_1, X'_1], Y_3) = \phi(0, Y_3) = 0,$$

because $\mathfrak{t}_1$ is abelian. Now (2.2.4) shows $\phi(X'_1, \mathfrak{t}_2) = \{0\}$. □

The orthogonal decomposition $\mathfrak{g} = \mathfrak{t}_1 \oplus \mathfrak{t}_1^\perp$ with respect to ϕ induces the projection

$$\pi_1 : \mathfrak{g} = \mathfrak{t}_1 \oplus \mathfrak{t}_1^\perp \to \mathfrak{t}_1.$$

Then we can identify $\pi_1\widetilde{\mu}(M)$ with the moment polytope of the T_1-action as usual.

Lemma 2.2.5. *For any T_1-invariant subset M' of M, we have $\pi_1\widetilde{\mu}(M') \subset \pi_{12}\widetilde{\mu}(M')$.*

Proof. Take any point p of M'. Then $\pi_{23}\widetilde{\mu}(p)$ is an element of $\mathfrak{t}_2 + \mathfrak{t}_3$. Lemma 2.2.2 (b) and Lemma 2.2.3 shows that there exists an element θ_1 of T_1 where $\pi_{23}\widetilde{\mu}(\theta_1 \cdot p)$ is an element of $\mathfrak{t}_3$. The definition of $\mathfrak{t}_2'$ and Lemma 2.2.4 shows that we have an orthogonal direct sum decomposition

$$\mathfrak{t}_2 + \mathfrak{t}_3 = \mathfrak{t}_2' \oplus \mathfrak{t}_3.$$

Since $\pi_{23}\widetilde{\mu}(\theta_1 \cdot p)$ is in $\mathfrak{t}_3$, the element $\widetilde{\mu}(\theta_1 \cdot p)$ is orthogonal to $\mathfrak{t}_2'$. By the similar orthogonal direct sum decomposition

$$\mathfrak{t}_1 + \mathfrak{t}_2 = \mathfrak{t}_1 \oplus \mathfrak{t}_2',$$

we can deduce that $\pi_{12}\widetilde{\mu}(\theta_1 \cdot p)$ is an element of $\mathfrak{t}_1$. Hence we we have

$$\pi_{12}\widetilde{\mu}(\theta_1 \cdot p) = \pi_1\widetilde{\mu}(\theta_1 \cdot p).$$

Thus, the T_1-invariance of M' shows that

$$\pi_1\widetilde{\mu}(p) = \pi_1\widetilde{\mu}(\theta_1 \cdot p) = \pi_{12}\widetilde{\mu}(\theta_1 \cdot p) \in \pi_{12}\widetilde{\mu}(M'). \quad \square$$

Let us prove the convexity of $\pi_{12}\widetilde{\mu}(M)$ with the aid of the above lemmas. Take two points p and p' of M and a point t of $[0, 1]$. Then $(1-t)\pi_{12}\widetilde{\mu}(p) + t\pi_{12}\widetilde{\mu}(p')$ is an element of $\mathfrak{t}_1 + \mathfrak{t}_2$, and so Lemma 2.2.2 (b) and lemma 2.2.3 show that there exists an element θ_3 of T_3 where $(1-t)\pi_{12}\widetilde{\mu}(\theta_3 \cdot p) + t\pi_{12}\widetilde{\mu}(\theta_3 \cdot p')$ is an element of $\mathfrak{t}_1$. Let us define q and q' by

$$q = \theta_3 \cdot p \quad \text{and} \quad q' = \theta_3 \cdot p'.$$

Let $\widetilde{\pi} : \mathfrak{t}_1 + \mathfrak{t}_2 \to \mathfrak{t}_1$ be the orthogonal projection with respect to ϕ, then we have

$$\widetilde{\pi}(Y_1) = Y_1 \quad \text{for } Y_1 \in \mathfrak{t}_1. \tag{2.2.5}$$

We also have $\widetilde{\pi}\pi_{12} = \pi_1$. Since $(1-t)\pi_{12}\widetilde{\mu}(q) + t\pi_{12}\widetilde{\mu}(q')$ is an element of $\mathfrak{t}_1$, we have

$$\begin{aligned}(1-t)\pi_{12}\widetilde{\mu}(q) + t\pi_{12}\widetilde{\mu}(q') &= \widetilde{\pi}((1-t)\pi_{12}\widetilde{\mu}(q) + t\pi_{12}\widetilde{\mu}(q')) \\ &= (1-t)\widetilde{\pi}\pi_{12}\widetilde{\mu}(q) + t\widetilde{\pi}\pi_{12}\widetilde{\mu}(q') \\ &= (1-t)\pi_1\widetilde{\mu}(q) + t\pi_1\widetilde{\mu}(q'),\end{aligned}$$

and this is an element of $\pi_1\widetilde{\mu}(M)$ by theorem 2.1.1. By lemma 2.2.5, we obtain

$$(1-t)\pi_{12}\widetilde{\mu}(q) + t\pi_{12}\widetilde{\mu}(q') \in \pi_{12}\widetilde{\mu}(M).$$

Thus $\pi_{12}\widetilde{\mu}(M)$ is convex.

Next, let us prove that $\pi_{12}\widetilde{\mu}(M)$ is equal to the convex hull of $\mathrm{Ad}(T_k)\pi_{12}\widetilde{\mu}(M^{T_i})$ by using the convexity of $\pi_{12}\widetilde{\mu}(M)$. By an argument similar to that of (2.2.5), we have

$$\pi_{12}\widetilde{\mu}(M) \cap \mathfrak{t}_1 = \widetilde{\pi}\left(\pi_{12}\widetilde{\mu}(M) \cap \mathfrak{t}_1\right) \subset \pi_1\widetilde{\mu}(M).$$

Hence we obtain

$$\pi_{12}\widetilde{\mu}(M) \cap \mathfrak{t}_1 = \pi_1\widetilde{\mu}(M) \tag{2.2.6}$$

by lemma 2.2.5. By applying $\mathrm{Ad}(T_3)$ to the this equality, we have

$$\begin{aligned}
\mathrm{Ad}(T_3)\pi_1\widetilde{\mu}(M) &= \mathrm{Ad}(T_3)(\pi_{12}\widetilde{\mu}(M) \cap \mathfrak{t}_1) \\
&= \bigcup_{\theta_3 \in T_3} (\mathrm{Ad}(\theta_3)\pi_{12}\widetilde{\mu}(M) \cap \mathrm{Ad}(\theta_3)\mathfrak{t}_1) \\
&= \bigcup_{\theta_3 \in T_3} (\pi_{12}\widetilde{\mu}(M) \cap \mathrm{Ad}(\theta_3)\mathfrak{t}_1) \\
&= \pi_{12}\widetilde{\mu}(M) \cap (\mathrm{Ad}(T_3)\mathfrak{t}_1) \\
&= \pi_{12}\widetilde{\mu}(M) \cap (\mathfrak{t}_1 + \mathfrak{t}_2) \\
&= \pi_{12}\widetilde{\mu}(M) \qquad (2.2.7)
\end{aligned}$$

by lemma 2.2.2 (b) and lemma 2.2.3. Since T_1 acts on M^{T_1} trivially, we obtain

$$\pi_1\widetilde{\mu}(M^{T_1}) \subset \pi_{12}\widetilde{\mu}(M^{T_1})$$

by Lemma 2.2.5. Thus we have

$$\begin{aligned}
\pi_{12}\widetilde{\mu}(M) &= \mathrm{Ad}(T_3)\pi_1\widetilde{\mu}(M) \\
&= \mathrm{Ad}(T_3)\mathrm{cvx}(\pi_1\widetilde{\mu}(M^{T_1})) \\
&\subset \mathrm{cvx}\left(\mathrm{Ad}(T_3)\pi_1\widetilde{\mu}(M^{T_1})\right) \\
&\subset \mathrm{cvx}(\mathrm{Ad}(T_3)\pi_{12}\widetilde{\mu}(M^{T_1})) \\
&= \mathrm{cvx}(\pi_{12}\widetilde{\mu}(T_3 \cdot M^{T_1})) \\
&\subset \mathrm{cvx}(\pi_{12}\widetilde{\mu}(M)) \\
&= \pi_{12}\widetilde{\mu}(M).
\end{aligned}$$

Here, we used the convexity of $\pi_{12}\widetilde{\mu}(M)$ at the last equality. This shows $\pi_{12}\widetilde{\mu}(M) = \mathrm{cvx}(\mathrm{Ad}(T_3)\pi_{12}\widetilde{\mu}(M^{T_1}))$. This completes the proof of Theorem 2.2.1 (and hence Theorem 2.1.3).

2.3 The characterization of the Lie subalgebras generated by $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{t}_3$

In this section, we characterize the Lie algebras $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{t}_3$ satisfying the condition (1.1). We use the classification of the compact simple Lie algebras for this purpose.

In the following, we give a proof of Proposition 2.1.4. First of all, let us prove that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Take any two elements $X = X_1 + X_2 + X_3\,(X_i \in \mathfrak{t}_i)$, $Y = Y_1 + Y_2 + Y_3\,(Y_i \in \mathfrak{t}_i)$ of $\mathfrak{h}$. Then we have

$$[X, Y] = \sum_{i,j=1}^{3} [X_i, Y_j]. \tag{2.3.1}$$

Since the condition (2.1.1) implies that

$$[\mathfrak{t}_i, \mathfrak{t}_j] \subset \mathfrak{t}_k \tag{2.3.2}$$

for any i, j, k satisfying $\{i, j, k\} = \{1, 2, 3\}$, the right hand side of (2.3.1) is an element of $\mathfrak{h}$. Thus $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Define

$$\mathfrak{h}' := \mathfrak{t}_1' + \mathfrak{t}_2' + \mathfrak{t}_3'. \tag{2.3.3}$$

where $\mathfrak{t}_i'\,(i = 1, 2, 3)$ are the ones defined in (2.2.3). Let us show that $\mathfrak{h}'$ is a Lie subalgebra of $\mathfrak{h}$. For this, it is suffices to show

$$[\mathfrak{t}_i', \mathfrak{t}_j'] \subset \mathfrak{t}_k' \tag{2.3.4}$$

for any i, j, k satisfying $\{i, j, k\} = \{1, 2, 3\}$. So let us show this property. Note that $[\mathfrak{t}_i', \mathfrak{t}_j']$ is the linear subspace of $\mathfrak{g}$ generated by the subset $\{[X_i, X_j] \in \mathfrak{g} \mid X_i \in \mathfrak{t}_i', X_j \in \mathfrak{t}_j'\}$. For any elements X_i' and X_j' of $\mathfrak{t}_i'$ and $\mathfrak{t}_j'$ respectively, we have

$$[X_i', X_j'] \in [\mathfrak{t}_i', \mathfrak{t}_j'] \subset [\mathfrak{t}_i, \mathfrak{t}_j] \subset \mathfrak{t}_k, \tag{2.3.5}$$

and for any element Z of $\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3$, we have

$$\phi([X_i', X_j'], Z) = \phi(X_i', [X_j', Z]) = \phi(X_i', 0) = 0.$$

Since this shows (2.3.4), we proved that $\mathfrak{h}'$ is a Lie subalgebra of $\mathfrak{h}$.

Next, let us show that we have an orthogonal decomposition

$$\mathfrak{h} = (\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3) \oplus \mathfrak{h}' \tag{2.3.6}$$

which is also a decomposition as Lie algebra. By the definition of $\mathfrak{t}_i'$, we have an orthogonal decomposition

$$\mathfrak{t}_i = (\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3) \oplus \mathfrak{t}_i' \quad (i = 1, 2, 3). \tag{2.3.7}$$

So we obtain $\mathfrak{h} \subset \mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3 + \mathfrak{h}'$. Since $\mathfrak{h}'$ and $\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3$ is orthogonal with respect to ϕ by the definitions of $\mathfrak{h}'$, their intersection is $\{0\}$. Thus we obtain the orthogonal decomposition $\mathfrak{h} = \mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3 \oplus \mathfrak{h}'$. This is in fact a decomposition of $\mathfrak{h}$ as a Lie algebra because each $\mathfrak{t}_i$ is abelian.

Lemma 2.3.1. $\mathfrak{t}_i'(i=1,2,3)$ *is a maximal abelian subalgebra of* $\mathfrak{h}'$.

Proof. We prove this only for $\mathfrak{t}_1$. Suppose that an element X' of $\mathfrak{h}'$ satisfies

$$[X', \mathfrak{t}_1'] = \{0\}.$$

Then, since $\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3$ commutes with $\mathfrak{h}'$, we obtain $[X', \mathfrak{t}_1] = \{0\}$. Hence, by the maximality of $\mathfrak{t}_1$, X' is an element of $\mathfrak{h}' \cap \mathfrak{t}_1$. Since $(\mathfrak{t}_1' + \mathfrak{t}_2' + \mathfrak{t}_3') \perp (\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3)$, we obtain

$$\mathfrak{h}' \cap \mathfrak{t}_1 = (\mathfrak{t}_1' + \mathfrak{t}_2' + \mathfrak{t}_3') \cap \mathfrak{t}_1 \subset \mathfrak{t}_1'.$$

So X' is an element of $\mathfrak{t}_1'$. □

Lemma 2.3.2. $\mathfrak{h}'$ *is a compact semi-simple Lie algebra.*

Proof. The reason for $\mathfrak{h}'$ being compact semi-simple is that $\mathfrak{h}'$ is a Lie subalgebra of the compact Lie algebra $\mathfrak{g}$ and that $\mathfrak{h}'$ has trivial center. Here, we give a proof for the reader. Let X' be an element of $\mathfrak{h}'$ which satisfies

$$\mathrm{Tr}(\mathrm{ad}_{\mathfrak{h}'}(X') \circ \mathrm{ad}_{\mathfrak{h}'}(Y')) = 0 \quad (Y' \in \mathfrak{h}'). \tag{2.3.8}$$

We show that $X' = 0$. Since $\mathfrak{g}$ has the G-invariant inner product ϕ, the adjoint action can be written by $\mathrm{Ad} : G \to O(\mathfrak{g})$ with respect to ϕ. Differentiating this, we obtain

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{o}(\mathfrak{g}).$$

Let us write the adjoint action of $\mathfrak{h}'$ by $\mathrm{ad}_{\mathfrak{h}'} : \mathfrak{h}' \to \mathfrak{gl}(\mathfrak{h}')$. The inner product ϕ on $\mathfrak{g}$ induces an inner product ϕ' on $\mathfrak{h}'$, and this map can be written by

$$\mathrm{ad}_{\mathfrak{h}'} : \mathfrak{h}' \to \mathfrak{o}(\mathfrak{h}') \tag{2.3.9}$$

with respect to this inner product ϕ'. Now, by choosing a basis of $\mathfrak{h}'$, $\mathrm{ad}_{\mathfrak{h}'}(X')$ is represented by a skew-symmetric matrix. Let us denote this matrix by (a_{ij}). Choosing X' as Y in (2.3.8), we obtain

$$\mathrm{Tr}(\mathrm{ad}_{\mathfrak{h}'}(X') \circ \mathrm{ad}_{\mathfrak{h}'}(X')) = 0.$$

Since the left hand side of this equation can be written by

$$\sum_{i,j=1}^{n} a_{ij} a_{ji} = -\sum_{i,j=1}^{n} {a_{ij}}^2 \tag{2.3.10}$$

with respect to the above representation, we obtain $a_{ij} = 0 (i, j = 1, \cdots, n)$. This means

$$\mathrm{ad}_{\mathfrak{h}'}(X') = 0. \tag{2.3.11}$$

Now let us show that (2.3.9) is an injection to prove $X' = 0$. Here, since the injectivity of (2.3.9) and the triviality of the center of $\mathfrak{h}'$ are equivalent, we show the latter. Let W be an element of the center of $\mathfrak{h}'$. That is, we have

$$[W, Y'] = 0 \quad \text{for } Y' \in \mathfrak{h}'.$$

Since $\mathfrak{t}'_1 \subset \mathfrak{h}'$, we have $[W, \mathfrak{t}'_1] = \{0\}$. Then Lemma 2.3.1 shows that W is an element of $\mathfrak{t}'_1$. Similarly, we can show $W \in \mathfrak{t}'_2$. Since we have $\mathfrak{t}'_1 \cap \mathfrak{t}'_2 = 0$ by Lemma 2.2.4, we obtain $W = 0$. This shows that the center of $\mathfrak{h}$ is 0 as desired, and (2.3.9) is injective. Thus we obtain $X' = 0$ by (2.3.11). This shows that $\mathfrak{h}'$ is semi-simple. Now by (2.3.10), $\mathfrak{h}'$ is a compact Lie algebra. $\square$

Since $\mathfrak{h}'$ is a compact semi-simple Lie algebra, there exists ideals $\mathfrak{h}'_1, \cdots, \mathfrak{h}'_n$ of $\mathfrak{h}'$ which satisfy

1) $\mathfrak{h}'_1, \cdots, \mathfrak{h}'_n$ are compact simple Lie algebras,
2) $\mathfrak{h}' = \mathfrak{h}'_1 \oplus \cdots \oplus \mathfrak{h}'_n$,

Then Lemma 2.2.4 and Lemma 2.3.1 shows that

$$\dim \mathfrak{h}' = \dim \mathfrak{t}'_1 + \dim \mathfrak{t}'_2 + \dim \mathfrak{t}'_3 = 3 \dim \mathfrak{t}'_1 = 3 \operatorname{rank} \mathfrak{h}'. \tag{2.3.12}$$

Also, the decomposition $\mathfrak{h}' = \mathfrak{h}'_1 \oplus \cdots \oplus \mathfrak{h}'_n$, means

$$\begin{aligned} \dim \mathfrak{h}' &= \dim \mathfrak{h}'_1 + \cdots + \dim \mathfrak{h}'_n \\ \operatorname{rank} \mathfrak{h}' &= \operatorname{rank} \mathfrak{h}'_1 + \cdots + \operatorname{rank} \mathfrak{h}'_n. \end{aligned}$$

Combining these equalities, we obtain

$$(\dim \mathfrak{h}'_1 - 3 \operatorname{rank} \mathfrak{h}'_1) + \cdots + (\dim \mathfrak{h}'_n - 3 \operatorname{rank} \mathfrak{h}'_n) = 0. \tag{2.3.13}$$

The classification of compact simple Lie algebras shows that, for any compact semi-simple Lie algebra $\mathfrak{a}$, we have

$$\dim \mathfrak{a} - 3 \operatorname{rank} \mathfrak{a} \geq 0,$$

and we have equality if and only if $\mathfrak{a} \cong \mathfrak{su}(2)$ (i.e. $\dim \mathfrak{a} = 3$ and $\operatorname{rank} \mathfrak{a} = 1$). Thus (2.3.13) shows that

$$\mathfrak{h}'_l \cong \mathfrak{su}(2) \quad \text{for } l = 1, \cdots, n.$$

Now the decomposition $\mathfrak{h} = \mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3 \oplus \mathfrak{h}'$ (as Lie algebras) shows that $\mathfrak{h} \cong \mathbb{R}^m \oplus \mathfrak{su}(2)^{\oplus n}$ where $\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3$ corresponds to $\mathbb{R}^m$.

In the following, we regard $\mathfrak{su}(2)^{\oplus n} = \mathfrak{h}' = \mathfrak{t}'_1 + \mathfrak{t}'_2 + \mathfrak{t}'_3 \subset \mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3$. Lemma 2.2.4 shows that

$$3 \dim \mathfrak{t}'_i = \dim(\mathfrak{t}'_1 + \mathfrak{t}'_2 + \mathfrak{t}'_3) = \dim(\mathfrak{su}(2)^{\oplus n}) = 3n.$$

So $\mathfrak{t}_i'$ is an n-dimensional abelian Lie subalgebra of $\mathfrak{su}(2)^{\oplus n}$ $(i = 1, 2, 3)$, and this shows that $m + n = \dim \mathfrak{t}_i = \operatorname{rank} G$. Let us define

$$X_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ X_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then each $(\mathrm{span}\{X_i\})^{\oplus n} \subset \mathfrak{su}(2)^{\oplus n}$ is a maximal abelian Lie subalgebra of dimension n for $i = 1, 2, 3$. This shows that $\mathfrak{t}_1'$ is also a maximal abelian Lie subalgebra of $\mathfrak{su}(2)^{\oplus n}$ because of its dimension. Now Cartan's theorem shows that there exists elements $g_{i1}, \cdots, g_{in}$ of $SU(2)$ such that

$$\mathrm{Ad}(g_{i1}, \cdots, g_{in})((\mathrm{span}\{X_i\})^{\oplus n}) = \mathfrak{t}_i'$$

So we obtain

$$\mathrm{Ad}(g_{i1})(\mathrm{span}\{X_i\}) \oplus \cdots \oplus \mathrm{Ad}(g_{in})(\mathrm{span}\{X_i\}) = \mathfrak{t}_i',$$

and this shows

$$\dim(\mathfrak{t}_i' \cap \mathfrak{su}(2)) = 1 \tag{2.3.14}$$

for each summand in $\mathfrak{su}(2)^{\oplus n}$. That is, we obtain

$$\mathfrak{t}_i' = (\mathfrak{t}_i' \cap \mathfrak{su}(2)) \oplus \cdots \oplus (\mathfrak{t}_i' \cap \mathfrak{su}(2)) \tag{2.3.15}$$

because of its dimension.

Fix a summand $\mathfrak{su}(2)$ of $\mathfrak{su}(2)^{\oplus n}$, and choose a basis e_i of $\mathfrak{t}_i' \cap \mathfrak{su}(2)$ $(i = 1, 2, 3)$ where each e_i is a unit vector with respect to the inner product $-2\mathrm{Tr}(\cdot, \cdot)$ on $\mathfrak{su}(2)$. Then (2.3.14), (2.3.15) and the condition (2.1.1) now state that $[e_1, e_2] \neq 0, [e_2, e_3] \neq 0, [e_3, e_1] \neq 0$ and

$$[e_i, e_j] \in \mathfrak{t}_k \cap \mathfrak{su}(2) \quad \text{for } \{i, j, k\} = \{1, 2, 3\}.$$

So there exists $a, b, c \in \mathbb{R} - \{0\}$ such that

$$[e_1, e_2] = ae_3, \ [e_2, e_3] = be_1, [e_3, e_1] = ce_2.$$

Since $[e_i, e_j]$ has to be orthogonal to e_i and e_j, this condition shows that $\{e_1, e_2, e_3\}$ forms an orthonormal basis of $\mathfrak{su}(2)$ (with respect to the above inner product on $\mathfrak{su}(2)$). Recalling that the adjoint action $\mathrm{Ad} : SU(2) \to O(\mathfrak{su}(2)) \cong SO(3)$ is a surjection, there exists en element $g \in SU(2)$ such that

$$\begin{gathered} \mathrm{Ad}(g)X_1 = e_1, \ \mathrm{Ad}(g)X_2 = e_2, \ \mathrm{Ad}(g)X_3 = e_3, \\ \text{or} \\ \mathrm{Ad}(g)X_1 = e_1, \ \mathrm{Ad}(g)X_3 = e_3, \ \mathrm{Ad}(g)X_2 = e_2, \end{gathered}$$

depending on their orientations. Since we have

$$[X_1, X_2] = X_3, \ [X_2, X_3] = X_1, \ [X_3, X_1] = X_2$$

by direct calculations, we obtain

$$[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2,$$
$$\text{or}$$
$$[e_1, e_2] = -e_3,\ [e_2, e_3] = -e_1,\ [e_3, e_1] = -e_2.$$

Retaking e_3 by $-e_3$ in the latter case, we obtain

$$[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2$$

as claimed in Proposition 2.1.4.

Let us prove the converse. Assume that $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{t}_3$ satisfy the conditions in Proposition 2.1.4. The standard basis of $\mathbb{R}^m = \mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3$ and the collection of e_i in each summand in $\mathfrak{su}(2)^{\oplus n}$ form a basis of $\mathfrak{t}_i$ $(i = 1, 2, 3)$ since $m + n = \operatorname{rank} G = \dim \mathfrak{t}_i$. Let i, j, k be integers such that $\{i, j, k\} = \{1, 2, 3\}$. Then the conditions $[e_j, e_k] = e_i$ on each summand in $\mathfrak{su}(2)^{\oplus n}$ insist that $[\mathfrak{t}_j, \mathfrak{t}_k]$ is contained in $\mathfrak{t}_i$ and we have $\dim[\mathfrak{t}_j, \mathfrak{t}_k] = n$. Take a G-invariant inner product ϕ on $\mathfrak{g}$. Then $\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3$ is orthogonal to $[\mathfrak{t}_j, \mathfrak{t}_k]$ because of G-invariance of ϕ. Hence we obtain

$$(\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3) + [\mathfrak{t}_j, \mathfrak{t}_k] = (\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3) \oplus [\mathfrak{t}_j, \mathfrak{t}_k] \subset \mathfrak{t}_i.$$

Since we have $m + n = \operatorname{rank} G = \dim \mathfrak{t}_i$, we obtain

$$(\mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3) + [\mathfrak{t}_j, \mathfrak{t}_k] = \mathfrak{t}_i.$$

This completes our proof of Proposition 2.1.4.

2.4 Examples

In this section, we give some examples which illustrate Theorem 2.1.3. It is convenient to use Theorem 2.2.1 rather than Theorem 2.1.3 for those examples. Observe that (2.2.6) and (2.2.7) show the following corollaries, and those will be useful to understand examples in this section.

Corollary 2.4.1. *Under the assumptions of Theorem 2.2.1, the following holds.*

$$(\pi_{12} \circ \widetilde{\mu}(M)) \cap \mathfrak{t}_1 = \pi_1 \circ \widetilde{\mu}(M)$$

Corollary 2.4.2. *Under the assumptions of Theorem 2.2.1, the following holds.*

$$\pi_{12} \circ \widetilde{\mu}(M) = \mathrm{Ad}(T_3)(\pi_1 \circ \widetilde{\mu}(M))$$

That is, the image $\pi_{12} \circ \widetilde{\mu}(M)$ is the $\mathrm{Ad}(T_3)$*-orbit of the moment polytope for the T_1 action.*

Remark 2.4.3. By the symmetry of the indexes in the assumption 2.1.1, the same statements hold after replacing $\pi_1 \circ \widetilde{\mu}(M)$ to $\pi_2 \circ \widetilde{\mu}(M)$ where $\pi_i : \mathfrak{t}_1 + \mathfrak{t}_2 \to \mathfrak{t}_i$ is the orthogonal projections.

2.4.1 Example 1: the 2-dimensional sphere

The 2dimensional sphere S^2 is a symplectic manifold whose symplectic form is the pull-back of the standard orientation form on $\mathbb{R}^3$. The standard $SO(3)$-action on S^2 is a hamiltonian action. Let us define a $SO(3)$-invariant inner product $\phi : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathbb{R}$ by

$$\phi(X, Y) = \mathrm{Tr}({}^t XY) \quad (X, Y \in \mathfrak{so}(3)).$$

Then the map $\widetilde{\mu} : S^2 \longrightarrow \mathfrak{so}(3)$ defined by

$$\widetilde{\mu}(x) = \frac{1}{2}\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \quad (x \in S^2)$$

is a moment map of the $SO(3)$-action on S^2. Now, define

$$\mathfrak{t}_1 = \left\{ \left(\begin{array}{c|cc} 0 & -\theta & 0 \\ \hline \theta & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \in \mathfrak{so}(3) \,\middle|\, \theta \in \mathbb{R} \right\},$$

$$\mathfrak{t}_2 = \left\{ \left(\begin{array}{c|cc} 0 & 0 & \eta \\ \hline 0 & 0 & 0 \\ -\eta & 0 & 0 \end{array}\right) \in \mathfrak{so}(3) \,\middle|\, \eta \in \mathbb{R} \right\},$$

$$\mathfrak{t}_3 = \left\{ \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & -\xi \\ 0 & \xi & 0 \end{array}\right) \in \mathfrak{so}(3) \,\middle|\, \xi \in \mathbb{R} \right\},$$

and define

$$T_1 = \exp \mathfrak{t}_1, T_2 = \exp \mathfrak{t}_2, T_3 = \exp \mathfrak{t}_3,$$

then T_1, T_2, and T_3 are maximal tori of $SO(3)$. Now, the map $\pi_{12} \circ \widetilde{\mu} : S^2 \longrightarrow \mathfrak{t}_1 + \mathfrak{t}_2$ is written by

$$\pi_{12} \circ \widetilde{\mu}(x) = \frac{1}{2}\left(\begin{array}{c|cc} 0 & -x_3 & x_2 \\ \hline x_3 & 0 & 0 \\ -x_2 & 0 & 0 \end{array}\right) \quad (x \in S^2).$$

From this explicit form, it is clear that the image $\pi_{12} \circ \widetilde{\mu}(S^2)$ is a closed disc. Let us calculate the image $\pi_{12} \circ \widetilde{\mu}(S^2)$ by Theorem 2.2.1. First, we have

$$(S^2)^{T_1} = \{(0, 0, 1), (0, 0, -1)\}.$$

So we obtain

$$\pi_{12} \circ \widetilde{\mu}((S^2)^{T_1}) = \left\{ \frac{1}{2}\left(\begin{array}{c|cc} 0 & -1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right), \frac{1}{2}\left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \right\}. \tag{2.4.1}$$

We also have

$$T_3 = \exp \mathfrak{t}_3 = \left\{ \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \cos\xi & -\sin\xi \\ 0 & \sin\xi & \cos\xi \end{array} \right) \in SO(3) \,\middle|\, \xi \in \mathbb{R} \right\}. \tag{2.4.2}$$

Now a direct calculation shows

$$\mathrm{Ad}(T_3)(\pi_{12} \circ \widetilde{\mu}((S^2)^{T_1})) = \left\{ \frac{1}{2} \left(\begin{array}{c|cc} 0 & -\cos\xi & -\sin\xi \\ \hline \cos\xi & 0 & 0 \\ \sin\xi & 0 & 0 \end{array} \right) \in \mathfrak{so}(3) \,\middle|\, \xi \in \mathbb{R} \right\}.$$

Since this is a circle in the plane $\mathfrak{t}_1 + \mathfrak{t}_2$, by Theorem 2.2.1, the image $\pi_{12} \circ \widetilde{\mu}(S^2)$ is a closed disk which is given by the convex hull of this circle (Figure2.4.1). Note that, from Figure 2.4.1, it is obvious that generic level sets of $\pi_{12} \circ \widetilde{\mu} : S^2 \longrightarrow \mathfrak{t}_1 + \mathfrak{t}_2$ are not connected.

Remark 2.4.4. In general, we always have $(\pi_{12} \circ \widetilde{\mu})^{-1}(0) = (\pi_1 \circ \widetilde{\mu})^{-1}(0) \cap (\pi_2 \circ \widetilde{\mu})^{-1}(0)$ under the assumption of Theorem 2.2.1 where $\pi_i \circ \widetilde{\mu} \to \mathfrak{t}_i$ is a moment map of the T_i-action for $i = 1, 2$.

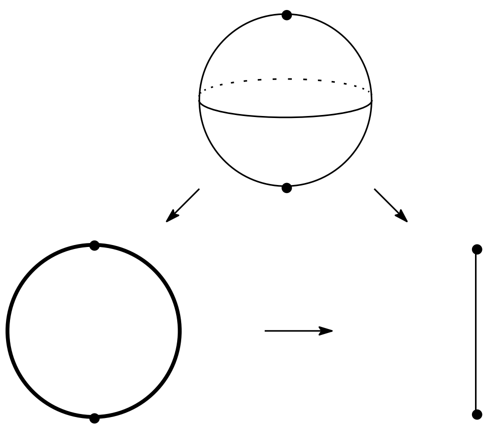


Figure 2.4.1: S^2, $\pi_{12} \circ \widetilde{\mu}(S^2)$ and the moment polytope for T_1 action

2.4.2 Example 2: the complex projective spaces

The complex projective space $\mathbb{C}P^n$ is a symplectic manifold whose symplectic form is the Fubini-Study form. The standard $SU(n+1)$-action on $\mathbb{C}P^n$ is a hamiltonian action with a moment map $\mu : \mathbb{C}P^n \to \mathfrak{su}(n+1)^*$. We identify $\mathfrak{su}(n+1)$ and $\mathfrak{su}(n+1)^*$ by the $SU(n+1)$-invariant inner product $-\mathrm{Tr}(\cdot \times \cdot)$ on $\mathfrak{su}(n+1)$. First, we construct $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{t}_3$ in $\mathfrak{su}(n+1)$. For simplicity, let us assume $n = 2k (k \geq 1)$. We note a similar construction for the case $n = 2k+1 (k \geq$

0) in the appendix. Define $X_1, \cdots, X_n, Y_1, \cdots, Y_n, Z_1, \cdots, Z_n, W_1, \cdots, W_k \in \mathfrak{su}(2k+1)$ by

$$X_1 = \begin{pmatrix} 0 &&&&&&& \\ & -i & 0 &&&&& \\ & 0 & i &&&&& \\ &&& 0 & 0 &&& \\ &&& 0 & 0 &&& \\ &&&&& \ddots && \\ &&&&&& 0 & 0 \\ &&&&&& 0 & 0 \end{pmatrix}, \cdots, X_k = \begin{pmatrix} 0 &&&&&&& \\ & 0 & 0 &&&&& \\ & 0 & 0 &&&&& \\ &&& 0 & 0 &&& \\ &&& 0 & 0 &&& \\ &&&&& \ddots && \\ &&&&&& -i & 0 \\ &&&&&& 0 & i \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 &&&&&&& \\ & 0 & -1 &&&&& \\ & 1 & 0 &&&&& \\ &&& 0 & 0 &&& \\ &&& 0 & 0 &&& \\ &&&&& \ddots && \\ &&&&&& 0 & 0 \\ &&&&&& 0 & 0 \end{pmatrix}, \cdots, Y_k = \begin{pmatrix} 0 &&&&&&& \\ & 0 & 0 &&&&& \\ & 0 & 0 &&&&& \\ &&& 0 & 0 &&& \\ &&& 0 & 0 &&& \\ &&&&& \ddots && \\ &&&&&& 0 & -1 \\ &&&&&& 1 & 0 \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} 0 &&&&&&& \\ & 0 & i &&&&& \\ & i & 0 &&&&& \\ &&& 0 & 0 &&& \\ &&& 0 & 0 &&& \\ &&&&& \ddots && \\ &&&&&& 0 & 0 \\ &&&&&& 0 & 0 \end{pmatrix}, \cdots, Z_k = \begin{pmatrix} 0 &&&&&&& \\ & 0 & 0 &&&&& \\ & 0 & 0 &&&&& \\ &&& 0 & 0 &&& \\ &&& 0 & 0 &&& \\ &&&&& \ddots && \\ &&&&&& 0 & i \\ &&&&&& i & 0 \end{pmatrix},$$

$$W_1 = \begin{pmatrix} -2i &&&&&&& \\ & i & 0 &&&&& \\ & 0 & i &&&&& \\ &&& 0 & 0 &&& \\ &&& 0 & 0 &&& \\ &&&&& \ddots && \\ &&&&&& 0 & 0 \\ &&&&&& 0 & 0 \end{pmatrix}, \cdots, \ W_k = \begin{pmatrix} -2i &&&&&&& \\ & 0 & 0 &&&&& \\ & 0 & 0 &&&&& \\ &&& 0 & 0 &&& \\ &&& 0 & 0 &&& \\ &&&&& \ddots && \\ &&&&&& i & 0 \\ &&&&&& 0 & i \end{pmatrix}.$$

We define the Lie subalgebras $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{t}_3$ of $\mathfrak{su}(2k+1)$ by

$$\begin{aligned} \mathfrak{t}_1 &= \operatorname{span}\{W_1, \cdots, W_k, X_1, \cdots, X_k\}, \\ \mathfrak{t}_2 &= \operatorname{span}\{W_1, \cdots, W_k, Y_1, \cdots, Y_k\}, \\ \mathfrak{t}_3 &= \operatorname{span}\{W_1, \cdots, W_k, Z_1, \cdots, Z_k\}. \end{aligned}$$

Then these satisfy the condition (2.1.1), and

$$T_1 = \exp \mathfrak{t}_1,\ T_2 = \exp \mathfrak{t}_2,\ T_3 = \exp \mathfrak{t}_3$$

are maximal tori of $SU(2k+1)$. Now we consider the map

$$\pi_{12} \circ \widetilde{\mu} : \mathbb{C}P^{2k} \longrightarrow \mathfrak{t}_1 + \mathfrak{t}_2.$$

Theorem 2.2.1 states that the image of this map is the convex hull of the $\mathrm{Ad}(T_3)$-orbit of the image of the fixed point set M^{T_1} of the T_1-action. Here, the fixed point set $(\mathbb{C}P^{2k})^{T_1}$ is

$$(\mathbb{C}P^{2k})^{T_1} = \{[1, 0, \cdots, 0], [0, 1, \cdots, 0], \cdots, [0, \cdots, 0, 1]\}.$$

In the following, we study the case $k = 1$ in detail.

The case $k = 1$ (the complex projective plane $\mathbb{C}P^2$):

Let us give the explicit description of the image of $\pi_{12} \circ \mu$ for the case of the complex projective plane $\mathbb{C}P^2$ with the standard hamiltonian $SU(3)$-action. We can take

$$\widetilde{\mu}([z]) = -\frac{i}{2|z|^2} \begin{pmatrix} |z_0|^2 - |z|^2/3 & z_0\overline{z_1} & z_0\overline{z_2} \\ z_1\overline{z_0} & |z_1|^2 - |z|^2/3 & z_1\overline{z_2} \\ z_2\overline{z_0} & z_2\overline{z_1} & |z_2|^2 - |z|^2/3 \end{pmatrix} \quad (z \in \mathbb{C}^3 - \{\mathbf{0}\})$$

as a moment map of this action. The above construction gives us

$$\mathfrak{t}_1 = \left\{ \left(\begin{array}{c|cc} -2i\zeta & 0 & 0 \\ \hline 0 & i\zeta & 0 \\ 0 & 0 & i\zeta \end{array}\right) + \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & i\theta & 0 \\ 0 & 0 & -i\theta \end{array}\right) \in \mathfrak{su}(3) \,\middle|\, \zeta, \theta \in \mathbb{R} \right\},$$

$$\mathfrak{t}_2 = \left\{ \left(\begin{array}{c|cc} -2i\zeta & 0 & 0 \\ \hline 0 & i\zeta & 0 \\ 0 & 0 & i\zeta \end{array}\right) + \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & -\eta \\ 0 & \eta & 0 \end{array}\right) \in \mathfrak{su}(3) \,\middle|\, \zeta, \eta \in \mathbb{R} \right\},$$

$$\mathfrak{t}_3 = \left\{ \left(\begin{array}{c|cc} -2i\zeta & 0 & 0 \\ \hline 0 & i\zeta & 0 \\ 0 & 0 & i\zeta \end{array}\right) + \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & i\xi \\ 0 & i\xi & 0 \end{array}\right) \in \mathfrak{su}(3) \,\middle|\, \zeta, \xi \in \mathbb{R} \right\},$$

Now, the map $\pi_{12} \circ \widetilde{\mu} : \mathbb{C}P^2 \longrightarrow \mathfrak{t}_1 + \mathfrak{t}_2$ is given by

$$\pi_{12} \circ \widetilde{\mu}([z]) = -\frac{i}{2|z|^2} \left(\begin{array}{c|cc} |z_0|^2 - |z|^2/3 & 0 & 0 \\ \hline 0 & |z_1|^2 - |z|^2/3 & i\mathrm{Im}(z_1\bar{z}_2) \\ 0 & -i\mathrm{Im}(z_1\bar{z}_2) & |z_2|^2 - |z|^2/3 \end{array}\right) \quad (z \in \mathbb{C}^3 - \{\mathbf{0}\}).$$

Let us calculate the image $\pi_{12} \circ \widetilde{\mu}(\mathbb{C}P^2)$. At first, we have

$$(\mathbb{C}P^2)^{T_1} = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\},$$

$$\pi_{12} \circ \widetilde{\mu}((\mathbb{C}P^2)^{T_1}) = \left\{ \frac{1}{6}\left(\begin{array}{c|cc} -2i & 0 & 0 \\ \hline 0 & i & 0 \\ 0 & 0 & i \end{array}\right), \frac{1}{6}\left(\begin{array}{c|cc} i & 0 & 0 \\ \hline 0 & -2i & 0 \\ 0 & 0 & i \end{array}\right), \frac{1}{6}\left(\begin{array}{c|cc} i & 0 & 0 \\ \hline 0 & i & 0 \\ 0 & 0 & -2i \end{array}\right) \right\}. \tag{2.4.3}$$

We also have

$$T_3 = \exp \mathfrak{t}_3 = \left\{ \left(\begin{array}{c|cc} e^{-2i\zeta} & 0 & 0 \\ \hline 0 & e^{i\zeta} & 0 \\ 0 & 0 & e^{i\zeta} \end{array}\right) \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \cos\xi & i\sin\xi \\ 0 & i\sin\xi & \cos\xi \end{array}\right) \in \mathfrak{su}(3) \,\middle|\, \zeta, \xi \in \mathbb{R} \right\}.$$

So we obtain

$$\begin{aligned} &\mathrm{Ad}(T_3)(\pi_{12} \circ \widetilde{\mu}((\mathbb{C}P^2)^{T_1})) \\ &\quad = \left\{ \frac{1}{6}\left(\begin{array}{c|cc} -2i & 0 & 0 \\ \hline 0 & i & 0 \\ 0 & 0 & i \end{array}\right) \right\} \\ &\qquad \cup \left\{ -\frac{1}{12}\left(\begin{array}{c|cc} -2i & 0 & 0 \\ \hline 0 & i & 0 \\ 0 & 0 & i \end{array}\right) - \frac{1}{4}\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & i\cos 2\xi & \sin 2\xi \\ 0 & -\sin 2\xi & -i\cos 2\xi \end{array}\right) \in \mathfrak{su}(3) \,\middle|\, \xi \in \mathbb{R} \right\}. \end{aligned}$$

Defining

$$a_1 = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & i & 0 \\ 0 & 0 & -i \end{array}\right), \; a_2 = \left(\begin{array}{c|cc} -2i & 0 & 0 \\ \hline 0 & i & 0 \\ 0 & 0 & i \end{array}\right), \; a_3 = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right),$$

we have $\mathfrak{t}_1 = \mathrm{span}\{a_1, a_2\}$, $\mathfrak{t}_1 + \mathfrak{t}_2 = \mathrm{span}\{a_1, a_2, a_3\}$. Now we can write

$$\begin{aligned} &\mathrm{Ad}(T_3)(\pi_{12} \circ \widetilde{\mu}((\mathbb{C}P^2)^{T_1})) \\ &\quad = \left\{ \frac{1}{6} a_2 \right\} \cup \left\{ -\frac{1}{12} a_2 - \frac{1}{4}(\cos 2\xi) a_1 - \frac{1}{4}(\sin 2\xi) a_3 \in \mathfrak{su}(3) \,\middle|\, \xi \in \mathbb{R} \right\}. \end{aligned}$$

This is a union of a point and a circle in $\mathfrak{t}_1 + \mathfrak{t}_2 + \mathfrak{t}_3$, and Theorem 2.2.1 states that the image $\pi_{12} \circ \widetilde{\mu}(\mathbb{C}P^2)$ is the closed cone with its interior which is given by the convex hull of the point and the circle (Figure2.4.2).

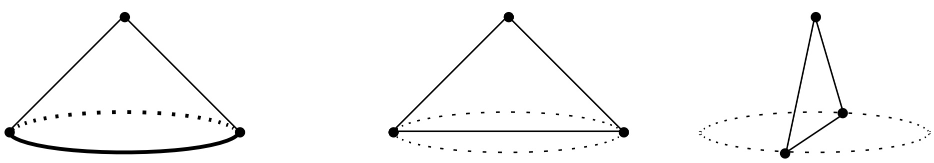


Figure 2.4.2: $\pi_{12} \circ \widetilde{\mu}(\mathbb{C}P^2)$ and the moment polytopes for T_1 and T_2 actions

2.4.3 Example 3: the flag manifolds

The flag manifold $Fl_n(\mathbb{C})$ is a symplectic manifold as a coadjoint orbit of $SU(n)$. The $SU(n)$-action on $Fl_n(\mathbb{C})$ is a hamiltonian action and the inclusion of the orbit into $\mathfrak{su}(n+1)^*$ is a moment map. Denote this inclusion by $\mu : Fl_n(\mathbb{C}) \to \mathfrak{su}(n)^*$ Taking the same Lie subalgebras $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{t}_3$ of $\mathfrak{su}(n)$ as in previous example, we can consider the map

$$\pi_{12} \circ \mu : Fl_n(\mathbb{C}) \longrightarrow \mathfrak{t}_1 + \mathfrak{t}_2.$$

Theorem 2.2.1 states that the image of this map is the convex hull of the $\mathrm{Ad}(T_3)$-orbit of the image of the fixed point set M^{T_1}. Here, the fixed point set M^{T_1} is the coordinate flags. In the following, we study the case $n = 3$ in detail.

The case $n = 3$ (the flag manifold $Fl_3(\mathbb{C})$):

Let X be an element of $\mathfrak{su}(3)$ of the form

$$X = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

We identify the flag manifold $Fl_3(\mathbb{C})$ with the $SU(3)$-adjoint orbit of X via the SU(3)-invariant inner product $-\mathrm{Tr}(\cdot,\cdot)$. Then we can take the inclusion $\iota : Fl_3(\mathbb{C}) \hookrightarrow \mathfrak{su}(3)$ as a moment map $\widetilde{\mu}$ of the $SU(3)$-action on $Fl_3(\mathbb{C})$.

Composing $\iota : Fl_3(\mathbb{C}) \hookrightarrow \mathfrak{su}(3)$ with the orthogonal projection $\pi_{12} : \mathfrak{su}(3) \to \mathfrak{t}_1 + \mathfrak{t}_2$, we obtain a map

$$\pi_{12} \circ \iota : Fl_3(\mathbb{C}) \to \mathfrak{t}_1 + \mathfrak{t}_2.$$

Let us calculate the image $\pi_{12} \circ \iota(Fl_3(\mathbb{C}))$. At first, we have

$$Fl_3(\mathbb{C})^{T_1} = \left\{ \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}, \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}, \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Let us define

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix},\ a_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -2i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix},\ a_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then we have

$$\mathfrak{t}_1 = \mathrm{span}\{a_1, a_2\},\ \mathfrak{t}_1 + \mathfrak{t}_2 = \mathrm{span}\{a_1, a_2, a_3\},$$

and

$$Fl_3(\mathbb{C})^{T_1} = \left\{ \frac{1}{2}a_1 - \frac{\sqrt{3}}{2}a_2,\ -\frac{1}{2}a_1 + \frac{\sqrt{3}}{2}a_2,\ a_1, \right.$$
$$\left. -a_1,\ \frac{1}{2}a_1 + \frac{\sqrt{3}}{2}a_2,\ -\frac{1}{2}a_1 - \frac{\sqrt{3}}{2}a_2 \right\}.$$

Now $\pi_{12} \circ \iota : Fl_3(\mathbb{C}) \to \mathfrak{t}_1 + \mathfrak{t}_2$ maps each of elements in M^{T_1} to itself since they are elements of $\mathfrak{t}_1$. That is to say, we obtain

$$\pi_{12} \circ \iota(Fl_3(\mathbb{C})^{T_1}) = \left\{ \frac{1}{2}a_1 - \frac{\sqrt{3}}{2}a_2,\ -\frac{1}{2}a_1 + \frac{\sqrt{3}}{2}a_2,\ a_1, \right.$$
$$\left. -a_1,\ \frac{1}{2}a_1 + \frac{\sqrt{3}}{2}a_2,\ -\frac{1}{2}a_1 - \frac{\sqrt{3}}{2}a_2 \right\}.$$

So the image $\pi_{12} \circ \iota(Fl_3(\mathbb{C})^{T_1})$ is the vertices of a hexagon in $\mathfrak{t}_1 \subset \mathfrak{t}_1 + \mathfrak{t}_2$. Now by an argument similar to that in previous example, Theorem 2.2.1 states that the image of the map $\pi_{12} \circ \iota : Fl_3(\mathbb{C}) \to \mathfrak{t}_1 + \mathfrak{t}_2$ is the convex set which is given by the trajectory of rotation of the hexagon along a_2-axis (Figure2.4.3).

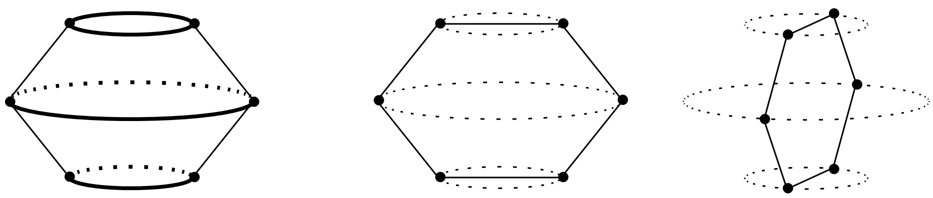


Figure 2.4.3: $\pi_{12} \circ \iota(Fl_3(\mathbb{C}))$ and the moment polytopes for T_1 and T_2 actions

2.5 A relation between Theorem 2.1.3 and super-integrable systems

In this section, we discuss a relation between Theorem 2.1.3 and super-integrable systems.

To begin, we first quote some definitions. In the following, we let (M, ω) be a symplectic manifold.

Definition 2.5.1. Smooth functions $f_1, \cdots, f_n$ on M are **independent** if

$$\{p \in M \mid (df_1)_p, \cdots, (df_n)_p \text{ are linearly independent.}\}$$

is an open dense subset of M.

Equivalently, we can also say that $f_1, \cdots, f_n$ on M are independent if and only if the vector-valued map $(f_1, \cdots, f_n) : M \to \mathbb{R}^n$ is a submersion on a open dense subset of M. Now let H be a smooth function on M and $n = \frac{1}{2} \dim M$ in the following.

Definition 2.5.2. A triple (M, ω, H) is an **integrable system** if there exists independent smooth functions $f_1(= H), \cdots, f_n$ such that these are pairwise Poisson commutative.

For a triple (M, ω, H), the function H is called the hamiltonian function, and a smooth function f on M which Poisson commutes with H is called an integral of motion.

Definition 2.5.3. A triple (M, ω, H) is a **super-integrable system** if there exists pairwise Poisson commutative smooth functions $f_1(= H), \cdots, f_n$, and there exists smooth functions $g_2, \cdots, g_k$ $(2 \le k \le n)$ where each of them is Poisson commutative with H, and the total functions $f_1(= H), \cdots, f_n, g_2, \cdots, g_k$ are independent.

Remark 2.5.4. We do *not* require the connectivity of the fibers of the map $(f_1, \cdots, f_n, g_2, \cdots, g_k) : M \to \mathbb{R}^{n+k-1}$, though some other authors([16], [44]) require this connectivity and some other propoerties to make this map into a torus bundle. Also, since n is the maximal number of pairwise Poisson commutative independent smooth functions, the total functions $f_1(= H), \cdots, f_n, g_2, \cdots, g_k$ can *not* be pairwise Poisson commutative.

We now study the critical point set of the map $\pi_{12}\mu : M \to \mathfrak{t}_1 + \mathfrak{t}_2$.

Proposition 2.5.5. *Under the assumption of Theorem 2.2.1, we obtain*

$$\mathrm{Cr}(\pi_{12}\mu) = T_3 \cdot \mathrm{Cr}(\pi_1\mu).$$

Proof. By the definition of critical points, we have $\mathrm{Cr}(\pi_{12}\mu) \supset \mathrm{Cr}(\pi_1\mu)$. Recall that the map $\pi_{12}\mu : M \to \mathfrak{t}_1 + \mathfrak{t}_2$ is T_3-equivariant. Thus, for any $\theta_3 \in T_3$, we have the following commutative diagram.

$$\begin{array}{ccc} M & \xrightarrow{\pi_{12}\mu} & \mathfrak{t}_1 + \mathfrak{t}_2 \\ {\scriptstyle \theta_3}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}(\theta_3)} \\ M & \xrightarrow{\pi_{12}\mu} & \mathfrak{t}_1 + \mathfrak{t}_2 \end{array}$$

For any point $p \in M$, this induces

$$\begin{array}{ccc} T_pM & \xrightarrow{(\pi_{12}\mu)_{*p}} & \mathfrak{t}_1 + \mathfrak{t}_2 \\ {\scriptstyle (\theta_3)_{*p}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Ad}(\theta_3)} \\ T_{\theta_3 p}M & \xrightarrow{(\pi_{12}\mu)_{*\theta_3 p}} & \mathfrak{t}_1 + \mathfrak{t}_2. \end{array}$$

Hence T_3-action restricts to $\mathrm{Cr}(\pi_{12}\mu)$. That is, we obtain

$$\mathrm{Cr}(\pi_{12}\mu) \supset T_3 \cdot \mathrm{Cr}(\pi_1\mu).$$

On the other hand, if the top map $(\pi_{12}\mu)_{*p}$ on the above diagram is not surjective, there exists a line l in $\mathfrak{t}_1 + \mathfrak{t}_2$ whose intersection with $(\pi_{12}\mu)_{*p}(T_pM)$ is the origin. By the lemma 2.2.2, there exists an element θ_3' of T_3 such that $\mathrm{Ad}(\theta_3')l \subset \mathfrak{t}_1$ (because l is 1-dimensional, l can be written as $l = \mathbb{R}a$ for some $a \in \mathfrak{t}_1 + \mathfrak{t}_2$). Now we have $(\pi_{12}\mu)_{*\theta_3' p}(T_{\theta_3' p}M) = \mathrm{Ad}(\theta_3')((\pi_{12}\mu)_{*p}(T_pM))$ and

$$\begin{aligned} \mathrm{Ad}(\theta_3')((\pi_{12}\mu)_{*p}(T_pM)) \cap \mathrm{Ad}(\theta_3')l &= \mathrm{Ad}(\theta_3')((\pi_{12}\mu)_{*p}(T_pM) \cap l) \\ &= \mathrm{Ad}(\theta_3')(\{0\}) \\ &= \{0\}. \end{aligned}$$

This shows that $(\pi_1\mu)_{*\theta_3' p} : M \to \mathfrak{t}_1 + \mathfrak{t}_2 \to \mathfrak{t}_1$ can not be surjective because the second map of this composed map is identity map on $\mathfrak{t}_1$. Hence we obtain $p = \theta_3'^{-1}\theta_3' p \in T_3 \cdot \mathrm{Cr}(\pi_1\mu)$ which justifies our claim. $\Box$

Now let us suppose

(i) T_1 acts on M effectively,
(ii) $T_3 \cong (\mathbb{R}/\mathbb{Z}) \times (T_1 \cap T_2 \cap T_3)_0$ (as Lie groups)

where $(T_1 \cap T_2 \cap T_3)_0$ is the identity component of $T_1 \cap T_2 \cap T_3$. By (ii), we identify T_3 and $(\mathbb{R}/\mathbb{Z}) \times (T_1 \cap T_2 \cap T_3)_0$ in the following. Since $\pi_1\mu$ is a moment map of T_1 action, it is T_1-equivariant, and the intersection $T_1 \cap T_2 \cap T_3$ $(\subset T_1)$ preserves $\mathrm{Cr}(\pi_1\mu)$. So we have

$$\begin{aligned} T_3 \cdot \mathrm{Cr}(\pi_1\mu) &= ((\mathbb{R}/\mathbb{Z}) \times (T_1 \cap T_2 \cap T_3)_0) \cdot \mathrm{Cr}(\pi_1\mu) \\ &= (\mathbb{R}/\mathbb{Z}) \cdot (T_1 \cap T_2 \cap T_3)_0 \cdot \mathrm{Cr}(\pi_1\mu) \\ &= (\mathbb{R}/\mathbb{Z}) \cdot \mathrm{Cr}(\pi_1\mu). \end{aligned}$$

By the first assumption, we can write

$$\mathrm{Cr}(\pi_1\mu) = \bigcup_{i=1}^{\infty} Z_i$$

where each Z_i is a proper closed symplectic submanifold of M. Especially, we have $\dim Z_i \leq \dim M - 2$. Hence we obtain

$$T_3 \cdot \mathrm{Cr}(\pi_1\mu) = \bigcup_{i=1}^{\infty}((\mathbb{R}/\mathbb{Z}) \cdot Z_i).$$

Consider a map $f : \mathbb{R}/\mathbb{Z} \times Z_i \to M$ via the G-action on M :

$$([a], p) \mapsto [a] \cdot p \qquad \text{for } [a] \in \mathbb{R}/\mathbb{Z}, p \in Z_i.$$

Since this map f is a restriction of the group action $G \times M \to M$, f is a smooth map. Note that $f(\mathbb{R}/\mathbb{Z} \times Z_i)$ is closed in M because its compact. Here we have $\dim(\mathbb{R}/\mathbb{Z} \times Z_i) \leq \dim M - 1$. Now Theorem 10.5 in [39] states that $M - (\mathbb{R}/\mathbb{Z}) \cdot Z_i (= M - f(\mathbb{R}/\mathbb{Z} \times Z_i))$ is open dense in M. Hence, by Baire's theorem, the subset

$$M - \mathrm{Cr}(\pi_{12}\mu) = M - T_3 \cdot \mathrm{Cr}(\pi_1\mu) = \bigcap_{i=1}^{\infty}(M - f(\mathbb{R}/\mathbb{Z} \times Z_i))$$

is dense in M, and is also open since $\mathrm{Cr}(\pi_{12}\mu)$ is closed in M. Now we obtain the following.

Proposition 2.5.6. *Under the assumptions of Theorem 2.2.1, if T_1 acts on M effectively and $T_3 \cong (\mathbb{R}/\mathbb{Z}) \times (T_1 \cap T_2 \cap T_3)_0$, then $\pi_{12}\mu$ is a submersion on an open dense subset of M.*

Corollary 2.5.7. *Under the assumptions of Theorem 2.1.3, if T_1 acts on M effectively and $T_3 \cong (\mathbb{R}/\mathbb{Z}) \times (T_1 \cap T_2 \cap T_3)_0$, then for any $X \in \mathfrak{t}_1 \cap \mathfrak{t}_2$, the triple (M, ω, μ^X) is a super-integrable system which has $\frac{1}{2}\dim M + 1$ independent integrals of motion.*

Proof. We have $\langle \mathcal{R}_{12} \circ \mu, X \rangle = \langle \mu, X \rangle = \mu^X$. Since $X \in \mathfrak{t}_1 \cap \mathfrak{t}_2$ commutes with $\mathfrak{t}_1 + \mathfrak{t}_2$, μ^X Poisson commutes with $\langle \mathcal{R}_{12} \circ \mu, Y \rangle (= \mu^Y)$ for all $Y \in \mathfrak{t}_1 + \mathfrak{t}_2$. Let n be $\dim \mathfrak{t}_1$. Then there exists a linear isomorphism $\mathfrak{t}_1 + \mathfrak{t}_2 \cong \mathbb{R}^{2n+1}$ where $\mathfrak{t}_1 \cap \mathfrak{t}_2$ corresponds to a coordinate linear subspace of $\mathbb{R}^{2n+1}$. Let $Y_1, \cdots, Y_{2n+1}$ be the elements corresponding to the standard basis of $\mathbb{R}^{2n+1}$. Then we have

$$\mathcal{R}_{12} \circ \mu = (\langle \mathcal{R}_{12} \circ \mu, Y_1 \rangle, \cdots, \langle \mathcal{R}_{12} \circ \mu, Y_{2n+1} \rangle) : M \to \mathbb{R}^{2n+1}.$$

Recall that we have a commutative diagram given above Theorem 2.2.1. Since one of $Y_1, \cdots, Y_{2n+1}$ is X, Proposition 2.5.6 provides our claim. $\square$

Example 2.4.1 is trivially a super-integrable system, because every two dimensional completely integrable system is always a super-integrable system in a trivial sense.

In Example 2.4.2, it is well known that a moment map $\pi_1\mu$ of the hamiltonian T_1-action on $\mathbb{C}P^n$ provides a completely integrable system with the hamiltonian function μ^W where W is

$$W = W_1 + \cdots + W_k.$$

We can understand this system as a compactification of the system of harmonic oscillators on $\mathbb{C}^n$. Now Corollary 2.5.7 provides a super-integrable system on

$\mathbb{C}P^{2k}$ with $k+1$ integrals of motion whose hamiltonian function is μ^W. This argument also works for the case $n = 2k+1$ ($k \geq 0$) in a similar way. In fact, it is known that these completely integrable systems are maximally super-integrable.

Appendix

In this appendix, we construct a triple $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{t}_3$ in $\mathfrak{su}(2k)$ $(k \geq 1)$. Let us define $X_1, \cdots, X_k, Y_1, \cdots, Y_k, Z_1, \cdots, Z_k, W_1, \cdots, W_{k-1} \in \mathfrak{su}(2k)$ by

$$X_1 = \begin{pmatrix} -i & 0 & & & & & \\ 0 & i & & & & & \\ & & 0 & 0 & & & \\ & & 0 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \end{pmatrix}, \cdots, X_k = \begin{pmatrix} 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & 0 & & & \\ & & 0 & 0 & & & \\ & & & & \ddots & & \\ & & & & & -i & 0 \\ & & & & & 0 & i \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 0 & & & \\ & & 0 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \end{pmatrix}, \cdots, Y_k = \begin{pmatrix} 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & 0 & & & \\ & & 0 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} 0 & i & & & & & \\ i & 0 & & & & & \\ & & 0 & 0 & & & \\ & & 0 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \end{pmatrix}, \cdots, Z_k = \begin{pmatrix} 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & 0 & & & \\ & & 0 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & i \\ & & & & & i & 0 \end{pmatrix},$$

$$W_1 = \begin{pmatrix} -i & 0 & & & & & \\ 0 & -i & & & & & \\ & & i & 0 & & & \\ & & 0 & i & & & \\ & & & & \ddots & & \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \end{pmatrix}, \cdots, W_{k-1} = \begin{pmatrix} -i & 0 & & & & & \\ 0 & -i & & & & & \\ & & 0 & 0 & & & \\ & & 0 & 0 & & & \\ & & & & \ddots & & \\ & & & & & i & 0 \\ & & & & & 0 & i \end{pmatrix}.$$

Now define Lie subalgebra $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{t}_3$ by

$$\begin{aligned}
\mathfrak{t}_1 &= \mathrm{span}\{W_1, \cdots, W_{k-1}, X_1, \cdots, X_k\}, \\
\mathfrak{t}_2 &= \mathrm{span}\{W_1, \cdots, W_{k-1}, Y_1, \cdots, Y_k\}, \\
\mathfrak{t}_3 &= \mathrm{span}\{W_1, \cdots, W_{k-1}, Z_1, \cdots, Z_k\}.
\end{aligned}$$

Then the condition (2.1.1) is clearly satisfied.

Chapter 3

Equivariant cohomology

3.1 Universal principal bundles

In this section, we study universla principal bundles. They will be used in the construction of equivariant cohomology in the next section.

Definition 3.1.1. An covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of a topological space B is *numerable* if it admits a refinement by a locally finite partition of unity, i.e., if there exists a locally finite partition of unity $\{u_\gamma : B \to [0,1]\}_{\gamma\in\Gamma}$ such that every set $\overline{u_\lambda^{-1}((0,1])}$ is contained in some U_λ. A topological fiber bundle $E \to B$ is *numerable* if B admits a numerable open covering $\{U_\lambda\}_\lambda$ such that $E|_{U_\lambda}$ is trivial bundle for each $\lambda \in \Lambda$.

Proposition 3.1.2. *A topological fiber bundle $E \to B$ is numerable if B is paracompact.*

Proof. See 2.1 of [14]. $\square$

Let G be a topological group. In this note, a principal G-bundle is defined with its *right* G-action.

Definition 3.1.3. A *universal principal G-bundle $EG \to BG$* is a numerable topological principal G-bundle which satisfies the following:

1. For any numerable topological principal G-bundle $E \to B$, there exists a continuous map $f : B \to BG$ such that E is isomorphic to the pull back bundle f^*EG.

2. Two continuous maps $f, g : B \to BG$ induces an isomorphism $f^*EG \cong g^*EG$ *if and only if* they are homotopic.

The base space BG of a universal principal G-bundle is called *classifying space* for G.

For the proof of next proposition, see 7.5 in [14].

Theorem 3.1.4. *A numerable topological principal G-bundle is a universal principal G-bundle if and only if the total space is contractible.*

3.1.1 Uniqueness of universal principal G-bundles

In this section, we use the following property which we refer [14] for a proof. Recall that *a vertical homotopy* of a topological fiber bundle $E \to B$ is a continuous map $H : I \times B \to E$ such that $p \circ H = \mathrm{id}_B$.

Proposition 3.1.5. *Let $E \to B$ be a topological fiber bundle. If the fiber is contractible, then a continuous section $s : B \to E$ exists unique up to vertical homotopy. As a corollary, the projection map $E \to B$ is a homotopy equivalence.*

Proof. Since the fiber is contractible, 3.2 in [14] shows that the map $E \to B$ is shrinkable. Now our claims follows from 1.5 (c) in [14]. □

Lemma 3.1.6. *Let $E \to B$ be a numerable topological principal G-bundle, Y be a topological G-space. If Y is contractible (not necessarily G-equivariantly), any two G-equivariant continuous maps $f, g : E \to Y$ are G-equivariantly homotopic.*

Proof. There is a natural one to one correspondence between G-equivariant continuous maps $E \to Y$ and continuous sections $B \to (E \times Y)/G$ of the associated Y-bundle (see the appendix for this section). So f and g induce continuous sections s_f and s_g of the associated Y-bundle. Since the fiber of this associated Y-bundle is contractible, sections of this associated Y-bundle are unique up to vertical homotopy (Proposition3.1.5). Thus there exists vertical homotopy between s_f and s_g. Since the correspondence written above also holds for G-equivariant homotopy $I \times E \to Y$ and vertical homotopy $I \times B \to (E \times Y)/G$ (again, see the appendix for this chapter), we obtain a G-equivariant homotopy between f and g. □

Let $EG \to BG$ and $EH \to BH$ be universal bundles of topological groups G and H. Let $\rho : G \to H$ be a homomorphism of topological groups.

Corollary 3.1.7. *A ρ-equivariant continuous map $EG \to EH$ exists unique up to ρ-equivariant homotopy.*

Proof. Consider the associated principal H-bundle $EG \times_G H \to BG$. By the universality of EH, there exists a pull back diagram

$$\begin{tikzcd} EG \times_G H \arrow[r] \arrow[d] & EH \arrow[d] \\ BG \arrow[r] & BH \end{tikzcd}$$

where the left vertical map is the associated principal H-bundle defined by the condition $[\alpha, h] = [\alpha g^{-1}, gh]$ for any $\alpha \in EG, h \in H$ and $g \in G$, and the top map is H-equivariant. Combining the top map and $EG \to EG \times_G H$ given by $\alpha \mapsto [\alpha, 1]$, we obtain a ρ-equivariant continuous map $EG \to EH$. The uniqueness of such maps is a direct consequence of the previous Lemma as follows. Think of EH as topological G-space via ρ. Then ρ-equivariance is equivalent to G-equivariance. Let $f, g : EG \to EH$ be ρ-equivariant continuous maps. Then f and g are G-equivariant continuous maps. By the previous lemma, there exists a G-equivariant continuous map $F : EG \times I \to EH$ such that $F(\alpha, 0) = f(x)$ and $F(\alpha, 1) = g(x)$. Now, of course, F is ρ-equivariant because of the definition of the G-action on EH. □

Uniqueness of universal bundle:

Proposition 3.1.8. *The G-equivariant homotopy type of contractible universal principal G-bundles is uniquely determined. As a result, the homotopy type of the classifying spaces is uniquely determined.*

Proof. Let $EG \to BG$ and $E'G \to B'G$ be universal principal G-bundles. By the property 1 in Definition 3.1.3, there exists a G-equivariant continuous map $\tilde{f} : E'G \to EG$. Similarly, there exists a G-equivariant continuous map $\tilde{g} : EG \to E'G$. Now the composition $\tilde{f} \circ \tilde{g} : EG \to EG$ have to be G-equivariantly homotopic to the identity map because of the previous Lemma, and the same holds for $\tilde{g} \circ \tilde{f}$. Thus EG and $E'G$ are G-equivariantly homotopic. As a result, BG and $B'G$ are homotopic. □

We list some useful properties (up to homotopy) of universal bundles. Let $EG \to BG$ and $EH \to BH$ be universal principal bundles for G and H, respectively.

— $EG \times EH \to BG \times BH$ is a universal principal bundle of $G \times H$.

— If H is a subgroup of G, the quotient map $EG \to EG/H$ is a universal principal H-bundle.

— A homomorphism $\rho : G \to H$ induces a ρ-equivariant continuous map $EG \to EH$ unique up to ρ-equivariant homotpy.

The first claim is follows because of the uniqueness of universal principal $G \times H$ bundles. For the second claim, we can use the locally trivializations of $EG \to EG/G = BG$ and its G-equivariance and the contractibility of the total space EG. The third one is Corollary 3.1.7. □

Existence of universal bundles:

Theorem 3.1.9. *Let G be a topological group. Then there exists a universal principal G-bundle $EG \to BG$.*

In fact, there is an explicit construction of a universal principal G-bundle (sometimes called the Milnor join of G) for any topological group G due to Milnor ([42]). Note that Theorem 3.1.4 ensures that the total space of universal bundle is contractible.

Appendix for section 3.1

Let G be a topological group, and $\pi : E \to B$ a topological principal G-bundle. Let X be a topological G-space, then we obtain the associated topological X-bundle $p : E \times_G X \to B$.

Proposition 3.1.10. *For any topological space W (without G-action), we have the following.*

1. *A continuous G-map $F : W \times E \to X$ induces the continuous map*

$$\tilde{F} : W \times B \to E \times_G X \quad ; \quad (w, [\alpha]) \mapsto [\alpha, F(w, \alpha)]$$

such that $p \circ \tilde{F} = W \times B \overset{proj}{\to} B$.

2. *A continuous map $H : W \times B \to E \times_G X$ satisfying $p \circ H = W \times B \overset{proj}{\to} B$ induces the continuous G-map*

$$\tilde{H} : W \times E \to X$$

defined by the condition $H(w, [\alpha]) = [\alpha, \tilde{H}(w, \alpha)]$.

3. *By 1. and 2., we obtain a natural correspondence:*

$$\{\textit{continuous } G\textit{-map } F : W \times E \to X\}$$
$$\updownarrow 1:1$$
$$\{\textit{continuous map } H : W \times B \to E \times_G X \text{ s.t. } p \circ H = W \times B \overset{proj}{\to} B\}$$

Corollary 3.1.11. *For any topological principal G-bundle $\pi : E' \to B'$, we have natural correspondences:*

$$\{\textit{continuous bundle map } F : E \to E'\}$$
$$\updownarrow 1:1$$
$$\{\textit{continuous section } H : B \to E \times_G E'\}$$

$$\{G\textit{-equivariant homotopy } H : I \times E \to E' \textit{ between } F_1 \textit{ and } F_2\}$$
$$\updownarrow 1:1$$
$$\{\textit{vertical homotopy } \tilde{H} : I \times B \to E \times_G E' \textit{ between } \tilde{F}_1 \textit{ and } \tilde{F}_2\}$$

(Note: A vertical homotopy is a homotopy $\tilde{H} : I \times B \to E \times_G E'$ such that $p \circ \tilde{H} = id_B$.)

3.2 Equivariant cohomology

In this section, all cohomologies are treated over a principal ideal domain with the unit, otherwise specified.

Let G be a topological group acting continuously on a topological space X from the left. Recall that, in this note, the G-action on a topological principal G-bundle is the one from the right.

Definition 3.2.1. Let EG be a universal principal G-bundle. The *Borel construction* of X for with respect to EG is the defined by

$$EG \times_G X := (EG \times X)/G$$

where the G-action on $EG \times X$ is given by

$$g \cdot (p, x) := (pg^{-1}, gx)$$

for all $g \in G$, $p \in EG$ and $x \in X$.

Observe that the Borel construction $EG \times_G X$ is the total space of the associated X-bundle over BG ;

$$EG \times_G X = (EG \times X)/G \to EG/G = BG. \tag{3.2.1}$$

Let H^* be the singular cohomology over a principal ideal domain.

Definition 3.2.2. The *G-equivariant cohomology* of X is defined by

$$H^*_G(X) := H^*(EG \times_G X) \tag{3.2.2}$$

where EG is a universal principal G-bundle. The map induced by (3.2.1) makes $H^*_G(X)$ into an $H^*(BG)$-algebra with the cup product.

This is well-defined up to isomorphism since the homotopy type of the Borel construction $EG \times_G X$ is uniquely determined up to homotopy (See Proposition 3.1.8).

Let K be a topological group, and $\rho : G \to K$ a homomorphism of topological groups. Then a ρ-equivariant continuous map (which exists uniquely up to ρ-equivariant homotopy) $EG \to EK$ induces a ring homomorphism

$$H^*_G(X) \to H^*_K(X)$$

As a special case, let $K = \{e\}$ the identity group. Then the induced map is

$$H^*_G(X) \to H^*(X). \tag{3.2.3}$$

This map coincides with the pull back homomorphism by an inclusion $X \hookrightarrow EG \times_G X$ (which does not depend on the choice of an inclusion). This map is called *forgetful map* or *non-equivariant limit.* Under some assumptions, (3.2.3) will be surjective and $H^*_G(X)$ can recover the ordinary cohomology $H^*(X)$ as $H^*_G(X)/(H^{>0}(BG)) \cong H^*(X)$.

3.2.1 Relations with the cohomology of the quotient space

In some nice situations (as the following), $H^*_G(X)$ and $H^*(X/G)$ are the same as rings. A similar statement in a more subtle situations will be explained in detail in section 3.4.

Suppose that a topological group G acts on a topological space X in a way that the quotient map $X \to X/G$ is a topological principal G-bundle. Under this assumption, the natural projection map

$$EG \times_G X \to X/G$$

is the associated EG-bundle where EG is a universal principal G-bundle. Since the fiber EG is contractible, the induced map

$$H^*_G(X) = H^*(EG \times_G X) \leftarrow H^*(X/G)$$

is an isomorphism by Proposition 3.1.5. This means that, if the quotient $X \to X/G$ is "good", then the $H^*_G(X)$ coincides with $H^*(X/G)$ as rings. However, the equivariant cohomology $H^*_G(X)$ is better than $H^*(X/G)$ in the sense that $H^*_G(X)$ can recover the ordinary cohomology in many situation in which $H^*(X/G)$ does not.

3.2.2 Finite dimensional approximations

The explanation in this section is essentially due to Fulton's lecture note [19]. Let $\pi : E \to B$ be a topological principal G-bundle such that

- $H_*(E)$ is a finitely generated free graded module (i.e. each $H_j(E)$ is a finitely generated free module and only finitely many $H_j(E)$'s are non-zero), and
- there exists an integer $k \geq 0$ such that $\tilde{H}^i(E) = 0$ for all $i < k$.

Then there is a *canonical* isomorphism

$$H^i(EG \times_G X) \cong H^i(E \times_G X) \quad \text{for all } i < k \tag{3.2.4}$$

in the following sense. The trivial G-equivariant EG-bundle $EG \times X \to X$ induces the associated EG-bundle

$$(EG \times E) \times_G X \longrightarrow E \times_G X$$

where G acts on $EG \times E$ diagonally. Similarly, the trivial G-equivariant E-bundle $E \times X \to X$ induces the associated E-bundle

$$(EG \times E) \times_G X \longrightarrow EG \times_G X.$$

Combining them, we obtain an E-bundle (on the left) and an EG-bundle (on the right)

$$EG \times_G X \longleftarrow (EG \times E)_G \times X \longrightarrow E \times_G X. \tag{3.2.5}$$

Here, E and EG are path-connected, and $H_*(EG)$ and $H_*(E)$ are finitely generated graded modules by the assumptions. Hence we can use the precise version of the Leray-Hirsh theorem in the appendix, and we obtain

$$H^i(EG \times_G X) \xrightarrow{\cong} H^i((EG \times E)_G \times X) \xleftarrow{\cong} H^i(E \times_G X) \tag{3.2.6}$$

for all $i < k$, which gives the desired isomorphism (3.2.4).

Remark 3.2.3. If we have a G-equivariant continuous map $E \to EG$, the canonical isomorphism (3.2.4) is equal to the pull back

$$H^i(EG \times_G X) \xrightarrow{\cong} H^i(E \times_G X) \quad (\text{for all } i < k).$$

This can be explained as follows. In this case, the right map in (3.2.5) (which is an EG-bundle) has an obvious section $s : (EG \times E)_G \times X \leftarrow E \times_G X$, and the composition of s and the left map in (3.2.5) gives the induced map $E \times_G X \to EG \times_G X$. It is easy to show that the inverse of the right map in (3.2.6) has to be the induced map s^* by using a property of a section (i.e. the composition of the projection and a section is the identity).

Example 3.2.4. Let $S^{2m+1}(\subset \mathbb{C}^{m+1})$ be the $2m+1$-dimensional sphere. Consider a sequence of inclusions of spheres $S^1 \subset S^3 \subset \cdots \subset S^{2m+1} \subset \cdots$ given by

$$S^{2m+1} \hookrightarrow S^{2m+3} \quad ; \quad (x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_m, 0),$$

and let $S^\infty := \cup_{m \in \mathbb{N}} S^{2m+1}$. This sequence induces a sequence of the inclusions of complex projective spaces $\mathbb{P}^1 \subset \mathbb{P}^2 \subset \cdots \subset \mathbb{P}^m \subset \cdots$ given by

$$\mathbb{P}^m \hookrightarrow \mathbb{P}^{m+1} \quad ; \quad [x_1, \cdots, x_m] \mapsto [x_1, \cdots, x_m, 0], \tag{3.2.7}$$

We endow S^∞ a topology as follows; a subset of S^∞ is an open set if and only if the intersection with S^{2m+1} is an open set of S^{2m+1} for each $m \in \mathbb{N}$. A topology on $\mathbb{P}^\infty := \cup_{m \in \mathbb{N}} \mathbb{P}^m$ is defined similarly.

The 1-dimensional torus $S^1(\subset \mathbb{C})$ acts on S^{2m+1} from the right by

$$z \cdot (x_1, \cdots, x_m) = (x_1 z, \cdots, x_m z).$$

for all $z \in S^1$ and $(x_1, \cdots, x_m) \in S^{2m+1}$. We think of this as a right action, and the natural projection map $S^{2m+1} \to \mathbb{P}^m$ is a principal S^1-bundle. This S^1-action extends to a continuous S^1-action on S^∞. Since each S^n is a Hausdorff space, we have

$$\varinjlim i_{2m+1*} : \varinjlim H_i(S^{2m+1} : \mathbb{Z}) \xrightarrow{\cong} H_i(S^\infty : \mathbb{Z})$$

for each $i \in \mathbb{Z}$, where each $i_{2m+1} : S^{2m+1} \to S^\infty$ is the inclusion. Also, the inclusion $S^{2m+1} \hookrightarrow S^{2m+3}$ induces $H_i(S^{2m+1}) \to H_i(S^{2m+3})$, and this is an isomorphism for $i < 2m+1$. Hence, we see that the induced map

$$i_{n*} : H_i(S^n : \mathbb{Z}) \to H_i(S^\infty : \mathbb{Z})$$

is an isomorphism for $i < 2m+1$. From this, we obtain

$$H_i(S^\infty : \mathbb{Z}) = \begin{cases} \mathbb{Z} & (i=0) \\ 0 & (i \neq 0) \end{cases} \quad \text{and} \quad H^i(S^\infty : \mathbb{Z}) = \begin{cases} \mathbb{Z} & (i=0) \\ 0 & (i \neq 0). \end{cases}$$

Hence, we have the canonical isomorphism

$$H^*_{S^1}(X : \mathbb{Z}) \cong H^*(S^\infty \times_{S^1} X : \mathbb{Z}). \tag{3.2.8}$$

Remark 3.2.5. In fact, it is know that S^∞ with the above topology is contractible. Hence S^∞ is a universal principal S^1-bundle, and $\mathbb{P}^\infty$ is a classifying space for S^1.

Example 3.2.6. Let $V_k(\mathbb{C}^{k+n})$ be the Stiefel manifold of unitary k-frames in $\mathbb{C}^{k+n}$. Then the unitary group $\mathrm{U}(k)$ naturally acts on $V_k(\mathbb{C}^{k+n})$ from the right. There is a standard way to see $V_k(\mathbb{C}^{k+n})$ as a finite CW-complex such that the boundary map of the chain complex of the CW complex $V_k(\mathbb{C}^{k+n})$ are zero maps. Hence the homomology $H_*(V_k(\mathbb{C}^{k+n}))$ is a finitely generated free graded module. Since this CW complex has no cells of dimension between 1 and n, we see that

$$\tilde{H}^i(V_k(\mathbb{C}^{k+n})) = 0 \quad \text{for all } i \leq n.$$

Hence we obtain the canonical isomorphism

$$H^i_{U(k)}(X) \cong H^i(V_k(\mathbb{C}^{k+n}) \times_{U(k)} X) \quad \text{for all } i \leq n.$$

Example 3.2.7. Let G be a compact Lie group. Recall that G can be embedded into the unitary group $\mathrm{U}(k)$ for a sufficiently large k as a closed subgroup. Then, G acts on $V_{k,n}(\mathbb{C})$ freely, via this embedding $G \hookrightarrow \mathrm{U}(k)$. Hence the quotient map $V_{k,n}(\mathbb{C}) \to V_{k,n}(\mathbb{C})/G$ is a principal G-bundle. Now we obtain the canonical isomorphism

$$H^i_G(X) \cong H^i(V_{k,n}(\mathbb{C}) \times_G X) \quad \text{for all } i \leq n.$$

Appendix for section 3.2

A graded module $\{C_k\}_{k\in\mathbb{Z}}$ (over a commutative ring with the unit) is said to be *finitely generated* if $\oplus_k C_k$ is finitely generated module (equivalently, each C_k is finitely generated module and only finite C_k's can be nonzero).

Let (E, E^0) and (F, F^0) be pair of topological spaces, B a topological space, $p : E \to B$ a continuous map. The tuple $\xi = ((E, E^0), p, B, (F, F^0))$ is said to be *a fiber bundle pair* if, for any $b \in B$, there exist an open neighborhood V of b in B and a homeomorphism $\varphi : V \times (F, F^0) \xrightarrow{\cong} (p^{-1}(V), p^{-1}(V) \cap E^0)$ satisfying

$$p \circ \varphi(b', y) = b' \quad (b' \in V,\, y \in F).$$

In the following Leray-Hirsch theorem, we use the singular cohomology over a principal ideal domain.

Theorem 3.2.8. *(Leray-Hirsch) Let $\xi = ((E, E^0), p, B, (F, F^0))$ be a fiber bundle pair. Suppose that $H_*(F, F^0)$ is a finitely generated free graded module, and there exist an integer $n \in \mathbb{Z}$ and a module homomorphism (of degree 0) $\theta : H^*(F, F^0) \to H^*(E, E^0)$ such that, for any $b \in B$, the map $j_b^* \circ \theta : H^k(F, F^0) \to H^k(E_b, E_b^0)$ is an isomorphism for any $k < n$ where $j_b : (E_b, E_b^0) \hookrightarrow (E, E^0)$ is the inclusion. Then, we have the following homomorphisms (as graded modules)*

$$\Phi : H_*(E, E^0) \longrightarrow H_*(B) \otimes H_*(F, F^0),$$
$$\Phi^* : H^*(B) \otimes H^*(F, F^0) \longrightarrow H^*(E, E^0)$$

defined by

$$\Phi(e) = \sum_{i=1}^{k} p_*(\theta(\alpha_i) \cap e)) \otimes a_i \quad (e \in H_*(E, E^0)),$$
$$\Phi^*(\beta \otimes \alpha) = p^*(\beta) \cup \theta(\alpha) \quad (\beta \in H^*(B),\, \alpha \in H^*(F, F^0)),$$

and both of Φ and Φ^* are isomorphism for any component of degree $k < n$. Here $\{a_1, \cdots, a_k\}$ is a basis of $H_*(F, F^0)$ and $\{\alpha_1, \cdots, \alpha_k\}$ is a basis of $H^*(F, F^0)$ satisfying $\langle \alpha_i, a_j \rangle = \delta_{ij}$, and the map Φ does not depend on the choice of these basis.

Corollary 3.2.9. *Let $\xi = (E, p, B, F)$ be a fiber bundle. Suppose that $H_*(F)$ is a finitely generated free graded module, and there exists an integer $n \in \mathbb{Z}$ such that $\tilde{H}^k(F) = 0\,(k < n)$. Then, $p^* : H^k(B) \longrightarrow H^k(E)$ is isomorphism for any $k < n$.*

3.3 Induced homomorphisms

Let G be a topological group acting on a topological space X, and K a topological group acting on a topological space Y. Let $\rho : G \to K$ be a homomorphism of topological groups, and $f : X \to Y$ a ρ-equivariant continuous map. For a universal principal bundles $EG \to BG$ and $EK \to BK$, there exists a ρ-equivariant continuous map $E\rho : EG \to EK$ unique up to ρ-equivariant homotopy (Corollary 3.1.7). So we obtain a commutative diagram of continuous map

$$\begin{tikzcd} EG \times_G X \arrow[r] \arrow[d] & EK \times_K Y \arrow[d] \\ BG \arrow[r] & BK \end{tikzcd}$$

which induces a commutative diagram

$$\begin{tikzcd} H_K^*(Y) \arrow[r, "f^*"] & H_G^*(X) \\ H^*(BK) \arrow[r] \arrow[u] & H^*(BG). \arrow[u] \end{tikzcd}$$

In this sense, the induced map $f^* : H_K^*(Y) \to H_G^*(X)$ is an algebra homomorphism with respect to the ring homomorphism $H^*(BK) \to H^*(BG)$. By the uniqueness (up to ρ-equivariant homotopy) of the map $E\rho$, this diagram does not depend on the choice of $E\rho$. Moreover, this diagram does not depend on the choice of universal principal bundles $EG \to BG$ and $EK \to BK$ in the following sense. For a universal principal bundles $E'G \to B'G$ and $E'K \to B'K$, there exists a ρ-equivariant continuous map $E'\rho : E'G \to E'K$ (unique up to ρ-equivariant homotopy), and we obtain a continuous map $E'G \times_G X \to E'K \times_K Y$ which induces $H^*(E'K \times_K Y) \to H^*(E'G \times_G X)$. Now we have a commutative diagram

$$\begin{tikzcd} H^*(EK \times_K Y) \arrow[r, "f^*"] \arrow[d, leftrightarrow, "\cong"'] & H^*(EG \times_G X) \arrow[d, leftrightarrow, "\cong"] \\ H^*(E'K \times_K Y) \arrow[r, "f^*"] & H^*(E'G \times_G X). \end{tikzcd}$$

where the vertical isomorphisms are the canonical isomorphisms given in (3.2.4) in section 3.2.2. The commutativity follows from the definition of the canonical isomorphism (3.2.4).

Furthermore, suppose that a topological principal G-bundle $E_1 \to B_1$ and a topological principal K-bundle $E_2 \to B_2$ satisfy the assumptions in section 3.2.2 and there is a ρ-equivariant continuous map $E_1 \to E_2$. Then we have a commutative diagram

$$\begin{tikzcd} H^*(EK \times_K Y) \arrow[r, "f^*"] \arrow[d, leftrightarrow, "\cong"'] & H^*(EG \times_G X) \arrow[d, leftrightarrow, "\cong"] \\ H^*(E_2 \times_K Y) \arrow[r, "f^*"] & H^*(E_1 \times_G X). \end{tikzcd} \tag{3.3.1}$$

3.3.1 Equivariant cohomology of a point

In this section, we study the equivariant cohomology of a point. Let G be a topological group, acting trivially on a point. Then we have

$$H_G^*(\mathrm{pt}) = H^*(EG \times_G \mathrm{pt}) = H^*(BG)$$

where $EG \to BG$ is a universal principal G-bundle. That is, the G-equivariant cohomology of a point is the cohomology of a classifying space for G.

Example 3.3.1. Let S^1 be the 1-dimensional torus and $ES^1 \to BS^1$ a universal principal S^1-bundle. Recall the principal S^1-bundle $S^\infty \to \mathbb{P}^\infty$. Then there exists a pull back diagram of principal S^1-bundles

$$\begin{tikzcd} S^\infty \arrow[r, "\phi"] \arrow[d] & ES^1 \arrow[d] \\ \mathbb{P}^\infty \arrow[r, "\overline{\phi}"] & BS^1 \end{tikzcd}$$

For a positive integer, let $\mathbb{C}_a = \mathbb{C}$ be an S^1-representation given by

$$g \cdot z := g^a z \quad \text{for all } g \in S^1 (\subset \mathbb{C}) \text{ and } z \in \mathbb{C}_a.$$

One can see that we obtain a pull back diagram of associated complex line bundles

$$\begin{tikzcd} S^\infty \times_{S^1} \mathbb{C}_1 \arrow[r, "\phi"] \arrow[d] & ES^1 \times_{S^1} \mathbb{C}_1 \arrow[d] \\ \mathbb{P}^\infty \arrow[r, "\overline{\phi}"] & BS^1. \end{tikzcd}$$

Here, the induced map $H^*(\mathbb{P}^\infty) \leftarrow H^*(BS^1)$ is an isomorphism (see (3.2.8) in Example 3.2.4). It is easy to show that the vector bundle $S^\infty \times_{S^1} \mathbb{C}_1$ is isomorphic to the tautological vector bundle over $\mathbb{P}^\infty$. Since the first Chern class of tautological line bundle generates the ring $H^*(\mathbb{P}^\infty)$, we conclude that

$$H^*(BS^1 : \mathbb{Z}) = \mathbb{Z}[t] \qquad \text{(as rings)}$$

where $t := c_1(ES^1 \times_{S^1} \mathbb{C}_1)$.

Let T be an n-dimensional torus, and $\mathfrak{t}^*_\mathbb{Z}$ the dual $\mathbb{Z}$-module of the integral lattice $\mathfrak{t}_\mathbb{Z}$ (which is the kernel of the exponential map $\mathfrak{t} \to T$) of T. An element $\lambda \in \mathfrak{t}^*_\mathbb{Z}$ induces a homomorphism $\lambda_* : T \to S^1$ and hence an S^1-representation $\mathbb{C}_\lambda = \mathbb{C}$ by

$$t \cdot z := \lambda_*(t) z \quad \text{for all } t \in S^1 \text{ and } z \in \mathbb{C}_\lambda.$$

So we get the associated complex line bundle

$$ET \times_T \mathbb{C}_\lambda \to BT \quad ; \quad [\alpha, z] \mapsto [\alpha]. \tag{3.3.2}$$

Taking its first Chern class, we obtain a map

$$\mathfrak{t}^*_\mathbb{Z} \to H^*(BT : \mathbb{Z}). \tag{3.3.3}$$

This is a homomorphism as additive groups because

$$\begin{aligned} c_1(ET \times_T \mathbb{C}_{\lambda+\lambda'}) &= c_1((ET \times_T \mathbb{C}_\lambda) \otimes (ET \times_T \mathbb{C}_{\lambda'})) \\ &= c_1(ET \times_T \mathbb{C}_\lambda) + c_1(ET \times_T \mathbb{C}_{\lambda'}) \end{aligned}$$

for any $\lambda, \lambda' \in \mathfrak{t}^*_\mathbb{Z}$. Since $H^*(BT : \mathbb{Z})$ is a commutative ring as explained above, this naturally induces a ring homomorphism

$$S(\mathfrak{t}^*_\mathbb{Z}) \to H^*(BT : \mathbb{Z}) \tag{3.3.4}$$

where $S(\mathfrak{t}^*_\mathbb{Z})$ is the symmetric algebra over $\mathbb{Z}$ of $\mathfrak{t}^*_\mathbb{Z}$.

Proposition 3.3.2. *The homomorphism* (3.3.4) *is an isomorphism.*

Proof. Without loss of generality, we can assume that $T = (S^1)^n$. Let $\mathrm{Lie}(S^1)_{\mathbb{Z}}(\cong \mathbb{Z})$ be the integral lattice of $(S^1)^n$. Then we have $\mathfrak{t}_{\mathbb{Z}} = \oplus_{i=1}^n \mathrm{Lie}(S^1)_{\mathbb{Z}}$ where we identify Let $\lambda_i \in \mathfrak{t}^*_{\mathbb{Z}}$ be the element given by $\lambda_i(X_1, \cdots, X_n) = X_i$. Then $\mathfrak{t}^*_{\mathbb{Z}} = \oplus_{i=1}^n \mathbb{Z}\lambda_i$, and we have

$$S(\mathfrak{t}^*_{\mathbb{Z}}) = \mathbb{Z}[\lambda_1, \cdots, \lambda].$$

Recall that there exists a T-equivariant homotopy equivalence $(ES^1)^n \to ET$. The homomorphism λ_i gives us the vector bundle (3.3.2), and we have the following pull-back diagrams

$$\begin{tikzcd}
ES^1 \times_{S^1} \mathbb{C}_{\lambda_i} \arrow[d] & (ES^1 \times_{S^1} \mathbb{C}_{\lambda_i}) \times (BS^1)^{n-1} \arrow[l] \arrow[r] \arrow[d] & ET \times_T \mathbb{C}_{\lambda_i} \arrow[d] \\
BS^1 & (BS^1)^n \arrow[l] \arrow[r, "\simeq"] & BT.
\end{tikzcd}$$

Since $H^*(BS^1 : \mathbb{Z})$ is a free $\mathbb{Z}$-module of finite type, Künneth theorem ensures that the map

$$H^*(BS^1 : \mathbb{Z}) \otimes \cdots \otimes H^*(BS^1 : \mathbb{Z}) \longrightarrow H^*((BS^1)^n : \mathbb{Z})$$

given by $(\alpha_1, \cdots, \alpha_n) \mapsto p_1^*\alpha_1 \cup \cdots \cup p_n^*\alpha_n$ is a ring isomorphism where $p_i : (BS^1)^n \to BS^1$ is the projection to the i-th BS^1. With Example 3.3.1, we see that $H^*((BS^1)^n) \cong \otimes_{i=1}^n H^*(BS^1)$ is the polynomial ring generated by the first Chern classes of the middle vector bundle for $\lambda_1, \cdots, \lambda_n$. Hence, we see

$$H^*(BT : \mathbb{Z}) = \mathbb{Z}[y_1, \cdots, y_n] \quad \text{where} \quad y_i = c_1(ET \times_T \mathbb{C}_{\lambda_i}).$$

Since (3.3.4) maps λ_i to y_i, we conclude that (3.3.4) is an isomorphism. □

Suppose that we have a homomorphism $\rho : T \to T'$ between tori. Then we have the obvious induced map

$$\rho^* : S(\mathfrak{t}^*_{\mathbb{Z}}) \leftarrow S(\mathfrak{t}'^*_{\mathbb{Z}}).$$

Observe that T acts on T' (from the left) via $\rho : T \to T'$. So we have the associated T'-bundle

$$ET \times_T T' \to BT \quad : \quad [\alpha, t'] \mapsto [\alpha]. \tag{3.3.5}$$

T' naturally acts on $ET \times_T T'$ on the T'-component from the right, and this T'-action makes (3.3.5) into a topological principal T'-bundle. Hence, there exists a continuous map

$$B\rho : BT \to BT' \tag{3.3.6}$$

such that the pull back of $ET' \to BT'$ gives (3.3.5) because BT' is a classifying space for T' (see the property 1 in Definition 3.1.3). Since such a continuous map $BT \to BT'$ is unique up to homotopy (the property 2 in Definition 3.1.3), we obtain a canonical homomorphism

$$\rho^* : H^*(BT : \mathbb{Z}) \leftarrow H^*(BT' : \mathbb{Z}) \tag{3.3.7}$$

which does not depend on the choice of the continuous map (3.3.6). The naming ρ^* makes sense because of the next proposition.

Proposition 3.3.3. *The following diagram commutes.*

$$\begin{tikzcd} S(\mathfrak{t}_{\mathbb{Z}}^*) \arrow[r, "\cong"] & H^*(BT : \mathbb{Z}) \\ S(\mathfrak{t'}_{\mathbb{Z}}^*) \arrow[r, "\cong"] \arrow[u, "\rho^*"] & H^*(BT' : \mathbb{Z}) \arrow[u, "\rho^*"'] \end{tikzcd}$$

Proof. It suffices to check the commutativity of the diagram

$$\begin{tikzcd} \mathfrak{t}_{\mathbb{Z}}^* \arrow[r] & H^*(BT : \mathbb{Z}) \\ \mathfrak{t'}_{\mathbb{Z}}^* \arrow[r] \arrow[u, "\rho^*"] & H^*(BT' : \mathbb{Z}). \arrow[u, "\psi_\rho"'] \end{tikzcd}$$

Let $\lambda' : \mathfrak{t'}_{\mathbb{Z}} \to \mathbb{Z}$, then we have the associated vector bundle

$$ET' \times_{T'} \mathbb{C}_{\lambda'} \to BT'. \tag{3.3.8}$$

By taking the pull back $\rho^*(\lambda') : \mathfrak{t}_{\mathbb{Z}} \to \mathbb{Z}$, we also have the associated vector bundle

$$ET \times_T \mathbb{C}_{\rho^*(\lambda')} \to BT. \tag{3.3.9}$$

Recall that $B\rho$ pulls back the principal T'-bundle $ET' \to BT'$ to the principal T'-bundle (3.3.5), i.e., we have a pull back diagram of principal T'-bundles

$$\begin{tikzcd} ET \times_T T' \arrow[r] \arrow[d] & ET' \arrow[d] \\ BT \arrow[r, "B\rho"] & BT'. \end{tikzcd}$$

So the following diagram of associated complex vector bundles is a pull back diagram:

$$\begin{tikzcd} (ET \times_T T') \times_{T'} \mathbb{C}_{\lambda'} \arrow[r] \arrow[d] & ET' \times_{T'} \mathbb{C}_{\lambda'} \arrow[d] \\ BT \arrow[r, "B\rho"] & BT'. \end{tikzcd}$$

Now, the map

$$(ET \times_T T') \times_{T'} \mathbb{C}_{\lambda'} \to ET \times_T \mathbb{C}_{\rho^*(\lambda')} \quad ; \quad [\alpha, t', v] \mapsto [\alpha, t'v]$$

is well-defined since

$$[\alpha g^{-1}, \rho(g)t'h'^{-1}, h'v] \mapsto [\alpha g^{-1}, \rho(g)t'h'^{-1}h'v] = [\alpha g^{-1}, \rho(g)t'v],$$

and is an isomorphism as complex vector bundles because each fiber is mapped isomorphically. Now, we see that the complex line bundle (3.3.9) is isomorphic to the pull back of (3.3.8) by $B\rho$ which completes the proof. $\square$

Remark 3.3.4. As in the proof of Proposition 3.3.2, $H^*(BT) \cong \mathbb{Z}[y_1, \cdots, y_n]$ is a polynomial ring where $y_i = c_1(ET \times_T \mathbb{C}_{\lambda_i})$. In this sense, any T equivariant cohomology $H_T^*(X)$ is an algebra over the polynomial ring $\mathbb{Z}[y_1, \cdots, y_n]$. Here, each y_i acts on $H_T^*(X)$ as the multiplication of the first Chern class of the complex vector bundle

$$(ET \times M \times \mathbb{C}_{\lambda_i})/T \to (ET \times M)/T.$$

3.4 Vietoris-Begle mapping theorem

In some nice situations, $H_G^*(X)$ and $H^*(X/G)$ describe the same ring as explained in section 3.2.1. In this section, we treat a more subtle situation, and we will obtain the similar statements in $\mathbb{Q}$-coefficient.

Let H_{AS}^* be the Alexander-Spanier cohomology. We refer [46] for the definition of H_{AS}^* and the proof of the following theorem.

Theorem 3.4.1. *(Vietoris-Begle mapping theorem) Let $f : X' \to X$ be a closed continuous surjective map between paracompact Hausdorff spaces. Assume that there exists an integer $n \geq 0$ such that $\tilde{H}_{AS}^q(f^{-1}(x)) = 0$ for all $x \in X$ and $q < n$. Then the induced map*

$$f^* : H_{AS}^q(X) \to H_{AS}^q(X')$$

is an isomorphism for $q < n$ and a monomorphism for $q = n$.

Since the Alexander-Spanier cohomology and the singular cohomology are naturally isomorphic for locally contractible paracompact Hausdorff spaces ([46], Cor.5, Sec.9, Chp.6), we obtain the similar claim (with an assumption about locally contractibility for then fibers $f^{-1}(x)$) for singular cohomology.

Corollary 3.4.2. *(Vietoris-Begle mapping theorem) Let $f : X' \to X$ be a closed continuous surjective map between locally contractible paracompact Hausdorff spaces. Assume that $f^{-1}(x)$ is locally contractible and there exists an integer $n \geq 0$ such that $\tilde{H}^q(f^{-1}(x)) = 0$ for all $x \in X$ and $q < n$. Then the induced map*

$$f^* : H^q(X) \to H^q(X')$$

is an isomorphism for $q < n$ and a monomorphism for $q = n$.

We apply this corollary to equivariant cohomolgy.

Proposition 3.4.3. *Let G be a compact Lie group acting on a compact manifold M with discrete stabilizers. Then the projection $\theta : EG \times_G M \to M/G$ induces a ring isomorphism*

$$\theta^* : H^*(M/G; \mathbb{Q}) \to H^*_G(M; \mathbb{Q}).$$

We can prove this by using Vietoris-Begle mapping theorem (Corollary 3.4.2) directly, but this Proposition follows from the following more general statement.

Proposition 3.4.4. *Let M be a compact manifold with a smooth action of a compact Lie group K. Let $G \subset K$ be a closed normal subgroup that acts on M with discrete stabilizers. Then the projection $\theta : M \to M/G$ induces an isomorphism as $H^*(B(K/G); \mathbb{Q})$-algebras*

$$\theta^* : H^*_{K/G}(M/G; \mathbb{Q}) \to H^*_K(M; \mathbb{Q})$$

where the ring $H^*_K(M; \mathbb{Q})$ is an $H^*(B(K/G); \mathbb{Q})$-algebra via the induced map $H^*(B(K/G); \mathbb{Q}) \to H^*(BK; \mathbb{Q})$.

Proof. Let $R := K/G$. We will show that $\theta^* : H^i_R(M/G; \mathbb{Q}) \longrightarrow H^i_K(M; \mathbb{Q})$ is a ring isomorphism for each $i > 0$. Let $EK \to BK$ and $ER \to BR$ be universal principal bundles for K and R, respectively. Since K is a compact Lie group, K can be embedded into $U(k)$ for a sufficiently large k as a closed subgroup. Let $V_{k,n} := V_{k,n}(\mathbb{C})$ be the Stiefel manifold of univtary k-frames in $\mathbb{C}^{k+n}$. Then, K acts on $V_{k,n}(\mathbb{C})$ freely, via this embedding $G \hookrightarrow U(k)$. Hence the quotient map $V_{k,n} \to V_{k,n}/K$ is a principal K-bundle. Now we obtain the canonical isomorphism

$$\phi : H^i_K(X; \mathbb{Q}) \cong H^i(V_{k,n} \times_K X; \mathbb{Q}) \qquad \text{for all } i \leq n.$$

as in Example 3.2.7. In the following, we fix $i \in \mathbb{Z}$ arbitrary, and take a sufficiently large $n (\geq i)$.

There exists an equivariant continuous map $EK \to ER$ with respect to the quotient homomorphism $K \to R$. Since $V_{k,n} \to V_{k,n}/K$ is a principal K-bundle, there is a K-equivariant continuous map $V_{k,n} \to EK$ by the universality of $EK \to BK$. So the composition map $\varphi : V_{k,n} \to EK \to ER$ is also equivariant with respect to the quotient $K \to R$. Denoting the natural maps as

$$\theta : EK \times_K M \to ER \times_R (M/G)$$
$$\theta_n : V_{k,n} \times_K M \to ER \times_R (M/G),$$

the diagram of pull backs

$$\begin{tikzcd} H^i_R(M/G; \mathbb{Q}) \arrow[r, "\theta^*"] \arrow[dr, "\theta_n^*"'] & H^i_K(M; \mathbb{Q}) \arrow[d, "\phi", "\cong"'] \\ & H^i(V_{k,n} \times_K M; \mathbb{Q}) \end{tikzcd}$$

is commutative (see (3.3.1) in section 3.2.2). Since ϕ is an isomorphism, it suffices to show that θ_n^* is an isomorphism. Observe that we can write $\theta_n = f \circ s$ where

$$\begin{aligned} s &: V_{k,n} \times_K M \to ER \times_R (V_{k,n} \times_G M) \quad ; \quad [\alpha, x]_K \mapsto [\varphi(\alpha), [\alpha, x]_G]_R, \\ f &: ER \times_R (V_{k,n} \times_G M) \to ER \times_R (M/G) \ ; \ [\beta, [\alpha, x]_G]_R \mapsto [\beta, [x]_G]_R. \end{aligned}$$

It is easy to see that s is a section of the ER-bundle $g : ER \times_R (V_{k,n} \times_G M) \to V_{k,n} \times_K M$ sending $[\beta, [\alpha, x]_G]_R \mapsto [\alpha, x]_K$ whose fiber is the contractible space ER. Therefore s^* is an isomorphism ([14], 1.5 (c) and 3.2). On the other hand, the preimage of f at $[\beta, [x]_G]_R$ is homeomorphic to $V_{k,n}/G_x$ where G_x is the isotropy of the G-action at $x \in M$. Then we have

$$\tilde{H}^p(V_{k,n}/G_x; \mathbb{Q}) = 0 \quad \text{for} \ \ 0 < p \leq n \tag{3.4.1}$$

(we can use Theorem 5.30 in [30] with Theorem 5.8 and the comment in Example 1 in p. 250 to prove this claim). The projection $f' : V_{k,n} \times_G M \to M/G$ is a closed map since it is a map from a compact space to a Hausdorff space (here, $V_{k,n}$ is the set of unitary k-frames in $\mathbb{C}^{k+n}$). This implies that f is a closed surjection, by chasing the following commutative diagram of projections

$$\begin{array}{ccc} ER \times (V_{k,n} \times_G M) & \xrightarrow{(id, f')} & ER \times (M/G) \\ \downarrow & & \downarrow \\ ER \times_R (V_{k,n} \times_G M) & \xrightarrow{f} & ER \times_R (M/G) \end{array}$$

where the vertical maps are closed since R is compact (c.f. Proposition 1.58 in [30]). Observe that M/G is locally contractible because of the equivariant tubular neighborhood theorem. Hence, $ER \times_R (V_{k,n} \times_G M)$ and $ER \times_R (M/G)$ are locally contractible, paracompact Hausdorff spaces. Also, $V_{k,n}/G_x$ is locally contractible again by the equivariant tublar neighborhood theorem. Together with (3.4.1), Vietoris-Begle mapping theorem (Corollary 3.4.2) shows that

$$f^* : H^i(ER \times_R (V_{k,n} \times_G M); \mathbb{Q}) \to H^i(ER \times_R (M/G); \mathbb{Q})$$

is an isomorphism since $n \geq i$. Finally we conclude that

$$\theta_n^* : H_R^i(M/G; \mathbb{Q}) \to H^i(V_{k,n} \times_K M; \mathbb{Q})$$

is also an isomorphism because $\theta_n = f \circ s$. □

3.5 Equivariant classes of invariant subvarieties

In this section, we will construct equivariant cohomology classes associated to irreducible invariant subvarieties of non-singular quasi-projective varieties. Each of those classes is supported on the corresponding subvariety. Throughout this section, we assume that G is a compact connected Lie group, and H^* always denotes the singular cohomology over $\mathbb{Z}$. We emphasize that ambient varieties need *not* to be projective.

Here, we summarize the results. Let G be a compact connected Lie group acting on a non-singular quasi-projective variety X in which each element of G acts on X as an automorphism of a quasi-projective variety. Let V be a G-stable irreducible subvariety of X. We will construct the *G-equivariant class* $[V]_G \in H^*_G(X)$ which is supported on V (strictly, the Borel construction of V). This satisfies the following property; for any G-invariant non-singular Zariski-open set $U \subset V$, the restriction map

$$H^*_G(X) \to H^*_G(U)$$

sends $[V]_G$ to the G-equivariant Euler class of the normal bundle of U in X.

3.5.1 Equivariant Thom isomorphism

Let G be a compact connected Lie group, N a complex G-manifold and M a complex G-submanifold of N of codimension c which is a closed subset of N. Then the normal bundle F of M in N is a G-equivariant complex vector bundle. Since G is a compact Lie group, there exists a G-equivariant tubular neighborhood U of M with an identification (equivariantly) with F. That is, we have an equivariant open embedding $F \to M$ whose image is U. Note that this induces an open embedding $F_G \to M_G$ whose image is U_G where $M_G = EG \times_G M$ is the Borel construction of M and similar for others. We call U_G a tubular neighborhood of M_G in N_G, though M_G and N_G are not finite dimensional manifolds. Since F_G is a complex vector bundle of rank $2c$ over M_G, we have the Thom isomorphism

$$H^i_G(M) = H^i(M_G) \longrightarrow H^{i+2c}(F_G, F_G \backslash M_G) = H^{i+2c}_G(F, F \backslash M).$$

Now we have a sequence of isomorphisms:

$$H^i_G(M) \xrightarrow{\text{Thom}} H^{i+2c}_G(F, F\backslash M) \longrightarrow H^{i+2c}_G(U, U\backslash M) \xleftarrow{\text{excision}} H^{i+2c}_G(N, N\backslash M). \tag{3.5.1}$$

Lemma 3.5.1. *The isomorphism (3.5.1) does not depend on choice of equivariant tubular neighborhood U.*

Proof. Suppose that we have G-equivariant tubular neighborhoods of M in N

$$\phi : E \to N, \quad \text{and} \quad \phi' : E \to N.$$

Here is a diagram which we want to show its commutativity:

$$\begin{tikzcd} H^i_G(M) \arrow[r, "\text{Thom}"] \arrow[d, leftrightarrow, "\text{id}"'] & H^{i+2c}_G(E, E\backslash M) & H^{i+2c}_G(N, N\backslash M) \arrow[l, "\phi^*"'] \arrow[d, leftrightarrow, "\text{id}"'] \\ H^i_G(M) \arrow[r, "\text{Thom}"] & H^{i+2c}_G(E, E\backslash M) & H^{i+2c}_G(N, N\backslash M) \arrow[l, "\phi'^*"'] \end{tikzcd} \tag{3.5.2}$$

Since G is a compact Lie group, there exists a positive integer p such that G can be embedded as a closed subgroup of the unitary group $\mathrm{U}(p)$. Let $V_{p,m} := V_{p,m}(\mathbb{C})$ be the Stiefel manifold of unitary p-frames in $\mathbb{C}^{p+m}$. Then G acts freely on $V_{p,m}$ via the inclusion $G \hookrightarrow \mathrm{U}(p)$ Consider $M_m := V_{p,m} \times_G M$ for a sufficiently large m, and similar for N_m, U_m and F_m. Observe that M_m and N_m are finite dimensional smooth manifolds (they may not be complex manifolds). Also, F_m is a complex vector bundle over M_m of rank $2c$ which coincides with the codimension of M_m in N_m as real manifolds. The maps ϕ and ϕ' obviously induce tubular neighborhoods

$$\phi_m : E_m \to N_m \quad \text{and} \quad \phi'_m : E_m \to N_m$$

of M_m in N_m. By the uniqueness theorem of tubular neighborhoods, there exists an isotopy (of embeddings) $\Psi : \mathbb{R} \times E_m \to N_m$ and an isomorphism $\lambda : E_m \to E_m$ of complex vector bundles such that $\Psi_0 = \phi$ and $\Psi_1 = \phi' \circ \lambda$ (See Lang, Differential Manifolds, IV-6, and the construction of λ). Hence we obtain a commutative diagram

$$\begin{tikzcd} H^i(M_m) \arrow[r, "\text{Thom}"] \arrow[d, leftrightarrow, "\text{id}"'] & H^{i+2c}(E_m, E_m\backslash M_m) & H^{i+2c}(N_m, N_m\backslash M_m) \arrow[l, "\phi_m^*"'] \arrow[d, leftrightarrow, "\text{id}"'] \\ H^i(M_m) \arrow[r, "\text{Thom}"] & H^{i+2c}(E_m, E_m\backslash M_m) \arrow[u, "\lambda", "\cong"'] & H^{i+2c}(N_m, N_m\backslash M_m). \arrow[l, "\phi_m'^*"'] \end{tikzcd} \tag{3.5.3}$$

Between the diagram (3.5.2) and (3.5.3), we have the isomorphisms $H_G^{i+2c}(M) \cong H^{i+2c}(M_m)$ etc for a sufficiently large m, constructed in Example 3.2.7. It is easy to see that these isomorphisms commute with ϕ_m^* and $\phi_m'^*$ (see section 3.3). They also commute with the Thom isomorphisms in (3.5.2) and (3.5.3) by considering the Thom isomorphism of the complex vector bundle

$$(EG \times V_p(\mathbb{C}^m)) \times_G M \leftarrow (EG \times V_p(\mathbb{C}^m)) \times_G E$$

since this vector bundle is given by both of the pull backs of the vector bundles

$$EG \times_G M \leftarrow EG \times_G E \quad \text{and} \quad V_p(\mathbb{C}^m) \times_G M \leftarrow V_p(\mathbb{C}^m) \times_G E.$$

by the maps induced by projections $EG \times V_p(\mathbb{C}^m) \to EG$ and $EG \times V_p(\mathbb{C}^m) \to V_p(\mathbb{C}^m)$, respectively. Combining all the commutativity, we obtain the commutativity of the diagram (3.5.2). $\square$

By this lemma, we obtain a *canonical isomorphism* (constructed as above)

$$\varphi_M^N : H_G^i(M) \longrightarrow H_G^{i+2c}(N, N\backslash M). \tag{3.5.4}$$

which is determined only by the equivariant closed embedding $M \hookrightarrow N$.

If M is connected, then we obtain $\mathbb{Z} \cong H_G^0(M) \overset{\cong}{\to} H_G^{2c}(N, N\backslash M)$. So $H_G^{2c}(N, N\backslash M)$ has a canonical generator called *orientation class* which corresponds to the unit $1 \in H_G^0(M)$. Considering the special case that N is a G-equivariant complex vector bundle over M, we see that this naming is reasonable because φ_M^N is nothing but the equivariant Thom isomorphism.

Backing to the original situation, let N be a complex G-manifold and M a complex G-submanifold of N which is a closed subset of N.

Proposition 3.5.2. *Consider*

$$H_G^i(M) \xrightarrow{\varphi_M^N} H_G^{i+2c}(N, N\backslash M) \longrightarrow H_G^{i+2c}(M)$$

where the right-map is the restriction map. This map is given by the cup product with the equivariant Euler class $\chi_G(E)$ of the normal bundle of M in N.

Proof. Take an equivariant tubular neighborhood U of M in N. We have the following commutative diagram

$$\begin{tikzcd}
H_G^i(M) \arrow[r, "\mathrm{Thom}"] & H_G^{i+2c}(E, E\backslash M) \arrow[r] \arrow[dr, "s_0^*"'] & H_G^{i+2c}(U, U\backslash M) \arrow[d] & H_G^{i+2c}(N, N\backslash M) \arrow[l] \arrow[dl] \\
 & & H_G^{i+2c}(M) &
\end{tikzcd}$$

where $s_0 : M \to (E, E\backslash M)$ is the zero section of the normal bundle. The composition of the horizontal maps is the canonical isomorphism φ_M^N. Now the claim follows because the pull-back of the equivariant Thom class by the zero section s_0 is the equivariant Euler class. $\square$

Let W be a G-invariant open subset of M, and set $P := M\backslash W$. Then $N\backslash P$ is a complex G-manifold, and W is a complex G-submanifold of $N\backslash P$ which is a closed subset of $N\backslash P$. So we have the canonical isomorphism

$$\varphi_W^{N\backslash P} : H_G^i(W) \longrightarrow H_G^{i+2c}(N\backslash P, (N\backslash P)\backslash W).$$

Lemma 3.5.3. *The following diagram is commutative:*

$$\begin{tikzcd}
H_G^i(M) \arrow[r, "\varphi_M^N"] \arrow[d] & H_G^{i+2c}(N, N\backslash M) \arrow[d] \\
H_G^i(W) \arrow[r, "\varphi_W^{N\backslash P}"] & H_G^{i+2c}(N\backslash P, (N\backslash P)\backslash W)
\end{tikzcd}$$

where the vertical maps are restriction maps.

Proof. Let $\phi : F \to N$ be a G-equivariant tubular neighborhood of M in N where F is the normal bundle of M in N. Since ϕ is inejctive and sends $P(\subset M \subset F)$ bijectively onto $P(\subset N)$, the image of $F|_W = F|_{M\backslash P} = F\backslash F|_P$ has no intersection with $P(\subset N)$. So, we obtain a map

$$F|_W \hookrightarrow F \xrightarrow{\phi} N\backslash P, \tag{3.5.5}$$

and this is a G-equivariant tubular neighborhood of W in $N\backslash P$. Let U be the image of ϕ and $U|_W$ the image of (3.5.5). Now, consider the following diagram:

$$\begin{tikzcd}
H^i_G(M) \arrow[r, "\mathrm{Thom}"] \arrow[d] & H^{i+2c}_G(U, U\backslash M) \arrow[r] \arrow[d] & H^{i+2c}_G(N, N\backslash M) \arrow[d] \\
H^i_G(W) \arrow[r, "\mathrm{Thom}"] & H^{i+2c}_G(U|_W, U|_W\backslash W) \arrow[r] & H^{i+2c}_G(N\backslash P, (N\backslash P)\backslash W)
\end{tikzcd}$$

where the vertical maps are restriction maps. It suffices to show that this diagram commutes.

The right box is commutative since the maps are all induced by inclusions. For the left box, the middle vertical map sends the equivariant Thom class of F to the equivariant Thom class of $F|_W$ because $F|_W$ is the pull back of F by the inclusion $W \hookrightarrow M$. Hence it also commutes. $\square$

3.5.2 Equivariant class of invariant subvarieties

Let G be a compact connected Lie group, X a quasi-projective variety with G-action where each element of G serves as an automorphism of the algebraic variety X, and V a G-subvariety of X. We consider singular equivariant cohomology of these varieties with respect to the Euclidean topologies. Note that a subvariety may not be irreducible in our convention.

Proposition 3.5.4. *If X is non-singular, then we have*

$$H^i_G(X, X\backslash V) \cong \begin{cases} \mathbb{Z}^{\oplus r} & (\text{if } i = 2c) \\ 0 & (\text{if } i < 2c) \end{cases}$$

where r is the number of d-dimensional irreducible components of V ($d = \dim V$).

Proof. We first show the claim for the case V is non-singular and pure-dimensional (for arbitrary non-singular ambient space X). In this case, V is a complex G-submanifold of X with r connected components. So we have the canonical isomorphism φ^X_V (see (3.5.4)) :

$$H^i_G(X, X\backslash V) \xleftarrow{\varphi^X_V} H^{i-2c}_G(V).$$

Since EG is connected, we have

$$H^{i-2c}_G(V) \cong \begin{cases} \mathbb{Z}^{\oplus r} & (\text{if } i - 2c = 0) \\ 0 & (\text{if } i - 2c < 0). \end{cases}$$

This shows the claim for the case V is non-singular.

Next, we prove the claim for general V by induction on the dimension of V. We first consider the case $\dim V = 0$ (i.e. $c = \dim X$). Since V has r irreducible components, we can write $V = \{x_1, \cdots, x_r\}$. Here, each x_i is a G-fixed point since G is path-connected. In this case, V is non-singular, and we obtain

$$H^i_G(X, X\backslash V) \cong \oplus_{i=1}^r H_G^{i-2c}(x_i) \cong \begin{cases} \mathbb{Z}^{\oplus r} & (\text{if } i - 2c = 0) \\ 0 & (\text{if } i - 2c < 0) \end{cases}$$

where the middle map is the canonical isomorphism (see (3.5.4)). Now we consider the case $\dim V \geq 0$. Let $Z(\subset V)$ be the union of the singular locus of V and all irreducible components of V of dimension smaller than $d = \dim V$. Then obviously Z is a G-invariant subvariety of X satisfying $\dim Z < \dim V$. By induction hypothesis, we have

$$H^i_G(X, X\backslash Z) = \begin{cases} \mathbb{Z} & (\text{if } i = 2c_Z) \\ 0 & (\text{if } i < 2c_Z) \end{cases}$$

where $c_Z := \dim X - \dim Z$. Observing $2c < 2c_Z$, the long exact sequence

$$\cdots \longrightarrow H^i_G(X, X\backslash Z) \longrightarrow H^i_G(X, X\backslash V) \longrightarrow H^i_G(X\backslash Z, X\backslash V) \longrightarrow \cdots$$

for a triple $(X, X\backslash Z, X\backslash V)$ gives the following isomorphism

$$H^i_G(X, X\backslash V) \longrightarrow H^i_G(X\backslash Z, X\backslash V) \quad (i \leq 2c). \tag{3.5.6}$$

Since $V\backslash Z \neq \emptyset$, it is not hard to check

$X\backslash Z$ is a non-singular quasi-projective variety, and

$V\backslash Z$ is non-singular irreducible pure dimensional subvariety of $X\backslash Z$, and

$X\backslash V = (X\backslash Z)\backslash(V\backslash Z)$ (since $Z \subset V \subset X$).

Moreover, the irreducible components of $V\backslash Z$ are exactly given by the restrictions of the d-dimensional irreducible components of V ($d = \dim V$), that is, $V\backslash Z$ has r irreducible components. Also, we have

$$\dim(X\backslash Z) = \dim X \quad \text{and} \quad \dim(V\backslash Z) = \dim V.$$

Hence, by the claim for non-singular irreducible pure-dimensional subvarieties (which we have already proved), we obtain

$$H^i_G(X\backslash Z, X\backslash V) = H^i_G(X\backslash Z, (X\backslash Z)\backslash(V\backslash Z)) \cong \begin{cases} \mathbb{Z}^{\oplus r} & (\text{if } i - 2c = 0) \\ 0 & (\text{if } i - 2c < 0) \end{cases}$$

Combining with the isomorphism (3.5.6), the claim is proved. □

In the rest of this section, we assume that

- X is non-singular in which G acts smoothly, and
- V is irreducible ($\dim V = d$ and $\operatorname{codim} V = c$).

Let S be the set of singular points of V, and put $V^\circ := V\backslash S$ and $X^\circ := X\backslash S$. In the proof of the previous lemma, we proved that the restriction map

$$H_G^{2c}(X, X\backslash V) \longrightarrow H_G^{2c}(X^\circ, X^\circ\backslash V^\circ) \tag{3.5.7}$$

is an isomorphism. Composing with the canonical isomorphism $\varphi_{V^\circ}^{X^\circ}$, we obtain an isomorphism

$$H_G^{2c}(X, X\backslash V) \longrightarrow H_G^{2c}(X^\circ, X^\circ\backslash V^\circ) \xleftarrow{\varphi_{V^\circ}^{X^\circ}} H_G^0(V^\circ). \tag{3.5.8}$$

Definition 3.5.5. The *G-equivariant refined class* $\eta_V \in H_G^{2c}(X, X\backslash V)$ is defined to be the element which is mapped to $1 \in H_G^0(V^\circ)$ under the isomorphism (3.5.8).

Note that, if V is non-singular, the refined class η_V is nothing but the G-equivariant orientation class discussed in section 3.5.1.

Definition 3.5.6. The *G-equivariant class* of V in X, denoted by $[V]_G$, is the image of η_V under the restriction map $H_G^{2c}(X, X\backslash V) \to H_G^{2c}(X)$.

Let U be a G-invariant Zariski-open subset of X. Then U is a non-singular quasi-projective variety. Also, $V_U := V \cap U$ is irreducible G-subvariety of U of codimension c.

Proposition 3.5.7. *Under the restriction map $H_G^{2c}(X, X\backslash V) \longrightarrow H_G^{2c}(U, U\backslash V_U)$, the equivariant refined class η_V is mapped to the equivariant refined class η_{V_U}.*

Proof. For brevity, we write $W := V \cap U (= V_U)$. Let $S' \subset W$ be the set of singular points of W, and put $U^\circ := U\backslash S'$ and $W^\circ := W\backslash S'$. We have

$$S' = S \cap W = S \cap U$$

where the right equality holds because $S \subset V$. Hence, we obtain $W^\circ \subset V^\circ$ since

$$W^\circ = W\backslash S' = (V \cap U)\backslash(S \cap U) = (V\backslash S) \cap U = V^\circ \cap U \subset V^\circ. \tag{3.5.9}$$

Now, consider the following diagram:

$$\begin{tikzcd}
H_G^{2c}(X, X\backslash V) \arrow[r] \arrow[d] & H_G^{2c}(X^\circ, X^\circ\backslash V^\circ) & H_G^0(V^\circ) \arrow[l, "\varphi_{V^\circ}^{X^\circ}"'] \arrow[d] \\
H_G^{2c}(U, U\backslash W) \arrow[r] & H_G^{2c}(U^\circ, U^\circ\backslash W^\circ) & H_G^0(W^\circ) \arrow[l, "\varphi_{W^\circ}^{U^\circ}"']
\end{tikzcd}$$

where the vertical maps are the restriction maps. Since $W^\circ \subset V^\circ$, we define $D^\circ (\subset V^\circ)$ by the condition

$$V^\circ = W^\circ \coprod D^\circ \quad \text{(as sets)}.$$

Then we have $U^\circ \cap D^\circ = \emptyset$ because

$$U^\circ \cap V^\circ = (U\backslash S') \cap (V\backslash S) \subset (U\backslash S') \cap V = (V \cap U)\backslash S' = W\backslash S' = W^\circ.$$

Noticing that

$$\begin{aligned}&(X^\circ\backslash D^\circ)\backslash W^\circ = X^\circ\backslash V^\circ,\\ &U^\circ \subset X^\circ\backslash D^\circ\end{aligned}$$

(where the second assertion follows since $U^\circ = U\backslash S' = U\backslash(S \cap U) = U\backslash S \subset X\backslash S = X^\circ$ and $U^\circ \cap D^\circ = \emptyset$), consider the following diagram:

$$\begin{tikzcd}
H^{2c}_G(X, X\backslash V) \arrow[r] \arrow[dd] & H^{2c}_G(X^\circ, X^\circ\backslash V^\circ) \arrow[d] & H^0_G(V^\circ) \arrow[l, "\varphi^{X^\circ}_{V^\circ}"'] \arrow[d] \\
 & H^{2c}_G(X^\circ\backslash D^\circ, (X^\circ\backslash D^\circ)\backslash W^\circ) \arrow[d] & H^0_G(W^\circ) \arrow[l, "\varphi^{X^\circ\backslash D^\circ}_{W^\circ}"'] \arrow[d, "\mathrm{id}"] \\
H^{2c}_G(U, U\backslash W) \arrow[r] & H^{2c}_G(U^\circ, U^\circ\backslash W^\circ) & H^0_G(W^\circ) \arrow[l, "\varphi^{U^\circ}_{W^\circ}"']
\end{tikzcd}$$

where the right-top diagram is exactly the one in Lemma 3.5.3 (and hence it is commutative) since

$$\begin{aligned}&V^\circ \text{ is a closed set of } X^\circ,\\ &W^\circ \text{ is an open set of } V^\circ \quad \text{(see (3.5.9))},\\ &W^\circ \text{ is the complement of } D^\circ \text{ in } V^\circ.\end{aligned}$$

Our goal is to show that this diagram is commutative. The left box is obviously commutative since all the maps are induced from inclusions, and so is the right-top diagram because of Lemma 3.5.3.

We show the commutativity of the right-bottom box. Recall that we have $U^\circ \subset X^\circ\backslash D^\circ \subset X^\circ$. Since U° and $X^\circ\backslash D^\circ$ are open subsets of X°, we obtain that U° is open in $X^\circ\backslash D^\circ$. So we see that

$$U^\circ \text{ is an open set of } X^\circ\backslash D^\circ.$$

Hence, by of the construction of the canonical map in (3.5.4), the composition

$$H^0_G(W^\circ) \xrightarrow{\varphi^{U^\circ}_{W^\circ}} H^{2c}_G(U^\circ, U^\circ\backslash W^\circ) \xleftarrow{\text{excision}} H^{2c}_G(X^\circ\backslash D^\circ, (X^\circ\backslash D^\circ)\backslash W^\circ)$$

is exactly the canonical map $\varphi^{X^\circ\backslash D^\circ}_{W^\circ} : H^0_G(W^\circ) \longrightarrow H^{2c}_G(X^\circ\backslash D^\circ, (X^\circ\backslash D^\circ)\backslash W^\circ)$. This completes our proof. □

Corollary 3.5.8. *If V_U is non-singular (i.e. $V_U \subset V\backslash S$), then the restriction $H^*_G(X) \to H^*_G(V_U)$ maps $[V]_G$ to the equivariant Euler class $\chi_G(E_{V_U})$, where E_{V_U} is the normal bundle of V_U in X.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
 & H_G^{2c}(X, X\backslash V) \arrow[r] \arrow[d] & H_G^{2c}(X) \arrow[d] \arrow[dr] & \\
H_G^0(V_U) \arrow[r] & H_G^{2c}(U, U\backslash V_U) \arrow[r] & H_G^{2c}(U) \arrow[r] & H_G^{2c}(V_U).
\end{tikzcd}$$

Proposition 3.5.7 shows that $[V]_G$ is mapped to $[V_U]_G$ by the right vertical map. Since V_U is non-singular by the assumption, the restriction $H_G^*(U) \to H_G^*(V_U)$ maps $[V_U]_G$ to the equivariant Euler class $\chi_G(E_{V_U})$ by Proposition 3.5.2. Observe that the normal bundle of V_U in U is the normal bundle of V_U in X since U is an open set in X. $\square$

3.5.3 Behavior under group homomorphisms

Let X and G as above. Let K be a compact connected Lie group. Suppose that we have a homomorphism $\phi : K \to G$ as Lie groups. Then K acts on X via ϕ. We obtain the pull back homomorphism $\phi^* : H_G^*(X) \to H_K^*(X)$ as constructed in section 3.3.

The G-invariant irreducible subvariety $V \subset X$ defines the G-equivariant class $[V]_G \in H_G^*(X)$ as above. Since K acts on X via ϕ, K acts on X smoothly and each element of K preserves the structure of algebraic variety X. The subvariety V is also K-invariant, and there is the K-equivariant class $[V]_K \in H_K^*(X)$.

Proposition 3.5.9. $\phi^*([V]_G) = [V]_K$.

Proof. By the definition of equivariant classes of subvarieties, it suffices to show that the G-equivariant refined class η_V^T is sent to the K-equivariant refined class η_V^K under the map

$$H_G^{2c}(X, X\backslash V) \to H_K^{2c}(X, X\backslash V).$$

With the notation in the previous section, this suffices to check that the map

$$H_G^{2c}(X^\circ, X^\circ\backslash V^\circ) \to H_K^{2c}(X^\circ, X^\circ\backslash V^\circ)$$

sends the G-equivariant orientation class of V° to the K-equivariant orientation class of V°. This suffices to show that the following diagram is commutative

$$\begin{tikzcd}
H_G^0(V^\circ) \arrow[r, "\varphi_{V^\circ}^{X^\circ}"] \arrow[d] & H_G^{2c}(X^\circ, X^\circ\backslash V^\circ) \arrow[d] \\
H_K^0(V^\circ) \arrow[r, "\varphi_{V^\circ}^{X^\circ}"] & H_K^{2c}(X^\circ, X^\circ\backslash V^\circ)
\end{tikzcd}$$

because the G-equivariant orientation class is the image of $1 \in H_G^{2c}(V^\circ)$ and similar for K-equivariant orientation class. By the definition of the canonical isomorphisms $\varphi^{X^\circ}_{V^\circ}$, it suffices to show that the following diagram commutes

$$\begin{tikzcd} H^0_G(V^\circ) \arrow[r, "\text{Thom}"] \arrow[d] & H^{2c}_G(N^\circ, N^\circ\backslash V^\circ) \arrow[d] \\ H^0_K(V^\circ) \arrow[r, "\text{Thom}"] & H^{2c}_K(N^\circ, N^\circ\backslash V^\circ) \end{tikzcd}$$

where N° is the normal bundle of V° in X°. This is equivalent to show that the map

$$H^{2c}_G(N^\circ, N^\circ\backslash V^\circ) \to H^{2c}_K(N^\circ, N^\circ\backslash V^\circ) \tag{3.5.10}$$

sends the G-equivariant Thom class to the K-equivariant Thom class. Consider the following commutative diagram.

$$\begin{tikzcd} EK \times_K N^\circ \arrow[r] \arrow[d] & EG \times_G N^\circ \arrow[d] \\ EK \times_K V^\circ \arrow[r] & EG \times_G V^\circ \end{tikzcd}$$

This is obviously a pull back diagram of complex vector bundles. Hence (3.5.10) sends the G-equivariant Thom class to the K-equivariant Thom class, as desired. □

3.5.4 Behavior under equivariant morphisms

Let G be a compact connected Lie group. Let X and Y be non-singular quasi-projective varieties in which each element of G acts as automorphism of the variety. Let F be an irreducible quasi-projective variety in which each element of G acts as automorphism of the variety. A G-invariant irreducible subvariety $V \subset X$ defines the G-equivariant class $[V]_G \in H^*_G(X)$ as above. Let $f : Y \to X$ be a G-equivariant fiber bundle. Then it follows that $f^{-1}(V) \subset Y$ is an irreducible G-subvariety since the fiber F is irreducible. So we have the G-equivariant class $[f^{-1}(V)]_G \in H^*_G(Y)$.

We assume that f is equivariantly locally trivialized. That is, for each point $x \in X$, there exists a G-invariant open neighborhood of x in X and a G-equivariant isomorphism $\varphi : f^{-1}(U) \to U \times F$ where G acts on the target in an obvious way. Under these assumptions, we have the following.

Proposition 3.5.10. $[f^{-1}(V)]_G = f^*([V]_G)$.

Proof. By the assumptions, Proposition 3.5.7 ensures that it suffices to assume that the bundle f is a trivial bundle. That is, denoting the fiber by F, we can write f as the projection

$$f : Y = X \times F \to X \quad (x, q) \mapsto x$$

where G acts on $Y = X \times F$ in an obvious way. Also, by the isomorphism (3.5.7) and the construction of the equivariant classes of subvarieties, Proposition 3.5.7 ensures that $V \subset X$ is non-singular. Hence, again by Proposition 3.5.7, we can assume that X is the normal bundle of V in X and $Y = X \times F$ is the pullback bundle of X by by the projection $V \times F \to V$. Now our claim follows from the naturality of the equivariant Thom classes with respect to pullbacks. $\square$

3.6 GKM theory for torus actions

In this section, we study a combinatorial presentation of a torus equivariant cohomology developed by Goresky-Kottwitz-MacPherson ([20]). This presentation will allow us to do elementary computations on the equivariant cohomology. In this section, the cohomology $H^* = H^*(\ \ ;\mathbb{Q})$ is of rational coefficient unless otherwise specified.

3.6.1 GKM theory for a general setting

Let $T = (S^1)^n$ be the n-dimensional torus acting on a locally contractible, Hausdorff space X. We also assume that

(i) X is compact, and

(ii) $H^*_T(X)$ is a free module over $H^*(BT)$

where H^*_T is the singular T-equivariant cohomology. Denote

$$X_0 := \{[x] \in X \mid \operatorname{corank} T_{[x]} = 0\} = X^T,$$
$$X_1 := \{[x] \in X \mid \operatorname{corank} T_{[x]} \leq 1\}$$

where X^T is the fixed point set of the T-action. There exists a natural isomorphism between the Čech cohomology theory and the singular cohomology theory for any closed pair of locally contractible, paracompact, Hausdorff spaces. Thus, the results in [10] applies for the singular T-equivariant cohomology: the restriction map $H^*_T(X) \to H^*_T(X_0)$ is injective, and so is the connecting homomorphism $H^*_T(X, X_0) \to H^*_T(X_1, X_0)$ of the exact sequence for the triple (X, X_1, X_0) because of the second assumption above. Combining the exact sequences for the pair (X, X_0) and the one for triple (X, X_1, X_0), we obtain the following exact sequence.

Proposition 3.6.1. *Under the above assumptions, the following sequence (of coefficient in $\mathbb{Q}$) is exact:*

$$0 \to H^*_T(X) \xrightarrow{i^*} H^*_T(X_0) \xrightarrow{\delta} H^{*+1}_T(X_1, X_0)$$

where the middle map is the restriction, and the right map is the connecting homomorphism of the exact sequence for the pair (X_1, X_0).

We now make some additional assumptions on the T-action on X:

(iii) $\#X_0$ and $\#\{$connected components of $X_1 - X_0\}$ are finite, and

(iv) for each connected component E of $X_1 - X_0$, its closure $\overline{E}$ is given by $E \cup \{x, y\}$ where $x, y \in X - E$, and is homeomorphic to the complex projective line $\mathbb{P}^1$, and

(v) the T-action on restricts on $\mathbb{P}^1 - \{\infty\} \cong \mathbb{C}$ (by $[z, 1] \mapsto z$) through the above homeomorphism, and is identified with the (complex) representation of T.

By Proposition 3.6.1, we have

$$\mathrm{Im}\iota^* = \mathrm{Ker}\delta. \tag{3.6.1}$$

Consider the inclusions $i_k : \{x_k, y_k\} \hookrightarrow X_0$ and $j_k : (\overline{E_k}, \{x_k, y_k\}) \hookrightarrow (X_1, X_0)$. Then we obtain a commutative diagram

$$\begin{tikzcd} H_T^*(X_0) \arrow[r, "\delta"] \arrow[d, "\oplus_k i_k^*"'] & H_T^{*+1}(X_1, X_0) \arrow[d, "\oplus_k j_k^*"] \\ \oplus_k H_T^*(\{x_k, y_k\}) \arrow[r, "\oplus_k \delta_k"] & \oplus_k H_T^{*+1}\left(\overline{E_k}, \{x_k, y_k\}\right). \end{tikzcd} \tag{3.6.2}$$

Combining the assumptions (iii)-(v) with (3.6.1), we obtain the following.

Proposition 3.6.2.

$$\mathrm{Ker}\delta = \mathrm{Ker}(\oplus_k \delta_k \circ \oplus_k i_k^*) = (\oplus_k i_k^*)^{-1} (\oplus_k \mathrm{Ker}\delta_k) = \bigcap_k (i_k^*)^{-1} \mathrm{Ker}\delta_k$$

Proof. We prove the left equality. Other equalities can be checked by direct calculation. It suffices to show that, in the diagram (3.6.2), the right vertical map $\oplus_k j_k^*$ is an isomorphism. Let $E_1, \cdots, E_l$ be the connected components of $X_1 \backslash X_0$. We first fix $1 \leq k \leq l$. Let us write $\overline{E_k} = E_k \cup \{x_k, y_k\}$. We have a homeomorphism $\varphi_k : \overline{E_k} \to \mathbb{P}^1$. For each i such that $x_k \in \overline{E_i}$, we define

$$D_i(x_k) = \begin{cases} \varphi_i^{-1}\{[1 : w] \in \mathbb{P}^1 \mid |w| \leq c\} & \text{if } \varphi_i(x_k) = [1 : 0], \\ \varphi_i^{-1}\{[z : 1] \in \mathbb{P}^1 \mid |w| \leq c\} & \text{if } \varphi_i(x_k) = [0 : 1] \end{cases}$$

where $c > 1$ is a real number, and define $D_i(y_k)$ similarly. Define, for each j,

$$A_k := \varphi_k^{-1}\{[1 : w] \in \mathbb{P}^1 \mid 1/M \leq |w| \leq M\},$$

$$B_j := \begin{cases} \emptyset & \text{if } i = k, \\ D_j(x_k) \cup D_j(y_k) & \text{if } i \neq k \text{ and } x_k, y_k \in \overline{E_j}, \\ D_j(x_k) & \text{if } x_k \in \overline{E_j} \text{ and } y_k \notin \overline{E_j}, \\ D_j(y_k) & \text{if } x_k \notin \overline{E_j} \text{ and } y_k \in \overline{E_j}, \\ \overline{E_j} & \text{if } x_k, y_k \notin \overline{E_j}. \end{cases}$$

Let us denote

$$Y_1 := X_1 - \bigcup_j B_j, \qquad Y_2 := X_1 - A_k$$

(here, Y_1 and Y_2 depend on k).

$Y_1 =$

$Y_2 =$

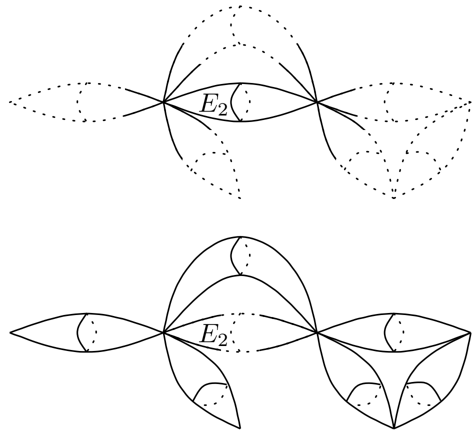


Figure 3.6.1: The case $k = 2$

Since A_k and the finite union $\bigcup_k B_k$ are compact, Y_1 and Y_2 are open in X_1 (recall that X is assumed to be a Hausdorff space). The condition (v) shows that T-action on X restricts on A_k and B_k. Also define

$$F_1 := \{x_k, y_k\}, \qquad F_2 := X_0.$$

Observing that X_0 is a finite set,

$$Y_1 \cup Y_2 = X_1, \quad F_1 \cup F_2 = X_0$$

are T-invariant open coverings of X_1 and X_0. So we have the Mayer-Vietoris exact sequence :

$$\cdots \to H_T^*(X_1, X_0) \to H_T^*(Y_1, F_1) \oplus H_T^*(Y_1, F_1) \to H_T^*(Y_1 \cap Y_2, F_1 \cap F_2) \to \cdots .$$

By assumption (v), we identify the corresponding T-action on $\mathbb{P}^1 - \{\infty\}$ with the the representation given by a weight in $\mathrm{Hom}(T, S^1)$. This is possible since irreducible (complex) representations of S^1 are all one dimensional given by a weight function. So, by shrinking T-equivariantly, we have

$$\begin{aligned} &H_T^*(Y_1 \cap Y_2, F_1 \cap F_2) \cong H_T^*(F_1, F_1) = 0, \\ &H_T^*(Y_1, F_1) \cong H_T^*\left(\overline{E_k}, \{x_k, y_k\}\right), \\ &H_T^*(Y_2, F_2) \cong H_T^*\left(X_1 - E_k, X_0\right). \end{aligned}$$

Hence, the exact sequence splits into

$$H_T^*(X_1, X_0) \xrightarrow{\cong} H_T^*\left(\overline{E_k}, \{x_k, y_k\}\right) \oplus H_T^*\left(X_1 - E_k, X_0\right).$$

Continuing this process, we obtain

$$H_T^*(X_1, X_0) \cong \oplus_k H_T^*\left(\overline{E_k}, \{x_k, y_k\}\right) \oplus H_T^*\left(X_1 - \cup_k E_k, X_0\right).$$

Since we have $X_1 - \cup_k E_k = X_0$ by definition, we get

$$\begin{aligned} H_T^*(X_1, X_0) &\cong \oplus_k H_T^*\left(\overline{E_k}, \{x_k, y_k\}\right) \oplus H_T^*\left(X_0, X_0\right) \\ &= \oplus_k H_T^*\left(\overline{E_k}, \{x_k, y_k\}\right). \end{aligned}$$

Recalling that this isomorphism is just a direct product of the homomorphism induced by inclusions, we complete the proof. □

For an one-dimensional orbit E_k, we have a homeomorphism $\varphi_k : \overline{E_k} \xrightarrow{\cong} \mathbb{P}^1$ by the assumption (iv). Let T_k be the isotropy subgroup of a point in E_k (which dose not depend on this point), and let $\mathfrak{t}_k$ be its Lie algebra. Also, let $\mathfrak{t}_{\mathbb{Q}}^* := \mathbb{Q} \otimes \mathfrak{t}_{\mathbb{Z}}^* (\subset \mathfrak{t}^*)$.

Proposition 3.6.3. *Under the identification $H_T^*(\mathrm{pt}) = H^*(BT) = \mathrm{Sym}(\mathfrak{t}_{\mathbb{Q}}^*)$, we have*

$$\mathrm{Ker}\delta_k = \left\{(f, g) \in \mathrm{Sym}(\mathfrak{t}_{\mathbb{Q}}^*) \oplus \mathrm{Sym}(\mathfrak{t}_{\mathbb{Q}}^*) \mid f|_{\mathfrak{t}_k} = g|_{\mathfrak{t}_k}\right\}. \tag{3.6.3}$$

Proof. Consider the long exact sequence of the pair $(\overline{E_k}, \{x_k, y_k\})$;

$$\cdots \to H_T^q(\overline{E_k}) \to H_T^q(x_k) \oplus H_T^q(y_k) \xrightarrow{\delta_k} H_T^{q+1}(\overline{E_k}, \{x_k, y_k\}) \to \cdots.$$

Recall that $\{x_k, y_k\}$ is the T-fixed point of the T-action on $\mathbb{P}^1$ by assumption (v). Since $H^{\mathrm{odd}}(\mathbb{P}^1) = 0$, the equivariant cohomology $H_T^q(\overline{E_k})$ is a free module over $H^*(BT)$. This shows that the left map is injective by Proposition 3.6.1. Also, by the Serre spectral sequence, we have $H_T^{odd}(\overline{E_k}) \cong H_T^{odd}(\mathbb{P}^1) = 0$. Hence, the above sequence splits into the following short exact sequence

$$0 \to H_T^q(\overline{E_k}) \to H_T^q(x_k) \oplus H_T^q(y_k) \xrightarrow{\delta_k} H_T^{q+1}(\overline{E_k}, \{x_k, y_k\}) \to 0.$$

Next, consider the Mayer-Vietoris exact sequence;

$$\cdots \to H_T^q(\overline{E_k}) \to H_T^q(U) \oplus H_T^q(V) \to H_T^q(U \cap V) \to \cdots$$

where $U := \overline{E_k} - \{x_k\}$ and $V := \overline{E_k} - \{y_k\}$ are T-invariant open subsets of $\overline{E_k} = \mathbb{P}^1$. By the same reason, this sequence also splits into the following short exact sequence

$$0 \to H_T^q(\overline{E_k}) \to H_T^q(U) \oplus H_T^q(V) \to H_T^q(U \cap V) \to 0.$$

Now, considering the homomorphisms induced by the inclusions, we obtain

$$\begin{tikzcd} 0 \arrow[r] & H^q_T(\overline{E_k}) \arrow[r] & H^q_T(x_k) \oplus H^q_T(y_k) \arrow[r, "\delta_k"] & H^{q+1}_T(\overline{E_k}, \{x_k, y_k\}) \arrow[r] & 0 \\ 0 \arrow[r] & H^q_T(\overline{E_k}) \arrow[r] \arrow[u] & H^q_T(U) \oplus H^q_T(V) \arrow[r] \arrow[u] & H^q_T(U \cap V) \arrow[r] & 0 \end{tikzcd}$$

Since the two horizontal sequences are exact, we obtain a homomorphism $H^q_T(U \cap V) \to H^{q+1}_T(\overline{E_k}, \{x_k, y_k\})$. This map is in fact an isomorphism since the the other vertical maps are isomorphism (since $\{0\} \hookrightarrow U$ is a T-homotopy equivalence). Now, by chasing the diagram, we obtain

$$\begin{aligned} &\operatorname{Ker}\left[H^q_T(x_k) \oplus H^q_T(y_k) \stackrel{\delta_k}{\to} H^{q+1}_T(\overline{E_k}, \{x_k, y_k\})\right] \\ &\quad = \operatorname{Ker}\left[H^q_T(x_k) \oplus H^q_T(y_k) \stackrel{\delta_k}{\to} H^{q+1}_T(\overline{E_k}, \{x_k, y_k\}) \stackrel{\cong}{\leftarrow} H^q_T(U \cap V)\right] \\ &\quad = \operatorname{Ker}\left[H^q_T(x_k) \oplus H^q_T(y_k) \stackrel{\cong}{\leftarrow} H^q_T(U) \oplus H^q_T(V) \to H^q_T(U \cap V)\right] \\ &\quad = \operatorname{Ker}\left[H^q_T(x_k) \oplus H^q_T(y_k) \to H^q_T(U \cap V)\right]. \end{aligned}$$

Here, the last map is $\psi - \phi$ where ψ and ϕ are the homomorphisms induced by the one-point maps $U \cap V \to \{x_k\}$ and $U \cap V \to \{y_k\}$.

By the assumption (v), there exists a T-homotopy equivalence $S^1 \hookrightarrow U \cap V$ where T acts on this S^1 transitively. So we have a homeomorphism $S^1 \cong T/K$ where K is the isotropy subgroup of a point of S^1. We have an obvious continuous map $ET \times_K \{\mathrm{pt}\} \to ET \times_T (T/K)$ sending $[v, \mathrm{pt}] \mapsto [v, [1]]$ where 1 is the identity element of T. This is a homeomorphism. In fact, the inverse continuous map is constructed as follows. Consider a continuous map $ET \times T \to ET \times \{\mathrm{pt}\}$ by $(v, t) \mapsto (vt, *)$. This induces $ET \times (T/K) \to ET \times_K \{\mathrm{pt}\}$ by $(v, [t]) \mapsto [vt, *]$. We finally obtain a continuous map $ET \times_T (T/K) \to ET \times_K \{\mathrm{pt}\}$ by $[v, [t]] \mapsto [vt, *]$, and this gives the inverse map. Recalling that we can take ET as EK, we obtain $H^*_T(T/K) \stackrel{\cong}{\to} H^*_K(\mathrm{pt})$. Now the map

$$H^{2q}_T(x_k) \stackrel{\psi}{\to} H^{2q}_T(U \cap V) \stackrel{\cong}{\to} H^{2q}_T(S^1) \stackrel{\cong}{\to} H^{2q}_T(T/K) \stackrel{\cong}{\to} H^{2q}_K(\mathrm{pt})$$

Since ψ is identified with the structure map of an $H^{2q}_T(x_k)$-algebra (i.e. the multiplication map of equivariant parameters), we see that this composition is the group-restriction map induced by $K \hookrightarrow T$. Considering similarly for ϕ, we obtain that

$$\begin{aligned} &\operatorname{Ker}\left[H^q_T(x_k) \oplus H^q_T(y_k) \stackrel{\delta_k}{\to} H^{q+1}_T(\overline{E_k}, \{x_k, y_k\})\right] \\ &\quad = \operatorname{Ker}\left[H^q_T(x_k) \oplus H^q_T(y_k) \to H^q_T(U \cap V)\right] \\ &\quad = \operatorname{Ker}\left[H^q_T(x_k) \oplus H^q_T(y_k) \to H^q_K(\mathrm{pt})\right] \\ &\quad = \operatorname{Ker}\left[(f, g) \mapsto (f - g)|_{\mathfrak{k}}\right] \end{aligned}$$

by Proposition 3.3.3. □

Now, with (3.6.1), we obtain the main theorem. Recall that we let $T = (S^1)^n$ be a torus acting on a locally contractible, Hausdorff space X.

Theorem 3.6.4. *(Goresky-Kottwitz-MacPherson)*
If the conditions (i)-(v) are satisfied, then the image of the restriction map $H_T^*(X;\mathbb{Q}) \hookrightarrow H_T^*(X^T;\mathbb{Q}) = \oplus_{X^T}\mathrm{Sym}(\mathfrak{t}_{\mathbb{Q}}^*)$ *is given by*

$$\left\{p \in \oplus_{X^T}\mathrm{Sym}(\mathfrak{t}_{\mathbb{Q}}^*) \mid p(x_k)|_{\mathfrak{t}_k} = p(y_k)|_{\mathfrak{t}_k} \text{ for all } k\right\}$$

where $\mathfrak{t}_k$ *is the Lie algebra of the isotropy subgroup of a point in a connected component* E_k *of the 1-dimensional orbits of* T*.*

Remark 3.6.5. In [24], they provide a similar theorem for generalized equivariant cohomology,

3.6.2 GKM theory for algebraic torus actions

There is a natural situation for Theorem 3.6.4. Let X be a (possibly singular) complex projective variety equipped with an algebraic action of a complex torus $T_{\mathbb{C}} = (\mathbb{C}^\times)^n$. Let $T = (S^1)^n \subset T_{\mathbb{C}}$ be the real torus. We consider the singular equivariant cohomology $H_T^*(X;\mathbb{Q})$ with respect to the Euclidean topology. The condition (i) is obviously satisfied. By Bialynicki-Birula's $T_{\mathbb{C}}$-invariant cell decomposition of X, we have $H^{\mathrm{odd}}(X;\mathbb{Q}) = 0$. Hence, the equivariant cohomology $H_T^*(X;\mathbb{Q})$ is a free $H^*(BK;\mathbb{Q})$-module, that is, the condition (ii) also holds. Recall that we denote

$$X^T = X_0 = \{[x] \in X \mid \mathrm{corank}\, T_{[x]} = 0\}, \quad X_1 = \{[x] \in X \mid \mathrm{corank}\, T_{[x]} \le 1\}.$$

We now assume that

(a) $\#X_0$ and $\#\{$connected components of $X_1 - X_0\}$ are finite, and

(b) for each connected component E of $X_1 - X_0$, its Zariski closure $\overline{E}$ is isomorphic to the complex projective line $\mathbb{P}^1$ (as algebraic varieties).

Let $E_1, \cdots, E_l$ be the connected component of 1-dimensional orbit $X_1 - X_0$. Recall that the automorphism group (as an algebraic variety) of $\mathbb{P}^1$ is $PSL_2(\mathbb{C})$. So the assumption (b) implies that there are exactly two $T_{\mathbb{C}}$-fixed points x_k and y_k in E_k, and the condition (iv) and (v) is automatically satisfied. Let T_k be the isotropy subgroup of a point in E_k (which dose not depend on this point), and let $\mathfrak{t}_k$ be its Lie algebra. Then Theorem 3.6.4 shows the following which is in fact the original statement provided in [20].

Theorem 3.6.6. *(Goresky-Kottwitz-MacPherson [20])*
If the conditions (a) and (b) are satisfied, then the image of the restriction map $H_T^*(X;\mathbb{Q}) \hookrightarrow H_T^*(X^T;\mathbb{Q}) = \oplus_{X^T}\mathrm{Sym}(\mathfrak{t}_{\mathbb{Q}}^*)$ *is given by*

$$\left\{p \in \oplus_{X^T}\mathrm{Sym}(\mathfrak{t}_{\mathbb{Q}}^*) \mid p(x_k)|_{\mathfrak{t}_k} = p(y_k)|_{\mathfrak{t}_k} \text{ for all } k\right\}.$$

As the closing of this section, we explain a back ground of this theorem in equivariant symplectic geometry as follows. Let X be a $T_{\mathbb{C}}$-invariant subvariety of a projective space $\mathbb{P}^N$ equipped with an Hamiltonian $T_{\mathbb{C}}$-action. Then the T-action on X is also a Hamiltonian action with a moment map $\mu : X \to \mathfrak{t}^*$. Let $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ be the weight of the $T_{\mathbb{C}}$-representation on $\mathbb{C} = \overline{E_k} \backslash \{x_k\}$ (equivalently, $-\lambda$ is the weight of the $T_{\mathbb{C}}$-representation on $\overline{E_k} \backslash \{y_k\}$). Then $\lambda_k \in \mathfrak{t}^*_{\mathbb{Z}} \subset \mathfrak{t}^*$ describe the direction (in $\mathfrak{t}^*$) of the edge $\mu(\overline{E_k})$ of the moment polytope $\mu(X)$. Also, $\mathfrak{t}_k$ can be written by

$$\mathfrak{t}_k = \{X \in \mathfrak{t} \mid \lambda_k(X) = 0\}.$$

This shows that

$$H^*_T(X; \mathbb{Q}) = \left\{p \in \oplus_{X^T} \mathrm{Sym}(\mathfrak{t}^*_{\mathbb{Q}}) \mid p(x_k) - p(y_k) \text{ is divisible by } \lambda_k \text{ for all } k\right\}.$$

Because of this equality, in many situation, it is easy to see the equivariant cohomology $H^*_T(X; \mathbb{Q})$ in terms of the *moment graph*; the vertex set of the moment graph is X^T, and the edges are the image $\mu(\overline{E_k})$ of the invariant projective lines equipped with the label $\pm\lambda_k \in \mathfrak{t}^*_{\mathbb{Z}}$ which is the weight of the T-representation on E_k (this has an ambiguity of signs, but this will not matter when we compute the equivariant cohomology). Then we can say that the combinatorial data of the moment graph determines the equivariant cohomology $H^*_T(X; \mathbb{Q})$.

Chapter 4

Schubert Calculus of Weighted Grassmannians

The cohomology ring of the complex Grassmannian manifold $\mathrm{Gr}(d,n)$ has been attracting mathematicians for several decades. This ring can be calculated very explicitly, and the *Schubert classes* play the important roles connecting geometry, algebraic topology, combinatorics and representation theory. The Schubert variety Ω_λ for each Young diagram λ (contained in the $d \times n$ rectangular box) is an irreducible subvariety of the Grassmannian $\mathrm{Gr}(d,n)$, and their fundamental classes, the Schubert classes, provides us a $\mathbb{Z}$-module basis of $H^*(\mathrm{Gr}(d,n);\mathbb{Z})$. There is a natural question to ask; the number $c^{\nu}_{\lambda\mu}$ in the product expansion

$$S_\lambda S_\mu = \sum_{\nu} c^{\nu}_{\lambda\mu} S_\nu \tag{4.0.1}$$

where the sum runs over all the Young diagram contained in the $d \times n$ rectangular box. Mathematics around these types of coefficients for other spaces (e.g., partial flag varieties, Hessenberg vaeities, etc), or other rings (e.g., K-theory, quantum cohomology, symmetric polynomial ring, etc) is called *Schubert calculus.* The most striking phenomenon of this problem which attract mathematicians is the fact that there is a surjective ring homomorphism

$$R_+(\mathrm{GL}(d,\mathbb{C})) \to H^*(\mathrm{Gr}(d,n);\mathbb{Z})$$

where $R_+(\mathrm{GL}(d,\mathbb{C}))$ is the subring of the representation ring of $\mathrm{GL}(d,\mathbb{C})$ generated by the polynomial representations. In fact, for each Young diagram λ, there is the irreducible representation V_λ of $\mathrm{GL}(d,\mathbb{C})$ whose highest weight is $\lambda = (\lambda_1, \cdots, \lambda_d)$ where each λ_i is the number of boxes in the i-th row. As a matter of fact, they form a $\mathbb{Z}$-module basis of the polynomial representation ring of $\mathrm{GL}(d,\mathbb{C})$, and the tensor product can be decomposed into irreducible representations

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} C^{\nu}_{\lambda\mu} V_\nu.$$

Here, if λ, μ and ν are contained in the $d \times n$ rectangular box, then

$$c^{\nu}_{\lambda\mu} = C^{\nu}_{\lambda\mu}.$$

In this sense, the Schubert classes are in an intersection of geometry, algebraic topology and representation theory.

As a generalization of this problem, Knutson-Tao studied the torus equivariant cohomology $H^*_T(\mathrm{Gr}(d,n),\mathbb{Z})$ of Grassmannians, and gave a beautiful combinatorial rule which describes the equivariant Schubert calculus where $T = (S^1)^n$ is the n-dimensional torus acting on $\mathrm{Gr}(d,n)$ as the induced action from the standard T-action on $\mathbb{C}^n$. Since each Schubert variety is a T-invariant irreducible suvariety of $\mathrm{Gr}(d,n)$, there is the T-equivariant fundamental class $\tilde{S}_\lambda$, called the T-equivariant Schubert calss, associated to this Schubert variety. They used so-called equivariant puzzle rule to calculate the equivariant structure constants $\tilde{c}^{\nu}_{\lambda\mu}$ with respect to the classes $\tilde{S}_\lambda$. See [35] for detail.

Recently, Tymoczko ([47]) studied a torus equivariant cohomology of weighted projective spaces in $\mathbb{Z}$-coefficient. The weighted projective space $\mathrm{w}\mathbb{P}^n$ is a very important example of *orbifolds* which admit mild singularities. Since they are singular as varieties in general, there are no nice notion of fundamental classes (in cohomology) for subvarieties of $\mathrm{w}\mathbb{P}^n$. In fact, she used a combinatorial Schubert-type module basis of $H^*_T(\mathrm{w}\mathbb{P}^n;\mathbb{Z})$. On the other hand, Corti-Reid ([11]) defined weighted partial flag varieties which can be thought as an orbifold version of the partial flag varieties for general Lie type. They provides a class of orbifolds which can be calculated explicitly. In this chapter, we will study the equivariant Schubert calculus of the *weighted Grassmannian* $\mathrm{wGr}(d,n)$, where our definition of Schubert classes is geometric in a sense of orbifolds (or stacks). This chapter is based on the paper [3] collaborated with Tomoo Matsumura.

4.1 Weighted Grassmannians and weighted Schubert varieties

In this section, we recall the definition of the *weighted Grassmannian* $\mathrm{wGr}(d,n)$, following [11]. We study the coordinate charts and obtain a *quasi-cell decomposition* which generalizes the usual Schubert cell decomposition of the ordinary Grassmannian $\mathrm{Gr}(d,n)$. This allows us to define the *weighted Schubert varieties* by taking the closure of each cell and also as a consequence, we show that the odd degree classes of the rational cohomology of $\mathrm{wGr}(d,n)$ vanish.

For positive integers d and n such that $d < n$, let $[n] := \{1, \cdots, n\}$, and

$$\{\begin{smallmatrix} n \\ d \end{smallmatrix}\} := \{\lambda \subset [n] \mid |\lambda| = d\}.$$

We denote the elements of λ by $\lambda_1, \cdots, \lambda_d$ where $\lambda_1 < \cdots < \lambda_d$. For $\lambda, \mu \in \{\begin{smallmatrix} n \\ d \end{smallmatrix}\}$, we define the *Bruhat order* by: $\lambda \leq \mu$ if

$$\lambda_i \geq \mu_i \text{ for all } i = 1, \cdots, d. \tag{4.1.1}$$

We define the *lexicographic order* by: $\lambda <_{lex} \mu$ if there exists an integer $1 \leq j \leq d$ such that

$$\lambda_i = \mu_i \text{ for al } i < j, \text{ and } \lambda_j < \mu_j. \tag{4.1.2}$$

An *inversion* (k, l) of λ is a pair of $k \in \lambda$ and $l \notin \lambda$ such that $k < l$. Let $\mathrm{inv}(\lambda)$ be the set of all inversions of λ. The *length* $l(\lambda)$ of λ is defined to be the cardinarity of $\mathrm{inv}(\lambda)$. For each $(k, l) \in \mathrm{inv}(\lambda)$, let $(k, l)\lambda$ be the element of $\{{n \atop d}\}$ obtained by replacing k in λ by l. Let

$$[\lambda] := \{\, \mu \in \{{n \atop d}\} \mid |\lambda \cap \mu| = d - 1 \} .$$

Then $[\lambda] = [\lambda]_+ \coprod [\lambda]_-$ where $[\lambda]_- := \{\mu \in [\lambda] \mid \mu \leq \lambda\}$ and $[\lambda]_+ := \{\mu \in [\lambda] \mid \mu \geq \lambda\}$. Note that there is a bijection $\mathrm{inv}(\lambda) \cong [\lambda]_-$, sending (k, l) to $(k, l)\lambda$. We say that λ *covers* μ if $\mu \in [\lambda]_-$ and $l(\lambda) = l(\mu) + 1$, and denote $\lambda \to \mu$.

4.1.1 The weighted Grassmannian

Let $\mathbb{C}^n$ be the complex n-plane with the standard basis $\{e_i, i \in [n]\}$ and $\bigwedge^d \mathbb{C}^n$ its d-th exterior product with the induced basis

$$\{e_\lambda := e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d}, \lambda \in \{{n \atop d}\}\}.$$

We identify $\bigwedge^d \mathbb{C}^n$ with the coordinate space $\mathbb{C}^{\{{n \atop d}\}}$ where each $x \in \bigwedge^d \mathbb{C}^n$ corresponds to the coordinate vector $(x_\lambda)_{\lambda \in \{{n \atop d}\}}$ with respect to e_λ's. We also denote each coordinate x_λ by $x(\lambda_1 \cdots \lambda_n)$. Let $T_{\mathbb{C}} := (\mathbb{C}^\times)^n$ and $(\mathbb{C}^\times)^{\{{n \atop d}\}}$ be the complex tori acting canonically on $\mathbb{C}^n$ and $\mathbb{C}^{\{{n \atop d}\}}$ respectively. Consider the following $T_{\mathbb{C}}$-equivariant map

$$\wedge^d : \overbrace{\mathbb{C}^n \times \cdots \times \mathbb{C}^n}^{d \ times} \to \textstyle\bigwedge^d \mathbb{C}^n \quad ; \quad (z_1, \cdots, z_d) \mapsto z_1 \wedge \cdots \wedge z_d$$

where $T_{\mathbb{C}}$ acts on the domain diagonally and, to the target through the map

$$\rho : T_{\mathbb{C}} \to (\mathbb{C}^\times)^{\{{n \atop d}\}} \quad ; \quad t = (t_1, \cdots, t_n) \mapsto \big(t_\lambda := \textstyle\prod_{l \in \lambda} t_l\big)_{\lambda \in \{{n \atop d}\}} . \tag{4.1.3}$$

Let $\mathrm{aPl}(d, n)$ be the image of $\wedge^d$ which is $T_{\mathbb{C}}$-invariant.

Let $w := (w_1, \cdots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $a \in \mathbb{Z}_{\geq 1}$. We introduce

$$\mathrm{w}D_{\mathbb{C}} := \{(t^{dw_1 + a}, \cdots, t^{dw_n + a}) \in T_{\mathbb{C}} \mid t \in \mathbb{C}^\times\} \quad \text{and} \quad \mathrm{w}R_{\mathbb{C}} := T_{\mathbb{C}}/\mathrm{w}D_{\mathbb{C}}.$$

Note that

$$\rho(\mathrm{w}D_{\mathbb{C}}) = \left\{ (t^{w_\lambda})_\lambda \in (\mathbb{C}^\times)^{\{{n \atop d}\}} \;\middle|\; t \in \mathbb{C}^\times \right\}, \quad \text{where} \ \ w_\lambda := a + \sum_{l \in \lambda} w_l.$$

In the case when $w = (0, \cdots, 0)$ and $a = 1$, we write $D_{\mathbb{C}}$ for the diagonal in $\mathbb{C}^{\{{n \atop d}\}}$ and $R_{\mathbb{C}} = T_{\mathbb{C}}/D_{\mathbb{C}}$. We denote the corresponding compact real tori in the complex tori by $T, \mathrm{w}R, \mathrm{w}D, R$ and D respectively.

Definition 4.1.1 (Corti-Reid [11])**.** Let $\mathrm{aPl}(d,n)^\times := \mathrm{aPl}(d,n) - \{0\}$. The *weighted Grassmannian* $\mathrm{wGr}(d,n)$ is the projective variety with at worst orbifold singularities, given by

$$\mathrm{wGr}(d,n) := \mathrm{aPl}(d,n)^\times / \mathrm{w}D_{\mathbb{C}}.$$

The quotient torus $\mathrm{w}R_{\mathbb{C}}$ acts on $\mathrm{wGr}(d,n)$. The ordinary Grassmannian $\mathrm{Gr}(d,n)$ is the special case when $w_1 = \cdots = w_n = 0$ and $a = 1$, i.e. $\mathrm{Gr}(d,n) = \mathrm{aPl}(d,n)^\times / D_{\mathbb{C}}$.

Remark 4.1.2. In [11], the $\mathbb{C}^\times$-action which defines $\mathrm{wGr}(d,n)$ as a quotient of $\mathrm{aPl}(d,n)^\times$ is actually given by the map $\mathbb{C}^\times \to \rho(\mathrm{w}D_{\mathbb{C}})$, $t \mapsto (t^{w_\lambda})_\lambda$, but obviously it defines the same algebraic variety.

4.1.2 The Charts for $\mathrm{aPl}(d,n)^\times$ and $\mathrm{wGr}(d,n)$

Extend the notation $x(l_1,\cdots,l_d)$ to any (not necessarily increasing) sequence $(l_1,\cdots,l_d)$ of integers in $[n]$ by the rule

$$x(l_1,\cdots,l_p,l_{p+1},\cdots,l_d) = -x(l_1,\cdots,l_{p+1},l_p,\cdots,l_d)$$

for any integer $1 \le p \le d-1$. It is known that $\mathrm{aPl}(d,n)^\times$ is a non-singular quasi-projective variety in $\mathbb{C}^{\{ {n \atop d} \}} - \{0\}$ defined by of the *Plücker relations* (c.f. [34]): for any sequence of integers $1 \le j_1,\cdots,j_{d-1}, l_1,\cdots,l_{d+1} \le n$,

$$\sum_{i=1}^{d+1} (-1)^{i-1} x(j_1,\cdots,j_{d-1},l_i) x(l_1,\cdots,\check{l_i},\cdots,l_{d+1}) = 0. \tag{4.1.4}$$

Consider the following $T_{\mathbb{C}}$-stable open neighborhood of e_λ in $\mathrm{aPl}(d,n)^\times$:

$$\mathrm{a}U^\lambda := \left\{ x \in \mathrm{aPl}(d,n)^\times \mid x_\lambda \neq 0 \right\}.$$

It is clear that $\mathrm{aPl}(d,n)^\times$ is covered by $\mathrm{a}U^\lambda$'s, and moreover we have the natural $T_{\mathbb{C}}$-equivariant coordinates on each $\mathrm{a}U^\lambda$. Let $\mathbb{C}^{[\lambda]}$ be the subspace of $\mathbb{C}^{\{ {n \atop d} \}}$ corresponding to the subspace generated by $\{e_\mu, \mu \in [\lambda]\}$ and consider the natural projection

$$\psi_\lambda : \mathrm{a}U^\lambda \to \mathbb{C}^\times \times \mathbb{C}^{[\lambda]} \quad ; \quad x \mapsto \left(x_\lambda, (x_\mu)_{\mu\in[\lambda]}\right). \tag{4.1.5}$$

This is a $T_{\mathbb{C}}$-equivariant homeomorphism where $T_{\mathbb{C}}$ acts on the target through the map

$$\rho_\lambda : T_{\mathbb{C}} \to (\mathbb{C}^\times)^{\{ {n \atop d} \}} \to \mathbb{C}^\times \times (\mathbb{C}^\times)^{[\lambda]} \;\; ; \;\; t \mapsto (t_\lambda, (t_\mu)_{\mu\in[\lambda]}).$$

Indeed, the inverse of ψ_λ is constructed as follows (c.f. [34, p.1065]). For each $i = 1,\cdots,d$ and $l \in [n]$, let

$$p_i(l) = \frac{x(\lambda_1 \cdots \lambda_{i-1} l \lambda_{i+1} \cdots \lambda_d)}{x_\lambda}$$

and consider the vectors $p_i := \sum_{l=1}^n p_i(l)e_l \in \mathbb{C}^n$. The numerator of each coefficient is $\pm x_\mu$ with $\mu \in [\lambda]$ if $l \in \{\lambda_1, \cdots, \lambda_d\}\backslash\{\lambda_i\}$, x_λ if $l = \lambda_i$, and zero otherwise. Thus we can define ψ_λ by assigning $y := x_\lambda p_1 \wedge \cdots \wedge p_d$ to each $\tilde{x} := \big(x_\lambda, (x_\mu)_{\mu\in[\lambda]}\big) \in \mathbb{C}^\times \times \mathbb{C}^{[\lambda]}$. It is straightforward to check that $y_\mu = x_\mu$ for all $\mu \in [\lambda]$ and $y_\lambda = x_\lambda$, i.e. $\psi_\lambda(y) = \tilde{x}$.

Passing to the quotient, we obtain the natural $\mathrm{w}R_\mathbb{C}$-equivariant affine charts of $\mathrm{wGr}(d,n)$. Let

$$\mathrm{w}U^\lambda := \mathrm{a}U^\lambda/\mathrm{w}D_\mathbb{C}.$$

Then ψ_λ induces a homeomorphism

$$\overline{\psi}_\lambda : \mathrm{w}U^\lambda \xrightarrow{\cong} (\mathbb{C}^\times \times \mathbb{C}^{[\lambda]})/\rho_\lambda(\mathrm{w}D_\mathbb{C}) \cong \mathbb{C}^{[\lambda]}/G_\lambda$$

where G_λ is a finite cyclic subgroup of $(\mathbb{C}^\times)^{[\lambda]}$ given by

$$G_\lambda = \Big\{(t^{w_\mu})_{\mu\in[\lambda]} \in (\mathbb{C}^\times)^{[\lambda]} \;\Big|\; t \in \mathbb{C}^\times \;\text{ and }\; t^{w_\lambda} = 1\Big\}.$$

4.1.3 The Schubert cell decompositions and Schubert varieties

Consider the $(\mathbb{C}^\times)^{\{ {n \atop d} \}}$-invariant decomposition of $\mathbb{C}^{\{ {n \atop d} \}} - \{0\}$

$$\mathbb{C}^{\{ {n \atop d} \}} - \{0\} := \coprod_{\lambda\in\{ {n \atop d} \}} C_\lambda$$

where

$$C_\lambda := \Big\{x \in \mathbb{C}^{\{ {n \atop d} \}} - \{0\} \;\Big|\; x_\lambda \neq 0 \;\text{ and }\; x_\mu = 0 \text{ for all } \mu >_{lex} \lambda\Big\}.$$

By restricting the above decomposition to $\mathrm{aPl}(d,n)^\times$, there is the $T_\mathbb{C}$-invariant decomposition

$$\mathrm{aPl}(d,n)^\times = \coprod_{\lambda\in\{ {n \atop d} \}} \mathrm{a}\Omega_\lambda^\circ \quad \text{where} \quad \mathrm{a}\Omega_\lambda^\circ := \mathrm{aPl}(d,n)^\times \cap C_\lambda. \tag{4.1.6}$$

Since $\mathrm{aPl}(d,n)^\times \cap C_\lambda \subset \mathrm{a}U^\lambda$, we have $\mathrm{a}\Omega_\lambda^\circ = \mathrm{a}U^\lambda \cap C_\lambda$. The following lemma helps us to describe the image of $\mathrm{a}\Omega_\lambda^\circ$ under the chart ψ_λ (Cor 4.1.4).

Lemma 4.1.3. *Let $x \in \mathrm{a}U^\lambda$. The following are equivalent:*

(i) $x_\mu = 0$ *for all* $\mu \in [\lambda]_-$.

(ii) $x_\mu = 0$ *for all* $\mu >_{lex} \lambda$ *(i.e.* $x \in \mathrm{a}\Omega_\lambda^\circ$*).*

(iii) $x_\mu = 0$ *for all* $\mu \ngeq \lambda$.

Proof. Since we have the implications $\nu \in [\lambda]_- \Rightarrow \nu >_{lex} \lambda \Rightarrow \nu \not\geq \lambda$, it is clear that $(iii) \Rightarrow (ii) \Rightarrow (i)$. We prove that (i) implies (iii). Assume (i) and let $\nu \not\geq \lambda$. We use induction on the number $k := d - |\nu \cap \lambda|$. If $k = 1$, $\nu \not\geq \lambda$ implies $\nu \in [\lambda]_-$. Thus $x_\nu = 0$. In general, choose an integer $1 \leq s \leq d$ such that $\nu_s \notin \lambda$. For $1 \leq i \leq d$ such that $\lambda_i \notin \nu$, let $\nu^{(i)} := (\nu \backslash \{\nu_s\}) \cup \{\lambda_i\}$ and $\lambda^{(i)} := (\lambda \backslash \{\lambda_i\}) \cup \{\nu_s\}$ in $\{{n \atop d}\}$. Then we claim that

$$\lambda \not\leq \lambda^{(i)} \text{ or } \lambda \not\leq \nu^{(i)}. \tag{4.1.7}$$

Indeed, $\lambda \leq \lambda^{(i)}$ implies $\nu_s < \lambda_i$, i.e. $\nu^{(i)} \leq \nu$. Therefore, together with $\lambda \leq \nu^{(i)}$, it implies $\nu \geq \lambda$. Thus the negation of (4.1.7) leads to a contradiction. Now, consider the Plücker relation for the sequences $\nu_1, \cdots, \check{\nu_s}, \cdots, \nu_d$ and $\lambda_1, \cdots, \lambda_d, \nu_s$

$$x_\nu x_\lambda = \sum_{\substack{1 \leq i \leq d \\ \lambda_i \notin \nu}} \pm x_{\nu^{(i)}} x_{\lambda^{(i)}} \tag{4.1.8}$$

where the signs are chosen appropriately according to (4.1.4). Since we have $d - |\nu^{(i)} \cap \lambda| < d - |\nu \cap \lambda|$ and $d - |\lambda^{(i)} \cap \lambda| = 1$, the induction hypothesis implies that $x_{\nu^{(i)}} = 0$ or $x_{\lambda^{(i)}} = 0$ by (4.1.7). Therefore (4.1.8) becomes $x_\nu x_\lambda = 0$. Since $x_\lambda \neq 0$ by $x \in \mathrm{a}U^\lambda$, we have $x_\nu = 0$. ☐

Corollary 4.1.4. *Under the chart* ψ_λ*, we have* $\mathrm{a}\Omega_\lambda^\circ \cong \mathbb{C}^\times \times \mathbb{C}^{[\lambda]_+} \times \{0\}^{[\lambda]_-}$.

Since the decomposition (4.1.6) is $T_\mathbb{C}$-invariants, it descends to the quotient $\mathrm{wGr}(d,n)$ and gives the $\mathrm{w}R_\mathbb{C}$-invariant decomposition. We call this decomposition a *quasi-cell decomposition* because each "cell" is actually homeomorphic to a Euclidean space modulo a finite group.

Proposition 4.1.5.

$$\mathrm{wGr}(d,n) = \coprod\nolimits_{\lambda \in \{{n \atop d}\}} \mathrm{w}\Omega_\lambda^\circ \quad \textit{where} \quad \mathrm{w}\Omega_\lambda^\circ := \mathrm{a}\Omega_\lambda^\circ / \mathrm{w}D_\mathbb{C}.$$

Under the chart $\overline{\psi}_\lambda$, $\mathrm{w}\Omega_\lambda^\circ \cong \mathbb{C}^{[\lambda]_+}/G_\lambda$.

Definition 4.1.6. For each $\lambda \in \{{n \atop d}\}$, we define the Schubert varieties in $\mathrm{aPl}(d,n)^\times$ and $\mathrm{wGr}(d,n)$ as Euclidean closures of $\mathrm{a}\Omega_\lambda^\circ$ and $\mathrm{w}\Omega_\lambda^\circ$ respectively, i.e.

$$\mathrm{a}\Omega_\lambda := \overline{\mathrm{a}\Omega_\lambda^\circ} \quad \text{and} \quad \mathrm{w}\Omega_\lambda := \overline{\mathrm{w}\Omega_\lambda^\circ}.$$

We will call $\mathrm{w}\Omega_\lambda$ the *weighted Schubert variety* corresponding to λ.

The next proposition seems well-known but for the sake of completeness we will give a proof.

Proposition 4.1.7.

$$\mathrm{a}\Omega_\lambda = \coprod_{\mu \geq \lambda} \mathrm{a}\Omega_\mu^\circ.$$

Proof. First, we show $\mathrm{a}\Omega_\lambda \subset \coprod_{\mu \geq \lambda} \mathrm{a}\Omega^\circ_\mu$. Let $x \in \mathrm{a}\Omega_\lambda$. Then there exists a sequence $\{x_N\}_{N=0}^\infty \subset \mathrm{a}\Omega^\circ_\lambda$ such that x_N converges to x as N goes to ∞. By Lemma 4.1.3, $(x_N)_\eta = 0$ for all $\eta \not\geq \lambda$. Therefore $x_\eta = 0$ for all $\eta \not\geq \lambda$, i.e. $x \notin \mathrm{a}\Omega^\circ_\eta$ for all $\mu \not\geq \lambda$. By the decomposition (4.1.6) of $\mathrm{aPl}(d,n)^\times$, we obtain $x \in \coprod_{\lambda \leq \mu} \mathrm{a}\Omega^\circ_\mu$. Next, we show $\mathrm{a}\Omega_\lambda \supset \coprod_{\mu \geq \lambda} \mathrm{a}\Omega^\circ_\mu$. If $\mu \gneqq \lambda$, then there is a covering sequence $\mu = \mu^s \to \mu^{s-1} \to \cdots \to \mu^1 \to \lambda$ where $s = l(\mu) - l(\lambda)$. Thus it suffices to show that $\mathrm{a}\Omega_\lambda \supset \mathrm{a}\Omega^\circ_\mu$ for any μ such that $\mu \to \lambda$, i.e. for some $1 \leq p \leq d$,

$$\mu_p = \lambda_p - 1 \quad \text{and} \quad \mu_q = \lambda_q \quad \text{for all} \quad q \neq p.$$

Let $y \in \mathrm{a}\Omega^\circ_\mu$. We construct a sequence $\{x_N\}_{N\in\mathbb{N}} \subset \mathrm{a}\Omega^\circ_\lambda$ which converges to y as N goes to ∞. For brevity, we omit the index N and write $x_N = x$. Since any point in $\mathrm{a}U^\lambda$ is determined by its coordinates of the indexes in $\{\lambda\}\coprod[\lambda]$, we define x to be the element of $\mathrm{a}U^\lambda$ uniquely given by

$$\begin{aligned}
&x_\lambda := N^{-1},\\
&x_\nu := y_\nu \quad \text{for all } \nu \in [\lambda] \cap (\{\mu\}\textstyle\coprod[\mu]),\\
&x_{\rho_{q,\alpha}} := \Big\{ y(\mu_1, \cdots, \check{\mu_q}, \cdots, \mu_d, \alpha) x(\mu_1, \cdots, \check{\mu_p}, \cdots, \mu_d, \lambda_p)\\
&\qquad\qquad - y(\mu_1, \cdots, \check{\mu_p}, \cdots, \mu_d, \alpha) y(\mu_1, \cdots, \check{\mu_q}, \cdots, \mu_d, \lambda_p) \Big\} \frac{(-1)^{\delta+r}}{y_\mu}
\end{aligned}$$

where $\delta = 0$ if $p < q$ and $\delta = 1$ if $q < p$, and

$$x_{\rho_{q,\alpha}} = (-1)^r x(\mu_1, \cdots, \check{\mu_q}, \cdots, \check{\mu_p}, \cdots, \mu_d, \alpha, \lambda_p)$$

for some integer r in the extended notation given at Section 4.1.2. Here we have used the decomposition of $[\lambda]$ into $[\lambda] \cap (\{\mu\}\coprod[\mu])$ and

$$[\lambda]\backslash(\{\mu\}\textstyle\coprod[\mu]) = \Big\{ \rho_{q,\alpha} := \{\lambda_1, \cdots, \lambda_d, \alpha\}\backslash\{\lambda_q\} \;\Big|\; q \neq p \text{ and } \alpha \notin \lambda \cup \{\mu_p\} \Big\}.$$

This x is an element of $\mathrm{a}\Omega^\circ_\lambda$ since $x_\eta = 0$ for all $\eta \in [\lambda]_-$ by Lemma 4.1.3. Indeed, if $\eta \in [\lambda]\cap(\{\mu\}\coprod[\mu])$, then $\eta \in [\mu]_-$ so that $x_\eta = y_\eta = 0$ by $y \in \mathrm{a}\Omega^\circ_\mu$. If $\eta = \rho_{q,\alpha}$, then $\mu_q = \lambda_q < \alpha$, i.e. $\{\mu_1, \cdots, \check{\mu_q}, \cdots, \mu_d, \alpha\} \in [\mu]_-$. Hence, the first term of $x_{\rho_{q,\alpha}}$ vanishes. Moreover, if $\mu_p < \alpha$, we have $\{\mu_1, \cdots, \check{\mu_p}, \cdots, \mu_d, \alpha\} \in [\mu]_-$, and if not, $\mu_q < \alpha < \mu_p < \lambda_p$, we have $\{\mu_1, \cdots, \check{\mu_q}, \cdots, \mu_d, \lambda_p\} \in [\mu]_-$. In both cases, the second term of $x_{\rho_{q,\alpha}}$ vanishes.

Since $x \in \mathrm{a}U^\mu$ by $x_\mu = y_\mu \neq 0$, we can compare x and y under the chart ψ_μ. In fact, $\psi_\mu(x)$ and $\psi_\mu(y)$ coincide except for the λ-component, therefore x goes to y when N goes to ∞, as desired. By the definition of x, it suffices to check that $x_\xi = y_\xi$ for any $\xi \in (\{\mu\}\coprod[\mu])\backslash(\{\lambda\}\coprod[\lambda])$. Observe that ξ can be written as $\{\mu_1, \cdots, \check{\mu_q}, \cdots, \mu_d, \alpha\}$ where $q \neq p$ and $\alpha \notin \lambda \cup \{\mu_p\}$. By the Plücker relation for the sequences $\mu_1, \cdots, \check{\mu_q}, \cdots, \check{\mu_p}, \cdots, \mu_d, \alpha$ and $\mu_1, \cdots, \check{\mu_p}, \cdots, \mu_d, \lambda_p, \mu_p$, we have

$$\begin{aligned}
&x(\mu_1,\cdots,\check{\mu_q},\cdots,\mu_d,\alpha)x(\mu_1,\cdots,\check{\mu_p},\cdots,\mu_d,\lambda_p)\\
&\quad=(-1)^{d-p+\delta}x(\mu_1,\cdots,\check{\mu_q},\cdots,\check{\mu_p},\cdots,\mu_d,\alpha,\mu_p)x(\mu_1,\cdots,\check{\mu_p},\cdots,\mu_d,\lambda_p)\\
&\quad=x(\mu_1,\cdots,\check{\mu_p},\cdots,\mu_d,\alpha)x(\mu_1,\cdots,\check{\mu_q},\cdots,\mu_d,\lambda_p)\\
&\qquad\qquad+(-1)^{\delta}x(\mu_1,\cdots,\check{\mu_q},\cdots,\check{\mu_p},\cdots,\mu_d,\alpha,\lambda_p)x(\mu_1,\cdots,\mu_d)\\
&\quad=y(\mu_1,\cdots,\check{\mu_p},\cdots,\mu_d,\alpha)y(\mu_1,\cdots,\check{\mu_q},\cdots,\mu_d,\lambda_p)+(-1)^{\delta+r}x_{\rho_{q,\alpha}}y_\mu\\
&\quad=y(\mu_1,\cdots,\check{\mu_q},\cdots,\mu_d,\alpha)x(\mu_1,\cdots,\check{\mu_p},\cdots,\mu_d,\lambda_p)
\end{aligned}$$

where we used the definition of $x_{\rho_{q,\alpha}}$ at the last equality. So we obtain

$$x(\mu_1,\cdots,\check{\mu_q},\cdots,\mu_d,\alpha)=y(\mu_1,\cdots,\check{\mu_q},\cdots,\mu_d,\alpha)$$

since $x(\mu_1,\cdots,\check{\mu_p},\cdots,\mu_d,\lambda_p)=\pm x_\lambda\neq 0$. □

The following proposition is a corollary.

Proposition 4.1.8.

$$\mathrm{w}\Omega_\lambda=\coprod_{\mu\geq\lambda}\mathrm{w}\Omega_\mu^\circ$$

Proof. It is clear from Proposition 4.1.7 that $\mathrm{w}\Omega_\lambda\supset\coprod_{\lambda\leq\mu}\mathrm{w}\Omega_\mu^\circ$. Let $[x]\in\mathrm{w}\Omega_\lambda$. There exists a sequence $\{[x^i]\}_{i=0}^\infty\subset\mathrm{w}\Omega_\lambda^\circ$ such that $[x^i]\to[x]$ as $i\to\infty$. Lemma 4.1.3 shows $x_\eta^i=0$ for all $\eta\not\geq\lambda$. This implies that $x_\eta=0$ for all $\eta\not\geq\lambda$. Thus x must lies in $\mathrm{w}\Omega_\mu^\circ$ for some $\mu\geq\lambda$. □

As corollaries, we can also write $\mathrm{a}\Omega_\lambda$ and $\mathrm{w}\Omega_\lambda$ explicitly as subvarieties of $\mathrm{aPl}(d,n)^\times$ and $\mathrm{wGr}(d,n)$ respectively.

Corollary 4.1.9. $\mathrm{a}\Omega_\lambda=\{x\in\mathrm{aPl}(d,n)^\times\mid x_\nu=0$ *for all* $\nu\not\geq\lambda\}$. *In particular, the complex codimension of* $\mathrm{a}\Omega_\lambda$ *in* $\mathrm{aPl}(d,n)^\times$ *is the length* $l(\lambda)$ *of* λ.

Proof. Let $x\in\mathrm{a}\Omega_\lambda$. By Proposition 4.1.7, x is contained in $\mathrm{a}\Omega_\mu^\circ$ for some $\mu\geq\lambda$. If $\nu\not\geq\lambda$, then $\nu\not\geq\mu$ so that $x_\nu=0$ by Lemma 4.1.3. On the other hand, suppose that x is in the RHS. Then $x\notin\mathrm{a}\Omega_\nu$ for all $\nu\not\geq\lambda$. Therefore $x\in\coprod_{\mu\geq\lambda}\mathrm{a}\Omega_\mu^\circ=\mathrm{a}\Omega_\lambda$. Finally $\dim\mathrm{a}\Omega_\lambda=\dim\mathrm{a}\Omega_\lambda^\circ=d(n-d)-l(\lambda)$ by Corollary 4.1.4. □

The previous corollary immediately implies the following.

Corollary 4.1.10. $\mathrm{w}\Omega_\lambda=\{[x]\in\mathrm{wGr}(d,n)\mid x_\nu=0$ *for all* $\nu\not\geq\lambda\}$. *In particular, the codimension of* $\mathrm{w}\Omega_\lambda^\circ$ *in* $\mathrm{wGr}(d,n)$ *is the length* $l(\lambda)$ *of* λ.

Remark 4.1.11. The varieties $\mathrm{a}\Omega_\lambda$ is irreducible in $\mathrm{aPl}(d,n)^\times$ since it is a closure of Ω_λ°. From this, it also follows that $\mathrm{w}\Omega_\lambda$ is an irreducible variety.

4.1.4 Vanishing of the odd degree

The quasi-cell decomposition in Proposition 4.1.5 allows us to show that the odd degree of the rational singular cohomology $H^*(\mathrm{wGr}(d,n))$ vanishes. As a consequence, the Serre spectral sequence for the fibration $E\mathrm{w}R \times_{\mathrm{w}R} \mathrm{wGr}(d,n) \to B\mathrm{w}R$ degenerates at E_2-stage and the wR-equivariant cohomology is free over $H^*(B\mathrm{w}R)$. *In this paper, all cohomologies are assumed to be over* $\mathbb{Q}$*-coefficients unless otherwise specified.*

Let us denote $\overline{H}_*$ the Borel-Moore homology over $\mathbb{Q}$. We first quote a lemma which will be used in the proof of the next proposition.

Lemma 4.1.12. *Let D^m be the closed unit disc in $\mathbb{R}^m$. Let $\iota : \{0\} \hookrightarrow D^m$ be the inclusion. Then $\iota_* : \overline{H}_*(\{0\}) \to \overline{H}_*(D^m)$ is an isomorphism.*

Let $a_1, \cdots, a_m, b$ be positive integers. Then $\mathbb{C}^\times$ acts on $\mathbb{C}^m$ by

$$g\cdot(z_1, \cdots, z_m) = (g^{a_1} z_1, \cdots, g^{a_m} z_m)$$

for any $g \in \mathbb{C}^\times$ and $(z_1, \cdots, z_m) \in \mathbb{C}^m$. Let $G := \{g \in \mathbb{C}^\times \mid g^b = e\}$. Observe that the Borel-Moore homology $\overline{H}_i(\mathbb{C}^m/G)$ is defined ($\mathbb{C}^m/G$ can be realized by an open subset of $\mathrm{w}\mathbb{P}(a_1, \cdots, a_m, b)$ which can also be realized as a closed subset of a projective space $\mathbb{P}^N$ for some integer N).

Proposition 4.1.13.

$$\overline{H}_i(\mathbb{C}^m/G) \cong \begin{cases} \mathbb{Q} & (i = 2m) \\ 0 & (\textit{otherwise}) \end{cases}$$

Proof. Let $D^{2m} = \{z \in \mathbb{C}^m \mid |z| \leq 1\}$. Then the G-action on $\mathbb{C}^m$ restricts on $\mathrm{int}D^{2m}$ since $G \subset S^1 \subset \mathbb{C}^\times$, and the G-equivariant homeomorphism $\mathrm{int}D^{2m} \cong \mathbb{C}^m$ defined by

$$z \mapsto \left(z_1/\sqrt{1-|z|^2}, \cdots, z_m/\sqrt{1-|z|^2}\right)$$

induces an homeomorphism $\mathrm{int}D^{2m}/G \cong \mathbb{C}^m/G$. We calculate $\overline{H}_i(\mathrm{int}D^{2m}/G)$ in the following.

The Borel-Moore homology $\overline{H}_*(D^{2m}/G)$ is also defined because D^{2m}/G is a closed subset of $\mathbb{C}^m/G$. Then, we have an exact sequence associated to the open embedding $\mathrm{int}D^{2m}/G \hookrightarrow D^{2m}/G$

$$\cdots \to \overline{H}_i(D^{2m}/G) \to \overline{H}_i(\mathrm{int}D^{2m}/G) \to \overline{H}_{i-1}(S^{2m-1}/G) \xrightarrow{\iota_*} \overline{H}_{i-1}(D^{2m}/G) \to \cdots$$

where $\iota : S^{2m-1} \hookrightarrow D^{2m}$ is the inclusion. Since the spaces D^{2m}/G and S^{2m-1}/G are compact, locally contractible spaces, we have

$$\overline{H}_i(D^{2m}/G) \cong H_i(D^{2m}/G) \cong \begin{cases} \mathbb{Q} & (i = 0) \\ 0 & (\text{otherwise}), \end{cases}$$

$$\overline{H}_i(S^{2m-1}/G) \cong H_i(S^{2m-1}/G) \cong \begin{cases} \mathbb{Q} & (i = 0, 2m-1) \\ 0 & (\text{otherwise}). \end{cases}$$

(see [46, Lem.14, sec.10, chap.6]). Hence, the above exact sequence shows

$$\overline{H}_i(\mathrm{int}D^{2m}/G) \cong \begin{cases} \mathbb{Q} & (i = 2m) \\ 0 & (i \neq 0, 1, 2m). \end{cases}$$

We prove $\overline{H}_0(\mathrm{int}D^{2m}/G) = \overline{H}_1(\mathrm{int}D^{2m}/G) = 0$. Since we have

$$0 \to \overline{H}_1(\mathrm{int}D^{2m}/G) \to \overline{H}_0(S^{2m-1}/G) \xrightarrow{\iota_*} \overline{H}_0(D^{2m}/G) \to \overline{H}_0(\mathrm{int}D^{2m}/G) \to 0,$$

it suffices to show that $\overline{H}_0(S^{2m-1}/G) \xrightarrow{\iota_*} \overline{H}_0(D^{2m}/G)$ is an isomorphism. Recall that D^{2m}/G is compact and can be embedded into $\mathbb{R}^M$ for some M. Without loss of generality, we can assume that there is a sequence of closed embeddings

$$\{0\} \to S^{2m-1}/G \hookrightarrow D^{2m}/G \to D(\hookrightarrow \mathbb{R}^M)$$

such that the compositions of embeddings coincides with the natural inclusion $j : \{0\} \hookrightarrow D$, where D is the closed unit disk in $\mathbb{R}^M$ and 0 is the origin of $\mathbb{R}^M$. Out of this sequence of closed embeddings, we obtain

$$\overline{H}_0(\{0\}) \to \overline{H}_0(S^{2m-1}/G) \to \overline{H}_0(D^{2m}/G) \to \overline{H}_0(D)$$

which coincides with $j_* : \overline{H}_0(\{0\}) \to \overline{H}_0(D)$. Since all the entries in this sequence are isomorphic to $\mathbb{Q}$, it is enough to show that j_* is an isomorphism which is proved by Lemma 4.1.12. □

Proposition 4.1.14.

$$H^i(\mathrm{wGr}(d,n)) \cong \begin{cases} \bigoplus_{2l(\lambda)=i} \mathbb{Q} & \textit{if } i \textit{ is even,} \\ 0 & \textit{if } i \textit{ is odd.} \end{cases}$$

Proof. The argument of Appendix B in [18] can be applied to the quasi-cell decomposition, and we obtain

$$\overline{H}_i(\mathrm{wGr}(d,n)) \cong \bigoplus_{\dim \mathrm{wGr}(d,n) - 2l(\lambda) = i} \overline{H}_i(\mathrm{w}\Omega^\circ_\lambda) \tag{4.1.9}$$

where $\overline{H}_*$ is the rational Borel-Moore homology. By Proposition 4.1.13, we obtain

$$\overline{H}_i(\mathrm{w}\Omega^\circ_\lambda) \cong \overline{H}_i(\mathbb{C}^{[\lambda]_-}/G_\lambda) \cong \begin{cases} \mathbb{Q} & \text{if } i = 2l(\lambda), \\ 0 & \text{if otherwise.} \end{cases} \tag{4.1.10}$$

Hence, we obtain

$$\overline{H}_i(\mathrm{wGr}(d,n)) \cong \begin{cases} \bigoplus_{\dim \mathrm{wGr}(d,n) - 2l(\lambda)=i} \mathbb{Q} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Since $\mathrm{wGr}(d,n)$ is covered by the locally contractible charts $\{\mathrm{w}U^\lambda\}$, we see that $\mathrm{wGr}(d,n)$ is a compact locally contractible space. Hence, the singular homology and the Borel-Moore homology agree ([46, Lem.14, sec.10, chap.6]):

$$H_i(\mathrm{wGr}(d,n)) \cong \overline{H}_i(\mathrm{wGr}(d,n)).$$

By applying the rational Poincaré duality (c.f. [5, Proposition 1.28]), we obtain the claim. □

Recall that the rational equivariant cohomology for the wR-action on $\mathrm{wGr}(d,n)$ is defined as the cohomology of the *Borel construction*, i.e. the total space of the fibration

$$\mathrm{wGr}(d,n) \overset{\zeta}{\hookrightarrow} E\mathrm{w}R \times_{\mathrm{w}R} \mathrm{wGr}(d,n) \to B\mathrm{w}R,$$

where $E\mathrm{w}R \to B\mathrm{w}R$ is a universal principal wR-bundle with the contractible total space and $E\mathrm{w}R \times_{\mathrm{w}R} \mathrm{wGr}(d,n) := (E\mathrm{w}R \times \mathrm{wGr}(d,n))/\mathrm{w}R$. The pullback of the projection to $B\mathrm{w}R$ defines the $H^*(B\mathrm{w}R)$-module structure of $H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n))$. Since the fiber $\mathrm{wGr}(d,n)$ is path-connected, the vanishing of odd degree classes implies that the Serre spectral sequence of this fibration collapses at E_2-stage. This implies the freeness of $H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n))$ as a $H^*(B\mathrm{w}R)$-module:

Proposition 4.1.15. *As $H^*(B\mathrm{w}R)$-modules,*

$$H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n)) \cong H^*(B\mathrm{w}R) \otimes_{\mathbb{Q}} H^*(\mathrm{wGr}(d,n)).$$

In particular, $H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n))$ is a free module over $H^*(B\mathrm{w}R)$.

4.2 Equivariant weighted Schubert classes

Recall that $T, \mathrm{w}R$ and R be the real tori in $T_{\mathbb{C}}, \mathrm{w}R_{\mathbb{C}}, R_{\mathbb{C}}$ respectively. In this section, we discuss the relations among the rational equivariant cohomologies $H^*_T(\mathrm{aPl}(d,n)^\times)$, $H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n))$, and $H^*_R(\mathrm{Gr}(d,n))$. *In fact, they are isomorphic as rings, while they are modules over different polynomial rings.* In $H^*_T(\mathrm{aPl}(d,n)^\times)$, there are geometrically defined cohomology classes $\mathrm{a}\tilde{S}_\lambda$ associated to the varieties $\mathrm{a}\Omega_\lambda$. We define our *equivariant weighted Schubert classes* $\mathrm{w}\tilde{S}_\lambda$ in $H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n))$ as the classes corresponding to $\mathrm{a}\tilde{S}_\lambda$ under the isomorphism.

The quotient maps from $\mathrm{aPl}(d,n)^\times$ to $\mathrm{wGr}(d,n)$ and $\mathrm{Gr}(d,n)$, and from T to wR and R, induce the following commutative diagram of the Borel constructions:

$$\begin{tikzcd}
ER \times_R \mathrm{Gr}(d,n) \arrow[d] & ET \times_T \mathrm{aPl}(d,n)^\times \arrow[l, "h"'] \arrow[r, "\mathrm{w}h"] \arrow[d] & E\mathrm{w}R \times_{\mathrm{w}R} \mathrm{wGr}(d,n) \arrow[d] \\
BR & BT \arrow[l] \arrow[r] & B\mathrm{w}R
\end{tikzcd}$$

By the functoriality, the pullback maps

$$h^* : H^*_R(\mathrm{Gr}(d,n)) \to H^*_T(\mathrm{aPl}(d,n)^\times) \quad \text{and} \quad \mathrm{w}h^* : H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n)) \to H^*_T(\mathrm{aPl}(d,n)^\times)$$

are homomorphism of rings over the polynomial rings $H^*(BR)$ and $H^*(B\mathrm{w}R)$. The proof of the following proposition is postponed until after we define the weighted Schubert classes.

Proposition 4.2.1. *The maps h^* and $\mathrm{w}h^*$ are isomorphisms as rings over the poylnomial rings $H^*(BR)$ and $H^*(B\mathrm{w}R)$ respectively.*

Proof. The claim follows essentially from the *Vietoris-Begle mapping theorem*, but we need to prepare the description of $\mathrm{wGr}(d,n)$ as the quotient of a compact space by a real torus.

Since the $\mathrm{w}D$-action on $\mathbb{C}^{\{n \atop d\}}$ factors through the canonical $(S^1)^{\{n \atop d\}}$-action, it is hamiltonian with the standard moment map. Since $\mathrm{aPl}(d,n)^\times$ is a $\mathrm{w}D$-invariant symplectic submanifold of $\mathbb{C}^{\{n \atop d\}}$, there is the induced moment map [1]

$$\Psi : \mathrm{aPl}(d,n)^\times \to \mathbb{R} \quad ; \quad x \mapsto -\frac{1}{2} \sum_{\lambda \in \{n \atop d\}} dw_\lambda |x_\lambda|^2.$$

For a regular value ξ, the preimage $M := \Psi^{-1}(\xi)$ is a compact T-invariant submanifold of $\mathrm{aPl}(d,n)^\times$. Moreover there is a T-equivariant deformation retraction from $\mathrm{aPl}(d,n)^\times$ to M given by the homotopy

$$F : \mathrm{aPl}(d,n)^\times \times I \to \mathrm{aPl}(d,n)^\times \ ; \ (x,s) \mapsto \Big((s\sqrt{\xi/\Psi(x)} + (1-s))x_\lambda \Big)_{\lambda \in \{n \atop d\}}.$$

Thus, the inclusion $\iota : M \hookrightarrow \mathrm{aPl}(d,n)^\times$ induces the isomorphism:

$$\iota^* : H_T^*(M) \longrightarrow H_T^*(\mathrm{aPl}(d,n)^\times). \tag{4.2.1}$$

Passing to the quotients, we obtain the $\mathrm{w}R$-equivariant map $\bar{\iota} : M/\mathrm{w}D \to \mathrm{wGr}(d,n)$. This map can be shown to be a homeomorphism by a direct computation (See also [32, Theorem 7.4]). Hence, we obtain the isomorphism:

$$\bar{\iota}^* : H_{\mathrm{w}R}^*(M/\mathrm{w}D) \longrightarrow H_{\mathrm{w}R}^*(\mathrm{wGr}(d,n)). \tag{4.2.2}$$

Let $\theta : ET \times_T M \to E\mathrm{w}R \times_{\mathrm{w}R} M/\mathrm{w}D$ be a map induced by the quotient maps $M \to M/\mathrm{w}D$ and $T \to \mathrm{w}R$. Then we have the following commutative diagram.

$$\begin{tikzcd} H_{\mathrm{w}R}^*(\mathrm{wGr}(d,n)) \arrow[r, "\mathrm{w}h^*"'] \arrow[d, "\cong"', "\bar{\iota}^*"] & H_T^*(\mathrm{aPl}(d,n)^\times) \arrow[d, "\cong"', "\iota^*"] \\ H_{\mathrm{w}R}^*(M/\mathrm{w}D) \arrow[r, "\theta^*"'] & H_T^*(M) \end{tikzcd} \tag{4.2.3}$$

Thus $\mathrm{w}h^*$ is an isomorphism if θ^* is an isomorphism, which we proved in Proposition 3.4.4. $\square$

[1] Here we identify $\mathrm{Lie}(\mathrm{w}D) \cong \mathrm{Lie}(S^1) = \mathbb{R}$ by the map $S^1 \to \mathrm{w}D(t \mapsto (t^{dw_1+a}, \cdots, t^{dw_n+a}))$.

Definition 4.2.2. Let $\lambda \in \{{n \atop d}\}$. Since variety $\mathrm{a}\Omega_\lambda$ is a closed T-invariant irreducible subvariety in a non-singular quasi-projective T-variety $\mathrm{aPl}(d,n)^\times$, there is the T-equivariant fundamental class $[\mathrm{a}\Omega_\lambda]_T$ associated to $\mathrm{a}\Omega_\lambda$ in $H_T^*(\mathrm{aPl}(d,n)^\times)$ (see section 3.5):

$$\mathrm{a}\tilde{S}_\lambda := [\mathrm{a}\Omega_\lambda]_T \in H_T^{2l(\lambda)}(\mathrm{aPl}(d,n)^\times).$$

We define the wR-*equivariant weighted Schubert class* by

$$\mathrm{w}\tilde{S}_\lambda := (\mathrm{w}h^*)^{-1}(\mathrm{a}\tilde{S}_\lambda) \in H_{\mathrm{w}R}^{2l(\lambda)}(\mathrm{wGr}(d,n)).$$

The equivariant Schubert class $\tilde{S}_\lambda$ for the ordinary Grassmannian $\mathrm{Gr}(d,n)$ is defined as the R-equivariant class associated to the ordinary Schubert variety Ω_λ as usual. We will see that $\tilde{S}_\lambda$ maps to $\mathrm{a}\tilde{S}_\lambda$ by h^* in Section 4.3.6.

Remark 4.2.3. As a natural generalization of [15] and [40], $H_T^*(\mathrm{aPl}(d,n)^\times)$ can be identified with the wR-equivariant cohomology of the quotient stack $[\mathrm{aPl}(d,n)^\times/\mathrm{w}D_\mathbb{C}]$ and the isomorphism $\mathrm{w}h^*$ is nothing but the identification of the (equivariant) rational cohomology rings of the *weighted Grassmannian orbifold stack* $[\mathrm{aPl}(d,n)^\times/\mathrm{w}D_\mathbb{C}]$ and its coarse moduli space $\mathrm{wGr}(d,n)$. In these identifications, $\mathrm{a}\tilde{S}_\lambda$ and $\mathrm{w}\tilde{S}_\lambda$ should be regarded as the class associated to the wR-invariant substack $[\mathrm{a}\Omega_\lambda/\mathrm{w}D_\mathbb{C}]$. It should then coincide with the Poincaré dual of the cycle $[\mathrm{w}\Omega_\lambda]$ *up to the multiplicity of the substack* $[\mathrm{a}\Omega_\lambda/\mathrm{w}D_\mathbb{C}]$ in $[\mathrm{aPl}(d,n)^\times/\mathrm{w}D_\mathbb{C}]$. We leave this aspect of the theory to elsewhere.

4.3 GKM descriptions and Schubert classes

In this section, we study the combinatorial presentations of $H_{\mathrm{w}R}^*(\mathrm{wGr}(d,n))$ and $H_T^*(\mathrm{aPl}(d,n)^\times)$, now known as the *GKM theory* developed in [20]. This allows us, in particular, to show that the equivariant weighted Schubert classes $\mathrm{w}\tilde{S}_\lambda, \lambda \in \{{n \atop d}\}$ form an $H^*(B\mathrm{w}R)$-module basis of $H_{\mathrm{w}R}^*(\mathrm{wGr}(d,n))$.

Recall that $H^*(BT)$ can be canonically identified with the symmetric algebra $\mathrm{Sym}(\mathrm{Lie}(T)_\mathbb{Z}^* \otimes \mathbb{Q})$ where $\mathrm{Lie}(T)_\mathbb{Z}^*$ is the space of $\mathbb{Z}$-linear functions on the integral lattice $\mathrm{Lie}(T)_\mathbb{Z} \subset \mathrm{Lie}(T)$. Since $T = (S^1)^n$, we identify its Lie algebra $\mathrm{Lie}(T)$ with $\mathbb{R}^n$, so that we have the $\mathbb{Z}$-basis $\{y_1, \cdots, y_n\}$ of $\mathrm{Lie}(T)_\mathbb{Z}^*$ dual to the standard basis of $\mathrm{Lie}(T)_\mathbb{Z}$. Let

$$\mathbb{Q}[T^*] := H^*(BT) = \mathrm{Sym}(\mathrm{Lie}(T)_\mathbb{Z}^* \otimes \mathbb{Q}) = \mathbb{Q}[y_1, \ldots, y_n].$$

We adapt the same notation for all other tori except that T is the only standard torus such that the canonical generators y_i's of $H^*(BT)$ are given. The quotient map $T \to \mathrm{w}R$ induces the injection $\mathrm{Lie}(\mathrm{w}R)_\mathbb{Z}^* \hookrightarrow \mathrm{Lie}(T)_\mathbb{Z}^*$ and hence we will identify $\mathbb{Q}[\mathrm{w}R^*]$ with the image of the induced embedding $\mathbb{Q}[\mathrm{w}R^*] \hookrightarrow \mathbb{Q}[T^*]$. Similarly we will identify $\mathbb{Q}[R^*]$ with the image of $\mathbb{Q}[R^*] \hookrightarrow \mathbb{Q}[T^*]$.

We shall start with the GKM theory for R-action on $\mathrm{Gr}(d,n)$ studied in [35]. The R-fixed points in $\mathrm{Gr}(d,n)$ are the points $[e_\mu]$ corresponding to the vector

e_μ in $\mathrm{aPl}(d,n)^\times$. We identify $H_R^*([e_\mu])$ with $\mathbb{Q}[R^*]$. The restriction map to the fixed points

$$H_R^*(\mathrm{Gr}(d,n)) \to \bigoplus_{\mu\in\{{n \atop d}\}} \mathbb{Q}[R^*]; \qquad \gamma \mapsto (\gamma|_\mu)_{\mu\in\{{n \atop d}\}} \tag{4.3.1}$$

is injective and the image is given by

$$\left\{ \alpha = (\alpha(\mu))_\mu \in \bigoplus_{\mu\in\{{n \atop d}\}} \mathbb{Q}[R^*] \;\middle|\; \begin{array}{c} \alpha(\lambda)-\alpha(\mu) \text{ is divisible by } y_\lambda - y_\mu \\ \text{for any } \lambda \text{ and } \mu \text{ such that } |\lambda\cap\mu| = d-1 \end{array} \right\}. \tag{4.3.2}$$

where $y_\mu := \sum_{i\in\mu} y_i$ for all $\mu \in \{{n \atop d}\}$. Note that $y_\lambda - y_\mu \in \mathrm{Lie}(R)^*_{\mathbb{Z}}$ since the linear function $y_\lambda - y_\mu$ restricted to $\mathrm{Lie}(D)_{\mathbb{Z}}$ is 0.

Remark 4.3.1. In [35], the authors consider the action of $T' = (S^1)^n$ on $\mathrm{Gr}(d,n)$ though a map $T' \to R$. Their presentation is valid in our set-up since $H_R^*(\mathrm{Gr}(d,n))$ injects to $H_{T'}^*(\mathrm{Gr}(d,n))$ as rings over $\mathbb{Q}[R^*]$.

Now we turn to $\mathrm{wGr}(d,n)$ and then $\mathrm{aPl}(d,n)^\times$. The fixed points of the $\mathrm{w}R$-action on $\mathrm{wGr}(d,n)$ are also the points $[e_\mu] \in \mathrm{wGr}(d,n)$ corresponding to the vectors e_μ. We can naturally identify $H_{\mathrm{w}R}^*([e_\mu]) \cong \mathbb{Q}[\mathrm{w}R^*]$ and so we have the restriction map

$$H_{\mathrm{w}R}^*(\mathrm{wGr}(d,n)) \longrightarrow \bigoplus_{\mu\in\{{n \atop d}\}} \mathbb{Q}[\mathrm{w}R^*], \qquad \gamma \mapsto (\gamma|_\mu)_{\mu\in\{{n \atop d}\}} \tag{4.3.3}$$

For $\mathrm{aPl}(d,n)^\times$, we restrict $H_T(\mathrm{aPl}(d,n)^\times)$ to complex 1-dimensional orbits of $T_{\mathbb{C}}$ instead of restricting to the fixed points. The complex 1-dimensional orbits of $T_{\mathbb{C}}$ are given by $\mathbb{C}^\times e_\mu$. For each $\mu \in \{{n \atop d}\}$, let T_μ be the isotropy subgroup at e_μ for the T-action on $\mathrm{aPl}(d,n)^\times$. It is the kernel of the map $T \to S^1$ sending $(t_1,\cdots,t_n)$ to $t_\mu := t_{\mu_1}\cdots t_{\mu_d}$ so that it is not hard to see that T_μ is connected. Thus, with the natural isomorphisms $H_T(\mathbb{C}^\times e_\mu) \cong H_{T_\mu}(e_\mu) \cong \mathbb{Q}[T_\mu^*]$, we obtain

$$H_T^*(\mathrm{aPl}(d,n)^\times) \longrightarrow \bigoplus_{\mu\in\{{n \atop d}\}} \mathbb{Q}[T_\mu^*], \qquad P \mapsto (P|_\mu)_{\mu\in\{{n \atop d}\}} \tag{4.3.4}$$

Putting (4.3.1), (4.3.3) and (4.3.4) together with h^* and $\mathrm{w}h^*$, we have the commutative diagram

$$\begin{tikzcd} H_R^*(\mathrm{Gr}(d,n)) \arrow[r] \arrow[d, "h^*"', "\cong"] & \bigoplus_\mu \mathbb{Q}[R^*] \arrow[d, "\kappa^*"', "\cong"] \\ H_T^*(\mathrm{aPl}(d,n)^\times) \arrow[r] & \bigoplus_\mu \mathbb{Q}[T_\mu^*] \\ H_{\mathrm{w}R}^*(\mathrm{wGr}(d,n)) \arrow[r] \arrow[u, "\mathrm{w}h^*", "\cong"'] & \bigoplus_\mu \mathbb{Q}[\mathrm{w}R^*] \arrow[u, "\mathrm{w}\kappa^*", "\cong"'] \end{tikzcd} \tag{4.3.5}$$

where the right vertical maps are induced from $\kappa_\mu : T_\mu \to T \to R$ and $\mathrm{w}\kappa_\mu : T_\mu \to T \to \mathrm{w}R$ and they are isomorphisms because κ_μ and $\mathrm{w}\kappa_\mu$ have finite (or trivial) kernels. Also note that the inclusion $T_\mu \to T$ induces the surjection $\mathrm{Lie}(T)^*_{\mathbb{Z}} \to \mathrm{Lie}(T_\mu)^*_{\mathbb{Z}}$ and $\mathrm{Lie}(T_\mu)^*_{\mathbb{Z}} \cong \mathrm{Lie}(T)^*_{\mathbb{Z}}/(y_\mu)$. Thus we identify

$$\mathbb{Q}[T_\mu^*] \cong \mathbb{Q}[T^*]/(y_\mu).$$

The following are obtained by translating (4.3.2) to $H_T^*(\mathrm{aPl}(d,n)^\times)$ and $H_{\mathrm{w}R}^*(\mathrm{wGr}(d,n))$ via the diagram (4.3.5).

Proposition 4.3.2 (GKM for $\mathrm{wGr}(d,n)$)**.** *The restriction map (4.3.3) is injective and the image is given by*

$$\left\{ \alpha \in \bigoplus_{\mu \in \binom{n}{d}} \mathbb{Q}[\mathrm{w}R^*] \;\middle|\; \begin{array}{l} \alpha(\lambda) - \alpha(\mu) \text{ is divisible by } w_\mu y_\lambda - w_\lambda y_\mu \\ \text{for any } \lambda \text{ and } \mu \text{ such that } |\lambda \cap \mu| = d-1 \end{array} \right\}.$$

Here note that $w_\mu y_\lambda - w_\lambda y_\mu \in \mathbb{Q}[\mathrm{w}R^*]$.

Proposition 4.3.3 (GKM for $\mathrm{aPl}(d,n)^\times$)**.** *The restriction map (4.3.4) is injective and the image is given by*

$$\left\{ P \in \bigoplus_{\mu \in \binom{n}{d}} \mathbb{Q}[T_\mu^*] \;\middle|\; \begin{array}{c} P(\lambda) = P(\mu) \quad \mathit{in} \;\; \mathbb{Q}[T^*]/(y_\lambda, y_\mu) \\ \text{for any } \lambda \text{ and } \mu \text{ such that } |\lambda \cap \mu| = d-1 \end{array} \right\}.$$

Proof of Proposition 4.3.3 and Proposition 4.3.2
The injectivity of the maps (4.3.3) and (4.3.4) follows from the injectivity of the map (4.3.1) by the commutativity of the diagram (4.3.5). What is left is to check that the GKM conditions are equivalent under the isomorphisms κ^* and $\mathrm{w}\kappa^*$. We prove it for $\mathrm{w}\kappa$ because κ is a special case of $\mathrm{w}\kappa$. First note that, in Proposition 4.3.2, $\alpha(\lambda) - \alpha(\mu)$ is divisible by $w_\mu y_\lambda - w_\lambda y_\mu$ if and only if $\alpha(\lambda) - \alpha(\mu) = 0$ in $\mathbb{Q}[\mathrm{w}R^*]/(w_\mu y_\lambda - w_\lambda y_\mu)$. Therefore the GKM conditions are equivalent under $\mathrm{w}\kappa^*$ if $\mathrm{w}\kappa_\lambda^*$ and $\mathrm{w}\kappa_\mu^*$ induce the isomorphism

$$\frac{\mathbb{Q}[\mathrm{w}R^*]}{(w_\mu y_\lambda - w_\lambda y_\mu)} \to \frac{\mathbb{Q}[T^*]}{(y_\lambda, y_\mu)}, \quad f \mapsto \mathrm{w}\kappa_\lambda^*(f) = \mathrm{w}\kappa_\mu^*(f).$$

This map is obviously well-defined. It is also easy to check that this is an isomorphism. □

Remark 4.3.4. Proposition 4.3.2 can be proved directly as a consequence of [20, Theorem 7.2] by studying the data of 0 and 1-dimensional $\mathrm{w}R$-orbits and Lie algebras of isotropic subgroups of $\mathrm{w}R$-action on $\mathrm{wGr}(d,n)$. For this alternative proof, see section 4.3.1. Proposition 4.3.3 can be also shown directly from Theorem 5.5 in [26] by using the description of $\mathrm{wGr}(d,n)$ as the symplectic quotient of $\mathrm{aPl}(d,n)^\times$ by the real torus $\mathrm{w}D$ explained in Section 4.2.

In the rest of the section, we compute the certain values of the Schubert classes $\mathrm{a}\tilde{S}_\lambda$ and $\mathrm{w}\tilde{S}_\lambda$ under the restriction maps. As a corollary, we show that the Schubert classes $\tilde{S}_\lambda$ in $H^*_R(\mathrm{Gr}(d,n))$ correspond to the classes $\mathrm{a}\tilde{S}_\lambda$ in $H^*_T(\mathrm{aPl}(d,n)^\times)$ under h^* as expected. Also we show that the weighted Schubert classes $\mathrm{w}\tilde{S}_\lambda$ will form an $H^*(B\mathrm{w}R)$-module basis of $H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n))$.

Proposition 4.3.5.

$$\mathrm{a}\tilde{S}_\lambda|_\mu = \begin{cases} 0 & \text{if } \mu \not\geq \lambda, \\ \prod_{(k,l)\in\mathrm{inv}(\lambda)} y_{(k,l)\lambda} & \text{if } \mu = \lambda. \end{cases} \qquad \text{in } \mathbb{Q}[T^*]/(y_\mu)$$

Proof. Let $Y := \mathrm{aPl}(d,n)^\times$ for brevity. By the construction (c.f. [18, Appedix B.3]), the class $\mathrm{a}\tilde{S}_\lambda$ maps to the equivariant Euler class $\chi_T(N^\circ)$ by the pullback along the inclusion $\mathrm{a}\Omega^\circ_\lambda \hookrightarrow Y$ where N° is the normal bundle of $\mathrm{a}\Omega^\circ_\lambda$ in Y. This maps further to $\chi_{T_\lambda}(N^\circ_{e_\lambda})$ via the pullback map $H^*_T(\mathrm{a}\Omega^\circ_\lambda) \to H^*_{T_\mu}(e_\mu)$ where $N^\circ_{e_\mu}$ is the fiber of N° at e_μ. Since the normal bundle N° is T-equivariantly identified with $\mathrm{a}U^\lambda$, the equivariant chart ψ_λ given at (4.1.5) allows us to find the weight of the representation $T_\lambda \curvearrowright N_{e_\lambda}$ to be

$$\prod_{(k,l)\in\mathrm{inv}(\lambda)} y_{(k,l)\lambda} \quad \text{as an element of } \mathrm{Lie}(T)^*_{\mathbb{Z}}/(y_\lambda) = \mathrm{Lie}(T_\lambda)^*_{\mathbb{Z}}. \tag{4.3.6}$$

This proves the case when $\mu = \lambda$. The case $\mu \not\geq \lambda$ follows from $e_\mu \notin \mathrm{a}\Omega_\lambda$ (Proposition 4.1.7). $\square$

It is a well-known fact that $\tilde{S}_\lambda|_\lambda = \prod_{(k,l)\in inv(\lambda)}(y_{(k,l)\lambda} - y_\lambda)$ and $\tilde{S}_\lambda|_\lambda = 0$ for all $\mu \not\geq \lambda$ (c.f. [35]). Also a class having such values at fixed points is unique [35, LEMMA 1]. Therefore Proposition 4.3.5, together with the diagram 4.3.5, has the following immediate corollary.

Corollary 4.3.6. *For each $\lambda \in \{{n \atop d}\}$, $h^*(\tilde{S}_\lambda) = \mathrm{a}\tilde{S}_\lambda$.*

The next proposition is also immediate from Proposition 4.3.5.

Proposition 4.3.7.

$$\mathrm{w}\tilde{S}_\lambda|_\mu = \begin{cases} 0 & \text{if } \mu \not\geq \lambda, \\ \prod_{(k,l)\in\mathrm{inv}(\lambda)} \left(y_{(k,l)\lambda} - \dfrac{w_{(k,l)\lambda}}{w_\lambda} y_\lambda\right) & \text{if } \mu = \lambda. \end{cases}$$

Proof. Since $\mathrm{w}\tilde{S}_\lambda := (\mathrm{w}h^*)^{-1}(\mathrm{a}\tilde{S}_\lambda)$ and $y_{(k,l)\lambda} - \frac{w_{(k,l)\lambda}}{w_\lambda} y_\lambda = y_{(k,l)\lambda}$ in $\mathbb{Q}[T^*]/\langle y_\lambda\rangle$, we only need to check that

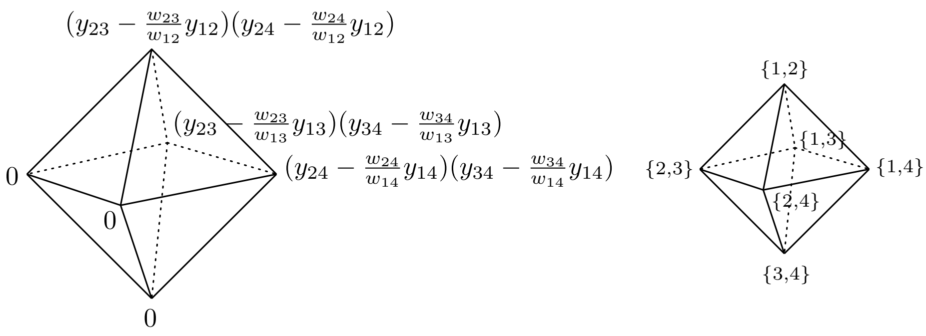
$$y_{(k,l)\lambda} - \frac{w_{(k,l)\lambda}}{w_\lambda} y_\lambda \in \mathrm{Lie}(\mathrm{w}R)^*_{\mathbb{Z}} \otimes \mathbb{Q}. \tag{4.3.7}$$

This can be checked by a straightforward calculation. $\square$

Having the upper-triangularity of the weighted Schubert classes as above, the proof of [35, Proposition 1] can be applied words by words to obtain

Proposition 4.3.8. *$\{\mathrm{w}\tilde{S}_\lambda\}_\lambda$ is an $H^*(B\mathrm{w}R)$-module basis of $H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n))$.*

Example 4.3.9. The followings is $\mathrm{w}\tilde{S}_{14}$ in $H^*_{\mathrm{w}R}(\mathrm{wGr}(2,4))$:



where the vertices are the elements of $\{{4 \atop 2}\}$ and there is an edge for each pair of λ and μ satisfying $|\lambda \cap \mu| = 1$.

4.3.1 An alternative proof of Proposition 4.3.2

We give an alternative proof of Proposition 4.3.2 as a direct consequence of [20, Theorem 7.2], by studying 0 and 1 dimensional orbits and Lie algebras of isotropy subgroups of wR.

We start with notations. For $\lambda \in \{{n \atop d}\}$ and $\gamma \in \mathrm{Lie}(T)$, we denote $\gamma_\lambda := \sum_{i\in\lambda} \gamma_i$. Since $\mathrm{Lie}(\mathrm{w}R) = \mathrm{Lie}(T)/\mathrm{Lie}(\mathrm{w}D)$, we write an element of Lie(wR) as $[\gamma]$ where $\gamma \in \mathrm{Lie}(\mathrm{w}R)$. Define

$$\mathcal{O}_{\lambda\mu} := \{[x] \in \mathrm{w}\mathbb{P}(\wedge^d \mathbb{C}^n) \mid x(\lambda) \neq 0, x(\mu) \neq 0, x(\eta) = 0\,(\eta \neq \lambda, \mu)\}.$$

Let $\mathbb{P}^1(w_\lambda, w_\mu)$ be the weighted projective line with weight w_λ and w_μ. Consider a continuous map $f : \mathbb{C}^2 \backslash \{0\} \to \mathrm{aPl}(d,n)^\times$ defined by

$$f(x,y)(\eta) = \begin{cases} x & (\text{if } \eta = \lambda) \\ y & (\text{if } \eta = \mu) \\ 0 & (\text{otherwise}). \end{cases}$$

Then the map f induces a continuous map $\overline{f} : \mathbb{P}^1(w_\lambda, w_\mu) \to \mathrm{wGr}(d,n)$ which is a homeomorphism onto $\overline{\mathcal{O}_{\lambda\mu}} = \mathcal{O}_{\lambda\mu} \cup \{[e_\lambda], [e_\mu]\}$.

For brevity, let $X := \mathrm{wGr}(d,n)$, and denote

$$X_0 := \{[x] \in \mathrm{wGr}(d,n) \mid \operatorname{corank} \mathrm{w}R_{[x]} = 0\},$$
$$X_1 := \{[x] \in \mathrm{wGr}(d,n) \mid \operatorname{corank} \mathrm{w}R_{[x]} \leq 1\}.$$

where $\mathrm{w}R_{[x]}$ is the isotropy subgroup of wR at $[x]$ and $\operatorname{corank} \mathrm{w}R_{[x]} := (n-1) - \operatorname{rank} \mathrm{w}R_{[x]}$. In other words, X_0 is the set of wR-fixed points, and X_1 is the set of 0 and 1 dimensional orbits of wR. The data in the next proposition will provide us the GKM description of $H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n))$ (Proposition 4.3.2).

Proposition 4.3.10. *For the* wR*-action on* wGr(d,n)*, the followings hold:*

(1) X_0 *consists of the points* e_λ *for all* $\lambda \in \{{n \atop d}\}$.
(2) X_1 *is the union of* $\overline{\mathcal{O}_{\lambda\mu}}$ *for all* λ *and* μ *such that* $|\lambda \cap \mu| = d-1$.
(3) *For any* $[x] \in \mathcal{O}_{\lambda\mu}$ *where* $|\lambda\cap\mu| = d-1$,
$$\mathrm{Lie}(\mathrm{w}R_{[x]}) = \{[\gamma] \in \mathrm{Lie}(\mathrm{w}R) \mid w_\mu\gamma_\lambda - w_\lambda\gamma_\mu = 0\}.$$

Proof. Let $[x] \in \mathrm{wGr}(d,n)$, and write $\{\lambda \in \{{n \atop d}\} \mid x_\lambda \neq 0\} = \{\lambda^{(1)}, \cdots, \lambda^{(p)}\}$. We show

$$\mathrm{Lie}(\mathrm{w}R_{[x]}) = \{[\gamma] \in \mathrm{Lie}(\mathrm{w}R) \mid (\check{w_{\lambda^{(1)}}} \cdots w_{\lambda^{(p)}})\gamma_{\lambda^{(1)}} = \cdots = (w_{\lambda^{(1)}} \cdots \check{w_{\lambda^{(p)}}})\gamma_{\lambda^{(p)}}\}. \tag{4.3.8}$$

By the definition of the isotropy subgroup, we have

$$\mathrm{w}R_{[x]} = \{[t] \in \mathrm{w}R \mid \text{ for some } \epsilon \in \mathbb{C}^\times, t_{\lambda^{(i)}} = \epsilon^{w_{\lambda^{(i)}}} (1 \leq i \leq p)\}. \tag{4.3.9}$$

Denoting

$$H := \{[t] \in \mathrm{w}R \mid (t_{\lambda^{(1)}})^{\check{w_{\lambda^{(1)}}} w_{\lambda^{(2)}} \cdots w_{\lambda^{(p)}}} = \cdots = (t_{\lambda^{(p)}})^{w_{\lambda^{(1)}} \cdots w_{\lambda_{p-1}} \check{w_{\lambda^{(p)}}}}\},$$

we have w$R_{[x]} \subset H$ which implies Lie(w$R_{[x]}$) $\subset$ Lie(H). Observe that Lie(H) is equal to the right hand side of (4.3.8). For any $\gamma \in \mathrm{Lie}(H)$, putting $E := t_{\lambda^{(1)}}/w_{\lambda^{(1)}}$, we have $t_{\lambda^{(i)}} = w_{\lambda^{(i)}}E$ $(1 \leq i \leq p)$. So, for any $s \in \mathbb{C}$, we have

$$(\exp(s\gamma))_{\lambda^{(i)}} = \exp(s\gamma_{\lambda^{(i)}}) = \exp(sw_{\lambda^{(i)}}E) = (\exp(sE))^{w_{\lambda^{(i)}}}$$

for all $1 \leq i \leq p$. This shows $\exp(s\gamma) \in \mathrm{w}R_{[x]}$ for any $s \in \mathbb{C}$ which implies $\gamma \in \mathrm{Lie}(\mathrm{w}R_{[x]})$. Hence, we obtain Lie(w$R_{[x]}$) = Lie H, i.e. the equality (4.3.8). Now we see

$$\mathrm{corank}\,\mathrm{w}R_{[x]} = 0 \text{ if and only if } |\{\lambda \in \{{n \atop d}\} \mid x_\lambda \neq 0\}| = 1,$$
$$\mathrm{corank}\,\mathrm{w}R_{[x]} = 1 \text{ if and only if } |\{\lambda \in \{{n \atop d}\} \mid x_\lambda \neq 0\}| = 2.$$

Hence, we obtain the claims (1) and (3). For (2), let $\lambda, \mu \in \{{n \atop d}\}$, and consider $\mathcal{O}_{\lambda\mu}$. Suppose that we have $|\lambda\cap\mu| = d-1$. Let us write $\lambda = \{\lambda_1, \cdots, \lambda_d\} \subset [n]$ and $\mu = \{\lambda_1, \cdots, \check{\lambda_s}, \cdots, \lambda_d, \alpha\} \subset [n]$ for some $\alpha \notin \lambda$. Then any $[x] \in \mathcal{O}_{\lambda\mu}$ can be written as

$$\begin{aligned}[x] &= [ae_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d} + be_{\lambda_1} \wedge \cdots \wedge e_{\check{\lambda_s}} \wedge \cdots \wedge e_{\lambda_d} \wedge e_\alpha] \\ &= [e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_{s-1}} \wedge (ae_{\lambda_s} + (-1)^{d-i}be_\alpha) \wedge e_{\lambda_{s+1}} \wedge \cdots \wedge e_{\lambda_d}]\end{aligned}$$

for some $a, b \in \mathbb{C}^\times$. Hence, we obtain $\mathcal{O}_{\lambda\mu} \subset \mathrm{wGr}(d,n)$ since aPl$(d,n)^\times$ = Im $\wedge^d -\{0\}$. On the other hand, suppose $|\lambda\cap\mu| < d-1$. For any element $[x] \in \mathcal{O}_{\lambda\eta} \cap \mathrm{wGr}(d,n)$, we have $x_\xi = 0$ for any $\xi \neq \lambda, \eta$ because of the definition of $\mathcal{O}_{\lambda\eta}$. The condition $|\lambda\cap\eta| < d-1$ means that all the coordinates of the orbifold chart $\overline{\psi}_\lambda$ of $[x]$ are zero. This means that $[x] = [e_\lambda]$ which contradicts to $[x] \in \mathcal{O}_{\lambda\eta}$, and we obtain $\mathcal{O}_{\lambda\eta} \cap \mathrm{wGr}(d,n) = \emptyset$. $\square$

There exists a natural isomorphism between the Čech cohomology theory and the singular cohomology theory for any closed pair of locally contractible, paracompact, Hausdorff spaces. Thus, the results in [10] applies for the singular wR-equivariant cohomology: the restriction map $H^*_{\mathrm{w}R}(X) \to H^*_{\mathrm{w}R}(X_0)$ is injective, and so is the connecting homomorphism $H^*_{\mathrm{w}R}(X, X_0) \to H^*_{\mathrm{w}R}(X_1, X_0)$ of the exact sequence for the triple (X, X_1, X_0) since $H^*_{\mathrm{w}R}(X)$ is a free module over $H^*(B\mathrm{w}R)$ (Proposition 4.1.15). Combining the exact sequences for the pair (X, X_0) and the one for triple (X, X_1, X_0), we obtain the following exact sequence.

Proposition 4.3.11. *The following sequence is exact:*

$$0 \to H^*_{\mathrm{w}R}(X) \to H^*_{\mathrm{w}R}(X_0) \to H^{*+1}_{\mathrm{w}R}(X_1, X_0)$$

where the middle map is the restriction, and the right map is the connecting homomorphism of the exact sequence for the pair (X_1, X_0).

Now, observing that $\mathbb{P}^1(w_\lambda, w_\mu) \cong \mathbb{CP}^1$ as algebraic varieties, the argument in the proof of Theorem 7.2 in [20] directly applies for the w$R_{\mathbb{C}}$-action on $\mathrm{wGr}(d, n)$, and we obtain Proposition 4.3.2.

4.4 Structure constants and positivity

Since $\{\mathrm{w}\tilde{S}_\lambda\}_\lambda$ is an $H^*(B\mathrm{w}R)$-module basis of $H^*_{\mathrm{w}R}(\mathrm{wGr}(d, n))$, we can expand their pairwise cup product uniquely over $H^*(B\mathrm{w}R)$:

$$\mathrm{w}\tilde{S}_\lambda \mathrm{w}\tilde{S}_\mu = \sum_\nu \mathrm{w}\tilde{c}^\nu_{\lambda\mu} \mathrm{w}\tilde{S}_\nu \quad \text{where } \mathrm{w}\tilde{c}^\nu_{\lambda\mu} \in H^*(B\mathrm{w}R). \tag{4.4.1}$$

Knutson-Tao [35] gave an explicit combinatorial formula for the equivariant structure constants $\tilde{c}^\nu_{\lambda\mu}$ of the R-equivariant cohomology of $\mathrm{Gr}(d, n)$ in terms of the equivariant puzzles. Their formula of $\tilde{c}^\nu_{\lambda\mu}$ is *manifestly positive* in a sense that $\tilde{c}^\nu_{\lambda\mu}$ is a polynomial in u_i's with non-negative coefficients where $\{u_i := y_{i+1} - y_i\}$ is a basis of $\mathrm{Lie}(R)^*_{\mathbb{Z}}$. In this section, we derive the formula for $\mathrm{w}\tilde{c}^\nu_{\lambda\mu}$ from their formula by passing it through $H_T(\mathrm{aPl}(d, n)^\times)$ via h^* and $\mathrm{w}h^*$. Also we find a $\mathbb{Q}$-basis $\{\mathrm{w}u_i\}$ of $\mathrm{Lie}(\mathrm{w}R)^*_{\mathbb{Z}} \otimes \mathbb{Q}$ such that our formula of $\mathrm{w}\tilde{c}^\nu_{\lambda\mu}$ is manifestly positive with respect to the basis $\{\mathrm{w}u_i\}$ when $w_1 \leq \cdots \leq w_n$. The manifestly positive formula for the structure constants $\{\mathrm{w}c^\nu_{\lambda\mu}\}$ of the ordinary cohomology $H^*(\mathrm{wGr}(d, n))$ is also obtained by specializing the one for $\mathrm{w}\tilde{c}^\nu_{\lambda\mu}$ at $\mathrm{w}u_1 = \cdots = \mathrm{w}u_{n-1} = 0$.

We start with introducing new terminologies which extend the ones provided in [35, p.227]. For every puzzle, we choose a total order on the set of equivariant pieces once and for all. Let P be a puzzle satisfying $\partial P = \Delta^\nu_{\lambda\mu}$. Let p be an equivariant piece in P, whose weight is $\mathrm{wt}(p) = y_j - y_i$ where $j > i$, i.e. p pokes out the i-th and j-th place on the south side of P. For each $\xi \in \binom{n}{d}$, we define the *w-weight of p with respect to ξ* by

$$\mathrm{wt}^\xi(p) := (y_j - y_i) - \frac{w(p)}{w_\xi} y_\xi \in \mathbb{Q}[\mathrm{w}R^*] \quad \text{where } w(p) := w_j - w_i. \tag{4.4.2}$$

Let $(p_1, \cdots, p_r)$ be the ordered set of equivariant pieces in P. For each covering sequence $\xi^k \to \cdots \to \xi^0$ in $\{^n_d\}$ with $k \leq r$, we define the *w-weight of* P to be an element of $\mathbb{Q}[\mathrm{w}R^*]$

$$\mathrm{wt}^{\xi^0,\ldots,\xi^k}(P) := \sum_{1\leq i_1<\cdots<i_k\leq r} \frac{w(p_{i_1})}{w_{\xi^0}} \cdots \frac{w(p_{i_k})}{w_{\xi^{k-1}}} \frac{\prod_{i=1}^r \mathrm{wt}^{\xi^{s(i)}}(p_i)}{\mathrm{wt}^{\xi^1}(p_{i_1})\cdots \mathrm{wt}^{\xi^k}(p_{i_k})}. \quad (4.4.3)$$

where s is a function on $\{1, \cdots, r\}$ defined by

$$s(i) := \begin{cases} 0 & \text{if } i < i_1 \\ l & \text{if } i_l \leq i < i_{l+1}, l = 1, \cdots, k-1 \\ k & \text{if } i_k \leq i \end{cases}$$

As a special case when $k = 0$, we have

$$\mathrm{wt}^{\xi}(P) = \mathrm{wt}^{\xi}(p_1)\cdots \mathrm{wt}^{\xi}(p_r). \quad (4.4.4)$$

Remark that this expression (4.4.3) depends on the order of the equivariant pieces in P in general.

Lemma 4.4.1. *Let* id *be the unique minimum in* $\{^n_d\}$ *with respect to the Bruhat order and* div *the unique element with* $l(\mathrm{id}) = 1$*. Let* $\nu \in \{^n_d\}$*. Then*

$$\mathrm{a}\tilde{S}_{\mathrm{div}} = y_{\mathrm{id}} \ ; \quad (4.4.5)$$

$$0 = -y_\nu \mathrm{a}\tilde{S}_\nu + \sum_{\nu'\to\nu} \mathrm{a}\tilde{S}_\nu \ . \quad (4.4.6)$$

Proof. Since $\tilde{S}_{\mathrm{div}}|_\mu = y_{\mathrm{id}} - y_\mu$ ([35, Lemma 3]) for each $\mu \in \{^n_d\}$, we have

$$\mathrm{a}\tilde{S}_{\mathrm{div}}|_\mu = y_{\mathrm{id}} - y_\mu = y_{\mathrm{id}}, \qquad \text{in } \mathbb{Q}[T^*]/(y_\mu).$$

Therefore (4.4.5) holds: $\mathrm{a}\tilde{S}_{\mathrm{div}} = y_{\mathrm{id}}\mathrm{a}\tilde{S}_{\mathrm{id}} = y_{\mathrm{id}} \cdot 1$. On the other hand, the equivariant Pieri-rule given in [35, Proposition 2] holds also in $H_T^*(\mathrm{aPl}(d, n)^\times)$ by the isomorphism h^*, and hence we have

$$\mathrm{a}\tilde{S}_{\mathrm{div}}\mathrm{a}\tilde{S}_\nu = (y_{\mathrm{id}} - y_\nu)\mathrm{a}\tilde{S}_\nu + \sum_{\nu'\to\nu} \mathrm{a}\tilde{S}_\nu \quad (4.4.7)$$

which, together with (4.4.5), implies (4.4.6). □

The following is the essential equation, immediate from (4.4.6), to relate the $\mathbb{Q}[R^*]$-action to the $\mathbb{Q}[\mathrm{w}R^*]$-action in $H_T^*(\mathrm{aPl}(d, n)^\times)$.

Proposition 4.4.2. *Let* p *be an equivariant piece of a puzzle. Then*

$$\mathrm{wt}(p)\mathrm{a}\tilde{S}_\nu = \mathrm{wt}^\nu(p)\mathrm{a}\tilde{S}_\nu + \sum_{\nu'\to\nu} \frac{w(p)}{w_\nu}\mathrm{a}\tilde{S}_{\nu'}, \qquad \textit{in } H_T^*(\mathrm{aPl}(d, n)^\times).$$

Proof. Let $\mathrm{wt}(p) = y_j - y_i$ be the weight of p. The formula is obtained by multiplying $(w_j - w_i)/w_\nu$ and then adding $(y_j - y_i)\mathrm{a}\tilde{S}_\nu$ to the equation (4.4.6). □

Now we are ready to prove the main theorem of this section.

Theorem 4.4.3. *For each* $\lambda, \mu, \nu \in \{{n \atop d}\}$*, the equivariant structure constant* $\mathrm{w}\tilde{c}^{\nu}_{\lambda\mu}$ *is given by*

$$\mathrm{w}\tilde{c}^{\nu}_{\lambda\mu} = \left(\sum_{\substack{\text{puzzle } P \\ \partial P = \Delta^{\nu}_{\lambda\mu}}} \mathrm{wt}^{\nu}(P) \right) + \sum_{\substack{\nu \to \nu^1 \to \\ \cdots \to \nu^k \geq \lambda,\mu}} \sum_{\substack{\text{puzzle } Q \\ \partial Q = \Delta^{\nu^k}_{\lambda\mu}}} \mathrm{wt}^{\nu^k, \cdots, \nu^1, \nu}(Q). \tag{4.4.8}$$

Remark that the w-weights in this expression depend on the orders of the equivariant pieces in P and Q, but $\mathrm{w}\tilde{c}^{\nu}_{\lambda\mu}$ doesn't depend on the order.
Proof. By the isomorphism h^*, Theorem 2 in [35] translates to

$$\mathrm{a}\tilde{S}_\lambda \mathrm{a}\tilde{S}_\mu = \sum_{\eta \geq \lambda,\mu} \overbrace{\sum_{\substack{\text{puzzle } P \\ \partial P = \Delta^{\eta}_{\lambda\mu}}} \mathrm{wt}(p_r) \cdots \mathrm{wt}(p_1)}^{\tilde{c}^{\eta}_{\lambda\mu}} \mathrm{a}\tilde{S}_\eta \qquad \text{in } H^*_T(\mathrm{aPl}(d,n)^\times),$$

where $(p_1, \cdots, p_r)$ denotes an ordered set of all equivariant pieces in P. Remark that the number of equivariant pieces in P must be $l(\lambda) + l(\mu) - l(\eta)$. For each $l \leq r$, by applying Proposition 4.4.2 repeatedly, we obtain

$$\begin{aligned}
\mathrm{wt}(p_l) \cdots \mathrm{wt}(p_1)\mathrm{a}\tilde{S}_\eta = \mathrm{wt}^\eta(p_1) \cdots \mathrm{wt}^\eta(p_l)\mathrm{a}\tilde{S}_\eta + \sum_{k=1}^{l} \sum_{\substack{\eta^k \to \cdots \\ \to \eta^1 \to \eta}} \Bigg(\sum_{1 \leq i_1 < \cdots < i_k \leq l} \\
\mathrm{wt}^\eta(p_1) \cdots \mathrm{wt}^\eta(p_{i_1 - 1}) \cdot \frac{w(p_{i_1})}{w_\eta} \cdot \mathrm{wt}^{\eta^1}(p_{i_1+1}) \cdots \mathrm{wt}^{\eta^1}(p_{i_2 - 1}) \cdot \frac{w(p_{i_2})}{w_{\eta^1}} \cdot \mathrm{wt}^{\eta^2}(p_{i_2+1}) \\
\cdots \mathrm{wt}^{\eta^{k-1}}(p_{i_k - 1}) \cdot \frac{w(p_{i_k})}{w_{\eta^{k-1}}} \cdot \mathrm{wt}^{\eta^k}(p_{i_k+1}) \cdots \mathrm{wt}^{\eta^k}(p_l) \Bigg) \mathrm{a}\tilde{S}_{\eta^k}.
\end{aligned}$$

It is straightforward to prove this formula by an induction on $l \leq r$. When $l = r$,

$$\mathrm{wt}(p_r) \cdots \mathrm{wt}(p_1)\mathrm{a}\tilde{S}_\eta = \mathrm{wt}^\eta(P)\mathrm{a}\tilde{S}_\eta + \sum_{k=1}^{l(\lambda)+l(\mu)-l(\eta)} \sum_{\substack{\eta^k \to \cdots \\ \to \eta^1 \to \eta}} \mathrm{wt}^{\eta, \eta^1, \cdots, \eta^k}(P) \cdot \mathrm{a}\tilde{S}_{\eta^k}.$$

Therefore

$$
\begin{aligned}
\mathrm{a}\tilde{S}_\lambda \mathrm{a}\tilde{S}_\mu &= \sum_{\eta\geq\lambda,\mu}\sum_{\substack{\text{puzzle } P\\ \partial P=\Delta^\eta_{\lambda\mu}}}\left(\mathrm{wt}^\eta(P)\cdot\mathrm{a}\tilde{S}_\eta + \sum_{k=1}^{l(\lambda)+l(\mu)-l(\eta)}\sum_{\substack{\eta^k\to\cdots\\ \to\eta^1\to\eta}}\mathrm{wt}^{\eta,\eta^1,\cdots,\eta^k}(P)\cdot\mathrm{a}\tilde{S}_{\eta^k}\right)\\
&= \sum_{\eta\geq\lambda,\mu}\sum_{\substack{\text{puzzle } P\\ \partial P=\Delta^\eta_{\lambda\mu}}}\mathrm{wt}^\eta(P)\cdot\mathrm{a}\tilde{S}_\eta + \sum_{\substack{\nu\to\eta^{k-1}\to\cdots\\ \to\eta^1\to\eta\geq\lambda,\mu}}\sum_{\substack{\text{puzzle } P\\ \partial P=\Delta^\eta_{\lambda\mu}}}\mathrm{wt}^{\eta,\eta^1,\cdots,\eta^{k-1},\nu}(P)\cdot\mathrm{a}\tilde{S}_\nu\\
&= \sum_{\nu\geq\lambda,\mu}\left(\left(\sum_{\substack{\text{puzzle } P\\ \partial P=\Delta^\nu_{\lambda\mu}}}\mathrm{wt}^\nu(P)\right)+\sum_{\substack{\nu\to\nu^1\to\\ \cdots\to\nu^k\geq\lambda,\mu}}\sum_{\substack{\text{puzzle } Q\\ \partial Q=\Delta^{\nu^k}_{\lambda\mu}}}\mathrm{wt}^{\nu^k,\cdots,\nu^1,\nu}(Q)\right)\cdot\mathrm{a}\tilde{S}_\nu.
\end{aligned}
$$

Since $\mathrm{wt}^\nu(P)$ and $\mathrm{wt}^{\nu^k,\cdots,\nu^1,\nu}(Q)$ are in $\mathbb{Q}[\mathrm{w}R^*]$, $\mathrm{w}h^*$ maps this equation to the desired equation in $H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n))$. $\square$

Let

$$\mathrm{w}u_i := (y_{i+1}-y_i) - \frac{w_{i+1}-w_i}{w_{\mathrm{id}}}y_{\mathrm{id}} \in \mathbb{Q}[\mathrm{w}R^*]. \tag{4.4.9}$$

We can easily check that $\{\mathrm{w}u_1,\cdots,\mathrm{w}u_{n-1}\}$ is a $\mathbb{Q}$-basis of $\mathrm{Lie}(\mathrm{w}R)^*_{\mathbb{Z}}\otimes\mathbb{Q}$. Then the next positivity theorem is a direct consequence of Theorem 4.4.3 and Proposition 4.4.5 which is proved right after.

Theorem 4.4.4. *If $w_1\leq w_2\leq\cdots\leq w_n$, then $\mathrm{w}\tilde{c}^\nu_{\lambda\mu}$ is a polynomial in $\mathrm{w}u_1,\cdots,\mathrm{w}u_{n-1}$ with non-negative coefficients.*

Proposition 4.4.5. *Let P be a puzzle whose south string is ν. Suppose that P involves an equivariant piece p. If $w_1\leq w_2\leq\cdots\leq w_n$, then $\mathrm{wt}^\nu(p)$ is a linear combination of $\mathrm{w}u_1,\cdots,\mathrm{w}u_{n-1}$ with non-negative coefficients.*

Proof. We prove, by induction on the length $l(\nu)$, that the linear polynomial $(y_j-y_i)-\frac{w_j-w_i}{w_\nu}y_\nu$ is a linear combination of $\mathrm{w}u_1,\cdots,\mathrm{w}u_{n-1}$ with non-negative coefficients for each $1\leq i<j\leq n$ and each $\nu\in\{{n \atop d}\}$.

If $l(\nu)=0$, the statement is obvious since $\nu=\mathrm{id}$. We assume that the claim holds for all ν' with $l(\nu')\leq m-1$ for some integer m. Let $l(\nu)=m$. Observe

$$
\begin{aligned}
&(y_j-y_i)-(w_j-w_i)\frac{y_\nu}{w_\nu}\\
&\quad=\sum_{k=i}^{j-1}\left[\left((y_{k+1}-y_k)-(w_{k+1}-w_k)\frac{y_{\mathrm{id}}}{w_{\mathrm{id}}}\right)+(w_{k+1}-w_k)\left(\frac{y_{\mathrm{id}}}{w_{\mathrm{id}}}-\frac{y_\nu}{w_\nu}\right)\right].
\end{aligned}
$$

We show that $\frac{y_{\mathrm{id}}}{w_{\mathrm{id}}}-\frac{y_\nu}{w_\nu}$ is written non-negatively. Let $\nu^1,\cdots,\nu^m\in\{{n \atop d}\}$ such that $\nu=\nu^m\to\cdots\to\nu^1\to\mathrm{id}$, and write

$$\frac{y_{\mathrm{id}}}{w_{\mathrm{id}}}-\frac{y_\nu}{w_\nu}=\sum_{s=0}^{m-1}\left(\frac{y_{\nu^s}}{w_{\nu^s}}-\frac{y_{\nu^{s+1}}}{w_{\nu^{s+1}}}\right)$$

where $\nu_0 := \mathrm{id}$. Since $\nu^{s+1} \to \nu^s$, there exists an integer $1 \leq a \leq n$ such that $\nu^{s+1} = (a, a+1)\nu^s$, and therefore

$$\frac{y_{\nu^s}}{w_{\nu^s}} - \frac{y_{\nu^{s+1}}}{w_{\nu^{s+1}}} = \frac{1}{w_{\nu^{s+1}}}\left((y_{a+1} - y_a) - (w_{a+1} - w_a)\frac{y_{\nu^s}}{w_{\nu^s}}\right).$$

By the induction hypothesis, the RHS is a linear combination of $\mathrm{w}u_i$'s with non-negative coefficients, and so is $\frac{y_{\mathrm{id}}}{w_{\mathrm{id}}} - \frac{y_\nu}{w_\nu}$ and $(y_j - y_i) - (w_j - w_i)\frac{y_\nu}{w_\nu}$. $\square$

As a corollary of Theorem 4.4.3, we give an explicit formula of the structure constants in $H^*(\mathrm{wGr}(d,n))$. For each $\lambda \in \{{n \atop d}\}$, define

$$\mathrm{w}S_\lambda := \zeta^*(\mathrm{w}\tilde{S}_\lambda) \in H^*(\mathrm{wGr}(d,n))$$

where $\zeta^* : H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n)) \to H^*(\mathrm{wGr}(d,n))$ is the surjection mentioned in Section 4.1.4. Under the natural isomorphism $H^*(\mathrm{wGr}(d,n)) \cong H^*_{\mathrm{w}D}(\mathrm{aPl}(d,n)^\times)$ that also follows from Proposition 3.4.4, this $\mathrm{w}S_\lambda$ corresponds to the $\mathrm{w}D$-equivariant cohomology class associated to $\mathrm{a}\Omega_\lambda$. By Proposition 4.1.15, those classes form a $\mathbb{Q}$-basis of $H^*(\mathrm{wGr}(d,n))$. The structure constants $\mathrm{w}c^\nu_{\lambda\mu}$ of $H^*(\mathrm{wGr}(d,n))$ are defined with respect to this basis $\{\mathrm{w}S_\lambda\}_\lambda$. Since ζ^* is the ring homomorphism given by

$$H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n)) \to H^*_{\mathrm{w}R}(\mathrm{wGr}(d,n)) \otimes_{\mathbb{Q}[\mathrm{w}R^*]} \mathbb{Q} \cong H^*(\mathrm{wGr}(d,n)),$$

these non-equivariant structure constants are obtained by evaluating $\mathrm{w}\tilde{c}^\nu_{\lambda\mu}$ at $\mathrm{w}u_1 = \cdots = \mathrm{w}u_{n-1} = 0$, i.e.

$$\mathrm{w}c^\nu_{\lambda\mu} = \mathrm{w}\tilde{c}^\nu_{\lambda\mu}(\mathrm{w}u_1 = \cdots = \mathrm{w}u_{n-1} = 0).$$

In particular, the structure constants $c^\nu_{\lambda\mu}$ of $H^*(\mathrm{Gr}(d,n))$ with respect to the ordinary Schubert classes S_λ, that are computed in [36, Theorem 1] also in terms of puzzles, can be obtained from $\tilde{c}^\nu_{\lambda\mu}$ evaluating at $u_1 = \cdots = u_{n-1} = 0$. Here we recall that $\tilde{c}^\nu_{\lambda\mu}$ is a polynomial in u_i's where $\{u_i = y_{i+1} - y_i, i = 1, \cdots, n-1\}$ is a basis of $\mathrm{Lie}(R)^*_{\mathbb{Q}}$.

Corollary 4.4.6. *Let $\lambda, \mu, \nu \in \{{n \atop d}\}$. The structure constant $\mathrm{w}c^\nu_{\lambda\mu}$ is given by*

$$\mathrm{w}c^\nu_{\lambda\mu} = c^\nu_{\lambda\mu} + \sum_{\substack{\nu \to \nu^1 \to \\ \cdots \to \nu^k \geq \lambda, \mu}} \frac{\tilde{c}^{\nu^k}_{\lambda\mu}(u_i = w_{i+1} - w_i, i = 1, \cdots, n-1)}{w_{\nu^1} \cdots w_{\nu^k}},$$

if $l(\lambda) + l(\mu) = l(\nu)$ and is 0 otherwise. If $w_1 \leq w_2 \leq \cdots \leq w_n$, $\mathrm{w}c^\nu_{\lambda\mu}$ is non-negative for all $\lambda, \mu, \nu \in \{{n \atop d}\}$.

Proof. After the evaluation, the first summation of 4.4.8 vanishes unless $l(\lambda) + l(\mu) - l(\nu) = 0$ since only the puzzles P without equivariant pieces can survive. Therefore, if $l(\lambda) + l(\mu) = l(\nu)$, by [36, Theorem 1] the first sum in the RHS of (4.4.8) becomes

$$\sum_{\substack{\text{puzzle } P \\ \partial P = \Delta^\nu_{\lambda\mu} \\ \text{no equivariant pieces}}} 1 = c^\nu_{\lambda\mu}.$$

In the second sum, only the puzzles Q with exactly k equivariant pieces survive after the evaluation, and so the summation vanishes unless $l(\lambda)+l(\mu)-l(\nu)=0$. Therefore the second term becomes, if $l(\lambda)+l(\mu)=l(\nu)$,

$$\begin{aligned}
&\sum_{\substack{\nu\to\nu^1\to\\ \cdots\to\nu^k\geq\lambda,\mu}}\ \sum_{\substack{\text{puzzle }Q\\ \partial Q=\Delta_{\lambda\mu}^{\nu^k}}} \mathrm{wt}^{\nu^k,\cdots,\nu^1,\nu}(Q)\Big|_{\mathrm{w}u_1=\cdots=\mathrm{w}u_{n-1}=0}\\
=&\sum_{\substack{\nu\to\nu^1\to\\ \cdots\to\nu^k\geq\lambda,\mu}}\ \sum_{\substack{\text{puzzle }Q\\ \partial Q=\Delta_{\lambda\mu}^{\nu^k}}} \frac{w(p_1)}{w_{\nu^1}}\cdots\frac{w(p_k)}{w_{\nu^k}}\\
=&\sum_{\substack{\nu\to\nu^1\to\\ \cdots\to\nu^k\geq\lambda,\mu}} \frac{\tilde{c}_{\lambda\mu}^{\nu^k}(u_i=w_{i+1}-w_i, i=1,\cdots,n-1)}{w_{\nu^1}\cdots w_{\nu^k}}.
\end{aligned}$$

Combining these terms, we obtain the desired formula. The positivity is a direct consequence of the equivaraiant positivity (Theorem 4.4.4). □

Remark 4.4.7. We can say that our positivity theorem holds for all weighted Grassmannians in a sense as follows: for a given $\mathrm{wGr}(d,n)$ with the weight $w=(w_1,\cdots,w_n)$, we can always perform a permutation on the basis $\{e_1,\cdots,e_n\}$ of $\mathbb{C}^n$ so that the new order on the weight is non-decreasing. Then we can redefine the Schubert classes $\{\mathrm{w}\tilde{S}_\lambda\}_\lambda$ to make sure that the structure constants are positive.

We conclude this section by listing the *equivariant weighted Pieri rule* and working out a few examples. First, by Equation (4.4.5) interpreted through $\mathrm{w}h^*$ and $\mathrm{w}\kappa^*$, we obtain the restriction of $\mathrm{w}\tilde{S}_{\mathrm{div}}$ to the fixed points:

$$\mathrm{w}\tilde{S}_{\mathrm{div}}|_\lambda = y_{\mathrm{id}} - \frac{w_{\mathrm{id}}}{w_\lambda}y_\lambda.$$

Then we apply the translation formula in Proposition 4.4.2 to the usual equavariant Pieri rule (4.4.7) and obtain the *equivariant weighted Pieri rule*:

Lemma 4.4.8.

$$\mathrm{w}\tilde{S}_{\mathrm{div}}\mathrm{w}\tilde{S}_\lambda = (\mathrm{w}\tilde{S}_{\mathrm{div}}|_\lambda)\mathrm{w}\tilde{S}_\lambda + \sum_{\lambda'\to\lambda}\frac{w_{\mathrm{id}}}{w_\lambda}\mathrm{w}\tilde{S}_{\lambda'}.$$

Remark 4.4.9. From the equivariant weighted Pieri rule, it is easy to show a recursive formula for the structure constants $\mathrm{w}\tilde{c}_{\lambda\mu}^\nu$, in the exactly same way shown in [35, Theorem 3]:

$$\left(\mathrm{w}\tilde{S}_{\mathrm{div}}|_\nu - \mathrm{w}\tilde{S}_{\mathrm{div}}|_\lambda\right)\mathrm{w}\tilde{c}_{\lambda\mu}^\nu = \left(\sum_{\lambda'\to\lambda}\frac{w_{\mathrm{id}}}{w_\lambda}\mathrm{w}\tilde{c}_{\lambda'\mu}^\nu - \sum_{\nu\to\nu'}\frac{w_{\mathrm{id}}}{w_{\nu'}}\mathrm{w}\tilde{c}_{\lambda\mu}^{\nu'}\right). \tag{4.4.10}$$

However this equation (4.4.10) plays no role in the derivation of our main formula, while the recursive formula in [35] plays a crucial role in their process of obtaining the original puzzle formula for $\tilde{c}_{\lambda\mu}^\nu$.

Example 4.4.10 ($\mathrm{wGr}(2,4)$)**.** Here we demonstrate the computation of the product $\mathrm{w}\tilde{S}_{23}\mathrm{w}\tilde{S}_{23}$. By the upper triangularity of the GKM description of $\mathrm{w}\tilde{S}_{23}$, the product must be written by

$$\mathrm{w}\tilde{S}_{23}\mathrm{w}\tilde{S}_{23} = \mathrm{w}\tilde{c}^{23}_{23,23}\mathrm{w}\tilde{S}_{23} + \mathrm{w}\tilde{c}^{13}_{23,23}\mathrm{w}\tilde{S}_{13} + \mathrm{w}\tilde{c}^{12}_{23,23}\mathrm{w}\tilde{S}_{12},$$

where

$$\begin{aligned}
\mathrm{w}\tilde{c}^{23}_{23,23} &= \sum_{\substack{\text{puzzle } P\\ \partial P=\Delta^{23}_{23,23}}} \mathrm{wt}^{23}(P)\\
\mathrm{w}\tilde{c}^{13}_{23,23} &= \sum_{\substack{\text{puzzle } P\\ \partial P=\Delta^{13}_{23,23}}} \mathrm{wt}^{13}(P) + \sum_{\substack{\text{puzzle } Q\\ \partial Q=\Delta^{23}_{23,23}}} \mathrm{wt}^{23,13}(Q)\\
\mathrm{w}\tilde{c}^{12}_{23,23} &= \sum_{\substack{\text{puzzle } P\\ \partial P=\Delta^{12}_{23,23}}} \mathrm{wt}^{12}(P) + \sum_{\substack{\text{puzzle } Q\\ \partial Q=\Delta^{13}_{23,23}}} \mathrm{wt}^{13,12}(Q) + \sum_{\substack{\text{puzzle } Q\\ \partial Q=\Delta^{23}_{23,23}}} \mathrm{wt}^{23,13,12}(Q).
\end{aligned}$$

We can compute the above from the product for ordinary Grassmannian

$$\tilde{S}_{23}\tilde{S}_{23} = (y_4-y_2)(y_4-y_3)\tilde{S}_{23} + (y_4-y_3)\tilde{S}_{13} + \tilde{S}_{12};$$

or equivalently by the fact that:

- there is exactly one puzzle P such that $\partial P = \Delta^{23}_{23,23}$ with two equivariant pieces p_1 and p_2 with the weights $\mathrm{wt}(p_1) = y_4 - y_3$ and $\mathrm{wt}(p_2) = y_4 - y_2$;
- there is exactly one puzzle P such that $\partial P = \Delta^{13}_{23,23}$ with a equivariant piece with the weight $y_4 - y_3$;
- there is exactly one puzzle P such that $\partial P = \Delta^{12}_{23,23}$ without equivariant pieces.

Here are the computation:

$$\begin{aligned}
\mathrm{w}\tilde{c}^{23}_{23,23} &= \left(y_4 - y_2 - (w_4-w_2)\frac{y_{23}}{w_{23}}\right)\left((y_4-y_3) - (w_4-w_3)\frac{y_{23}}{w_{23}}\right)\\
\mathrm{w}\tilde{c}^{13}_{23,23} &= (y_4-y_3) - (w_4-w_3)\frac{y_{13}}{w_{13}} + \frac{w_4-w_2}{w_{23}}\left((y_4-y_3)-(w_4-w_3)\frac{y_{13}}{w_{13}}\right)\\
&\quad + \left((y_4-y_2)-(w_4-w_2)\frac{y_{23}}{w_{23}}\right)\frac{w_4-w_3}{w_{23}}\\
\mathrm{w}\tilde{c}^{12}_{23,23} &= 1 + \frac{w_4-w_3}{w_{13}} + \frac{w_4-w_2}{w_{23}}\frac{w_4-w_3}{w_{13}}
\end{aligned}$$

Similarly we can also work out

$$\mathrm{w}\tilde{S}_{23}\mathrm{w}\tilde{S}_{14} = \mathrm{w}\tilde{c}^{13}_{23,14}\mathrm{w}\tilde{S}_{13} + \mathrm{w}\tilde{c}^{12}_{23,14}\mathrm{w}\tilde{S}_{12}$$

from $\tilde{S}_{23}\tilde{S}_{14} = (y_4 - y_1)\tilde{S}_{13}$:

$$\begin{aligned}
\mathrm{w}\tilde{c}_{23,14}^{13} &= \sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{23,14}^{13}}} \mathrm{wt}^{13}(P) = (y_4 - y_1) - (w_4 - w_1)\frac{y_{13}}{w_{13}} \\
\mathrm{w}\tilde{c}_{23,14}^{12} &= \sum_{\substack{\text{puzzle } Q \\ \partial Q = \Delta_{23,14}^{13}}} \mathrm{wt}^{13,12}(Q) = \frac{w_4 - w_1}{w_{13}}.
\end{aligned}$$

4.5 Factorial Schur functions

We discuss the relation between equivariant weighted Schubert classes and the factorial Schur functions. More precisely, we show that the restriction $\mathrm{w}\tilde{S}_\lambda|_\mu$ of the weighted Schubert classes can be obtained by specializing the factorial Schur functions.

Let $x = (x_1, \cdots, x_d)$ and $a = (a_i)_{i\in\mathbb{Z}}$ be sequences of variables. Let

$$(x_j|a)^k := (x_j - a_1)\cdots(x_j - a_k) \quad (1 \le j \le d).$$

The Young diagram $\underline{\lambda}$ corresponds to each $\lambda \in \{{n \atop d}\}$ by setting the number of boxes in the i-th row to be $\underline{\lambda}^i := n - d + i - \lambda_i$ where $i = 1, \cdots, d$. The *factorial Schur function* corresponding to λ (c.f. [43]) is defined by

$$s_\lambda(x|a) := \frac{\det\left[(x_j|a)^{\underline{\lambda}^i + d - i}\right]_{1\le i,j\le d}}{\prod_{i<j}(x_i - x_j)}. \tag{4.5.1}$$

For any sequence $b = (b_i)_{i\in\mathbb{Z}}$, let $\overline{b} = (\overline{b}_i)_{i\in\mathbb{Z}}$ be defined by $\overline{b}_i := b_{n+1-i}$. For each $\mu \in \{{n \atop d}\}$, let

$$b(\mu) = (b_{\mu_1}, \cdots, b_{\mu_d}). \tag{4.5.2}$$

The vanishing theorem ([45], [43, Theorem 2.1], [35, Section 6]) shows that the restriction of the equivariant Schubert class $\tilde{S}_\lambda$ to $[e_\mu]$ is given by

$$\tilde{S}_\lambda|_\mu = s_\lambda(-y(\mu)| - \overline{y}). \tag{4.5.3}$$

To generalize this equation to the weighted Schubert classes, we introduce the *μ-shifted sequence* associated to each sequence $b = (b_i)_{i\in\mathbb{Z}}$ by

$$b^\mu := \left(b_i - w_i\frac{b_\mu}{w_\mu}\right)_{i\in\mathbb{Z}} \qquad \text{where } b_\mu = \sum_{k\in\mu} b_k.$$

Theorem 4.5.1. *For all $\lambda, \mu \in \{{n \atop d}\}$, we have*

$$\mathrm{w}\tilde{S}_\lambda|_\mu = s_\lambda(-y^\mu(\mu)| - \overline{y^\mu}).$$

Proof. We rewrite (4.5.3) as

$$\tilde{S}_\lambda|_\mu \times \prod_{i<j}(-y_{\mu_i} + y_{\mu_j}) = \det\left[(-y_{\mu_j}| - \overline{y})^{\underline{\lambda}^i+d-i}\right]_{1\le i,j\le d}.$$

Now recall the diagram (4.3.5). By the isomorphism κ_μ^*, we can regard this equality as in $\mathbb{Q}[T^*]/(y_\mu)$ so that we can shift it by multiples of y_μ to obtain

$$\mathrm{a}\tilde{S}_\lambda|_\mu \times \prod_{i<j}(-(y^\mu)_{\mu_i} + (y^\mu)_{\mu_j}) = \det\left[\prod_{p=1}^{\underline{\lambda}^i+d-i}(-(y^\mu)_{\mu_j} + (y^\mu)_{n+1-p})\right]_{1\le i,j\le d}.$$

Since $-(y^\mu)_{\mu_i} + (y^\mu)_{\mu_j}$ and $-(y^\mu)_{\mu_j} + (y^\mu)_{n+1-p}$ are elements of $\mathbb{Q}[\mathrm{w}R^*]$, this becomes, under the isomorphism $\mathrm{w}\kappa_\mu^*$,

$$\mathrm{w}\tilde{S}_\lambda|_\mu \times \prod_{i<j}(-(y^\mu)_{\mu_i} + (y^\mu)_{\mu_j}) = \det[(-(y^\mu)_{\mu_j}| - \overline{y^\mu})^{\underline{\lambda}^i+d-i}]_{1\le i,j\le d}.$$

which is the desired equation. □

Remark 4.5.2. In fact, $\mathrm{w}\tilde{S}_\lambda|_\mu$ can also be obtained by specializing the *weighted factorial Schur functions* that will be introduced and studied in the next chapter.

Chapter 5

Weighted Schur functions

5.1 Introduction

Let $\mathcal{P}(\mathsf{d})$ be the set of partitions with at most d rows. For every $\lambda \in \mathcal{P}(\mathsf{d})$, the Schur function is a polynomial in the variables $(x_1, \cdots, x_{\mathsf{d}})$ defined by

$$s_\lambda(x) := \frac{\det[x_j^{b_i^\lambda + \mathsf{d} - \mathsf{i}}]_{1 \le i,j \le \mathsf{d}}}{\prod_{i<j}(x_i - x_j)}.$$

They form a $\mathbb{Z}$-module basis of $\mathbb{Z}[x]^{\mathfrak{S}_{\mathsf{d}}}$ which is the ring of symmetric polynomials in x-variables. One of the important aspects of Schur functions is that they are the characters of irreducible polynomial representations of $\mathrm{GL}(\mathsf{d}, \mathbb{C})$. Let $R_+(\mathrm{GL}(\mathsf{d}, \mathbb{C}))$ be the polynomial representation ring. Then there is a ring isomorphism

$$R_+(\mathrm{GL}(\mathsf{d}, \mathbb{C})) \stackrel{\text{char}}{\longrightarrow} \mathbb{Z}[x]^{\mathfrak{S}_{\mathsf{d}}} \tag{5.1.1}$$

by taking the character. Let $\lambda = (\lambda_1, \cdots, \lambda_{\mathsf{d}})$ be a partition with at most d rows. Then there is the irreducible polynomial representation V_λ of $\mathrm{GL}(\mathsf{d}, \mathbb{C})$ corresponding to λ (see [18] for an explicit construction). Now, the character of V^λ is given by the Schur function $s_\lambda(x)$, i.e., the isomorphism sends V^λ to $s_\lambda(x)$. Let $\mathsf{n}(> \mathsf{d})$ be an integer. Since $\mathbb{Z}[x]^{\mathfrak{S}_{\mathsf{d}}}$ is generated by the Schur polynomials $s_{(k)}(x)$ (where $(k) \in \mathcal{P}(\mathsf{d})$ is the partition given by $(k)_1 = k, (k)_2 = 0, \cdots, (k)_{\mathsf{d}} = 0$), the coincidence of Pieri-rules in $\mathbb{Z}[x]^{\mathfrak{S}_{\mathsf{d}}}$ and $H^*(\mathrm{Gr}(d,n)(\mathsf{d},\mathsf{n}); \mathbb{Z})$ ensures that, there is a surjective ring homomorphism

$$\mathbb{Z}[x]^{\mathfrak{S}_{\mathsf{d}}} \to H^*(\mathrm{Gr}(d,n)(\mathsf{d},\mathsf{n}); \mathbb{Z}) \tag{5.1.2}$$

which sends $s_\lambda(x)$ to the Schubert class S_λ if $\lambda \in \mathcal{P}(\mathsf{d}, \mathsf{n})$, or 0 otherwise. In this sense, the Schubert classes in the cohomology are incarnations of Schur functions as Ikeda mentioned.

In fact, the surjective ring homomorphism generalizes to an equivariant setting. For every $\lambda \in \mathcal{P}(\mathsf{d})$, the factorial Schur function $s_\lambda(x|a)$ ([9]) is defined as a polynomial in the variables $(x_1, \cdots, x_{\mathsf{d}})$ and $(a_l)_{l\in\mathbb{N}}$ (see (5.2.1) for the defining formula). They form a $\mathbb{Z}[a]$-module basis of $\mathbb{Z}[a][x]^{\mathfrak{S}_{\mathsf{d}}}$ which is the ring of symmetric polynomials in x-variables with the coefficients in the polynomial ring of a-variables. The standard n-torus T^{n} action on $\mathbb{C}^{\mathsf{n}}$ induces the T^{n}-action on the Grassmannian $\mathrm{Gr}(d,n)(\mathsf{d},\mathsf{n})$ and we have the equivariant cohomology $H^*_{\mathsf{T}^{\mathsf{n}}}(\mathrm{Gr}(d,n)(\mathsf{d},\mathsf{n});\mathbb{Z})$ which is an algebra over the polynomial ring $H^*(B\mathsf{T}^*;\mathbb{Z}) = \mathbb{Z}[y_1, \cdots, y_{\mathsf{n}}]$. The Schubert varieties in $\mathrm{Gr}(d,n)(\mathsf{d},\mathsf{n})$ are T^{n}-invariant subvarieties indexed by the partitions $\lambda \in \mathcal{P}(\mathsf{d},\mathsf{n})$ where $\mathcal{P}(\mathsf{d},\mathsf{n})$ is the set of the partitions contained in the $\mathsf{d} \times (\mathsf{n}-\mathsf{d})$ rectangle. The corresponding equivariant Schubert classes $\widetilde{S}_\lambda$ form a $\mathbb{Z}[y_1, \cdots, y_{\mathsf{n}}]$-module basis of the equivariant cohomology. The factorial Schur functions represent the equivariant cohomology rings of Grassmannians in a sense that there is a surjective $\mathbb{Z}[a]$-algebra homomorphism

$$\mathbb{Z}[a][x]^{\mathfrak{S}_{\mathsf{d}}} \to H^*_{\mathsf{T}^{\mathsf{n}}}(\mathrm{Gr}(d,n)(\mathsf{d},\mathsf{n});\mathbb{Z}) \tag{5.1.3}$$

that sends $s_\lambda(x|a)$ to $\widetilde{S}_\lambda$ if $\lambda \in \mathcal{P}(\mathsf{d},\mathsf{n})$, or 0 otherwise ([35]). Here the $\mathbb{Z}[a]$-action on the RHS is given by the projection $\mathbb{Z}[a] \to \mathbb{Z}[y_1, \cdots, y_{\mathsf{n}}]$ sending $a_i \mapsto -y_{\mathsf{n}+1-i}$ if $1 \le i \le \mathsf{n}$, or 0 otherwise. This picture specializes to the Schur functions and the ordinary cohomology of Grassmannians by setting a-variables and y-variables to zero, i.e. the map (5.1.3) induces a ring surjection (5.1.2). One of the advantages of these correspondences is that we can study the structure constants by multiplying actual polynomials. Similar examples include the (double/quantum) Schubert polynomials ([17, 38, 41]) for (equivariant/quantum) cohomolgoy of full flag varieties and (factorial) Schur Q-polynomials ([27, 28, 29]) for (equivariant) cohomology of Lagrangian Grassmannians, e.t.c.

In this chapter, we introduce the *weighted (factorial) Schur functions* that are obtained as a variant of the factorial Schur functions $s_\lambda(x|a)$. In the sense mentioned above, these functions will present the equivariant cohomology of the weighted Grassmannians introduced by Corti-Reid [11]. The contents are based on the paper [4] collaborated with Tomoo Matsumura.

5.2 Weighted (Factorial) Schur Functions

5.2.1 Preliminaries

Fix a positive integer d. Let $\mathcal{P}(\mathsf{d})$ be the set of partitions with at most d rows. For each $\lambda \in \mathcal{P}(\mathsf{d})$, the number of boxes at the i-th row is denoted by b_i^λ where $i = 1, \cdots, \mathsf{d}$. Let $x = (x_1, \cdots, x_{\mathsf{d}})$ and $a = (a_l)_{l\in\mathbb{N}}$ be sequences of variables. Let $\mathbb{Z}[a]$ be the polynomial ring in a_l's, by which we mean the ring of finite linear combinations of monomials in a_l's with finite degrees. Let $\mathbb{Z}[x]^{\mathfrak{S}_{\mathsf{d}}}$ be the symmetric polynomial ring where $\mathfrak{S}_{\mathsf{d}}$ denotes the permutation group on d letters. Recall that the *factorial Schur function* $s_\lambda(x|a)$ is defined as follows (c.f. [43]). For each $k > 0$, let

$$(y|a)^k := (y-a_1)\cdots(y-a_k).$$

Define, for each partition $\lambda \in \mathcal{P}(\mathsf{d})$,

$$s_\lambda(x|a) := \frac{\det[(x_j|a)^{b_i^\lambda + d - i}]_{1 \le i,j \le \mathsf{d}}}{\prod_{i<j}(x_i - x_j)}. \tag{5.2.1}$$

Although $s_\lambda(x|a)$ is a rational function a priori, it is actually a positive degree polynomial function that involves finitely many variables. In fact, we have the following combinatorial formula

$$s_\lambda(x|a) = \sum_T \prod_{\alpha \in \lambda} (x_{T(\alpha)} - a_{T(\alpha)+c(\alpha)})$$

where T runs over all semi-standard tableaux of the shape λ with entries in $\{1, \cdots, \mathsf{d}\}$, $T(\alpha)$ is the entry of the box α in λ, and $c(\alpha) := j - i$ if α is in the i-th row and the j-th column. The ordinary Schur functions $s_\lambda(x)$ can be obtained by specializing $s_\lambda(x|a)$ at $a_l = 0$ for all $l \in \mathbb{N}$. The factorial Schur functions form a $\mathbb{Z}[a]$-module basis of $\mathbb{Z}[a] \otimes_{\mathbb{Z}} \mathbb{Z}[x]^{\mathfrak{S}_\mathsf{d}}$ and the Littlewood-Richardson type formula for the structure constants $c_{\lambda\mu}^{\nu}(a) \in \mathbb{Z}[a]$ is obtained by Molev-Sagan [43]. They actually computed more general structure constants $c_{\lambda\mu}^{\nu}(a,b) \in \mathbb{Z}[a,b]$ defined by

$$s_\lambda(x|b) \cdot s_\mu(x|a) = \sum_{\nu \in \mathcal{P}(\mathsf{d})} c_{\lambda\mu}^{\nu}(a,b) s_\nu(x|a), \tag{5.2.2}$$

where $b = (b_l)_{l \in \mathbb{N}}$ is another infinite sequence of variables.

Definition 5.2.1. For each $\lambda \in \mathcal{P}(\mathsf{d})$, let $\bar{\lambda} = (\bar{\lambda}_1 > \cdots > \bar{\lambda}_\mathsf{d})$ be the strictly decreasing sequence of integers defined by

$$\bar{\lambda}_i := b_i^\lambda + (\mathsf{d} - i + 1) \quad \text{for all } i = 1, \cdots, \mathsf{d}. \tag{5.2.3}$$

For each $\mu \in \mathcal{P}(\mathsf{d})$, we introduce a $\mathbb{Z}[a]$-algebra homomorphism

$$\psi_\mu : \mathbb{Z}[a] \otimes_{\mathbb{Z}} \mathbb{Z}[x]^{\mathfrak{S}_\mathsf{d}} \to \mathbb{Z}[a] \quad \text{by} \quad x_i \mapsto a_{\bar{\mu}_i} \quad \text{for all } i = 1, \cdots, \mathsf{d}. \tag{5.2.4}$$

Lemma 5.2.2 (Vanishing Lemma, [45], c.f. [43])**.** *Let* $a_\lambda := \sum_{i=1}^{\mathsf{d}} a_{\bar{\lambda}_i}$ *for each* $\lambda \in \mathcal{P}(\mathsf{d})$. *For each* $\lambda, \mu \in \mathcal{P}(\mathsf{d})$, *we have*

$$\psi_\mu(s_\lambda(x|a)) = s_\lambda(a_{\bar{\mu}_\mathsf{d}}, \cdots, a_{\bar{\mu}_\mathsf{d}}|a) = \begin{cases} 0 & \text{if } \mu \not\supseteq \lambda \\ \prod_{\rho \in [\lambda]_-} (a_\lambda - a_\rho) & \text{if } \mu = \lambda, \end{cases} \tag{5.2.5}$$

where $[\lambda]_-$ *is the set of partitions* ρ *such that* $\rho \subset \lambda$ *and* $|\{\bar{\rho}_1, \cdots, \bar{\rho}_\mathsf{d}\} \cap \{\bar{\lambda}_1, \cdots, \bar{\lambda}_\mathsf{d}\}| = \mathsf{d} - 1$.

5.2.2 Definition of Weighted (Factorial) Schur Functions

In order to define the weighted Schur functions, we introduce new sets of variables $\mathsf{w} := (\mathsf{w}_l)_{l \in \mathbb{N}}$ and $\mathsf{v} = (\mathsf{v}_1, \cdots, \mathsf{v}_\mathsf{d})$. Let $\mathbb{Z}[\mathsf{w}]$ and $\mathbb{Z}[\mathsf{v}]$ be the corresponding polynomial rings. Let $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}$ be the localization of the ring $\mathbb{Z}[\mathsf{w}]$ at the multiplicative subset

$$\{\mathsf{w}_{\lambda^{(1)}}\cdots\mathsf{w}_{\lambda^{(k)}} \mid \lambda^{(1)},\cdots,\lambda^{(k)}\in\mathcal{P}(\mathsf{d}), k\in\mathbb{Z}_{\geq 0}\}$$

where $\mathsf{w}_\lambda := \sum_{i=1}^{\mathsf{d}} \mathsf{w}_{\bar{\lambda}_i}$ for each $\lambda\in\mathcal{P}(\mathsf{d})$. Similarly let $\mathbb{Z}[\mathsf{v}]_{\mathrm{loc}}$ be the localization of $\mathbb{Z}[\mathsf{v}]$ at the multiplicative subset $\{\mathsf{v}_{\mathrm{ch}}^l \mid l\in\mathbb{Z}_{\geq 0}\}$ where $\mathsf{v}_{\mathrm{ch}} := \mathsf{v}_1+\cdots\mathsf{v}_{\mathsf{d}}$. We denote $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a] := \mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}\otimes_{\mathbb{Z}}\mathbb{Z}[a]$. Let $(\mathbb{Z}[\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_d}$ be the invariant ring of $\mathbb{Z}[\mathsf{v}]_{\mathrm{loc}}[x] := \mathbb{Z}[\mathsf{v}]_{\mathrm{loc}}\otimes_{\mathbb{Z}}\mathbb{Z}[x]$ under the simultaneous permutations on the variable sets x and v. We denote $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]([\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}$ for $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]\otimes_{\mathbb{Z}}(\mathbb{Z}[\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}$. We adapt the same notational convention for $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}([\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}$ and so on.

We introduce the *shifted sequences* $x^{\mathsf{v}} = (x_1^{\mathsf{v}},\cdots,x_{\mathsf{d}}^{\mathsf{v}})$ and $a^{\mathsf{vw}} = (a_i^{\mathsf{vw}})_{i\in\mathbb{N}}$ by

$$x_i^{\mathsf{v}} := x_i - \frac{\mathsf{v}_i}{\mathsf{v}_{\mathrm{ch}}}x_{\mathrm{ch}} \quad \text{for all } i=1,\cdots,\mathsf{d} \qquad \text{and} \qquad a_l^{\mathsf{vw}} := a_l - \frac{\mathsf{w}_l}{\mathsf{v}_{\mathrm{ch}}}x_{\mathrm{ch}} \quad \text{for all } l\in\mathbb{N}$$

where $x_{\mathrm{ch}} := x_1+\cdots+x_{\mathsf{d}}$. We denote $0_l^{\mathsf{vw}} := a_l^{\mathsf{vw}}|_{a_l=0}, l\in\mathbb{N}$, and the corresponding sequence by 0^{vw}.

Definition 5.2.3. For each $\lambda\in\mathcal{P}(\mathsf{d})$, the *weighted factorial Schur function* $s_\lambda^{\mathsf{w}}(\mathsf{v};x|a)\in\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a][\mathsf{v}]_{\mathrm{loc}}[x]$ is defined by

$$s_\lambda^{\mathsf{w}}(\mathsf{v};x|a) := s_\lambda(x^{\mathsf{v}}|a^{\mathsf{vw}})$$

Similarly for each $\lambda\in\mathcal{P}(\mathsf{d})$, the *weighted Schur function* $s_\lambda^{\mathsf{w}}(\mathsf{v};x)\in\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[\mathsf{v}]_{\mathrm{loc}}[x]$ is defined by

$$s_\lambda^{\mathsf{w}}(\mathsf{v};x) := s_\lambda(x^{\mathsf{v}}|0^{\mathsf{vw}}).$$

Since $s_\lambda(x|a)$ is invariant under the permutations on x-varianbles, it is not hard to see from this definition that $s_\lambda^{\mathsf{w}}(\mathsf{v};x|a)$ is invariant under the simultaneous permutations on x and v variables, i.e.

$$s_\lambda^{\mathsf{w}}(\mathsf{v};x|a)\in\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]([\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}} \quad \text{and} \quad s_\lambda^{\mathsf{w}}(\mathsf{v};x)\in\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}([\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}.$$

Example 5.2.4. We list a few examples of $s_\lambda^{\mathsf{w}}(\mathsf{v};x|a)$ and $s_\lambda^{\mathsf{w}}(\mathsf{v};x)$. Let $\mathsf{w}_{\mathrm{ch}} := \sum_{i=1}^{\mathsf{d}}\mathsf{w}_i$ and $a_{\mathrm{ch}} := \sum_{i=1}^{\mathsf{d}} a_i$. Let $\square\in\mathcal{P}(\mathsf{d})$ be the partition given by $b_1^\square = 1$ and $b_2^\square=\cdots=b_{\mathsf{d}}^\square = 0$.

$$s_\square^{\mathsf{w}}(\mathsf{v};x|a) = \frac{\mathsf{w}_{\mathrm{ch}}}{\mathsf{v}_{\mathrm{ch}}}x_{\mathrm{ch}} - a_{\mathrm{ch}} \tag{5.2.6}$$

$$s_\square^{\mathsf{w}}(\mathsf{v};x) = \frac{\mathsf{w}_{\mathrm{ch}}}{\mathsf{v}_{\mathrm{ch}}}x_{\mathrm{ch}} \tag{5.2.7}$$

Let $\lambda\in\mathcal{P}(\mathsf{d})$ be a partition with one box only at the first and second row, i.e. $b_1^\lambda = b_2^\lambda = 1$ and $b_3^\lambda=\cdots=b_{\mathsf{d}}^\lambda=0$. Then

$$s_\lambda^{\mathsf{w}}(\mathsf{v};x|a) = \sum_{1\leq i<j\leq d}\big(x_i - a_i - \frac{\mathsf{v}_i-\mathsf{w}_i}{\mathsf{v}_{\mathrm{ch}}}x_{\mathrm{ch}}\big)\big(x_j - a_{j-1} - \frac{\mathsf{v}_j-\mathsf{w}_{j-1}}{\mathsf{v}_{\mathrm{ch}}}x_{\mathrm{ch}}\big). \tag{5.2.8}$$

Definition 5.2.5. We extend the homomorphism ψ_μ to a $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$-algebra homomorphism ψ_μ^{vw}

$$\psi_\mu^{\mathsf{vw}} : \mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]([\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}} \to \mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a];$$

by sending x_i to $a_{\bar{\mu}_i}$ and v_i to $\mathsf{w}_{\bar{\mu}_i}$ for all $i = 1, \cdots, \mathsf{d}$. Observe that this is well-defined since v_{ch} maps to $\mathsf{w}_\mu \neq 0$. We denote ψ_μ^{vw} by ψ_μ when there is no confusion.

Lemma 5.2.6 (Vanishing Lemma)**.** *For each $\lambda, \mu \in \mathcal{P}(\mathsf{d})$, we have*

$$\psi_\mu(s_\lambda^{\mathsf{w}}(\mathsf{v}; x|a)) = \begin{cases} 0 & \text{if } \mu \not\supset \lambda \\ \prod_{\rho \in [\lambda]_-} \left(\frac{\mathsf{w}_\rho}{\mathsf{w}_\lambda} a_\lambda - a_\rho\right) & \text{if } \mu = \lambda. \end{cases}$$

Proof. Let $a^\mu = ((a^\mu)_l)_{l \in \mathbb{N}}$ be the *μ-shifted sequence of a* defined by

$$(a^\mu)_l := a_l - \frac{\mathsf{w}_l}{\mathsf{w}_\mu} a_\mu.$$

Since $\psi_\mu(x_i^{\mathsf{v}}) = (a^\mu)_{\bar{\mu}_i}$ for $i = 1, \cdots, \mathsf{d}$ and $\psi_\mu(a_l^{\mathsf{vw}}) = (a^\mu)_l$ for $l \in \mathbb{N}$, we find from (5.2.5) that

$$\begin{aligned} \psi_\mu(s_\lambda(x^{\mathsf{v}}|a^{\mathsf{vw}})) &= s_\lambda((a^\mu)_{\bar{\mu}_1}, \ldots, (a^\mu)_{\bar{\mu}_{\mathsf{d}}}|a^\mu) \\ &= \begin{cases} 0 & \text{if } \mu \not\supseteq \lambda \\ \prod_{\rho \in [\lambda]_-} ((a^\lambda)_\lambda - (a^\lambda)_\rho) & \text{if } \mu = \lambda. \end{cases} \end{aligned}$$

We finish the proof by computing

$$(a^\lambda)_\lambda - (a^\lambda)_\rho = \left(a_\lambda - \frac{\mathsf{w}_\lambda}{\mathsf{w}_\lambda} a_\lambda\right) - \left(a_\rho - \frac{\mathsf{w}_\rho}{\mathsf{w}_\lambda} a_\lambda\right) = \frac{\mathsf{w}_\rho}{\mathsf{w}_\lambda} a_\lambda - a_\rho.$$

5.2.3 Algebras of weighted (factorial) Schur functions

Let $\widetilde{\mathsf{w}\mathrm{Sch}}$ be the $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$-submodule of $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]([\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}$ generated by $s_\lambda^{\mathsf{w}}(\mathsf{v}; x|a)$'s:

$$\widetilde{\mathsf{w}\mathrm{Sch}} := \sum_{\lambda \in \mathcal{P}(d)} \mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a] \cdot s_\lambda^{\mathsf{w}}(\mathsf{v}; x|a).$$

Similarly let $\mathsf{w}\mathrm{Sch}$ be the $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}$-submodule of $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}([\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}$ generated by $s_\lambda^{\mathsf{w}}(\mathsf{v}; x)$'s:

$$\mathsf{w}\mathrm{Sch} := \sum_{\lambda \in \mathcal{P}(d)} \mathbb{Z}[\mathsf{w}]_{\mathrm{loc}} \cdot s_\lambda^{\mathsf{w}}(\mathsf{v}; x).$$

Our goal in this section is to show that these submodules $\widetilde{\mathsf{w}\mathrm{Sch}}$ and $\mathsf{w}\mathrm{Sch}$ are acutually subalgebras and also to prove that the weighted factorial Schur functions form a $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$-module basis of $\widetilde{\mathsf{w}\mathrm{Sch}}$. The linear independency of weighted Schur fucntions will be postponed until Section 5.4.

To begin with, observe that we have

$$s_{\lambda}^{\mathsf{w}'}(\mathsf{v};x|a')\cdot s_{\mu}^{\mathsf{w}}(\mathsf{v};x|a)=\sum_{\nu\in\mathcal{P}(\mathsf{d})}c_{\lambda\mu}^{\nu}(a^{\mathsf{vw}},a'^{\mathsf{vw}'})s_{\nu}^{\mathsf{w}}(\mathsf{v};x|a) \tag{5.2.9}$$

by substituting $a\mapsto a^{\mathsf{vw}}$, $a'\mapsto a'^{\mathsf{vw}'}$ and $x\mapsto x^{\mathsf{v}}$ in (5.2.2). This is *not* an expansion formula of a product of two weighted factorial Schur functions over a- and a'-variables: one should notice that each $c_{\lambda\mu}^{\nu}(a'^{\mathsf{vw}'},a^{\mathsf{vw}})$ contains x-variables. Here, the product is taken in the ring $\mathbb{Z}[\mathsf{w}]_{\text{loc}}[a]\otimes\mathbb{Z}[\mathsf{w}']_{\text{loc}}[a']\otimes(\mathbb{Z}[\mathsf{v}]_{\text{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}$.

Lemma 5.2.7 (Weighted Pieri Rule)**.**

$$s_{\square}^{\mathsf{w}'}(\mathsf{v};x|a')\cdot s_{\lambda}^{\mathsf{w}}(\mathsf{v};x|a)=\left(\frac{\mathsf{w}'_{\text{ch}}}{\mathsf{w}_{\lambda}}a_{\lambda}-a'_{\text{ch}}\right)s_{\lambda}^{\mathsf{w}}(\mathsf{v};x|a)+\sum_{\lambda'\to\lambda}\frac{\mathsf{w}'_{\text{ch}}}{\mathsf{w}_{\lambda}}s_{\lambda'}^{\mathsf{w}}(\mathsf{v};x|a) \tag{5.2.10}$$

$$s_{\square}^{\mathsf{w}'}(\mathsf{v};x)\cdot s_{\lambda}^{\mathsf{w}}(\mathsf{v};x)=\sum_{\lambda'\to\lambda}\frac{\mathsf{w}'_{\text{ch}}}{\mathsf{w}_{\lambda}}s_{\lambda'}^{\mathsf{w}}(\mathsf{v};x) \tag{5.2.11}$$

Proof. From Theorem 3.1. in [43], we find that

$$s_{\square}(x|a')\cdot s_{\lambda}(x|a)=(a_{\lambda}-a'_{\text{ch}})s_{\lambda}(x|a)+\sum_{\lambda'\to\lambda}s_{\lambda'}(x|a).$$

By substituting $a\mapsto a^{\mathsf{vw}}$, $a'\mapsto a'^{\mathsf{vw}'}$ and $x\mapsto x^{\mathsf{v}}$ as in (5.2.9), we obtain

$$s_{\square}(x^{\mathsf{v}}|a'^{\mathsf{vw}'})\cdot s_{\lambda}(x^{\mathsf{v}}|a^{\mathsf{vw}}) \;=\; (a_{\lambda}^{\mathsf{vw}}-a'^{\mathsf{vw}'}_{\text{ch}})s_{\lambda}(x^{\mathsf{v}}|a^{\mathsf{vw}})+\sum_{\lambda'\to\lambda}s_{\lambda'}(x^{\mathsf{v}}|a^{\mathsf{vw}})$$

By the definition of a^{vw} and (5.2.6), we have

$$a_{\lambda}^{\mathsf{vw}}-a'^{\mathsf{vw}'}_{\text{ch}}=(a_{\lambda}-a'_{\text{ch}})-\frac{\mathsf{w}_{\lambda}-\mathsf{w}'_{\text{ch}}}{\mathsf{v}_{\text{ch}}}x_{\text{ch}}=(a_{\lambda}-\frac{\mathsf{w}_{\lambda}}{\mathsf{w}'_{\text{ch}}}a'_{\text{ch}})-\frac{\mathsf{w}_{\lambda}-\mathsf{w}'_{\text{ch}}}{\mathsf{w}'_{\text{ch}}}s_{\square}^{\mathsf{w}'}(\mathsf{v};x|a')$$

After substituting this to the previous equation, it is straightforward to obtain the desired formula.

Lemma 5.2.8. *As submodules of* $\mathbb{Z}[\mathsf{w}]_{\text{loc}}[a]([\mathsf{v}]_{\text{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}$*, we have*

$$\widetilde{\mathsf{w}\text{Sch}}=\mathbb{Z}[a]\otimes\mathsf{w}\text{Sch}\,.$$

Proof. First we prove that $\widetilde{\mathsf{w}\text{Sch}}\subset\mathbb{Z}[a]\otimes\mathsf{w}\text{Sch}$, i.e. for each λ, $s_{\lambda}^{\mathsf{w}}(\mathsf{v};x|a)\in\mathbb{Z}[a]\otimes\mathsf{w}\text{Sch}$. Setting $\mu=\emptyset$, $a_i=0$ and $\mathsf{w}'_i=\mathsf{w}_i$ for all $i\in\mathbb{N}$, and substituting $a'_i\mapsto a_i$ for all $i\in\mathbb{N}$ in (5.2.9), we obtain

$$s_{\lambda}^{\mathsf{w}}(\mathsf{v};x|a)=\sum_{\nu\in\mathcal{P}(\mathsf{d})}c_{\lambda\emptyset}^{\nu}(0^{\mathsf{vw}},a^{\mathsf{vw}})s_{\nu}^{\mathsf{w}}(\mathsf{v};x).$$

Since $a_l^{\mathsf{vw}}=a_l-\mathsf{w}_l\frac{x_{\text{ch}}}{\mathsf{v}_{\text{ch}}}$ and $0_l^{\mathsf{vw}}=-\mathsf{w}_l\frac{x_{\text{ch}}}{\mathsf{v}_{\text{ch}}}$, $c_{\lambda\emptyset}^{\nu}(a^{\mathsf{vw}},0^{\mathsf{vw}})$ is a polynomial in $x_{\text{ch}}/\mathsf{v}_{\text{ch}}$ with coefficients in $\mathbb{Z}[\mathsf{w}]_{\text{loc}}[a]$. Since we have (5.2.7), the weighted Pieri rule

(5.2.11) implies that $s^{\mathsf{w}}_{\lambda}(\mathsf{v};x|a)$ is a linear combination of $s^{\mathsf{w}}_{\nu}(\mathsf{v};x)$ over $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$. Thus we conclude that $s^{\mathsf{w}}_{\lambda}(\mathsf{v};x|a) \in \mathbb{Z}[a] \otimes \mathsf{w}\mathrm{Sch}$.

To prove that $\widetilde{\mathsf{w}\mathrm{Sch}} \supset \mathbb{Z}[a] \otimes \mathsf{w}\mathrm{Sch}$, it suffices to show that $s^{\mathsf{w}}_{\lambda}(\mathsf{v};x) \in \widetilde{\mathsf{w}\mathrm{Sch}}$ for each λ. We use the similar argument as above. After setting $\mu = \emptyset$ and $a'_l = 0$ for all $l \in \mathbb{N}$ in (5.2.9), the equation (5.2.6) and the weighted Pieri rule (5.2.10) imply that $s^{\mathsf{w}}_{\lambda}(\mathsf{v};x) \in \widetilde{\mathsf{w}\mathrm{Sch}}$.

Proposition 5.2.9. $\mathsf{w}\mathrm{Sch}$ *is a* $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}$*-subalgebra of* $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}([\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}$*. In particular,* $\widetilde{\mathsf{w}\mathrm{Sch}}$ *is a* $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$*-subalgebra of* $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]([\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}$*.*

Proof. By evaluating $a_l = a'_l = 0$ and $\mathsf{w}'_l = \mathsf{w}_l$ for all $l \in \mathbb{N}$ in (5.2.9), we have

$$s^{\mathsf{w}}_{\lambda}(\mathsf{v};x) \cdot s^{\mathsf{w}}_{\mu}(\mathsf{v};x) = \sum_{\nu \in \mathcal{P}(\mathsf{d})} c^{\nu}_{\lambda\mu}(0^{\mathsf{vw}}, 0^{\mathsf{vw}}) s^{\mathsf{w}}_{\nu}(\mathsf{v};x).$$

Since $c^{\nu}_{\lambda\mu}(a,a)$ is a polynomial in $\{a_k - a_l, k, l \in \mathbb{N}\}$, $c^{\nu}_{\lambda\mu}(0^{\mathsf{vw}}, 0^{\mathsf{vw}})$ is a polynomial in

$$-\mathsf{w}_k \frac{x_{\mathrm{ch}}}{\mathsf{v}_{\mathrm{ch}}} - \left(-\mathsf{w}_l \frac{x_{\mathrm{ch}}}{\mathsf{v}_{\mathrm{ch}}}\right) = -(\mathsf{w}_k - \mathsf{w}_l)\frac{x_{\mathrm{ch}}}{\mathsf{v}_{\mathrm{ch}}} = -(\mathsf{w}_k - \mathsf{w}_l)\frac{s^{\mathsf{w}}_{\square}(\mathsf{v};x)}{\mathsf{w}_{\mathrm{ch}}}.$$

Therefore by the weighted Pieri rule (5.2.11), the product $s^{\mathsf{w}}_{\lambda}(\mathsf{v};x) \cdot s^{\mathsf{w}}_{\mu}(\mathsf{v};x)$ is a linear combination of $\{s^{\mathsf{w}}_{\nu}(\mathsf{v};x)\}_{\nu \in \mathcal{P}(\mathsf{d})}$ over $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}$. Now the latter claim follows from Lemma 5.2.8.

Proposition 5.2.10. $\{s^{\mathsf{w}}_{\lambda}(\mathsf{v};x|a)\}_{\lambda \in \mathcal{P}(\mathsf{d})}$ *is a* $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$*-basis of* $\widetilde{\mathsf{w}\mathrm{Sch}}$*.*

Proof. By the definition of $\widetilde{\mathsf{w}\mathrm{Sch}}$, it is sufficient to show the linear independency. Suppose

$$\sum_{\lambda \in \mathcal{P}(\mathsf{d})} f_{\lambda}(\mathsf{w}, a) \cdot s^{\mathsf{w}}_{\lambda}(\mathsf{v};x|a) = 0$$

for some f_{λ}'s in $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$. Let μ be a minimal (with respect to the inclusion) partition among those λ such that f_{λ} is not identically zero, i.e. there is no λ such $\mu \subset \lambda$ and $f_{\lambda} \neq 0$ except μ itself. Thus by the Vanishing Lemma 5.2.6, we have

$$0 = \psi_{\mu}\left(\sum_{\lambda \in \mathcal{P}(\mathsf{d})} f_{\lambda} \cdot s^{\mathsf{w}}_{\lambda}(\mathsf{v};x|a)\right) = f_{\mu}\psi_{\mu}(s^{\mathsf{w}}_{\mu}(\mathsf{v};x|a)) = f_{\mu} \prod_{\rho \in [\mu]_-} \left(\frac{\mathsf{w}_{\rho}}{\mathsf{w}_{\mu}} a_{\mu} - a_{\rho}\right).$$

Since $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$ has no zero divisor, we have $f_{\lambda} = 0$. This is a contradiction.

5.3 (Equivariant) Cohomology of Weighted Grassmannians

In the rest of the paper, all cohomologies are over $\mathbb{Q}$-coefficients unless otherwise specified.

5.3.1 Review of Weighted Grassmannians and Weighted Schubert Classes

Let us fix an infinite sequence $\{w_i\}_{i\in\mathbb{N}}$ of non-negative integers and a positive integer u. In this section, we recollect from the previous chapter a few facts about the cohomology of the weighted Grassmannians $\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})$ with the weight $(\bar{w}_1,\cdots,\bar{w}_{\mathsf{n}})=(w_{\mathsf{n}},\cdots,w_1)$.

For a natural number $\mathsf{n}>\mathsf{d}$, let $\mathcal{P}(\mathsf{d},\mathsf{n})$ be the set of partitions that are contained in the $\mathsf{d}\times(\mathsf{n}-\mathsf{d})$ rectangle. Upon a choice of n, we identify $\mathcal{P}(\mathsf{d},\mathsf{n})$ with the set $\{{\mathsf{n} \atop \mathsf{d}}\}$ of subsets of $\{1,\cdots,\mathsf{n}\}$ with cardinality d by

$$\lambda\mapsto\{\lambda_1<\cdots<\lambda_{\mathsf{d}}\}\quad\text{where}\quad\lambda_i:=\mathsf{n}+1-\bar{\lambda}_i,\quad i=1,\cdots,\mathsf{d}\qquad(5.3.1)$$

where $\bar{\lambda}_i$ is defined at (5.2.3).

Let $\{\mathsf{e}_1,\cdots,\mathsf{e}_{\mathsf{n}}\}$ be the standard basis of $\mathbb{C}^{\mathsf{n}}$ for each $\mathsf{n}>\mathsf{d}$. We define $\mathrm{aPl}(\mathsf{d},\mathsf{n})$ to be the image of

$$\underbrace{\mathbb{C}^{\mathsf{n}}\times\cdots\times\mathbb{C}^{\mathsf{n}}}_{\mathsf{d}}\to\wedge^{\mathsf{d}}\mathbb{C}^{\mathsf{n}},\quad(\alpha_1,\cdots,\alpha_{\mathsf{d}})\mapsto\alpha_1\wedge\cdots\wedge\alpha_{\mathsf{d}}$$

and let $\mathrm{aPl}(\mathsf{d},\mathsf{n})^{\times}:=\mathrm{aPl}(\mathsf{d},\mathsf{n})\backslash\{0\}$. The standard action of the n-dimensional complex torus $\mathsf{T}^{\mathsf{n}}_{\mathbb{C}}$ on $\mathbb{C}^{\mathsf{n}}$ induces an action of $\mathsf{T}^{\mathsf{n}}_{\mathbb{C}}$ on $\mathrm{aPl}(\mathsf{d},\mathsf{n})^{\times}$. The twisted diagonal subgroup $\overline{\mathrm{w}}\mathsf{D}_{\mathbb{C}}$ of $\mathsf{T}^{\mathsf{n}}_{\mathbb{C}}$ is defined by

$$\overline{\mathrm{w}}\mathsf{D}_{\mathbb{C}}:=\{(t^{\mathsf{d}\bar{w}_1+u},\cdots,t^{\mathsf{d}\bar{w}_{\mathsf{n}}+u})\mid t\in\mathbb{C}^{\times}\}.$$

The weighted Grassmannian $\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})$ for the weights $(\bar{w}_1,\cdots,\bar{w}_{\mathsf{n}})$ is defined by

$$\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}):=\mathrm{aPl}(\mathsf{d},\mathsf{n})^{\times}/\,\overline{\mathrm{w}}\mathsf{D}^{\mathsf{n}}_{\mathbb{C}}.$$

The real subtorus T^{n} in $\mathsf{T}^{\mathsf{n}}_{\mathbb{C}}$ acts on $\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})$ through the quotient map $\mathsf{T}^{\mathsf{n}}\to\mathrm{w}R^{\mathsf{n}-1}:=\mathsf{T}^{\mathsf{n}}/\mathsf{T}^{\mathsf{n}}\cap\mathrm{w}D^{\mathsf{n}}_{\mathbb{C}}$. There is the bijection between $\mathcal{P}(\mathsf{d},\mathsf{n})$ and the fixed point set F_{n} for the T^{n}-action on $\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})$ sending $\lambda\in\{{\mathsf{n} \atop \mathsf{d}}\}=\mathcal{P}(\mathsf{d},\mathsf{n})$ to the equivalent class of $\mathsf{e}_{\lambda_1}\wedge\cdots\wedge\mathsf{e}_{\lambda_{\mathsf{d}}}$ which we denote by $[\mathsf{e}_\lambda]$.

The linear inclusion $\wedge^{\mathsf{d}}\mathbb{C}^{\mathsf{n}}\mapsto\wedge^{\mathsf{d}}\mathbb{C}^{\mathsf{n}+1}$ sending $\mathsf{e}_{\lambda_1}\wedge\cdots\wedge\mathsf{e}_{\lambda_{\mathsf{d}}}\mapsto\mathsf{e}_{\lambda_1+1}\wedge\cdots\mathsf{e}_{\lambda_{\mathsf{d}}+1}$ for each $\{\lambda_1<\cdots<\lambda_{\mathsf{d}}\}\in\{{\mathsf{n} \atop \mathsf{d}}\}$ induces a map $\iota_{\mathsf{n}}:\mathrm{aPl}(\mathsf{d},\mathsf{n})^{\times}\to\mathrm{aPl}(\mathsf{d},\mathsf{n}+1)^{\times}$. This is equivariant under the homomorphism $\rho_{\mathsf{n}}:\mathsf{T}^{\mathsf{n}}_{\mathbb{C}}\to\mathsf{T}^{\mathsf{n}+1}_{\mathbb{C}}$ which sends $(t_1,\cdots,t_{\mathsf{n}})$ to $(1,t_1,\cdots,t_{\mathsf{n}})$, and hence induces the ρ_n-equivariant map

$$\mathrm{w}\iota_{\mathsf{n}}:\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})\to\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}+1).$$

This restricts to the inclusion $F_n\hookrightarrow F_{\mathsf{n}+1}$ which corresponds to the natural inclusion $\mathcal{P}(\mathsf{d},\mathsf{n})\subset\mathcal{P}(\mathsf{d},\mathsf{n}+1)$ and the inclusion $\{{\mathsf{n} \atop \mathsf{d}}\}\hookrightarrow\{{\mathsf{n}+1 \atop \mathsf{d}}\}$ given by $\{\lambda_1<\cdots<\lambda_{\mathsf{d}}\}\mapsto\{\lambda_1+1<\cdots<\lambda_{\mathsf{d}}+1\}$. Let $y_1,\cdots,y_n$ be the standard basis of the weight lattice $\mathrm{Lie}(\mathsf{T}^{\mathsf{n}})^*_{\mathbb{Z}}$ for every n. We identify $H^*(B\mathsf{T}^{\mathsf{n}})=\mathbb{Q}[y_1,\cdots,y_{\mathsf{n}}]$ and ρ_{n} induces the projection

$$\rho^*_{\mathsf{n}}:\mathbb{Q}[y_1,\cdots,y_{\mathsf{n}+1}]\to\mathbb{Q}[y_1,\cdots,y_n]$$

which maps x_i to $a_{\bar{\mu}_i}$ and v_i to $\mathsf{w}_{\bar{\mu}_i}$ for all $i = 1, \cdots, \mathsf{d}$. Recall that for each $\lambda \in \mathcal{P}(\mathsf{d}, \mathsf{n})$, the equivariant weighted Schubert class $\overline{\mathrm{w}}\widetilde{S}_\lambda^{\mathsf{n}}$ is defined as a class in $H^*_{\overline{\mathrm{w}}\mathsf{R}^{\mathsf{n}-1}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n}))$ and that they form a $H^*(B\,\overline{\mathrm{w}}\mathsf{R}^{\mathsf{n}-1})$-basis. We use the same symbol $\overline{\mathrm{w}}\widetilde{S}_\lambda^{\mathsf{n}}$ for the image of $\overline{\mathrm{w}}\widetilde{S}_\lambda^{\mathsf{n}}$ under the map $H^*_{\overline{\mathrm{w}}\mathsf{R}^{\mathsf{n}-1}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n})) \hookrightarrow H^*_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n}))$ induced by the quotient map $\mathsf{T}^{\mathsf{n}} \to \overline{\mathrm{w}}\mathsf{R}^{\mathsf{n}-1}$. The following are easy to check and will be used in the rest of the paper.

(i) The equivariant weighted Schubert classes $\overline{\mathrm{w}}\widetilde{S}_\lambda, \lambda \in \mathcal{P}(\mathsf{d}, \mathsf{n})$ form a basis of $H^*_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n}))$ as $\mathbb{Q}[y_1, \cdots, y_{\mathsf{n}}]$-module.

(ii) The restriction map

$$H^*_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n})) \to H^*_{\mathsf{T}^{\mathsf{n}}}(F_{\mathsf{n}}) = \bigoplus_{[\mathsf{e}_\mu] \in F_{\mathsf{n}}} H^*_{\mathsf{T}^{\mathsf{n}}}([\mathsf{e}_\mu]) = \bigoplus_{\mu \in \mathcal{P}(\mathsf{d}, \mathsf{n})} \mathbb{Q}[y_1, \cdots, y_{\mathsf{n}}]$$

is a $\mathbb{Q}[y_1, \cdots, y_{\mathsf{n}}]$-algebra homomorphism, and it is in fact injective. The image $\overline{\mathrm{w}}\widetilde{S}_\lambda^{\mathsf{n}}|_\mu$ of $\overline{\mathrm{w}}\widetilde{S}_\lambda^{\mathsf{n}}$ at the fixed point $[\mathsf{e}_\mu] \in F_{\mathsf{n}}$ is computed in Theorem 4.5.1 in the previous chapter ;

$$\overline{\mathrm{w}}\widetilde{S}_\lambda^{\mathsf{n}}|_\mu = s_\lambda(-(y^\mu)_{\mu_1}, \cdots, -(y^\mu)_{\mu_{\mathsf{d}}} | -(y^\mu)_{\mathsf{n}}, \cdots, -(y^\mu)_1)$$

where $y_\mu := \sum_{i=1}^d y_{\mu_i}$ and $\bar{w}_\mu := \sum_{i=1}^d \bar{w}_{\mu_i}$ and $(y^\mu)_i = y_i - \frac{\bar{w}_i}{\bar{w}_\mu} y_\mu$.

(iii) The pullback map $\widetilde{\mathrm{w}\iota}_{\mathsf{n}}^* : H^*_{\mathsf{T}^{\mathsf{n}+1}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n}+1)) \to H^*_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n}))$ is a $\mathbb{Q}[y_1, \cdots, y_{\mathsf{n}+1}]$-algebra homomorphism with respect to ρ_{n}^* and for each $\mu \in \mathcal{P}(\mathsf{d}, \mathsf{n})$, we have

$$\widetilde{\mathrm{w}\iota}_{\mathsf{n}}^*(\overline{\mathrm{w}}\widetilde{S}_\lambda^{\mathsf{n}+1}) = \begin{cases} \overline{\mathrm{w}}\widetilde{S}_\lambda^{\mathsf{n}} & \text{if } \lambda \in \mathcal{P}(\mathsf{d}, \mathsf{n}), \\ 0 & \text{if } \lambda \notin \mathcal{P}(\mathsf{d}, \mathsf{n}). \end{cases}$$

In particular, $\widetilde{\mathrm{w}\iota}_{\mathsf{n}}^*$ is surjective.

(iv) For each $\lambda \in \mathcal{P}(\mathsf{d}, \mathsf{n})$, the weighted Schubert class $\overline{\mathrm{w}}S_\lambda$ is the image of $\overline{\mathrm{w}}\widetilde{S}_\lambda$ under the natural map $H^*_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n})) \to H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n}))$ and they form a $\mathbb{Q}$-basis of $H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n}))$. The pullback map $\mathrm{w}\iota_{\mathsf{n}}^* : H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n}+1)) \to H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n}))$ satisfies

$$\mathrm{w}\iota_{\mathsf{n}}^*(\overline{\mathrm{w}}S_\lambda^{\mathsf{n}+1}) = \begin{cases} \overline{\mathrm{w}}S_\lambda^{\mathsf{n}} & \text{if } \lambda \in \mathcal{P}(\mathsf{d}, \mathsf{n}), \\ 0 & \text{if } \lambda \notin \mathcal{P}(\mathsf{d}, \mathsf{n}). \end{cases}$$

In particular, $\mathrm{w}\iota_{\mathsf{n}}^*$ is surjective.

5.3.2 Cohomology of $\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \infty)$

By using the inclusions $\{\mathrm{w}\iota_{\mathsf{n}}, \mathsf{n} \in \mathbb{N}\}$, we define

$$\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \infty) := \varinjlim \overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n}) = \bigcup_{\mathsf{n} \in \mathbb{N}} \overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d}, \mathsf{n}).$$

Since $\mathrm{w}\iota_{\mathsf{n}}^*$ is surjective for each n, we have $\underset{\longleftarrow}{\lim}^1 H^k(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})) = 0$ for each k. Therefore there is the $\mathbb{Q}$-linear isomorphism

$$H^k(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)) \to \underset{\leftarrow}{\lim}\, H^k(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})) \quad \text{for each } k \geq 0.$$

The cup products on $H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}))$ for all $\mathsf{n} \in \mathbb{N}$ canonically define a structure of $\mathbb{Q}$-algebra on the direct sum of $\underset{\leftarrow}{\lim}\, H^k(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}))$ over all $k \geq 0$ and we have

Proposition 5.3.1. *The inclusions induces a canonical* $\mathbb{Q}$*-algebra isomorphism*

$$H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)) \to \bigoplus_{k\geq 0} \underset{\leftarrow}{\lim}\, H^k(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})).$$

The property (iv), together with this proposition, defines the classes $\overline{\mathrm{w}}S_\lambda^\infty$ in $H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty))$ such that the pullback $H^k(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)) \to H^k(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}))$ sends $\overline{\mathrm{w}}S_\lambda^\infty$ to $\overline{\mathrm{w}}S_\lambda^{\mathsf{n}}$ if $\lambda \in \mathcal{P}(\mathsf{d},\mathsf{n})$ and 0 otherwise.

Proposition 5.3.2. $\{\overline{\mathrm{w}}S_\lambda^\infty\}_{\lambda\in\mathcal{P}(\mathsf{d})}$ *forms a* $\mathbb{Q}$*-basis of* $H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty))$.

Proof. For each $k \geq 0$, the pull back gives us an isomorphism

$$H^k(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)) \cong H^k(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})) = \bigoplus_{\substack{\lambda\in\mathcal{P}(\mathsf{d},\mathsf{n})\\ l(\lambda)=k}} \mathbb{Q}\cdot\overline{\mathrm{w}}S_\lambda^{\mathsf{n}}$$

for a sufficiently large $\mathsf{n} > k$ where $l(\lambda) = \sum_{i=1}^d b_i^\lambda$ is the number of boxes in λ. By the definition of the elements $\{\overline{\mathrm{w}}S_\lambda^\infty\}$, they correspond to $\{\overline{\mathrm{w}}S_\lambda^{\mathsf{n}}\}$ which is a $\mathbb{Q}$-basis of the image, the claim follows.

5.3.3 Cohomology of equivariant analogue of $\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)$

Let $E\mathsf{T}^{\mathsf{n}} \to B\mathsf{T}^{\mathsf{n}}$ is a universal principal T^{n}-bundle in which $E\mathsf{T}^{\mathsf{n}}$ is contractible. We choose a ρ_{n}-equivariant continuous map $E\mathsf{T}^{\mathsf{n}} \to E\mathsf{T}^{\mathsf{n}+1}$ for each n. Hence we have the induced maps $\rho'_{\mathsf{n}} : B\mathsf{T}^{\mathsf{n}} \to B\mathsf{T}^{\mathsf{n}+1}$ whose cohomology pullbacks are exactly the surjection ρ_{n}^*. Let $B\mathsf{T}_\infty := \underset{\rightarrow}{\lim}\, B\mathsf{T}^{\mathsf{n}}$ be the corresponding inductive limit. As in the last section, there is no $\lim^1$ and hence we have a $\mathbb{Q}$-algebra isomorphism

$$H^*(B\mathsf{T}_\infty) \cong \bigoplus_{k\geq 0} \underset{\leftarrow}{\lim}\, \mathbb{Q}[y_1,\cdots,y_{\mathsf{n}}]^{(k)} \tag{5.3.2}$$

where $\mathbb{Q}[y_1,\cdots,y_{\mathsf{n}}]^{(k)}$ is the component of the cohomological degree $2k$. Let $(\overline{y}_l)_{l\in\mathbb{N}}$ be an infinite sequence of variables and let $\widetilde{\mathbb{Q}[\overline{y}]}$ be the ring of polynomials in $\overline{y}_l$'s which are possibly infinite linear combinations of finite degree monomials. Then the RHS of (5.3.2) can be identified with $\widetilde{\mathbb{Q}[\overline{y}]}$ through the homomorphisms

$$\theta_{\mathsf{n}} : \widetilde{\mathbb{Q}[\overline{y}]} \to \mathbb{Q}[y_1,\cdots,y_{\mathsf{n}}]; \quad \overline{y}_l \mapsto \begin{cases} y_{\mathsf{n}+1-l} & \text{if } 1 \leq l \leq \mathsf{n} \\ 0 & \text{if } l > \mathsf{n}. \end{cases}$$

By using the ρ_{n}-equivariant map $\mathrm{w}\iota_{\mathsf{n}}$, define

$$\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)_{\mathsf{T}} := \lim_{\rightarrow}(E\mathsf{T}^{\mathsf{n}} \times_{\mathsf{T}^{\mathsf{n}}} \overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})),$$

then we have the commutative diagrams for all n:

$$\begin{tikzcd} \overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)_{\mathsf{T}} \arrow[d] & E\mathsf{T}^{\mathsf{n}} \times_{\mathsf{T}^{\mathsf{n}}} \overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}) \arrow[l] \arrow[d] \\ B\mathsf{T}_{\infty} & B\mathsf{T}^{\mathsf{n}} \arrow[l] \end{tikzcd} \tag{5.3.3}$$

Note that the cohomology $H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)_{\mathsf{T}})$ and the pullback $\widetilde{\mathrm{w}\iota}_{\mathsf{n}}^*$ do not depend on the choices of $E\mathsf{T}^{\mathsf{n}}$'s and the maps $E\mathsf{T}^{\mathsf{n}} \to E\mathsf{T}^{\mathsf{n}+1}$ up to isomorphisms.

As in the last section, we have no $\lim^1$ term for the propjective system $\{H^k_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})), \widetilde{\mathrm{w}\iota}_{\mathsf{n}}^*\}$ for each k, therefore the top map in the above diagram induces the $\mathbb{Q}$-algebra isomorphism

$$H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)_{\mathsf{T}}) \to \bigoplus_{k\geq 0} \lim_{\leftarrow} H^k_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})). \tag{5.3.4}$$

Since the right vertical maps of (5.3.3) commute with ρ'_{n} and $\widetilde{\mathrm{w}\iota_{\mathsf{n}}} : E\mathsf{T}^{\mathsf{n}} \times_{\mathsf{T}^{\mathsf{n}}} \overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}) \to E\mathsf{T}^{\mathsf{n}+1} \times_{\mathsf{T}^{\mathsf{n}+1}} \overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}+1)$, the ring $\widetilde{\mathbb{Q}[\overline{y}]}$ acts on the RHS of (5.3.4). Thus, by the commutativity of (5.3.3), the map (5.3.4) is actually a $\widetilde{\mathbb{Q}[\overline{y}]}$-algebra isomorphism. With the property (iii), the isomorphism (5.3.4) defines the classes $\overline{\mathrm{w}}\widetilde{S}^{\infty}_{\lambda}$ in $H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)_{\mathsf{T}})$ such that the pullback $H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)_{\mathsf{T}}) \to H^*_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}))$ sends $\overline{\mathrm{w}}\widetilde{S}^{\infty}_{\lambda}$ to $\overline{\mathrm{w}}\widetilde{S}^{\mathsf{n}}_{\lambda}$ if $\lambda \in \mathcal{P}(\mathsf{d},\mathsf{n})$ and 0 otherwise. Finally it is not difficult to see from the RHS of (5.3.4) that $\overline{\mathrm{w}}\widetilde{S}^{\infty}_{\lambda}, \lambda \in \mathcal{P}(\mathsf{d})$ form a $\widetilde{\mathbb{Q}[\overline{y}]}$-module basis of $H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)_{\mathsf{T}})$. We conclude this section by summarizing above as follows.

Proposition 5.3.3. *$H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)_{\mathsf{T}})$ is a $\widetilde{\mathbb{Q}[\overline{y}]}$-algebra and there is a $\widetilde{\mathbb{Q}[\overline{y}]}$-module basis $\{\overline{\mathrm{w}}\widetilde{S}^{\infty}_{\lambda},\ \lambda \in \mathcal{P}(\mathsf{d})\}$ such that $\overline{\mathrm{w}}\widetilde{S}^{\infty}_{\lambda}$ maps to $\overline{\mathrm{w}}\widetilde{S}^{\mathsf{n}}_{\lambda}$ under the pullback $H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)_{\mathsf{T}}) \to H^*_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}))$ for each n.*

5.4 Correspondences of functions and cohomology

Recall from Proposition 5.2.9, $\widetilde{\mathsf{wSch}}$ is a $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$-subalgebra of $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]([\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}$. Let $\{w_l\}_{l\in\mathbb{N}} \subset \mathbb{Z}_{\geq 0}$ and $u \in \mathbb{Z}_{>0}$. For each $\mathsf{n} \in \mathbb{N}$, define a ring homomorphism

$$\varphi^w_{\mathsf{n}} : \mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a] \to \mathbb{Q}[y_1, \cdots, y_{\mathsf{n}}];\ \ \mathsf{w}_l \mapsto w_l + u/\mathsf{d} \ \text{ and } \ a_l \mapsto \begin{cases} -y_{\mathsf{n}+1-l} & \text{if } 1 \leq l \leq \mathsf{n} \\ 0 & \text{if } l > \mathsf{n} \end{cases}$$

where l runs all the natural numbers $\mathbb{N}$. It is well-defined since

$$\mathsf{w}_{\mu} = \sum_{i=1}^{\mathsf{d}} \mathsf{w}_{\bar{\mu}_i} \mapsto \overline{w}_{\mu} = \sum_{i=1}^{\mathsf{d}} \overline{w}_{\mu_i} + u \neq 0.$$

These φ_{n}^{w}'s induce the algebra homomorphism

$$\varphi_{\infty}^{w} : \mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a] \to \widetilde{\mathbb{Q}[\overline{y}]}; \quad a_l \mapsto -\overline{y}_l \quad \text{and} \quad \mathsf{w}_l \mapsto w_l + u/\mathsf{d} \quad \text{for all } l \in \mathbb{N}.$$

Theorem 5.4.1. *There is a* $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$*-algebra homomorphism*

$$\widetilde{\Phi}_{\mathsf{n}} : \widetilde{\mathsf{w}\mathrm{Sch}} \to H^*_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})); \qquad s^{\mathsf{w}}_{\lambda}(\mathsf{v};x|a) \mapsto \begin{cases} \overline{\mathrm{w}}\widetilde{S}^{\mathsf{n}}_{\lambda} & \text{if } \lambda \in \mathcal{P}(\mathsf{d},\mathsf{n}), \\ 0 & \text{otherwise,} \end{cases}$$

where the action of $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$ *on the RHS is given by* φ^{w}_{n}*. In particular, this defines a* $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$*-algebra homomorphism*

$$\widetilde{\Phi}_{\infty} : \widetilde{\mathsf{w}\mathrm{Sch}} \to H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)_{\mathsf{T}}); \qquad s^{\mathsf{w}}_{\lambda}(\mathsf{v};x|a) \mapsto \overline{\mathrm{w}}\widetilde{S}^{\infty}_{\lambda}$$

where the action $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$ *on the RHS is given by* φ^{w}_{∞}*.*

Proof. Consider the $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]$-algebra homomorphism

$$\widetilde{\mathsf{w}\mathrm{Sch}} \xrightarrow[(\psi^{\mathsf{vw}}_{\mu})_{\mu\in\mathcal{P}(\mathsf{d})}]{} \prod_{\mu\in\mathcal{P}(\mathsf{d})} \mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a] \longrightarrow \prod_{\mu\in\mathcal{P}(\mathsf{d},\mathsf{n})} \mathbb{Q}[y_1,\cdots,y_{\mathsf{n}}] \tag{5.4.1}$$

where the second map is given by φ^{w}_{n} if $\mu \in \mathcal{P}(\mathsf{d},\mathsf{n})$ and a trivial map if otherwise. If $\lambda \notin \mathcal{P}(\mathsf{d},\mathsf{n})$, then for all $\mu \in \mathcal{P}(\mathsf{d},\mathsf{n})$, we have $\mu \not\supseteq \lambda$ and therefore the image of $s^{\mathsf{w}}_{\lambda}(\mathsf{v};x|a)$ under (5.4.1) is 0. If $\lambda \in \mathcal{P}(\mathsf{d},\mathsf{n})$, then for all $\mu \in \mathcal{P}(\mathsf{d},\mathsf{n})$, under the map (5.4.1) we have

$$\begin{aligned} s^{\mathsf{w}}_{\lambda}(\mathsf{v};x|a) &\overset{\psi_{\mu}}{\mapsto} s_{\lambda}((a^{\bar{\mu}})_{\bar{\mu}_1},\cdots,(a^{\bar{\mu}})_{\bar{\mu}_{\mathsf{d}}}|a^{\bar{\mu}}), \qquad (a^{\bar{\mu}})_l = a_l - \frac{\mathsf{w}_l}{\mathsf{w}_{\bar{\mu}}}a_{\bar{\mu}} \\ &\overset{\varphi^{w}_{\mathsf{n}}}{\mapsto} s_{\lambda}(-(y^{\mu})_{\mu_1},\cdots,-(y^{\mu})_{\mu_{\mathsf{d}}}|-(y^{\mu})_{\mathsf{n}},\cdots,-(y^{\mu})_1,0,0,\cdots) \end{aligned}$$

where $(y^{\mu})_i = y_i - \frac{\bar{w}_i}{\bar{w}_{\mu}}y_{\mu}$. Here for the second map, we have used the fact that $s_{\lambda}(x|a)$ does not involve a_i for all $i > \mathsf{n}$ if $\lambda \in \mathcal{P}(\mathsf{d},\mathsf{n})$ by definition. The property (ii) in Section 5.3.1 implies that the image of $s^{\mathsf{w}}_{\lambda}(\mathsf{v};x|a)$ is exactly the restriction of $\overline{\mathrm{w}}\widetilde{S}^{\mathsf{n}}_{\lambda}$ to the fixed points if $\lambda \in \mathcal{P}(\mathsf{d},\mathsf{n})$ and 0 otherwise. Moreover it follows from the injectivity in (ii) that the map (5.4.1) factors through the desired map $\widetilde{\Phi}_{\mathsf{n}} : \widetilde{\mathsf{w}\mathrm{Sch}} \to H^*_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}))$. By introducing $\deg x_i = \deg a_l = 2$ (and $\deg \mathsf{v}_i = \deg \mathsf{w}_l = 0$) for all $i = 1,\cdots,\mathsf{d}$ and $l \in \mathbb{N}$, the map $\widetilde{\Phi}_{\mathsf{n}}$ is a homomorphism as graded $\mathbb{Q}$-algebras. The obvious commutativity $\widetilde{\mathrm{w}\iota}^*_{\mathsf{n}} \circ \widetilde{\Phi}_{\mathsf{n}+1} = \widetilde{\Phi}_{\mathsf{n}}$ allow us to take the projective limit of the maps $\widetilde{\Phi}_{\mathsf{n}}$ on each degree, and taking their direct sum, we obtain the desired map $\widetilde{\Phi}_{\infty}$.

By tensoring $\mathbb{Z}$ over $\mathbb{Z}[a]$ with respect to the homomorphism $\mathbb{Z}[a] \to \mathbb{Z}, (a_i \mapsto 0)$, we obtain the $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}$-algebra homomorphism $\Phi_{\mathsf{n}} := \mathbb{Z} \otimes_{\mathbb{Z}[a]} \widetilde{\Phi}_{\mathsf{n}}$

$$\Phi_{\mathsf{n}} : \mathbb{Z} \otimes_{\mathbb{Z}[a]} \widetilde{\mathsf{w}\mathrm{Sch}} \to \mathbb{Z} \otimes_{\mathbb{Z}[a]} H^*_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})).$$

The LHS is exactly $\mathsf{w}\mathrm{Sch}$ by Lemma 5.2.8. The action of $\mathbb{Z}[a]$ on $H^*_{\mathsf{T}^{\mathsf{n}}}(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})$ is through $\mathbb{Z}[a] \to \mathbb{Q}[y_1,\cdots,y_{\mathsf{n}}]$ that sends a_i to $y_{\mathsf{n}+1-i}$ if $1 \le i \le \mathsf{n}$ and 0 otherwise. Therefore the RHS is $H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n}))$. Thus the following is an immediate consequence.

Theorem 5.4.2. *For each* $\mathsf{n} \in \mathbb{N}$*, the map* $\widetilde{\Phi}_{\mathsf{n}}$ *induces the* $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}$*-algebra homomorphism*

$$\Phi_{\mathsf{n}} : \mathsf{w}\mathrm{Sch} \to H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\mathsf{n})); \qquad s^{\mathsf{w}}_{\lambda}(\mathsf{v};x) \mapsto \begin{cases} \overline{\mathrm{w}}S^{\infty}_{\lambda} & \text{if } \lambda \in \mathcal{P}(\mathsf{d},\mathsf{n}), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this defines a $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}$*-algebra homomorphism*

$$\Phi_{\infty} : \mathsf{w}\mathrm{Sch} \to H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)); \qquad s^{\mathsf{w}}_{\lambda}(\mathsf{v};x) \mapsto \overline{\mathrm{w}}S^{\infty}_{\lambda}.$$

Proposition 5.4.3. *The weighted Schur functions* $s^{\mathsf{w}}_{\lambda}(\mathsf{v};x), \lambda \in \mathcal{P}(\mathsf{d})$*, form a* $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}$*-module basis of* $\mathsf{w}\mathrm{Sch}$.

Proof. It is enough to check the linear independency. Suppose that

$$\sum_{\lambda \in \mathcal{P}(\mathsf{d})} f_{\lambda}(\mathsf{w}) s^{\mathsf{w}}_{\lambda}(\mathsf{v};x) = 0$$

for some $f_{\lambda}(\mathsf{w}) \in \mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}$. There exists n such that, for all λ appearing in the sum, $f_{\lambda}(\mathsf{w})$ involves only $\mathsf{w}_1, \cdots, \mathsf{w}_{\mathsf{n}}$. Then Φ_{n} send the equality to

$$\sum_{\lambda \in \mathcal{P}(\mathsf{d})} f_{\lambda}(w_1 + u/\mathsf{d}, \cdots, w_{\mathsf{n}} + u/\mathsf{d}) \overline{\mathrm{w}}S^{\mathsf{n}}_{\lambda} = 0.$$

This holds for arbitrary $w_1, \cdots, w_{\mathsf{n}} \in \mathbb{Z}_{\geq 0}$ and $u \in \mathbb{Z}_{>0}$ and since $\{\overline{\mathrm{w}}S^{\mathsf{n}}_{\lambda}\}_{\lambda \in \mathcal{P}(\mathsf{d},\mathsf{n})}$ is a linearly independent set, we conclude that $f_{\lambda}(\mathsf{w})$ is identically 0 for all λ appearing in the sum.

Remark 5.4.4. The homomorphisms $\widetilde{\Phi}_{\infty}$ in Theorem 5.4.1 and Φ_{∞} in Theorem 5.4.2 can be made into isomorphisms by evaluating the w-variables. Namely, let

$$\begin{aligned} w\mathrm{Sch} &:= \mathbb{Q} \otimes_{\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}} \mathsf{w}\mathrm{Sch} \subset (\mathbb{Q}[\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}} \\ w\widetilde{\mathrm{Sch}} &:= \widetilde{\mathbb{Q}[a]} \otimes_{\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}}[a]} \mathsf{w}\widetilde{\mathrm{Sch}} \subset \widetilde{\mathbb{Q}[a]}([\mathsf{v}]_{\mathrm{loc}}[x])^{\mathfrak{S}_{\mathsf{d}}}. \end{aligned}$$

where the tensor products are given by $\mathbb{Z}[\mathsf{w}]_{\mathrm{loc}} \to \mathbb{Q} \quad (\mathsf{w}_i \mapsto w_i + u/\mathsf{d})$. Then clearly Φ_{∞} and $\widetilde{\Phi}_{\infty}$ induce the isomorphisms

$$\Phi^{w}_{\infty} : w\mathrm{Sch} \overset{\cong}{\to} H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)) \qquad \text{and} \qquad \widetilde{\Phi}^{w}_{\infty} : w\widetilde{\mathrm{Sch}} \overset{\cong}{\to} H^*(\overline{\mathrm{w}}\mathrm{Gr}(\mathsf{d},\infty)_{\mathsf{T}}).$$

Here $\widetilde{\Phi}^{w}_{\infty}$ is an algebra isomorphism with respect to $\widetilde{\mathbb{Q}[a]} \cong \widetilde{\mathbb{Q}[\overline{y}]}$ defined by $a_l \mapsto -\overline{y}_l$ for all $l \in \mathbb{N}$. Then these isomorphisms send the evaluated weighted (factorial) Schur functions

$$s^{w}_{\lambda}(\mathsf{v};x) := s^{\mathsf{w}}_{\lambda}(\mathsf{v};x)|_{\mathsf{w}_l = w_l + u/d, l \in \mathbb{N}} \quad \text{and} \quad s^{w}_{\lambda}(\mathsf{v};x|a) := s^{\mathsf{w}}_{\lambda}(\mathsf{v};x|a)|_{\mathsf{w}_l = w_l + u/d, l \in \mathbb{N}}.$$

to $\overline{\mathrm{w}}S^{\infty}_{\lambda}$ and $\overline{\mathrm{w}}\widetilde{S}^{\infty}_{\lambda}$ respectively.

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