Complex Contact Structures on Nilmanifolds

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Contents

Overview

Part I

The main object of Part I is to study complex contact manifolds. We introduce the Iwasawa Lie group $\mathcal{L}$ and construct holomorphic group actions on $\mathcal{L}$.

Chapter 1. We explain a background and a history of complex contact manifolds and state our main results.

Chapter 2. We study the notion of *developing maps* which is an essential tool to prove our main results and collect several previous results on similarity geometry.

Chapter 3. We construct the *Iwasawa nilpotent Lie group* $\mathcal{L}$. Considering similarity group actions on $\mathcal{L}$, we obtain *complex contact similarity manifolds* and give a classification on compact complex contact similarity manifolds under a certain assumption.

Chapter 4. Let $\mathcal{L}/\Gamma$ be a complex contact infranilmanifold which is a holomorphic torus bundle over the quaternionic euclidean orbifold. Then we prove that the connected sum of $\mathcal{L}/\Gamma$ with the complex projective space $\mathbb{CP}^{2n+1}$ admits a complex contact structure.

Chapter 5. We verify $\mathcal{L}$ from the viewpoint of geometric structure. The *quaternionic Heisenberg Lie group* $\mathcal{M}$ is known to posses a quateronic Carnot-Carathéodory structure (qCC-structure). In fact $\mathcal{M}$ occurs as a central group extension $1 \to \mathbb{R}^3 \to \mathcal{M} \longrightarrow \mathbb{H}^n \to 1$ where $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ is the imaginary part of the quaternion field $\mathbb{H}$. Taking a quotient of $\mathcal{M}$ by $\mathbb{R}(= \mathbb{R}i)$, it turns out that the quotient group is the Iwasawa nilpotent group $\mathcal{L}(= \mathcal{L}_{2n+1})$ which admits a complex contact structure induced from the qCC-structure on $\mathcal{M}$.

Part II

The main object of Part II is to study the *mapping class groups* on a surface, the group of isotopy class of the diffeomorphisms on a surface. We study a certain filtration of the mapping class group called the *Johnson filtration* and give the Johnson filtration version of the *Birman exact sequence*.

Chapter 6. We introduce some basic results concerning the mapping class groups and *Dehn twists*. We show that the *Humphrie generator* which is an generator of the mapping class group is obtained from the *Lickorish generator*.

Chapter 7. We recall the *Torelli group* which is the kernel of the symplectic representation of the mapping class group, and consider the Johnson filtration. We also give some lemmas to prove the main result of Part II.

Chapter 8. We study homomorphisms from the mapping class group of a subsurface S' of S to the mapping class group of S which is induced by the inclusion of the subsurface $S' \hookrightarrow S$. We introduce a classical tool for studying the mapping class group called the Birman exact sequence, and give the shape of the kernel of this short exact sequence which restricts it to the Johnson filtrations.

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Part I

Complex Contact Structures on Nilmanifolds

Chapter 1

Introduction to Part I

1.1 Background and Results

The notion of a complex contact manifold was first introduced by Kobayashi[32] in 1959 as a complex analogue of (real) contact manifold.

There are constructions of three different types of complex contact structure. Given a $4n$-dimensional quaternionic Kähler manifold M which locally has a quaternion structure $\{\mathcal{I}, \mathcal{J}, \mathcal{K}\}$, we can obtain the twistor space Z locally described as

$$Z = \{x\mathcal{I} + y\mathcal{J} + z\mathcal{K} \mid x^2 + y^2 + z^2 = 1\} \subset End(TM).$$

This is the total space of the S^2 bundle

$$S^2 \to Z \to M.$$

Salamon[52] proved that the twistor space of a quaternionic Kähler manifold of nonzero scalar curvature admits a complex contact structure.

Similarly, a quaternionic Kähler manifold M^{4n} of positive (resp. negative) scalar curvature induces a Sasakian 3-structure (resp. pseudo-Sasakian 3-structure) on the total space P^{4n+3} of the principal bundle: $S^3 \to P \to M$. Dividing by the free action induced by a Reeb vector field, we obtain the following diagram.

$$\begin{array}{ccccc} S^3 & \longrightarrow & P & \longrightarrow & M \\ {\scriptstyle /S^1}\big\downarrow & & {\scriptstyle /S^1}\big\downarrow & & \big\| \\ S^2 & \longrightarrow & P/S^1 & \longrightarrow & M \end{array}$$

The total space P/S^1 of the quotient bundle $S^2 \to P/S^1 \to N$ admits a complex contact structure. (See Ishihara & Konishi[22]; Moroianu & Semmelmann[46]; Tanno[53].) However, these constructions cannot produce complex contact manifolds for quaternionic Kähler manifolds of vanishing scalar curvature.

On the other hand, if N^{4n} is a complex symplectic manifold with a complex symplectic form $\Omega = \Omega_1 + \mathbf{i}\Omega_2$ such that $[\Omega_i] \in H^2(N;\mathbb{Z})$ is an integral class ($i = 1, 2$), then the *complex Boothby-Wang fibration* induces a compact complex contact manifold M which has a connection bundle: $T^2 \to M \to N$ (cf. Foreman[16]; Blair[9]). If N^{4n} happens to be a quaternionic Kähler manifold with vanishing scalar curvature, then we have a new example of compact complex manifold. In fact, Foreman[16] shows that a complex nilmanifold M which is the total space of a principal torus bundle over a complex torus $T^{2n}_{\mathbb{C}}$ admits a complex contact structure. The universal covering $\tilde{M}$ is endowed with a complex nilpotent Lie group structure which is called the *generalized complex Heisenberg group* in Foreman[16].

In Part I, we study *complex contact transformation groups* by taking into account this specific nilpotent Lie group.

In Chapter 3, we give the definition of a complex contact manifold and construct *complex contact similarity manifolds*. We are mainly interested in constructing examples of *compact* complex contact manifolds which are not known previously. Let $\mathrm{Sim}(\mathcal{L})$ be the group of complex contact similarity transformations. It is defined to be the semidirect product $\mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot \mathbb{C}^*)$ $(\mathbb{C}^* = S^1 \times \mathbb{R}^+)$. The pair $(\mathrm{Sim}(\mathcal{L}), \mathcal{L})$ is said to be *complex contact similarity geometry*. A manifold M locally modelled on this geometry is called a complex contact similarity manifold. Denote by $\mathrm{Aut}_{cc}(M)$ the group of complex contact transformations of M.

A similarity manifold is defined in Fried[17]. We define this object in Chapter 2.

Theorem 1.1.1 (Fried[17]). *Let M be a closed similarity manifold. Then M is finitely covered by either a flat torus or a Hopf manifold.*

On the other hand, Miner proved the following CR analogue of the theorem by Fried.

Theorem 1.1.2 (Miner[41]). *Let M be a compact spherical CR-manifold with amenable holonomy. Then M is finitely covered by S^{2n+1}, $S^1 \times S^{2n-2}$ or a compact quotient of the Heisenberg group $\mathcal{H}/\Gamma$ where Γ is lattice in $\mathcal{H}$.*

He showed that, if M is complete, then M is covered by S^{2n+1}; otherwise M has the structure of a Heisenberg similarity manifold.

We prove the following characterization of compact complex contact similarity manifolds in Chapter 3. This theorem can be thought of as a quaternion version of the above theorems.

Theorem A. *Let M be a compact complex contact similarity manifold of complex dimension $2n+1$. If $S^1 \leq \mathrm{Aut}_{cc}(M)$ acts on M without fixed points, then M is holomorphically isomorphic to a complex contact infranilmanifold $\mathcal{L}/\Gamma$ or a complex contact infra-Hopf manifold $\mathcal{L} - \{0\}/\Gamma$ which is finitely covered by a Hopf manifold $S^{4n+1} \times S^1$. Here Γ is a discrete cocompact virtually nilpotent subgroup in $\mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot S^1)$ or isomorphic to the product of a cyclic group with an infinite cyclic subgroup $\mathbb{Z}_p \times \mathbb{Z}^+$ of $\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$.*

Proposition B. *The following hypothesis* (i) *or* (ii) *yields the same conclusion of Theorem A.*

(i) The holonomy group is virtually nilpotent.

(ii) The holonomy group is discrete.

In Chapter 4, we can perform a connected sum of our complex contact infranilmanifolds $\mathcal{L}/\Gamma$ $(\Gamma \leq \mathrm{E}(\mathcal{L}))$.

Theorem C. *The connected sum* $\mathbb{CP}^{2n+1}\#\mathcal{L}/\Gamma$ *admits a complex contact structure.*

By iteration of this procedure there exists a complex contact structure on the connected sum of a finite number of complex contact similarity manifolds and $\mathbb{CP}^{2n+1}$'s.

Corollary D. *Any connected sum* $M_1\#\cdots\#M_k\#\ell\mathbb{CP}^{2n+1}$ *admits a complex contact structure for a finite number of complex contact similarity manifolds* $M_1,\ldots,M_k$ *and* ℓ*-copies of* $\mathbb{CP}^{2n+1}$.

These examples are different from those admitting S^2 (resp. T^2)-fibrations.

In Chapter 5, we verify $\mathcal{L}$ from the viewpoint of geometric structure. In fact the sphere S^{4n+3} admits a canonical quaternionic CR-structure. The sphere S^{4n+3} with one point ∞ removed is isomorphic to the $4n+3$-dimensional quaternionic Heisenberg Lie group $\mathcal{M}$ as a quaternionic CR-structure. $\mathcal{M}$ has a central group extension : $1 \to \mathbb{R}^3 \to \mathcal{M} \xrightarrow{p} \mathbb{H}^n \to 1$ where $\mathbb{R}^3 = \mathrm{Im}\mathbb{H}$ is the imaginary part of the quaternion field $\mathbb{H}$. Taking a quotient of $\mathcal{M}$ by $\mathbb{R}\,(=\mathbb{R}\mathbf{i})$, we obtain a complex nilpotent Lie group $\mathcal{L}\,(=\mathcal{L}_{2n+1})$ which supports a holomorphic principal bundle $\mathbb{C} \to \mathcal{L} \xrightarrow{p} \mathbb{C}^{2n}$. The canonical quaternionic CR-structure on S^{4n+3} restricts a Carnot-Carathéodory structure B to $\mathcal{M}$. Using this bundle B, a left invariant complex contact structure on $\mathcal{L}$ is obtained (cf. Alekseevsky & Kamishima[1]; Kamishima[29]).

Results of Part I are from Kamishima & Tanaka[30].

Chapter 2

Similarity Manifolds

2.1 Developing Maps

Definition 2.1.1. Let X be a connected smooth manifold, and suppose a group G acts on X via diffeomorphisms transitively. Then a manifold M satisfying following conditions is called a (G, X)*-manifold*.

1. There exists an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of M and a family of maps

$$\phi_\alpha : U_\alpha \to V_\alpha$$

   each of which is a diffeomorphism onto its image $V_\alpha := \phi_\alpha(U_\alpha)$ in X.

2. If $U_\alpha \cap U_\beta \neq \emptyset$, then there exists an element $g \in G$ s.t.

$$g = \phi_\alpha \circ \phi_\beta^{-1}, \text{ on } V_\alpha \cap V_\beta.$$

The second condition of Definition 2.1.1 says that, if we think of G as diffeomorphisms of X, each transition functions

$$\gamma_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : V_\alpha \cap V_\beta \to V_\alpha \cap V_\beta$$

is the restriction of an element of G to $V_\alpha \cap V_\beta$.

Example. If G is the group of isometries of Euclidean space $\mathbb{E}^n$, then $(G, \mathbb{E}^n)$-manifold is called a Euclidean, or flat, manifold. If G is the group of affine transformations of $\mathbb{R}^n$, then $(G, \mathbb{R}^n)$-manifold is called an affine manifold.

Take a (G, X)-manifold M. Let (U_α, ϕ_α) and (U_β, ϕ_β) be two charts of M and suppose $x \in U_\alpha \cap U_\beta$. By the definition of (G, X)-manifold, on $U_\alpha \cap U_\beta$, ϕ_α and ϕ_β differ by $\gamma_{\alpha\beta}$. Thus composing $\gamma_{\alpha\beta}$ to ϕ_β, we obtain ϕ_α extended to $U_\alpha \cup U_\beta$. Fix a family of charts of M and consider a path $\alpha(t), t \in [0, 1]$ on M s.t $\alpha(0) \in U_\alpha$. Then we go along α and modify the function at each intersection of charts. And we successively extend ϕ_α. But in general we can not extend ϕ_α to whole M in this way, since it is necessary to coincide ϕ_α and modified function when $\alpha(1)$ is again in U_α. One of sufficient conditions to extend to M is that M is simply connected. So we consider the universal covering of $\pi : \tilde{M} \to M$ and define a map on it as follows. First we think of $\tilde{M}$ as the space of homotopy classes of paths in M which start at the fixed basepoint in U_0 . We take a path α representing a point $[\alpha] \in \tilde{M}$ s.t. $\alpha(1) = \pi([\alpha])$. Then we extend ϕ_0 along α and obtain

$$\psi = \gamma_{01}\gamma_{12}\cdots\gamma_{n-1,n}\phi_n.$$

It can be shown that ψ depends on only the first choice of charts and homotopy class of α. So we set $\phi^{[\alpha]} = \psi$.

Definition 2.1.2 (see Thurston[55])**.** Fix a basepoint and initial chart ϕ_α. The developing map of a (G, X)-manifold M is the map

$$\mathrm{dev} : \tilde{M} \to X$$

defined as $\mathrm{dev} = \phi^{[\alpha]} \circ \pi$ in a neighborhood of $[\alpha] \in \tilde{M}$.

Note that other choices of the basepoint and the initial chart cause the difference by composition in the range with an element of G.

We use dev to give the classification of complex contact similarity manifolds. For this purpose, we state the following proposition.

Proposition 2.1.3 (see Cheeger & Ebin[11])**.** *If $f : (M, g_M) \to (N, g_N)$ is a local isometry and M is complete, then $f : M \to N$ is a covering map.*

By this proposition, if we find a metric on the manifold which is domain of dev s.t. dev becomes a local isometry, then dev becomes a covering map.

2.2 Similarity Manifolds

2.2.1 Euclidean Similarity Manifolds

A Euclidean similarity manifold M is an affine manifold whose holonomy group lies in the group $\mathrm{Sim}(\mathbb{E}^n) := \mathbb{R}^n \times (\mathrm{O}(n) \times \mathbb{R}^+)$ of Euclidean similarity transformations of $\mathbb{E}^n$. A developing map determines similarities $\rho(g) \in \mathrm{Sim}(\mathbb{E}^n)$, where $g \in \pi_1(M)$, with the property that

$$\rho(g) \circ \mathrm{dev} = \mathrm{dev} \circ g : \tilde{M} \to \mathbb{E}.$$

So we have

$$\rho : \pi_1(M) \to \mathrm{Sim}(\mathbb{E}^n).$$

We call the pair (ρ, dev) developing pair. If $\rho(\pi_1(M))$ fixes a single point, then we say the similarity manifold is radiant. Fried[17] showed the following theorem.

Theorem 2.2.1 (Fried[17])**.** *A compact incomplete Euclidean similarity manifold is radiant.*

Choosing the fixed point as origin, radiant similarity structures are based on the group of linear similarities $\mathrm{Sim}_0(\mathbb{E}^n) := \mathrm{Sim}(\mathbb{E}^n) \cap \mathrm{GL}(n, \mathbb{R})$. Fried also showed geodesically complete similarity manifolds are Euclidean. So Theorem 2.2.1 shows that the structure group of a similarity manifold is always reduced to $\mathrm{Sim}_0(\mathbb{E}^n)$ or the Euclidean group $\mathrm{Euc}(\mathbb{E}^n)$. Thus we have Theorem 1.1.1. In this section, we explain Theorem 2.2.1 along Miner[41].

First, we need some preparation. Fix $\mathrm{dev} : \tilde{M} \to \mathbb{E}^n$. This defines the pull-back metric $\mathrm{dev}^* g_{\mathbb{E}}$ on $\tilde{M}$ and scale factors $\alpha : \mathrm{Sim}(\mathbb{E}) \to \mathbb{R}^+$ by $|gv| = \alpha(g)|v|$ for $g \in \mathrm{Sim}(\mathbb{E}), v \in \mathbb{E}$.

The exponential map for M is the affine map $\exp : \mathcal{E} \to M$ defined on an open subset $\mathcal{E} \subset TM$ s.t. for each $v \in \mathcal{E}$ the path $\gamma(t) = \exp(tv)$ for $0 \leq t \leq 1$ is the affinely parametrized geodesic on M with velocity vector v at $t = 0$. For each $m \in M$ we define its exponential domain $\mathcal{E}_m$ to be $\mathcal{E} \cap T_m M$.

We define

$$\mathrm{Exp} : T\mathbb{E}(= \mathbb{E}) \to \mathbb{E}$$

to be the exponential map for $T\mathbb{E}$ and the following diagram is commutative.

$$\begin{tikzcd}
T\tilde{M} \arrow[r, "d\,\mathrm{dev}"] \arrow[d, "\exp"'] & T\mathbb{E} \arrow[d, "\mathrm{Exp}"] \\
\tilde{M} \arrow[r, "\mathrm{dev}"'] & \mathbb{E}
\end{tikzcd}$$

Since M is locally modelled on $\mathbb{E}$, we have $D : T\tilde{M} \to \mathbb{E}$ s.t. $D|_{T_{\tilde{m}}\tilde{M}} = \mathrm{Exp} \circ d\,\mathrm{dev}$.

Proposition 2.2.2. *Let M be an affine manifold, g be an element in the group of affine transformations* $\mathrm{Aff}(\mathbb{E})$ *and $\mathcal{U}$ be a convex subset of* $\mathrm{dev}(\tilde{M})$. *If $\mathcal{U} \subset g(\mathrm{dev}(\tilde{M}))$ and $g(\mathcal{V}) \subset \mathcal{U}$, then $(D|_{T_{\tilde{m}}\tilde{M}})^{-1}(\mathcal{V}) \subset \mathcal{E}_{\tilde{m}}$.*

By the assumption that M is an incomplete similarity manifold, we define a function $R : \tilde{M} \to \mathbb{R}^+$ by $R(\tilde{m})$ to be the radius of the largest open ball in $T_{\tilde{m}}\tilde{M}$ on which exp is defined. More precisely,

$$R(\tilde{m}) = \sup\{r | \mathcal{E}_m \text{ contains a ball of radius } r \text{ with respect to } \mathrm{dev}^* g_{\mathbb{E}}\}.$$

We denote the ball as $D_{\tilde{m}}$.

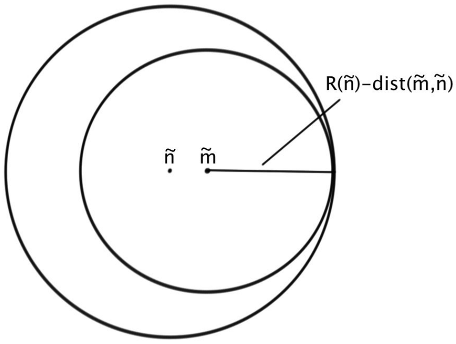

Fig. 2.1

Fig. 2.1 shows that $R(\tilde{m}) \geq R(\tilde{n}) - \mathrm{dist}(\tilde{m}, \tilde{n})$. Thus we have the following.

Lemma 2.2.3. *R satisfies $|R(x) - R(Y)| \leq dist(x, y)$. Namely R is Lipschitz.*

So we have the following lemma.

Lemma 2.2.4. *Let $g \in \pi_1(M)$ and $\rho(g)$ is a similarity transformation. Then $D_g(\tilde{m}) = dg(D_{\tilde{m}})$ and $R(g\tilde{m}) = \alpha(\rho(g))R(\tilde{m})$.*

From this lemma,

$$g_{\tilde{M}}(\tilde{m}) := \mathrm{dev}^* \, g_{\mathbb{E}}(\tilde{m})/R^2(x)$$

defines a Riemannian metric $g_{\tilde{M}}$ on $\tilde{M}$ which is invariant under $\pi_1(M)$. This metric is conformal to the flat metric $dev^* g_{\mathbb{E}}$, and $D_{\tilde{m}}$ is the unit disk with respect to this metric. Since $g_{\tilde{M}}$ is $\pi_1(M)$-invariant, there exists a Riemannian metric g_M on M s.t.

$$\pi^* g_M = f_{\tilde{M}}.$$

Hence we have the following.

Proposition 2.2.5. *Let M be an incomplete similarity manifold. Then there exists a unique continuous conformal Riemannian metric for which the unit ball in $T_m M$ is the maximum ball contained in the exponential domain $\mathcal{E}_m$ for each $m \in M$*

For fixed $\tilde{m} \in \tilde{M}$, let $v_0 \in \partial D_{\tilde{m}}$ be a vector s.t. the affine geodesic $\tilde{\gamma} = \exp(tv_0)$ is defined for $0 \leq t < 1$ but not for $t = 1$. By the definition, $\tilde{\gamma}(t)$ does not converge as $t \to 1$ although it does in the developing image, $\mathrm{dev}(\tilde{\gamma}(t)) \to \mathrm{Exp}(d\,\mathrm{dev}(v_0))$.

Since M is compact, $\gamma(t) := \pi \circ \tilde{\gamma}$ has a point of accumulation $y \in M$ and there exists an ϵ-ball B centers at y which $\gamma(t)$ must exits every time it enters. Pick a sequence of times;

$$0 < t_0 < t_1 < \cdots < t_n < \cdots$$

with $t_n \nearrow 1$ s.t. $\gamma(t_i)$ is in B but $\gamma([t_i, t_{i+1}])$ is not in B. Namely at each interval, $\gamma(t)$ exits B. For each i, let η_i be the radial geodesic from y to $\gamma(t_i)$ contained in B. Then we can define elements $g_{ij} \in \pi_1(M, y)$ as follows. Starting at y and we go along η_i, $\gamma|_{[t_i,t_j]}$, and η_j^{-1}. (See Fig. 2.2.)

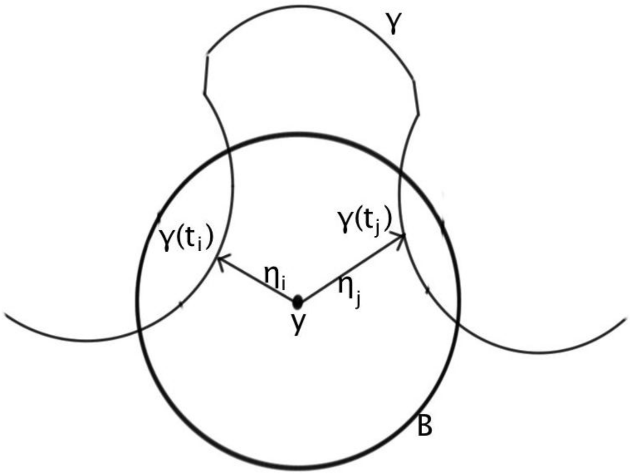

Fig. 2.2

Thus we obtain a family $\{g_{ij}\}$ in $\pi_1(M)$. Now we consider the actions of these elements in the developing image.

Let g be an element of $\mathrm{Sim}(\mathcal{L})$, then the action is written as

$$g(x) = c + \alpha A(x - c)$$

where $c \in \mathbb{R}^n, \alpha \in \mathbb{R}^+$ and $A \in \mathrm{O}(n)$. If $g(X) = Y$, then c can be written as

$$c = (I - \alpha A)^{-1}(Y - \alpha A(X)).$$

When $X = \mathrm{dev}(\tilde{\gamma}(t_i)), Y = \mathrm{dev}(\tilde{\gamma}(t_j)), A = I$ and $\alpha = (1 - t_j)/(1 - t_i)$, it can be shown that for $j >> i >> 0$, the above equation reduces to $c = v_0$. On the other hand, since $\mathrm{O}(n)$ is compact, the matrices $\{A_{ij}\}$ must accumulate. And $\rho(g_{jk}) \circ \rho(g_{ij}) = \rho(g_{ik})$ implies that $A_{jk}A_{ij} = A_{ik}$. Hence $|A_{ij} - I| < \epsilon$ for infinitely many $\{i, j\}$. So we have the following lemma.

Lemma 2.2.6. *For sufficiently large i and j, we can choose $\rho(g_{ij})$ to be a similarity centered arbitrarily close to v_0 with almost no rotation.*

From Lemma 2.2.4, there exists a half space neighborhood of the origin in each tangent space on which the exponential map is defined. Let

$$H_{\tilde{m}} = \{\tilde{w} \in T_{\tilde{m}}\tilde{M} \mid \tilde{\mu}(\tilde{w}, \tilde{v}_0) \leq 1\}.$$

Decompose the boundary of $H_{\tilde{m}}$ into visible set

$$J_{\tilde{m}} = \{\tilde{w} \in \partial H_{\tilde{m}} \mid \exp_{\tilde{m}}(\tilde{w}) \text{ is defined}\}$$

and invisible set

$$I_{\tilde{m}} = \{\tilde{w} \in \partial H_{\tilde{m}} \mid \exp_{\tilde{m}}(\tilde{w}) \text{ is undefined}\}.$$

As mentioned above, we can choose $\rho(g_{ij})$ close to a homothety centered at v_0. The set $D_{ij} := \rho(g_{ij})^{-1}(\rho(g_{ij})(\mathrm{D}(D_{\tilde{m}})) \cap \mathrm{D}(D_{\tilde{m}}))$ satisfies the hypotheses of Proposition 2.2.2. $\mathrm{D}^{-1}(D_{ij})$ can be made to approximate $D_{\tilde{m}}$ arbitrarily well. Thus for sufficiently large i, j, k, $\mathrm{D}(H_{\tilde{m}})$ is in $\rho(g_{ij})^{-k}(D_{ij})$ and apply Proposition 2.2.2, we have $H_{\tilde{m}} \subset \mathcal{E}_{\tilde{m}}$.

Lemma 2.2.7. *$I_{\tilde{m}}$ is an affine subspace of $\partial H_{\tilde{m}}$.*

Proof. Suppose there exists an $\vec{n} \in J_{\tilde{m}}$, and let $\tilde{n} = \exp \vec{n}$. Then $\tilde{n}$ has a neighborhood $H_{\tilde{n}}$ on which $\exp_{\tilde{n}}$ is defined. We claim that $\mathrm{D}(v_0)$ is in $\partial(\mathrm{D}(H_{\tilde{n}}))$. If $\mathrm{D}(v_0) \notin \partial(\mathrm{D}(H_{\tilde{n}}))$, consider the action of $\rho(g_{ij})^{-1}$. This takes $\mathrm{D}(H_{\tilde{n}})$ into itself, and so by Proposition 2.2.2, $\exp_{\tilde{n}}$ is defined on $\rho(g_{ij})(H_{\tilde{n}})$. But the $\partial(\mathrm{D}(H_{\tilde{n}}))$ is contained in $\rho(g_{ij})(H_{\tilde{n}})$ which is contradiction.

For any point in $I_{\tilde{m}}$, there exists an associated family of contraction $\{\rho(\gamma_{ij})\}$ and we can use the same argument as above. Thus we have

$$I_{\tilde{m}} \subset \partial(\mathrm{D}(H_{\tilde{m}})) \cap \partial(\mathrm{D}(H_{\tilde{n}}))$$

This is an affine subspace of dimension $\dim(\partial H_{\tilde{m}}) - 1$. If there are no visible points in this intersection, it is in fact $I_{\tilde{m}}$ and the lemma is proved. Otherwise, repeat the process to show $I_{\tilde{m}}$ is contained in an affine subspace of $\dim(\partial H_{\tilde{m}})-2$. Since this procedure must terminate, the lemma is proved.

□

It can be shown that $I := \mathrm{D}(I_{\tilde{m}})$ is locally constant, so $I \notin \mathrm{dev}(\tilde{M})$. Define the vector field $\tilde{X}$ on $\tilde{M}$ by specifying its value at each point to be the shortest vector for which exp is undefined. By Lemma2.2.4, this vector field is $\pi_1(M)$ invariant and thus defines a vector field X on M. X is seen to correspond to the vector field Y on $\mathbb{E}^d$ which assigns to $x \in \mathbb{E}^d$ the vector $Y(x)$ from x to I which is perpendicular to I. Let $\tilde{\omega}$ be the standard volume form on $\tilde{M}$ and $R(p)^{-1}\tilde{\omega}$ is $\pi_1(M)$ invariant. Thus we can obtain ω which is a volume form on M. Computation shows $\mathrm{div}_\omega(X) = dim(I)$. By Green's theorem, we have $\dim(I) = 0$ and so X is a radiant vector field. This proves Theorem 2.2.1.

Take a closed connected euclidean similarity manifold M, As a consequence of Theorem 2.2.1, Fried gave the following classification.

Theorem 2.2.8 (Fried[17]). *If M is complete, then M is Euclidean. Otherwise M is radiant.*

2.2.2 Heisenberg Similarity Manifolds

Let $\mathcal{H}$ be the Lie group $\mathbb{C}^n \times \mathbb{R}$ with group law:

$$(\zeta, v)(\xi, w) = (\zeta + \xi, v + w + Im\langle \zeta, \xi \rangle).$$

Here $\langle \zeta, \xi \rangle = \sum_{i=1}^{n} \zeta_i \bar{\xi}_i$. And $\alpha \cdot A \in \mathrm{U}(n) \times \mathbb{R}^+$ acts on $\mathcal{H}$ as:

$$\alpha \cdot A(\zeta, v) = (\alpha A \zeta, \alpha^2 v),$$

where $\alpha \in \mathbb{R}^+$ and $A \in U(n)$. We denote the Heisenberg similarity group $\mathcal{H} \times (\mathrm{U}(n) \times \mathbb{R}^+)$ as $\mathrm{Sim}(\mathcal{H})$.

CR-structure is a real codimension 1 subbundle of the tangent bundle with an integrable complex structure on it. The canonical CR-structure on $\partial B^n \subset \mathbb{C}^n$ is given at each point by taking the maximal complex subspace $J(T_x \partial B^n) \cap T_x \partial B^n$ of $T\partial B^n$. Since we can think of the boundary of complex hyperbolic space as the one point compactification of the Heisenberg group, we can think of $\mathcal{H}$ as a subset of ∂B^n. So $\mathcal{H}$ inherits this CR-structure.

A (real) contact structure on a $2n-1$-dimensional manifold X is a 1-form ω s.t. $\omega \wedge (d\omega)^{n-1} \neq 0$. Let ω be a contact form on ∂B^n and set $E = \{x \in T(\partial B^n) \mid \omega(x) = 0\}$. If E is the subbundle giving the CR-structure on ∂B^n, then we say ω is a calibration for the CR-structure. We call $\mathrm{Ker}\, \omega|_x$ the contact plane at x which defines a notion of horizontal at a point. We say a vector field X is CR-horizontal if $\omega(X) = 0$. In Heisenberg coordinates, the contact plane at the origin corresponds to $\{(\zeta, v) \in \mathcal{H} \mid v = 0\}$.

We explain the following generalization of Theorem 2.2.1.

Theorem 2.2.9 (Miner[41])**.** *A compact incomplete Heisenberg similarity manifold is radiant.*

We can pull back a metric $d_{\mathcal{H}}$ on $\mathcal{H}$ to $T_{\tilde{m}}\tilde{M}$ via the map D and denote it as $d_{\mathcal{H}}$ too. Define $R_{\mathcal{H}}(\tilde{m})$ to be the radius of the largest Heisenberg ball $D_{\tilde{m}} \subset T_{\tilde{m}}\tilde{M}$ centered at the origin on which the exponential map is defined. And let $\alpha_{\mathcal{H}} : \mathrm{Sim}(\mathcal{H}) \to \mathbb{R}^+$ be a homomorphism defined by $|g(\zeta, v)|_{\mathcal{H}} = \alpha_{\mathcal{H}}(g)|(\zeta, v)|_{\mathcal{H}}$for $g \in \mathrm{Sim}(\mathcal{H})$. Then we can check that Lemma 2.2.3, Lemma 2.2.4 and Proposition 2.2.6 still hold in the Heisenberg case.

Lemma 2.2.10. *$R_{\mathcal{H}}$ is Lipschitz with respect to $d_{\mathcal{H}}$. $D_g(\tilde{m}) = dg(D_{\tilde{m}})$ and $R_{\mathcal{H}}(g\tilde{m}) = \alpha_{\mathcal{H}}(\rho(g))R(\tilde{m})$ for $g \in \mathrm{Aut}(\tilde{M})$.*

Fix a point $\tilde{m} \in \tilde{M}$ and $(\zeta_0, v_0) \in \partial D_{\tilde{m}}$ s.t. $\bar{\gamma}(t) = \exp_{\tilde{m}}(t\zeta_0, tv_0)$ is incomplete. In the same way that we did in the Euclidean case, we can choose a family $\{g_{ij}\}$ in $\pi_1(M)$ associated to any incomplete geodesic which has a point of accumulation in M. Then we have the following lemma.

Lemma 2.2.11. *For sufficiently large i and j, we can choose $\rho(g_{ij})$ to be a Heisenberg similarity centered arbitrarily close to $D(\zeta_0, v_0)$ with almost no rotation.*

By this lemma and Proposition 2.2.2, we can expand the region $D_{\tilde{m}} \subset T_{\tilde{m}}\tilde{M}$ to a half space $H_{\tilde{m}}$ on which exp is defined. Recall that $\partial H_{\tilde{m}}$ is divided into sets $J_{\tilde{m}}$ and $I_{\tilde{m}}$ of visible and invisible points, and $I_{\tilde{m}}$ is the intersection of planes $\partial H_{\tilde{n}}$ as $\tilde{n}$ ranges over various points in $J_{\tilde{m}}$. It follows that $I_{\tilde{m}}$ is an affine subspace of $\partial H_{\tilde{m}}$. Let I be the constant image $\operatorname{dev}(I_{\tilde{m}})$ as $\tilde{m}$ ranges over $\tilde{M}$. Miner showed the following lemma.

Lemma 2.2.12 (Miner[41]). *I is CR-horizontal.*

So suppose I is CR-horizontal and assume the developing map was chosen so that I passes through the origin without loss of generality. It can be checked that the maximal CR-horizontal subspace passing through the origin in Heisenberg space is conjugate to the n dimensional subspace $\mathbb{E}$ of real points of $\mathbb{C}^n$ thinking of $\mathcal{H}$ as $\mathbb{C}^n \times \mathbb{R}$.

Since $\rho(\pi_1(M))$ must stabilize I, $\rho(\pi_1(M))$ lies in the subgroup of $\operatorname{Sim}(\mathcal{H})$ which stabilizes the real points. The subgroup of $\mathrm{U}(n)$ which stabilizes the real points is isomorphic to $\mathrm{O}(n)$. The Heisenberg translations stabilizing the real points are those of the form $(\zeta, 0)$ where ζ is a real vector. The Heisenberg dilation stabilizing the real points are those centers lie in the set of real points. Thus the action of $\operatorname{Stab}(\mathbb{E}, \operatorname{Sim}(\mathcal{H}))$ agrees with the action of $\operatorname{Sim}(\mathbb{R}^{2n-1})$ on $\mathbb{E}$.

Define

$$h(x) = x|x|$$

and

$$\phi(\mathbf{z}, v) = (\mathbf{z}, h^{-1}(v + \mathbf{x} \cdot \mathbf{y}))$$

where $x \in \mathbb{R}$ and $\mathbf{z} = \mathbf{x} + i\mathbf{y}$. Then $\operatorname{Stab}(\mathbb{E}, \operatorname{Sim}(\mathcal{H}))$ is conjugate to $\operatorname{Stab}(\mathbb{E}, \mathbb{R}^{2n-1})$ by ϕ. Let $H(\zeta, v)$ be the Heisenberg translation by (ζ, v), then for a real vector $\mathbf{r}$, we have

$$\phi \circ H(\mathbf{r}, 0) \circ \phi^{-1}(\mathbf{z}, v) = (\mathbf{z} + \mathbf{r}, v).$$

Similarly, let D_α be the Heisenberg dilation by α centered at the origin, then

$$\phi \circ D_\alpha \circ \phi^{-1}(\mathbf{z}, v) = (\alpha \mathbf{z}, \alpha v).$$

If $A \in \mathrm{U}(n)$, then we have

$$\phi \circ A \circ \phi^{-1}(\zeta, v) = (A\zeta, v).$$

Thus we have

$$\phi \circ \operatorname{Stab}(\mathbb{E}) \circ \phi^{-1} \subset \operatorname{Sim}(\mathbb{R}^{2n-1}).$$

So $\phi \circ \operatorname{dev}$ gives M a Euclidean similarity structure:

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{\operatorname{dev}} & \mathcal{H} & \xrightarrow{\phi} & \mathbb{R}^{2n+1} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \rho(g)} & & \big\downarrow{\scriptstyle \phi\circ\rho(g)\circ\phi^{-1}} \\ \tilde{M} & \xrightarrow{\operatorname{dev}} & \mathcal{H} & \xrightarrow{\phi} & \mathbb{R}^{2n+1} \end{array}$$

Since $\phi(I) = I$, this structure is also incomplete. By applying Theorem 2.2.8, we have Theorem 2.2.9 and as a consequence of Theorem 2.2.9, Theorem 1.1.2 follows.

Chapter 3

Complex Contact Structure on the Nilpotent Group

3.1 Definition of Complex Contact Structure

Recall that a complex contact structure on a complex manifold M in complex dimension $2n+1$ is a collection of local forms $\{U_\alpha, \omega_\alpha\}_{\alpha\in\Lambda}$ which satisfies that (1) $\bigcup_{\alpha\in\Lambda} U_\alpha = M$. (2) Each ω_α is a holomorphic 1-form defined on U_α. Then $\omega_\alpha \wedge (d\omega_\alpha)^n \neq 0$ on U_α. (3) If $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a nonzero holomorphic function $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ such that $f_{\alpha\beta} \cdot \omega_\alpha = \omega_\beta$. Unlike contact structures on orientable smooth manifolds, it does not always exist a holomorphic 1-form globally defined on M. Note that if the first Chern class $c_1(M)$ vanishes, then there is a global existence of a complex contact form ω on M. (See Kobayashi[32]; Lebrun[35])

Let $h : M \to M$ be a biholomorphism. Suppose that $h(U_\alpha) \cap U_\beta \neq \emptyset$ for some $\alpha, \beta \in \Lambda$. If there exists a holomorphic function $f_{\alpha\beta}$ on an open subset in U_α such that $h^*\omega_\beta = f_{\alpha\beta}\omega_\alpha$, then we call h a *complex contact transformation* of M. Denote $\mathrm{Aut}_{cc}(M)$ the group of complex contact transformations. It is not necessarily a finite dimensional complex Lie group.

3.2 The Iwasawa Lie Group

Let $\mathcal{L}_{2n+1}$ be the product $\mathbb{C}^{2n+1} = \mathbb{C} \times \mathbb{C}^{2n}$ with group law ($n \geq 1$):

$$(x, z) \cdot (y, w) = (x + y + \sum_{i=1}^{n} z_{2i-1}w_{2i} - z_{2i}w_{2i-1}, z + w)$$
$$\text{where } z = (z_1, \ldots, z_{2n}), w = (w_1, \ldots, w_{2n}).$$

Put $\mathcal{L} = \mathcal{L}_{2n+1}$. It is easy to see that $[(x,z),(y,w)] = (2\sum_{i=1}^{n} z_{2i-1}w_{2i} - z_{2i}w_{2i-1}, 0)$ so $[\mathcal{L},\mathcal{L}] = (\mathbb{C},(0,\dots,0)) = \mathbb{C}$ is the center of $\mathcal{L}$. Thus there is a central group extension: $1 \to \mathbb{C} \to \mathcal{L}_{2n+1} \longrightarrow \mathbb{C}^{2n} \to 1$. It is easy to check that $\mathcal{L}_3$ is isomorphic to the Iwasawa group consisting of 3×3-upper triangular unipotent complex matrices by the following correspondence.

$$(c,(a,b)) \mapsto \begin{pmatrix} 1 & a & \frac{ab+c}{2} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

Definition 3.2.1. A complex $2n+1$-dimensional complex nilpotent Lie group $\mathcal{L}_{2n+1}$ is said to be the Iwasawa Lie group.

See Foreman[16], pp.193-195 for more general construction of this kind of Lie group.

Now we construct a complex contact structure on $\mathcal{L}_{2n+1}$. Choose a coordinate $(z_0, z_1, \dots, z_{2n}) \in \mathcal{L}_{2n+1}$, we define a complex 1-form η:

$$\begin{aligned} \eta =& dz_0 - (\sum_{i=1}^{n} z_{2i-1}\cdot dz_{2i} - z_{2i}\cdot dz_{2i-1}) \\ =& \, dz_0 - (z_1,\dots,z_{2n})\, \mathsf{J}_n \begin{pmatrix} dz_1 \\ \vdots \\ dz_{2n} \end{pmatrix} \end{aligned}$$

where $\mathsf{J}_n = \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix}$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Since $\eta \wedge (d\eta)^n$ is a non-vanishing form $2n(-2)^n dz_0 \wedge \cdots \wedge dz_{2n}$ on $\mathcal{L}_{2n+1}$, η is a complex contact structure on $\mathcal{L}_{2n+1}$ by Definition 2.1.1.

3.3 Complex Contact Transformations

Let $\mathrm{hol}(\mathcal{L}_{2n+1})$ be the group of biholomorphic transformations of $\mathcal{L} = \mathcal{L}_{2n+1}$. The group of complex contact transformations on $\mathcal{L}$ with respect to η is denoted by

$$\mathrm{hol}(\mathcal{L},\eta) = \{f \in \mathrm{hol}(\mathcal{L}) \mid f^*\eta = \tau\cdot\eta\}$$

where τ is a holomorphic function on $\mathcal{L}$.

Let $\mathrm{Sp}(n,\mathbb{C}) = \{A \in M(2n,\mathbb{C}) \mid {}^t A \mathsf{J}_n A = \mathsf{J}_n\}$ be the complex symplectic group. As $\mathrm{Sp}(n,\mathbb{C}) \cap \mathbb{C}^* = \{\pm 1\}$, denote $\mathrm{Sp}(n,\mathbb{C})\cdot\mathbb{C}^* = \mathrm{Sp}(n,\mathbb{C}) \times \mathbb{C}^*/\{\pm 1\}$. Put

$$\mathrm{A}(\mathcal{L}) = \mathcal{L} \rtimes (\mathrm{Sp}(n,\mathbb{C})\cdot\mathbb{C}^*)$$

which forms a group as follows; write elements $\lambda\cdot A, \mu\cdot B \in \mathrm{Sp}(n,\mathbb{C})\cdot\mathbb{C}^*$ for $A, B\in \mathrm{Sp}(n,\mathbb{C}), \lambda,\mu\in\mathbb{C}^*$. Let $(a,w),(b,z)\in\mathcal{L}$. Define

$$\begin{aligned}&\big((a,w),\lambda\cdot A\big)\cdot\big((b,z),\mu\cdot B\big)\\ &=\big((a+\lambda^2 b+{}^t w\,\mathsf{J}_n(\lambda Az), w+\lambda Az),\lambda\mu\cdot AB\big).\end{aligned}$$

Here ${}^t w\,\mathsf{J}_n(\lambda Az)=\sum_{i=1}^{n} w_{2i-1}\cdot(\lambda Az)_{2i}-w_{2i}\cdot(\lambda Az)_{2i-1}$ as before.

Let $\big((a,w),\lambda\cdot A\big)\in \mathrm{A}(\mathcal{L}), (z_0,z)\in\mathcal{L}$. $\mathrm{A}(\mathcal{L})$ acts on $\mathcal{L}$ as

$$\begin{aligned}\big((a,w),\lambda\cdot A\big)\cdot(z_0,z)&=(a,w)\cdot(\lambda^2 z_0,\lambda Az)\\ &=(a+\lambda^2 z_0+{}^t w\,\mathsf{J}_n(\lambda Az),\ w+\lambda Az).\end{aligned}\tag{3.1}$$

If $h=((b,w),\mu\cdot B)\in\mathrm{A}(\mathcal{L})$ is an element, then it is easy to see that

$$h^*\eta=\mu^2\cdot\eta.\tag{3.2}$$

Thus $\mathrm{A}(\mathcal{L})$ preserves the complex contact structure on $\mathcal{L}$ defined by η.

Let $\mathrm{Aff}(\mathbb{C}^{2n+1})=\mathbb{C}^{2n+1}\rtimes\mathrm{GL}(2n+1,\mathbb{C})$ be the complex affine group which is a subgroup of $\mathrm{hol}(\mathcal{L})$ since $\mathcal{L}_{2n+1}=\mathbb{C}^{n+1}$ (biholomorphically). We assign to each $\big((a,w),\lambda\cdot A\big)\in\mathrm{A}(\mathcal{L})$ an element

$$\left(\begin{bmatrix} a\\ w\end{bmatrix},\left(\begin{array}{c|c}\lambda^2 & \lambda\,{}^t w\,\mathsf{J}_n A\\ \hline 0 & \lambda A\end{array}\right)\right)\in\mathrm{Aff}(\mathbb{C}^{2n+1}).$$

Then the action (3.1) of $\big((a,w),\lambda\cdot A\big)$ on $\mathcal{L}$ coincides with the above affine transformation of $\mathbb{C}^{2n+1}$. Moreover, it is easy to check that this correspondence is an injective homomorphism:

$$\mathrm{A}(\mathcal{L})\le\mathrm{Aff}(\mathbb{C}^{2n+1}).\tag{3.3}$$

As a consequence it follows

$$\mathrm{A}(\mathcal{L})\le\mathrm{hol}(\mathcal{L},\eta).$$

Let M be a smooth manifold. Suppose that there exists a maximal collection of charts $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in\Lambda}$ whose coordinate changes belong to $\mathrm{A}(\mathcal{L})$. More precisely, $M=\bigcup_{\alpha\in\Lambda}U_\alpha$, $\varphi_\alpha: U_\alpha\to\mathcal{L}$ is a diffeomorphism onto its image. If $U_\alpha\cap U_\beta\neq\emptyset$, then there exists a unique element $g_{\alpha\beta}\in\mathrm{A}(\mathcal{L})$ such that $g_{\alpha\beta}=\varphi_\beta\cdot\varphi_\alpha^{-1}$ on $\varphi_\alpha(U_\alpha\cap U_\beta)$. We say that M is locally modelled on $(\mathrm{A}(\mathcal{L}),\mathcal{L})$. (Compare Kulkarni[34].)

Here is a sufficient condition for the existence on complex contact structure.

Proposition 3.3.1. *If a $(4n+2)$-dimensional smooth manifold M is locally modelled on $(\mathrm{A}(\mathcal{L}),\mathcal{L})$, then M is a complex contact manifold. Moreover, M is also a complex affinely flat manifold.*

Proof. First of all, we define a complex structure on M. Let J_0 be the standard complex structure on $\mathcal{L} = \mathbb{C}^{2n+1}$. Define a complex structure J_α on U_α by setting $\varphi_{\alpha *}J_\alpha = J_0\varphi_{\alpha *}$ on U_α for each $\alpha \in \Lambda$. When $g_{\alpha\beta} \in \mathrm{A}(\mathcal{L})$, note that $g_{\alpha\beta *}J_0 = J_0 g_{\alpha\beta *}$ from (3.3). On $U_\alpha \cap U_\beta$, a calculation shows that $\varphi_{\beta *}J_\alpha = g_{\alpha\beta *}\varphi_{\alpha *}J_\alpha = J_0\varphi_{\beta *}$. Since $\varphi_{\beta *}J_\beta = J_0\varphi_{\beta *}$ by the definition, it follows $J_\alpha = J_\beta$ on $U_\alpha \cap U_\beta$. This defines a complex structure J on M. In particular, each $\varphi_\alpha : (U_\alpha, J) \to (\mathcal{L}, J_0)\,(= \mathbb{C}^{2n+1})$ is a holomorphic embedding. Let η be the holomorphic 1-form on $\mathcal{L}$ as before. Define a family of local holomorphic 1-forms $\{\omega_\alpha, U_\alpha\}_{\alpha\in\Lambda}$ by

$$\omega_\alpha = \varphi_\alpha^*\eta \text{ on } U_\alpha.$$

If $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a unique element $g_{\alpha\beta} \in \mathrm{A}(\mathcal{L})$ such that $g_{\alpha\beta} = \varphi_\beta \cdot \varphi_\alpha^{-1}$. From (3.2), $g_{\alpha\beta}^*\eta = \mu_{\alpha\beta}^2 \cdot \eta$ for some $\mu_{\alpha\beta} \in \mathbb{C}^*$. It follows $\omega_\beta = \mu_{\alpha\beta}^2 \cdot \omega_\alpha$. Thus the family $\{\omega_\alpha, U_\alpha\}_{\alpha\in\Lambda}$ is a complex contact structure on (M, J).

Apart from the complex contact structure, since $\mathrm{A}(\mathcal{L}) \leq \mathrm{Aff}(\mathbb{C}^{2n+1})$ from (3.3), M is also modelled on $(\mathrm{Aff}(\mathbb{C}^{2n+1}), \mathbb{C}^{2n+1})$ where $\mathcal{L} = \mathbb{C}^{2n+1}$. M is a complex affinely flat manifold. □

Remark 3.3.2.

1. *When a subgroup $\Gamma \leq \mathrm{A}(\mathcal{L})$ acts properly discontinuously and freely on a domain Ω of $\mathcal{L}$ with compact quotient, we obtain a compact complex contact manifold Ω/Γ by this proposition. In fact let $p : \Omega \to \Omega/\Gamma$ be a covering holomorphic projection. Take a set of evenly covered neighborhoods $\{U_\alpha\}_{\alpha\in\Lambda}$ of Ω/Γ. Choose a family of open subsets $\tilde{U}_\alpha$ such that $p_\alpha = p_{|\tilde{U}_\alpha} : \tilde{U}_\alpha \to U_\alpha$ is a biholomorphism. Put $\omega_\alpha = (p_\alpha^{-1})^*\eta$. Then the family $\{U_\alpha, \omega_\alpha\}_{\alpha\in\Lambda}$ is a complex contact structure on Ω/Γ.*

2. *When $\Omega = \mathcal{L}$, $\mathcal{L}/\Gamma$ is said to be a compact complete affinely flat manifold. Concerning the Auslandr-Milnor conjecture, we do not know whether the fundamental group Γ is virtually polycyclic.*

When M is a complex manifold, we assume that the complex structure on M coincides with the one constructed in Proposition 3.3.1.

3.4 Complex Contact Similarity Geometry

It is in general difficult to find such a properly discontinuous group Γ as in Remark 3.3.2. $\mathrm{Sp}(n, \mathbb{C})$ contains a maximal compact symplectic subgroup $\mathrm{Sp}(n) = \{A \in \mathrm{U}(2n) \mid {}^t A \mathrm{J}_n A = \mathrm{J}_n\}$ where $\mathrm{Sp}(n, \mathbb{C}) \cong \mathrm{Sp}(n) \times \mathbb{R}^{n(2n+1)}$.

Definition 3.4.1. Put $\mathrm{Sim}(\mathcal{L}) = \mathcal{L} \rtimes (\mathrm{Sp}(n)\cdot\mathbb{C}^*) \leq \mathrm{A}(\mathcal{L})$. The pair $(\mathrm{Sim}(\mathcal{L}), \mathcal{L})$ is called complex contact similarity geometry. If a manifold M is locally modelled on this geometry, M is said to be a complex contact similarity manifold. The euclidean subgroup of $\mathrm{Sim}(\mathcal{L})$ is defined to be $\mathrm{E}(\mathcal{L}) = \mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot S^1)$.

For example, choose $c \in \mathbb{C}^*$ with $|c| \neq 1$ and $A \in \mathrm{Sp}(n)$. Put $r = \big((0,0), c \cdot A\big) \in \mathrm{Sim}(\mathcal{L})$. Let $\mathbb{Z}^+$ be an infinite cyclic group generated by r. Then it is easy to see that $\mathbb{Z}^+$ acts freely and properly discontinuously on the complement $\mathcal{L} - \{0\}$. Here $0 = (0,0) \in \mathcal{L}_{2n+1} = \mathcal{L}$. The quotient $\mathcal{L} - \{0\}/\mathbb{Z}^+$ is diffeomorphic to $S^1 \times S^{4n+1}$. By Proposition 3.3.1 (1 of Remark 3.3.2), $S^1 \times S^{4n+1}$ is a complex contact similarity manifold.

Let $\mathbb{H}^n$ be the $4n$-dimensional quaternionic vector space. The quaternionic similarity group $\mathrm{Sim}(\mathbb{H}^n) = \mathbb{H}^n \rtimes ((\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)) \times \mathbb{R}^+)$ (resp. quaternionic euclidean group $\mathrm{E}(\mathbb{H}^n) = \mathbb{H}^n \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)))$ has a special subgroup $\widehat{\mathrm{Sim}}(\mathbb{H}^n) = \mathbb{H}^n \rtimes ((\mathrm{Sp}(n) \cdot S^1) \times \mathbb{R}^+)$ (resp. $\widehat{\mathrm{E}}(\mathbb{H}^n) = \mathbb{H}^n \rtimes (\mathrm{Sp}(n) \cdot S^1))$. When we identify $\mathbb{H}^n$ with the complex vector space $\mathbb{C}^{2n}$ by the correspondence $(a + b\mathbf{j}) \mapsto (\bar{a}, b)$, $\widehat{\mathrm{Sim}}(\mathbb{H}^n)$ is canonically isomorphic to the complex similarity subgroup $\mathbb{C}^{2n} \rtimes (\mathrm{Sp}(n) \cdot \mathbb{C}^*)$ where $\mathbb{C}^* = S^1 \times \mathbb{R}^+$. Then there are commutative exact sequences:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{Sim}(\mathcal{L}) & \xrightarrow{\mathsf{p}} & \widehat{\mathrm{Sim}}(\mathbb{H}^n) & \longrightarrow & 1 \\
 & & || & & \cup & & \cup & & \\
1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(\mathcal{L}) & \xrightarrow{\mathsf{p}} & \widehat{\mathrm{E}}(\mathbb{H}^n) & \longrightarrow & 1.
\end{array}
$$

Choosing a torsionfree discrete cocompact subgroup Γ from $\mathrm{E}(\mathcal{L})$, we obtain an infranilmanifold $\mathcal{L}/\Gamma$ of complex dimension $2n+1$. In particular, $\Gamma \cap \mathcal{L}$ is discrete uniform in $\mathcal{L}$ by the Auslander-Bieberbach theorem. As $\mathbb{C}$ is the central subgroup of $\mathcal{L}$, $\Gamma \cap \mathbb{C}$ is discrete uniform in $\mathbb{C}$ and so $\Delta = \mathsf{p}(\Gamma)$ is a discrete uniform subgroup in $\widehat{\mathrm{E}}(\mathbb{H}^n)$. We obtain a Seifert singular fibration over a quaternionic euclidean orbifold $\mathbb{H}^n/\Delta$: $T^1_{\mathbb{C}} \to \mathcal{L}/\Gamma \longrightarrow \mathbb{H}^n/\Delta$. By 1 of Remark 3.3.2, $\mathcal{L}/\Gamma$ is a complex contact manifold.

Remark 3.4.2. *When we take a finite index nilpotent subgroup Γ' of Γ admitting a central extension : $1 \to \mathbb{Z}^2 \to \Gamma' \longrightarrow \mathbb{Z}^{4n} \to 1$, a nilmanifold $\mathcal{L}/\Gamma'$ admits a holomorphic principal $T^1_{\mathbb{C}}$-bundle over a complex torus $T^{2n}_{\mathbb{C}} = \mathbb{H}^n/\mathbb{Z}^{4n}$. This holomorphic example is a special case of Foreman's T^2- connection bundle over $T^{2n}_{\mathbb{C}}$ (Foreman[16]).*

We give rise to a classification of compact complex contact similarity manifolds under the existence of S^1-actions. (Compare Fried[17]; Miner[41] for the related results of similarity manifolds.) Recall that $\mathrm{Aut}_{cc}(M)$ is the group of complex contact transformations defined in section 3.1.

Theorem 3.4.3. *Let M be a compact complex contact similarity manifold of complex dimension $2n+1$. If $S^1 \leq \mathrm{Aut}_{cc}(M)$ acts on M without fixed points, then M is holomorphically isomorphic to a complex contact infranilmanifold $\mathcal{L}/\Gamma$ or a complex contact infra-Hopf manifold $\mathcal{L}-\{0\}/\Gamma$ which is finitely covered by a Hopf manifold $S^{4n+1}\times S^1$. Here Γ is a discrete cocompact virtually nilpotent subgroup in $\mathcal{L}\rtimes(\mathrm{Sp}(n)\cdot S^1)$ or isomorphic to the product of a cyclic group with an infinite cyclic subgroup $\mathbb{Z}_p\times\mathbb{Z}^+$ of $\mathrm{Sp}(n)\cdot S^1\times\mathbb{R}^+$.*

3.5 Proof of the Main Theorem

Let J be a complex structure on M. Given a collection of charts $\{U_\alpha,\varphi_\alpha,J_\alpha\}$ on M with $J_\alpha = J_{|U_\alpha}$ such that $\varphi_\alpha : (U_\alpha,J_\alpha)\to(\mathcal{L},J_0)$ is a holomorphic diffeomorphism onto its image. Recall that the monodromy argument shows that there is a developing pair:

$$(\rho,\mathrm{dev}) : (\mathrm{Aut}_{cc}(\tilde{M}),\tilde{M})\to(\mathrm{Sim}(\mathcal{L}),\mathcal{L})$$

where $\tilde{M}$ is the universal covering and $\tilde{J}$ is a lift of J to $\tilde{M}$, and $\pi=\pi_1(M)\leq \mathrm{Aut}_{cc}(\tilde{M})$. Let J be a complex structure on $\tilde{M}$. Then dev is a holomorphic immersion $\mathrm{dev}_*J = J_0\mathrm{dev}_*$ and $\rho:\mathrm{Aut}_{cc}(\tilde{M})\to\mathrm{Sim}(\mathcal{L})$ is a holonomony homomorphism. Put $\Gamma=\rho(\pi)$. Let $\tilde{S}^1$ be a lift of S^1 to $\tilde{M}$ so that $\rho(\tilde{S}^1)\leq \mathrm{Sim}(\mathcal{L})$.

Case 1. If $\Gamma\leq \mathrm{E}(\mathcal{L})$, then there is a $\mathrm{E}(\mathcal{L})$-invariant Riemannian metric on $\mathcal{L}$. As M is compact, the pullback metric on $\tilde{M}$ by dev is (geodesically) complete, $\mathrm{dev}:\tilde{M}\to\mathcal{L}$ is an isometry. As dev becomes a complex contact diffemorphism, M is holomorphically isomorphic to a complex contact infranilmanifold $\mathcal{L}/\Gamma$.

Case 2. Suppose that some $\rho(\gamma)$ has a nontrivial summand in $\mathbb{R}^+\leq\mathcal{L}\rtimes(\mathrm{Sp}(n)\cdot S^1\times\mathbb{R}^+)=\mathrm{Sim}(\mathcal{L})$. In view of the affine representation $\rho(\gamma)=(p,P)$ where $P=\left(\begin{array}{c|c}\lambda^2 & \lambda^t w\,\mathsf{J}_n A\\ \hline 0 & \lambda A\end{array}\right)$ from (3.3), we note $|\lambda|\neq 1$, i.e. P has no eigenvalue 1. Then there exists an element $z_0\in\mathcal{L}$ such that the conjugate $(z_0,I)\rho(\gamma)(-z_0,I)=(0,P)$. We may assume that $\rho(\gamma)=(0,P)\in\mathrm{Aff}(\mathcal{L})$ from the beginning. As $\rho(\tilde{S}^1)$ centralizes Γ, if $\rho(t)=(q,Q)\in\rho(\tilde{S}^1)$, then the equation $\rho(t)\rho(\gamma)=\rho(\gamma)\rho(t)$ implies that $Pq=q$ and so $q=0$. Thus $\rho(t)=(0,Q)=\big((0,0),\mu_t\cdot B_t\big)\in\mathrm{Sp}(n)\cdot S^1\times\mathbb{R}^+\leq\mathrm{Sim}(\mathcal{L})$. It follows $\rho(\tilde{S}^1)\leq\mathrm{Sp}(n)\cdot S^1\times\mathbb{R}^+$. In particular, $\rho(\tilde{S}^1)$ has a non-empty fixed point set $\mathcal{S}$ in $\mathcal{L}$. If $\mathrm{dev}(x)\in\mathcal{S}$, then $\mathrm{dev}(\tilde{S}^1x)=\rho(\tilde{S}^1)\,\mathrm{dev}(x)=x$. Since dev is an immersion, $\tilde{S}^1x=x$. As S^1 has no fixed points on M, it is noted that $\mathrm{dev}(\tilde{M})\subset\mathcal{L}-\mathcal{S}$. Let $\mathrm{Sim}(\mathcal{L}-\mathcal{S})$ be the subgroup of $\mathrm{Sim}(\mathcal{L})$ whose elements leave $\mathcal{S}$ invariant. Note that $\Gamma\leq\mathrm{Sim}(\mathcal{L}-\mathcal{S})$.

We determine $\mathcal{S}$ and $\mathrm{Sim}(\mathcal{L}-\mathcal{S})$. Since $\rho(\tilde{S}^1)$ belongs to the maximal abelian group $T^{2n}\cdot S^1\times\mathbb{R}^+$ up to conjugate in $\mathrm{Sp}(n)\cdot S^1\times\mathbb{R}^+$, we can put $\langle\lambda_t\rangle\leq S^1\times\mathbb{R}^+$,

$\langle s_t \rangle = S^1$ and

$$\rho(\tilde{S}^1) = \{\big((0,0), \mu_t \cdot B_t)\big) = \Big(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Big(\begin{array}{c|c} \mu_t^2 & 0 \\ \hline 0 & \mu_t B_t \end{array} \Big) \Big) \}$$

$$B_t = \begin{pmatrix} s_t & & & & & \\ & \ddots & & & & \\ & & s_t & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in T^{2n} \leq \mathrm{Sp}(n)$$

where $\mathrm{Sp}(n) \leq \mathrm{U}(2n)$ is canonically embedded so that $2k$-numbers of s_t's and 2ℓ-numbers of 1's. Recall that $\rho(\tilde{S}^1)$ acts on $\mathcal{L}$ by $\rho(t)(z_0, z) = (\mu_t^2 z_0, \mu_t B_t z)$.

Case I. $\mu_t \neq 1$. Suppose that $\mu_t \lambda_t = 1$. Then $\mathcal{S} = \mathrm{Fix}(\rho(\tilde{S}^1), \mathcal{L}) = \{(0, (z, 0)) \in \mathcal{L} \,|\, z \in \mathbb{C}^{2k}\}$ $(0 \leq k \leq n)$. As the element $\big((a, w), \lambda \cdot A\big) \in \mathrm{Sim}(\mathcal{L})$ acts by $\big((a, w), \lambda \cdot A\big)(0, (z, 0)) = (a + \lambda^t w\, \mathsf{J}_n Az,\ w + \lambda Az) \in \mathcal{S}$ (cf. (3.1)), we can check that $a = 0$, $w \in \mathbb{C}^{2k}$ and so $\lambda Az \in \mathbb{C}^{2k}$. In particular, $A \in \mathrm{Sp}(k)$. From $w\, \mathsf{J}_n Az = 0$, it follows $w = 0$.

$$\mathrm{Sim}(\mathcal{L} - \mathcal{S}) = \{\big((0,0), \lambda \cdot A) \,|\, A \in \mathrm{Sp}(k)\} = \mathrm{Sp}(k) \cdot S^1 \times \mathbb{R}^+.$$

Case II. $\mu_t = 1$. Then $\mathcal{S} = \{(z_0, (0, z)) \in \mathcal{L} \,|\, z \in \mathbb{C}^{2\ell}\} = \mathcal{L}_{2\ell+1}$ $(0 \leq \ell \leq n-1)$. It follows as above

$$\begin{aligned} \mathrm{Sim}(\mathcal{L} - \mathcal{S}) &= \{\big((a, w), \lambda \cdot A) \,|\, w \in \mathbb{C}^{2\ell}, A \in \mathrm{Sp}(\ell)\} \\ &= \mathcal{L}_{2\ell+1} \rtimes (\mathrm{Sp}(\ell) \cdot S^1 \times \mathbb{R}^+) = \mathrm{Sim}(\mathcal{L}_{2\ell+1}). \end{aligned}$$

We need the following lemma.

Lemma 3.5.1. $\mathrm{Sim}(\mathcal{L} - \mathcal{S})$ *acts properly on* $\mathcal{L} - \mathcal{S}$.

Proof. **Case I.** There is an equivariant inclusion

$$(\mathrm{Sp}(k) \cdot S^1 \times \mathbb{R}^+, \mathcal{L} - \mathcal{S}) \subset (\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+, \mathcal{L} - \{0\}).$$

As there is an $\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$-invariant Riemannian metric on $\mathcal{L} - \{0\}$ and $\mathrm{Sp}(k) \cdot S^1 \times \mathbb{R}^+$ is a closed subgroup, it acts properly on $\mathcal{L} - \mathcal{S}$.
Case II. Let $G = \mathbb{C}^{2\ell} \rtimes (\mathrm{Sp}(\ell) \cdot S^1 \times \mathbb{R}^+)$ be the semidirect group which preserves the complement $\mathbb{C}^{2n} - \mathbb{C}^{2\ell}$. Then there is an equivariant principal bundle:

$$(\mathbb{C}, \mathbb{C}) \to (\mathrm{Sim}(\mathcal{L}_{2\ell+1}), \mathcal{L} - \mathcal{L}_{2\ell+1}) \longrightarrow (G, \mathbb{C}^{2n} - \mathbb{C}^{2\ell}). \tag{3.4}$$

We note that G acts properly on $\mathbb{C}^{2n} - \mathbb{C}^{2\ell}$. For this, we observe that

$$\mathbb{C}^{2n} - \mathbb{C}^{2\ell} = S^{4n} - S^{4\ell} = \mathbb{H}_{\mathbb{R}}^{4\ell+1} \times S^{4n-4\ell-1}$$

in which

$$G \leq \mathbb{R}^{4\ell} \rtimes (\mathrm{O}(4\ell) \times \mathbb{R}^+) = \mathrm{Sim}(\mathbb{R}^{4\ell}) \leq \mathrm{PO}(4\ell+1,1).$$

As $\mathrm{PO}(4\ell+1,1) \times \mathrm{O}(4n-4\ell) = \mathrm{Isom}(\mathbb{H}_{\mathbb{R}}^{4\ell+1} \times S^{4n-4\ell-1})$ and G is a closed subgroup of $\mathrm{PO}(4\ell+1,1)$, G acts properly on $\mathbb{C}^{2n} - \mathbb{C}^{2\ell}$.

Since $\mathbb{C}$ acts properly on $\mathcal{L} - \mathcal{L}_{2\ell+1}$, the above principal bundle (3.4) implies that $\mathrm{Sim}(\mathcal{L}_{2\ell+1})$ acts properly on $\mathcal{L} - \mathcal{L}_{2\ell+1}$. $\square$

We continue the proof of Theorem 3.4.3. For **Case I**, there is an $\mathrm{Sp}(k) \cdot S^1 \times \mathbb{R}^+$-invariant Riemannian metric on $\mathcal{L} - \mathcal{S}$. Put $H = \mathrm{Sp}(k) \cdot S^1 \times \mathbb{R}^+$. As $\mathcal{L} - \mathcal{S} = \mathbb{C}^{2n+1} - \mathbb{C}^{2k} = \mathbb{H}_{\mathbb{R}}^{4k+1} \times S^{4n-4k+1}$ where $H \leq \mathrm{Sim}(\mathbb{R}^{4k}) \leq \mathrm{PO}(4k+1,1)$, note that the quotient $\mathcal{L} - \mathcal{S}/H$ is a Hausdorff space. On the other hand,

$$\begin{aligned}
\mathcal{L} - \mathcal{S}/H &= \mathbb{H}_{\mathbb{R}}^{4k+1}/H \times S^{4n-4k+1} \\
&= \mathbb{R}^{4k} \rtimes \mathbb{R}^+/H \times S^{4n-4k+1} \\
&= \mathbb{R}^{4k}/(\mathrm{Sp}(k) \cdot S^1) \times S^{4n-4k+1}.
\end{aligned}$$

$\mathcal{L} - \mathcal{S}/H$ cannot be compact unless $k = 0$.

On the other hand, as M is compact and $\Gamma \leq \mathrm{Sim}(\mathcal{L} - \mathcal{S})$, using Lemma 3.5.1, $\mathrm{dev} : \tilde{M} \to \mathcal{L} - \mathcal{S}$ is a covering map. $\mathcal{L} - \mathcal{S}$ is simply connected unless $k = n$. Then $M \cong \mathcal{L} - \mathcal{S}/\Gamma$ is compact ($k \neq n$). If we consider the fiber space $\mathcal{L} - \mathcal{S}/\Gamma \to \mathcal{L} - \mathcal{S}/H$, $\mathcal{L} - \mathcal{S}/H$ must be compact, which cannot occur except for $k = 0$.

If $\mathcal{L} - \mathcal{S} = \mathbb{H}_{\mathbb{R}}^{4n+1} \times S^1$ ($k = n$), then there is a lift of dev, $\widetilde{\mathrm{dev}} : \tilde{M} \to \mathbb{H}_{\mathbb{R}}^{4n+1} \times \mathbb{R}$ which is a diffeomorphism. The group $\tilde{\Gamma} = \widetilde{\mathrm{dev}} \circ \pi \circ \widetilde{\mathrm{dev}}^{-1}$ acts properly discontinuously and freely on $\mathbb{H}_{\mathbb{R}}^{4n+1} \times \mathbb{R}$ such that $\mathbb{H}_{\mathbb{R}}^{4n+1} \times \mathbb{R}/\tilde{\Gamma}$ is compact. As there is the canonical projection:

$$\mathbb{H}_{\mathbb{R}}^{4n+1} \times \mathbb{R}/\tilde{\Gamma} \to \mathbb{H}_{\mathbb{R}}^{4n+1} \times S^1/H,$$

$\mathbb{H}_{\mathbb{R}}^{4n+1} \times S^1/H$ is compact. This case is also impossible.

For $k = 0$, $\mathcal{S} = \{0\}$, $\mathrm{dev} : \tilde{M} \to \mathcal{L} - \{0\}$ is a diffeomorphism. As $\Gamma \leq S^1 \times \mathbb{R}^+$ acting freely on $\mathcal{L} - \{0\} = \mathbb{H}_{\mathbb{R}}^1 \times S^{4n+1}$, M is biholomorphic to $\mathcal{L} - \{0\}/\Gamma$. On the other hand, Γ is discrete in $\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Sp}(n) \cdot S^1 & \longrightarrow & \mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+ & \longrightarrow & \mathbb{R}^+ & \longrightarrow & 1 \\
 & & \cup & & \cup & & \cup & & \\
1 & \longrightarrow & F & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}^+ & \longrightarrow & 1.
\end{array}$$

The image of Γ to $\mathbb{R}^+$ is nontrivial (and so an infinite cyclic subgroup $\mathbb{Z}^+$) because $\mathcal{L} - \{0\}/\Gamma$ is compact. Choosing an infinite cyclic subgroup $\mathbb{Z}^+$ from Γ which maps onto $\mathbb{Z}^+ \leq \mathbb{R}^+$, it follows that $\Gamma = F \rtimes \mathbb{Z}^+$. Consider further the exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Sp}(n) & \longrightarrow & \mathrm{Sp}(n) \cdot S^1 & \xrightarrow{\mathsf{p}} & S^1 & \longrightarrow & 1 \\
 & & \cup & & \cup & & \cup & & \\
1 & \longrightarrow & F \cap \mathrm{Sp}(n) & \longrightarrow & F & \longrightarrow & \mathsf{p}(F) & \longrightarrow & 1.
\end{array}$$

As $F \leq \Gamma$ acts freely on $\mathcal{L} - \{0\}$ in which $A \in \mathrm{Sp}(n)$ acts on $\mathcal{L} - \{0\}$ by $A(z_0, z) = (z_0, Az)$ it follows that $F \cap \mathrm{Sp}(n) = \{1\}$. In particular, $F \cong \mathsf{p}(F)$ is a finite cyclic subgroup of S^1. Put $F = \mathbb{Z}_p$ ($^\exists\, p \in \mathbb{Z}$). As a consequence,

$$\Gamma = \mathbb{Z}_p \times \mathbb{Z}^+ \leq T^{2n+1} \times \mathbb{R}^+$$

which acts on $\mathcal{L} - \{0\}$ by

$$\lambda(z_0, z_1, \ldots, z_{2n}) = (\lambda^2 z_0, \lambda z_1, \ldots, \lambda z_{2n}).$$

As the $\mathbb{Z}_p$-action preserves the standard sphere $S^{4n+1} \subset \mathcal{L} - \{0\} = S^{4n+1} \times \mathbb{R}^+$, the Γ-action on $\mathcal{L} - \{0\}$ induces a properly discontinuous action of $\mathbb{Z}^+$ on $S^{4n+1}/\mathbb{Z}_p \times \mathbb{R}^+$. (Here $S^{4n+1}/\mathbb{Z}_p$ is the lens space $L(p, 2, 1, \ldots, 1)$.) Hence $\mathcal{L} - \{0\}/\Gamma$ is biholomorphic to an infra-Hopf complex contact manifold $S^{4n+1}/\mathbb{Z}_p \times S^1$. In other words, M is finitely covered by a Hopf manifold $S^{4n+1} \times S^1$.

For **Case II**, $\mathcal{L} - \mathcal{L}_{2\ell+1}$ is always simply connected (cf. (3.4)). Then M is diffeomorphic to $\mathcal{L} - \mathcal{L}_{2\ell+1}/\Gamma$ so that $\Gamma \leq \mathrm{Sim}(\mathcal{L}_{2\ell+1}) = \mathcal{L}_{2\ell+1} \rtimes (\mathrm{Sp}(\ell) \cdot S^1 \times \mathbb{R}^+)$ is a discrete subgroup. As there is a fiber space

$$\mathrm{Sim}(\mathcal{L}_{2\ell+1})/\Gamma \to \mathcal{L} - \mathcal{L}_{2\ell+1}/\Gamma \longrightarrow \mathcal{L} - \mathcal{L}_{2\ell+1}/\mathrm{Sim}(\mathcal{L}_{2\ell+1}),$$

it follows that $\mathrm{Sim}(\mathcal{L}_{2\ell+1})/\Gamma$ is compact. Since $\mathcal{L}_{2\ell+1}$ is a maximal nilpotent subgroup of $\mathrm{Sim}(\mathcal{L}_{2\ell+1})$, $\mathcal{L}_{2\ell+1} \cap \Gamma$ is discrete uniform in $\mathcal{L}_{2\ell+1}$. As $\mathbb{R}^+$ acts on $\mathcal{L}$ as multiplication, Γ cannot have a nontrivial summand in $\mathbb{R}^+$. This contradicts the hypothesis of **Case 2**. So **Case II** does not occur. This proves the theorem.

3.6 Variations

Theorem 3.6.1. *Let M be a $4n+2$-dimensional compact complex contact similarity manifold. If the holonomy group is virtually nilpotent, then M is holomorphically diffeomorphic to a complex contact infranilmanifold $\mathcal{L}/\Gamma$ or a complex contact infra-Hopf manifold $S^{4n+1}/\mathbb{Z}_p \times S^1$.*

Another hypothesis yields the same conclusion of Theorem 3.6.1.

Corollary 3.6.2. *If the holonomy group is discrete, then the same conclusion holds for a $4n+2$-dimensional compact complex contact similarity manifold M.*

The following corollary was proved in section 3.5 but we prove along the scheme of Theorem 3.6.1.

Corollary 3.6.3. *M be a compact complex contact similarity manifold of complex dimension $2n+1$. If $S^1 \leq \mathrm{Aut}_{cc}(M)$ acts on M without fixed points, then M is holomorphically diffeomorphic to a complex contact infranilmanifold $\mathcal{L}/\Gamma$ or a complex contact infra-Hopf manifold $S^1 \times S^{4n+1}/\mathbb{Z}_p$.*

Here Γ is a discrete cocompact subgroup in $\mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot S^1)$ or $\mathbb{Z}^+$ is an infinite cyclic subgroup of $\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$.

3.6.1 Proof of Theorem 3.6.1

First, we review the settings. J is a complex structure on M and $\{U_\alpha, \varphi_\alpha, J_\alpha\}$ is a collection of charts on M with $J_\alpha = J_{|U_\alpha}$ s.t. $\varphi_\alpha : (U_\alpha, J_\alpha) \to (\mathcal{L}, J_0)$ is a holomorphic diffeomorphism onto its image. We have a developing pair:

$$(\rho, \mathrm{dev}) : (\mathrm{Aut}_{cc}(\tilde{M}), \tilde{M}, \tilde{J}) \to (\mathrm{Sim}(\mathcal{L}), \mathcal{L}, J_0)$$

where $\tilde{M}$ is the universal covering and $\tilde{J}$ is a lift of J to $\tilde{M}$. dev is a holomorphic immersion satisfying $\mathrm{dev}_* J = J_0 \mathrm{dev}_*$ and $\rho : \mathrm{Aut}_{cc}(\tilde{M}) \to \mathrm{Sim}(\mathcal{L})$ is a holonomy homomorphism. Since $\pi = \pi_1(M) \leq \mathrm{Aut}_{cc}(\tilde{M})$, the holonomy group $\Gamma = \rho(\pi)$ is virtually nilpotent by the hypothesis. Taking a finite index subgroup we may assume Γ is nilpotent.

Assertion (I). Suppose that some $\gamma = ((a, w), \lambda \cdot A) \in \Gamma$ has a nontrivial summand $\lambda\,(\neq 1)$ in $\mathbb{C}^* \leq \mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot \mathbb{C}^*) = \mathrm{Sim}(\mathcal{L})$. Then $\Gamma \leq \mathrm{Sp}(n) \cdot \mathbb{C}^*$.

Proof. As $\mathbb{C}^* = S^1 \times \mathbb{R}^+$ acts as multiplication on $\mathcal{L}$, it is noted that $[\mathcal{L}, \mathbb{R}^+] = \mathcal{L}$ (and also $[\mathcal{L}, S^1] = \mathcal{L}$). So if Γ has a nontrivial summand x in $\mathcal{L}$, then $(1 - \lambda \cdot A)x \in \Gamma_1 = [\Gamma, \Gamma]$ and so $(1 - \lambda \cdot A)^i x \in \Gamma_i = [\Gamma_{i-1}, \Gamma]$ (possibly $A = I$). So $\Gamma_i = [\Gamma_{i-1}, \Gamma]$ cannot be trivial for any $i > 0$. This is impossible because Γ is nilpotent. Thus Γ has no summand in $\mathcal{L}$ and hence

$$\Gamma \leq \mathrm{Sp}(n) \cdot \mathbb{C}^* \leq \mathrm{Sim}(\mathcal{L}).$$

In particular, Γ has the fixed point 0 at the origin $(0, 0) \in \mathcal{L}$. (In this case M is radiant.)

We can assume that there exists an element $\gamma = (k, t) \in \Gamma$ such that

$$k \in \mathrm{Sp}(n) \cdot S^1 \ \text{ and } \ t \in \mathbb{R}^+ \ \text{ with } \ t < 1. \tag{3.5}$$

□

Assertion (II). Under Assertion (I), dev misses $\{0\}$, i.e. $\{0\} \notin \mathrm{dev}(\tilde{M})$.

Proof. Assume $\{0\} \in \mathrm{dev}(\tilde{M})$.

Case 1. If the complement $\mathcal{L} - \mathrm{dev}(\tilde{M})$ is not empty, then it is Γ-invariant closed subset which satisfies that $(\mathcal{L} - \mathrm{dev}(\tilde{M})) \cap \{0\} = \emptyset$. Since both $\{0\}$ and $\mathcal{L} - \mathrm{dev}(\tilde{M})$ are Γ-invariant, it follows

$$\gamma^i(\mathcal{L} - \mathrm{dev}(\tilde{M})) \cap \{0\} = \emptyset.$$

Choose a point $p = (z_0, z) \in \mathcal{L} - \mathrm{dev}(\tilde{M})$. Then $\gamma^i p = (t^{4i} z_0, t^{2i} k^{2i} z)$ by the definition. It follows that $\lim_{i\to\infty} \gamma^i p = (0, 0) = 0 \in \mathcal{L}$ by (3.5). Since $\gamma^i p \in \mathcal{L} - \mathrm{dev}(\tilde{M})$ which is a closed subset, it follows $0 \in \mathcal{L} - \mathrm{dev}(\tilde{M})$. This yields a contradiction. Case 1 does not occur.

Case 2. Suppose $\mathcal{L} - \mathrm{dev}(\tilde{M}) = \emptyset$, i.e. dev is surjective. Recall that M is viewed as an affinely flat manifold; $(\rho, \mathrm{dev}) : (\pi, \tilde{M}) \to (\Gamma, \mathbb{C}^{2n+1})$ where

$$\Gamma = \{\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \left(\begin{array}{c|c} \lambda^2 & 0 \\ \hline 0 & \lambda A \end{array}\right)\right)\} \leq \mathrm{Aff}(\mathbb{C}^{2n+1}).$$

If we note that Γ is nilpotent, then it follows from Theorem A of Fried, Goldman & Hirsch[18] that M is complete, i.e. the developing map $\mathrm{dev} : \tilde{M} \to \mathbb{C}^{2n+1}$ is a diffeomorphism. Then $\Gamma \leq \mathrm{Sp}(n) \cdot \mathbb{C}^* \leq \mathrm{Sim}(\mathcal{L})$ is discrete and so a finite index subgroup of Γ is an infinite cyclic subgroup. Then $\mathbb{C}^{2n+1}/\Gamma$ cannot be compact and so Case 2 does not occur. As a consequence, combining Case 1 and Case 2 show that dev misses $\{0\}$. □

Assertion (III). Under Assertion (I), M is holomorphically isomorphic to $S^{4n+1} \times \mathbb{R}^+/\Gamma$ which is diffeomorphic to an infra-Hopf manifold $S^{4n+1}/\mathbb{Z}_p \times S^1$.

Proof. First note that the complement $\mathcal{L} - \{0\}$ admits a $\mathrm{Sp}(n) \cdot \mathbb{C}^*$-invariant Riemannian metric. In fact, $\mathbb{R}^+ \leq \mathbb{C}^*$ acts on $\mathcal{L} - \{0\}$ as

$$\lambda(z_0, z) = (\lambda^2 z_0, \lambda z)$$

so $\mathbb{R}^+$ acts properly. Since $\mathrm{Sp}(n) \cdot S^1$ is compact, $\mathrm{Sp}(n) \cdot \mathbb{C}^* = \mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$ acts properly on $\mathcal{L} - \{0\}$. Then there exists a $\mathrm{Sp}(n) \cdot \mathbb{C}^*$-invariant Riemannian metric on $\mathcal{L} - \{0\}$. (See Koszul[33] for example.) By Assertion (II) (under Assertion (I)), the developing image $\mathrm{dev}(\tilde{M})$ misses 0, i.e. $\mathrm{dev}(\tilde{M}) \subset \mathcal{L} - \{0\}$. If we take a pullback metric of this metric by the developing map, then $\mathrm{dev} : \tilde{M} \to \mathcal{L} - \{0\}$ is a local isometry.

Since this metric is Γ-invariant and dev is equivariant with respect to ρ, the pullback metric on $\tilde{M}$ is π-invariant and so it induces a Riemannian metric on the quotient M.
As M is compact, M is complete with respect to this metric so is $\tilde{M}$.
By Proposition 2.1.3, $(\rho, \mathrm{dev}) : (\pi, \tilde{M}) \to (\Gamma, \mathcal{L} - \{0\})$ becomes an equivariant covering map. Since $\mathcal{L} - \{0\} \cong S^{4n+1} \times \mathbb{R}^+$ is simply connected, $\mathrm{dev} : \tilde{M} \to \mathcal{L} - \{0\}$ is a diffeomorphism. Taking the quotient, M is holomorphically diffeomorphic to $\mathcal{L} - \{0\}/\Gamma$. As we saw in the proof of Theorem 3.4.3, we have

$$\Gamma = \mathbb{Z}_p \times \mathbb{Z}^+ \leq T^{2n+1} \times \mathbb{R}^+. \tag{3.6}$$

Hence $\mathcal{L} - \{0\}/\Gamma$ is biholomorphic to an infra-Hopf complex contact manifold $S^{4n+1}/\mathbb{Z}_p \times S^1$. □

There is the remaining case to Assertion (I).

Assertion (IV). Suppose $\Gamma \leq \mathcal{L} \rtimes \mathrm{Sp}(n) \cdot S^1 \leq \mathrm{Sim}(\mathcal{L})$, i.e. every γ has no summand in $\mathbb{R}^+$. Then M is holomorphically diffeomorphic to an infranilmanifold $\mathcal{L}/\Gamma$.

Proof. Since $\mathcal{L}\rtimes \mathrm{Sp}(n)\cdot S^1$ acts properly on $\mathcal{L}$, there is a $\mathcal{L}\rtimes \mathrm{Sp}(n)\cdot S^1$-invariant Riemannian metric on $\mathcal{L}$. Taking the pullback metric, it follows similarly as above that $(\rho, \mathrm{dev}) : (\pi, \tilde{M}) \to (\mathcal{L} \rtimes \mathrm{Sp}(n)\cdot S^1, \mathcal{L})$ becomes an equivariant isometry for which $\Gamma \leq \mathcal{L}\rtimes \mathrm{Sp}(n)\cdot S^1$. Hence M is holomorphically diffeomorphic to $\mathcal{L}/\Gamma$. The Auslander-Bieberbach theorem implies that $\mathcal{L}/\Gamma$ is finitely covered by a nilmanifold $\mathcal{L}/\Delta$ where $\Delta = \Gamma\cap\mathcal{L}$ is a finite index normal subgroup of Γ. Hence $\mathcal{L}/\Gamma$ is an infranilmanifold. $\square$

All together with Assertions (I),(II),(III),(IV), this finishes the proof of Theorem 3.6.1.

Proof of Corollary 3.6.2. Since $\mathrm{Sim}(\mathcal{L}) = \mathrm{Sp}(n)\cdot S^1\times\mathbb{R}^+$ is an amenable Lie group, any discrete subgroup is virtually polycyclic. (See Milnor[40].) We may assume that Γ is polycyclic. Consider the exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{L}\rtimes(\mathrm{Sp}(n)\cdot\mathbb{C}^*) & \xrightarrow{L} & \mathrm{Sp}(n)\cdot\mathbb{C}^* & \longrightarrow & 1\\
 & & \cup & & \cup & & \cup & & \\
1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \xrightarrow{L} & L(\Gamma) & \longrightarrow & 1
\end{array}$$

where we put $\Delta = \mathcal{L}\cap\Gamma$. As $L(\Gamma)$ is solvable, it follows that $L(\Gamma)\leq T^n\times S^1\times\mathbb{R}^+$. In particular, $[\Gamma,\Gamma]\leq\Delta$. Hence we can assume that Δ is nontrivial. Otherwise, $[\Gamma,\Gamma]=\{1\}$, Γ is abelian. So it follows from Theorem 3.6.1 that some finite cover of M is either a complex contact nilmanifold $\mathcal{L}/\Gamma$ or a complex contact Hopf manifold $S^{4n+1}\times S^1$. As Γ is abelian, some finite cover of M must be a complex contact Hopf manifold $S^{4n+1}\times S^1$.

Suppose that some $L(\gamma)$ has a nontrivial summand λ in $\mathbb{R}^+$. We may assume that $\lambda<1$. Put $\gamma = ((a,w),\lambda A)$ and $n=(b,z)\in\Delta$. In view of (3.3), we can check that

$$\begin{aligned}
\gamma^i n\gamma^{-i} &= (\lambda^{2i}b + 2\sum_{k=1}^{i}\lambda^{2i-k}\,({}^t wJ_nA^kz), \lambda^iA^iz)\\
&= (\lambda^i(\lambda^ib + 2\sum_{k=1}^{i}\lambda^{i-k}\,({}^t wJ_nA^kz)), \lambda^iA^iz).
\end{aligned}$$

As $\gamma^i n\gamma^{-i}\in\Delta$, it follows

$$\lim_{i\to\infty}\gamma^i n\gamma^{-i} = (0,0) = 1\in\Delta,$$

which contradicts that Δ is discrete.

Hence $L(\Gamma)\leq T^n\times S^1$ and so $\Gamma\leq\mathcal{L}\rtimes T^n\times S^1\leq\mathrm{Sp}(n)\cdot S^1$. As in the proof of Assertion (IV), M is holomorphically diffeomorphic to an infranilmanifold $\mathcal{L}/\Gamma$. This prove the corollary.

Proof of Corollary 3.6.3. Let $\tilde{M}$ be the universal covering of M endowed with a complex structure $\tilde{J}$ which is a lift of J on M. Put $\pi=\pi_1(M)\leq$

$\mathrm{Aut}_{cc}(\tilde{M})$. There is the developing pair:

$$(\rho, \mathrm{dev}) : (\pi, \tilde{M}, \tilde{J}) \to (\mathrm{Sim}(\mathcal{L}), \mathcal{L}, J_0).$$

Put $\Gamma = \rho(\pi)$. Let $\tilde{S}^1 \leq \mathrm{Aut}_{cc}(\tilde{M})$ be a lift of S^1 to $\tilde{M}$ so that $\rho(\tilde{S}^1) \leq \mathrm{Sim}(\mathcal{L})$.
Case (i). If every element of Γ has no summand in $\mathbb{R}^+ \leq \mathrm{Sim}(\mathcal{L}) = \mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+)$, i.e. $\Gamma \leq \mathrm{E}(\mathcal{L}) = \mathcal{L} \rtimes \mathrm{Sp}(n) \cdot S^1$, then there is a $\mathrm{E}(\mathcal{L})$-invariant Riemannian metric on $\mathcal{L}$. As M is compact, the pullback metric on $\tilde{M}$ by dev is (geodesically) complete, dev : $\tilde{M} \to \mathcal{L}$ is an isometry. As dev becomes a complex contact diffemorphism, M is holomorphically isomorphic to a complex contact infranilmanifold $\mathcal{L}/\Gamma$.

Case (ii). Suppose that some $\gamma \in \Gamma$ has a nontrivial summand in $\mathbb{R}^+ \leq \mathrm{Sim}(\mathcal{L})$. Write γ as the affine representation, $\rho(\gamma) = (p, P)$ where $P = \left(\begin{array}{c|c} \lambda^2 & \lambda^t w\, \mathsf{J}_n A \\ \hline 0 & \lambda A \end{array}\right)$ with $|\lambda| \neq 1$, i.e. P has no eigenvalue 1. Then there exists an element $z_0 \in \mathcal{L}$ such that the conjugate $(z_0, I)\rho(\gamma)(-z_0, I) = (0, P)$. We may assume that $\rho(\gamma) = (0, P) \in \mathrm{Aff}(\mathcal{L})$ up to conjugate. Let $(q, Q) \in \rho(\tilde{S}^1)$. As $\rho(\tilde{S}^1)$ centralizes Γ, the equation $(q, Q) \cdot \gamma = \gamma \cdot (q, Q)$ implies that $Pq = q$ and so $q = 0$. Put $\rho(\tilde{S}^1) = \{\rho(t)\}_{t \in \mathbb{R}}$. Then it is easy to see that

$$\rho(t) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \left(\begin{array}{c|c} \mu_t^2 & 0 \\ \hline 0 & \mu_t B_t \end{array}\right) \right) \in \mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$$

where $\mu_t \in S^1 \times \mathbb{R}^+$ and $B_t \in \Delta T^n \leq \mathrm{U}(2n)$. Here

$$\Delta T^n = \{(t_1, \bar{t}_1, t_2, \bar{t}_2, \ldots, t_n, \bar{t}_n) \in T^{2n}\}$$

is the maximal torus of $\mathrm{Sp}(n)$ embedded in $\mathrm{U}(2n)$.

Choose an arbitrary element

$$\gamma' = \left(\begin{bmatrix} b \\ z \end{bmatrix}, \left(\begin{array}{c|c} \nu^2 & \nu\, {}^t z\, \mathsf{J}_n C \\ \hline 0 & \nu C \end{array}\right) \right) \in \Gamma.$$

Again the equation $\rho(t) \cdot \gamma' = \gamma' \cdot \rho(t)$ implies the following equalities:

$$\begin{aligned} &\mu_t^2 b = b, \;\; \mu_t B_t z = z. \\ &\nu^2\, {}^t z\, \mathsf{J}_n C = \nu \mu_t\, {}^t z\, \mathsf{J}_n C B_t. \end{aligned} \tag{3.7}$$

The last equality is equivalent to

$$\nu \mu_t\, {}^t z\, \mathsf{J}_n C (B_t - \mu_t I) = 0. \tag{3.8}$$

As $\mu_t \in S^1 \times \mathbb{R}^+$, note that $\mu_t \neq 1$ for $t \neq 0$. The first equality shows

$$b = 0.$$

Note that $\nu \mu_t\, {}^t z\, \mathsf{J}_n C = 0$ implies $z = 0$. If $z \neq 0$ for some $\gamma' \in \Gamma$, then $\nu \mu_t\, {}^t z\, \mathsf{J}_n C \neq 0$ which shows by the above equality (3.8) that $B_t = \mu_t I$. As

$\mu_t B_t z = z$ from (3.7), it follows $\mu_t^2 z = z$ and so $\mu_t^2 = 1$. Then $\mu_t B_t = \mu_t^2 I = I$. Therefore

$$\rho(t) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \left(\begin{array}{c|c} \mu_t^2 & 0 \\ \hline 0 & \mu_t B_t \end{array} \right) \right) = (0, I),$$

which is impossible. Thus all the corresponding z's are 0 for every $\gamma' \in \Gamma$. Combing the fact $b = 0$ shows

$$\gamma' = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \left(\begin{array}{c|c} \nu^2 & 0 \\ \hline 0 & \nu C \end{array} \right) \right)$$

and hence

$$\Gamma \leq \mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+.$$

Since $\rho(\tilde{S}^1) \leq \mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$, $\rho(\tilde{S}^1)$ fixes the origin $0 \in \mathcal{L}$. It is easy to check that $\tilde{S}^1$ has no fixed point in $\tilde{M}$ by the hypothesis. As dev is an immersion, $\{0\}$ is outside the image of dev, i.e. $\{0\} \notin \mathrm{dev}(\tilde{M})$. As in the argument of Assertion (III), $\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$ acts properly on $\mathcal{L} - \{0\}$ so that dev : $\tilde{M} \to \mathcal{L} - \{0\}$ is a diffeomorpshim. The remaining proof is the same as that of Assertion (III). It follows from (3.6) that $\Gamma = \mathbb{Z}_p \times \mathbb{Z}^+ \leq T^{2n+1} \times \mathbb{R}^+$ and $\mathcal{L} - \{0\}/\Gamma$ is biholomorphic to an infra-Hopf complex contact manifold $S^{4n+1}/\mathbb{Z}_p \times S^1$. This prove the corollary.

Chapter 4

Connected Sum

4.1 Connected Sum

In Kobayashi[32], there is a complex contact structure on the complex projective space $\mathbb{CP}^{2n+1}$; let $\omega = \sum_{i=1}^{n+1}(z_{2i-1} \cdot dz_{2i} - z_{2i} \cdot dz_{2i-1})$ be a holomorphic 1-form on $\mathbb{C}^{2n+2}$. Put $U_i = \{[w_0, \ldots, w_{2n+1}] \,|\, w_i \neq 0\}$ which forms a cover $\{U_i\}$ of $\mathbb{CP}^{2n+1}$. If s_i is a holomorphic cross-section of the principal bundle $\mathbb{C}^* \to \mathbb{C}^{2n+2} - \{0\} \longrightarrow \mathbb{CP}^{2n+1}$ restricted to U_i, setting $\omega_i = s_i^* \omega$, $\{\omega_i\}$ defines a complex contact structure on $\mathbb{CP}^{2n+1}$. For example, let $\iota : U_0 \to \mathbb{C}^{2n+1}$ be the local coordinate system defined by $\iota([w_0, \ldots, w_{2n+1}]) = (z_0, \ldots, z_{2n})$ such that $w_{i+1}/w_0 = z_i$. A holomorphic map $s_0 : U_0 \to \mathbb{C}^{2n+2} - \{0\}$ may be defined as

$$s_0 \circ \iota^{-1}(z_0, \ldots, z_{2n}) = (1, z_0, -z_1, z_2, -z_3, z_4, \ldots, -z_{2n-1}, z_{2n}).$$

$$\begin{array}{ccccc} & & \mathbb{C}P^{2n+1} & & \mathcal{L} \cong \mathbb{C}^{2n+1} \\ & & \uparrow & & \uparrow \\ \mathbb{C}^{2n+2} - 0 & \xleftarrow{s_0} & U_0 & \xrightarrow{\iota} & \iota(U_0) \end{array}$$

Then the holomorphic 1-form $(s_0 \circ \iota^{-1})^* \omega$ on $\iota(U_0)$ is described as

$$(s_0 \circ \iota^{-1})^* \omega = dz_0 - \sum_{i=1}^{n}(z_{2i-1} \cdot dz_{2i} - z_{2i} \cdot dz_{2i-1}).$$

For this,

$$\begin{aligned} (s_0 \circ \iota^{-1})^* \omega &= (s_0 \circ \iota^{-1})^* \big((z_1 dz_2 - z_2 dz_1) + (z_3 dz_4 - z_4 dz_3) + \cdots \\ &\qquad \cdots + (z_{2n+1} dz_{2n+2} - z_{2n+2} dz_{2n+1})\big) \\ &= dz_0 - (z_1 dz_2 - z_2 dz_1) - \cdots - (z_{2n-1} \cdot dz_{2n} - z_{2n} dz_{2n-1}). \end{aligned}$$

So $(s_0 \circ \iota^{-1})^* \omega$ is equivalent with $\eta_{|\iota(U_0)}$ the complex contact 1-form on $\mathcal{L}$ defined in section 3.2.

Let $p : \mathcal{L} \to \mathcal{L}/\Gamma$ be the holomorphic covering map. Put $V_0 = p(\iota(U_0))$ and $p(0) = x$. Then the map $p \circ \iota : U_0 \to V_0$ is a biholomorphic diffeomorphism with $p \circ \iota([1, 0, \dots, 0]) = x$. Choose a neighborhood $U_0' \subset U_0$ such that $\iota(U_0')$ is a closed ball B at the origin in $\mathbb{C}^{2n+1}$. Put $p(B) = V_0' \subset V_0$. Summarizing this, we have the following.

$$
\begin{array}{ccccccc}
 & & \mathbb{C}P^{2n+1} & & \mathcal{L} \cong \mathbb{C}^{2n+1} & \xrightarrow{p} & \mathcal{L}/\Gamma \\
 & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{C}^{2n+2} - 0 & \xleftarrow{s_0} & U_0 & \xrightarrow{\iota} & \iota(U_0) & \longrightarrow & V_0
\end{array}
$$

Then a connected sum $\mathbb{CP}^{2n+1} \# \mathcal{L}/\Gamma$ is obtained by glueing $\mathbb{CP}^{2n+1} - \mathrm{int} U_0'$ and $\mathcal{L}/\Gamma - \mathrm{int} V_0'$ along the boundaries $\partial U_0'$ and $\partial V_0'$ by $p \circ \iota$.

Theorem 4.1.1. *The connected sum $\mathbb{CP}^{2n+1} \# \mathcal{L}/\Gamma$ admits a complex contact structure.*

Proof. As above $(s_0 \circ \iota^{-1})^* \omega = \eta$ on $\iota(U_0)$. Note that $\omega_0 = s_0^* \omega = \iota^* \eta$ on U_0. On the other hand, the complex contact structure $\{\eta_i\}$ on $\mathcal{L}/\Gamma$ satisfies that $p^* \eta_0 = \eta$ on $\iota(U_0)$. The holomorphic map $p \circ \iota : U_0 \to V_0$ satisfies that $(p \circ \iota)^* \eta_0 = \omega_0$. Since $J(p \circ \iota)_* = (p \circ \iota)_* J$ on U_0, the complex structure J is naturally extended to a complex structure on $\mathbb{CP}^{2n+1} \# \mathcal{L}/\Gamma$ along the boundary $\partial U_0'$. $\square$

Since any complex contact similarity manifold M is locally modelled on $(\mathrm{Sim}(\mathcal{L}), \mathcal{L})$ by the definition, every point of M has a neighborhood U on which the complex contact structure is equivalent to a restriction of $(\eta, \mathcal{L})$. Similarly to the above proof, we have

Corollary 4.1.2. *Any connected sum $M_1 \# \cdots \# M_k \# \ell \mathbb{CP}^{2n+1}$ admits a complex contact structure for a finite number of complex contact similarity manifolds $M_1, \dots, M_k$ and ℓ-copies of $\mathbb{CP}^{2n+1}$.*

Chapter 5

Contact Structure from Quaternionic Heisenberg Lie Group

5.1 Quaternionic Heisenberg Geometry

Denote $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ which is the imaginary part of the quaternion field $\mathbb{H}$. $\mathcal{M}$ is the product $\mathbb{R}^3 \times \mathbb{H}^n$ with group law:

$$(\alpha, u) \cdot (\beta, v) = (\alpha + \beta + \operatorname{Im}\langle u, v\rangle, u + v).$$

Here $\langle u, v\rangle = {}^t\bar{u} \cdot v = \sum_{i=1}^{n} \bar{u}_i v_i$ is the Hermitian inner product where $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$ is the quaternion conjugate. $\mathcal{M}$ is nilpotent because $[\mathcal{M}, \mathcal{M}] = \mathbb{R}^3$ which is the center consisting of the form $((a, b, c), 0)$ $(a, b, c \in \mathbb{R})$. $\mathcal{M}$ is called *quaternionic Heisenberg Lie group.* The similarity subgroup $\operatorname{Sim}(\mathcal{M})$ is defined to be the semidirect product $\mathcal{M} \rtimes (\operatorname{Sp}(n) \cdot \operatorname{Sp}(1) \times \mathbb{R}^+)$. The action of $\operatorname{Sim}(\mathcal{M})$ on $\mathcal{M}$ is given as follows; for $h = \big((\alpha, u), (A \cdot g, t)\big) \in \mathcal{M} \rtimes (\operatorname{Sp}(n) \cdot \operatorname{Sp}(1) \times \mathbb{R}^+)$, $(\beta, v) \in \mathcal{M}$,

$$h \circ (\beta, v) = (\alpha + t^2 g\beta g^{-1} + \operatorname{Im}\langle u, tAvg^{-1}\rangle,\ u + t \cdot Avg^{-1}).$$

The pair $(\operatorname{Sim}(\mathcal{M}), \mathcal{M})$ is called *quaternionic Heisenberg geometry.*

Let $u_i = z_i + w_i\mathbf{j} \in \mathbb{H}$ $(z_i, w_i \in \mathbb{C})$. It is easy to check that the correspondence $\mathsf{p} : \mathcal{M} \to \mathcal{L}$ defined by

$$\big(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, (u_1, \ldots, u_n)\big) \mapsto \big(b + c\mathbf{i}, (\bar{z}_1, w_1, \bar{z}_2, w_2, \ldots, \bar{z}_n, w_n)\big) \tag{5.1}$$

is a Lie group homomorphism. Let $\widehat{\operatorname{Sim}}(\mathcal{M}) = \mathcal{M} \rtimes (\operatorname{Sp}(n) \cdot S^1 \times \mathbb{R}^+)$ be the subgroup of $\operatorname{Sim}(\mathcal{M})$. Then $\mathsf{p} : \mathcal{M} \to \mathcal{L}$ induces a homomorphism $\mathsf{q} : \widehat{\operatorname{Sim}}(\mathcal{M}) \to \operatorname{Sim}(\mathcal{L})$ for which $(\mathsf{q}, \mathsf{p}) : (\widehat{\operatorname{Sim}}(\mathcal{M}), \mathcal{M}) \to (\operatorname{Sim}(\mathcal{L}), \mathcal{L})$ is equivariant.

Take the coordinates $(a, b, c) \in \mathbb{R}^3, u = (u_1, \ldots, u_n) \in \mathbb{H}^n$. Define a $\operatorname{Im}\mathbb{H}$-valued 1-form on $\mathcal{M}$ to be

$$\omega = da\mathbf{i} + db\mathbf{j} + dc\mathbf{k} - \operatorname{Im}\langle u, du\rangle. \tag{5.2}$$

We may put

$$\omega = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k} \tag{5.3}$$

for some real 1-forms $\omega_1, \omega_2, \omega_3$ on $\mathcal{M}$. Noting (5.1), $\mathsf{p}^*\eta \cdot \mathsf{j}$ is a $\mathbb{C}\mathsf{j}\,(\leq \mathbb{H})$-valued 1-form on $\mathcal{M}$. A calculation shows that

$$\omega - \mathsf{p}^*\eta \cdot \mathsf{j} = da\mathsf{i} + \sum_{i=1}^{n}(\bar{z}_i dz_i - z_i d\bar{z}_i + w_i d\bar{w}_i - \bar{w}_i dw_i)$$

which is an $\mathbb{R}\mathsf{i}$-valued 1-form. Then we have from (5.3) that

$$\omega - \mathsf{p}^*\eta \cdot \mathsf{j} = \omega_1 \cdot \mathsf{i}.$$

In particular when $\mathsf{p}_* : T\mathcal{M} \to T\mathcal{L}$ is the differential map, this equality shows

$$\mathsf{p}_*(\operatorname{Ker}\omega) = \operatorname{Ker}\eta. \tag{5.4}$$

5.2 Quaternionic Carnot-Carathéodory Structure on $\mathcal{M}$

Let $\nu : \mathcal{M} \to \mathbb{H}^n$ be the projection defined by $\nu((a, b, c), u) = u$. Then it is easy to check that $\nu_* : \operatorname{Ker}\omega \to T\mathbb{H}^n$ is an isomorphism at each point. By the pull-back of this isomorphism, the standard quaternionic structure $\{J_1, J_2, J_3\}$ on $\mathbb{H}^n$ induces an almost quaternionic structure on $\operatorname{Ker}\omega$. (We write it as $\{J_1, J_2, J_3\}$ also.) As $[\operatorname{Ker}\omega, \operatorname{Ker}\omega] = \mathbb{R}^3$, $(\operatorname{Ker}\omega, \{J_\alpha\}_{\alpha=1,2,3})$ is said to be *quaternionic Carnot-Carathéodory* structure on $\mathcal{M}^{4n+3}$ (cf. Alekseevsky & Kamishima[1]).

Set $u_i = z_i + w_i\mathbf{j} = x_i + y_i\mathbf{i} + (p_i + q_i\mathbf{i})\mathbf{j}$ so that

$$g = |du|^2 = \sum_{i=1}^{n}(dx_i^2 + dy_i^2 + dp_i^2 + dq_i^2)$$

is the standard positive definite symmetric bilinear form on $\operatorname{Ker}\omega$. Since $d\omega = -d\bar{u}\wedge du = d\omega_1\mathbf{i} + d\omega_2\mathbf{j} + d\omega_3\mathbf{k}$ from (5.2), (5.3), a reciprocity of the quaternionic structure shows that

$$\begin{aligned} d\omega_1(J_1X, Y) = d\omega_2(J_2X, Y) = d\omega_3(J_3X, Y) = -g(X, Y). \\ (\forall\, X, Y \in \operatorname{Ker}\omega). \end{aligned} \tag{5.5}$$

Let J_0 be the complex structure on $\mathcal{L}$ and $\mu : \mathcal{L} \to \mathbb{C}^{2n}$ the canonical projection. Since η is a holomorphic 1-form, $\mu_* : (\operatorname{Ker}\eta, J_0) \to (T\mathbb{C}^{2n}, J_0)$ is an equivariant isomorphism. If $\mathsf{q} : \mathbb{H}^n \to \mathbb{C}^{2n}$ is an isomorphism defined by $\mathsf{q}(u_1, \ldots, u_n) = (\bar{z}_1, w_1, \ldots, \bar{z}_n, w_n)$, then there is the commutative diagram:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\nu} & \mathbb{H}^n \\ {\scriptstyle \mathsf{p}}\Big\downarrow & & {\scriptstyle \mathsf{q}}\Big\downarrow \\ \mathcal{L} & \xrightarrow{\mu} & \mathbb{C}^{2n}, \end{array} \tag{5.6}$$

By the definition of J_1, $\mathsf{q}_* \circ J_1 = J_0 \circ \mathsf{q}_*$ on $T\mathbb{H}^n$.

Note that $\operatorname{Ker}\omega_1 = \operatorname{Ker}\omega \oplus \langle \frac{d}{db}, \frac{d}{dc} \rangle$ with $\omega_1(\frac{d}{da}) = 1$ and $T\mathcal{L} = \operatorname{Ker}\eta \oplus \langle \frac{d}{db}, \frac{d}{dc} \rangle$. Since $\mathsf{p}_* \langle \frac{d}{db}, \frac{d}{dc} \rangle = \langle \frac{d}{db}, \frac{d}{dc} \rangle$ (cf. (5.1)) and by (5.4), $\mathsf{p}_* : \operatorname{Ker}\omega_1 \to T\mathcal{L}$ is an isomorphism.

5.3 Complex Contact Bundle on $\mathcal{L}$

As $\mathbb{R}^3$ acts as translations on $\mathcal{M}$, $\mathbb{R}^3$ leaves ω (resp. ω_i $(i = 1, 2, 3)$) invariant. $\mathbb{R}^3$ induces the distribution of vector fields $\langle \frac{d}{da}, \frac{d}{db}, \frac{d}{dc} \rangle$ on $\mathcal{M}$. Define an almost complex structure $\bar{J}_1$ on $\operatorname{Ker}\omega_1$ as

$$\bar{J}_1 | \operatorname{Ker}\omega_1 = J_1, \ \bar{J}_1 \frac{d}{db} = \frac{d}{dc}, \ \bar{J}_1 \frac{d}{dc} = -\frac{d}{db}.$$

Lemma 5.3.1. $\mathsf{p}_* \circ \bar{J}_1 = J_0 \circ \mathsf{p}_*$ *on* $\operatorname{Ker}\omega_1$.

Proof. Let $X \in \operatorname{Ker}\omega$. By the commutativity of (5.6)

$$\mu_*(\mathsf{p}_*(J_1 X)) = \mathsf{q}_* \nu_*(J_1 X) = J_0 \mathsf{q}_* \nu_*(X) = \mu_*(J_0 \mathsf{p}_*(X)),$$

so $\mathsf{p}_*(J_1 X) = J_0 \mathsf{p}_*(X)$. Obviously, $\mathsf{p}_*(\bar{J}_1(\frac{d}{db}, \frac{d}{dc})) = J_0 \mathsf{p}_*(\frac{d}{db}, \frac{d}{dc})$. $\square$

Lemma 5.3.2. $\bar{J}_1$ *is integrable on* $\operatorname{Ker}\omega_1$.

Proof. Let $\operatorname{Ker}\omega_1 \otimes \mathbb{C} = T^{1,0}_{\omega_1} \oplus T^{0,1}_{\omega_1}$ be the eigenspace decomposition. Then $T^{1,0}_{\omega_1} = T^{1,0}_{\omega} \oplus \langle \frac{d}{db} - \frac{d}{dc}\mathbf{i} \rangle$. If we note that $d\omega_1(\bar{J}_1 X, \bar{J}_1 Y) = d\omega_1(X, Y)$ $(X, Y \in \operatorname{Ker}\omega_1)$ from (5.5), then $[T^{1,0}_{\omega_1}, T^{1,0}_{\omega_1}] \subset \operatorname{Ker}\omega_1 \otimes \mathbb{C}$. Then $\mathsf{p}_*([T^{1,0}_{\omega_1}, T^{1,0}_{\omega_1}]) = [T^{1,0}(\mathcal{L}), T^{1,0}(\mathcal{L})]$. Since J_0 is the complex structure on $\mathcal{L}$, $[T^{1,0}(\mathcal{L}), T^{1,0}(\mathcal{L})] \subset T^{1,0}(\mathcal{L})$. It follows

$$[T^{1,0}_{\omega_1}, T^{1,0}_{\omega_1}] \subset T^{1,0}_{\omega_1}.$$

$\square$

Remark 5.3.3. *The pair* $(\operatorname{Ker}\omega_1, \bar{J}_1)$ *is not a strictly pseudoconvex CR-structure on* $\mathcal{M}$ *unlike Sasakian* 3*-structures. For this,* $[\frac{d}{db}, \frac{d}{dc}] = 0$ *in* $\operatorname{Ker}\omega_1 = \operatorname{Ker}\omega \oplus \langle \frac{d}{db}, \frac{d}{dc} \rangle$*, so* $d\omega_1(\frac{d}{db}, \frac{d}{dc}) = 0$*. However,* $d\omega_1 : \operatorname{Ker}\omega \times \operatorname{Ker}\omega \to \mathbb{R}$ *is nondegenerate from* (5.5).

We put $\operatorname{Ker}\eta\otimes\mathbb{C} = T^{1,0}_{\eta}\oplus T^{0,1}_{\eta}$. Let $\mathsf{p}_* : \operatorname{Ker}\omega_1\otimes\mathbb{C}\to T\mathcal{L}\otimes\mathbb{C}$ be an isomorphism so that $\mathsf{p}_*(\dfrac{d}{db}-\dfrac{d}{dc}\mathbf{i}) = \dfrac{d}{db}-\dfrac{d}{dc}\mathbf{i}$. By Lemma 5.3.1, we have $\mathsf{p}_*(T^{1,0}_{\omega}) = T^{1,0}_{\eta}$.

Theorem 5.3.4. *The complex $2n$-dimensional holomorphic subbundle $T^{1,0}_{\eta}$ is a complex contact subbundle on $\mathcal{L}$.*

Proof. Let $T^{1,0}_{\omega_1}\otimes\mathbb{C} = T^{1,0}_{\omega}\oplus\langle\dfrac{d}{db}-\dfrac{d}{dc}\mathbf{i}\rangle$ and $T^{1,0}(\mathcal{L}) = T^{1,0}_{\eta}\oplus\langle\dfrac{d}{db}-\dfrac{d}{dc}\mathbf{i}\rangle$ as above. From Remark 5.3.3, $d\omega_1 : T^{1,0}_{\omega}\times\bar{T}^{1,0}_{\omega}\to\mathbb{C}$ is nondegenerate. Since $J_1(J_3X) = -\mathbf{i}(J_3X)$, $J_3X\in\bar{T}^{1,0}_{\omega}$. Then $d\omega_1(J_3X,Y) = -d\omega_2(X,Y) = \omega_2([X,Y])$ from (5.5). Thus $\omega_2([T^{1,0}_{\omega},T^{1,0}_{\omega}]) = \mathbb{C}$. In particular, $[T^{1,0}_{\omega},T^{1,0}_{\omega}]\neq\{0\}$. As $[T^{1,0}_{\omega},T^{1,0}_{\omega}]\subset T^{1,0}_{\omega_1} = T^{1,0}_{\omega}\oplus\langle\dfrac{d}{db}-\dfrac{d}{dc}\mathbf{i}\rangle$ by Lemma 5.3.2, it follows

$$[T^{1,0}_{\eta},T^{1,0}_{\eta}]\equiv\langle\frac{d}{db}-\frac{d}{dc}\mathbf{i}\rangle \bmod T^{1,0}_{\eta}.$$

Hence $T^{1,0}_{\eta}$ is a complex contact subbundle on $\mathcal{L}$. $\square$

Part II

Mapping Class Group

Chapter 6

Introduction to Part II

Let $S^m_{g,n}$ be a closed oriented surface of genus g with n boundaries and m punctures where $g, m, n \geqq 0$. Let $\mathcal{M}^m_{g,n}$ be the *mapping class group* of $S^m_{g,n}$ defined as $\pi_0(\mathrm{Diff}_+(S^m_{g,n}))$, the group of path components of $\mathrm{Diff}_+(S^m_{g,n})$, where $\mathrm{Diff}_+(S^m_{g,n})$ is the group of orientation preserving diffeomorphisms of $S^m_{g,n}$ which are identity on boundaries of M and fix the set of punctures. In other words, $\mathcal{M}^m_{g,n}$ is the group of isotopy classes of $\mathrm{Diff}_+(S^m_{g,n})$. Isotopies are required to fix boundaries pointwise. If $m, n = 0$, we write the surface as S_g. If decorations are not important, then we write the surface as S.

There is a natural action of $\mathcal{M}_g$ on $H_1(S_g, \mathbb{Z})$. This action preserves the algebraic intersection number, and induced representation is surjective. The *Torelli group* $\mathcal{I}_g$ is defined as the kernel of this action. So we have the following short exact sequence

$$1 \to \mathcal{I}_g \to \mathcal{M}_g \to Sp(2g, \mathbb{Z}) \to 1.$$

Johnson[25] defined a filtration of $\mathcal{M}_g$ called the *Johnson filtration* $\{\mathcal{N}_g(k)\}_k$ as a generalization of the Torelli group. By the definition of the Johnson filtration, $\mathcal{I}_g = \mathcal{N}_g(1)$. We see the definition and properties later.

On the other hand, there is a useful tool to study mapping class groups called the *Birman exact sequence*

$$1 \to \pi_1(S_g) \to \mathcal{M}^1_g \to \mathcal{M}_g \to 1$$

and the *relative Birman exact sequence*

$$1 \to \pi_1(UT(S_g)) \to \mathcal{M}_{g,1} \to \mathcal{M}_g \to 1$$

where $UT(S_g)$ is the unit tangent bundle for S_g.

This gives the inductive step for the number of punctures and boundaries. To study Johnson filtrations, it is convenient to know the intersection of the kernel of the Birman exact sequence and $\mathcal{N}_g(k)$. Let $K_k = \pi_1(UT(S_g)) \cap \mathcal{N}_g(k)$. In this part we prove the following theorem.

TheoremE. *$K_2 \simeq \mathbb{Z} \times F$ where F is a free group. K_k is a free group of infinite rank for $k \geqq 3$*

6.1 Dehn Twists

In this section, we introduce basic property of *Dehn twists*, first introduced by Dehn[13]. Proofs of following facts can be found in Farb & Margalit[15].

Unless otherwise specified, simple closed curves are always essential i.e. it does not contract to a point. We write the geometric intersection number of a and b by $i(a, b)$, which is the minimal number of intersections of free isotopy classes of a and b, and algebraic intersection number of a and b by $\tilde{i}(a, b)$. Consider an oriented surface S and simple closed curves on S. Roughly speaking, the Dehn twist along a simple closed curve a is the diffeomorphism of S obtained by the following steps. First we cut S along a, then we obtain two boundaries and twist one of them through an angle of 2π. Finally we reglue them. We write the Dehn twist along a as T_a. Figure 6.1 illustrates how the Dehn twist along a acts on c. Dehn twists are well-defined, namely if a and b are isotopic, then T_a and T_b are the same element in the mapping class group. Moreover it is known that if $T_a = T_b$, then a and b are isotopic.

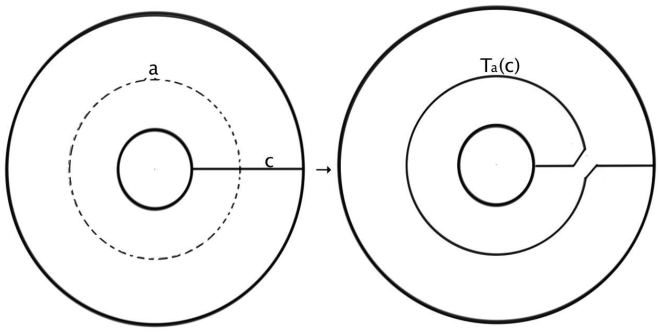

Fig. 6.1. The Dehn twist along a.

If we consider a regular neighborhood of a simple closed curve which we twist, it is obvious that the Dehn twist along the curve does nothing to the complement. Thus we have

Fact 6.1.1. *If $i(a, b) = 0$, then T_a and T_b commute.*

We state another fact.

Fact 6.1.2. *For any $f \in \mathcal{M}_g$ and the isotopy class of a simple closed curve a, we have*

$$T_{f(a)} = fT_af^{-1}.$$

To see this fact, let ϕ be a representative of f, α be a representative of a and ψ_a be a representative of T_a. Then $\phi\psi_a\phi^{-1}$, which is a representative of fT_af^{-1}, takes a regular neighborhood of $\phi(\alpha)$ to a regular neighborhood of α and twists the neighborhood of α, then takes this twisted neighborhood of α

back to a neighborhood of $\phi(\alpha)$. So the resulting diffeomorphism is a Dehn twist along $\phi(\alpha)$.

Next, suppose we have isotopy classes of simple closed curves a and b and choose their representations α and β. Consider the shape of $T_a^k(b)$. We gather intersections of α and β, then locally $\alpha \cup \beta$ looks like $i(a,b)$ copies of α and one copy of β. If we twist β along α, at each intersection $i(a,b)$ copies of α are added. Fig. 6.2 shows $i(a,b) = 2, k = 1$ case.

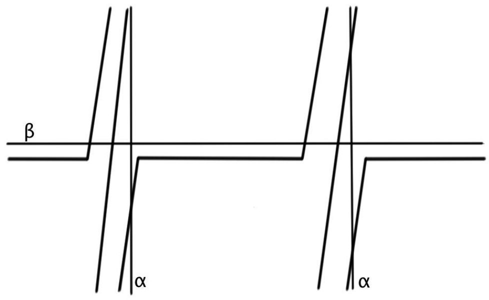

Fig. 6.2. The case $i(a,b) = 2$ and $k = 1$.

Hence we have following Lemma.

Lemma 6.1.3. *Let a and b are any simple closed curves and k be an any integer. Then we have*

$$i(T_a^k(b), b) = |k| i(a,b)^2.$$

As a corollary for this lemma, we have a following basic property of Dehn twists.

Corollary 6.1.4. *Dehn twists have infinite order.*

Since we can put any simple closed curve which is parallel to the boundary component of $S_{g,1}$ so that it does not intersect any other simple closed curve, the Dehn twist along a simple closed curve which is parallel to the boundary and any Dehn twist commute. By the fact that mapping class group is generated by (finite number of) Dehn twists (Theorem 6.2.1) and Corollary 6.1.4, we have the following fact.

Lemma 6.1.5. *The subgroup generated by the Dehn twist along a simple closed curve which is parallel to the boundary, which is isomorphic to $\mathbb{Z}$, is the center of $\mathcal{M}_{g,1}$.*

Note that it is known that $\mathcal{M}_g$ is centerless.

For later use, we state one more basic property of Dehn twists. The action of T_a on $H_1(S,\mathbb{Z})$ is determined by the homology class of a by following formula.

Lemma 6.1.6. $T_a(x) = x + \tilde{i}(a \cdot x)a$ *for* $x \in H_1(S,\mathbb{Z})$.

6.2 Finite Generating Sets for the Mapping Class Group

In this section, we introduce two famous finite generating set for the mapping class group; the *Lickorish generators* and the *Humphries generators*.

The proof are based on investigating the stabilizer of the action on a certain simplicial complex.

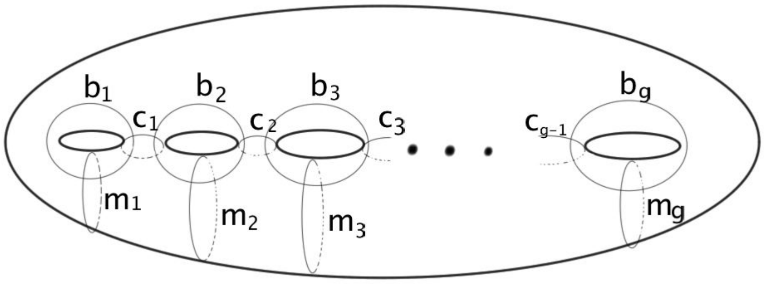

Fig. 6.3. The Lickorish generators.

Theorem 6.2.1 (Lickorish[36]). *For $g \geq 1$, $\mathcal{M}_g$ is generated by Dehn twists along the isotopy classes of*

$$b_1, \ldots, b_g, m_1 \ldots, m_g, c_1, \ldots, c_{g-1}$$

in Fig. 6.3.

This can be shown by induction on g. Note that when $g = 1$, S_2 is the torus T^2, it is known that the symplectic representation of the mapping class group

$$1 \to \mathcal{I}_g \to \mathcal{M}_g \to Sp(2g, \mathbb{Z})$$

reduce to isomorphism

$$\mathcal{M}_1 \cong \mathrm{SL}(2, \mathbb{Z}),$$

and Dehn twists along a basis of $H_1(T^2, \mathbb{Z})$ which are the lickorish generators in the $g = 1$ case corresponding to the generator of $SL(2, \mathbb{Z})$.

Humphries[21] showed that the number of the Lickorish generators can be reduced to $2g + 1$.

Theorem 6.2.2 (Humphries[21]). *Let $g \geq 2$. Then the mapping class group is generated by the Dehn twists along isotopy classes of*

$$b_1, \ldots, b_g, m_1, m_2, c_1, \ldots, c_{g-1}$$

in Fig. 6.3.

Proof. We will show that $T_{m_3}, \ldots, T_{m_g}$ can be written in terms of the other Lickorish generators. To achieve this purpose, we will find a product h_i of Dehn twists along a_i, c_i and m_{i+1} for $1 \leq i \leq g-2$ which takes m_i to m_{i+2}. If we obtain such h, by Fact 6.1.2, $T_{m_{i+2}} = T_{h_i(m_i)} = h_i T_{m_i} h_i^{-1}$ and obtain generators we wanted.

The following figures are in clockwise order from top left. Fig. 6.4 shows the ingredients of such h_i. Fig. 6.5 and Fig. 6.6 shows the image of $T_{m_{i+1}} T_{b_{i+1}} T_{c_i} T_{b_i}(m_i)$ and denote the image by m_i'. Fig. 6.7 shows the image of $T_{c_{i+1}} T_{b_{i+2}} T_{b_{i+1}} T_{c_{i+1}}(m_i')$. This curve is symmetric with respect to the $(i+1)$-th hole so we can construct a similar product of Dehn twists which takes this curve to m_{i+1}. In this product, no Dehn twists along m_i's are contained except m_{i+1}. Hence the claim was proved.

□

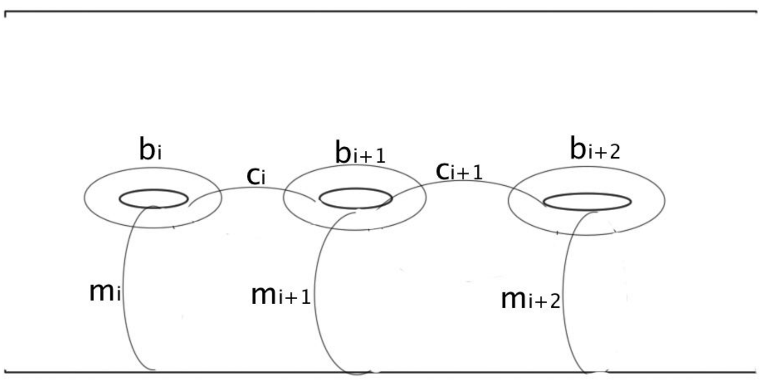

Fig. 6.4

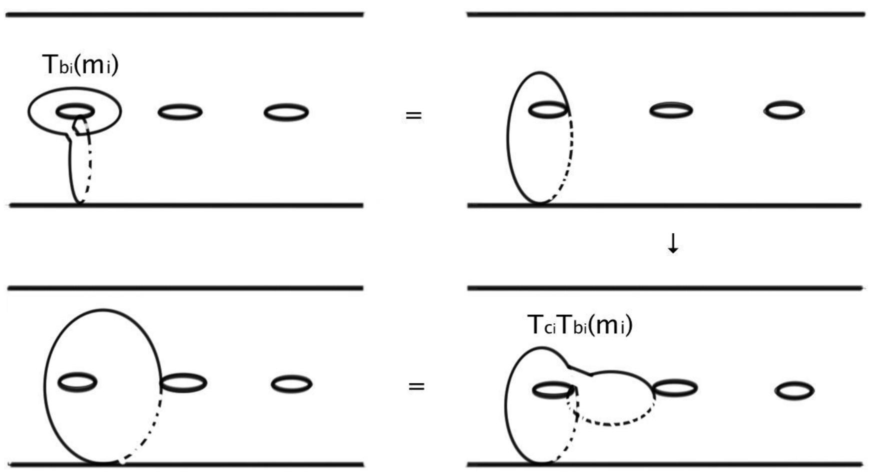

Fig. 6.5

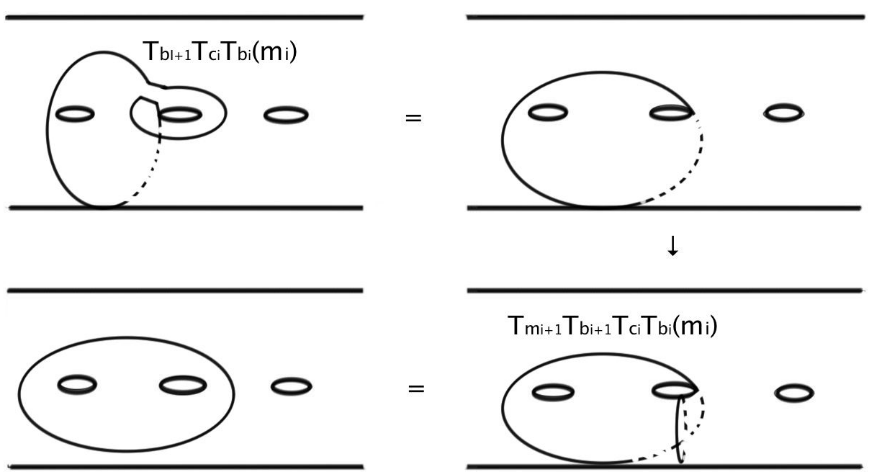

Fig. 6.6

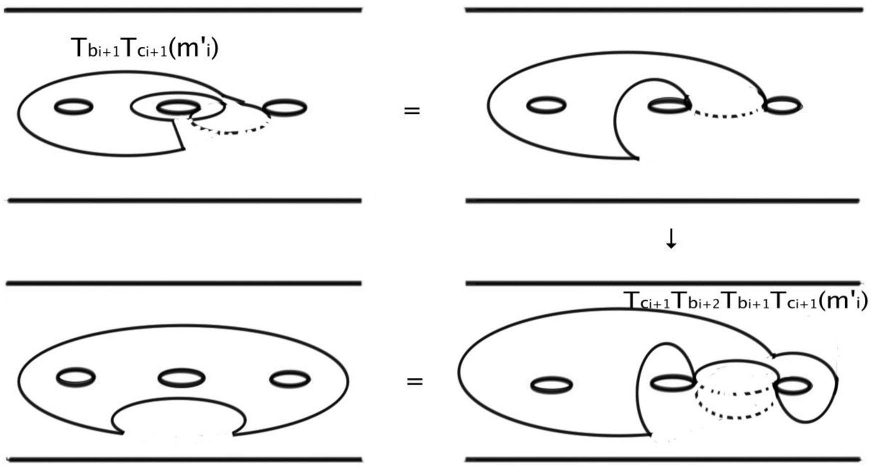

Fig. 6.7

Chapter 7

Torelli Group and Johnson Filtration

In this chapter, we define the Torelli group and the Johnson filtration. Also we introduce some theorems used in later chapters and give some proofs.

7.1 The Torelli Group

There is the natural action of $\mathcal{M}_g$ on $H_1(S_g, \mathbb{Z})$. This action preserves the algebraic intersection number, and induced representation is surjective. The Torelli group $\mathcal{I}_g$ is defined as the kernel of this action. In other words, $\mathcal{I}_g$ is a subgroup of $\mathcal{M}_g$ which acts trivially on $H_1(S_g, \mathbb{Z})$. So we have the following short exact sequence

$$1 \to \mathcal{I}_g \to \mathcal{M}_g \to Sp(2g, \mathbb{Z}) \to 1.$$

The Torelli group of a surface with boundaries or punctures are also can be defined (See Putman[50], Church[12]).

One of the early topological motivation for studying Torelli group is the following fact. Let $\phi : S_g \to S^3$ be a fixed Heegaard embedding of S^3 and $\psi \in \mathcal{M}_{g,1}$. Cutting S^3 along $\phi(S_g)$ and reglue by a diffeomorphism representing ψ, we obtain a new manifold M_ψ. It is known that the homology of M_ψ depends only on the image in $Sp(2g, \mathbb{Z})$ of ψ. Thus if $\psi \in \mathcal{I}_{g,1}$, then M_ψ is a homology 3-sphere. Moreover, any homology 3-sphere can be constructed in this way (Morita[44]).

In the following section, we see the mapping class group has the filtration called the Johnson filtration: $\mathcal{M}_g \supset \mathcal{N}_g(1) \supset \mathcal{N}_g(2) \supset \cdots$. It is known that $\mathcal{N}_g(1) = \mathcal{I}_g$ and $\mathcal{N}_g(2) = \mathcal{K}_g$ which is called the Johnson kernel. Morita[44] proved that the elements of $\mathcal{N}_{g,1}(2)$ suffice to obtain the every homology 3-sphere. More recently Pitsch[48] showed that every homology 3-sphere arises as M_ψ where $\psi \in \mathcal{N}_{g,1}(3)$ for $g \geq 9$.

In this section, we see some properties of the Torelli group for later use.

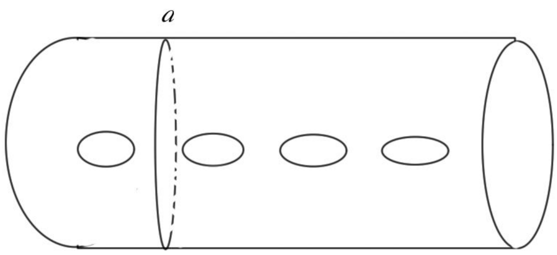

Fig. 7.1. A bounding simple closed curve of genus 1.

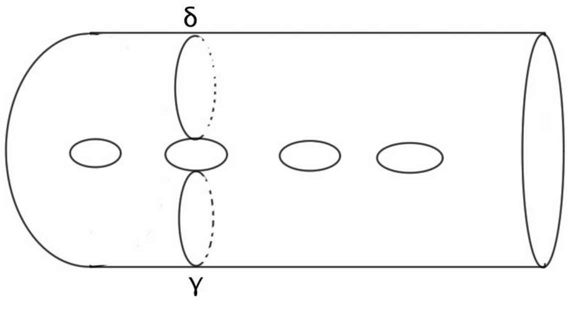

Fig. 7.2. A bounding pair of genus 1.

First we introduce two types of elements in the Torelli group. A simple closed curve a is a *bounding simple closed curve* (denote *BSCC*) if a bounds subsurfaces in S. Namely a separates S into two pieces. When $S = S_{g,1}$, we obtain two subsurfaces $S_{k,1}$ and $S_{g-k,2}$. Then we define the genus of a to be

k. Since BSCC is homologically trivial, a Dehn twist along BSCC is contained in the Torelli group by Lemma 6.1.6. Another type is called *bounding pair map* (*BP map*). A bounding pair is a pair of simple closed curves (γ, δ) s.t. each γ, δ is homologically nontrivial (i.e. non-bounding) and they are disjoint each other and their homology classes are homologous. Then by Lemma 6.1.6, T_γ and T_δ act the same on $H_1(S, \mathbb{Z})$ so $T_\gamma T_\delta^{-1}$ is contained in the Torelli group. Note that nontrivial BP maps do not exist when $g \leq 2$.

Next we see generators for the Torelli group. Let

$$1 \longrightarrow K \longrightarrow G \xrightarrow{p} Q \longrightarrow 1$$

be a short exact sequence of groups. Suppose $\{g_1, \dots, g_n\}$ be a finite set of generators for G and Q has a presentation in terms of the set $\{p(g_i)\}$. Let $\{r_j\}$ be relators for Q which are words written in $\{p(g_i)\}$. For each j, let $\tilde{r}_j$ be the corresponding words in the $\{g_i\}$ which is in K. It is a well known fact that all $\tilde{r}_j$ and their conjugates forms a generating set for K (See Magnus, Karass & Solitar[37], Theorem 2.1). Applying this fact to following short exact sequence

$$1 \to \mathcal{I}_g \to \mathcal{M}_g \to Sp(2g, \mathbb{Z}) \to 1.$$

Birman gave generators for $\mathcal{I}_g$ and Powell gave a geometric interpretation of the Birman's generators as follows.

Theorem 7.1.1 (Powell [49])**.** *$\mathcal{I}_g$ is generated by all BSCC maps of genus 1 or 2 and all BP maps which bound a subsurface of genus 1 in S_g.*

If a group G acts on a simply connected simplicial complex X and X/G is also simply connected, then G is generated by stabilizers of simplices. From this fact, other proofs of this theorem were given in Putman[50] and Hatcher[20] by investigating the action on different complexes.

Building on the Powell's work, Johnson showed that any BP map can be expressed as a product of BP maps when $g \geq 3$. Hence we have following theorem.

Theorem 7.1.2 (Johnson [23])**.** *$\mathcal{I}_g$ (reap. $\mathcal{I}_{g,1}$) is generated by all BP maps which bound a subsurface of genus 1 in S_g(resp. $S_{g,1}$).*

In fact, the Torelli group is finitely generated for $g \geq 3$. This was first proved by Johnson[26]. He produced a list of finite BP maps (including genus 1 BP maps) and consider the group generated by those BP maps. By checking the conjugate of each BP map by Humphries generators is again in the list, he showed that the group generated by those BP maps in the list is a normal subgroup of the mapping class group. Since the Torelli group is generated by all genus 1 BP maps and they are conjugate in the mapping class group, this becomes a finite generating set. The number of generators is a exponential function of g.

More recently, Putman[51] produced a finite generating set for the Torelli group with its number is a cubic function of g. As we will see in section 7, Theorem 7.5.2, an abelianization of the Torelli group $H(\mathcal{I}_{g,1},\mathbb{Z})$ is isomorphic to $\wedge^3 H \times (\mathbb{Z}/2\mathbb{Z})^N$ where $N = \binom{2g}{2} + \binom{2g}{1} + \binom{2g}{0}$. The number of a generating set for the Torelli group must be an at least cubic function of g. In this sense, the Putman's generator is very sharp. Note that when $g = 2$, the Torelli group is not finitely generated. McCullough & Miller[38] showed that $\mathcal{I}_2$ is not finitely generated and Mess[39] showed $\mathcal{I}_2$ is an infinitely generated free group. Since there does not exist any BP in S_2 and $\mathcal{K}_g$ is the subgroup generated by BSCC's, we have $\mathcal{I}_2 = \mathcal{K}_2$ and so $\mathcal{K}_2$ is not finitely generated. We do not know if $\mathcal{I}_g$ for $g \geq 3$ is finitely presented or not, and $\mathcal{K}_g$ for $g \geq 3$ is finitely generated or not (Biss & Farb[7],[8]).

7.2 Johnson Filtrations

By the *Dehn-Nielsen-Baer theorem*, for $g \geq 1$ we have an isomorphism

$$\mathcal{M}_g^{\pm} \cong \mathrm{Out}(\pi_1(S_g))$$

where $\mathcal{M}_g^{\pm}$ is the extended mapping class group: the group of isotopy classes of all diffeomorphism of S_g (including the orientation-reversing ones) and $\mathrm{Out}(\pi_1(S_g))$ is the outer automorphism group of $\pi_1(S_g)$. Thus the mapping class group $\mathcal{M}_g$ is isomorphic to $\mathrm{Out}_+(\pi_1(S_g))$ which is the orientation preserving outer automorphism group of $\pi_1(S_g)$; index two subgroup of $\mathrm{Out}(\pi_1(S_g))$.

Let $\Gamma = \pi_1(S_g)$ and $\Gamma(k)$ be the k-th term of the lower central series of Γ, namely $\Gamma(0) := \Gamma$ and $\Gamma(k+1) := [\Gamma, \Gamma(k)]$. We define $\Gamma_{g,n}^m$ to be $\pi_1(S_{g,n}^m)$, but to simplify the notation, we use the same notation Γ as long as there is no risk of confusion. The above isomorphism induces the homomorphism

$$\mathcal{M}_g \to \mathrm{Out}(\Gamma/\Gamma(k))$$

The Johnson filtration $\{\mathcal{N}_g(k)\}_k$ is defined as

$$\mathcal{N}_g(k) = \ker\{\mathcal{M}_g \to \mathrm{Out}(\Gamma/\Gamma(k))\}.$$

It is known that the first term of the Johnson filtration $\mathcal{N}_g(1) = \mathcal{I}_g$ and the second term of the Johnson filtration $\mathcal{N}_g(2) = \mathcal{K}_g$ which is called the *Johnson kernel*. $\mathcal{K}_g$ is the subgroup of $\mathcal{M}_g$ generated by Dehn twists along BSCC's in S_g. $\mathcal{K}_g$ is called the Johnson kernel because $\mathcal{K}_g$ is precisely the kernel of the Johnson homomorphism.

We denote $\mathcal{K}_{g,1}$ and $\mathcal{N}_{g,1}(k)$ for the Johnson kernel and the Johnson filtration of a surface with one boundary component respectively.

Since each $\mathcal{N}_{g,1}(k)$ is the kernel of the homomorphism $\mathcal{M}_{g,1} \to \mathrm{Out}(\Gamma/\Gamma(k))$. So $\mathcal{N}_g(k)$ is a normal subgroup of $\mathcal{M}_{g,1}$. On the other hand, $\mathcal{N}_g(k+1)$ is the kernel of the homomorphism from $\mathcal{N}_g(k)$ to certain abelian group, so we have a filtration of normal subgroups $\mathcal{M}_g \rhd \mathcal{N}_g(1) \rhd \mathcal{N}_g(2) \rhd \cdots$. Note that the Johnson filtration is residually nilpotent i.e. $\cap_{k=1}^{\infty} \mathcal{N}_g(k) = 1$ (Bass & Lubotzky[3]).

Let M be a multi curve in $S_{g,1}$ which is a collection of disjoint simple closed curves and $G(M)$ be the group generated by the Dehn twists in the curves of M. By Fact 6.1.1 and Lemma 6.1.2, $G(M)$ is free abelian group of rank $|M|$. In this section, we introduce some facts and the theorems by Bestvina, Bux & Margalit[4].

The following formula is known as Witt-Hall identity.

Fact 7.2.1 (Magnus, Karass & Solitar[37])**.**

$$[ab, c] = a[b, c]a^{-1}[a, c]$$

Lemma 7.2.2. *Suppose $g \geq 2$. Let x and y be an element of Γ represented by homologically nontrivial curves, and suppose representing elements of $H_1(S_{g,1}, \mathbb{Z})$ are distinct. Then the commutator $[x, y]$ is not contained in $\Gamma(2)$.*

Proof. It is well known fact that the free graded Lie algebra generated by $H_1(S_{g,1}, \mathbb{Z})$ (We simply write H.) is isomorphic to the graded Lie algebra $\bar{\Gamma} := \oplus_i \Gamma(i-1)/\Gamma(i)$. We denote $\Gamma(i-1)/\Gamma(i)$ as $\bar{\Gamma}_i$. Recall that we defined $\pi_1(S_{g,1}) = \Gamma = \Gamma(0)$, so $\bar{\Gamma}_1$ is H. We write the image of x, an element of Γ, in H as $\bar{x}$. From the assumption, $\bar{x}$ and $\bar{y}$ are nontrivial element of H. So x and y are not in $\Gamma(1)$. Since we consider the surface has one boundary, Γ is a free group. Thus $[x, y]$ can not be in $\Gamma(2)$. □

Lemma 7.2.3. *Let x be an element of Γ represented by a homologically nontrivial curve, and suppose y is an element of Γ that is not homologous to x and is not contained in $\Gamma(2)$. Then the commutator $[x, y]$ is not in $\Gamma(3)$.*

Proof. As mentioned in the proof of Lemma 7.2.2, x is not in $\Gamma(1)$. If y is represented by a homologically nontrivial curve, then we apply Lemma 7.2.2. So we assume y is an element of $\Gamma(1)$. By the assumption that y is not contained in $\Gamma(2)$ and the fact Γ is free group, $[x.y]$ can not be in $\Gamma(3)$. □

Using these facts, we give proofs of the following theorems.

Theorem 7.2.4. *If M is a multicurve consisting of BSCC's, then $G(M) \cap \mathcal{N}_{g,1}(3)$ is trivial.*

Proof. Let f be a multitwist supported in M. Without loss of generality, we assume that each curve of M is used by f. Let ϕ be a representative for f supported in a regular neighborhood $N \subset S_{g,1}$ of M. Suppose c be a curve of M s.t. c bounds a subsurface S' of $S_{g,1}$ which is bounded by c and contains no curves of M in its interior. Put the base point for Γ on any point in the component of ∂N which is parallel to c but not contained in S'. Let γ be an element of Γ which is isotopic to this component of ∂N, and let α be an element of Γ contained in $S' \cup N$, and which is represented by a homologically nontrivial curve. Take β to be an element of Γ which is represented by a homologically nontrivial curve and it intersects N only at the base point.

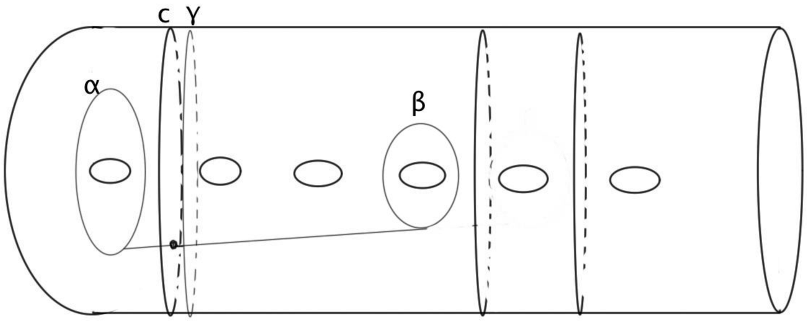

Fig. 7.3. An example of curves for Theorem 7.2.4.

Then $\phi(\alpha)$ can be written as $\phi(\alpha) = \gamma\alpha\gamma^{-1}$, so $\phi(\alpha)\alpha^{-1} = [\gamma, \alpha]$. Let δ be an any element of Γ, let i_δ be the corresponding inner automorphism of Γ. Let $\psi = i_\delta \circ \phi$. Then we have

$$\begin{aligned}\psi(\alpha)\alpha^{-1} &= \delta\gamma\alpha\gamma^{-1}\delta^{-1}\alpha^{-1} \\ &= [\delta\gamma, \alpha]\end{aligned}$$

and, since c does not intersects β, we have

$$\begin{aligned}\psi(\beta)\beta^{-1} &= \delta\beta\delta^{-1}\beta^{-1} \\ &= [\delta, \beta].\end{aligned}$$

To see f is not in $\mathcal{N}_{g,1}(3)$, we must show that f is a nontrivial element in $\mathrm{Out}(\Gamma/\Gamma(3))$. So our goal is to show that at least one of these two commutators is not in $\Gamma(3)$ for any choice of δ.

Since β is represented by a homologically nontrivial curve, $[\delta, \gamma]$ is in $\Gamma(3)$ if and only if δ is in $\Gamma(2)$ as follows. $\delta \in \Gamma(2) \Rightarrow [\delta, \gamma] \in \Gamma(3)$ is obvious. $[\delta, \gamma] \in \Gamma(3) \Rightarrow \delta \in \Gamma(2)$ is a nothing but Lemma 7.2.3. Thus we assume that $\delta \in \Gamma(2)$ and must show that $[\delta\gamma, \alpha]$ is not in $\Gamma(3)$. By Fact 7.2.1, we have

$$[\delta\gamma, \alpha] = \delta[\gamma, \alpha]\delta^{-1}[\delta, \alpha]$$

Since δ is in $\Gamma(2)$ and α is represented by a homologically nontrivial curve, the commutator $[\delta, \alpha]$ is in $\Gamma(3)$, and by Lemma 7.2.3, $\delta[\gamma, \alpha]\delta^{-1}$ is not in $\Gamma(3)$. Hence $[\delta\gamma, \alpha]$ is not in $\Gamma(3)$. $\square$

Fig. 7.4 below shows an example of position of curves and by the same argument, we can show following theorem.

Theorem 7.2.5. *If M is a multicurve consisting of BP's, then $G(M) \cap \mathcal{N}_{g,1}(2)$ is trivial.*

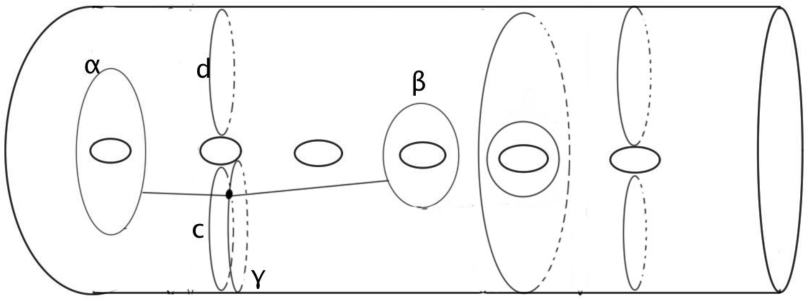

Fig. 7.4. An example of curves for Theorem 7.2.5.

7.3 The Lower Central Series of the Torelli Group

Let $\mathcal{I}_g(k)$ denote the k-th term of the lower central series of the Torelli group and $\mathcal{I}_{g,1}(k)$ be the k-th term of the lower central series of the Torelli group of a surface with one boundary component.

Theorem 7.3.1 (Johnson[25])**.** *$\mathcal{N}_{g,1}(k+1)$ contains $\mathcal{I}_{g,1}(k)$ for $k \geqq 1$.*

We prove this theorem by using results of Morita.

Lemma 7.3.2 (Morita[23])**.** *For any element $f \in \mathcal{N}_{g,1}(k)$ and $\gamma \in \Gamma_i$, then $f(\gamma)\gamma^{-1} \in \Gamma_{k+i}$.*

Proof. We prove this by the induction on i. If $i = 0$, then the claim follows from the definition of $\mathcal{N}_{g,1}(k)$. Suppose the claim holds up to $i-1$. Let γ be an element of $\Gamma(i)$. Then we can describe γ as $\gamma = [\gamma_1, \gamma_2]$ where $\gamma_1 \in \Gamma(i-1)$ and $\gamma_2 \in \Gamma(0)$. By the assumption, $f(\gamma_1) = \gamma_1\alpha$ and $f(\gamma_2) = \gamma_2\beta$ for some $\alpha \in \Gamma(k+i-1)$ and $\beta \in \Gamma(k)$. Then we have

$$\begin{aligned}
f(\gamma)\gamma^{-1} &= f([\gamma_1, \gamma_2])[\gamma_1, \gamma_2]^{-1} \\
&= [f(\gamma_1), f(\gamma_2)][\gamma_1, \gamma_2]^{-1} \\
&= \gamma_1\alpha\gamma_2\beta\alpha^{-1}\gamma_1^{-1}\beta^{-1}\gamma_2^{-1}\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1} \\
&= \gamma_1\alpha\gamma_2\beta\alpha^{-1}\beta^{-1}\gamma_2^{-1}\gamma_2\beta\gamma_1^{-1}\beta^{-1}\gamma_1\gamma_2^{-1}\gamma_1^{-1} \\
&= \gamma_1[\alpha, \gamma_2\beta]\gamma_2[\beta, \gamma_1^{-1}]\gamma_2^{-1}\gamma_1^{-1}.
\end{aligned}$$

Since $[\Gamma(k),\Gamma(i)] < \Gamma(k+l+1)$, $[\alpha,\gamma_2\beta]$ and $[\beta,\gamma_1^{-1}]$ are in $\Gamma(k+i)$ as required. $\square$

Lemma 7.3.3 (Morita[23]). *For any $f \in \mathcal{N}_{g,1}(k)$ and $g \in \mathcal{N}_{g,1}(l)$, $[f,g]$ is contained in $\mathcal{N}_{g,1}(k+l)$*

Proof. We can assume $k \geq l$. Let $\gamma \in \Gamma(0)$ be any element. By the assumption we can write $f(\gamma) = \gamma_1\gamma$ and $g(\gamma) = \gamma_2\gamma$ for some $\gamma_1 \in \Gamma(k)$ and $\gamma_2 \in \Gamma(l)$. Then $f^{-1}(\gamma) = f^{-1}(\gamma_1^{-1})\gamma^{-1}$ and $g^{-1}(\gamma) = g^{-1}(\gamma_2^{-1})\gamma^{-1}$. Then we have

$$\begin{aligned}
[f,g](\gamma) &= fgf^{-1}g^{-1}(\gamma)\\
&= fgf^{-1}(g^{-1}(\gamma_2^{-1})\gamma)\\
&= fg(f^{-1}g^{-1}(\gamma_2^{-1})f^{-1}(\gamma_1^{-1})\gamma)\\
&= [f,g](\gamma_2^{-1})fgf^{-1}(\gamma_1^{-1})f(\gamma_2)\gamma_1\gamma\\
&= [f,g](\gamma_2^{-1})\gamma_2\gamma_2^{-1}fgf^{-1}(\gamma_1^{-1})\gamma_1\gamma_2\gamma_2^{-1}\gamma_1^{-1}f(\gamma_2)\gamma_2^{-1}\gamma_1\gamma_2[\gamma_2^{-1},\gamma_1^{-1}]\gamma\\
[f,g](\gamma)\gamma^{-1} &= [f,g](\gamma_2^{-1})\gamma_2\gamma_2^{-1}fgf^{-1}(\gamma_1^{-1})\gamma_1\gamma_2\gamma_2^{-1}\gamma_1^{-1}f(\gamma_2)\gamma_2^{-1}\gamma_1\gamma_2[\gamma_2^{-1},\gamma_1^{-1}].
\end{aligned}$$

By Lemma 7.3.2, we can check that $[f,g](\gamma)\gamma^{-1} \in \Gamma(k+l)$ as required. $\square$

Now we finish the proof of Theorem 7.3.1.

Proof. We prove the theorem by using the induction on k. If $k=1$, $\mathcal{N}_{g,1}(2) > \mathcal{I}_{g,1}(1) = [\mathcal{I}_{g,1},\mathcal{I}_{g,1}] = [\mathcal{N}_{g,1}(1),\mathcal{N}_{g,1}(1)]$ by Lemma 7.3.3. Suppose $\mathcal{N}_{g,1}(k) > \mathcal{I}_{g,1}(k-1)$, then $\mathcal{I}_{g,1}(k) = [\mathcal{I}_{g,1}(k-1),\mathcal{I}_{g,1}] < [\mathcal{N}_{g,1}(k),\mathcal{N}_{g,1}(1)]$. Thus the theorem was proved. $\square$

We also have that $\mathcal{N}_g(k+1)$ contains $\mathcal{I}_g(k)$ for $k \geqq 1$.

Note that it is known that there are terms of $\mathcal{N}_g(k)$ which is not contained in any term of $\mathcal{I}_g(k)$ (Hain[19]).

As mentioned above, $\mathcal{N}_g(k)$ $(resp.\mathcal{N}_{g,1}(k+1)$ are normal subgroup of $\mathcal{M}_g$ $(resp.\mathcal{M}_{g,1})$ and $\mathcal{I}_g = \mathcal{N}_g(1)$, so we have $\mathcal{N}_g(k+1) \triangleright \mathcal{I}_g(k)$ and $\mathcal{N}_{g,1}(k+1) \triangleright \mathcal{I}_{g,1}(k)$.

7.4 The Johnson Homomorphism

In this section, we introduce the Johnson homomorphism and give the proof of following theorem by Johnson[24].

Theorem 7.4.1 (Johnson[24]). $\mathcal{I}_{g,1}(1) = \mathcal{N}_{g,1}(2) \cap \mathcal{I}_{g,1}^2$.

Let Γ be $\pi_1(S_{g,1})$, H be $H_1(S_{g,1},\mathbb{Z})$ and Γ' be the commutator subgroup of Γ. Define $E := \Gamma/[\Gamma,\Gamma']$ and $N := \Gamma'/[\Gamma,\Gamma']$. Then we have a central extension

$$0 \longrightarrow N \xrightarrow{\ i\ } E \xrightarrow{\ p\ } H \longrightarrow 0,$$

Let g be an element of Γ, then

$$E = \Gamma/[\Gamma, \Gamma'] \ni [g] \xrightarrow{p} \langle g \rangle \in H = \Gamma/[\Gamma, \Gamma].$$

Since Γ' is a normal subgroup of Γ, $[\Gamma, \Gamma'] \subset [\Gamma, \Gamma]$. If $gg'^{-1} \in [\Gamma, \Gamma']$ for $g' \in \Gamma$, then $gg'^{-1} \in [\Gamma, \Gamma]$. So this homomorphism is well-defined. By the definitions of N and E, this is a central extension.

Let $\wedge^k H$ be a k-th exterior product of H. Since H is a free abelian group of rank $2g$, $\wedge^k H$ is a free abelian group of rank $\begin{pmatrix} 2g \\ k \end{pmatrix}$.

Let x, y are elements of H. We take lifts of them $\tilde{x}, \tilde{y}$ in E. Then $[\tilde{x}, \tilde{y}] \in N$ and this does not depend on the lifting. Let $p(\tilde{x}) = p(\tilde{x}') = x$ and $p(\tilde{y}) = p(\tilde{y}') = y$. To see $[\tilde{x}, \tilde{y}]$ and $[\tilde{x}', \tilde{y}']$ defines the same element in N, it suffices to see $[\tilde{x}, \tilde{y}][\tilde{x}', \tilde{y}']^{-1} = 0$ in N. Since $\tilde{x}$ and $\tilde{x}'$ are lifts of x and $\tilde{y}$ and $\tilde{y}'$ are lifts of y, we can describe $\tilde{x}' = \tilde{w}\tilde{x}, \tilde{y}' = \tilde{z}\tilde{y}$ for some $\tilde{w}, \tilde{z} \in E$. Then

$$\begin{aligned} [\tilde{x}, \tilde{y}][\tilde{x}', \tilde{y}']^{-1} &= \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}\tilde{y}'\tilde{x}'\tilde{y}'^{-1}\tilde{x}'^{-1} \\ &= \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}\tilde{z}\tilde{y}\tilde{w}\tilde{x}\tilde{y}^{-1}\tilde{z}^{-1}\tilde{x}^{-1}\tilde{w}^{-1}. \end{aligned}$$

By the centrality of N, this element equals to $[\tilde{x}, \tilde{y}][\tilde{x}, \tilde{y}]^{-1} = 0$. Thus $[\tilde{x}, \tilde{y}]$ does not depend on lifts and we denote this element $\{x, y\}$. So we can define a map $\{,\} : H \times H \to N$ by $(x, y) \mapsto \{x, y\}$.

Lemma 7.4.2. *The map $\{,\} : H \times H \to N$ is bilinear and antisymmetric i.e. $\{x, y + z\} = \{x, y\}\{x, z\}$ and $\{x, y\} = \{y, z\}^{-1}$. Hence this map defines a surjective homomorphism $j : \wedge^2 H \to N$ by $j(x \wedge y) = \{x, y\}$.*

Proof. Let $\tilde{x}, \tilde{y}, \tilde{z}$ be lifts of x, y, z. Then $\tilde{y}\tilde{x}$ is a lift of $y + z$. Thus

$$\begin{aligned} \{x, y + z\} &= [\tilde{x}, \tilde{y}\tilde{z}] \\ &= \tilde{x}\tilde{y}\tilde{z}\tilde{x}^{-1}\tilde{z}^{-1}\tilde{y}^{-1} \\ &= \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}\tilde{y}\tilde{x}\tilde{z}\tilde{x}^{-1}\tilde{z}^{-1}\tilde{y}^{-1} \\ &= [\tilde{x}, \tilde{y}]\tilde{y}[\tilde{x}, \tilde{z}]\tilde{y}^{-1} \\ &= [\tilde{x}, \tilde{y}][\tilde{x}, \tilde{z}] \\ &= \{x, y\}\{x, z\} \end{aligned}$$

$\{x, y\} = \{y, z\}^{-1}$ is from the definition of the bracket product. Since N is generated by all $[\tilde{x}, \tilde{y}]$ for $\tilde{x}, \tilde{y} \in E$, j is surjective. ☐

N is a free abelian group of rank $\begin{pmatrix} 2g \\ 2 \end{pmatrix}$ (see Magnus, Karass & Solitar[37], Theorems 5.11 and 5.12). And $\wedge^2 H$ is also a free abelian group of rank $\begin{pmatrix} 2g \\ 2 \end{pmatrix}$. A surjective homomorphism between finitely generated free abelian groups of the same rank must be an isomorphism. Hence j is an isomorphism. We shall identify N and $\wedge^2 H$ via j.

We use the same notation for $f \in \mathcal{M}_{g,1}$ and for induced maps on Γ, H and E.

Lemma 7.4.3. *Let $f \in \mathcal{I}_{g,1}$ and $p(e) = x \in H$. Then $f(e)e^{-1} \in N$ and this does not depend on lifts.*

Proof. Since $\mathcal{I}_{g,1}$ acts on H trivially, $\mathcal{I}_{g,1}$ acts on $N = \wedge^2 H$ trivially. Let e' be another lift of x, then we can write $e' = en$ for some $n \in N$. Then we have

$$\begin{aligned} f(e') &= f(en) \\ &= f(e)f(n) \\ &= f(e)n. \end{aligned}$$

Hence

$$\begin{aligned} f(e')e'^{-1} &= f(e)n(en)^{-1} \\ &= f(e)e^{-1}. \end{aligned}$$

□

Denote $\delta f(x) := f(e)e^{-1}$ and we have the following lemma.

Lemma 7.4.4. *$\delta f : H \to N$ is a homomorphism.*

Proof. Suppose $x_1, x_2 \in H$ and e_1, e_2 be there lifts. Since N is central in E, we have

$$\begin{aligned} \delta f(x_1 + x_2) &= f(e_1 e_2)(e_1 e_2)^{-1} \\ &= f(e_1)(f(e_2)e_2^{-1})e_1^{-1} \\ &= f(e_1)e_1^{-1} f(e_2)e_2^{-1} \\ &= \delta f(x_1)\delta f(x_2). \end{aligned}$$

□

By the identification $N = \wedge^2 H$, we have $\delta f \in Hom(H, \wedge^2 H)$.

Lemma 7.4.5. *$\delta : \mathcal{I}_{g,1} \to Hom(H, \wedge^2 H)$ is a homomorphism.*

Proof. Suppose e is a lift of x. Since f acts trivially on N, we have

$$\begin{aligned} \delta(fg)x &= fg(e)e^{-1} \\ &= f(g(e)e^{-1}e)e^{-1} \\ &= f(g(e)e^{-1})f(e)e^{-1} \\ &= f(\delta g(x))\delta f(x) \\ &= \delta g(x)\delta f(x). \end{aligned}$$

□

There is the natural isomorphisms

$$Hom(H, \wedge^2 H) \cong \wedge^2 H \otimes H^* \cong \wedge^2 H \otimes H,$$

by the correspondence

$$\delta \mapsto \sum_{i=1}^{g} (\delta(a_i) \otimes b_i - \delta(b_i) \otimes a_i)$$

where $\{a_i, b_i\}$ is a fixed symplectic basis of H.

Now we define the Johnson homomorphism. Suppose $f \in \mathcal{I}_{g,1}$, then we define

$$\tau(f) := \sum_{i=1}^{g} (\delta f(a_i) \otimes b_i - \delta f(b_i) \otimes a_i).$$

The homomorphism $\tau : \mathcal{I}_{g,1} \to \wedge^2 H \otimes H$ is called the Johnson homomorphism. Computation shows that τ takes Powell's generators into $\wedge^3 H \subset \wedge^2 H \otimes H$ and a basis of $\wedge^3 H$ is in Imτ.

Theorem 7.4.6 (Johnson[24])**.** *For $g \geq 2$, $Im\tau = \wedge^3 H$.*

By the definition, Ker$\tau = \mathcal{K}_{g,1}$. Johnson[27] showed that $\mathcal{I}_{g,1}/\mathcal{T}$ is isomorphic to $\wedge^3 H$ where $\mathcal{T}$ is the subgroup of $\mathcal{I}_{g,1}$ generated by Dehn twists along BSCC's. Namely $\mathcal{K}_{g,1}$ is generated by Dehn twists along BSCC's.

Let $\mathcal{I}^2_{g,1}$ be the subgroup generated by all squares in $\mathcal{I}_{g,1}$ and $\mathcal{I}'_{g,1}$ $(= \mathcal{I}_{g,1}(1))$ be the commutator subgroup of $\mathcal{I}_{g,1}$.

Lemma 7.4.7.

$$\begin{array}{ccc} \mathcal{I}_{g,1}/\mathcal{I}'_{g,1} & \xrightarrow{\tau} & \wedge^3 H \\ {\scriptstyle \otimes\mathbb{Z}_2}\big\downarrow & & {\scriptstyle \otimes\mathbb{Z}_2}\big\downarrow \\ \mathcal{I}_{g,1}/\mathcal{I}^2_{g,1} & \xrightarrow{\tau} & \wedge^3 H \mathit{mod}\ 2 \end{array}$$

is a pullback diagram.

Proof. We see the map $\mathcal{I}_{g,1}/\mathcal{I}'_{g,1} \to \mathcal{I}_{g,1}/\mathcal{I}^2_{g,1} \oplus \wedge^3 H$ given by $x \mapsto (x \otimes \mathbb{Z}_2, \tau(x))$ is one to one and the image of the map is all elements (u, λ) satisfies that $\tau(u) = \lambda \otimes \mathbb{Z}_2$. First we see the injectivity of the map. From the extension

$$1 \to \mathcal{K}_{g,1} \to \mathcal{I}_{g,1} \to \wedge^3 H \to 1$$

and Lemma 7.3.1, we divide the above extension by $\mathcal{I}'_{g,1}$ and we have

$$1 \to \mathcal{K}_{g,1}/\mathcal{I}'_{g,1} \to \mathcal{I}_{g,1}/\mathcal{I}'_{g,1} \to \wedge^3 H \to 1.$$

For the simplicity, we write this extension as

$$0 \to T \to A \to \wedge^3 H \to 0.$$

We tensor $0 \to T \to A \to \wedge^3 H \to 0$ with $\mathbb{Z}_2$. Since T is already a $\mathbb{Z}_2$-vector space, $T \otimes \mathbb{Z}_2$. By the definition $A \otimes \mathbb{Z}_2 = A/2A$ is $U = \mathcal{I}_{g,1}/\mathcal{I}^2_{g,1}$. Thus we have another exact sequence

$$0 \to T \to U \to \wedge^3 H \bmod 2 \to 0.$$

Suppose $(x \otimes \mathbb{Z}_2, \tau(x)) = (0,0)$ namely x is contained in the kernel of the map. $\tau(x) = 0$ implies $x \in T$. On the other hand, from the extension

$$0 \to T \to U \to \wedge^3 H \bmod 2 \to 0,$$

$\tau(x) = 0$ means $x \in \mathcal{I}'_{g,1}$ namely $x = 0$ in A. Thus the injectivity of $A \to U \oplus \wedge^3 H$ was proved. Next, for any element (u, λ) satisfying $\tau(u) \equiv \lambda$ mod 2, we construct the element of A which maps to (u, λ). Since $A \to \wedge^3 H$ is surjective, we can choose an element f_1 of A s.t. $\tau(f_1) = \lambda$. Then $\tau(f_1) \equiv \tau(u)$ mod 2. Namely there exists an element t in A s.t.

$$A \ni t = f_1 \otimes \mathbb{Z}_2 - u \mapsto \tau(t) = 0 \in \wedge^3 H \text{mod } 2.$$

From the extension $0 \to T \to U \to \wedge^3 H\text{mod } 2 \to 0$, t is in T which is a subspace of U. we can think t is an element of A and put $f := f_1 - t \in A$. By the definition, the image of f in U is u and $\tau(f) = \tau(f_1) = \lambda$. □

From this lemma, once we understand U, then we can determine A.

7.5 The Birman-Craggs Homomorphism

For a fixed Heegaard embedding $h : S \to S^3$ and any $f \in \mathcal{I}_g$, we can construct a homology sphere M_f by cutting along Imh and reglue them by f as stated before. Birman & Craggs[6] constructed a homomorphism $\mathcal{I}_g \to \mathbb{Z}_2$ called the Birman-Craggs homomorphism. It takes f to the Rochlin invariant of M_f. This homomorphism depends on the choice of the embedding h but they showed that different choices of f make only finitely many different homomorphisms.

Based on their works, Johnson constructed a homomorphism σ which is also called the Birman-Craggs homomorphism or the Birman-Craggs-Johnson homomorphism. In this section, we introduce this homomorphism and state results of Johnson which is needed in this paper.

First we define the Boolean polynomial algebra $B_{g,1} = B$, the target group of σ. This is $\mathbb{Z}_2$-algebra with unit 1 and has a generator for each nonzero element $x \in H_1(S_{g,1}, \mathbb{Z})$. We denote this element by $\bar{x} \in B$. Furthermore, we require the following relations:

1. $\overline{x + y} = \bar{x} + \bar{y} + x \cdot y$ where $x \cdot y$ is the mod 2 intersection number.

2. $\bar{x}^2 = \bar{x}$.

From the relation 2, $a^2 = a$ for any $a \in B$ and all monomials $e_{i_1} e_{i_2} \cdots e_{i_k}$, $(0 \leq k \leq 2g; 1 \leq i_1 < i_2 < \cdots < i_k \leq 2g)$, where e_i is a basis for $H_1(S_{g,1}, \mathbb{Z})$, forms a $\mathbb{Z}_2$-basis for B. Let $B^k_{g,1} = B^k$ be the subspace generated by all monomials of degree $\leq k$. By theorem 7.1.2, it suffices to define $\sigma : \mathcal{I}_{g,1} \to B$ on a BP map of genus 1. If $f = T_\gamma T_\delta^{-1}$ and $S_{1,2}$ is a subsurface of $S_{g,1}$ whose boundary is $\gamma \cup \delta$, we have $\sigma(f) = \bar{a}\bar{b}(\bar{c} + 1)$ where c is the homology class of γ, oriented so that $S_{1,2}$ is on its left, and a, b are any homology classes of $H_1(S_{1,2}, \mathbb{Z}) \subset H_1(S_{g,1}, \mathbb{Z})$ s.t. $a \cdot b = 1$.

Johnson proved that $\sigma : U_{g,1} \to B^3_{g,1}$ is an isomorphism by investigating kernels of the module homomorphisms $\sigma : U \to B^3/B^i, i = 2, 1, 0$. Let T, S, R be the kernel of following homomorphisms respectively.

$$1 \to T \to U \to B^3/B^2 \to 1$$

$$1 \to S \to U \to B^3/B^1 \to 1$$

$$1 \to R \to U \to B^3/B^0 \to 1$$

Then $S = \mathrm{Ker}\{U \to B^3/B^1\} = \mathrm{Ker}\{T \to B^2/B^1\}$ and $R = \mathrm{Ker}\{U \to B^3/B^0\} = \mathrm{Ker}\{S \to B^1/B^0\}$. In Johnson[24], it was proved that $q : B^k_{g,1} \to \wedge^k H \mathrm{mod}\ 2$: the $\mathbb{Z}_2$-linear map which sends B^{k-1} to zero and the k-nomial $\bar{e}_{i_1} \bar{e}_{i_2} \cdots \bar{e}_{i_k}$ to $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$, identifies B^k/B^{k-1} and $\wedge^k H$ mod 2. Combining these facts and a technique to compute generators of the kernel (Johnson[28],Lemma 10.), he obtained generators of R and showed $R = \mathbb{Z}_2$ for $g \geq 3$ and $\sigma : R \to B^0$ is an isomorphism. Hence we have the following theorem.

Theorem 7.5.1. *$\sigma : U_{g,1} \to B^3_{g,1}$ is an isomorphism for $g \geq 3$.*

Recall that we put $A = \mathcal{I}_{g,1}/\mathcal{I}'_{g,1}$ and $U = \mathcal{I}_{g,1}/\mathcal{I}^2_{g,1}$.

Theorem 7.5.2.

$$\begin{array}{ccc} A & \xrightarrow{\tau} & \wedge^3 H \\ \sigma \downarrow & & \downarrow \otimes\mathbb{Z}_2 \\ B^3 & \xrightarrow{q} & \wedge^3 H \mathit{mod}\ 2 \end{array}$$

is a pullback diagram. Hence $\mathcal{I}'_{g,1} = \mathrm{Ker}\,\tau \cap \mathrm{Ker}\,\sigma$.

Proof. The statement follows from Lemma 7.4.7 and commutativity of the following diagram.

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d, "quotient"'] & B^3 \arrow[d, "\otimes\mathbb{Z}_2"] \\ U \arrow[ur, "\sigma", "\cong"'] \arrow[r, "g"'] & \wedge^3 H \mathrm{mod}\ 2 \end{tikzcd}$$

□

Theorem 7.5.3. *The kernel of $\sigma : \mathcal{I}_{g,1} \to B^3$ is precisely $\mathcal{I}^2_{g,1}$*

Summarizing these theorems, we have following corollary.

Corollary 7.5.4. $\mathcal{I}'_{g,1} = \mathcal{K}_{g,1} \cap \mathcal{I}^2_{g,1}$.

Chapter 8

Birman Exact Seqence

The Birman exact sequence is a basic and useful tool to study the mapping class group. In this chapter, we introduce the Birman exact sequence and investigate its kernel restricted to each of the Johnson filtration.

8.1 Forget Maps and Push Maps

Recall that S_g^m is a surface with m punctures and genus g. We denote the set of punctures by $P = \{p_1, \ldots, p_m\}$. We define the pure mapping class group $P\mathcal{M}_{g,n}^m$ as the subgroup of $\mathcal{M}_{g,n}^m$ which fix each puncture and boundary component individually. Note that $\mathcal{M}_{g,n}^m$ is required to fix P setwise.

The Birman exact sequence is described as follows:

$$1 \longrightarrow \Gamma^1 \xrightarrow{Push} \mathcal{M}_g^1 \xrightarrow{Forget} \mathcal{M}_g \longrightarrow 1.$$

The forgetful map $Forget$ is the map induced by the inclusion $S_g^1 \hookrightarrow S_g$, and x be the puncture of S_g^1. Let f be an element of the Kernel of $Forget$, ϕ be its representation in $\mathcal{M}_g^1$. Since $Forget(f) = 1$, there exists an isotopy from $Forget(f)$ to the identity map of S_g. During this isotopy, the image of the puncture, which is a point in S_g and we denote it as x too, traces a loop α based at x. We can push x along α^{-1} and recover $f \in \mathcal{M}_g^1$. So the kernel of $Forget$ is isomorphic to Γ.

Theorem 8.1.1 (see Birman[5]). *Suppose $\chi(S_g^1) < 0$, then the following sequence is exact:*

$$1 \longrightarrow \Gamma \xrightarrow{Push} \mathcal{M}_g^1 \xrightarrow{Forget} \mathcal{M}_g \longrightarrow 1.$$

Similarly, we can generalize to m punctured version.

$$1 \longrightarrow \Gamma^{m-1} \xrightarrow{Push} P\mathcal{M}_g^m \xrightarrow{Forget} P\mathcal{M}_g^{m-1} \longrightarrow 1$$

This gives a inductive step for the number of punctures.

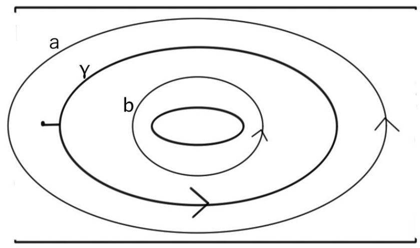

Fig. 8.1

We have a useful description of $Push$ in terms of Dehn twists. Let γ be a simple element in Γ i.e. represented by simple closed curve, and a, b are simple closed curves as illustrated in Fig. 8.1. We look at neighborhood of γ and consider the action of $Push(\gamma)$.

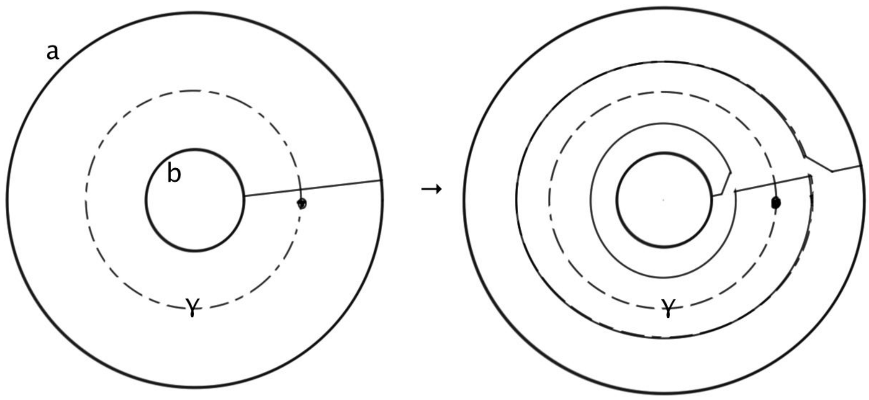

Fig. 8.2. The push map as Dehn twits.

Fig. 8.2 shows that $Push(\gamma) = T_a T_b^{-1}$. This is a BP map, and so Γ is contained in the Torelli group.

We have another related short exact sequence called the relative Birman exact sequence. Take a surface with one boundary component. Let D be a disc in S_g s.t. $\partial D = \partial S_{g,1}$. If we think of $\mathcal{M}_{g,1}$ as all diffeomorphism of S_g which are identity on D modulo isotopies which fix D pointwise, then we obtain a homomorphism $\mathcal{M}_{g,1} \to \mathcal{M}_g$. Since every diffeomorphism of S_g is isotopic to one which is identity on D, this homomorphism is surjective. Johnson[26] determined the kernel of this homomorphism.

Theorem 8.1.2 (Johnson[26])**.**

$$1 \longrightarrow \pi_1(UT) \longrightarrow \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_g \longrightarrow 1,$$

where $\pi_1(UT)$ is the fundamental group of the unit tangent bundle of S_g.

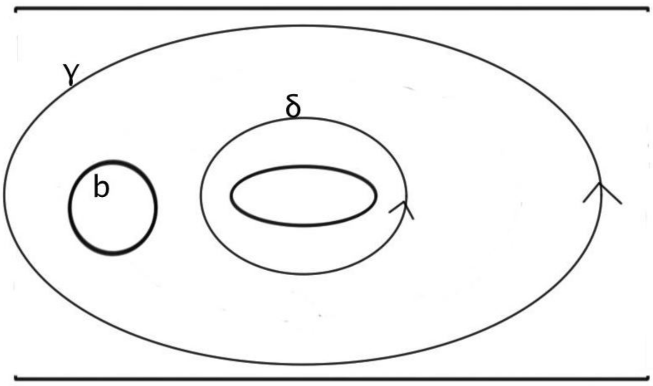

Fig. 8.3. A generator for $\pi_1(UT)$.

We can think $\pi_1(UT)$ as a subgroup of $\mathcal{M}_{g,1}$ which is generated by elements come from $Push$ and the Dehn twist along the boundary which is in $\mathcal{K}_{g,1}$. More precisely, it is known that $\pi_1(UT)$ is generated by the Dehn twist along the boundary T_b and BP maps $T_\gamma T_\delta^{-1}$ s.t. BP (γ, δ) bound a subsurface with genus $g-1$ as shown in Fig. 8.3. $\pi_1(UT)$ is also in the Torelli group.

By Theorem 7.2.5, $\pi_1(UT)/(\mathcal{N}_{g,1}(2) \cap \pi_1(UT))$ has a subgroup which is isomorphic to $\mathbb{Z}$ generated by a fixed BP map. For later use, we state following fact.

Lemma 8.1.3. *$\mathcal{N}_{g,1}(2) \cap \pi_1(UT)$ is a infinite index subgroup of $\pi_1(UT)$*

When we study the Johnson filtration, it is convenient to know the kernel of these exact sequence restricted to $\mathcal{N}_g(k)$ and we will observe it in the next section.

On the other hand, we have the following theorem on the homomorphisms induced by inclusions of subsurfaces by Paris & Rolfsen[47]

Theorem 8.1.4. *Let S be a subsurface of $S^m_{g,n}$. Suppose S is not a disk nor an annulus, and each component of $S^m_{g,n} - S$ is once punctured disk or annulus with no puncture. Let $a_1, \ldots, a_r$ be the boundary components of S which bound once punctured disks, and b_j, b'_j, $j = 1, \ldots, s$ be the pairs of boundary components of S which cobound annulus with no puncture. Then the kernel of the homomorphism induced by the inclusion* $\mathrm{Ker}\{i : \mathcal{M} \to \mathcal{M}^m_{g,n}\}$ *is generated by* $\{T_{a_1}, \ldots, T_{a_r}, T^{-1}_{b_1}T_{b'_1}, \ldots, T^{-1}_{b_s}T_{b'_s}\}$ *where $\mathcal{M}$ is the mapping class group of S.*

We give the sketch of the proof. Fig. 8.4 shows an example of the position of curves and subsurfaces.

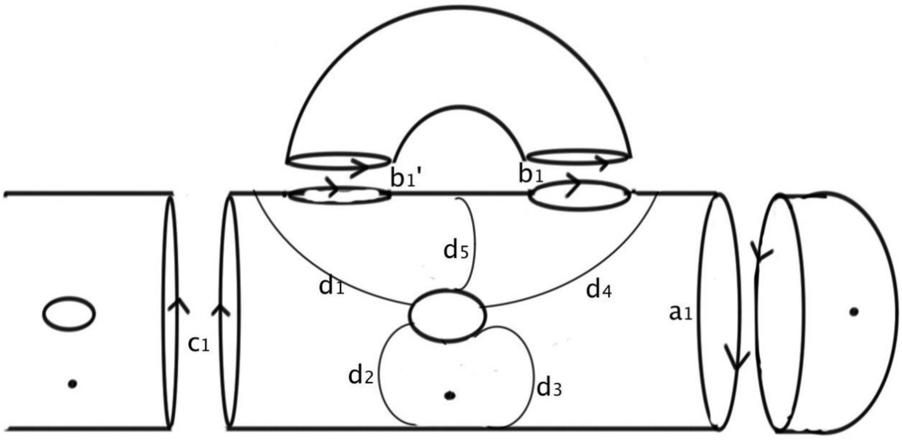

Fig. 8.4

Let Σ_P be the group of permutations of the set of punctures P in $S^m_{g,n}$. By the definition of the pure mapping class group, we have the exact sequence;

$$1 \to P\mathcal{M}^m_{g,n} \to \mathcal{M}^m_{g,n} \to \Sigma_P \to 1.$$

Let P' be the set of punctures in S. Then we have the following commutative diagram.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P\mathcal{M} & \longrightarrow & \mathcal{M} & \longrightarrow & \Sigma_{P'} & \longrightarrow & 1 \\ & & \downarrow & & {\scriptstyle i}\downarrow & & \downarrow & & \\ 1 & \longrightarrow & P\mathcal{M}^m_{g,n} & \longrightarrow & \mathcal{M}^m_{g,n} & \longrightarrow & \Sigma_P & \longrightarrow & 1 \end{array}$$

Suppose $f \in \mathrm{Ker}\, i$. Since $\Sigma_{P'} \to \Sigma_P$ is an injective homomorphism, and the above diagram is commutative, $f = 1$ in $\Sigma_{P'}$. Namely f is in $P\mathcal{M}$. Let $c_1, \ldots, c_t$ be the components of ∂S different from the a_i and b_j, b'_j, and $d_1, \ldots, d_u$ be simple closed curves which define a pants decomposition of S. A pants decomposition is a collection of simple closed curves on a surface which cut the surface into $S^3_{0,0}$'s. Such curves can be always found when the surface has a negative Euler characteristic. Since f is in $\operatorname{Ker} i$, f is isotopic to identity in $S^m_{g,n}$. So each $f(d_i)$ is isotopic to d_i in $S^m_{g,n}$. This is true in S if $\overline{S^m_{g,n} - S}$ does not have disks with no puncture as its components(Paris & Rolfsen [47], Proposition 3.5). Thus we assume f is identity on the boundary of each pants, and so f is a product of Dehn twists which does not intersect the boundary of each pants. Hence we have

$$f = T_{a_1}^{\alpha_1} \cdots T_{a_r}^{\alpha_r} T_{b_1}^{\beta_1} T_{b_1'}^{\beta_1'} \cdots T_{b_s}^{\beta_s} T_{b_s'}^{\beta_s'} T_{c_1}^{\gamma_1} \cdots T_{c_t}^{\gamma_t} T_{d_1}^{\delta_1} \cdots T_{d_u}^{\delta_u}.$$

Since each T_{a_i} is trivial in $\mathcal{M}^m_{g,n}$ by the assumption and T_{b_j} and $T_{b_j'}$ is the same element in $\mathcal{M}^m_{g,n}$,

$$1 = i(f) = T_{b_1}^{\beta_1+\beta_1'} \cdots T_{b_s}^{\beta_s+\beta_s'} T_{c_1}^{\gamma_1} \cdots T_{c_t}^{\gamma_t} T_{d_1}^{\delta_1} \cdots T_{d_u}^{\delta_u}.$$

Hence we have

$$\beta_1 + \beta_1' = \cdots = \beta_s + \beta_s' = \gamma_1 = \cdots = \gamma_t = \delta_1 = \cdots = \delta_u = 0.$$

Therefor

$$f = T_{a_1}^{\alpha_1} \cdots T_{a_r}^{\alpha_r} (T_{b_1}^{-1} T_{b_1'})^{\beta_1'} \cdots (T_{b_s}^{-1} T_{b_s'})^{\beta_s'}.$$

Conversely, any element in this form is in $\ker i$.

We use the following fact that is a special case of the above Theorem.

Lemma 8.1.5. $1 \to \mathbb{Z} \to \mathcal{M}_{g,1} \to \mathcal{M}^1_g \to 1$ *is a short exact sequence where* $\mathbb{Z}$ *is the group generated by the Dehn twist along* $\partial\mathcal{M}_{g,1}$.

8.2 The Birman Exact Sequence for the Johnson filtrations

In this section, we observe the kernel of the maps $\mathcal{N}_{g,1}(k) \to \mathcal{N}_g(k)$ and $\mathcal{I}_{g,1}(k) \to \mathcal{I}_g(k)$. Γ is always $\pi_1(S_g)$ in this section.

In the previous section, we saw three short exact sequences;

$$1 \to \pi_1(S_g) \to \mathcal{M}^1_g \to \mathcal{M}_g \to 1,$$

$$1 \to \pi_1(UT) \to \mathcal{M}_{g,1} \to \mathcal{M}_g \to 1,$$

and

$$1 \to \mathbb{Z} \to \mathcal{M}_{g,1} \to \mathcal{M}^1_g \to 1.$$

Combining a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(UT) \xrightarrow{\ p\ } \pi_1(S_g) \longrightarrow 1,$$

we have the following diagram (Farb & Margalit[15]).

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & \mathbb{Z} \arrow[r, "\simeq"] \arrow[d] & \mathbb{Z} \arrow[d] & & \\
1 \arrow[r] & \pi_1(UT) \arrow[r] \arrow[d, "p"] & \mathcal{M}_{g,1} \arrow[r] \arrow[d, "f"] & \mathcal{M}_g \arrow[r] \arrow[d, "\simeq"] & 1 \\
1 \arrow[r] & \pi_1(S_g) \arrow[r] \arrow[d] & \mathcal{M}_{g,*} \arrow[r] \arrow[d] & \mathcal{M}_g \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}$$

From the relative Birman exact sequence, we have

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(UT) \arrow[r] & \mathcal{M}_{g,1} \arrow[r] & \mathcal{M}_g \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(UT) \cap \mathcal{N}_{g,1}(k) \arrow[r] \arrow[u] & \mathcal{N}_{g,1}(k) \arrow[r] \arrow[u] & \mathcal{N}_g(k) \arrow[r] \arrow[u] & 1.
\end{tikzcd}$$

We denote $K_k := \pi_1(UT) \cap \mathcal{N}_{g,1}(k)$.

As we saw in the previous section, the image of $\pi_1(UT)$ is in $\mathcal{I}_{g,1} = \mathcal{N}_{g,1}(1)$. So $K_1 = \pi_1(UT)$.

Fact 8.2.1. *If G is a infinite index subgroup of surface group then G is a free group.*

Theorem 8.2.2. *$K_2 \simeq \mathbb{Z} \times F$ where F is a free group.*

Proof. From the above central extension, we have the following diagram.

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \arrow[r] & \pi_1(UT) \arrow[r, "p"] & \pi_1(S_g) \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Z} \cap K_2 \arrow[r] \arrow[u] & K_2 \arrow[r] \arrow[u] & p(K_2) \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

Let $Z(K_k)$ be the center of K_k. As mentioned in Corollary 7.5.4, $\mathcal{N}_{g,1}(2) \supset \mathcal{I}_{g,1}(1) = \mathcal{N}_{g,1}(2) \bigcap \mathcal{I}_{g,1}^2$. Hence the group generated by the square of the Dehn twist along the boundary component of $S_{g,1}$, denote Z, is in $\mathcal{N}_{g,1}(2)$ and $\pi_1(UT)$. So Z is in K_2. By Lemma 6.1.5, Z is central and isomorphic to $\mathbb{Z}$. Therefore $Z(K_2)$ has a subgroup isomorphic to $\mathbb{Z}$.

Next, we consider the following diagram.

$$
\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \arrow[r] & \pi_1(UT) \arrow[r, "p"] & \pi_1(S_g) \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Z} \cap K_2 \arrow[r] \arrow[u] & K_2 \arrow[r] \arrow[u] & p(K_2) \arrow[r] \arrow[u] & 1 \\
 & & Z(K_2) \arrow[r] \arrow[u] & p(Z(K_2)) \arrow[u] &
\end{tikzcd}
$$

Since $p(Z(K_2))$ is the subgroup of $\pi_1(S_g)$ and a surface group has no non-trivial center, $p(Z(K_2)) = 1$. Combining the commutativity of the above diagram, $Z(K_2)$ is in $\mathbb{Z} \cap K_2$. Hence $\mathbb{Z} \cap K_2 \simeq \mathbb{Z}$. We have

$$K_2 \simeq \mathbb{Z} \times p(K_2).$$

By Lemma 8.1.3, K_2 is a infinite index subgroup of $\pi_1(UT)$, so $p(K_2)$ is a infinite index subgroup of $\pi_1(S_g)$. From Fact 8.2.1, $p(K_2)$ is a free group. $\square$

Next, we see the case $k \geqq 3$.

Lemma 8.2.3. *The natural homomorphism $f : \mathcal{N}_{g,1}(k) \to \mathcal{N}_{g,*}(k)$ is isomorphism for $k \geq 3$.*

Proof. By Lemma 8.1.4, we have the short exact sequence

$$1 \to \mathbb{Z} \to \mathcal{M}_{g,1} \to \mathcal{M}_g^1 \to 1$$

where $\mathbb{Z}$ is the group generated by the Dehn twist along $\partial\mathcal{M}_{g,1}$. By Theorem 7.2.4, the intersection of this group and $\mathcal{N}_{g,1}(k)$ is trivial. Hence the above extension restrict to the isomorphism

$$\mathcal{N}_{g,1}(k) \to \mathcal{N}_g^1(k).$$

$\square$

From this lemma, if we restrict ourself to the $k \geq 3$ case, then the Birman exact sequence and relative Birman exact sequence are the same.

Lemma 8.2.4 (Asada & Kaneko[2])**.** $\pi_1(S_g) \cap \mathcal{N}_g^1(k+1) = \Gamma(k)$

Lemma 8.2.5 (Karrass & Solitar[31])**.** *Let F be a free group and H be an infinite index subgroup of F. If H contains a nontrivial normal subgroup of F, then H is a free group of infinite rank.*

Theorem 8.2.6. *K_k is a free group of infinite rank for $k \geqq 3$*

Proof. Suppose $k \geq 3$. By Lemma 8.2.3, we have the following diagram.

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \arrow[r] & \mathcal{M}_{g,1} \arrow[r, "f"] & \mathcal{M}_g^1 \arrow[r] & 1 \\
 & 1 \arrow[r] & \mathcal{N}_{g,1}(k) \arrow[r, "\simeq"] \arrow[u] & \mathcal{N}_g^1(k) \arrow[r] \arrow[u] & 1 \\
 & 1 \arrow[r] & K_k \arrow[r, "\simeq"] \arrow[u] & f(K_k) \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

Hence, we can identify K_k with a subgroup of $\pi_1(S_g)$. Combining Lemma 8.2.3 and Lemma 8.2.4, we have

$$\begin{aligned}
K_k &= \pi_1(UT) \cap \mathcal{N}_{g,1}(k) \\
&= \pi_1(S_g) \cap \mathcal{N}_{g,1}(k) \\
&= \pi_1(S_g) \cap \mathcal{N}_g^1(k) \\
&= \Gamma(k-1).
\end{aligned}$$

Γ is not nilpotent, so K_k is not trivial for $k < \infty$.

Since $\Gamma/\Gamma(1) = H_1(S_g, \mathbb{Z})$ is infinite, $\Gamma(1)$ is an infinite index subgroup of Γ. From Fact 8.2.1, $\Gamma(1)$ is a free group. As mentioned in the proof of Lemma 7.2.2, the successive quotients $\bar{\Gamma}_k = \Gamma(k-1)/\Gamma(k)$ forms the graded Lie algebra generated by $H = \Gamma/\Gamma(1)$, and $\Gamma(k)/\Gamma(k+1)$ is infinite. Hence, $K_k = \Gamma(k-1)$ is an infinite index subgroup of $\Gamma(1)$. Combining the fact K_{k+1} is a normal subgroup of K_k and Lemma 8.2.5, K_k is a free group of infinite rank. □

Similarly, let the kernel of the homomorphism $\mathcal{I}_{g,1}(k) \to \mathcal{I}_g(k)$ be K'_k. By the same argument in the proof of the Theorem 8.2.2, we have following.

Corollary 8.2.7. *$K'_2 \simeq \mathbb{Z} \times F$ where F is a free group.*

From the argument in the proof of Theorem 8.2.6, we have following.

Corollary 8.2.8. *K'_k is a free group of infinite rank for $k \geqq 3$.*

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