On the existence of a crepant resolution and the McKay correspondence for Gorenstein toric quotients of the conifold

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Contents

Chapter 1

Introduction

In the following, we work over $\boldsymbol{C}$. Let (X, x) be a normal $\boldsymbol{Q}$-Gorenstein singularity and $f : Y \to X$ be a resolution of a singularity with exceptional divisors E_i $(i = 1, 2, \ldots, r)$. Then the adjunction formula $K_Y = f^*K_X + \sum_{i=1}^{r} \mathrm{discr}(E_i)$ holds for rational numbers $\mathrm{discr}(E_i)$, which are called *discrepancies*. If $\mathrm{discr}(E_i) = 0$ for all i, f is called a *crepant* resolution. (X, x) is called *canonical* (resp. *terminal*) if the inequality $\mathrm{discr}(E_i) \geq 0$ (resp. > 0) holds for all i.

Varieties with nonsmooth terminal singularities do not admit any crepant resolutions by definition. Although, canonical Gorenstein quotients of those varieties, if they exist, may admit a crepant resolution. In this thesis, we give a series of toric canonical Gorenstein quotients of the toric conifold by toric group actions which admit crepant resolution. Moreover, we consider a McKay correspondence for those quotients. For three dimensional canonical Gorenstein quotient singularities, the existence of crepant resolution and the McKay correspondence are well known (see Theorem 1.0.1 and Theorem 1.0.2), but for three dimensional canonical Gorenstein quotients of singular varieties, these problems have been left untouched.

The crepant resolution plays an important role in the study of the McKay correspondence. The McKay correspondence is often expressed as "a bridge" connecting the representation theory and the geometry. In [21], J. McKay found out a strange coincidence of two graphs for two-dimensional quotient singularities $\boldsymbol{C}^2/G$ where G is a finite subgroup of $GL(2,\boldsymbol{C})$. One of the graphs is given by the nontrivial irreducible representations of G and the other is the dual graph of exceptional set, where the vertices are the exceptional divisors of the minimal resolution of $\boldsymbol{C}^2/G$ and the edges are the intersections. The coincidence of graphs was interpreted as an isomorphism between the G-equivariant K-theory of $\boldsymbol{C}^2$ and the K-theory of the minimal resolution $\widetilde{\boldsymbol{C}^2/G}$ of $\boldsymbol{C}^2/G$ in [11]. A finer interpretation is given by [18] as an equivalence between the derived category of G-equivariant coherent sheaves on $\boldsymbol{C}^2$ and the derived category of coherent sheaves on $\widetilde{\boldsymbol{C}^2/G}$. The McKay correspondence was generalized to three-dimensional Gorenstein quotient singularities by using crepant resolutions. The three-dimensional set-theoretical McKay correspondence was given by Y. Ito and M. Reid as follows.

Theorem 1.0.1 ([14])**.** *Let Y be a crepant resolution of three-dimensional Gorenstein quotient singularity $\boldsymbol{C}^3/G$. Then there exists a correspondence between the set of canonical bases of $H^{2i}(Y,\boldsymbol{Q})$ and the set of conjugacy classes of G with weight i where i is in $\boldsymbol{Z}\cap[0,3]$.*

On the other hand, the existence problem of crepant resolution for three-dimensional Calabi-Yau varieties raised by physicists on 1980's. Y. Ito, D. G. Markushevich and S. S. Roan answered to the problem in the case of quotient singularities.

Theorem 1.0.2 ([12][19][25])**.** *Any three-dimensional Gorenstein quotient singularity $\boldsymbol{C}^3/G$ admits a crepant resolution.*

For three-dimensional Gorenstein quotient singularities, T. Bridgeland, A. King and M. Reid gave a construction of crepant resolution and the McKay correspondence by using Hilbert scheme of G-orbits and Serre functor of derived category in [1]. In the case that the dimension is higher than three, crepant resolutions of Gorenstein quotient singularities do not always exist. Nevertheless, for some special cases, sufficient conditions for the existence of a crepant resolution was found out by [7][26] and other papers.

In the studies of the McKay correspondence and the existence problem of crepant resolutions, quotient singularities have been main objects. By the way, if the dimension is greater than two, a minimal model has terminal singularities in general. So, it will be natural to consider a generalization to quotient spaces of terminal 3-folds.

We consider the existence of toric crepant resolutions for Gorenstein singularities which are given as quotients of the toric conifold X by finite groups G acting *toroidally* (see Definition 2.2.1) and compute the Euler number of the crepant resolutions. Moreover, we consider an analogy to Theorem 1.0.1 for X/G. Throughout the thesis, we always take a toric model of the singularity, and we sometimes denote a singularity (X, x) by X for simplicity. It is known that any affine toric terminal 3-fold X is smooth or isomorphic to either of the following two:

(i) the quotient singularity of type $\frac{1}{r}(a, -a, 1)$ where a and r are coprime,

(ii) the hypersurface singularity $\mathrm{Spec}(\boldsymbol{C}[x, y, z, w]/(xz - yw))$.

See Theorem 2.2.2 and Theorem 2.2.3. If X is smooth and X/G is a Gorenstein singularity, then there exists a crepant resolution for X/G by Theorem 1.0.2. In the case that X is a quotient singularity of type $\frac{1}{r}(a, -a, 1)$, then there exists a crepant resolution for X/G. This is because the existence problem for X/G can be reduced to the existence problem for $\boldsymbol{C}^3/G'$ where G' is a small finite subgroup of $SL(3, \boldsymbol{C})$. For details, see Section 2.3. In the case of the hypersurface $\mathrm{Spec}(\boldsymbol{C}[x, y, z, w]/(xz - yw))$, which we call *the conifold* in the following, we assume that the quotient X/G has a Gorenstein singularity. In Section 2.4, we give a classification of the toroidal group actions on the conifold. In Section 2.5, we show that X/G admits a toric crepant resolutions and compute the Euler number. The main result of this thesis is as follows.

Theorem 1.0.3. *Let X be the conifold and G be a finite group acting on X toroidally. Assume X/G is a Gorenstein singularity. Then X/G admits a toric crepant resolution $\widetilde{X/G}$. The Euler number of $\widetilde{X/G}$ is $2|G|$ where $|G|$ is the order of G.*

Corollary 1.0.1. *Let X be an affine toric terminal 3-fold and G be a finite group acting on X toroidally. Assume X/G is a Gorenstein singularity. Then X/G admits a toric crepant resolution $\widetilde{X/G}$.*

We remark that, in the case that X is the conifold, the Euler number of a crepant resolution $\widetilde{X/G}$ is $2|G|$. This implies that the usual McKay correspondence on $\widetilde{X/G}$ does not hold. Indeed, every conjugacy class of G corresponds to two toric prime divisors on $\widetilde{X/G}$. However this complexity can be interpreted by *the string theoretic Hodge theory* and *the strong McKay correspondence* advocated by V. V. Batyrev and D. I. Dais in [3]. Roughly, the strong McKay correspondence is the McKay correspondence limited on the exceptional divisors, and that is hold for *GV-varieties*: varieties which have a stratification by affine charts with at most Gorenstein toric or Gorenstein quotient singularities. Therefore, in the case of GV-varieties, we can construct the McKay correspondence on every affine chart. In our case, a small resolution $\widehat{X/G}$ is covered by two Gorenstein quotient singularities which are isomorphic to each other. There exists a crepant resolution $\widetilde{X/G}$ which is covered by two crepant resolutions of the Gorenstein quotient singularities and go through the small resolution. Therefore, for $\widetilde{X/G}$, the strong McKay correspondence holds on every affine chart. For the detail, see Section 3.

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1.1 Basic knowledge on toric varieties

In this section, we shall recall basic knowledge of affine toric varieties which is the main object of our study. Toric varieties are suitable for constructing examples of the McKay correspondence because it is easy to observe the cohomologies on a toric crepant resolution.

Definition 1.1.1. We define $\boldsymbol{T}^n$ as follows and call it an n-dimensional torus.

$$\boldsymbol{T}^n := \underbrace{\boldsymbol{C}^* \times \boldsymbol{C}^* \times \cdots \times \boldsymbol{C}^*}_{n-times}$$

Sometimes, $\boldsymbol{T}^n$ is denoted by $\boldsymbol{T}$ for simplicity when the dimension of the torus is clear.

Definition 1.1.2. A normal irreducible variety X is said to be toric if $\boldsymbol{T}$ is contained in X as a Zariski open dense subset and the group action of $\boldsymbol{T}$ on itself extends to an algebraic group action of $\boldsymbol{T}$ on X, which we denote by $\pi_{\boldsymbol{T}}$.

In this thesis, we call $\pi_{\boldsymbol{T}}$ a *torus action* on X. For $a = (a_1, a_2, \ldots, a_n) \in \boldsymbol{Z}^n$, a group homomorphism $\mathbf{e}(a) : \boldsymbol{T} \to \boldsymbol{C}^*$ which is called a *character* is given as follows.

$$\mathbf{e}(a)(t_1, t_2, \ldots, t_n) = t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$$

It is known that all characters of $\boldsymbol{T}$ is given by this way. Characters of $\boldsymbol{T}$ form a free abelian $\boldsymbol{Z}$-module M which is called the *character lattice*. We may identify M with $\boldsymbol{Z}^n$. Let N be the dual $\boldsymbol{Z}$-module of M, i.e., $N = \mathrm{Hom}_{\boldsymbol{Z}}(M, \boldsymbol{Z})$. We note that N is naturally isomorphic to $\boldsymbol{Z}^n$. In the case that X is of finite type, the orbit space of X by $\pi_{\boldsymbol{T}}$ corresponds to a finite set (N, Δ) of the faces of a *rational strongly convex polyhedral* cone σ in $N_{\boldsymbol{R}} := N \otimes \boldsymbol{R}$, which is called a *finite fan*. When N is clear, we sometime denote a finite fan (N, Δ) by Δ for simplicity. The intersection of the dual cone of σ and M is a semi-group in M, which we denote by S_X. Let $\{\check{e_1}, \check{e_2}, \ldots, \check{e_n}\}$ be the canonical $\boldsymbol{Z}$-basis of M. By using the canonical pairing $\langle\ ,\ \rangle$, we have the dual $\boldsymbol{Z}$-basis of N denoted by $\{e_1, e_2, \ldots, e_n\}$. For $(b_1, b_2, \ldots, b_n) \in N$, a group homomorphism $\gamma_n : \boldsymbol{C}^* \to \boldsymbol{T}$ which is called a *one parameter subgroup* of $\boldsymbol{T}$ is defined as follows.

$$\gamma_n(t) = (t^{b_1}, t^{b_2}, \ldots, t^{b_n})$$

By the above discussion, an affine toric variety X has coordinates induced by the coordinates $\{\mathbf{e}(e_1), \mathbf{e}(e_2), \ldots, \mathbf{e}(e_n)\}$ of $\boldsymbol{T}$.

Let $\{m_i \in M \mid 1 \leq i \leq s\}$ be a system of minimal generators of S_X, hence $S_X = \sum_{i=1}^{s} \boldsymbol{Z}_{\geq 0} m_i$. We have local coordinates $(\boldsymbol{e}(m_1), \cdots, \boldsymbol{e}(m_s))$ on X, and the action $\pi_{\boldsymbol{T}}$ can be written as follows:

$$\pi_{\boldsymbol{T}}(t, (\boldsymbol{e}(m_1), \cdots, \boldsymbol{e}(m_s))) = (t(m_1)\boldsymbol{e}(m_1), \cdots, t(m_s)\boldsymbol{e}(m_s))$$

where t is an element in $\boldsymbol{T}$ and $\boldsymbol{e}(m_i)$ is the character of $\boldsymbol{T}$ for $m_i \in M$.

1.2 Toric quotient singularities

In this section, we shall recall a relation between age and discrepancy in case of toric quotient singularities. The main references are [10] and [22].

For a finite fan (N, Δ), we denote the corresponding toric variety by $X(N, \Delta)$, which is written as $X(N, \sigma)$ if Δ consists of the faces of a cone σ.

Proposition 1.2.1. *A toric variety $X(N, \Delta)$ is nonsingular if and only if each $\sigma \in \Delta$ is generated by a part of a basis of N. (See p.15 of [22].)*

We shall use the following.

Corollary 1.2.1. *Let σ be an n-dimensional simplicial convex cone generated by n primitive elements $x_1, \ldots, x_n$ in N. Then $X(N, \sigma)$ is nonsingular, if and only if $\{x_1, \ldots, x_n\}$ is a basis of N.*

Let g be an element of finite order in $GL(n, \boldsymbol{C})$. Then g is diagonalizable and there exists $h \in GL(n, \boldsymbol{C})$ such that

$$hgh^{-1} = \begin{pmatrix} e^{2\pi i \theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i \theta_n} \end{pmatrix}$$

where $\theta_1, \theta_2, \cdots, \theta_n$ are rational numbers in $[0, 1)$. We define the *age* of g as

$$\operatorname{age}(g) := \theta_1 + \theta_2 + \cdots + \theta_n.$$

The age is independent of the choice of h. The age of g is an integer if g is in $SL(n, \boldsymbol{C})$.

We shall use the following vector notation:

$$\frac{1}{s}(t_1, t_2, \cdots, t_n) := \begin{pmatrix} e^{2\pi i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i\theta_n} \end{pmatrix}$$

where θ_i equals to $\frac{t_i}{s}$ and $t_1, t_2, \cdots, t_n$ are nonnegative integers which are less than s. The age of g equals to $\frac{1}{s}(t_1 + \cdots + t_n)$. We denote the vector $\frac{1}{s}(t_1, t_2, \ldots, t_n)$ by $v(g)$.

Let $g_1, g_2, \cdots, g_m$ be elements in $SL(n, \boldsymbol{C})$. If $g_1, g_2, \cdots, g_m$ are commutative each other, then the elements $g_1, g_2, \cdots, g_m$ are simultaneously diagonalizable.

Let $G \subset SL(n, \boldsymbol{C})$ be an abelian finite subgroup. We may assume that G is diagonalized. We consider the natural action of G on $\boldsymbol{C}^n$. In this case, the action of groups is represented as the following theorem by toric technique.

Theorem 1.2.1. *Let $\{e_1, e_2, \cdots, e_n\}$ be the canonical basis of N. Let $N' = N + \sum_{g \in G} v(g)\boldsymbol{Z}$. Then N is a submodule of N' with finite index.*

Let Δ be the finite fan which is generated by $\sigma := \langle e_1, e_2, \cdots, e_n \rangle_{\boldsymbol{R}_{\geq 0}}$ and $\psi : (N, \Delta) \to (N', \Delta)$ be the natural morphism of finite fans. Then ψ corresponds to the morphism of toric varieties denoted by $X(\psi) : X(N, \Delta) \to X(N', \Delta)$ and $X(\psi)$ is the quotient map by $N'/N \simeq G$.

It is known that any toric variety admits an equivariant resolution of singularities.

Theorem 1.2.2. *Let Δ' be a locally finite nonsingular subdivision of a fan Δ in N. Then the equivariant holomorphic map $\mathrm{id}_* : X(N, \Delta') \to X(N, \Delta)$ corresponding to the natural map $(N, \Delta') \to (N, \Delta)$ is proper birational and is an equivariant resolution of singularities for $X(N, \Delta)$.*

Moreover, for a primitive vector $v \in N'$ such that $v\boldsymbol{R}_{\geq 0}$ is in Δ', there exists an element g in G such that $v(g) = v$ by the quotient map in Theorem 1.2.1. For an exceptional divisor E_g, the following formula holds:

$$\mathrm{discr}(E_g) = \mathrm{age}(g) - 1,$$

where $E_g = \overline{\mathrm{orb}(\boldsymbol{R}_{\geq 0} v(g))}$.

For the details of the above proposition, theorems or corollary, see [22] (especially Section 1.4 and 1.5) and see [24] for the assertions with respect to age and discrepancy.

Next, we shall review a projectivity condition for toric morphisms.

Definition 1.2.1. An $\boldsymbol{R}$-valued function h on the support $|\Delta|$ is called *a* Δ*-linear support function* if h is $\boldsymbol{Z}$-valued on $N \cap |\Delta|$ and linear on each $\sigma \in \Delta$ where $|\Delta|$ means $\cup_{\sigma \in \Delta} \sigma$.

The set consisting of all Δ-linear support functions becomes an additive group. The group is denoted by $\mathrm{SF}(N, \Delta)$. In Definition 1.2.1, if Δ is a finite fan and h is $\boldsymbol{Q}$-valued on $N \cap |\Delta|$, h is also Δ-linear support function by taking some multiple. Let $\Delta(1)$ be the set of all 1-dimensional cones in Δ. For $\rho \in \Delta(1)$, we denote the primitive element in $N \cap \rho$ by $n(\rho)$.

Proposition 1.2.2. *There exists an injective homomorphism*

$$\mathrm{SF}(N, \Delta) \hookrightarrow \boldsymbol{Z}^{\Delta(1)}$$

$$h \mapsto (h(n(\rho)))_{\rho \in \Delta(1)}.$$

A support function h is determined by integers $h(n(\rho))$.

If X is nonsingular, there exists an isomorphism such that

$$\mathrm{SF}(N, \Delta) \xrightarrow{\sim} \boldsymbol{Z}^{\Delta(1)}.$$

Proposition 1.2.3. *A toric resolution* $\phi : X(N, \tilde{\Delta}) \to X(N, \Delta)$ *is complete if and only if* $|\tilde{\Delta}|$ *equals to* $|\Delta|$.

Definition 1.2.2. Suppose $|\tilde{\Delta}|$ equals to $|\Delta|$. A $\tilde{\Delta}$-linear support function h is said to be *strictly upper convex on* $\tilde{\Delta}$ if h satisfies the following conditions.
(*a*) $\langle l_\sigma, x \rangle \geq h(x)$ for all $\sigma \in \tilde{\Delta}$ and for all $x \in N_{\boldsymbol{R}}$,
(*b*) $\langle l_\sigma, x \rangle = h(x)$ if and only if $x \in \sigma$,
where l_σ is an element in M such that $\langle l_\sigma, x \rangle$ equals to $h(x)$ if x is in σ and $\langle l_\sigma, x \rangle$ equals to $\langle l_\tau, x \rangle$ for $x \in \tau$ if τ is a face of σ.

Let (N, Δ) be a finite fan and $\Delta(n)$ be the set of the n-dimensional cones in Δ. A set $\{l_\sigma;\ \sigma \in \Delta(n)\} \subset M$ is determined uniquely by $h \in \mathrm{SF}(N, \Delta)$.

Proposition 1.2.4. *For a complete toric resolution $\phi : X(N, \tilde{\Delta}) \to X(N, \Delta)$, the following conditions are equivalent.*
(a) ϕ is projective.
(b) There exists $h \in \mathrm{SF}(N, \tilde{\Delta})$ such that h is strictly upper convex on $\tilde{\Delta}$.

1.3 Complete intersection quotient singularities

In this section, we shall describe a part of the results of [29] and [28]; which give the criterion for some quotient singularities to be complete intersection. Notations used here is similar to the previous sections.

Definition 1.3.1. Let R be the ring $\boldsymbol{C}[X_1, X_2, \cdots, X_n]$ and I be the index set $\{1, 2, 3, \cdots, n\}$ of the variables, D be a set consisting of subsets of I and ω be a map from D to the set of the positive integers $\boldsymbol{Z}_{>0}$. The pair (D, ω) is said to be a *special datum*, if D and I satisfy the following conditions.
(a) the subset $\{i\}$ is an element in D for any $i \in I$,
(b) if J and J' are elements in D, then J and J' satisfy the condition $J \subset J'$, $J' \subset J$ or $J \cap J' = \varnothing$,
(c) if J is a maximal set in D, then $\omega(J)$ equals to 1,
(d) if J and J' are elements in D and if J' contains J properly, then $\omega(J')$ divides $\omega(J)$ and $\omega(J)$ is bigger than $\omega(J')$,
(e) if J_1, J_2 and J are elements in D and if $J_i \prec J$ $(i = 1, 2)$, then $\omega(J_1)$ equals to $\omega(J_2)$, where the notation $\prec$ means that J_i is a subset of J and there exist no element in D between J_i and J.

Definition 1.3.2. Let $\boldsymbol{D} = (D, \omega)$ be a special datum, we put $R_{\boldsymbol{D}}$ to be the subring $\boldsymbol{C}[X_J \mid J \in D]$ of R where $X_J = (\prod_{i \in J} X_i)^{\omega(J)}$.

We denote the diagonal matrix whose (i, i) component is a (resp. (i, i) component is a and (j, j) component is b) and the other diagonal components are 1 by $(a; i)$ (resp. $(a, b; i, j)$) here.

Definition 1.3.3. Let $\boldsymbol{D} = (D, \omega)$ be a special datum. A group $G_{\boldsymbol{D}}$ is the one generated by the following elements: $\{(e_\omega,\ e_\omega^{-1}; i, j) \mid J_1, J_2, J \in D, i \in J_1, j \in J_2, J_1 \prec J, J_2 \prec J \text{ and } \omega = \omega(J_1) = \omega(J_2)\}$.

Proposition 1.3.1. *If $\boldsymbol{D} = (D, \omega)$ is a special datum, then*
(1) *the ring $R_{\boldsymbol{D}}$ is a complete intersection,*
(2) *$R_{\boldsymbol{D}}$ is the invariant subring under the action of the group $G_{\boldsymbol{D}}$.*

Theorem 1.3.1. *If G is a finite abelian subgroup of $SL(n, \boldsymbol{C})$ and if the invariant ring R^G is a complete intersection, then there is a special datum $\boldsymbol{D}$ such that $R^G = R_{\boldsymbol{D}}$ and $G = G_{\boldsymbol{D}}$.*

Chapter 2

Existence problem of crepant resolution

It is known that a crepant resolution always exists for a quotient singularity by a finite subgroup of $SL(n, \boldsymbol{C})$ if n is equal to three or less [25][12][13][19][20], but not in the case in general when n is greater than three. Since around 1990, arithmetic conditions for the existence of a crepant resolution have been shown for some series of cyclic quotient singularities [8][6][5]. It is also shown that a crepant resolution exists for c.i. singularities [7][4].

2.1 Crepant resolutions of quotient singularities

For Gorenstein quotient singularities with dimension smaller than four, existence problem solved affirmatively.

Theorem 2.1.1 ([25][12][13][19][20]). *Any n-dimensional Gorenstein quotient singularity $\boldsymbol{C}^n/G$ admits a crepant resolution when n is smaller than four.*

In special cases, sufficient conditions for the existence of a crepant resolution are known.

Theorem 2.1.2 ([7]). *All Gorenstein cyclic quotient singularities $\boldsymbol{C}^n/G$ of type*

$$\frac{1}{\left(\frac{k^n-1}{k-1}\right)}(1, k, k^2, k^3, \ldots, k^{n-2}, k^{n-1})$$

admit toric projective crepant resolutions for all $n \geq 3$ and all $k \geq 2$.

Theorem 2.1.3 ([7]). *All abelian quotient c.i.-singularities admit projective crepant resolutions in all dimensions.*

We note that the quotient singularities in Theorem 2.1.2 are non-c.i.-singularities.

In the remaining part of this section, we introduce the result in [26]. We have found some infinite series of noncyclic and non-c.i. finite subgroups G of $SL(4, \boldsymbol{C})$ such that $\boldsymbol{C}^4/G$ admits a toric projective crepant resolution.

Through this section, the coordinate ring of $\boldsymbol{C}^4$ and the invariant ring under the action of the group G are denoted by R and R^G respectively.

$$R := \boldsymbol{C}[X_1, X_2, X_3, X_4]$$

Proposition 2.1.1. *Let p be a prime number and G be an abelian finite subgroup of $SL(4, \boldsymbol{C})$ generated by order p elements. Then G is a vector space over the prime field of order p. The dimension of G as a vector space is at most three and G is conjugate in $SL(4, \boldsymbol{C})$ to one of the followings:*
(1) $(\dim G = 1)$ *A cyclic group*

$\langle \frac{1}{p}(a, b, c, d) \rangle \cong \boldsymbol{Z}/p\boldsymbol{Z}$

(2) $(\dim G = 2)$ *Noncyclic groups with two generators*

$\langle \frac{1}{p}(1, 0, a, p-a-1), \frac{1}{p}(0, 1, b, p-b-1) \rangle \cong (\boldsymbol{Z}/p\boldsymbol{Z})^2 \qquad (2_{12})$
$\langle \frac{1}{p}(1, a, 0, p-a-1), \frac{1}{p}(0, 0, 1, p-1) \rangle \cong (\boldsymbol{Z}/p\boldsymbol{Z})^2, \ (a \neq 0) \quad (2_{13})$
$\langle \frac{1}{p}(0, 1, 0, p-1), \frac{1}{p}(0, 0, 1, p-1) \rangle \cong (\boldsymbol{Z}/p\boldsymbol{Z})^2 \qquad (2_{23})$

(3) $(\dim G = 3)$ *Noncyclic groups with three generators*

$\langle \frac{1}{p}(1, 0, 0, p-1), \frac{1}{p}(0, 1, 0, p-1), \frac{1}{p}(0, 0, 1, p-1) \rangle \cong (\boldsymbol{Z}/p\boldsymbol{Z})^3$

where p is a prime number and a, b, c, d are integers in $[0, p)$.

Proof. Let G be an abelian finite subgroup of $SL(4, \boldsymbol{C})$ generated by diagonal matrices. We define an injective homomorphism of groups $\varphi : (\boldsymbol{R}/\boldsymbol{Z})^{\oplus 4} \hookrightarrow GL(4, \boldsymbol{C})$ as

$$(x, y, z, w) \mapsto \begin{pmatrix} e^{2\pi i x} & & & 0 \\ & e^{2\pi i y} & & \\ & & e^{2\pi i z} & \\ 0 & & & e^{2\pi i w} \end{pmatrix}$$

and $\psi : (\boldsymbol{R}/\boldsymbol{Z})^{\oplus 3} \hookrightarrow SL(4, \boldsymbol{C})$ as

$$(x, y, z) \mapsto \begin{pmatrix} e^{2\pi i x} & & & 0 \\ & e^{2\pi i y} & & \\ & & e^{2\pi i z} & \\ 0 & & & e^{2\pi i(1-x-y-z)} \end{pmatrix}.$$

Then we have the following diagram.

$$\begin{array}{ccc} (\boldsymbol{R}/\boldsymbol{Z})^{\oplus 3} & \hookrightarrow & SL(4, \boldsymbol{C}) \\ \downarrow & & \downarrow \\ (\boldsymbol{R}/\boldsymbol{Z})^{\oplus 4} & \hookrightarrow & GL(4, \boldsymbol{C}) \end{array}$$

And we also have the following diagram where the vertical arrows are canonical quotient maps.

$$\begin{array}{ccc} G + \boldsymbol{Z}^{\oplus 3} & \hookrightarrow & \boldsymbol{R}^{\oplus 3} \\ \downarrow & & \downarrow \\ G & \hookrightarrow & (\boldsymbol{R}/\boldsymbol{Z})^{\oplus 3} \end{array}$$

We denote $G + \boldsymbol{Z}^{\oplus 3}$ by $\tilde{G}$. $\tilde{G}$ is a free abelian group of rank at most three. We choose an isomorphism $\rho : \tilde{G} \xrightarrow{\sim} \boldsymbol{Z}^{\oplus 3}$. Then the composition of the inclusion map $\boldsymbol{Z}^{\oplus r} \to \tilde{G} (\cong \boldsymbol{Z}^{\oplus r})$ and ρ is $\boldsymbol{Z}$-linear. We write the composition: $\boldsymbol{Z}^{\oplus 3} \hookrightarrow \tilde{G} \xrightarrow{\sim} \boldsymbol{Z}^{\oplus 3}$ as ν.

We may assume the image of ν equals to $d_1\boldsymbol{Z} \oplus d_2\boldsymbol{Z} \oplus d_3\boldsymbol{Z}$ where d_1, d_2 and d_3 are integers with $d_1|d_2|d_3$. Clearly, G is isomorphic to $\tilde{G}/\boldsymbol{Z}^{\oplus 3}$. Hence, G is isomorphic to Coker $\nu = (\boldsymbol{Z}/d_1\boldsymbol{Z}) \oplus (\boldsymbol{Z}/d_2\boldsymbol{Z}) \oplus (\boldsymbol{Z}/d_3\boldsymbol{Z})$.

By assumption, d_i equals to p or 1 where i is a positive integer smaller than four. Hence an abelian finite subgroup of $SL(4, \boldsymbol{C})$ generated by order p elements becomes the type (1), (2_{12}), (2_{13}), (2_{23}) or (3) using a change of bases. □

Theorem 2.1.4. *There exists a projective toric crepant resolution for the following types:*
(a) *Type* (2_{12}) *with* $a = b = 1$,
(b) *Type* (2_{13}) *with* $a = 1$, $\frac{p-1}{2}$, $p-2$ *or* $p-1$,
(c) *Type* (2_{23}),
(d) *Type* (3).

Proof. The case (a).

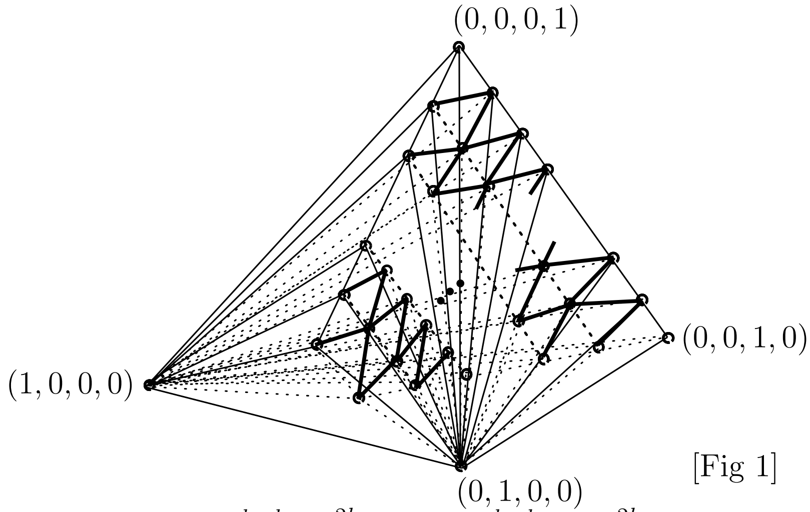
We will prove this case at the latter part of the proof of the case (b), $a = p-2$.

The case (b).

We define the plane section of the cone spanned by the elements $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$ as H. Let G be a group of the type (2_{13}).

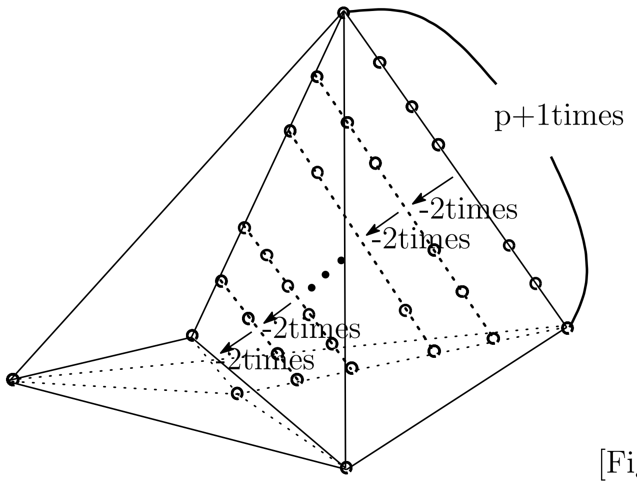
If a equals to 1 and $p \neq 2$, the quotient space $\boldsymbol{C}^4/G$ corresponds to the toric variety $X(N', \Delta)$ where the lattice set N' is $\boldsymbol{Z}^4 + \frac{1}{p}(1,1,0,p-2)\boldsymbol{Z} + \frac{1}{p}(0,0,1,p-1)\boldsymbol{Z}$ and Δ is the finite fan which consists of the faces of the cone generated by the points $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$. We define the subset $\{\frac{1}{p}(i,i,j,p-2i-j) \mid i \in [0, \frac{p-1}{2}] \cap \boldsymbol{Z}, j \in [0, p-2i] \cap \boldsymbol{Z}\} \subset N'$ as P. The age of all the points in P equals to 1.

We give a resolution for the singularity $X(N', \Delta)$ by subdividing $\Delta \cap H$ as the figure [Fig 1].



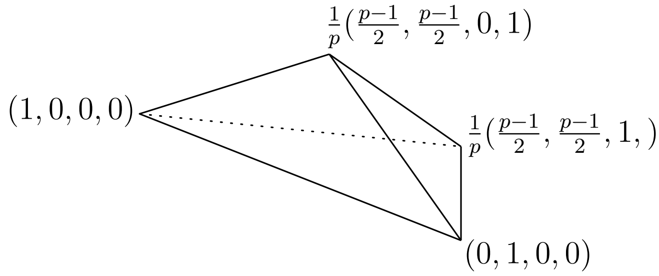
[Fig 1]

On the edge connecting $(\frac{k}{p}, \frac{k}{p}, \frac{p-2k}{p}, 0)$ and $(\frac{k}{p}, \frac{k}{p}, 0, \frac{p-2k}{p})$, there appear $(p-2k+1)$ points where the integer k satisfies the condition $0 \leq k \leq \frac{p+1}{2} - 1$. See [Fig 2].

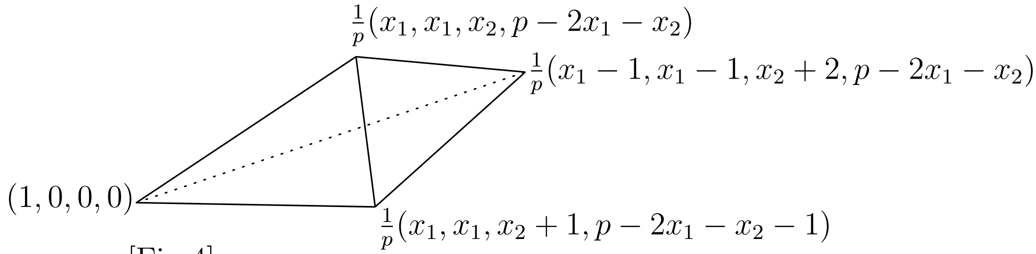


[Fig 2]

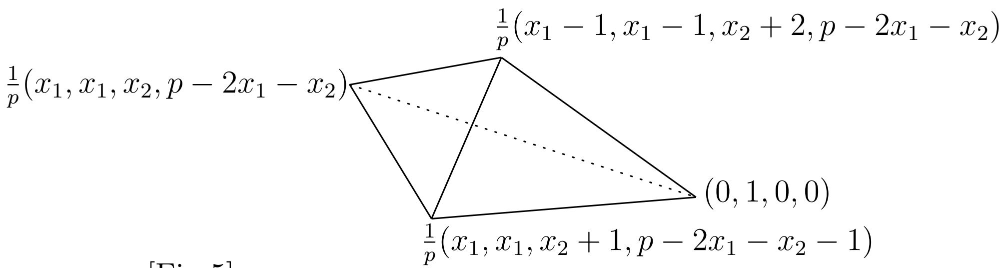
The figure [Fig 1] includes p^2 triangular pyramids of the following types: [Fig 3], [Fig 4], [Fig 5], [Fig 6], [Fig 7], [Fig 8] and [Fig 9].



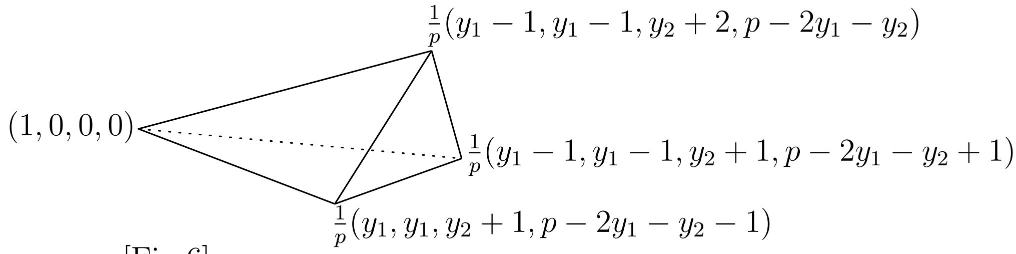
[Fig 3]



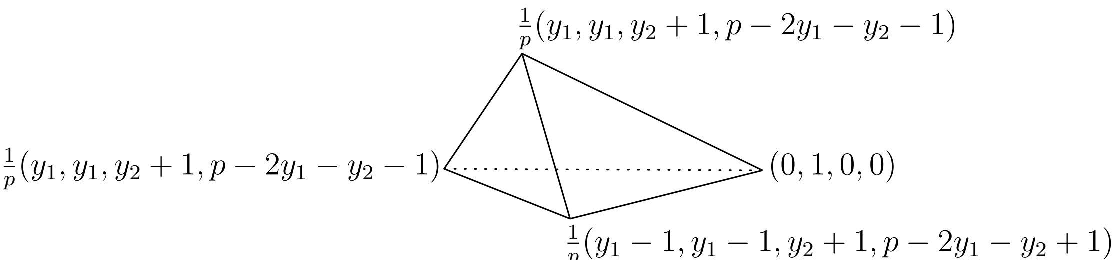
[Fig 4]



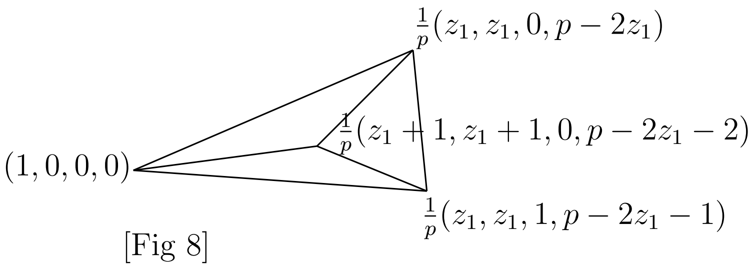
[Fig 5]



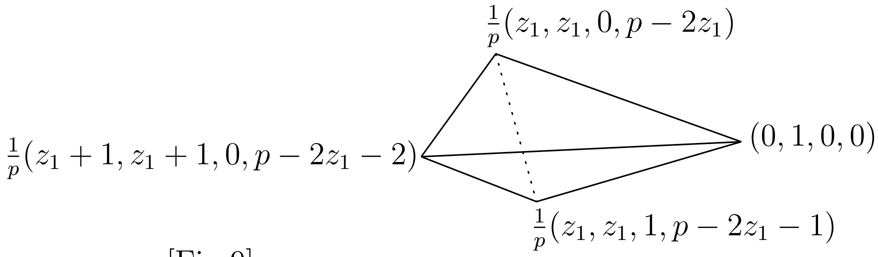
[Fig 6]



[Fig 7]



[Fig 8]



[Fig 9]

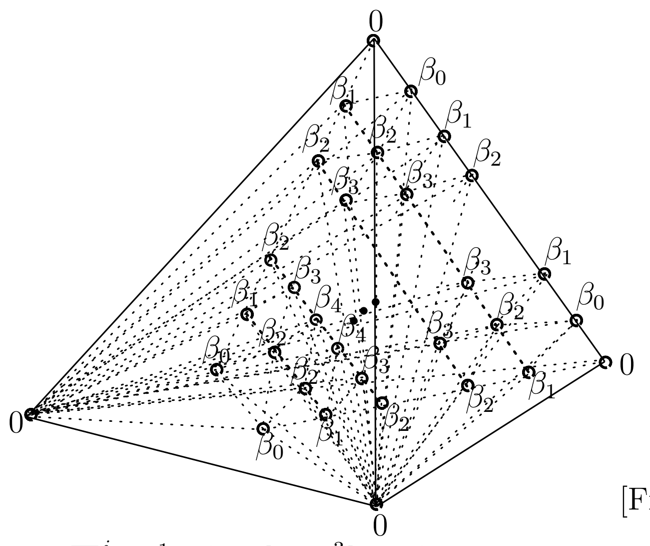
The variables satisfy the conditions $x_1, y_1, z_1 \in [1, \frac{p-1}{2}] \cap \boldsymbol{Z}$, $x_2, y_2 \in [0, \frac{p-1}{2}] \cap \boldsymbol{Z}$, $p - 2x_1 - x_2 \geq 1$ and $p - 2y_1 - y_2 \geq 0$.

All the determinants of the matrices made by the generators of the triangular pyramids equal to $\frac{1}{p^2}$ for the [Fig 3], [Fig 4], $\cdots$, [Fig 9]. Therefore, the cones generated by the four vertices of each triangular pyramid are non-singular and the variety corresponding to the fan $(N', \tilde{\Delta})$ is a resolution for $X(N', \Delta)$ where $\tilde{\Delta}$ is the finite fan decomposed as the figure.

The age of every lattice point corresponding to the exceptional divisors for this resolution equals to 1. Hence, the resolution is crepant.

Next, we prove that this resolution is projective.

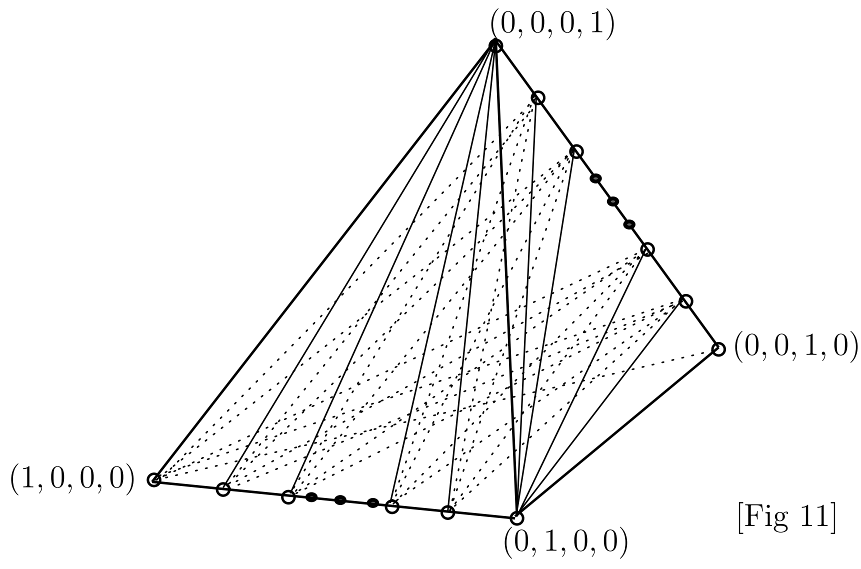
We shall define $\tilde{\Delta}$-linear support function h which is strictly upper convex on $\tilde{\Delta}$ by giving a $\boldsymbol{Q}$-value for each lattice point in $\Delta \cap H$. If h is strictly upper convex on $\tilde{\Delta} \cap H$, h is strictly upper convex on $\tilde{\Delta}$. Let h have the following $\boldsymbol{Q}$-value β_i at each lattice point in $\Delta \cap H$, then h is strictly upper convex on $\tilde{\Delta}$,



[Fig 10]

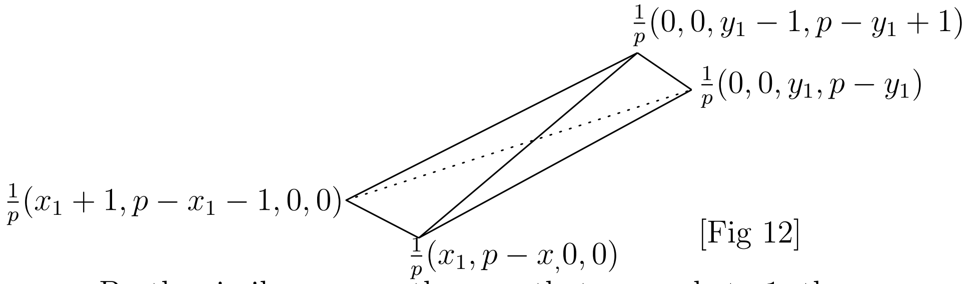
where $\beta_i := 1 + \sum_{k=0}^{i} \frac{1}{2^i}$, $(i \in [0, \frac{p-3}{2}] \cap \boldsymbol{Z})$.

From here, we shall construct a projective crepant resolution for the case a equals to $p-1$. We also treat the case $p=2$. The lattice set N' is $\boldsymbol{Z}^4 + \frac{1}{p}(1, p-1, 0, 0)\boldsymbol{Z} + \frac{1}{p}(0, 0, 1, p-1)\boldsymbol{Z}$. We define a subset $\{\frac{1}{p}(i, p-i, 0, 0), \frac{1}{p}(0, 0, j, p-j) \mid i, j \in [0, p] \cap \boldsymbol{Z}\} \subset N'$ as P. The age of the elements in P is always 1. We shall give a resolution for $X(N', \Delta)$ by subdividing $\Delta \cap H$ as the figure [Fig 11].



[Fig 11]

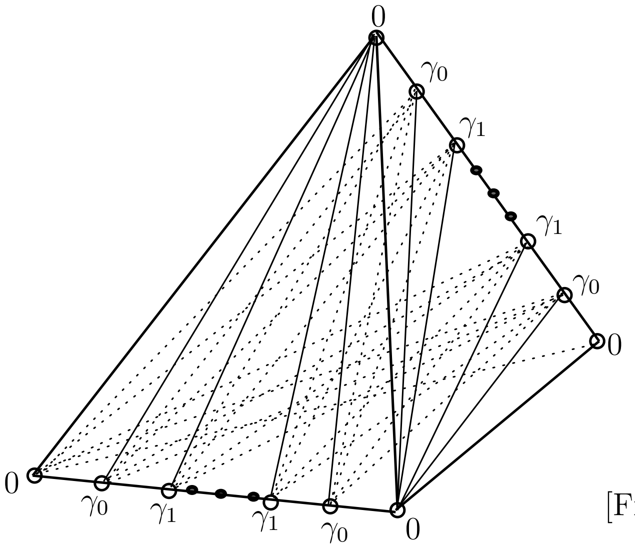
The figure [Fig 11] only contains the triangular pyramids as shown in [Fig 12].



[Fig 12]

By the similar way as the case that a equals to 1, the cone as shown in [Fig 12] is nonsingular and the variety $X(N', \tilde{\Delta})$ is a crepant resolution for $X(N', \Delta)$ where the finite fan $\tilde{\Delta}$ is decomposed as the figure [Fig 11].

Next, we prove that this resolution is projective. If h is determined by the value at each point as follows, then the $\tilde{\Delta}$-linear support function h is strictly upper convex on $\tilde{\Delta}$ where $\gamma_i := 2 + \sum_{k=0}^{i} \frac{1}{2^i}$, $(i \in [0, \frac{p-3}{2}] \cap \boldsymbol{Z})$.



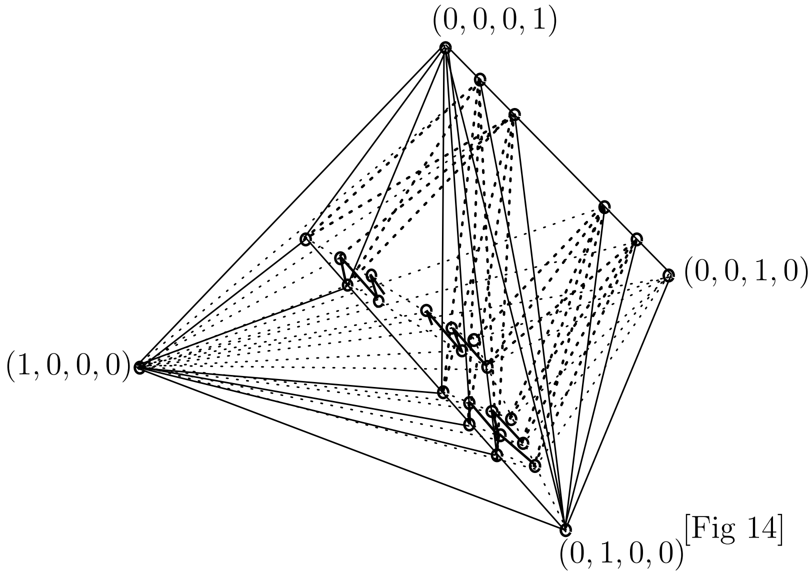
[Fig 13]

In the remaining case $a = 1$ and $p = 2$, the condition that G is the type (2_{13}) and $a = 1$, $p = 2$ is a equivalent one that G is the type (2_{13}) and $a = p-1$, $p = 2$. So, we have proved the case $a = 1,\ p-1$ of the type (2_{13}).

In the following, we shall prove the case $a = \frac{p-1}{2},\ p-2$. First, we consider the case $a = p-2$ where $p \neq 2$. The lattice set N' is $\boldsymbol{Z}^4 +$

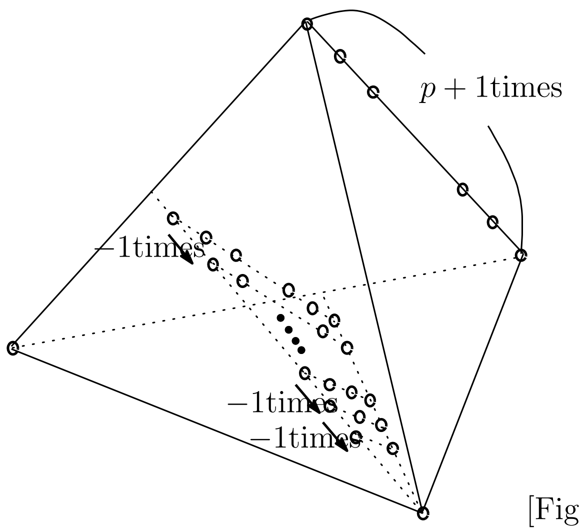
$\frac{1}{p}(1,p-2,0,1)\boldsymbol{Z}+\frac{1}{p}(0,0,1,p-1)\boldsymbol{Z}$ and Δ is as above. We define the subset $\{(1,0,0,0),\frac{1}{p}(0,0,i,p-i),\frac{1}{p}(j,k,l,j-l)\mid i\in[0,p]\cap\boldsymbol{Z},j\in[0,\frac{p-1}{2}]\cap\boldsymbol{Z},k=p-2j,l\in[0,j]\cap\boldsymbol{Z}\}\subset N'$ as P.

The age of all the points in P equals to 1. We will give a resolution for the singularity $X(N',\Delta)$ by subdividing $\Delta\cap H$ as the figure [Fig 14].



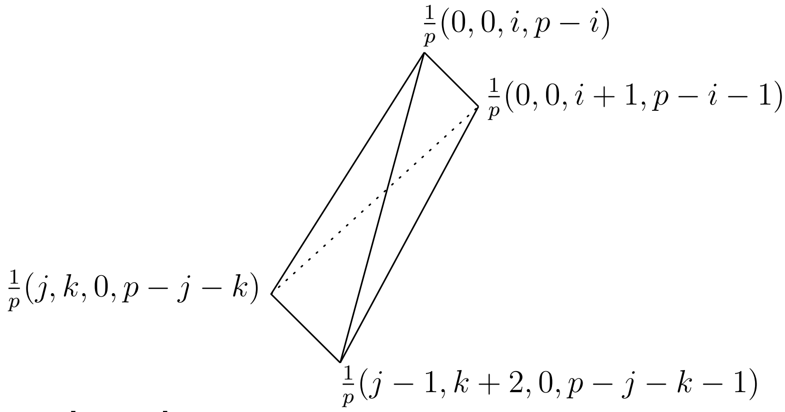
[Fig 14]

On the edge connecting $(\frac{j}{p},\frac{p-2j}{p},\frac{j}{p},0)$ and $(\frac{j}{p},\frac{p-2j}{p},0,\frac{j}{p})$, there appear $j+1$ points where the integer j satisfies the condition $0\leq j\leq\frac{p-1}{2}$. See [Fig 15].

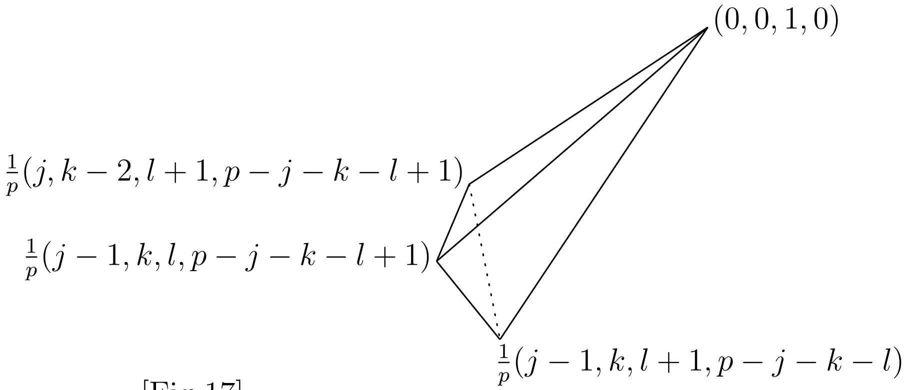


[Fig 15]

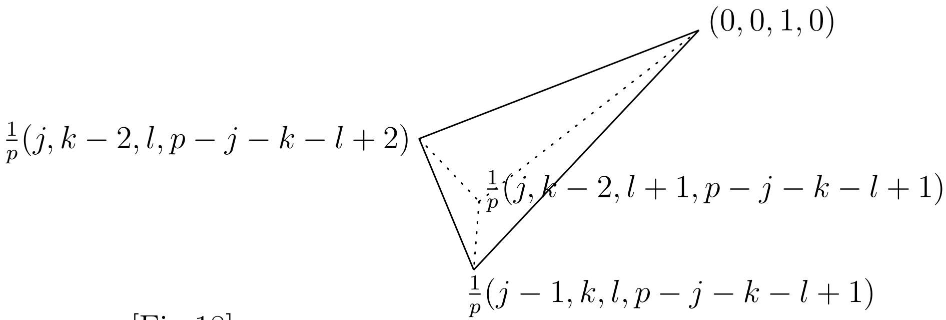
The figure [Fig 14] includes p^2 triangular pyramids of the following types: [Fig 16], [Fig 17], [Fig 18], [Fig 19], [Fig 20], [Fig 21] and [Fig 22].



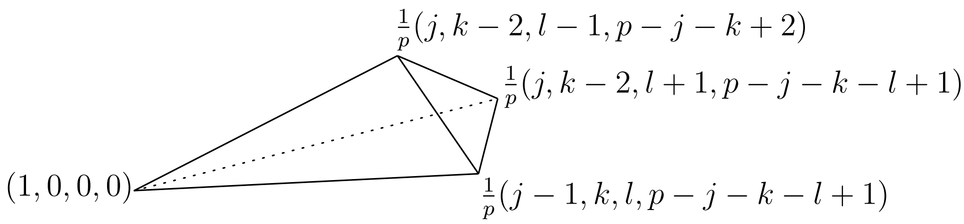
[Fig 16]



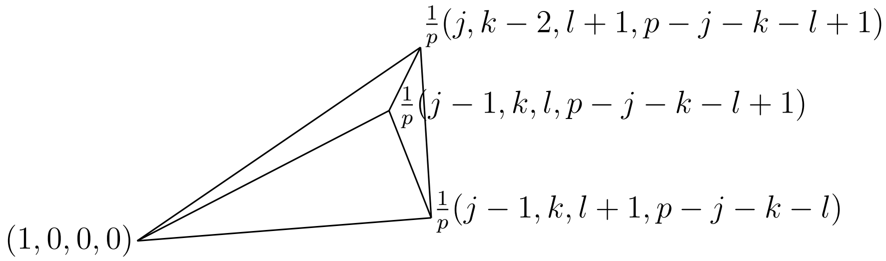
[Fig 17]



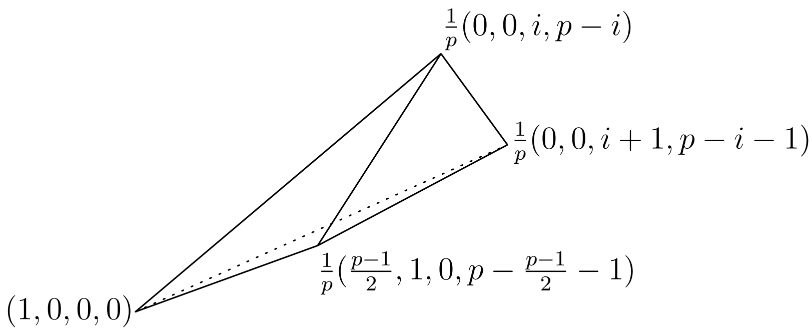
[Fig 18]



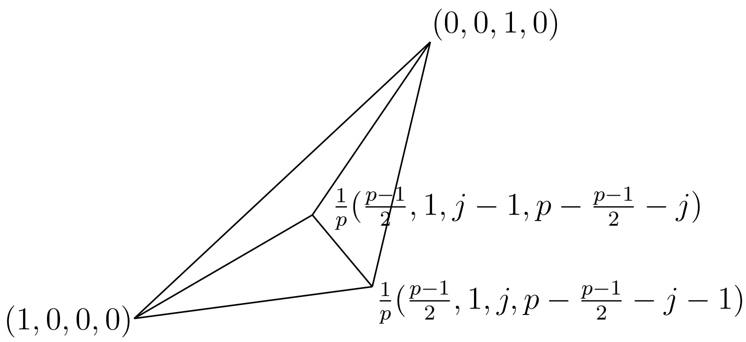
[Fig 19]



[Fig 20]



[Fig 21]



[Fig 22]

The variables satisfy the conditions $j \in [1, \frac{p-1}{2}] \cap \boldsymbol{Z}, k = p-2j, l \in [0, p-j-k] \cap \boldsymbol{Z}$ and $i \in [0, p-1] \cap \boldsymbol{Z}$.

All the determinants of the matrices made by the generators of the triangular pyramids equal to $\frac{1}{p^2}$ for the [Fig 16], [Fig 17], $\cdots$, [Fig 22]. Therefore, the cone generated by the four vertices of each triangular pyramid is non-singular and the variety corresponding to the fan $(N', \tilde{\Delta})$ is a resolution for $X(N', \Delta)$, where $\tilde{\Delta}$ is the finite fan decomposed as the figure.

The age of every lattice point corresponding to the exceptional divisors for this resolution equals to 1. Hence, the resolution is crepant.

We define h as the similar way for the case $a = 1$. Then it is confirmed that the crepant resolution is projective.

Here, we consider the case $p = 2$. The invariant ring is as follows.

$$R^G = \boldsymbol{C}[X_1^2, X_2, X_3^2, X_4^2, X_1X_2X_4]$$

This case is clearly complete intersection type (see Section 3) and there exist projective crepant resolutions.

Finally, We get the case $a = \frac{p-1}{2}$ from the case $a = p - 2$ by interchanging bases $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ and also the type (a) from this case by changing bases and by changing generators of the group. So we have proved the case (a) and (b).

<u>The case (c) and (d).</u>

The quotient singularities of this type are c.i. singularity. In the case (c) (resp. (d)), the invariant ring is

$$R^G = \boldsymbol{C}[X_1, X_2^p, X_3^p, X_4^p, X_2X_3X_4]$$

$$(\text{resp. } R^G = \boldsymbol{C}[X_1^p, X_2^p, X_3^p, X_4^p, X_1X_2X_3X_4]).$$

We can define a special datum $\boldsymbol{D}$ as $D := \{\{1\}, \{2\}, \{3\}, \{4\}, \{2, 3, 4\}\}$ (resp. $\{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3, 4\}\}$) and a map ω as $\{1\} \to 1$, $\{i\} \to p$ $(i = 2, 3, 4)$ and $\{2, 3, 4\} \to 1$ (resp. $\{i\} \to p$ $(i = 1, 2, 3, 4)$ and $\{1, 2, 3, 4\} \to 1$). So there exist projective crepant resolutions. □

Comment

For the group G of the type (2_{13}), if a equals to $p - 1$ then $\boldsymbol{C}^4/G$ is c.i. singularity. We shall show $\boldsymbol{C}^4/G$ is non-c.i. if a equals to $1, \frac{p-1}{2}, p - 2$ by showing that there are no special datum $\boldsymbol{D}$ for the group. Let I be the set $\{1,2,3,4\}$. Assume that group G is the type (2_{13}) and a equals to 1. Then the invariant ring is as follows:

$$R^G = \boldsymbol{C}[{X_1}^p, {X_2}^p, {X_3}^p, {X_4}^p, {X_1}^i {X_2}^{p-i}, {X_1}^2 X_3 X_4, {X_2}^2 X_3 X_4, X_1 X_2 X_3 X_4]$$

where $1 \leq i \leq p-1$. We define a set D as follows.

$$D := \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}$$

Let ω be a map from D to $\boldsymbol{Z}_{>0}$. Then the pair (D,ω) never becomes a special datum, since the images of ω for the elements $\{1,2\}$, $\{1,3,4\}$ and $\{2,3,4\}$ do not satisfy the condition that $\omega(E)$ equals to 1 or p where E is $\{1,2\}$, $\{1,3,4\}$ or $\{2,3,4\}$. Hence, in this case, we have the fact that G is non-c.i. type.

We conjecture that there exists a projective crepant resolution for type (2_{13}) for any p and a.

2.2 Quotients of affine toric terminal 3-folds

Let V be a toric variety. The notations follows Chapter 1. If V is an affine toric variety, then there exists a semi-group $S_V = \sum_{i=1}^{s} \boldsymbol{Z}_{\geq 0} m_i$ in M, and we have local coordinates $(\boldsymbol{e}(m_1), \cdots, \boldsymbol{e}(m_s))$ on V, and the action $\pi_{\boldsymbol{T}}$ can be written as follows:

$$\pi_{\boldsymbol{T}}(t, (\boldsymbol{e}(m_1), \cdots, \boldsymbol{e}(m_s))) = (t(m_1)\boldsymbol{e}(m_1), \cdots, t(m_s)\boldsymbol{e}(m_s))$$

where t is an element in $\boldsymbol{T}$ and $\boldsymbol{e}(m_i)$ is the character of $\boldsymbol{T}$ for $m_i \in M$.

On the other hand, let H be a finite abelian group acting on V and π_H be the action. If $(V/H, \bar{v})$ is an s-dimensional quotient singularity, then we may assume that V is a vector space over $\boldsymbol{C}$ and G is a subgroup of $GL(s, \boldsymbol{C})$ by the following theorem. See Theorem 6.4.5 in [15].

Lemma 2.2.1. *Let (W, w) be an n-dimensional quotient singularity. Then there exists a finite subgroup H of $GL(n, \boldsymbol{C})$ such that $(W, w) \cong (\boldsymbol{C}^n/H, 0)$ as germs. In particular, quotient singularities are algebraic.*

We introduce one more theorem on isomorphisms of quotient singularities (see also [23]).

Theorem 2.2.1 (Prill's isomorphism criterion)**.** *Let $H_1, H_2 \subset GL(s, \boldsymbol{C})$ be two small finite subgroups where s is equal to or greater than 2. Then there exists an analytic isomorphism $(\boldsymbol{C}^s/H_1, 0) \cong (\boldsymbol{C}^s/H_2, 0)$ if and only if H_1 and H_2 are conjugate to each other within $GL(s, \boldsymbol{C})$.*

By using the simultaneous diagonalization of all elements in H and Prill's isomorphism criterion, we may assume that the action π_H of H is given as follows:

$$\pi_H(h, c) := \mathrm{diag}(\alpha_1, \cdots, \alpha_s)\ {}^t(c_1, \cdots, c_s)$$

where $\mathrm{diag}(\alpha_1, \cdots, \alpha_s)$ is a diagonalization of the matrix $h \in H$, α_i $(i = 1, \ldots, s)$ are the eigenvalues of h and $(c_1, \cdots, c_s)$ are local coordinates on $\boldsymbol{C}^s$. We note that $\boldsymbol{C}^s$ contains the algebraic torus $(\boldsymbol{C}^*)^s$ and the semigroup $S_{\boldsymbol{C}^s}$ can be written as $S_{\boldsymbol{C}^s} = \sum_{i=1}^{s} \boldsymbol{Z}_{\geq 0}\check{e_i}$ where $\{\check{e_i} \mid i = 1, \ldots, s\}$ is a $\boldsymbol{Z}$-basis of M. Therefore we can recognize H as a finite subgroup of $\boldsymbol{T}$, and we have the formula:

$$\pi_{\boldsymbol{T}}(t, \pi_H(h, c)) = \pi_H(h, \pi_{\boldsymbol{T}}(t, c)),\ \forall t \in \boldsymbol{T}, \forall h \in H, \forall c \in V.$$

This formula says that π_H is equivariant with respect to $\pi_{\boldsymbol{T}}$. In this paper, we consider group actions on affine toric varieties having such equivariance.

Definition 2.2.1. Let X be an affine toric variety and G be an abelian finite group acting on X. The action π_G is said to be *toroidal* if the action π_G is equivariant with respect to $\pi_{\boldsymbol{T}}$, i.e., π_G satisfies the following formula:

$$\pi_{\boldsymbol{T}}(t, \pi_G(g, x)) = \pi_G(g, \pi_{\boldsymbol{T}}(t, x)),\ \forall t \in \boldsymbol{T}, \forall g \in G, \forall x \in X.$$

In addition, we say that G acts on X *toroidally.*

In the following, we assume that all group actions on X are toroidal. The usefulness of toroidal actions is that X/G is also a toric variety. Hence we can describe, explicitly, the toric equivariant geometry of $(X/G, \bar{x})$ with the language of fans.

Let us introduce a classification of affine toric terminal 3-folds. Theorem 2.2.2 was proved by G. K. White, D. Morrison, G. Stevens, V. Danilov and M. Frumkin and Theorem 2.3 was in [9]. The definition of type $\frac{1}{r}(a,-a,1)$ follows [24].

Theorem 2.2.2. *Let X be an affine toric $\boldsymbol{Q}$-factorial 3-fold. Then X is terminal if and only if X is of type $\frac{1}{r}(a,-a,1)$ where a is an integer coprime to r. In particular, if X is Gorenstein, then X is smooth.*

Theorem 2.2.3. *Let X be an affine toric non-$\boldsymbol{Q}$-factorial 3-fold. Then X has a terminal singularity if and only if $X \cong \mathrm{Spec}(\boldsymbol{C}[x,y,z,w]/(xz-yw))$.*

2.3 Gorenstein quotients of the singularity of type $\frac{1}{r}(a,-a,1)$

Let X be an affine toric terminal 3-fold of type $\frac{1}{r}(a,-a,1)$ and G be a group acting on X toroidally. Then X/G has a toric quotient singularity. We assume that the singularity is Gorenstein. For three-dimensional Gorenstein quotient singularities, the following theorems are known.

Theorem 2.3.1 ([12][19][25])**.** *All three-dimensional Gorenstein quotient singularities admit a crepant resolution.*

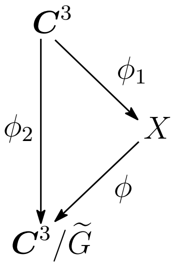
By Theorem 2.3.1, the quotient singularity $(X/G, \bar{x})$ admits a crepant resolution. The remaining problem is whether there really exist quotient morphisms from X to X/G which has a Gorenstein singularity. In the following, we shall give a way to construct an isolated Gorenstein quotient singularity $(X/G, \bar{x})$.

Theorem 2.3.2 ([17])**.** *Let n be an odd prime number. Let H be a finite subgroup of $GL(n,\boldsymbol{C})$ which is small. Assume that the $\boldsymbol{C}^n/H$ is Gorenstein with an isolated singularity. Then $\boldsymbol{C}^n/H$ has a cyclic quotient singularity.*

This theorem is a generalization of Theorem 23 in [27]. By Theorem 2.2.1 and Theorem 2.3.2, the singularity $(X/G, \bar{x}) \cong (\boldsymbol{C}^3/\widetilde{G}, 0)$ is a cyclic quotient singularity where $\widetilde{G}$ is a small finite subgroup of $GL(3, \boldsymbol{C})$. So it is enough to prove that, for any cyclic quotient $\boldsymbol{C}^3/\widetilde{G}$, there exists a quotient morphism from X to $\boldsymbol{C}^3/\widetilde{G}$ by a group acting on X toroidally. Let G' be the finite subgroup of $GL(3, \boldsymbol{C})$ generated by

$$\begin{pmatrix} \varepsilon_r^a & 0 & 0 \\ 0 & \varepsilon_r^{-a} & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix}$$

where ε_r is a primitive r-th root of unity and a, r are positive integers which are coprime. Let ϕ_1 be the quotient morphism from $\boldsymbol{C}^3$ to $\boldsymbol{C}^3/G'$, and let ϕ_2 be the quotient morphism from $\boldsymbol{C}^3$ to a cyclic quotient $\boldsymbol{C}^3/\widetilde{G}$. See [Fig 23]. In the following, we shall construct a quotient morphism ϕ such that $\phi \circ \phi_1 = \phi_2$.



[Fig 23]

We fix the coordinate ring $R := \boldsymbol{C}[x, y, z]$ of $\boldsymbol{C}^3$ on the top of [Fig 23]. The torus action $\pi_{\boldsymbol{T}}$ on $\boldsymbol{C}^3$ defining the coordinate ring $\boldsymbol{C}[x, y, z]$ is as follows:

$$\pi_{\boldsymbol{T}}((t_1, t_2, t_3), (x, y, z)) = (t_1 x, t_2 y, t_3 z)$$

where (t_1, t_2, t_3) is an element in $(\boldsymbol{C}^*)^3$. So it is clear that the action of G' is toroidal for $\pi_{\boldsymbol{T}}$. The following example gives a construction of the quotient morphism ϕ.

Example 2.3.1. Let $G'' \subset GL(3, \boldsymbol{C})$ be the subgroup generated by the matrices

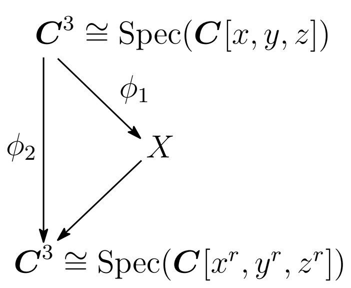
$$\begin{pmatrix} \varepsilon_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix}.$$

In [Fig 24], let ϕ_1 and ϕ_2 be the quotient morphisms of $\boldsymbol{C}^3$ by G' and G'' respectively. Since G' is a normal subgroup of G'', there is a quotient morphism from X to $\mathrm{Spec}(\boldsymbol{C}[x^r, y^r, z^r])$.

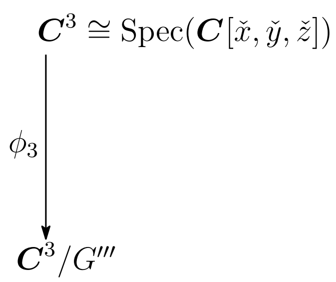
We define a small finite subgroup $G''' \subset GL(3, \boldsymbol{C})$ as

$$G''' := \langle \begin{pmatrix} \varepsilon_{r'}^{a'} & 0 & 0 \\ 0 & \varepsilon_{r'}^{b'} & 0 \\ 0 & 0 & \varepsilon_{r'}^{c'} \end{pmatrix} \rangle$$

where $\varepsilon_{r'}$ is a primitive r'-th root of unity, a', b', c' are elements in $\boldsymbol{Z} \cap [0, r')$ and satisfy $\mathrm{GCD}(a', b', c', r') = 1$. G''' clearly acts on $\mathrm{Spec}(\boldsymbol{C}[\check{x}, \check{y}, \check{z}])$, and we denote the quotient morphism form $\mathrm{Spec}(\boldsymbol{C}[\check{x}, \check{y}, \check{z}])$ to $\mathrm{Spec}(\boldsymbol{C}[\check{x}, \check{y}, \check{z}])/G'''$ by ϕ_3. The image of ϕ_3 is of type $\frac{1}{r'}(a', b', c')$.



[Fig 24]

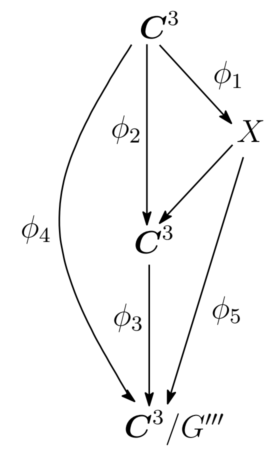


[Fig 25]

We shall consider the composition of [Fig 24] and [Fig 25] by the changing of variables

$$x^r \mapsto \check{x}, \ y^r \mapsto \check{y}, \ z^r \mapsto \check{z}.$$

See [Fig 26]. Let ϕ_4 be the composition of ϕ_2 and ϕ_3. In the following, we ascertain that there really exists ϕ_5 which is the quotient morphism by a group acting on X toroidally.



[Fig 26]

We define a subgroup $\widetilde{G}$ of $GL(3, \boldsymbol{C})$ as the direct sum of G'' and G''', i.e., $\widetilde{G}$ is as follows:

$$\widetilde{G} = \langle \begin{pmatrix} \varepsilon_{rr'}^{r'} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_{rr'}^{r'} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_{rr'}^{r'} \end{pmatrix}, \begin{pmatrix} \varepsilon_{rr'}^{ra'} & 0 & 0 \\ 0 & \varepsilon_{rr'}^{rb'} & 0 \\ 0 & 0 & \varepsilon_{rr'}^{rc'} \end{pmatrix} \rangle$$

where $\varepsilon_{rr'}$ is a primitive rr'-th root of unity. Clearly, $\widetilde{G}$ acts on $\mathrm{Spec}(R)$, and $R^{\widetilde{G}}$ coincides with $\boldsymbol{C}[\check{x}, \check{y}, \check{z}]^{G'''}$ via the changing of variables. This is why ϕ_4 is the quotient morphism from $\mathrm{Spec}(R)$ by $\widetilde{G}$. Since G' is a normal subgroup of $\widetilde{G}$, there exist a quotient map ϕ_5 and a group G acting on X such that $(\boldsymbol{C}^3/\widetilde{G}, 0) \cong (X/G, \bar{x})$. Moreover $(X/G, \bar{x})$ is a quotient singularity of type $\frac{1}{r'}(a', b', c')$, and G is isomorphic to $\widetilde{G}/G'$. Let $\{u_1(x, y, z), \cdots, u_s(x, y, z)\}$ be a system of minimal generators of $R^{G'}$, i.e., $R^{G'} = \boldsymbol{C}[u_1(x, y, z), \cdots, u_s(x, y, z)]$. The torus action on X is as follows:

$$\begin{aligned} &\pi_{\boldsymbol{T}/G'}((t_1, t_2, t_3), (u_1, \cdots, u_s)) \\ &\quad = \ (u_1(t_1, t_2, t_3)u_1(x, y, z), \cdots, u_s(t_1, t_2, t_3)u_s(x, y, z)) \end{aligned}$$

where (t_1, t_2, t_3) is an element in $\boldsymbol{T}$. Similarly, the action of G on X is as follows:

$$\begin{aligned}&\pi_G((\varepsilon_{rr'}^{a_1}, \varepsilon_{rr'}^{a_2}, \varepsilon_{rr'}^{a_3}), (u_1, \cdots, u_s))\\ &\quad = (u_1(\varepsilon_{rr'}^{a_1}, \varepsilon_{rr'}^{a_2}, \varepsilon_{rr'}^{a_3})u_1(x,y,z), \cdots, u_s(\varepsilon_{rr'}^{a_1}, \varepsilon_{rr'}^{a_2}, \varepsilon_{rr'}^{a_3})u_s(x,y,z))\end{aligned}$$

where $\varepsilon_{rr'}^{a_i}$ $(i = 1,2,3)$ are diagonal components of an element in $\widetilde{G}$. Therefore π_G is toroidal for $\pi_{\boldsymbol{T}/G'}$.

In summary, we have the following.

Proposition 2.3.1. *Let X be a quotient singularity of type $\frac{1}{r}(a,-a,1)$ where r and a are coprime. Then there exists a finite group G acting on X toroidally such that X/G has an cyclic quotient singularity of type $\frac{1}{r'}(a',b',c')$ where $\mathrm{GCD}(a',b',c',r') = 1$.*

Additionally, we can prove that X/G has an isolated Gorenstein singularity if and only if $a'+b'+c' \equiv 0 \pmod{r'}$ and $\mathrm{GCD}(a',r') = \mathrm{GCD}(b',r') = \mathrm{GCD}(c',r') = 1$. See Chapter 3 of [27].

2.4 Toroidal group actions on the conifold

Let X be the conifold $\mathrm{Spec}(\boldsymbol{C}[x,y,z,w]/(xz-yw))$. In this section, we shall classify groups G acting on X toroidally such that X/G has a Gorenstein singularity via a finite fan corresponding to X. The conifold X is a toric variety, and there is a torus action $\pi_{\boldsymbol{T}}$ on X by the algebraic torus $\boldsymbol{T} = (\boldsymbol{C}^*)^3$. The character lattice M is $\boldsymbol{Z}^3$ and S_X is a semi-group $\sum_{i=1}^4 \boldsymbol{Z}_{\geq 0} m_i \subset M$ where m_i $(i = 1,2,3,4)$ are elements in M satisfying $m_1 + m_3 = m_2 + m_4$. In this paper, we choose the canonical $\boldsymbol{Z}$-basis $\check{e_1}, \check{e_2}, \check{e_3}$ of M as m_1, m_2, m_3. Then the semigroup ring $\boldsymbol{C}[S_X]$ is $\boldsymbol{C}[\check{x}, \check{y}, \check{z}, \frac{\check{x}\check{z}}{\check{y}}] \cong \boldsymbol{C}[x,y,z,w]/(xz-yw)$. Let N be the dual $\boldsymbol{Z}$-module of M and $\{e_1, e_2, e_3\}$ be the canonical basis of N. Let Δ be the finite fan which consists of all the faces of the rational strongly convex polyhedral cone $\sigma := \boldsymbol{R}_{\geq 0}e_1 + \boldsymbol{R}_{\geq 0}e_3 + \boldsymbol{R}_{\geq 0}(e_1+e_2) + \boldsymbol{R}_{\geq 0}(e_2+e_3)$ in $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. In this case, the finite fan corresponding to X is (N, Δ). We first introduce a lemma on group actions on hypersurfaces (see also Lemma 7.3.8 in [15]).

Lemma 2.4.1. *Let $(V, 0) \subset (\boldsymbol{C}^{n+1}, 0)$ be an n-dimensional hypersurface singularity, H be a finite group acting on V and (W, w) be the quotient of $(V, 0)$ by H. Then there exists a small finite subgroup H' of $GL(n+1, \boldsymbol{C})$ such that H' acts on V and $(V, 0)/H' \cong (W, w)$.*

By Lemma 2.4.1, since the quotient X/G matters, we may assume that the group G is a small finite subgroups of $GL(4, \boldsymbol{C})$. We denote the action of G on X by π_G. The torus action on X is given as $(x, y, z, w) \mapsto (t_1 x, t_2 y, t_3 z, \frac{t_1 t_3}{t_2} w)$ where (x, y, z, w) is a point in $X \subset \boldsymbol{C}^4$ and $t_1, t_2, t_3 \in \boldsymbol{C}^*$. Therefore, if the action of G is toroidal, then all the elements in G are diagonal 4×4-matrices.

Proposition 2.4.1. *Let $G \subset GL(4, \boldsymbol{C})$ be a group acting on the conifold X toroidally and g be a generator of G. If the quotient X/G is Gorenstein, then g can be written as*

$$\begin{pmatrix} \varepsilon_r^a & 0 & 0 & 0 \\ 0 & \varepsilon_r^b & 0 & 0 \\ 0 & 0 & \varepsilon_r^{-a} & 0 \\ 0 & 0 & 0 & \varepsilon_r^{-b} \end{pmatrix} \tag{2.1}$$

where ε_r is a primitive r-th root of unity, $a, b \in [0, r) \cap \boldsymbol{Z}$ and $\mathrm{GCD}(a, b, r) = 1$.

Proof. All elements in G are diagonal matrices in $GL(4, \boldsymbol{C})$, and $g \in G$ can be written $\mathrm{diag}(\varepsilon_r^a, \varepsilon_r^b, \varepsilon_r^c, \varepsilon_r^d)$ where r is a positive integer, $a, b, c, d \in [0, r) \cap \boldsymbol{Z}$ and $\mathrm{GCD}(a, b, c, d, r) = 1$.

Since G acts on X, the defining function $xz - yw$ is semi-invariant under π_G. Moreover, since X/G is Gorenstein, the top form $\frac{1}{xz-yw} dx \wedge dy \wedge dz \wedge dw$ is invariant under π_G. By the action of g, $xz - yw$ and $\frac{1}{xz-yw} dx \wedge dy \wedge dz \wedge dw$ are transformed to $\varepsilon^{a+c} xz - \varepsilon^{b+d} yw$ and $\frac{\varepsilon^{a+b+c+d}}{\varepsilon^{a+c} xz - \varepsilon^{b+d} yw} dx \wedge dy \wedge dz \wedge dw$ respectively. Therefore, we have the equations

$$a + c \equiv b + d \quad \textit{and} \quad a + b + c + d \equiv a + c \pmod{r}.$$

In short, the formula

$$a + c \equiv 0 \quad \textit{and} \quad b + d \equiv 0 \pmod{r}$$

holds. Therefore, g can be written as (2.1), and, clearly, the action π_G is toroidal. $\square$

If G is generated by the matrices

$$\begin{pmatrix} \varepsilon_{r_1}^{a_1} & 0 & 0 & 0 \\ 0 & \varepsilon_{r_1}^{b_1} & 0 & 0 \\ 0 & 0 & \varepsilon_{r_1}^{-a_1} & 0 \\ 0 & 0 & 0 & \varepsilon_{r_1}^{-b_1} \end{pmatrix}, \cdots, \begin{pmatrix} \varepsilon_{r_k}^{a_k} & 0 & 0 & 0 \\ 0 & \varepsilon_{r_k}^{b_k} & 0 & 0 \\ 0 & 0 & \varepsilon_{r_k}^{-a_k} & 0 \\ 0 & 0 & 0 & \varepsilon_{r_k}^{-b_k} \end{pmatrix}$$

where r_i is a positive integer, ε_{r_i} is a primitive r_i-th root of unity and $a_i, b_i \in [0, r_i) \cap \boldsymbol{Z}$ for $i = 1, \ldots, k$, then, by changing r_i into $r := \mathrm{LCM}(r_1, \cdots, r_k)$ for all i, these generators of G can be written as follows:

$$\begin{pmatrix} \varepsilon_{r}^{l_1 a_1} & 0 & 0 & 0 \\ 0 & \varepsilon_{r}^{l_1 b_1} & 0 & 0 \\ 0 & 0 & \varepsilon_{r}^{-l_1 a_1} & 0 \\ 0 & 0 & 0 & \varepsilon_{r}^{-l_1 b_1} \end{pmatrix}, \cdots, \begin{pmatrix} \varepsilon_{r}^{l_k a_k} & 0 & 0 & 0 \\ 0 & \varepsilon_{r}^{l_k b_k} & 0 & 0 \\ 0 & 0 & \varepsilon_{r}^{-l_k a_k} & 0 \\ 0 & 0 & 0 & \varepsilon_{r}^{-l_k b_k} \end{pmatrix}$$

where l_i is $\frac{r}{r_i}$ for all i. In the following, we assume that all elements in G are the ones after the above commonization with respect to r_i. Hence it is enough to classify G for an fixed integer r.

Let us fix an integer r. Let $\mathcal{G}_r$ be the subset of $GL(4, \boldsymbol{C})$ which consists of all diagonal matrices in the form of (2.1) where r is fixed and $a, b \in [0, r) \cap \boldsymbol{Z}$. In this section, we denote the matrix (2.1) by the vector notation $\frac{1}{r}(a, b)$. By the isomorphism $\varphi : \mathcal{G}_r \to (\boldsymbol{Z}/r\boldsymbol{Z})^2$ defined by $\frac{1}{r}(a, b) \mapsto (a, b)$, a finite subgroup G of $\mathcal{G}_r$ corresponds to a finite $(\boldsymbol{Z}/r\boldsymbol{Z})$-submodule. We denote the submodule of $(\boldsymbol{Z}/r\boldsymbol{Z})^2$ corresponding to G by $\bar{G}$. We shall classify subgroups $G \subset \mathcal{G}_r$ into two families: (*i*) $\{G \subset \mathcal{G}_r : X/G$ is an isolated Gorenstein singularity$\}$, (*ii*) $\{G \subset \mathcal{G}_r : X/G$ is not so $\}$ via submodules of $(\boldsymbol{Z}/r\boldsymbol{Z})^2$.

Let ψ be the natural surjection $\boldsymbol{Z}^2 \to (\boldsymbol{Z}/r\boldsymbol{Z})^2$. The inverse image $\psi^{-1}(\bar{G})$ is a discrete submodule in $\boldsymbol{R}^2$. Hence, the rank of $\psi^{-1}(\bar{G})$ as a module is at most two, also the cardinality of a system of minimal generators of $\bar{G}$ is at most two. We denote the cardinality of a system of minimal generators of $\bar{G}$ by $\#\mathrm{SMG}(\bar{G})$. For instance, if G is a cyclic group, then $\#\mathrm{SMG}(\bar{G})$ is one. Assume systems of minimal generators $\{(a,b)\}$ (resp. $\{(a,b),(c,d)\}$) of $\bar{G}$ satisfy $\mathrm{GCD}(a,b,r)=1$ (resp. $\mathrm{GCD}(a,b,c,d,r)=1$).

Proposition 2.4.2. *For all submodules $\bar{G}$ of $(\boldsymbol{Z}/r\boldsymbol{Z})^2$ which satisfy*

$$\#\mathrm{SMG}(\bar{G})=2,$$

there exists a system of minimal generators which is in the form of either of the following two:

$$\{(a',b'),(0,d')\}, \tag{2.2}$$

$$\{(a',b'),(c',0)\} \tag{2.3}$$

where a',b',c',d' are integers in $[0,r)\cap\boldsymbol{Z}$ and $\mathrm{GCD}(a',b',c',d',r)=1$.

Proof. Assume that the set

$$\{(a,b),(c,d)\}\subset(\boldsymbol{Z}/r\boldsymbol{Z})^2$$

is a system of minimal generators of $\bar{G}$ where the integers $a,b,c,d\in[0,r)\cap\boldsymbol{Z}$ and $\mathrm{GCD}(a,b,c,d,r)=1$. Because $\#\mathrm{SMG}(\bar{G})$ is two, there do not exist non-zero elements $k_1,k_2\in\boldsymbol{Z}/r\boldsymbol{Z}$ such that $k_1\cdot(a,b)=(c,d)$ and $k_2\cdot(c,d)=(a,b)$.

Let α be $\mathrm{GCD}(a,c)$. Then there exist integers $s_1,s_2\in\boldsymbol{Z}\cap[0,r)$ such that $a=s_1\alpha$ and $c=s_2\alpha$. The integers s_1 and s_2 are coprime. Therefore, there exist integers c_1 and c_2 such that $c_1s_1+c_2s_2=1$, and we have the equation

$$c_1a+c_2c=\alpha\ .$$

By the above discussion, the formula

$$\begin{pmatrix} a & c\\ b & d\end{pmatrix}\begin{pmatrix} -s_2 & c_1\\ s_1 & c_2\end{pmatrix}\begin{pmatrix} 1 & 0\\ -k & 1\end{pmatrix}\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\equiv\begin{pmatrix} \alpha & 0\\ d' & \beta\end{pmatrix}\pmod{r} \tag{2.4}$$

holds where $k\equiv s_1-s_2$, $d'\equiv bc_1+dc_2$, $\beta\equiv kd'-b'$ and $b'\equiv bs_2-ds_1$ modulo r. In the formula (2.4), the second matrix, the third one and the fourth one are elementary transformations because those matrices are regular. The integers α,d' and β are elements in $[0,r)\cap\boldsymbol{Z}$. Since GCD is not changed under the elementary transformations of matrices, so we have the equation

$$\mathrm{GCD}(a,b,c,d,r)=\mathrm{GCD}(\alpha,d',\beta,r)=1.$$

Clearly, (α,d') and $(0,\beta)$ generate $\bar{G}$. ∎

We note that if a set $\{(a,0),(0,d) \in \bar{G} : ad \equiv 0 \pmod r\}$ generates $\bar{G}$, then $\#\mathrm{SMG}(\bar{G})$ is one and $\{(a,d) \mid ad \equiv 0 \pmod r\}$ is a system of minimal generators of $\bar{G}$. Moreover, the converse is also true. We admit interchanging variables x and y while classifying the group actions, and, hence we do not have to distinguish between (2.2) and (2.3). For all submodules $\bar{G}$, there, uniquely, exists a system of minimal generators

$$\mathcal{C}(\bar{G}) := \{(a,b),(0,d)\} \text{ (or } \{(a,b)\}) \subset \bar{G} \tag{2.5}$$

where the integers a, b, d are integers in $[0, r)$ which satisfy $\mathrm{GCD}(a,b,d,r) = 1$ (or $\mathrm{GCD}(a,b,r) = 1$) and the conditions: (i) $\mathrm{GCD}(a,r) = a$, (ii) $b = \min\{b' \in [0, r-1] \mid (a,b') \in \bar{G}\}$, (iii) $\mathrm{GCD}(d,r) = d$. We call the set $\mathcal{C}(\bar{G})$ the *canonical form* of $\bar{G}$ in this paper. In fact, the order of the element (a',b') in (2.2) is $\frac{r}{\mathrm{GCD}(a',r)}$, and, hence, $\mathrm{GCD}(a',r)$ is the minimum of the set $\{a'' \in [1, r-1] : (a'', *) \in \bar{G}\}$. Similarly, $\mathrm{GCD}(d',r)$ is the minimum of the set $\{d'' \in [1, r-1] \mid (0,d'') \in \bar{G}\}$. Therefore, by the minimality of the integers a, b, d, the uniqueness of the canonical form of $\bar{G}$ follows.

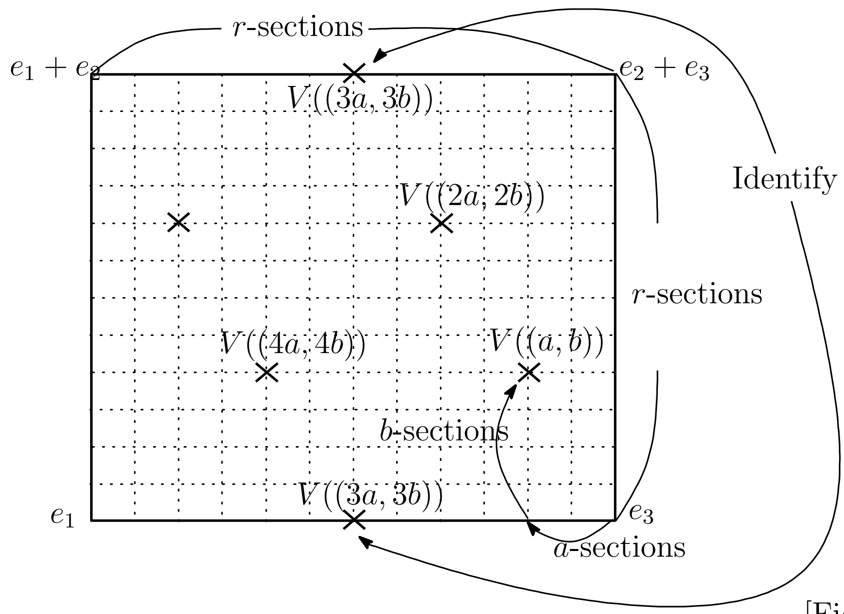
We define submodules C_1 and C_2 of $\bar{G}$ as follows:

$$C_1 := \{(a',0) \in \bar{G}\}, \ C_2 := \{(0,d') \in \bar{G}\}.$$

We shall explain a relation between the orders of C_1, C_2 and the isolatedness of a singularity X/G by using a finite fan (N', Δ) corresponding to X/G. If the canonical form of $\bar{G}$ is $\mathcal{C}(\bar{G}) = \{(a,b),(0,d)\}$ (resp. $\{(a,b)\}$), then the formula

$$N' = \frac{1}{r}(a,b,-a)\boldsymbol{Z} + \frac{1}{r}(0,d,0)\boldsymbol{Z} + N \text{ (resp. } \frac{1}{r}(a,b,-a)\boldsymbol{Z} + N) \tag{2.6}$$

holds. It is known that there is an isomorphism $G \cong N'/N$ as groups (see [22]). We define $S \subset N_{\boldsymbol{R}}$ as the quadrangle $\{l_1e_1 + l_2(e_1+e_2) + l_3(e_2+e_3) + l_4e_3 \mid l_1,l_2,l_3,l_4 \in \boldsymbol{R}, \sum_{i=1}^4 l_i = 1\}$. Since $\bar{G}$ is isomorphic to G, there is a surjection ν from N' to $\bar{G}$. We have the restriction of ν on $N' \cap S$ as $\nu|_{N'\cap S}(\frac{1}{r}(s,t,r-s)) \mapsto (s,t)$. We denote the points in N' which are in the inverse image $\nu|_{N'\cap S}^{-1}(\bar{g})$ by $V(\bar{g})$. See [Fig 27] where the largest square is S.



[Fig 27]

Each cone τ in Δ corresponds to a toric subvariety W of X, and the dimension of τ is equal to the codimension of W. Therefore X has an isolated singularity if and only if the maximal cone σ in Δ is singular and the other cones are not singular. We have the following.

Proposition 2.4.3. *X/G has an isolated singularity if and only if the canonical form of $\bar{G}$ is as follows:*

$$\mathcal{C}(\bar{G}) = \{(a,b)\} \tag{2.7}$$

where the integers a, b, r are pairwise coprime.

Proof. Assume that G is given by (2.7). Let σ^i be an i-dimensional cone in Δ where $i = 0,1,2,3$. Clearly, σ^i is generated by part of a basis of N' for $i = 0,1$, but the maximal cone $\sigma^3 = \sigma$ is not. There are four two-dimensional cones $\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2$ in Δ which are generated by $\{e_2+e_3, e_3\}, \{e_1+e_2, e_2+e_3\}, \{e_1, e_1+e_2\}, \{e_3, e_1\}$ respectively. We denote generators of σ_j^2 $(j = 1,2,3,4)$ by $\{g_{1j}, g_{2j}\}$. Since $\mathrm{GCD}(a,r) = \mathrm{GCD}(b,r) = 1$, there is an element $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3) \in N'$ of which a component α_k is one where $k = 1, 2, 3$. Let g_{3k} be an element of which k-th component α_k is one. By the formula (2.6), we may assume that the components α_1, α_3 of g_{3k} satisfy the equation $\alpha_1 = -\alpha_3$. By computations, we have the following equations:

$$|\det(g_{1k}g_{2k}g_{3k})| = \frac{1}{r}, \; |\det(g_{14}g_{24}g_{32})| = \frac{1}{r}$$

where $k = 1, 2, 3$. Hence $\{g_{1k}, g_{2k}, g_{3k}\}$ and $\{g_{14}, g_{24}, g_{32}\}$ are bases of N' where $k = 1, 2, 3$, and all two-dimensional cones in Δ are smooth. Therefore X/G has an isolated singularity.

Conversely, we assume that G is given by another canonical form. Let g_1 and g_2 be generators of σ^2. To show that X/G has no isolated singularities, it is enough to prove that there exist a two-dimensional cone $\sigma^2 \in \Delta$ which has an element $h \in N' \cap \sigma^2$ which can not be written as a linear combination of g_1 and g_2 over $\boldsymbol{Z}$. In fact, assume that there is an element $g_3 \in N'$ such that $\{g_1, g_2, g_3\}$ is a basis of N'. Then h can be written a linear combination of g_1, g_2 and g_3 over $\boldsymbol{Z}$. Since h, g_1 and g_2 are elements in σ^2, the element g_3 belongs to σ^2. This is a contradiction. Therefore there do not exist elements $g_3 \in N'$ such that $\{g_1, g_2, g_3\}$ is a basis of N'.

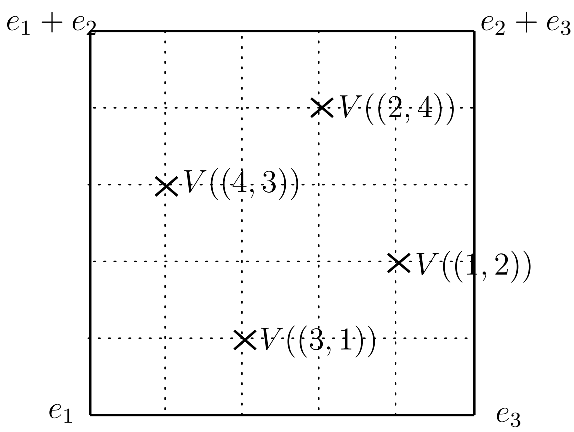
If $\#\mathrm{SMG}(\bar{G})$ is two, then N' is given by the equation of the left side of (2.6). The element $\frac{1}{r}(r, d, 0)$ (resp. $\frac{1}{r}(0, d, r)$) belongs to σ_3^2 (resp. σ_1^2), and this element can not be written as a linear combination of g_{13} and g_{23} (resp. g_{11} and g_{21}) over $\boldsymbol{Z}$. Hence the cones σ_3^2 and σ_1^2 are singular. Hence there are toric subvarieties of X/G which are singular, and the singular locus $\mathrm{Sing}(X/G)$ is not isolated.

Assume that $\#\mathrm{SMG}(\bar{G})$ is one. In this case, by the definition of the canonical form of $\bar{G}$ (see (2.5)), the formula $\mathrm{GCD}(a, b) = 1$ holds. So the remaining are two cases: (i) $\mathrm{GCD}(a, r) \neq 1$, (ii) $\mathrm{GCD}(b, r) \neq 1$. First of all, we consider the case of (i). We denote the integer $\frac{r}{a}$ by l. Since $\mathrm{GCD}(a, b) = 1$, we have an element $l(a, b) = (0, b')$ in $\bar{G}$, and b' is not zero. The lattice point $\frac{1}{r}(0, b', r)$ (resp. $\frac{1}{r}(r, b', 0)$) in the two-dimensional cone σ_1^2 (resp. σ_3^2) can not be written as a linear combination of g_{11} and g_{21} (resp. g_{13} and g_{23}) over $\boldsymbol{Z}$. So the singular locus $\mathrm{Sing}(X/G)$ is not isolated. Finally we treat the case of (ii). Let l' be the integer $\frac{r}{b}$. There is an element

$l'(a,b) = (a',0)$ in $\bar{G}$, and a' is not zero. The lattice point $\frac{1}{r}(a',0,r-a')$ (resp. $\frac{1}{r}(a',r,r-a')$) in the two-dimensional cone σ_2^2 (resp. σ_4^2) can not be written as a linear combination of g_{12} and g_{22} (resp. g_{14} and g_{24}) over $\boldsymbol{Z}$. □

In other words, X/G has an isolated singularity if and only if both of the orders of the submodules C_1 and C_2 are one. The following is an example that X/G has an isolated singularity $\mathrm{Sing}(X/G)$.

Example 2.4.1. Let X be the conifold and $G \subset GL(4, \boldsymbol{C})$ be a group acting on X toroidally. We assume that $r = 5$ and $\mathcal{C}(\bar{G}) = \{(1,2)\}$. Let (N', Δ) be the finite fan corresponding to X/G. Then $\mathrm{Sing}(X/G)$ is an isolated singularity. The quadrangle S is as follows.



[Fig 28]

Excepting for the generators of the maximal cone, there are no lattice points on the edge of S. By Proposition 2.4.3, X/G has an isolated singularity at the origin.

2.5 The proof of Theorem 1.0.3

Let X be the conifold and G be a finite group acting on X toroidally. For Note 2.5.1, see [22] and [24].

Note 2.5.1. *Let* $f : (\widetilde{\Delta}, N') \to (\Delta, N')$ *be a resolution corresponding to a triangulation of* S *and* φ *be a support function on* $|(\Delta, N')|$*. Then we have the formula*

$$K_Y = K_X + \sum_{\bar{g}\in\bar{G}} (1 - \varphi(V(\bar{g})))E_{\bar{g}}$$

where X *and* Y *are the toric varieties corresponding to* (Δ, N') *and* $(\widetilde{\Delta}, N')$ *respectively and* $E_{\bar{g}}$ *is the exceptional divisor* $\overline{\mathrm{orb}(V(\bar{g}))}$*. Moreover, the value of* φ *is given by* $\frac{1}{r}(a+c)$ *for* $\frac{1}{r}(a,b,c) \in N'$*, and* $\mathrm{discr}(E_{\bar{g}}) = 0$ *for all* $\bar{g} \in \bar{G}$*.*

Note 2.5.2. *Let* i *be an even number, and let* P_i *be an two-dimensional lattice polygon with* i *lattice points on its boundary. For any maximal triangulation* P_i' *of* P_i *by all lattice points in* P_i*, the equation*

$$2E = 3F + i$$

holds where E *(resp.* F*) is the number of edges (resp. faces) of* P_i'*.*

Theorem 2.5.1. *Let* X *be the conifold* $\mathrm{Spec}(\boldsymbol{C}[x,y,z,w]/(xz-yw))$*, and let* G *be a finite group acting on* X *toroidally. Assume that the quotient* X/G *has a Gorenstein singularity. Then the quotient admits a toric crepant resolution. Moreover, the Euler number of the crepant resolution is equal to* $2|G|$*.*

Proof. Isolated case.

Assume that X/G has an isolated singularity. By Proposition 2.4.3 and [Fig 27], there are no lattice points on ∂S except for $e_1, e_3, e_1{+}e_2, e_2{+}e_3$ where ∂S is the boundary of S. By the relation between $\bar{g} \in \bar{G}$ and $V(\bar{g}) \in N' \cap S$ in the discussion of [Fig 27], the number of the elements in $N' \cap S$ is $|G| + 3$. Let S' be a maximal triangulation of $S \subset N_{\boldsymbol{R}}$ by using all lattice points in $N' \cap S$. By Note 2.5.2, we have the equation

$$E = \frac{2}{3}F + 2$$

where E (resp. F) is the number of edges (resp. faces) in S'. By combining with the Euler's polyhedral theorem, the formula

$$|G| + 3 - \frac{2}{3}F - 2 + F = 1$$

holds. Therefore, the number of the triangles in S' is $2|G|$.

Let T be a minimal triangle in S' and t_1, t_2, t_3 be the vertices of T. Then the formulas

$$\mathrm{Vol}(T) = |\det(t_1, t_2, t_3)| \geq \frac{1}{r} \ ,$$

$$\mathrm{Vol}(S) = |\det(e_1, e_2 + e_1, e_2 + e_3)| + |\det(e_2 + e_3, e_3, e_1)| = 2$$

hold. Therefore, for all minimal triangles T in S', we have the equation

$$\mathrm{Vol}(T) = \frac{1}{r} \ .$$

This says that every maximal cone in Δ which is generated by the vertices of T is smooth. By Note 2.5.1 and the above discussion, the triangulation S' determines a toric crepant resolution, and the Euler number of the crepant resolution is equal to $2|G|$.

<u>Non-isolated case.</u>

Suppose that X/G has no isolated singularities. Then the order of C_1 or C_2 is not one, and there are lattice points $V(C_i)$ on ∂S where $i = 1, 2$. The number of lattice points in $N' \cap S$ is $|G| + |C_1| + |C_2| + 1$. Since there are $2(|C_1| + |C_2|)$ edges on ∂S, S can be seen as a lattice polygon $P_{2(|C_1|+|C_2|)}$ (see Note 2.5.2). By Note 2.5.2, we have the formula

$$E = \frac{3}{2}F + |C_1| + |C_2|.$$

By the Euler's polyhedral theorem, the equation

$$|G| + |C_1| + |C_2| + 1 - (\frac{3}{2}F + |C_1| + |C_2|) + F = 1$$

holds. Therefore, there are $2|G|$ minimal triangles in S'. By the same reason as the isolated case, this triangulation determines a toric crepant resolution, and its Euler number is $2|G|$. □

Chapter 3

McKay correspondence for Gorenstein toric quotients of the conifold

Let X and G be as in Theorem 2.5.1. In this section, we shall consider the strong McKay correspondence for X/G. We note that X/G is a *GV-variety* in Remark 6.4 in [3].

3.1 Small resolutions of Gorenstein toric quotients of the conifold

For X, there are two small resolutions $\pi_i : \widehat{X}^i \to X$ $(i = 1, 2)$. Since G acts on X toroidally and the small resolutions are toric, the group action lifts on $\widehat{X}^i$. Moreover, π_i is compatible with the quotient map by G. The small partial resolution of X/G denoted by $\pi' : \widehat{X}/G \to X/G$ $(i = 1, 2)$ is covered by two affine charts X^i_j/G^i_j, $(j = 1, 2)$ on the exceptional set isomorphic to $\boldsymbol{P}^1$ where X^i_j is isomorphic to $\boldsymbol{C}^3$ and G^i_j is isomorphic to a finite subgroup of $SL(3,\boldsymbol{C})$. i.e., X^i_j/G^i_j is a Gorenstein quotient singularity.

$$\widehat{X}^i/G = \bigcup_{j=1}^{2} X^i_j/G^i_j$$

Therefore, for every affine chart X^i_j/G^i_j, there is a crepant resolution f_{ij} : $\widetilde{X^i_j/G^i_j} \to X^i_j/G^i_j$. These two morphisms patch together to give a crepant resolution of the hole X/G.

Proposition 3.1.1. *Let X, G and X^i_j/G^i_j $(i=1,2,\ j=1,2)$ be as above. Then X^i_1/G^i_1 is isomorphic to X^i_2/G^i_2.*

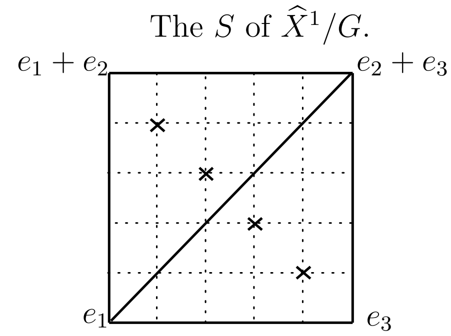
Proof. By computing Laurent polynomial rings of affine charts of small resolutions of X, we have the fact that a small resolution of X is covered by two affine charts $\mathrm{Spec}(\boldsymbol{C}[\frac{y}{z}, z, \frac{xz}{y}])$ and $\mathrm{Spec}(\boldsymbol{C}[x, y, \frac{z}{y}])$ or $\mathrm{Spec}(\boldsymbol{C}[\frac{x}{y}, y, z])$ and $\mathrm{Spec}(\boldsymbol{C}[x, \frac{y}{x}, \frac{xz}{y}])$. We set the notations as follows:

$$X^1_1 = \mathrm{Spec}\left(\boldsymbol{C}\left[\frac{y}{z}, z, \frac{xz}{y}\right]\right),\ X^1_2 = \mathrm{Spec}\left(\boldsymbol{C}\left[x, y, \frac{z}{y}\right]\right),$$

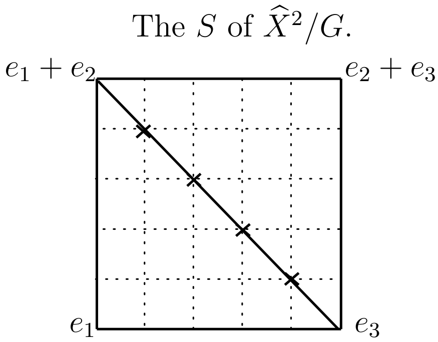
$$X^2_1 = \mathrm{Spec}\left(\boldsymbol{C}\left[\frac{x}{y}, y, z\right]\right),\ X^2_2 = \mathrm{Spec}\left(\boldsymbol{C}\left[x, \frac{y}{x}, \frac{xz}{y}\right]\right).$$

We assume that the order of G is r and the canoniccal form of G is $\mathcal{C}(\bar{G}) = \{(a,b)\}$. It is easy to check that X^1_j $(j=1,2)$ is the Gorenstein quotient singularities of type $\frac{1}{r}(a,b,-a-b)$ and X^2_j $(j=1,2)$ is the one of type $\frac{1}{r}(a,-b,b-a)$. ☐

We note that, in general, X^1_j/G^1_j is not isomophic to X^2_j/G^2_j. If (i) X/G is a non-isolated singularity or (ii) G is cyclic and $|G|$ is even number, then X^i_j/G^i_j is non-isolated singularity. X^i_j/G^i_j is not necessary isolated singularity even if (iii) X/G is an isolated singularity and $|G|$ is odd number, however either X^1_j/G^1_j or X^2_j/G^2_j is always an isolated singularity. Fig 7 (resp. Fig 8) is the quadrangle $S := \{l_1e_1 + l_2(e_1+e_2) + l_3(e_2+e_3) + l_4e_3 \mid l_1, l_2, l_3, l_4 \in \boldsymbol{R}, \sum_{i=1}^{4} l_i = 1\} \subset N'_{\boldsymbol{R}}$ of $\widehat{X}^1/G$ (resp. $\widehat{X}^2/G$) and explains the relation between the way of small resolutions and the isolatedness of X^i_j/G^i_j in the case of (iii).



[Fig 29]



[Fig 30]

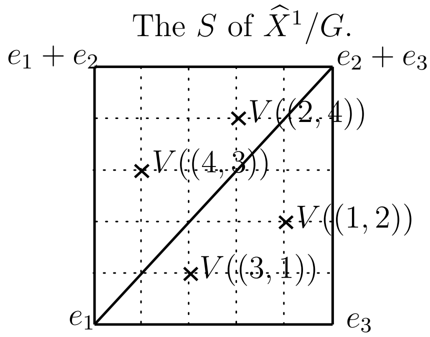
Two triangles in [Fig 29] (resp. [Fig 30]) are corresponding to the quotient singularities of type $\frac{1}{r}(a, b, -a-b)$ (resp. $\frac{1}{r}(a, -b, b-a)$). In the following, we assume that, if G satisfies the condition (iii), then $\widehat{X}/G$ is covered by affine charts with an isolated singularity.

In general, crepant resolution of X/G is not unique. However, by using toric flops, arbitrary crepant resolution of X/G can be reduced to the special ones which are coverd by $\widetilde{X^i_j/G^i_j}$. For every j, the McKay correspondence on $\widetilde{X^i_j/G^i_j}$ holds. It is proved that, if two smooth irreducible projective Calabi-Yau algebraic varieties are birational, then the i-th cohomologies with coefficient $\boldsymbol{C}$ of these varieties are isomorphic. See [2], [16]. Therefore, when we consider the strong McKay correspondence for $\widetilde{X/G}$, it is natural to take the above reduction by toric flops.

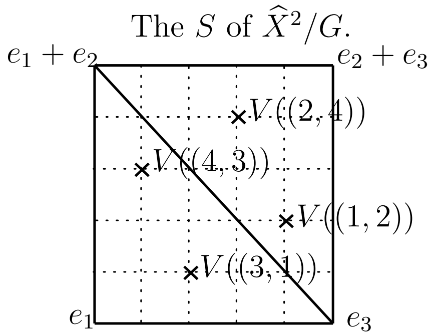
3.2 Example of Mckay correspondence

Finally, we give an example of the strong McKay correspondence in the case (iii).

Example 3.2.1. Let X and G be as in Example 2.4.1. Then the quadrangle S of a small resolution of X/G is as follows.



[Fig 31]



[Fig 32]

In this case, $\widehat{X}^1/G$ (resp. $\widehat{X}^2/G$) has two isolated singularities of type $\frac{1}{5}(1,2,2)$ (resp. $\frac{1}{5}(1,3,1)$).

Let H be an abelian subgroup of $SL(3,\boldsymbol{C})$ generated by the diagonal matrix $\mathrm{diag}(\varepsilon_3,\varepsilon_3^2,\varepsilon_3^2)$. The McKay correspondence for the quotient singularity of type $\frac{1}{5}(1,2,2)$ is as follows:

$$\Gamma_1=\left\{\frac{1}{5}(1,2,2),\frac{1}{5}(3,1,1)\right\}\leftrightarrow H^2(Y,\boldsymbol{Q})$$

$$\Gamma_1^{(0)}=\left\{\frac{1}{5}(1,2,2),\frac{1}{5}(3,1,1)\right\}\leftrightarrow \Gamma_2=\left\{\frac{1}{5}(2,4,4),\frac{1}{5}(4,3,3)\right\}$$
$$\leftrightarrow H^4(Y,\boldsymbol{Q})\leftrightarrow H_c^2(Y,\boldsymbol{Q})$$

where Y is a crepant resolution of the singularity. We follow notation in [14]. The Euler number $e(Y)$ is as follows:

$$e(Y)=h^0(Y,\boldsymbol{Q})+h^2(Y,\boldsymbol{Q})+h^4(Y,\boldsymbol{Q})=1+2+2=5.$$

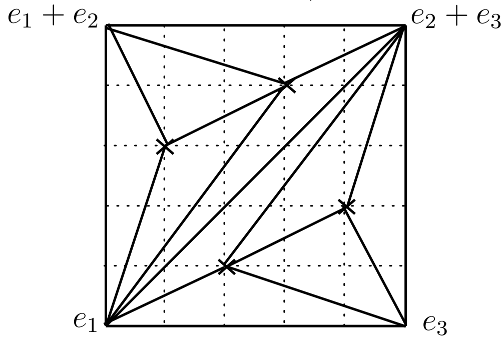
On the other hand, there is a correspondence between H and G.

$$H\supset\Gamma_1=\left\{\frac{1}{5}(1,2,2),\frac{1}{5}(3,1,1)\right\}\leftrightarrow\left\{\frac{1}{5}(1,2,4,3),\frac{1}{5}(3,1,2,4)\right\}\subset G$$

$$H\supset\Gamma_2=\left\{\frac{1}{5}(2,4,4),\frac{1}{5}(4,3,3)\right\}\leftrightarrow\left\{\frac{1}{5}(2,4,3,1),\frac{1}{5}(4,3,1,2)\right\}\subset G$$

The figure [Fig 33] is a crepant resolution $\widetilde{X/G}$ of X/G which is covered by crepant resolutions of $\boldsymbol{C}^3/H$.

The S of $\widetilde{X/G}$.



[Fig 33]

Bibliography

[1] T. Bridgeland, A. King and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001) 535–554.

[2] V. V. Batyrev, *Birational Calabi-Yau n-folds have equal Betti numbers*, New trends in algebraic geometry (Warwick, 1996), London Math. Soc. Lecture Note Ser. **264**, Cambridge Univ. Press, Cambridge (1999) 1–11.

[3] V. V. Batyrev and D. I. Dais, *Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry*, Topology **35** (4) (1996) 901–929.

[4] D. I. Dais, C. Haase and G. M. Ziegler, *All toric local complete intersection singularities admit projective crepant resolutions*, Tohoku Math. J. (2) **53** (2001) 95–107.

[5] D. I. Dais, U. U. Haus and M. Henk, *On crepant resolutions of 2-parameter series of Gorenstein cyclic quotient singularities*, Results in Math. **33** (1998) 208–265.

[6] D. I. Dais and M. Henk, *On a series of Gorenstein cyclic quotient singularities admitting a unique projective crepant resolution*, alg-geom/9803094.

[7] D. I. Dais, M. Henk and G. M. Ziegler, *All abelian quotient c.i.-singularities admit projective crepant resolutions in all dimensions*, Adv. in Math. **139** (1998) 194–239.

[8] D. I. Dais, M. Henk, and G. M. Ziegler, *On the existence of crepant resolutions of Gorenstein abelian quotient singularities in dimensions*$\geq$ 4, Contemp. Math., **423**, Amer. Math. Soc., Providence, RI, 2006.

[9] O. Fujino, H. Sato, Y. Takano, H. Uehara, *Three-dimensional terminal toric flips*, Cent. Eur. J. Math. **7** (2009) 46–53.

[10] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, Vol. **131**, Princeton University Press, 1993.

[11] G. Gonzalez-Sprinberg and J.-L. Verdier, *Construction geometrique de la correspondance de McKay*, Ann. Sci. Ecole Norm. Sup. (4) , **16**(3), (1984) 409–449, 1983.

[12] Y. Ito, *Crepant resolution of trihedral singularities and the orbifold Euler characteristic*, Intern. Jour. of Math. **6** (1995) 33–43.

[13] Y. Ito, *Gorenstein quotient singularities of monomial type in dimension three*, Jour. of Math. Sciences, University of Tokyo **2** (1995) 419–440.

[14] Y. Ito, M. Reid, *The McKay correspondence for finite subgroups of* $SL(3, \boldsymbol{C})$, Higher Dimensional Complex Varieties, Proc. of Internat. Conference, Trento 1994, de Gruyter, (1996), pp. 221–240.

[15] S. Ishii, *Tokuiten Nyuumon* (in Japanese), Gendai suugaku series, Springer-Verlag Tokyo (1997).

[16] M. Kontsevich, Lecture at Orsay (December 7, 1995).

[17] K. Kurano, S. Nishi, *Gorenstein isolated quotient singularities of odd prime dimension are cyclic*, Comm. in Algebra. **40** (2012) 3010–3020.

[18] M. Kapranov and E. Vasserot, *Kleinian singularities, derived categories and Hall algebras*, Math. Ann. **316** (2000), no. 3, 565–576.

[19] D. G. Markushevich, *Resolution of* $\boldsymbol{C}^3/H_{168}$, Math. Ann. **308** (1997) 279–289.

[20] D. G. Markushevich, M. A. Olshanetsky, A. M. Perelomov, *Description of a class of superstring compactifications related to semi-simple Lie algebras*, Commun. in Math. Phys. **111** (1987) 247–274.

[21] J. McKay, *Graphs singularities and finite groups*, Proc. of 1979 Santa Cruz group theory conference, AMS Symposia in Pure Mathematics **37**, (1980) pp. 183–186.

[22] T. Oda, *Convex bodies and algebraic geometry, An introduction to the theory of toric varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge, **15**, Springer-Verlag, 1988.

[23] D. Prill, *Local classification of quotients of complex manifolds by discontinuous groups*, Duke Math. Jour. **34** (1967) 375–386.

[24] M. Reid, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 345–414, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[25] S. S. Roan, *Minimal resolutions of Gorenstein orbifolds in dimension three*, Topology **35** (1996) 53–72.

[26] K. Sato, *Existence of crepant resolution for abelian quotient singularities by order* p *elements in dimension* 4, Saitama Math. J. **27** (2010) 9–23.

[27] S. S.-T. Yau and Y. Yu, *Gorenstein quotient singularities in dimension three*, Mem. Amer. Math. Soc. **105** (1993), No. 505.

[28] K. Watanabe, *Invariant subrings which are complete intersections. I. Invariant subrings of finite abelian groups*, Nagoya Math. J. **77** (1980) 89–98.

[29] K. Watanabe and H. Nakajima, *The classification of quotient singularities which are complete intersections*, Lecture notes in mathematics, Vol. **1092**, Springer-Verlag, 1984, 102–120.