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Doctoral Course in Mathematics, Graduate School of Tokyo Metropolitan Univ., **Abstract** (2011/02/11), pp.1–4

Historical development of classical fluid dynamics

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Abstract

Aims.

In our thesis, we discuss *the classical theory of mathematical fluid dynamics*, with interest in the theoretical formulation of the microscopically-descriptive [*MD*] hydrodynamical [*HD*] equations, above all, the Navier-Stokes [*NS*] equations, up to the fixed formulation, from the viewpoint of the mathematical history. We want to study the fluid dynamics in particular, not from all-inclusive history of topics, but from the mathematical deductions of the classical theories. Our initial motivation of study had been to seek from the classical theories for something new of deductive method of the *MDNS* equations.

Main results.

We treat the following kernel problems of theories, discussed in order from the viewpoint of theoretical and mathematical history, viz. :

(1) exact differential / complete differential
(2) the "two-constant" theory
(3) tensor function
(4) rapidly decreasing function [*RDF*]
(5) collision in gas theory
(6) solutions of the NS equations

We believe, in particular, the following discovering approaches :

- comparative and detailed descriptions of the various deductions of the *MDNS* equations by Navier, Cauchy, Poisson, Saint-Venant and Stokes, above all, the contribution of Saint-Venant to the universal form of tensor for the linear *NS* equations and our mention of the "two-constant" theory by Laplace as a progenitor of it
- theoretical deduction of the *MDNS* equations, including the "two-constant" theory, tensor function and rapidly decreasing function
- as a contemporary of an epoch of the formulation of *NS* equations, we pay attention to Gauss' contributions to the fluid mechanics including some mathematical achievements,
- the consistent follow-up of the *MDHD* equations after the formulation of the *NS* equations, including the gas theory, up to fixed formulation of our equations,

and these results from it, may be original. These total problems are the main themes we would like to present now.

The authorities for our originalities of results.

We have the following authorities for our originalities of our results.

- We refer O.Darrigol [1, 2] and C.Truesdell [3, 4, 5] as the introductions to start our study. They doesn't refer many of our results, in particular, *HD* equations by Euler, Lagrange, Laplace, Gauss, Maxwell, Kirchhoff, Boltzmann, and so on, while we mentioned them.
- Darrigol [1, 2] doesn't show his definition of the two-constant theory, but discusses from the point of view of exact science, and citing only one paragraph in his book. We owe our motivation to enhance this theme to Darrigol, however, we could adopt our own study, above all, from the mathematical viewpoint in **Part 2**, **Part 3** and **Part 4** .
- We couldn't cite some important persons including Newton, D.Bernoulli, and so on for fluid dynamics, however, we have intention to cover the important problems for the *NS* equations and the clasical theories of the *HD* equations in the 18-19 and the first half of 20 centuries. [1]
- We have scarecely heard about the history of mathematics on the NS equations up to Ladyzhenskaya.

Contents.

The contents of our thesis consist of the following three parts entitled with :

Part 1. (pp. 1-26) contains the problem (1) :

In the classical fluid mechanics, it had been an important principle to see whether equilibrium or motion, that in three variables, for $udx + vdy + wdz$ to be satisfied with an *exact differentiability* or a *complete differentiability*. By Maupertuis, Clairaut, d'Alembert, Euler, Lagrange, Laplace, Cauchy, Poisson and Stokes succeeded its theoretical side. From the geometrical point of view, Gauss and Riemann applied it. Gauss proposed a general principle between equilibrium and motion. Moreover Helmholtz and W.Thomson applied it to the theory of vorticity. To Helmholtz's vorticity equation, Bertrand criticized but Saint-Venant sided with Helmholtz. We would like to report on their studies of exact differential from the historical view of fluid dynamics.

In §2, we show the proofs of the eternal existence of once-occurred exact differential by Lagrange, Cauchy and Stokes.

On the other hand, the formulations of two-constant theory in equilibrium/motion was deduced by Poisson, Navier, Cauchy, Saint-Venant and Stokes, and today's *NS* equations were formulated and used in practice. The studies of it up to the present are shown in the following papers.

Part 2. (pp. 27-164) consists two parts of :

- The main or general remarks titled *the "two-constant" theory and tensor function underlying the NS equations.* (pp. 27-52)
- The particulars appending detailed contents to the main (pp. 53-164) :
  - A : Detailed commentary of deduction of the *NS* equations, the "two-constant" theory and tensor function
  - B : The "two-constant" theory in the capillary action
  - C : Laplace and Gauss

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[1] (⇓) To establish a time line of these contributor, we list for easy reference the year of their birth and death: Sir I.Newton(1643-1727), D.Bernoulli(1700-1782), Euler(1707-1783), d'Alembert(1717-1783), Lagrange(1736-1813), Laplace(1749-1827), Fourier(1768-1830), Gauss(1777-1855), Navier(1785-1836), Poisson(1781-1840), Cauchy(1789-1857), Saint-Venant(1797-1886), Stokes(1819-1903).

D : Abstract of Gauss' papers on *Disquisitiones generales circa superficies curvas. (General survey on the curved surface)*
E : Gauss' papers on *Principia generalia theoriae figurae fluidrum in statu aequilibrii (General principles of theory on fluid figure in equilibrium state)*
F : Poisson's "two-constant" theory in the capillary action
G : Figures.

The contents of **Part 2** are of the fundamental problems (2), (3), and (4) of our main results , in which we treat the process of the formulation of the *NS* equations and the contained-in-itself, fundamentally mathematical theories such as the "two-constant" theory, tensor and *RDF*.

The "two-constant" theory introduced for the first time by Laplace in 1805 still forms the basis of current theory describing isotropic, linear elasticity. The *NS* equations in incompressible case :

$$\partial_t \mathbf{u} - \mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div }\mathbf{u} = 0.$$

as presented in final form by Stokes in 1845, were derived in the course of the development of the "two-constant" theory.

Following in historical order the various contributions of Navier, Cauchy, Poisson, Saint-Venant and Stokes over the intervening period, we trace the evolution of the equations, and note concordances and differences between each contributor. In particular, from the historical perspective of these equations we look for evidence for the notion of tensor.

Also in the formulation of equilibrium equations, we obtain the competing theories of the "two-constant" theory in capillary action of Laplace and Gauss.

Finally, we uncover reasons for the practice in naming these fundamental equations of motion as the *NS* equations.

In the appendices, we show the process of formulation citing their main papers of Navier, Cauchy, Poisson, Laplace and Gauss with our commentary.

Part 3. (pp. 165-192) discuss the problem (5) :

The *MDHD* equations are followed by the description of equations of gas theory by Maxwell, Kirchhoff and Boltzmann. Above all, in 1872, Boltzmann formulated the Boltzmann equations, expressed by the follwing today's formulation :

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = Q(f,g), \quad t > 0, \quad \mathbf{x},\mathbf{v} \in \mathbb{R}^n (n \geq 3), \quad \mathbf{x} = (x,y,z), \quad \mathbf{v} = (\xi,\eta,\zeta), \tag{1}$$

$$Q(f,g)(t,x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*,\sigma)\{g(v'_*)f(v') - g(v_*)f(v)\}d\sigma dv_*, \quad g(v'_*) = g(t,x,v'_*), \text{ etc.} \tag{2}$$

These equations are able to be reduced for the general form of the *HD* equations, after the formulations by Maxwell and Kirchhoff, and from which the Euler equations and the *NS* equations are reduced as the special case.

After Stokes' linear equations, the equations of gas theories were deduced by Maxwell in 1865, Kirchhoff in 1868 and Boltzmann in 1872. They contributed to formulate the fluid equations and to fix the *NS* equations, when Prandtl stated the today's formulation in using the nomenclature as the "so-called *NS* equations" in 1934, in which Prandtl included the three terms of nonlinear and two linear terms with the ratio of two coefficients as 3 : 1, which arose Poisson in 1831, Saint-Venant in 1843, and Stokes in 1845. Prandtl says, "The following differential equation, known as the equation of Navier-Stokes, is the fundamental equation of hydrodynamics,"

$$\frac{D\mathbf{w}}{dt} = \mathbf{g} - \frac{1}{\rho}\text{grad } p + \frac{1}{3}\nu \text{ grad div } \Delta\mathbf{w} + \nu\Delta\mathbf{w},$$

$$\text{where,} \qquad \frac{D\mathbf{w}}{dt} \equiv \frac{\partial\mathbf{w}}{\partial t} + \mathbf{w}\cdot\nabla\mathbf{w}, \ \nu = \frac{\mu}{\rho}, \ \mathbf{w} = (u,v,w), \ \mathbf{g} = (X,Y,Z)$$

Part 4. (pp. 193-235) sketches the early studies of solutions of the *NS* equations.

In this part, we discuss the weak solutions by Leray and Hopf, the generalized solutions/ the strong solutions by Kieselev, Ladyzhenskaya in the course of history of the studies of solutions of the *NS* equations.

Conclusions.

- We had owed the development of classical fluid dynamics including *NS* equations, to the various results for centuries past, based on such the fundamental and mathematical theories as of the kinetic equation by Newton, of the exact differential, of the molecular activities in motion and equilibrium including the capillary action, of "two-constant" and of the *RDF*, by many progenitors of the theories[2] who studied the the formulation of *HD* equations and the mathematical theories. And even now, the studies for the open problems are in progress. We believe that the studies from the viewpoint of mathematical history are also able to play a part in contributing to verify these past facts and past development, and to make out the future development of the fluid dynamics.

- The "two-constant" defined at first in the kinetic equations of elasticity was applied to that of fluid by the *MD* method by Navier, Cauchy and Poisson, but later it was fixed as "two-coefficient" in the *HD* equations since Poisson's fluid equations. The former's ratio of coefficient attached to the tensor function with the main axis (the normal stress) of Laplacian to that of grad div : $\frac{\text{coefficient of tensor}}{\text{coefficient of grad div}} = \frac{1}{2}$, and the latter's is 3.
- The original *RDF* was deduced in the course of formulation of the equation of fluid dynamics, including the equations of capillary action by Laplace and Gauss, in particular, *Gaussian function* in the equations of capillary action was deduced over a hundred years ago before Schwartz' *distribution and hyperfunction.*

Remarks. Throughout our thesis, in citation of bibliographical sources, our are delimited by ($\Downarrow$) and (if necessary) ($\Uparrow$). And by $=^*$, we detail the statement by original authors, because we would like to discriminate and to avoid confusion from the descriptions by the original authors. The mark : $\Rightarrow$ means transformation of the statements in brevity by author. And all the frames surrounding the statements are inserted for important remark by author. Of course, when the descriptions are explicitly distinct without these marks, these are not the descriptions in citation of bibliographical sources.

TABLE 1. Comment descriptions by marks in our thesis

	case of comment	mark in paragraph	mark in equation or statement
1	comments by an original author	(usual description)	$=$, $\Rightarrow$
2	comments by another person to the original author	(usual description)	
3	comments by author	($\Downarrow$) $\cdots$ our comment $\cdots$ ($\Uparrow$)	$=^*$, $\Rightarrow^*$
4	additional comments to our comments	($\Downarrow$) $\cdots$ our comment $\cdots$ ($\Uparrow$)	$=^*$, $\Rightarrow^*$

The abridgements mean :

- *NS* : Navier-Stokes, (ex. the *NS* equations.)
- *MD* : microscopically-descriptive, (ex. the *MD* equations.)
- *HD* : hydrodynamical, (ex. the *HD* equations.)
- *RDF* : rapidly decreasing function, (ex. the *RDF*s.)

The introductions refered for beginning our study.
Except for over a hundred primary sources which we show in the references of each part, we show only the following introductions :

REFERENCES

[1] O.Darrigol, *Between hydrodynamics and elasticity theory : the first five births of the Navier-Stokes equation*, Arch. Hist. Exact Sci., **56**(2002), 95–150.
[2] O.Darrigol, *Worlds of flow: a history of hydrodynamics from the Bernoullis to Prandtl*, Oxford Univ. Press, 2005.
[3] C.Truesdell, *Notes on the History of the general equations of hydrodynamics*, Amer. Math. Monthly **60**(1953), 445-458.
[4] C.Truesdell, *The rational fluid mechanics. 1687-1765. Introduction to Leonhard Euleri Opera Omnia. Vol XII seriei secundae*, Auctoritate et impensis societatis scientiarum naturalium helveticae, **2-12** 1954, 10–125.
[5] C.Truesdell, *Editor's introduction to Leonhard Euleri Opera Omnia Vol. XIII seriei secundae*, ibid., **2-13** 1955, 9–105.

[2]($\Downarrow$) We mean at least, progenitors such as Newton, D.Bernoulli, Maupertuis, Clairaut, Euler, d'Alembert, Lagrange, Laplace, Navier, Cauchy, Gauss, Poisson, Saint-Venant, Stokes, Helmholtz, W. Thomson, Maxwell, Kirchhoff, Boltzmann, Prandtl et al., who we mention in our papers.

Doctoral Course in Mathematics, Graduate School of Tokyo Metropolitan Univ., **Part 1.**(2011/02/11), pp.1–26

Exact differentials as criteria for equilibrium/motion and irrotational motion/rotary motion

ABSTRACT. Exact differentials in fluid dynamics are important quantities in any mathematical analysis of continuous systems; for example, we may need to know if $udx + vdy + wdz$ satisfies *exact*, or equivalently *complete, differentiability* in three dimensions. In the hands of d'Alembert, Euler, Lagrange, Laplace, Cauchy, Poisson and Stokes, these practitioners have succeeded in developing its theoretical consequences. From the geometric point of view, Gauss and Riemann had applied such constructs, while Helmholtz and W. Thomson applied these to the theory of vortices. Although Helmholtz's vorticity equation was strongly criticized by Bertrand, Saint-Venant sided with Helmholtz. Here, we would like to review from the historical viewpoint the study of exact differential in fluid mechanics.

In §2, we present proofs of the eternal existence of unique exact differentials by Lagrange, Cauchy and Stokes.

From a separate development, the formulation of the two-constant theory in equilibrium/motion had been deduced by Navier, Poisson, Cauchy, Saint-Venant and Stokes. Today's Navier-Stokes equations were formulated and used in practice. An up-to-the present study is given in papers to follow.

Mathematics Subject Classification 2010 : 01Axx, 76A02, 76Mxx, 76-02, 76-03, 33A15, 35Qxx 35-xx.
Key words. Exact differential, complete differential, fluid dynamics, fluid mechanics, microscopically-descriptive equations, hydrostatics, hydrodynamics, hydromechanics, mathematical history.

CONTENTS

1. Introduction - the mathematical historic view of exact differentials

[1] In the early development of fluid mechanics, exact differentials of the form $udx + vdy + wdz$ had been used in theories of equiliblium, in various applications and appeared in numerious discussions. We present a summary of this development from a historical viewpoint in Table 1, under the following topic headings: condition of equilibrium of fluid, proof of the eternal continuity in time and space of exact differentials, curvature, electromagnetism, topology, vorticity , discussion of Helmholtz's papers, other applications.

Our motivation to study these topics arises from the last pages of Poisson's article 73 [37, pp.173-4], in which he remarked that although the exact differential may hold at some initial time in the motion, it doesn't follow that it always holds at later times.[2] We would like to reveal the mistake behind "*Poisson's conjecture*" and the fact that the Navier-Stokes equations can be formulated following this train of ideas.

2. Observations from the exact differential to the vortex

2.1. Maupertuis' principle of minimum action.

P.L.Maupertuis' paper is famous for its stating of the *Principle of least action*, notwithstanding its application to geometrical optics. The paper on the law of equilibrium was read to members of *l'Académie Royale des Sciences de Paris* in 1740:

> Ce n'est que dans ces derniers temps qu'on a découvert une loi dont on ne sauroit trop vanter la beauté & l'utilité, c'est que dans tout systême de corps élastiques en mouvement, qui aggisent les uns sur les autres, la somme des produits de chaque masse par le quarré de sa vitesse, ce qu'on appelle la force vive, demuere inaltérablement la même. $\cdots$
>
> Soit un systême de corps qui pesent, ou qui sont titrés vers des centres par des forces qui agissent chacune sur chacun, comme une puissance N de leurs distances aux centres: pour que tous ces corps demeurent en repos, il faut que la somme des produits de chaque masse, par l'intensité de sa force, & par la puissance $N+1$ de sa distance au centre de sa force (qu'on peut appeller la somme des forces du repos) fasse un maximum ou un minimum. [31, pp.47-48]

In the proof of the above propositions, he concluded that: for a system in equilibrium, it is necessary that the following holds:

$$mfz^n dz + m'f'z'^n dz' + m''f''z''^n dz'' = 0, \tag{1}$$

> where m, m', m'' are masses and f, f', f'' are forces. Hence, the value of $mfz^{n+1}dz + m'f'z'^{n+1}dz' + m''f''z''^{n+1}dz''$ is then either a maximum or minimum. [31, p.52]

As an aside, if homogeneous, we can substitute z, z', z'' with x, y, z and $mf, m'f', m''f''$ with P, Q, R then (1) becomes Euler's form of an equation for which he had cited Maupertuis: $dS = Pdx+Qdy+Rdz = 0$.

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[1]($\Downarrow$) To establish a time line of these contributor, we list for easy reference the year of their birth and death: Sir I.Newton(1643-1727), D.Bernoulli(1700-1782), Euler(1707-1783), d'Alembert(1717-1783), Lagrange(1736-1813), Laplace(1749-1827), Fourier(1768-1830), Gauss(1777-1855), Navier(1785-1836), Poisson(1781-1840), Cauchy(1789-1857), Saint-Venant(1797-1886), Stokes(1819-1903).

[2]($\Downarrow$) Poisson stated: Mais la démonstration qu'on donne de cette proposition suppose que les valeurs de u, v, w, doivent satisfaire non seulment aux équations différentielles du mouvement, mais encore à toutes celles qui s'en déduisent en les différentiant par rapport à t; ce qui n'a pas toujours lieu à l'egard des expressions de u, v, w, en séries d'exponentielles et de sinus ou cosinus dont les posans et les arcs sont proportionnels au temps; et la demonstration étant alors en défaut, il peut arriver que la formule $udx + vdy + wdz$ soit une différentielle exacte à l'origine du mouvement, et qu'elle ne soit plus à toute autre époque. [37, p.174]

TABLE 1. Theories, applications and discussions about the exact differentiability of $udx + vdy + wdz$ in fluid mechanics

equilibrium	proof	curvature	electromagnetics	topology	vorticity	discussion	application
Maupertuis 1740,68 [31, 32]							
Clairaut 1741-43,1808(2ed.) [6]					Clairaut 1741-43,1808(2ed.) [6]		
d'Alembert 1749-52 [8]							
Euler 1752-55 [13]					Euler[E226] 1755-57 [13]		
d'Alembert 1761 [9]					d'Alembert 1761 [9]		
Lagrange 1781-1869 [26]	Lagrange 1781-1869 [26]				Lagrange 1781-1869 [26]		
Laplace 1806/07-29 [29]							
Cauchy 1815-27 [5]	Cauchy 1815-27 [5]						
Navier 1822-27 [34]							
Gauss 1813 [15], 1827 [17], 1830 [18]		Gauss 1828 [16]					
Poisson 1829-31 [37]							
	Power 1842-42 [39]						
Stokes 1845-49 [43]	Stokes 1845-49 [43]				Stokes 1845-49 [43]		
			Green 1850 [20]				
				Riemann 1857 [40]			
					Helmholtz 1858 [21]	Helmholtz 1868 [22, 23, 24]	
							Clebsch 1858-1859 [7]
	Thomson 1867-69 [47]				Thomson 1867-69 [47]	Thomson 1867-69 [47]	
						Bertrand 1868 [1, 2, 3, 4]	
						Saint-Venant 1868 [42]	
	Lamb 1879 [28]						
							Leray 1934 [30]

I cite below

Gauss proposed the general principle of both static state and motion in a note in 1827 generalized Maupertuis' principle of minimum action, which we mention below in §3.

2.2. Clairaut's *effort* and *exact differential*.

Writing on hydrostatics in 1740, Clairaut had already used*effort* (response) and *exact differential*. In his thesis, *Théorie de la figure de la terre, tirée des principes de l'hydrostatique* (Theory of the shape of the Earth, derived from the principle of hydrostatics), he proposed the term *exact differential* earlier than Euler.

> Si on voulait présentement faire usage de cette quantité, pour trouver en termes finis la valeur du poids du canal ON, en supposant que la courbure de ce canal fût donnée par une équation entre x et y, on commencerait par faire évanouir y et dy de $Pdy + Qdx$; cette différentielle n'ayant plus que des x et dx, on intégrait en observant de compléter l'intégrale, c'est-à-dire

d'ajouter la constante nécessaire, afin que le poids fût nul, lorsque x serait égal à CG ; on ferait ensuite $X = CI$, et l'on aurait le poids total de ON. [6, §16, p.35-37]

Mais comme l'equilibre du fluide demande que le poids de ON ne dépende pas de la courbure de OSN, c'est-à-dire de la valeur particulière de y en x, il faut donc que $Pdy + Qdx$ puisse s'intégrer sans connaître la valeur de x, c'est-à-dire qu'il faut que $Pdy + Qdx$ soit une *différentielle complète*, afin qu'il puisse y avoir équilibre dans le fluide. [6, §16, p.35-37].

In a footnote, Clairaut commented on exact differentials [3] as follows:

J'entends par *différentielle complète*, une quantité qui a pour intégrale une function de x et de y. $ydx + xdy$, $\frac{ydx+xdy}{2\sqrt{a^2+xy}}$ sont des *différentielle complètes*, parce qu'elles ont pour intégrales xy, $\sqrt{a^2+xy}$, $\frac{xdy-ydx}{x^2+y^2}$ est aussi une *différentielle complète*, parce que son intégrale est représentée par l'arc dont la tangente est $\frac{y}{x}$, le rayon étant 1. Mais $y^3dx + x^3dy$, $y^2dx + x^2dy$, ne sont pas des *différentielles complètes*, parce qu'aucunes fonctions de x et de y n'en sauraient être les intégrales. [6, p.37, footnote]. [4]

2.3. D'Alembert's exact differential.

D'Alembert[8] teaches us various types of the exact differential of the complex value. (Now we show here e.d. in brief.)

If $Mdx + Ndz$ and $Ndx - Mdz$ are the exact differentials, we propose to find the quantity of M and N.

- $Mdx + Ndz$ is e.d. $\Rightarrow Mdx + N\sqrt{-1}\frac{dz}{\sqrt{-1}}$.
- $Ndx - Mdz$ is e.d. $\Rightarrow N\sqrt{-1}dx - Mdz\sqrt{-1}$ or $N\sqrt{-1}dx + M\frac{dz}{\sqrt{-1}}$ is e.d.
  $\Rightarrow \left(M + N\sqrt{-1}\right)\left(dx + \frac{dz}{\sqrt{-1}}\right)$ and $\left(M - N\sqrt{-1}\right)\left(dx - \frac{dz}{\sqrt{-1}}\right)$ are e.d.s
- $dx + \frac{dz}{\sqrt{-1}} = du \quad \Rightarrow \quad$ (a function of F) $+ x + \frac{z}{\sqrt{-1}} = u$,
- $dx - \frac{dz}{\sqrt{-1}} = dt \quad \Rightarrow \quad$ (a function of G) $+ x - \frac{z}{\sqrt{-1}} = t$,
- $M + N\sqrt{-1} = \alpha \quad \& \quad M - N\sqrt{-1} = \beta$,
  $\Rightarrow$
  - α is a function of u, i.e. $M + N\sqrt{-1} =$ a function of $F + x + \frac{z}{\sqrt{-1}}$,
  - β is a function of t, i.e. $M - N\sqrt{-1} =$ a function of $G + x - \frac{z}{\sqrt{-1}}$.

D'Alembert proposes the following simple format.

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[3]($\Downarrow$) It is called the condition for *exact differentibility* as follows. Now, for brevity, we treat only a two variable case. In the domain K of the xy-plane, where the two functions $\varphi(x,y) \in C^1$ and $\psi(x,y) \in C^1$ are given, and we suppose

$$\varphi(x,y)dx + \psi(x,y)dy \tag{2}$$

is the total differential of an arbitrary function $F(x,y)$, namely $dF = \varphi dx + \psi dy$. Hence, $F_x = \varphi, \quad F_y = \psi$.
Then by the assumption, we obtain $F_{xy} = \varphi_y$ and $F_{yx} = \psi_x$, namely,

$$\varphi_y = \psi_x. \tag{3}$$

(3) is the necessary condition that (2) becomes an *exact differential*, and if the domain K is simply-connected, (3) immediately becomes also a sufficient condition. We treat below*exact differential* and *complete differential* as being equivalent.

[4]Two examples of exact differentials given by Clairaut are simple: if we consider $\frac{xdy-ydx}{x^2+y^2}$ and set $P = -\frac{y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$, then we obtain:

$$\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$$

Considering also $\frac{ydx+xdy}{2\sqrt{a^2+xy}}$and put $P = \frac{y}{2\sqrt{a^2+xy}}$ and $Q = \frac{x}{2\sqrt{a^2+xy}}$, then we obtain:

$$\frac{\partial P}{\partial y} = \frac{2\sqrt{a^2+xy} - y(a^2+xy)^{-\frac{1}{2}}x}{(2\sqrt{a^2+xy})^2} = \frac{\partial Q}{\partial x}.$$

In contrast, as two examples of inexact differentials, we find for $x^2dy + y^2dx$

$$\frac{\partial P}{\partial y} = 2y \neq \frac{\partial Q}{\partial x} = 2x,$$

and for $x^3dy + y^3dx$, we get

$$\frac{\partial P}{\partial y} = 3y^2 \neq \frac{\partial Q}{\partial x} = 3x^2.$$

- $\frac{dp}{dz} = -\frac{dq}{dx}, \quad \frac{dp}{dx} = \frac{dq}{dz} \quad \Rightarrow \quad qdx + pdz, \quad pdx - qdz$ are e.d.s
  $\Rightarrow$
  - $q + p\sqrt{-1} = F + x + \frac{z}{\sqrt{-1}}$ & $q - p\sqrt{-1} = G + x - \frac{z}{\sqrt{-1}}$
    $$\Rightarrow \quad \begin{cases} q = \frac{1}{2}\Big\{\Big(F + x + \frac{z}{\sqrt{-1}}\Big) + \Big(G + x - \frac{z}{\sqrt{-1}}\Big)\Big\}, \\ p = \frac{1}{2\sqrt{-1}}\Big\{\Big(F + x + \frac{z}{\sqrt{-1}}\Big) - \Big(G + x - \frac{z}{\sqrt{-1}}\Big)\Big\} \end{cases}$$
  - p and q are the real numbers
    $\Rightarrow$ we must suppose $G = F$,
    $$\Rightarrow \quad \begin{cases} q = \Big\{\xi\Big(F + x + \frac{z}{\sqrt{-1}}\Big) + \xi\Big(F + x - \frac{z}{\sqrt{-1}}\Big)\Big\} + \sqrt{-1}\Big\{\zeta\Big(F + x + \frac{z}{\sqrt{-1}}\Big) - \zeta\Big(F + x - \frac{z}{\sqrt{-1}}\Big)\Big\}, \\ p = \frac{1}{\sqrt{-1}}\Big\{\xi\Big(F + x + \frac{z}{\sqrt{-1}}\Big) - \xi\Big(F + x - \frac{z}{\sqrt{-1}}\Big)\Big\} + \Big\{\zeta\Big(F + x + \frac{z}{\sqrt{-1}}\Big) + \zeta\Big(F + x - \frac{z}{\sqrt{-1}}\Big)\Big\}, \end{cases}$$
    where $\xi\Big(F + x \pm \frac{z}{\sqrt{-1}}\Big)$ and $\zeta\Big(F + x \pm \frac{z}{\sqrt{-1}}\Big)$ denote the differential functions.

2.4. **Euler's study on the exact differential.**
Euler investigated the nature of exact differentials in the following papers:

- (E258) *Principia motus fluidorum* (Principles of the motion of fluids) [1752], (1756/57), 1761.
- (E225) *Principes généraux de l'état d'équilibre des fluides* [1753], (1755), 1757.
- (E226) *Principes généraux du mouvement des fluides* [1755], (1755), 1757.
- (E227) *Continuation des recherches sur théorie du mouvement des fluides* [1755], (1755), 1757.
- (E375) *Sectio prima de statu aequiliblii fluidorum* (Section 1. On the state of equilibrium of fluids) (1768), 1769.
- (E396) *Sectio secunda de principiis motus fluidorum* (Section 2. On the principles of motion of fluids) (1769), 1770.

  where (E...) denotes the *Eneström Index*, while the years appearing at the end of the item are respectively:
  - in square brackets, the year commented on by C.Truesdell [48],
  - in parenthesis, the year commented on by *Eneström* in *Euleri Opera Omnia* [13], and
  - unbracketed, the published year commented on by *Eneström* in *Euleri Opera Omnia* [13],

2.4.1. **Development of the exact differential by Euler.**
Of the many papers in which Euler discussed exact differentials, we shall take a closer look atone of these. In (E396), Euler posed Problem 34:

> §88. Si cuiusque fluidi elementi ternae celeritates u, v, w ita sint comparatae, ut formula $udx + vdy + wdz$ integrationem admittat, aequationem, qua pressio fluidi exprimitur, evolvere. (E396) [13, p.127].
>
> (Translation) $\Rightarrow$ If the three elements of the velocity of an arbitrary fluid element: u, v and w are proportional to each other and the expression: $udx + vdy + wdz$ is integrable, derive the equation by which the fluid pressure can be expressed.

Euler solved his problem as follows:[5]

$$dI = udx + vdy + wdz + \Phi dt.$$

$$U = u\Big(\frac{du}{dx}\Big) + v\Big(\frac{du}{dy}\Big) + w\Big(\frac{du}{dz}\Big) + \Big(\frac{du}{dt}\Big).$$

where, it holds the exact differential, then $\dfrac{du}{dy} = \dfrac{dv}{dx}, \quad \dfrac{du}{dz} = \dfrac{dw}{dx}, \quad \dfrac{du}{dt} = \dfrac{d\Phi}{dx}.$

By substituting these terms for U, we get the following expression of U :

$$U = u\Big(\frac{du}{dx}\Big) + v\Big(\frac{dv}{dx}\Big) + w\Big(\frac{dw}{dx}\Big) + \Big(\frac{d\Phi}{dx}\Big) \tag{4}$$

Similarly we get the followings :

$$\begin{cases} \frac{dv}{dz} = \frac{dw}{dy}, \quad \frac{dv}{dx} = \frac{du}{dy}, \quad \frac{dv}{dt} = \frac{d\Phi}{dy} \quad \Rightarrow \quad V = u\Big(\frac{du}{dy}\Big) + v\Big(\frac{dv}{dy}\Big) + w\Big(\frac{dw}{dy}\Big) + \Big(\frac{d\Phi}{dy}\Big), \\ \frac{dw}{dx} = \frac{du}{dz}, \quad \frac{dw}{dy} = \frac{dv}{dz}, \quad \frac{dw}{dt} = \frac{d\Phi}{dz} \quad \Rightarrow \quad W = u\Big(\frac{du}{dz}\Big) + v\Big(\frac{dv}{dz}\Big) + w\Big(\frac{dw}{dz}\Big) + \Big(\frac{d\Phi}{dz}\Big). \end{cases} \tag{5}$$

[5]($\Downarrow$) The term Φdt is the originality by Euler.

Now we postulate that the outer forces P, Q, R act such that:

$$\int (Pdx + Qdy + Rdz) = S.$$

Staying with Euler's notation, we consider pressure $= p$ and density $= q$ in the fluid element, so that then

$$\frac{2gdp}{q} = 2gdS - Udx - Vdy - Wdz, \tag{6}$$

in which we assume the time t is constant, for the hypothesis is as follow :

$$dx\Big(\frac{du}{dt}\Big) + dy\Big(\frac{dv}{dt}\Big) + dz\Big(\frac{dw}{dt}\Big) = dx\Big(\frac{d\Phi}{dx}\Big) + dy\Big(\frac{d\Phi}{dy}\Big) + dz\Big(\frac{d\Phi}{dz}\Big) = d\Phi,$$

Here, from (4) and (5), the reduction on the other elements of (6) are observed

$$Udx + Vdy + Wdz = udu + vdv + wdw + d\Phi. \tag{7}$$

When we integrate the above pressure formula (6), then we get from (7) the following :

$$2g\int \frac{dp}{q} = 2gS - \frac{1}{2}\big(u^2 + v^2 + w^2\big) - \Phi + f : t,$$

where $f : t$ is Euler's notation for $f(t)$. Here, the density q is assumed to be a function of p only; for other reasons, if this equation also satisfies positivity requirements, and q is a function depending on p,[6] then this quantity becomes

$$2g\int \frac{dp}{q} = 2gS - \frac{1}{2}\big(u^2 + v^2 + w^2\big) - \Phi.$$

Euler proposes the Euler's equations in (E226) ¶21 [7]:

> ¶21. Nous n'avons donc qu'à égaler ces forces accélératrices avec les accélérations actuelles que nous venons de trouver, et nous obtiendrons les trois équations suivantes :
>
> $$\begin{cases} P - \frac{1}{q}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ Q - \frac{1}{q}\frac{dp}{dy} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ R - \frac{1}{q}\frac{dp}{dz} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz} \end{cases}$$
>
> Si nous ajoutons à ces trois équations premierment celle, que nous a fournie la considération de la continuité du fluide :
>
> $$\frac{dq}{dt} + \frac{d.qu}{dx} + \frac{d.qv}{dy} + \frac{d.qw}{dz} = 0$$

2.5. **Lagrange's velocity potential φ.**

Citing Euler's method, Lagrange however was the first to use φ for the velocity potential, the symbol widely reserved for this in modern conventions.

> §14. nous supposerons de plus que les forces accélératrices P, Q, R du fluide soient telles, que
>
> $$Pdx + Qdy + Rdz$$
>
> soit une différentielle complète ; ce qui a lieu, en général, lorsque ces forces viennent d'une ou de plusieurs attractions propotionelles à des fonctions quelconques des distances.
>
> De cette manière, si l'on fait
>
> $$dV = Pdx + Qdy + Rdz,$$
>
> la équation proposée étant divisée par Δ se réduira à cette forme
>
> $$\Big(\frac{dp}{dt} + p\frac{dp}{dx} + q\frac{dp}{dy} + r\frac{dp}{dz}\Big)dx + \Big(\frac{dq}{dt} + p\frac{dq}{dx} + q\frac{dq}{dy} + r\frac{dq}{dz}\Big)dy + \Big(\frac{dr}{dt} + p\frac{dr}{dx} + q\frac{dr}{dy} + r\frac{dr}{dz}\Big)dz = dV - \frac{d\Pi}{\Delta}.$$
>
> Ainsi le premier membre de cette équation devra être en particulier une différentielle complète relativement à x, y, z, puisque le second en est une.
>
> Qu'on retranche de part et d'autre la différentielle de
>
> $$\frac{p^2 + q^2 + r^2}{2}$$

[6]($\Downarrow$) This is called a barotropic fluid, for which $q = f(p)$.

[7][12, p.65]

prise relativement à x, y, z, laquelle est

$$\left(p\frac{dp}{dx}+q\frac{dp}{dy}+r\frac{dp}{dz}\right)dx+\left(p\frac{dq}{dx}+q\frac{dq}{dy}+r\frac{dq}{dz}\right)dy+\left(p\frac{dr}{dx}+q\frac{dr}{dy}+r\frac{dr}{dz}\right)dz;$$

on aura, en ordonnant les termes, cette transformée

$$\begin{aligned}&\frac{dp}{dt}dx+\frac{dq}{dt}dy+\frac{dr}{dt}dz\\ +\ &\left(\frac{dp}{dy}-\frac{dq}{dx}\right)(qdx-pdy)+\left(\frac{dp}{dz}-\frac{dr}{dx}\right)(rdx-pdz)+\left(\frac{dq}{dz}-\frac{dr}{dy}\right)(rdy-qdz)=dV-\frac{d\Pi}{\Delta}-\frac{p^2+q^2+r^2}{2}.\end{aligned}$$

Donc le primier membre de cette équation devra être pareillement une différentielle exacte.

§15. li est visible que, si l'on suppose que la quantité

$$pdx+qdy+rdz$$

soit elle-même la différentielle exacte d'une fonction quelconque φ composé de x, y, z et t, on aura

$$p=\frac{d\varphi}{dx},\quad q=\frac{d\varphi}{dy},\quad r=\frac{d\varphi}{dz}.$$

Donc

$$\frac{dp}{dt}=\frac{d^2\varphi}{dtdx},\quad \frac{dq}{dt}=\frac{d^2\varphi}{dtdy},\quad \frac{dr}{dt}=\frac{d^2\varphi}{dtdz},$$

$$\frac{dp}{dy}=\frac{d^2\varphi}{dxdy},\quad \frac{dq}{dx}=\frac{d^2\varphi}{dydx},\quad ,\cdots,$$

Ainsi l'équation précédente deviendra par ces substitutions

$$\frac{d^2\varphi}{dtdx}dx+\frac{d^2\varphi}{dtdy}dy+\frac{d^2\varphi}{dtdz}dz=dV-\frac{d\Pi}{\Delta}-\frac{p^2+q^2+r^2}{2},$$

laquelle est évidement intégrable par raport à x, y, z ; de sorte qu'en intégrant, on aura

$$\frac{d\varphi}{dt}=V-\int\frac{d\Pi}{\Delta}-\frac{p^2+q^2+r^2}{2}.$$

[26, pp.710-711]

2.6. Laplace's necessary and sufficient conditions of fluid equilibrium.

Laplace stated the *exact differential* as the necessary and sufficient condition for fluid equilibrium:

> Therefore, to support the equilibrium of a homogeneous mass of fluid, for which exterior surface is free and contains within it a fixed solid nucleus, of any figure whatever, it is requisite, and it is sufficient ;
>
> - First, that $P\delta x+Q\delta y+R\delta z$ should be an *exact differential* ;
> - Second, that the resultant forces acting on the exterior surface should be perpendicular to the surface and should be directed toward the inner part of the fluid.
>
> [29, Chap. IV p.95].

2.7. Navier's equation of fluid equilibrium.

Navier deduced an expression for the forces of molecular interaction between fluid elements under a state of motion as follows: [8]

Paraphrasing from Navier's work, we consider two molecules M and M'. Let x, y, z be the rectangular coordinates of M and $x+\alpha, y+\beta, z+\gamma$ be the rectangular coordinates of M'. The length of a rayon emitted from M : $\rho=\sqrt{\alpha^2+\beta^2+\gamma^2}$. The velocity components of the molecule M are u, v, w and that of the molecule M' are

$$\delta x+\frac{d\delta x}{dx}\alpha+\frac{d\delta x}{dy}\beta+\frac{d\delta x}{dz}\gamma,\quad \delta y+\frac{d\delta y}{dx}\alpha+\frac{d\delta y}{dy}\beta+\frac{d\delta y}{dz}\gamma,\quad \delta z+\frac{d\delta z}{dx}\alpha+\frac{d\delta z}{dy}\beta+\frac{d\delta z}{dz}\gamma,$$

$$\delta\alpha=\frac{d\delta x}{dx}\alpha+\frac{d\delta x}{dy}\beta+\frac{d\delta x}{dz}\gamma,\quad \delta\beta=\frac{d\delta y}{dx}\alpha+\frac{d\delta y}{dy}\beta+\frac{d\delta y}{dz}\gamma,\quad \delta\gamma=\frac{d\delta z}{dx}\alpha+\frac{d\delta z}{dy}\beta+\frac{d\delta z}{dz}\gamma.$$

$$\delta\rho=\frac{\alpha\delta\alpha+\beta\delta\beta+\gamma\delta\gamma}{\rho}.$$

$$\delta\rho=\frac{1}{\rho}\left(\frac{d\delta x}{dx}\alpha^2+\frac{d\delta x}{dy}\alpha\beta+\frac{d\delta x}{dz}\alpha\gamma+\frac{d\delta y}{dx}\alpha\beta+\frac{d\delta y}{dy}\beta^2+\frac{d\delta y}{dz}\beta\gamma+\frac{d\delta z}{dx}\alpha\gamma+\frac{d\delta z}{dy}\beta\gamma+\frac{d\delta z}{dz}\gamma^2\right)$$

[8](⇓) Navier ([34, pp.391-398]), §II. *Équations de l'équilibre des fluides.*

where

$$\frac{d\delta x}{dy}\alpha\beta+\frac{d\delta y}{dx}\alpha\beta=0,\quad \frac{d\delta y}{dz}\beta\gamma+\frac{d\delta z}{dy}\beta\gamma=0,\quad \frac{d\delta x}{dz}\alpha\gamma+\frac{d\delta z}{dx}\alpha\gamma=0.$$

We introduce, as does Navier, a function $f(\rho)$ depending on the distance ρ between M and M'. We denote by ψ the angle between Navier's "rayon" ρ and its projection onto the $\alpha\beta$-plane, and φ the angle which this projection forms with the α-axis, and then we can evaluate only the terms as follows:

$$\frac{8f(\rho)}{\rho}\Big(\frac{d\delta x}{dx}\alpha^2+\frac{d\delta y}{dy}\beta^2+\frac{d\delta z}{dz}\gamma^2\Big).$$

Here, we assume that the components of the rayon in the above polar coordinate system are:

$$\begin{cases}\alpha=\rho\cos\psi\cos\varphi,\\ \beta=\rho\cos\psi\sin\varphi,\\ \gamma=\rho\sin\psi,\end{cases}\tag{8}$$

and then evaluate finally the following (8)

$$8\int_0^\infty d\rho\rho^3 f(\rho)\int_0^{\frac{\pi}{2}}d\psi\int_0^{\frac{\pi}{2}}d\varphi\Big(\frac{d\delta x}{dx}\cos^3\psi\cos^2\varphi+\frac{d\delta y}{dy}\cos^3\psi\sin^2\varphi+\frac{d\delta z}{dz}\sin^2\psi\cos\psi\Big).$$

We use the following formulae:

$$\begin{cases}\int\cos^n x\,dx=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}\int\cos^{n-2}x dx,\\ \int\sin^m x\cos x\,dx=\frac{\sin^{m+1}x}{m+1},\\ \int\sin^2 x\,dx=\frac{1}{2}x-\frac{1}{4}\sin 2x,\\ \int\cos^2 x\,dx=\frac{1}{2}x+\frac{1}{4}\sin 2x.\end{cases}$$

from which we obtain:

$$\int_0^{\frac{\pi}{2}}d\psi\cos^3\psi=\frac{2}{3},\quad \int_0^{\frac{\pi}{2}}d\psi\sin^2\psi\cos\psi=\frac{1}{3},\quad \int_0^{\frac{\pi}{2}}d\varphi\cos^2\varphi=\int_0^{\frac{\pi}{2}}d\varphi\sin^2\varphi=\frac{\pi}{4},$$

Equation (8) simplifies to:

$$8\frac{2}{3}\frac{\pi}{4}\int_0^\infty d\rho\rho^3 f(\rho)\Big(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}+\frac{d\delta z}{dz}\Big).$$

Here for brevity we write

$$\frac{4\pi}{3}\int_0^\infty d\rho\rho^3 f(\rho)\equiv p$$

where p [9] depends not on the distance ρ but only on the coordinates of x,y,z which determine the position of the molecule M. Hence we have

$$p\Big(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}+\frac{d\delta z}{dz}\Big).$$

The equation describing the equilibrium condition of the system is:

$$0=\iiint dxdydz\Big[p\Big(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}+\frac{d\delta z}{dz}\Big)+P\delta x+Q\delta y+R\delta z\Big].$$

By partial integration we obtain

$$\begin{aligned}0\ =\ &\iiint dxdydz\Big[\Big(P-\frac{dp}{dx}\Big)\delta x+\Big(Q-\frac{dp}{dy}\Big)\delta y+\Big(R-\frac{dp}{dz}\Big)\delta z\Big]\\ -\ &\iint dydz\Big(p'\delta x'-p''\delta x''\Big)-\iint dxdz\Big(p'\delta y'-p''\delta y''\Big)-\iint dxdy\Big(p'\delta z'-p''\delta z''\Big).\end{aligned}$$

[9](⇓) In Part 2 of our following papers, we would like to introduce a universal method for the two-constant theory including Navier's p, showed in Table 2, 3 and 4.

2.7.1. **Two conditions deduced from the indeterminate equations.**
Navier reduced the indeterminate equations for fluid equilibrium into two parts.

- Exact differential for the equilibrium conditions of an arbitrary interior point of the fluid,

$$\frac{dp}{dx} = P, \quad \frac{dp}{dy} = Q, \quad \frac{dp}{dz} = R,$$

$$dp = Pdx + Qdy + Rdz$$

$$p = \int (Pdx + Qdy + Rdz) + const.$$

As a result, Navier explained exact differentials for the conditions of fluid equilibrium as follows:

> formule où la fonction sous le signe $\int$ doit être nécessairement susceptible d'une *intégration exacte*, pour que le fluide soumis à l'action des forces représentées par P, Q, R, puisse demeurer en équilibre. [34, p.396].

- The boundary condition at the surface,

Citing Lagrange [27, pp.221-223,§29-30], Navier explained the mathematical method as follows: if we substitute

- $dydz \quad \rightarrow \quad ds^2 \cos l$, $\; l$: the angles by which the tangent plane makes on the surface frame with the plane yz,
- $dxdz \quad \rightarrow \quad ds^2 \cos m$, $\; m$: similarly, the angles with the plane xz,
- $dxdy \quad \rightarrow \quad ds^2 \cos n$, $\; n$: similarly, the angles with the plane xy,
- $\iint dydz, \iint dxdz, \iint dxdy \quad \rightarrow \quad \mathbf{S}ds^2$

where l, m, n are the angles the tangent plane on the surface makes with the planes yz, yz, and xy respectively. Hence, noting the conditions manifesting at the points on the surface of the fluid element, we get the indeterminate equations as follows:

$$0 = \mathbf{S}ds^2[(p' \cos l' \delta x' - p'' \cos l'' \delta x'') + (p' \cos m' \delta y' - p'' \cos m'' \delta y'') + (p' \cos n' \delta z' - p'' \cos n'' \delta z'')],$$

$$0 = \int (Pdx + Qdy + Rdz) + const.$$

Therefore, we get the differential equation:

$$0 = Pdx + Qdy + Rdz$$

and among the variations $\delta x, \delta y, \delta z$, we derive the following relation:

$$0 = \delta x \cos l + \delta y \cos m + \delta z \cos n.$$

Navier cited Laplace for the molecule idea and chose consistently a repulsive force in his own papers [33, 34] for the function depending on the distance between molecules:

> Les lois de l'équilibre des fluides, énoncées ci-dessus, sont conformes à celles que les géomètres ont établies d'après le principe de l'équilibre des canaux, ou en supposant le fluide décomposé en élements rectangulaires infiniment petits, et exprimant que chacun de ces éléments, soumis à l'action des pressions exercées sur ses faces, et des forces accélératrices appliquées aux molécules, doit être en équilibre. La considération des forces répulsives que la pression développe entre les molécules, dont M.Laplace avait déja déduit les équations générales du mouvement des fluides, dans le XII^e livre de la Mécanique céleste, paraît dépendre plus immédiatement des notions physiques que l'on peut se former sur la nature de ces corps. [34, p.398]

However, N.Bowditch[10] pointed out that Laplace had rethought the repulsion theory and changed it, in 1819: $\varphi(f) = A(f) - R(f)$, where $\varphi(f)$: a function depending on the distance f between the molecules, $A(f)$: attractive force, $R(f)$: repulsive force.

---

[10]($\Downarrow$) N.Bowditch[29, p.685] commented as follows:

> This theory of capillary attraction was first published by La Place in 1806 ; and in 1807 he gave a supplement. In neither of these works is the repulsive force of the heat of fluid taken into consideration, because he supposed it to be unnecessary. But in 1819 he observed, that this action could be taken into account, by supposing the force $\varphi(f)$ to represent the difference between the attractive force of the particles of the fluid $A(f)$, and the repulsive force of the heat $R(f)$ so that the combined action would be expressed by, $\varphi(f) = A(f) - R(f)$; $\cdots$

2.8. Poisson's deduction of the equilibrium and an exact differential.

We cited Poisson's deduction [37, pp.90-124].[11]

2.8.1. Poisson's condition for fluid equiliblium.

Poisson proposed his tensor:

$$(\S 24.12_P) \quad \begin{cases} X\rho = \frac{dP_1}{dz} + \frac{dP_2}{dy} + \frac{dP_3}{dx}, \\ Y\rho = \frac{dQ_1}{dz} + \frac{dQ_2}{dy} + \frac{dQ_3}{dx}, \\ Z\rho = \frac{dR_1}{dz} + \frac{dR_2}{dy} + \frac{dR_3}{dx}, \end{cases} \tag{9}$$

Poisson explained his function R of the molecular action shown in Tables 2, 3 and 4 as follows: [12]

> R étant une fonction convenable de r, x, y, z, insensible pour toute valeur sensible de r, nous exprimerons généralement l'action mutuelle de μ et μ' par la formule:
>
> $$(\S 43.1_P) \quad R + \frac{1}{2}\frac{dR}{dx}\Big(u+u'\Big) + \frac{1}{2}\frac{dR}{dy}\Big(v+v'\Big) + \frac{1}{2}\frac{dR}{dz}\Big(w+w'\Big).$$
>
> Nous supposerons toujours cette force dirigée suivant la droite qui joint les deux points μ et μ' ; et nous la regarderons comme positive ou comme négative, selon qu'elle tendra à les écarter ou à les rapprocher l'un de l'autre. [37, p.97]

$$(\S 46.4_P) \quad p \equiv \frac{1}{6\varepsilon^3}\sum rR, \quad q \equiv -\frac{1}{4\varepsilon^3}\sum \frac{r_i^2 z' R}{r}, \tag{10}$$

$$N = p + q\Big(\frac{1}{\lambda} + \frac{1}{\lambda'}\Big),$$

$$(\S 51.10_P) \quad \rho X - \frac{dp}{dx} = 0, \quad \rho Y - \frac{dp}{dy} = 0, \quad \rho Z - \frac{dp}{dz} = 0,$$

$$(\S 51.11_P) \quad dp = \rho(X dx + Y dy + Z dz). \tag{11}$$

> Pour que l'équilibre du fluide soit possible, il faudra donc que les forces données soient telles que la formule $Xdx + Ydy + Zdz$, multipliée par la densité ρ, soit la *différentielle exacte* d'une fonction de trois variables indépendantes x, y, z. Quand cette condition sera remplie, l'équation (11) fera connaître la pression p en chaque point intérieur du fluide ; et la grandeur de l'intervalle moyen ε, qui est la seule inconnue du problème, se trouvera implicitement déterminée en founction de x, y, z, au moyen de la primière équation (10).

$$X dx + Y dy + Z dz = d\varphi.$$

$$dp = \rho d\varphi,$$

where N is the vertical force, and λ, λ' are the radii of the principal curvature. In equation (9),

$$P_3 = Q_2 = R_1 = p,$$

while the other 6 forces in the tensor are zeros.

2.9. Helmholtz's vorticity equations.

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[11]($\Downarrow$) §V. Calcul des pressions dans les Fluides en équilibre ; Équations differentielles de cet équilibre.

[12]($\Downarrow$) We would like to introduce a universal method for the two-constant theory including C_1, C_2, C_3, C_4, showed in Table 2, 3 and 4, in the following papaer of Part 2.

2.9.1. **Helmholtz's definition of irrotational motion.**
Helmholtz used Euler's equations (1_H), because he had not known at that time of Navier's equations.

$$(1_H)\quad \begin{cases} X-\frac{1}{h}\frac{dp}{dx}=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}, \\ Y-\frac{1}{h}\frac{dp}{dy}=\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}, \\ Z-\frac{1}{h}\frac{dp}{dz}=\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}, \\ \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0. \end{cases} \quad \Rightarrow \begin{cases} F-\frac{1}{h}\nabla p=\frac{d\mathbf{u}}{dt}+\mathbf{u}\cdot\nabla\mathbf{u}, \\ \nabla\cdot\mathbf{u}=0, \\ \text{where } F\equiv(X,Y,Z), \quad \mathbf{u}\equiv(u,v,w). \end{cases} \qquad (12)$$

We consider the forces X, Y and Z of the potential V :

$$(1a_H)\quad X=\frac{dV}{dx}, \quad Y=\frac{dV}{dy}, \quad Z=\frac{dV}{dz},$$

and moreover, the *Geschwindigkitespotential* φ (velocity potential), so that:

$$(1b_H)\quad u=\frac{d\varphi}{dx}, \quad v=\frac{d\varphi}{dy}, \quad w=\frac{d\varphi}{dz}, \qquad (13)$$

From the conservative law of (12) ($=1_H$), we also get the divergence of the null value as follows:

$$\Delta\varphi=0,$$

Helmholtz did not mention explicitly *vollständigen Differentialien* (*exact differential* or *complete differential*), however from (13) we can deduce the condition for the *exact differential* as follows:

$$(1c_H)\quad \frac{du}{dy}-\frac{dv}{dx}=0, \quad \frac{dv}{dz}-\frac{dw}{dy}=0, \quad \frac{dw}{dx}-\frac{du}{dz}=0, \quad \Rightarrow \nabla\times\mathbf{u}=\mathbf{0}$$

To study these three conditions ($1c_H$), Helmholtz, by considering an infinitely small volume of water in a time period dt, made a comprehensive investigation into the variation from the following three various motions:

(1) einer Fortfüfrung des Wassertheilchens durch den Raum hin,
($\Rightarrow$ a carrying away of the small particle of water through the volume,)
(2) einer Ausdehnung oder Zusammenziehung des Teilchen nach drei Hauptdilationsrichtungen, wobei ein jedes aus Wasser gebildete rechtwinklige Parallelepipedon, dessen Seiten den Hauptdilationsrichtungen parallel sind, rechtwinkelig bleibt, während seine Seiten zwar ihre Länge ändern, aber ihren früheren Richtungen parallel bleiben,
($\Rightarrow$ a stretching or contraction of the particle in the three main axis directions, where, each from water of the rectangle parallelepiped, whose sides are parallel to the direction of main axis, while the length of their side is changed, however the side remains in the parallel direction,)
(3) einer Drehung um eine beliebig gerichtete temporäre Rotationsaxe, welche Drehung nach einem bekannten Satze immer alsResultante dreier Drehungen um die Coordinataxen angesehen werden kann,
($\Rightarrow$ when a rotary motion around an arbitrary oriented, temporary axis of rotary motion exists, whatever the rotary motion is able to be considered according to a famous theorem as a resultant force of three rotary motions around the coordinate axis.) [21, p.29]

$$\begin{cases} u\equiv A, & \frac{du}{dx}\equiv a, \\ v\equiv B, & \frac{dv}{dy}\equiv b, \\ w\equiv C, & \frac{dw}{dz}\equiv c, \end{cases} \quad \begin{cases} \frac{dw}{dy}=\frac{dv}{dz}\equiv\alpha, \\ \frac{du}{dz}=\frac{dw}{dx}\equiv\beta, \\ \frac{dv}{dx}=\frac{du}{dy}\equiv\gamma \end{cases} \quad \cdots \text{ exact differential conditions}$$

When we now consider a molecule with coordinates x, y and z, is at an infinitely small distance from coordinate point $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$, then

$$\begin{cases} u=A+a(x-\tilde{x})+\gamma(y-\tilde{y})+\beta(z-\tilde{z}), \\ v=B+\gamma(x-\tilde{x})+b(\tilde{y}-y)+\alpha(z-\tilde{z}), \\ w=C+\beta(x-\tilde{x})+\alpha(y-\tilde{y})+c(z-\tilde{z}), \end{cases} \Rightarrow \begin{bmatrix} u \\ v \\ w \end{bmatrix}=\begin{bmatrix} A \\ B \\ C \end{bmatrix}+\begin{bmatrix} a & \gamma & \beta \\ \gamma & -b & \alpha \\ \beta & \alpha & c \end{bmatrix}\begin{bmatrix} x-\tilde{x} \\ y-\tilde{y} \\ z-\tilde{z} \end{bmatrix}. \qquad (14)$$

When we put

$$\begin{aligned}\varphi = & \; A(x-\tilde{x}) + B(y-\tilde{y}) + C(z-\tilde{z}) \\ + & \; \frac{1}{2}a(x-\tilde{x})^2 + \frac{1}{2}b(\tilde{y}-y)^2 + \frac{1}{2}c(z-\tilde{z})^2 \\ + & \; \alpha(y-\tilde{y})(z-\tilde{z}) + \beta(x-\tilde{x})(z-\tilde{z}) + \gamma(x-\tilde{x})(y-\tilde{y}),\end{aligned}$$

then

$$u = \frac{d\varphi}{dx}, \quad v = \frac{d\varphi}{dy}, \quad w = \frac{d\varphi}{dz}.$$

Moreover, if we choose suitable coordinates x_1, y_1 and z_1 centered on point $(\tilde{x}, \tilde{y}, \tilde{z})$:

$$\begin{aligned}\varphi = & \; A_1x_1 + B_1y_1 + C_1z_1 + \frac{1}{2}a_1{x_1}^2 + \frac{1}{2}b_1{y_1}^2 + \frac{1}{2}c_1{z_1}^2 \\ = & \; \underbrace{(A_1 + \frac{1}{2}a_1x_1)}_{u_1} x_1 + \underbrace{(B_1 + \frac{1}{2}b_1y_1)}_{v_1} y_1 + \underbrace{(C_1 + \frac{1}{2}c_1z_1)}_{w_1} z_1\end{aligned}$$

The velocity components u_1, v_1, and w_1, resolved in this new coordinate system are:

$$u_1 = A_1 + a_1x_1, \quad v_1 = B_1 + b_1y_1, \quad w_1 = C_1 + c_1z_1.$$

Here, Helmholtz observed two points of view as follows:

- The velocity u_1, parallel with the x_1-axis, is the same for all water particles of the same x_1 value, or the water particles that lie in the y_1z_1-plane at the begining of the time period dt , are also in the same plane at the end of the time period dt. This also holds for the x_1y_1- and x_1z_1-planes.
- If we therefore consider a parallelepiped bounded by the three parallel planes and their infinitesimal neighboring planes, therein are the enclosed water particles formed, even after the passage of time period dt, from the same parallel coordinate planes by the surfaces of a right-angled parallelepiped.

Given the above, Helmholtz finally summarized as follows: Of all motions satisfying condition (1_{c_H}), there can only be

- translations, and
- extensions or contractions along an edge,

and does not have any "Drehung" (rotary motion / rotation).

2.9.2. **Helmholtz's deduction of rotary motion in vorticity equations. - Helmholtz's decomposition.**

Next, Helmholtz assumed conditions for rotational motion as follows:

- We consider the rotational motion of an infinitely small mass of water located at the point $(\tilde{x}, \tilde{y}, \tilde{z})$.
- The rotary motion is around the axis on a parallel with the x, y and z.
- The mass goes through the point $(\tilde{x}, \tilde{y}, \tilde{z})$, directed at angles ξ, η and ζ.

We derive the resultant velocity components parallel with coordinate axes (x, y, z) as follows:

$$\begin{bmatrix} 0 & (z-\tilde{z})\xi & -(y-\tilde{y})\xi \\ -(z-\tilde{z})\eta & 0 & (x-\tilde{x})\eta \\ (y-\tilde{y})\zeta & -(x-\tilde{x})\zeta & 0 \end{bmatrix} \Rightarrow^* \begin{bmatrix} 0 & (y-\tilde{y})\zeta & -(z-\tilde{z})\eta \\ -(x-\tilde{x})\zeta & 0 & (z-\tilde{z})\xi \\ (x-\tilde{x})\eta & -(y-\tilde{y})\xi & 0 \end{bmatrix}$$

$$\Rightarrow^* \begin{bmatrix} 0 & \zeta & -\eta \\ -\zeta & 0 & \xi \\ \eta & -\xi & 0 \end{bmatrix} \begin{bmatrix} x-\tilde{x} \\ y-\tilde{y} \\ z-\tilde{z} \end{bmatrix} \qquad (15)$$

Combining (14) with (15) we the obtain the response tensor:

$$\begin{bmatrix} a & \gamma & \beta \\ \gamma & -b & a \\ \beta & a & c \end{bmatrix} + \begin{bmatrix} 0 & \zeta & -\eta \\ -\zeta & 0 & \xi \\ \eta & -\xi & 0 \end{bmatrix} = \begin{bmatrix} a & (\gamma+\zeta) & (\beta-\eta) \\ (\gamma-\zeta) & -b & (\alpha+\xi) \\ (\beta+\eta) & (\alpha-\xi) & c \end{bmatrix}$$

$$\begin{cases} u = A + a(x-\tilde{x}) + (\gamma+\zeta)(y-\tilde{y}) + (\beta-\eta)(z-\tilde{z}), \\ v = B + (\gamma-\zeta)(x-\tilde{x}) + b(\tilde{y}-y) + (\alpha+\xi)(z-\tilde{z}), \\ w = C + (\beta+\eta)(x-\tilde{x}) + (\alpha-\xi)(y-\tilde{y}) + c(z-\tilde{z}), \end{cases}$$

$$\Rightarrow \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} + \begin{bmatrix} a & (\gamma+\zeta) & (\beta-\eta) \\ (\gamma-\zeta) & -b & (\alpha+\xi) \\ (\beta+\eta) & (\alpha-\xi) & c \end{bmatrix} \begin{bmatrix} x-\tilde{x} \\ y-\tilde{y} \\ z-\tilde{z} \end{bmatrix}$$

By differentiating u, v and w with respect to x, y and z respectively, then the following vorticity equations result:

$$\begin{bmatrix} a & (\gamma+\zeta) & (\beta-\eta) \\ (\gamma-\zeta) & -b & (\alpha+\xi) \\ (\beta+\eta) & (\alpha-\xi) & c \end{bmatrix} \Rightarrow \quad (2_H) \quad \begin{cases} \frac{dv}{dz} - \frac{dw}{dy} = 2\xi, \\ \frac{dw}{dx} - \frac{du}{dz} = 2\eta, \\ \frac{du}{dy} - \frac{dv}{dx} = 2\zeta. \end{cases} \Rightarrow \quad \frac{1}{2}(\nabla \times \mathbf{u}) = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \equiv \mathbf{W} \qquad (16)$$

2.10. Thomson's circulation theorem and the criterion of the irrotational motion on the complete differential.

Thomson defined the Helmholtz-like *velocity potential* as follows:

> §31. Let now the "velocity potential" (as we shall call it, in conformity with a German usage which has been adopted by Helmholtz), be denoted by ϕ ; that is (§16) , let ϕ be such a function of (x, y, x, t) that
>
> $$(3_T) \quad u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz},$$
>
> and let $\dot{\phi}$ (or $\frac{d\phi}{dt}$) denote its rate of variation per unit of time at any instant t, for the point (x, y, z) regarded as fixed.
> Also, let q denote the resultant fluid velocity, so that
>
> $$(4_T) \quad q^2 = u^2 + v^2 + w^2 = \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2.$$
>
> The ordinary hydrodynamical formula gives
>
> $$(5_T) \quad p = \Pi - \dot{\phi} - \frac{1}{2}q^2,$$
>
> where Π denotes the constant pressure in all sensibly quiescent parts of the fluid. [47, p.26]

Thomson's propositions, now called collectively Thomson's circulation theorems, are as follows:

Prop 2.1. *The line-integral of the tangential component velocity round any closed curve of a moving fluid remains constant through all time.* [47, p.50]

Prop 2.2. *The rate of augmentation, per unit of time, of the space integral of the velocity along any terminated arc of the fluid is equal to the excess of the value of $\frac{1}{2}q^2 - p$, at the end towards which tangential velocity is reckoned as positive, above its value at the other end.* [47, p.50]

He explained the condition "complete differential" as a criterion for irrotational motion[13] as follows:

> §59(e). The condition that $udx + vdy + wdz$ is a complete differential [proved above (§13) to be the criterion of irrotational motion] means simply
>
> - *That the flow* [defined §60 (a)] *is the same in all different mutually reconcilable lines from one to another of any two points in the fluid* ; or which is the same thing,
> - *That the circulation* [§60 (a)] *is zero round every closed curve capable of being contracted to a point without passing out of a portion of the fluid through which the criterion holds.* [47, p.50]

His definitions are as follows:

> §60. *Definitions and elementary propositions.*
>
> - (a) The line-integral of the tangential component velocity along any finite line, straight or curved, in a moving fluid, is called the *flow* in that line. If the line is endless (that is, if it forms a closed curve or polygon), the *flow* is called *circulation*. [47, p.51]

---

[13]($\Downarrow$) Irrotational motion means laminar flow, having no rotary motion.

2.11. Disputes over Helmholtz's paper.

2.11.1. Bertrand's criticism of Helmholtz's definition of *rotary motion*.

In various articles Bertrand [1, 2, 3, 4] and Saint-Venant [42] discussed Helmholtz's theorem. In particular, Bertrand had always criticized Helmholtz on this point. As the *decisive* example of the motion along only the z-axis Bertrand stated: $\xi = 0,\ \eta = 0$ and $\zeta = \frac{1}{2}$.

> La possibilité de cette décomposition n'est nullement justifiée. M.Helmholtz l'adopte comme évidente en s'assurant sur le nombre de constants introduites dans l'expression générale du deplacement des points infiniment voisins. Acceptons ces assertions, et suivons-en les conséquences: en désignant par $\xi,\ \eta,\ \zeta$ les composantes de la rotation, il en calcule l'expression par les formules (2) de la page 31 de son Mémoire,[14] et $\cdots$
>
> Supposons, par exemple, en adoptant la notation de M. Helmholtz, $\cdots$ Les formules de M.Helmholtz nous donnent cependant, dans ce cas, $\xi = 0,\ \eta = 0$ and $\zeta = \frac{1}{2}$, et feraient croire que chaque molécule tourne uniformément autour d'un parallèle à l'axe des z.
>
> Un tel exemple n'est-il pas décisif ? [2, p.268].

2.11.2. Helmholtz's responses to Bertrand.

Helmholtz responsed to Bertrand as follows:

> Par la méthode de décomposition choisie par moi, j'ai aussi fixé, comme on voit, le sens dans lequel il faut prendre le terme *rotation* dans mon Mémoire.
>
> Nommons u, v, w les composantes de la vitesse parallèles aux axes des coordonnées x, y, z. Alors le résultat de mon analyse préliminaire, qui semble être l'object de la critique de M.Bertrand, est celui-ci.

Si l'expression $(udx + vdy + wdz)$ est une différentielle exacte, il n'y a pas de rotation dans la partie du fluid correspondant. Si cette expression n'est pas une différentielle exacte, il y a rotary motion.

> M.Bertrand, au contraire, a démontré que, dans un nombre trés-considérable de cas, on peut construire des parallélipipèdes obliques ayant une direction déterminée pour leur arêtes, qui se transforment en d'autres parallélipipèdes dont les arêtes restent parallèles à celles des premiers ; et l'illustre géomètre suppose que j'ai omis ce cas dans mon analyse, parce que je n'ai parlé que des parallélipipèdes rectangles. [22, pp.136-137]

3. Gauss' note on the general principle of both static state and motion

In 1827, Gauss [17] porposed "*ein neues allgemeines Grundgesetz der Mechanik*" (a new general principle of machanics) referring the equation on minimum action (1) by Maupertuis [31], to which Bertrand refers in his note edited in the Lagrange's works [27, Vol.**12**, pp.365-368, Note 8]. [15] Gauss asserted that we can't distinguish the static state from the moving state according to the principle of d'Alembert[16], and proposed his general principle. We cite here the introduction and the translation of top paragraph from German to French by Bertrand as follows :

> M. Gauss a fait connaître, dans le Tome IV du *Journal de M.Crelle*, un beau théorème qui comprend à las fois les lois générales de l'équilibre et du mouvement, et samble l'expression la plus générale et la plus élégante qu'on soit parvenu à leur donner ; les lecteurs français nous sauront gré de reproduire ici la traduction des quelques pages consacrées par illustre géomètre à l'exposition du nouveau principe.
>
> ≪ Le principe des vitesses virtuelles transforme, comm on sait, tout problème de Statique en une question de Mathématiques pures, et, par le principe d'Alembert, la Dynamique est, à son tour, ramenée à la Statique. Il résulte de là qu'aucun principe fondamental de l'equilibre et du mouvement ne peut être essentielment distinct de ceux que nous venons de citer, et que l'on pourra toujours, quel qu'il soit, le regarder comme leur conséquence plus ou moins immédiate. [27, p.365]

Gauss proposed his principle as follows :

---

[14]($\Downarrow$) (2_H) $(= (16))$.

[15]($\Downarrow$) Lagrange had already passed away in 1813. This note was written not by Lagrange but by Bertrand.

[16]($\Downarrow$) In 1758, from the Newton's kinetic equation (the second law of motion) : $\mathbf{F} = m\mathbf{r}$, d'Alembert proposed $\mathbf{F} - m\mathbf{r} = 0$, where, $\mathbf{F}$: the force, m : the gravity, $\mathbf{r}$: the acceleration. According to his assertion, the problem of kinetic dynamics turns into that of the static dynamics.

Das neue Princip ist nun folgendes. Die Bewegung eines Systems materieller, auf was immer für eine Art unter sich verknüpfter Punkte, deren Bewegungen zugleich an was immer für äusßere Beschränkungen gebunden sind, geschieht in jedem Augeblick in möglich grüßter Übereinstimmung mit der freien Bewegung, order unter möglich kleinsten Zwange, in dem man als Maße des Zwanges, den das ganze System in jedem zeittheilchen erleidet, die Summe der Produkte aus dem Quadrate der Ablenkung jedes Punkts von seiner freien Bewegung in seiner Maße betracht. [17, p.233]

Bertrand translated Gauss' note from German to French as follows :

≪ Le nouveau principe est suivant :
Le mouvement d'un systéme de points materiels liés entre eux d'une manière quelconque et soumis à des influences quelconques se fait, à chaque instant, dans le plus parfait accord possible avec le mouvement qu'ils auraient s'ils devenaient tous libres, c'est-à-dire avec la plus petite contrainte possible, en prennant pour mesure de la contrainte subie pendant un instant infiniment petit la somme des produits de la masse de chaque point par le carré de la quantité dont il sécarte de la position qu'il aurait pris s'il eût été libre. [27, p.366]

The sum of the product of mass at the every point by the square of the distance between two molecules. We assume that $m,\ m',\ m''$: mass of point, $a,\ a',\ a''$: position, $b,\ b',\ b''$: position after infinitesimal small interval of dt, by the force which the points are activated and the initial velocity of the time interval. Hence, we assume that it allows $b,\ b',\ b''$ to conbine with $c,\ c',\ c''$. Then

$$m(bc)^2 + m'(b'c')^2 + m''(b''c'')^2 \cdots$$

will be minimum. The equilibrium is a particular case of the general law.

$$m(ab)^2 + m'(a'b')^2 + m''(a''b'')^2 \cdots$$

is a minimum, namely, the conservation of system at rest, which lies nearer to free motion with each point than the possible deplacements which we consider. Here may be the important part for Gauss to assert eagarly, so we cite Gauss' original to which Bertrand refers as follows :

selbst ein Minimum sei, oder daß das Beharren des Systems im Zustande der Ruhe, der freien Bewegung der einzelnen Punkt näher liege, als jedes mögliche Heraustreten aus demselben. [17, p.234]

≪ sera un minimum, ou, en d'autre termes, lorsque la conservation du systèm dans l'état de repos sera *plus près* du mouvement libre que chacun tend à prendre que tout déplacement possible qu'on imaginerait. [27, p.367]

We assume that the force which operate on the point m in the time interval dt is clearly composed :

(1) a force, which recieves in addition to the effect of the velocity, moves the point a at c,
(2) a force, which operates on the point at rest at c, moves instantly from c to b.

These assumptions are applied to another point in the same manner.

Gauss proved his assertion as follow : we assume that $\gamma\ \gamma',\ \gamma'',\cdots$ are the positions which $m,\ m',\ m''$ can take without any obstacles to combine with, and $\theta,\ \theta',\ \theta''$ are the angles which $c\gamma,\ c'\gamma',\ c''\gamma''$ makes with $cb,\ c'b',\ c''b''$.

$$\gamma b^2 = cb^2 + c\gamma^2 - 2cb.c\gamma\cos\theta$$

$$\sum m\gamma b^2 - \sum mcb^2 = \sum mc\gamma^2 - 2\sum mcb.c\gamma\cos\theta \geq 0,$$

then $\sum m\gamma b^2 \geq \sum mcb^2 \Rightarrow \sum m\gamma b^2$ must be the maximum, or $\sum mcb^2$ must be the minimum. □

Gauss concludes as follows :

Es ist sehr merkwürdig, daß die frien Bewegungen, wenn sie mit notwendigen Bedingung nicht besteben können, von der Natur gerade auf disselbe Art modificirt werden, wie der rechnende Mathematiker,[17] nach der Methode der kleinsten Quadrate, Erfahrungen ausgleicht, die sich auf unter einander durch notwendige Abhängigkeit verknüpfte Größen bezieben. Diese Analogie ließsich noch weiter verfolgen, was jedoch genwärtig nicht zu meiner Absicht gehört. [17, p.234] (do.)

Bertrand translates Gauss' conclusion as follows :

≪ Il est bien remarquable que les mouvement libres, lorsqu'ils sont imcompatibles avec la nature du système, sont précissément modifiés de la même manière que les géomètres, dans leurs calculs, modifient les résultants obtenus directement en leur appliquant la méthode des moindres carrés pour les rendre compatibles avec les conditions nécessaires qui leur sont imposées par la nature de la question.

On pourrait poursuivre cette analogie, mais cela n'entre pas dans le but que je me propose en ce moment. [27, p.368] (do.)

It is very remarkable to be able to explain the free movement, which was incompatible with the static state, by the same method as the mathematicians had already calculated the problem, we can do it in applying the least square method to show to be compatible with the necessary conditions imposed on the characteristic of the question.

4. Proofs of the eternal continuity in time and space of an exact differential

4.1. Lagrange's first proof.

Historically, Lagrange proved, for the first time, the exernity of time for *exact differentials* in 1781 and in the process used φ to denote the velocity potential.

$$\begin{cases} p = p' + p''t + p'''t^2 + \cdots, \\ q = q' + q''t + q'''t^2 + \cdots, \\ r = r' + r''t + r'''t^2 + \cdots, \end{cases} \qquad \begin{cases} \alpha = \alpha' + \alpha''t + \alpha'''t^2 + \cdots, \\ \beta = \beta' + \beta''t + \beta'''t^2 + \cdots, \\ \gamma = \gamma' + \gamma''t + \gamma'''t^2 + \cdots, \end{cases}$$

where

$$\begin{cases} \frac{dp}{dy} - \frac{dq}{dx} \equiv \alpha, \\ \frac{dp}{dz} - \frac{dr}{dz} \equiv \beta, \\ \frac{dq}{dz} - \frac{dr}{dy} \equiv \gamma, \end{cases} \qquad \begin{cases} \frac{dp'}{dy} - \frac{dq'}{dx} \equiv \alpha', \\ \frac{dp'}{dz} - \frac{dr'}{dz} \equiv \beta', \\ \frac{dq'}{dz} - \frac{dr'}{dy} \equiv \gamma', \end{cases} \qquad \begin{cases} \frac{dp''}{dy} - \frac{dq''}{dx} \equiv \alpha'', \\ \frac{dp''}{dz} - \frac{dr''}{dz} \equiv \beta'', \\ \frac{dq''}{dz} - \frac{dr''}{dy} \equiv \gamma'', \end{cases} \qquad \cdots$$

$$\frac{dp}{dt}dx + \frac{dq}{dt}dy + \frac{dr}{dt}dz + \alpha(qdx - pdy) + \beta(rdx - pdz) + \gamma(rdy - qdz).$$

[17](⇓) Maupertuis et al. Gauss says above :

Der große Geometer, der das Gebäude der Mechanik auf dem Grunde des Princips der virtuellen Geschwindigkeiten, auf eine so glänzende Art aufgefürt hat, hat es nicht verschmäht, Maupertuis Princip der kleinsten Wirkung zu größerer Bestimmtheit und Allgemeinheit zu erheben, ein Princip, dessen man sich zuweilen mit vielem Vortheil bedinnen kann. [17, p.232]

Bertrand translates above as follow :

≪ Le grand géomètre qui a si brillamment fait reposer la science du mouvement sur la principe des vitesses virtuelles n'a pas dédaigné de perfectionner et de généraliser le principe de Maupertuis relatif à la *moindre action*, et l'on sait que ce principe est employé souvant par les géomètres d'une manière très avantageuse. [27, p.365]

Here, we can summarize this paragraph by Gauss as follows : if the great mathematicians had regard the science of movement on the principle virtual velocity, without paying no attention to perfect or to generalize the principle on the minimum action by Maupertuis, then we observe that this princeple is used often by the mathematicians with a very useful manner, which is Gauss' selling point mentioning in this note.

Substituting the time-series expansions and rearranging with respect to powers of t, the differential becomes:

$$
\begin{aligned}
& \Big[\quad (p''dx + q''dy + r''dz) \\
+ & \qquad \alpha'(q'dx - p'dy) + \beta'(r'dx - p'dz) + \gamma'(r'dy - q'dz) \Big] \\
+ & \; t \Big[2(p'''dx + q'''dy + r'''dz) \\
+ & \qquad \alpha'(q''dx - p''dy) + \beta'(r''dx - p''dz) + \gamma'(r''dy - q''dz) \\
+ & \qquad \alpha''(q'dx - p'dy) + \beta''(r'dx - p'dz) + \gamma''(r'dy - q'dz) \Big] \\
+ & \; t^2 \Big[3(p^{(4)}dx + q^{(4)}dy + r^{(4)}dz) \\
+ & \qquad \alpha'(q'''dx - p'''dy) + \beta'(r'''dx - p'''dz) + \gamma'(r'''dy - q'''dz) \\
+ & \qquad \alpha''(q''dx - p''dy) + \beta''(r''dx - p''dz) + \gamma''(r''dy - q''dz) \\
+ & \qquad \alpha'''(q'dx - p'dy) + \beta'''(r'dx - p'dz) + \gamma'''(r'dy - q'dz) \Big] \\
+ & \qquad \cdots
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
= & \; \Big\{ (p''dx + q''dy + r''dz) + \; 2t(p'''dx + q'''dy + r'''dz) + \; 3t^2(p^{(4)}dx + q^{(4)}dy + r^{(4)}dz) + \cdots \Big\}, \\
+ & \quad (\alpha' + \alpha''t + \alpha'''t^2 + \cdots)\Big\{ (q'dx - p'dy) + (q''dx - p''dy)t + (q'''dx - p'''dy)t^2 + \cdots \Big\} \\
+ & \quad (\beta' + \beta''t + \beta'''t^2 + \cdots)\Big\{ (r'dx - p'dz) + (r''dx - p''dz)t + (r'''dx - p'''dz)t^2 + \cdots \Big\} \\
+ & \quad (\gamma' + \gamma''t + \gamma'''t^2 + \cdots)\Big\{ (r'dy - q'dz) + (r''dx - q''dz)t + (r'''dx - q'''dz)t^2 + \cdots \Big\}
\end{aligned}
\tag{18}
$$

For this expression to become an exact differential that is independent of t, the coefficient of t must become an exact differential. If we suppose that $p'dx + q'dy + r'dz$ is an exact differential, then $\alpha' = \beta' = \gamma' = 0$. Hence,

- the first term in (17), $p''dx + q''dy + r''dz$, must be the exact differential. If we suppose that $p''dx + q''dy + r''dz$ is the exact differential, then the conditions $\alpha'' = \beta'' = \gamma'' = 0$ are necessary.
- the coefficient of t in the second term of (17) must be an exact differential and must reduce to $2(p'''dx + q'''dy + r'''dz)$, requiring that $\alpha''' = \beta''' = \gamma''' = 0$.
- the coefficient of t^2 in the third term of (17) must be an exact differential and will reduce to $3(p^{(4)}dx + q^{(4)}dy + r^{(4)}dz)$, and thus $\alpha^{(4)} = \beta^{(4)} = \gamma^{(4)} = 0$.
- by successive iterations higher-order exact differentials are generated to any order.

Hence, if we suppose that $p'dx + q'dy + r'dz$ be an exact differential,

$$p''dx + q''dy + r''dz, \quad p'''dx + q'''dy + r'''dz, \quad p^{(4)}dx + q^{(4)}dy + r^{(4)}dz \quad \cdots,$$

must be exact differentials, when time t is assumed infinitesimally small. We cite Lagrange [26, §19, pp.716-717] as follows:

> Il s'ensuit de là que, si la quantité
>
> $$pdx + qdy + rdz$$
>
> est une *différentielle exacte* lorsque $t = 0$, elle devra l'être aussi lorsque t aura une value quelconque très-petit ; d'ou l'on peut conclure, en général, que cette quantité devra être toujours une *différentielle exacte*, quelle que soit la valeur de t. Car puisqu'elle doit l'être depuis $t = 0$ jusqu'à $t = \theta$ (θ étant une quantité quelconque donnée très-petit) , si l'on y substitue partout $\theta + t'$ à la place de t, on prouvera de même qu'elle devra être une *différentielle exacte* depuis $t' = 0$ jusqu'à $t' = \theta$ par conséquent elle le sera depuis $t = 0$ jusqu'à $t = 2\theta$; et ainsi de suite.
>
> Donc, en général, comme l'origine des t est arbitraire, et qu'on peut prendre également t positif ou négatif, il s'ensuit que si la quantité
>
> $$pdx + qdy + rdz$$
>
> est une *différentielle exacte* dans un instant quelconque, elle devra l'être pour tous les autres instants. Par conséquent, s'il y a un seul instant dans lequel elle ne soit pas une *différentielle exacte*, elle ne pourra jamais l'être pendant tout le mouvement ; car si elle l'étant dans un autre instant quelconque, elle devrait l'être aussi dans le premier. [26, §19, pp.716-717].

Lagrange's claim is as follows: initially, we suppose θ is a small value and that t is in the interval $0 \le t \le \theta$. Next, we substitute t with $\theta + t'$, and setting t' in the interval $0 \le t' \le \theta$, we then get $0 \le t \le 2\theta$. We substitute t similarly and reiterate. Finally, we find that if $pdx + qdy + rdz$ is an exact differential at $t = 0$, then this conjecture holds also for all t such that $0 \le t \le \infty$.

4.2. **Cauchy's proof.**

$$(1_C) \quad u_0\delta + \frac{\partial q_0}{\partial a} = 0, \quad v_0\delta + \frac{\partial q_0}{\partial b} = 0, \quad w_0\delta + \frac{\partial q_0}{\partial c} = 0. \tag{19}$$

From (19), we get (3_C) :

$$(3_C) \quad \frac{\partial u_0}{\partial b} = \frac{\partial v_0}{\partial a}, \quad \frac{\partial u_0}{\partial c} = \frac{\partial w_0}{\partial a}, \quad \frac{\partial v_0}{\partial c} = \frac{\partial w_0}{\partial b}. \tag{20}$$

$$\begin{cases} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{1}{S(\pm\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\frac{\partial z}{\partial c})}\left[\left(\frac{\partial u_0}{\partial b} - \frac{\partial v_0}{\partial a}\right)\frac{\partial z}{\partial c} + \left(\frac{\partial w_0}{\partial a} - \frac{\partial u_0}{\partial c}\right)\frac{\partial z}{\partial b} + \left(\frac{\partial v_0}{\partial c} - \frac{\partial w_0}{\partial b}\right)\frac{\partial z}{\partial a}\right], \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \frac{1}{S(\pm\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\frac{\partial y}{\partial c})}\left[\left(\frac{\partial u_0}{\partial b} - \frac{\partial v_0}{\partial a}\right)\frac{\partial y}{\partial c} + \left(\frac{\partial w_0}{\partial a} - \frac{\partial u_0}{\partial c}\right)\frac{\partial y}{\partial b} + \left(\frac{\partial v_0}{\partial c} - \frac{\partial w_0}{\partial b}\right)\frac{\partial y}{\partial a}\right], \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \frac{1}{S(\pm\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\frac{\partial y}{\partial c})}\left[\left(\frac{\partial u_0}{\partial b} - \frac{\partial v_0}{\partial a}\right)\frac{\partial x}{\partial c} + \left(\frac{\partial w_0}{\partial a} - \frac{\partial u_0}{\partial c}\right)\frac{\partial x}{\partial b} + \left(\frac{\partial v_0}{\partial c} - \frac{\partial w_0}{\partial b}\right)\frac{\partial x}{\partial a}\right]. \end{cases} \tag{21}$$

$$i.e. \quad \Rightarrow^* \quad \begin{bmatrix} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \end{bmatrix} = \frac{1}{S(\pm\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\frac{\partial z}{\partial c})} \begin{bmatrix} \frac{\partial z}{\partial c} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial a} \\ \frac{\partial y}{\partial c} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial a} \\ \frac{\partial x}{\partial c} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial a} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial u_0}{\partial b} - \frac{\partial v_0}{\partial a}\right) \\ \left(\frac{\partial w_0}{\partial a} - \frac{\partial u_0}{\partial c}\right) \\ \left(\frac{\partial v_0}{\partial c} - \frac{\partial w_0}{\partial b}\right) \end{bmatrix}$$

where S is the relative sign of the permutation of a, b, c. Stokes explained Cauchy's S as follows:

> S is a function of the differential coefficients of x, y and z with respect to a, b and c, which by the condition of continuity is shown to be equal to $\frac{\rho_0}{\rho}$, ρ_0 being the initial density about the particle whose density at the time considered is ρ.

$$\frac{1}{S(\pm\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\frac{\partial z}{\partial c})} = 1$$

then (21) becomes (22) as follows:

$$\begin{cases} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \left(\frac{\partial u_0}{\partial b} - \frac{\partial v_0}{\partial a}\right)\frac{\partial z}{\partial c} + \left(\frac{\partial w_0}{\partial a} - \frac{\partial u_0}{\partial c}\right)\frac{\partial z}{\partial b} + \left(\frac{\partial v_0}{\partial c} - \frac{\partial w_0}{\partial b}\right)\frac{\partial z}{\partial a}, \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \left(\frac{\partial u_0}{\partial b} - \frac{\partial v_0}{\partial a}\right)\frac{\partial y}{\partial c} + \left(\frac{\partial w_0}{\partial a} - \frac{\partial u_0}{\partial c}\right)\frac{\partial y}{\partial b} + \left(\frac{\partial v_0}{\partial c} - \frac{\partial w_0}{\partial b}\right)\frac{\partial y}{\partial a}, \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \left(\frac{\partial u_0}{\partial b} - \frac{\partial v_0}{\partial a}\right)\frac{\partial x}{\partial c} + \left(\frac{\partial w_0}{\partial a} - \frac{\partial u_0}{\partial c}\right)\frac{\partial x}{\partial b} + \left(\frac{\partial v_0}{\partial c} - \frac{\partial w_0}{\partial b}\right)\frac{\partial x}{\partial a}. \end{cases} \tag{22}$$

$$i.e. \quad \Rightarrow^* \quad \begin{bmatrix} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial c} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial a} \\ \frac{\partial y}{\partial c} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial a} \\ \frac{\partial x}{\partial c} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial a} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial u_0}{\partial b} - \frac{\partial v_0}{\partial a}\right) \\ \left(\frac{\partial w_0}{\partial a} - \frac{\partial u_0}{\partial c}\right) \\ \left(\frac{\partial v_0}{\partial c} - \frac{\partial w_0}{\partial b}\right) \end{bmatrix}$$

Stokes [43] evaluated Cauchy's proof and developed his own proof with **Lemma 4.1** as follows:

> §11 $\cdots$ Since $\frac{dx}{da}$, & are finite, (for to suppose them infinite would be equivalent to supposing a discontinuity to exist in the field,) it follows at once from the preceding equations that if $\omega_0' = 0$, $\omega_0'' = 0$, $\omega_0''' = 0$, that is if $u_0da + v_0db + w_0dc$ be the exact differential, either for the whole fluid or for any portion of it, then shall $\omega' = 0$, $\omega'' = 0$, $\omega''' = 0$, i.e. $udx + vdy + wdz$ will be the exact differential, at any subsequent time, either for the whole mass or for the above portion of it.
>
> §12 It is not from seeing the smallest flaw in M.Cauchy's proof that I propose a new one, but because it is well to view the subject in different lights, and because the proof which I am about to give does not require such long equations. $\cdots$ [43, p.108]

4.3. **Stokes' proof.**

Stokes[43] stated in his abstract of Section 2,

- Objections to Lagrange's proof of the theorem that if $udx + vdy + wdz$ is the exact differential at any one instant it is always so, the pressure being supposed equal in all directions.
- Principles of M.Cauchy's proof.
- A new proof of the theorem.

• A physical interpretation of the circumstance of the above expression being the exact differential. Stokes proposed his new proof, comprising Power's method [39] showed in (18) of Lagrange and criticizing Newton[35], Lagrange[26], Cauchy[5] and Poisson[37, pp.173-4].[18] As an aside, Stokes cited Newton's proposition XL, Theorem XIII.[35].

> Si corpus cogente vi quacunque centripeta, moveatur utcunque, & corpus aliud recta ascendat vel desendat, sintque eorum velocitates in aliquo aequlium altitudinum casu aequales, velocitates eorum in omnibus aequalibus altitudinibus erunt aequales.
> ⇒ If the body moving with an arbitrary centripetal force, or another bodies ascending straightforword or decending straightforword, it takes the equal velocities at any same altitude in everywhere.

Stokes stated:

> I confess I cannot see that Newton in his *Principia* Lib.I, Prop. 40, has proved more than that if the velocities of the two bodies are equal increments of the distances are ultimately equal: at least something additional seems required to put the proof quite out of the reach of objection.

He claimed a lemma to prove that $udx+vdy+wdz$ will always remain an *exact differential* over intervals of finite time. Stokes posed the lemma as follows:

Lemma 4.1. (Stokes) *If $\omega_1, \omega_2, \cdots, \omega_n$ are n functions of t, which satisfy the n differential equations*

$$(25_S)\quad \begin{cases} \frac{d\omega_1}{dt} = P_1\omega_1 + Q_1\omega_2 \cdots + V_1\omega_n, \\ \cdots \\ \frac{d\omega_n}{dt} = P_n\omega_1 + Q_n\omega_2 \cdots + V_n\omega_n, \end{cases}$$

where $P_1, Q_1, \cdots V_n$ may be functions of $t, \omega_1, \cdots \omega_n$, and if when $\omega_1 = 0, \omega_2 = 0, \cdots, \omega_n = 0$, none of the quantities $P_1, \cdots, V_n$ is infinite for any value of t from 0 to T, and if $\omega_1, \cdots \omega_n$ are each zero when $t = 0$, then shall each of these quantities remain zero for all values of t from 0 to T.

Proof. First step : we evaluate the behavior of $\omega_1, \omega_2, \cdots, \omega_n$ in the interval of $0 \le t \le \tau \ll 1$ such that: at the time of τ,

> it may be taken so small that the values of $\omega_1, \omega_2, \cdots, \omega_n$ are sufficiently small to exclude all the values which might render any one of the quantities $P_1,\ Q_1, \cdots, V_1, \cdots, P_n,\ Q_n, \cdots, V_n$ infinite.

Defining L such that:

$$L \equiv \max\ (P_1,\ Q_1, \cdots, V_1, \cdots, P_n,\ Q_n, \cdots, V_n),$$

then (25_S) becomes:

$$(26_S)\quad \begin{cases} \frac{d\omega_1}{dt} = L(\omega_1 + \omega_2 + \cdots + \omega_n), \\ \cdots \\ \frac{d\omega_n}{dt} = L(\omega_1 + \omega_2 + \cdots + \omega_n), \end{cases} \qquad 0 \le^{\forall} t \le \tau.$$

Setting Ω :

$$\Omega \equiv \omega_1 + \omega_2 \cdots + \omega_n,$$

we obtain

$$\frac{d\Omega}{dt} = nL\Omega, \quad \Omega = Ce^{nLt}, \quad 0 \le^{\forall} t \le \tau,$$

> ...but no value of C different from zero will allow Ω to vanish when $t = 0$.

Hence, we arrive at $C = 0$, and then

$$\omega_1 = \omega_2 = \cdots = \omega_n = 0.$$

> Since then $\omega_1, \omega_2, \cdots, \omega_n$ would have to be equal to zero for all values of t from 0 to τ even if they satisfied equation (26_S), they must à *fortiori* be equal to zero in the actual case, since they satisfy equation (25_S).

Second step : we evaluate ω, in the interval of $0 \le t \le T$.

---

[18](⇓) We introduced "Poisson's conjecture" in the introduction §1.

This lemma might be extended to the case in which $n = \infty$, with certain restrictions as to the convergence of the series. We may also, instead of the integers $1, 2, \cdots, n$ have a continuous variable α which varies from 0 to a, so that ω is a function of the independent variables α and t, satisfying the differential equation:

$$\frac{d\omega}{dt} = \int_0^a \Psi(\alpha, \omega, t)\ \omega\ d\alpha,$$

where we suppose $\Psi(\alpha, 0, t) < \infty, \quad 0 \leq^\forall \alpha \leq a, \quad 0 \leq^\forall t \leq T.$

When $t = 0$, we obtain $\omega = 0, \quad 0 \leq^\forall t \leq T$. Finally, we consider this integral equation in putting $a = \infty$

$$\frac{d\omega}{dt} = \int_0^\infty \Psi(\alpha, \omega, t)\ \omega\ d\alpha, \quad 0 \leq^\forall \alpha < \infty, \quad 0 \leq^\forall t \leq T.$$

The proposition might be further extended to cases for which $\alpha = \infty$, with the equations (25_S) are already more general than I[19] shall have occasion to employ. □

We suppose ρ to be a function of p and $\frac{1}{f'(p)}$, namely, here we suppose the barotropic fluid, then

$$(27_S) \quad \frac{df(p)}{dx} = X - \frac{Du}{Dt}, \quad \frac{df(p)}{dy} = Y - \frac{Dv}{Dt}, \quad \frac{df(p)}{dz} = Z - \frac{Dw}{Dt},$$

The force X, Y, Z will here be supposed to be such that $Xdx + Ydy + Zdz$ is an *exact differential*, this being the case for any forces emanating from centers, and varying as any functions of the distances. Differentiating the first equation (27_S) with respect to y, and the second with respect to x, subtracting, putting for Du/Dt and Dv/Dt their values, adding and subtracting, $du/dz.dv/dz$[20] and employing the notation of Art. 2, we obtain

$$(28_S) \quad \begin{cases} \frac{D\omega'}{Dt} = -\left(\frac{dv}{dw} + \frac{dy}{dz}\right)\omega' + \frac{du}{dx}\omega'' + \frac{dv}{dx}\omega''', \\ \frac{D\omega''}{Dt} = \frac{du}{dy}\omega' - \left(\frac{du}{dx} + \frac{dw}{dz}\right)\omega'' + \frac{dw}{dy}\omega''', \\ \frac{D\omega'''}{Dt} = \frac{du}{dz}\omega' + \frac{dv}{dz}\omega'' - \left(\frac{du}{dx} + \frac{dv}{dy}\right)\omega'''. \end{cases}$$

By treating the first and third, and then the second and third of equation (27_S) in the same manner, we should obtain two more equations, $\cdots$ [43, p.111]

According to Stokes' explanation, from (27_S), we get:

$$\begin{aligned} \frac{D\omega'}{Dt} &= \frac{D}{Dt}\Big\{\frac{1}{2}\Big(\frac{dw}{dy} - \frac{dv}{dz}\Big)\Big\} \\ &= -\Big(\frac{dv}{dy} + \frac{dw}{dz}\Big)\Big\{\frac{1}{2}\Big(\frac{dw}{dy} - \frac{dv}{dz}\Big)\Big\} + \frac{dv}{dx}\Big\{\frac{1}{2}\Big(\frac{du}{dz} - \frac{dw}{dx}\Big)\Big\} + \frac{dw}{dx}\Big\{\frac{1}{2}\Big(\frac{dv}{dx} - \frac{du}{dy}\Big)\Big\} \\ &= \frac{1}{2}\Big[-\Big(\frac{dv}{dy} + \frac{dw}{dz}\Big)\Big(\frac{dw}{dy} - \frac{dv}{dz}\Big) + \frac{dv}{dx}\frac{du}{dz} - \frac{dvdw}{dx^2} + \frac{dwdv}{dx^2} - \frac{du}{dx}\frac{dw}{dy}\Big] \\ &= \frac{1}{2}\Big[-\Big(\frac{dv}{dy} + \frac{dw}{dz}\Big)\Big(\frac{dw}{dy} - \frac{dv}{dz}\Big) + \frac{du}{dx}\Big\{\frac{dv}{dz} - \frac{dw}{dy}\Big\}\Big] \\ &= -\frac{1}{2}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big(\frac{dw}{dy} - \frac{dv}{dz}\Big) \\ &= -\omega' \text{ div } \mathbf{u}. \end{aligned}$$

---

[19]($\Downarrow$) Stokes.

[20]($\Downarrow$) sic.

$$
\begin{aligned}
\frac{D\omega''}{Dt} &= \frac{D}{Dt}\Big\{\frac{1}{2}\Big(\frac{du}{dz}-\frac{dw}{dx}\Big)\Big\} \\
&= \frac{du}{dy}\Big\{\frac{1}{2}\Big(\frac{dw}{dy}-\frac{dv}{dz}\Big)\Big\} - \Big(\frac{du}{dx}+\frac{dw}{dz}\Big)\Big\{\frac{1}{2}\Big(\frac{du}{dz}-\frac{dw}{dx}\Big)\Big\} + \frac{dw}{dy}\Big\{\frac{1}{2}\Big(\frac{dv}{dx}-\frac{du}{dy}\Big)\Big\} \\
&= \frac{1}{2}\Big[\frac{dudw}{dy^2} - \frac{du}{dy}\frac{dv}{dz} + \frac{dw}{dy}\frac{dv}{dx} - \frac{dwdu}{dy^2} - \Big(\frac{du}{dx}+\frac{dw}{dz}\Big)\Big(\frac{du}{dz}-\frac{dw}{dx}\Big)\Big] \\
&= \frac{1}{2}\Big[-\frac{dv}{dy}\Big(\frac{du}{dz}-\frac{dw}{dx}\Big) - \Big(\frac{du}{dx}+\frac{dw}{dz}\Big)\Big(\frac{du}{dz}-\frac{dw}{dx}\Big)\Big] \\
&= -\frac{1}{2}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big(\frac{du}{dz}-\frac{dw}{dx}\Big) \\
&= -\omega'' \operatorname{div} \mathbf{u}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{D\omega'''}{Dt} &= \frac{D}{Dt}\Big\{\frac{1}{2}\Big(\frac{dv}{dx}-\frac{du}{dy}\Big)\Big\} \\
&= \frac{du}{dz}\Big\{\frac{1}{2}\Big(\frac{dw}{dy}-\frac{dv}{dz}\Big)\Big\} + \frac{dv}{dz}\Big\{\frac{1}{2}\Big(\frac{du}{dz}-\frac{dw}{dx}\Big)\Big\} - \Big(\frac{du}{dx}+\frac{dv}{dy}\Big)\Big\{\frac{1}{2}\Big(\frac{dv}{dx}-\frac{du}{dy}\Big)\Big\} \\
&= \frac{1}{2}\Big[\frac{du}{dz}\frac{dw}{dy} - \frac{dudv}{dz^2} + \frac{dvdu}{dz^2} + \frac{dv}{dz}\frac{dw}{dx} - \Big(\frac{du}{dx}+\frac{dv}{dy}\Big)\Big(\frac{dv}{dx}-\frac{du}{dy}\Big)\Big] \\
&= \frac{1}{2}\Big[-\frac{dw}{dz}\Big(\frac{dv}{dx}-\frac{du}{dy}\Big) - \Big(\frac{du}{dx}+\frac{dv}{dy}\Big)\Big(\frac{dv}{dx}-\frac{du}{dy}\Big)\Big] \\
&= -\frac{1}{2}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big(\frac{dv}{dx}-\frac{du}{dy}\Big) \\
&= -\omega''' \operatorname{div} \mathbf{u}.
\end{aligned}
$$

We can then arrange (28_S) by the array:

$$
\begin{aligned}
(28_S) \quad \Rightarrow \quad \begin{bmatrix} \frac{D\omega'}{Dt} \\ \frac{D\omega''}{Dt} \\ \frac{D\omega'''}{Dt} \end{bmatrix} &= \begin{bmatrix} -\Big(\frac{dv}{dy}+\frac{dw}{dz}\Big) & \frac{dv}{dx} & \frac{dw}{dx} \\ \frac{du}{dy} & -\Big(\frac{uw}{dx}+\frac{dw}{dz}\Big) & \frac{dw}{dy} \\ \frac{du}{dz} & \frac{dv}{dz} & -\Big(\frac{du}{dx}+\frac{dv}{dy}\Big) \end{bmatrix} \begin{bmatrix} \omega' \\ \omega'' \\ \omega''' \end{bmatrix} \\
&= \begin{bmatrix} -\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big) & 0 & 0 \\ 0 & -\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big) & 0 \\ 0 & 0 & -\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big) \end{bmatrix} \begin{bmatrix} \omega' \\ \omega'' \\ \omega''' \end{bmatrix} \\
&\Rightarrow \quad \frac{DW}{Dt} = -W \operatorname{div} \mathbf{u}, \qquad (23)
\end{aligned}
$$

where

$$
\omega' = \frac{1}{2}\Big(\frac{dw}{dy}-\frac{dv}{dz}\Big), \quad \omega'' = \frac{1}{2}\Big(\frac{du}{dz}-\frac{dw}{dx}\Big), \quad \omega''' = \frac{1}{2}\Big(\frac{dv}{dx}-\frac{du}{dy}\Big), \quad W = (\omega', \omega'', \omega''').
$$

> Now for points in the interior of the mass the differential coefficients $\frac{du}{dz}, \cdots$ will not be infinite, on acount of the continuity of the motion, and therefore the three equations just obtained are a particular case of equations (25_S).

Stokes concluded this problem with the following:

> If then $udx + vdy + wdz$ is an *exact differential* for any portion of the fluid when $t = 0$, that is, if ω', ω'' and ω''' are each zero when $t = 0$, it follows from the lemma of the last article that ω', ω'' and ω''' will be zero for any value of t, and therefore $udx + vdy + wdz$ will always remain an *exact differential*. [43, p.111].

Thus the proof of this problem, demonstrating the eternal continuity in time and the space of an exact differential, had been solved by Stokes or Cauchy.

TABLE 2. C_1, C_2, C_3, C_4 : the constant of definitions and computing of total moment of molecular actions by Poisson, Navier, Cauchy, Saint-Venant & Stokes

no	name	elastic solid	moment of elastic fluid	equiliblium of fluid
1	Poisson	$C_1 = k \equiv \frac{2\pi}{15}\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}$ $C_2 = K \equiv \frac{2\pi}{3}\sum \frac{r^3}{\alpha^5} fr$ $C_3 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta \sin\beta d\beta \ g_3$ $\Rightarrow \{\frac{2\pi}{5}, \frac{2\pi}{15}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta \sin\beta d\beta \ g_4 \Rightarrow \frac{2\pi}{3}$ Remark: C_3 is choiced as the common factor of $\{\cdot, \cdot\}$	$C_1 = -k \equiv -\frac{1}{30\varepsilon^3}\sum r^3 \frac{d.\frac{1}{r}fr}{dr}$ $= -\frac{2\pi}{15}\sum \frac{r^3}{4\pi\varepsilon^3}\frac{d.\frac{1}{r}fr}{dr}$ $C_2 = -K \equiv -\frac{1}{6\varepsilon^3}\sum rfr$ $= -\frac{2\pi}{3}\sum \frac{r}{4\pi\varepsilon^3} fr$ $C_3 : \begin{cases} G = \frac{1}{10}\sum r^3 \frac{d.\frac{1}{r}fr}{dr}, \\ E = F = \frac{1}{30}\sum r^3 \frac{d.\frac{1}{r}fr}{dr} \end{cases}$ $\Rightarrow \{\frac{1}{10}, \frac{1}{30}\} \Rightarrow \frac{1}{30}$ $C_4 : (3{-}2)_{Pf} \quad N = \frac{1}{6\varepsilon^3}\sum rfr \Rightarrow \frac{1}{6}$	$C_1 = -q \equiv \frac{1}{4\varepsilon^3}\sum \frac{r_1^2 z' R}{r}$ $C_2 = p \equiv \frac{1}{6\varepsilon^3}\sum rR$ $N = p + q\left(\frac{1}{\lambda} + \frac{1}{\lambda'}\right)$ where N : the vertical force, λ, λ' : the radii of the principal curvature
2	Navier	$C_1 = \varepsilon \equiv \frac{2\pi}{15}\int_0^\infty d\rho \cdot \rho^4 f\rho$ $C_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} \cos\varphi d\varphi \ g_3 \Rightarrow \{\frac{16}{15}, \frac{4}{15}, \frac{2}{5}\}$ $\Rightarrow \frac{1}{2}\frac{\pi}{4}\frac{16}{15} = \frac{2\pi}{15}$	$C_1 = \varepsilon \equiv \frac{2\pi}{15}\int_0^\infty d\rho \cdot \rho^4 f(\rho)$ $C_2 = E \equiv \frac{2\pi}{3}\int_0^\infty d\rho \cdot \rho^2 F(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi \ g_3$ $\Rightarrow \{\frac{\pi}{10}, \frac{\pi}{30}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi \ g_4 \Rightarrow \frac{2\pi}{3}$	$C_1 = p \equiv \frac{4\pi}{3}\int_0^\infty d\rho\rho^3 f(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\psi \int_0^{\frac{\pi}{2}} d\varphi g_3$ $\Rightarrow \{\frac{2}{3}, \frac{1}{3}, \frac{\pi}{4}\} \Rightarrow \frac{8\pi}{6} = \frac{4\pi}{3}$
3	Cauchy	$C_1 = R = \frac{2\pi\Delta}{15}\int_0^\infty r^3 f(r)dr$ $= \pm\frac{2\pi\Delta}{15}\int_0^\infty [r^4 \mathrm{f}'(r) - r^3 \mathrm{f}(r)]dr$ $C_2 = G = \pm\frac{2\pi\Delta}{3}\int_0^\infty r^3 \mathrm{f}(r)dr$ $C_3 = \frac{1}{2}\int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2\alpha \cos^2\beta dp$ $= \frac{1}{2}\int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2 p \sin^2 p \sin p dp$ $= \frac{2\pi}{15}$, $C_4 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi \cos^2\alpha \sin p dq dp$ $= \pi\int_0^\pi \cos^2 p \sin\ p dp = \frac{2\pi}{3}$,		
4	Saint-Venant		$C_1 = \varepsilon, \quad C_2 = \frac{\varepsilon}{3}$	
5	Stokes	$C_1 = A, \quad C_2 = B$	$C_1 = \mu, \quad C_2 = \frac{\mu}{3}$	

5. Conclusions

We state our conclusions:

(1) The study of exact differentials began with a discussion of the equilibrium condition given by Maupertuis [31] in 1740 and Clairaut [6] in 1743 and developed by Euler [14] in 1769-70 in extending the now-called Euler equations. Following that, various points of view were discussed (cf. Table 1). We saw that one of the ideas had come largely from fluid mechanics, for which Navier, Cauchy, Poisson, and others, had proposed equations of equilibrium and motion of fluids. When considering the classical topics of mathematical physics as applied to fluids, exact differentials are necessary in these endeavors.

(2) Gauss [17] propose the general principle on both static and motional state, to which Lagrange [27] refered as the most general and elegent principle in the ever heard. According to Gauss' principle, we can't distinguish the static state from the motinal state, and the former is one of the latter. Gauss applied this principle to his later studies, such as the capillary action, which we discuss in Part 2 of our following papar.

(3) The proof of the conservation in time and space of an exact differential was discussed by Lagrange, Cauchy, Stokes, and others. The herein-called "Poisson conjecture" in 1831, cited in the Introduction (§1) as one of our main motivations for this study, had its beginnings with the incomplete proof by Lagrange [27]. However, thereafter, Cauchy [5] had presented a proof as early as 1815, while Power [39] and Stokes [43] had tried by other methods. To date Cauchy's proof is still considered to be the best.

(4) In another approach to exact differentials, Helmholtz [21] and Thomson [45, 46, 47] proposed vortices and related concepts, and Bertrand [1, 2, 3, 4] and Saint-Venant [41], and other, discussed the relationship or distinction between rotational motion and irrotational motion with the exact differential with Helmholtz [21] proposing a criterion for it.

TABLE 3. The expression of the total moment of molecular actions by Poisson, Navier, Cauchy, Saint-Venant & Stokes

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Poisson [36]	elastic solid	k		$\frac{2\pi}{15}$		$\sum\frac{1}{\alpha^5}$	r^5		$\frac{d.\frac{1}{r}fr}{dr}$		
				K		$\frac{2\pi}{3}$	$\sum\frac{1}{\alpha^5}$		r^3		fr	
2	Poisson [37]	motion of fluid	k		$\frac{1}{30}$		$\sum\frac{1}{\varepsilon^3}$	r^3		$\frac{d.\frac{1}{r}fr}{dr}$		$C_3=\frac{1}{4\pi}\frac{2\pi}{15}=\frac{1}{30}$
				K		$\frac{1}{6}$	$\sum\frac{1}{\varepsilon^3}$		r		fr	$C_4=\frac{1}{4\pi}\frac{2\pi}{3}=\frac{1}{6}$
3	Poisson [37]	equiliblium of fluid	q		$\frac{1}{4}$		$\sum\frac{1}{\varepsilon^3}$	$\frac{1}{r}$		$r_i^2z'R$		$r_i=\sqrt{x'^2+y'^2}$
				p		$\frac{1}{6}$	$\sum\frac{1}{\varepsilon^3}$		r		R	
4	Navier [33]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$		ρ : radius
5	Navier fluid [34]	motion of fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$		ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$	
6	Navier fluid [34]	equiliblium of fluid	p		$\frac{4\pi}{3}$		$\int_0^\infty d\rho$	ρ^3		$f(\rho)$		ρ : radius
7	Saint-Venant [41]	fluid	ε	$\frac{\varepsilon}{3}$								
8	Stokes [43]	fluid	μ	$\frac{\mu}{3}$								
9	Stokes [43]	elastic solid	A	B								

TABLE 4. C_1, C_2 and equation of equilibrium of fluid containing exact differential by Poisson & Navier

no	name	C_1, C_2 of equilibrium	equation of equilibrium with exact differential term
1	Poisson [37]	$C_1=-q\equiv\frac{1}{4\varepsilon^3}\sum\frac{r_i^2z'R}{r}$ $C_2=p\equiv\frac{1}{6\varepsilon^3}\sum rR$	$N=p+q\left(\frac{1}{\lambda}+\frac{1}{\lambda'}\right)$ where N : the vertical force, λ,λ' : the radii of the principal curvature
2	Navier fluid [34]	$C_1=p\equiv\frac{4\pi}{3}\int_0^\infty d\rho\rho^3f(\rho)$ $C_3=\int_0^{\frac{\pi}{2}}d\psi\int_0^{\frac{\pi}{2}}d\varphi g_3$ $\Rightarrow\{\frac{2}{3},\frac{1}{3},\frac{\pi}{4}\}\Rightarrow\frac{8\pi}{6}=\frac{4\pi}{3}$	$0=\iiint dxdydz\left[p\left(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}+\frac{d\delta z}{dz}\right)+P\delta x+Q\delta y+R\delta z\right].$ By integration by parts $0=\iiint dxdydz\left[\left(P-\frac{dp}{dx}\right)\delta x+\left(Q-\frac{dp}{dy}\right)\delta y+\left(R-\frac{dp}{dz}\right)\delta z\right]$ $-\iint dydz\left(p'\delta x'-p''\delta x''\right)-\iint dxdz\left(p'\delta y'-p''\delta y''\right)-\iint dxdy\left(p'\delta z'-p''\delta z''\right),$ $\Rightarrow$. condition of inner point and exact differential $\frac{dp}{dx}=P,\quad\frac{dp}{dy}=Q,\quad\frac{dp}{dz}=R.\quad\Rightarrow\quad dp=Pdx+Qdy+Rdz$ $\Rightarrow$. boundary condition and relation of variation $\delta x,\delta y,\delta z$ $0=Pdx+Qdy+Rdz\quad\Rightarrow\quad 0=\delta x\cos l+\delta y\cos m+\delta z\cos n$

These have had a major effect on the development of the equations of fluid mechanics, including the Navier-Stokes equations.

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The "two-constant" theory and tensor function underlying the Navier-Stokes equations

ABSTRACT. The "two-constant" theory introduced first by Laplace in 1805 still forms the basis of current theory describing isotropic, linear elasticity. The Navier-Stokes equations in incompressible case :

$$\partial_t \mathbf{u} - \mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0.$$

as presented in final form by Stokes in 1845, were derived in the course of the development of the "two-constant" theory.

Following in historical order the various contributions of Navier, Cauchy, Poisson, Saint-Venant and Stokes over the intervening period, we trace the evolution of the equations, and note concordances and differences between each contributor. In particular, from the historical perspective of these equations we look for evidence for the notion of tensor.

Also in the formulation of equilibrium equations, we obtain the competing theories of the "two-constant" theory in capillary action of Laplace and Gauss.

After Stokes' linear equations, the equations of gas theories were deduced by Maxwell in 1865, Kirchhoff in 1868 and Boltzmann in 1872. They contributed to formulate the fluid equations and to fix the *NS* equations, when Prandtl stated the today's formulation in using the nomenclature as the "so-called *NS* equations" in 1934, in which Prandtl included the three terms of nonlinear and two linear terms with the ratio of two coefficients as 3 : 1, which arose Poisson in 1831, Saint-Venant in 1843, and Stokes in 1845. Prandtl says, "The following differential equation, known as the equation of Navier-Stokes, is the fundamental equation of hydrodynamics,"

$$\frac{D\mathbf{w}}{dt} = \mathbf{g} - \frac{1}{\rho}\text{grad } p + \frac{1}{3}\nu \text{ grad div } \Delta\mathbf{w} + \nu\Delta\mathbf{w},$$

where, $$\frac{D\mathbf{w}}{dt} \equiv \frac{\partial\mathbf{w}}{\partial t} + \mathbf{w}\cdot\nabla\mathbf{w}, \; \nu = \frac{\mu}{\rho}, \; \mathbf{w} = (u, v, w), \; \mathbf{g} = (X, Y, Z)$$

In the appendices, we show the process of formulation citing their main papers of Navier, Cauchy, Poisson, Laplace and Gauss with our commentary.
In addition to, from the viewpoint of mathematics, several important topics such as integral theory in §E.17 and §E.23 which is Gauss' selling point. We show his unique *RDF* and reduction of integral from sextuple to quadruple, in the sections §E.2, §E.16 and §E.17. In and after §E.18, we show his calculus of variations in the capillarity against the *RDF* and calculation of the capillarity by Laplace.

Finally, for the question to be solved by variational equation introduced in §E.18 and §E.19, we sketch his method deduced from the previous work of theory in curved surface [15], to the capillary problems including the height of fluid and the tangent angle made between the fluid surface and the wall in §E.28 and §E.29.

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Key words. The Navier-Stokes equations, two-constant theory, tensor function, fluid dynamics, fluid mechanics, hydrostatics, hydrodynamics, hydromechanics, microscopically-descriptive equations, rapidly decreasing function, mathematical history.

Contents

1. Introduction

[1] In the early 19th century, many investigators contributed to the development of the Navier-Stokes (NS) equations, the basis for the description of viscous incompressible fluid flow. From their inception with work of Laplace, the main contributors were Navier, Cauchy, Poisson, Saint-Venant and Stokes. [2] We study the original contributions of each of these investigators and the form of the NS equations as formulated by their authors, and endeavor to ascertain their aims and conceptual thoughts in developing the then new equations. Historical order is followed as determined by date of proposal or publication.

In 1805, Laplace introduced the "two-constant theory", so-called because of the prominence of two constants in his theory, in regard to capillary action with constants denoted by H and K. [3] (cf. Table 2, 3). Thereafter, contributing investigators in formulating NS equations, i.e. equations describing equilibrium or capillary situations, have presented various pairs of constants. The original two-constant theory is commonly accepted as describing isotropic, linear elasticity. [11, p.121]. However, the persistence of just two constants in later developments is to be particularly noted. We believe that Poisson was one of few who were aware of this aspect when he introduced Laplace's deductions when, in 1831, he states,

> "elles renferment les deux constantes spéciales donc j'ai parlé tout à l'heure" [62, p.4].
>
> (Engl. Transl.) "they incorporate the two special constants of which I mentioned just a while ago."

With this viewpoint in mind, we retrace the evolution of the two-constant theory over the subsequent four decades culminating in 1845 with the presentation of the NS equations in the work of Stokes. We especially pay attention to how contributors to this development introduce their two constants. To facilitate comparisons of each contributor, we develop a universal notation that helps in expressing the kinematic equations that are contained in the NS equations. The need for this is highlighted by two separate developments represented by Navier and Poisson. Indeed, at the time, there were heated arguments over Navier's use of integration and Poisson's use of summation.

Moreover, we trace the evolution of the stress tensor term that conventionally describes viscous forces. In so doing, we endeavor to ascertain if the notion of tensor, which is usually thought of as a later mathematic development stemming from the work on differential geometry, is present in any of the earlier formative works on elasticity and fluid dynamics.

Another topic discussed in the final section is the rapidly decreasing functions [$RDFs$] which were included in the "two-constant" and which provided the common, mathematical interpretation of fluid properties among the then progenitors, in particular by Gauss, a contemporary of the progenitors of the NS equations, who contributed to the formulation of fluid mechanics in the development of Laplace's capillarity.

Finally, we uncover reasons for the practice in naming these fundamental equations of fluid motion "NS equations". In Table 6, we present a chronology outlining this practice. The last entry from 1934 by Prandtl [64, p.259] grouped the equations containing three terms:
1) the nonlinear term
2) the tensor function with the main axis (the normal stress) of Laplacian multiplied by ν

[1]($\Downarrow$) Throughout this paper, in citation of bibliographical sources, we show our own paragraph or sentences of commentaries by surrounding between ($\Downarrow$) and ($\Uparrow$). (($\Uparrow$) is used only when not following to next section,). And by $=^*$, we detail the statement by original authors, because we would like to discriminate and to avoid confusion from the descriptions by original authors. The mark : $\Rightarrow$ means transformation of the statements in brevity by ours. And all the frames surrounding the statements are inserted for important remark of ours. Of course, when the descriptions are explicitly distinct without these marks, these are not the descriptions in citation of bibliographical sources.

[2]($\Downarrow$) To establish a time line of these contributor, we list for easy reference the year of their birth and death: Sir I.Newton(1643-1727), D.Bernoulli(1700-1782), Euler(1707-1783), d'Alembert(1717-1783), Lagrange(1736-1813), Laplace(1749-1827), Fourier(1768-1830), Gauss(1777-1855), Navier(1785-1836), Poisson(1781-1840), Cauchy(1789-1857), Saint-Venant(1797-1886), Stokes(1819-1903).

[3]($\Downarrow$) Of capillary action, Laplace [34, V.4, *Supplément* p.2] acknowledges Clailaut [8, p.22], and Clailaut cites Maupertuis [42].

3) the gradient term of divergence multiplied by $\frac{\nu}{3}$ and used the term "the Navier-Stokes equations" for this set of equations.

These equations with the two coefficients in the ratio of 1 : 3 originated from Poisson [16] in 1831. Moreover, these equations contained both a linear and a nonlinear term developed earlier in Navier's equations [20] in 1827. Still earlier, the nonlinear term was introduced by Euler [12] in 1752-5.

Cauchy [7] in 1828, with $(46)_C$, expressed the linear term as two terms, one the tensor function with the main axis (the normal stress) of Laplacian and the other a gradient of divergence term, with two coefficients or constants, which are our main theme in our paper.

2. Preliminary Remarks

In this paper, we use the following definition according to Cauchy of the second-rank tensor in Euclidean three-space, following closely the presentation of I. Imai [22, p.178]: we call a 3×3 array denoted here by $\mathrm{P} \equiv (P_{ij})$ a second-rank tensor if it returns a new vector $\mathbf{P}_n$ when contracted from the right by the unit vector $\mathbf{n}$ represented by the column matrix of directional cosines. Thus the vector $\mathbf{P}_n \equiv \mathrm{P} \cdot \mathbf{n}$ has column matrix

$$\begin{bmatrix} P_{nx} \\ P_{ny} \\ P_{nz} \end{bmatrix} = \begin{bmatrix} p_{xx} & p_{yx} & p_{zx} \\ p_{xy} & p_{yy} & p_{zy} \\ p_{xz} & p_{yz} & p_{zz} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

In what follows, "tensor" means the "stress tensor" as introduced above.

Throughout this paper, we display for brevity a tensor by specifying only its components, P_{ij}. If the tensor satisfies $P_{ij} = P_{ji}$ for all $i, j = x, y, z$ then this tensor is said to be symmetric. An example of a symmetric tensor is the well-known Kronecker-delta δ_{ij}. Alternatively, if $P_{ij} = -P_{ji}$ then the tensor is said to be anti-symmetric or skew-symmetric.

In addition, we have employed the Einstein summation convention where summation is implied over twice repeated indices. For example, we can write $\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$ simplified as $\nabla \cdot \vec{v} = \partial_k v_k = v_{k,k}$.

In labeling some equations we provide two numbering schemes. Numbers on the right-hand-side correspond to our normal indexing while numbers on the left-hand-side of equations refer to those given by the author in his original paper. The subscript to the original indexing, is in the format for example N^e/N^f, where the capital letter is an author designation and the lower case superscript gives the type of theory; the above example then signifies "elasticity/fluid by Navier. For equations indexed by section in the original papers, the citation is then in the format "section no.-no. by author". When referring to a "fluid", an "elastic fluid" is implied.

3. A universal method for the two-constant theory

In this section, we propose a universal method to describe the kinetic equations that arise in isotropic, linear elasticity. This method is outlined as follows:

- The partial differential equations describing waves in elastic solids or flows in elastic fluids are expressed by using one constant or a pair of constants C_1 and C_2 such that:

$$\text{for elastic solids:} \qquad \frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1 T_1 + C_2 T_2) = \mathbf{f},$$

$$\text{for elastic fluids:} \qquad \frac{\partial \mathbf{u}}{\partial t} - (C_1 T_1 + C_2 T_2) + \cdots = \mathbf{f},$$

where T_1, $T_2, \cdots$ are the terms depending on tensor quantities constituting our equations. For example, the NS equations corresponding to incompressible fluids consist of the kinetic equation along with the continuity equation and are conventionally written, in modern vector notation, as follows:

$$\frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0. \tag{1}$$

Here $\mathbf{u}$ is the velocity, $\mathbf{f}$ accounts for the body forces present, p the pressure and $\Delta \equiv \nabla \cdot \nabla$ the *Laplacian* or *Laplace operator.*

- The two coefficients C_1 and C_2 associated with the tensor terms are the two constants of the theory, definitions of which depend on the contributing author. For example, ε and E were introduced by Navier, R and G by Cauchy, k and K in elastic and $(K+k)\alpha$ and $\frac{(K+k)\alpha}{3}$ in fluid by Poisson, ε and $\frac{\varepsilon}{3}$ by Saint-Venant, and μ and $\frac{\mu}{3}$ by Stokes. Since Poisson, the ratio of coefficient attached to the term of the tensor function with the main axis (the normal stress) of Laplacian to that of grad div : $\frac{\text{coefficient of tensor}}{\text{coefficient of grad div}} = 3$ was fixed. Moreover, C_1 and C_2 can be expressed in the following form:

$$\begin{cases} C_1 \equiv \mathcal{L} r_1 g_1 S_1, \\ C_2 \equiv \mathcal{L} r_2 g_2 S_2, \end{cases} \quad \begin{cases} S_1 = \iint g_3 \to C_3, \\ S_2 = \iint g_4 \to C_4, \end{cases} \quad \Rightarrow \quad \begin{cases} C_1 = C_3 \mathcal{L} r_1 g_1 = \frac{2\pi}{15} \mathcal{L} r_1 g_1, \\ C_2 = C_4 \mathcal{L} r_2 g_2 = \frac{2\pi}{3} \mathcal{L} r_2 g_2. \end{cases}$$

Here $\mathcal{L}$ corresponds to either $\sum_0^\infty$ as argued for by Poisson or $\int_0^\infty$ as argued for by Navier. A heated debate had developed between the two over this point. It is a matter of personal preference as to how the two constants should be expressed.

- The two constants depend on two radial functions r_1 and r_2 related to the radius of the active sphere of the molecules, raised to some power of n for Poisson's and Navier's cases; the relationship between these functions can be expressed by a logarithm with base r such that: $\log_r \frac{r_1}{r_2} = 2$.
- g_1 and g_2 are the kernel functions having both
  - the physical caracteristics come from the fluid dynamics described by the microscopically basic relations of the attraction and/or repulsion and
  - the mathematical requirements for the rapidly decreasing function.
- S_1 and S_2 are two expressions which determine the angular dependence on the surface of the active unit-sphere centered on a molecule through application of the double integral (or single sum in the case of Poisson's fluid).
- g_3 and g_4 are certain compound spherical harmonic functions determining the momentum over the unit sphere.
- C_3 and C_4 are indirectly determined as the common coefficients derived from the invariant tensor. With the exception of Poisson's fluid case, C_3 of C_1 is $\frac{2\pi}{3}$, and C_4 of C_2 is $\frac{2\pi}{15}$, which are evaluated over the unit spheres for each molecule, and which are independent of the preference in using integrals or summations. In Poisson's case, we obtain the same values as the above after multiplying by $\frac{1}{4\pi}$. The integrals are calculated from the total momentum of the active sphere surrounding the molecule.
- The ratio of C_3 to C_4 : $\frac{C_3}{C_4} = \frac{1}{5}$ including Poisson's case.

4. Genealogy and settlement of the stress tensor

In Figure 1, we have traced the genealogy of the tensor terms, in particular noting the form of each tensor t_{ij} appearing in the NS equations. These tensors are listed in Table 5, where we have differentiated those tensors associated with elastic solids or elastic fluids. From this genealogy, it could be asserted that Cauchy [6, 7] was the first user of "tensors" and arguably its inventor. This view is supported by the admission of Poisson [60] that he received the idea of a "symmetric tensor" from Cauchy. Moreover, the idea of tensor by Saint-Venant concurs with the work of Stokes. Here, we denote the two routes as NCP and PSS, both of which are portrayed in our figure, and by which we can explain the genealogy of tensor as it applies to the NS equations. cf. Table 5.

Fig.1: *A genealogy of the stress tensors in the prototypical Navier-Stokes equations*

Navier[46, 47]$^{\otimes}$
$\nearrow$ $\|$ $\Updownarrow$ $\searrow$
(Euler)$\Rightarrow$ $\|$ Poisson[59, 60]$^{\bullet}$ $\Rightarrow$ Saint-Venant[67]† $\Rightarrow$ Stokes[74]‡$\Rightarrow$ Maxwell[43]$^{\flat}$ $\Rightarrow$Kirchhoff$^{\sharp}$$\Rightarrow$Boltzmann$^{\natural}$
$\searrow$ $\Downarrow$ $\Uparrow^{\odot}$ $\nearrow\Rightarrow$
Cauchy[6, 7]§
$\leftarrow$MD deduction$\rightarrow$ $\|\leftarrow$ Non-MD deduction $\rightarrow$ $\leftarrow$ Gas theory by MD deduction
$\leftarrow$ NCP pattern $\rightarrow$ $\|\leftarrow$ PSS pattern $\rightarrow$

Legend for superscripted marks:

$^{\otimes}$ Navier:	$t^e_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i}),\ t^f_{ij} = (p - \varepsilon u_{k,k})\delta_{ij} - \varepsilon(u_{i,j} + u_{j,i})$
$^{\bullet}$ Poisson:	$t^e_{ij} = -\frac{a^2}{3}(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i}),\ t^f_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$
§ Cauchy:	$t^{e,f}_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$
† Saint-Venant:	$t^f_{ij} = (\frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}),\ \frac{1}{3}(P_{xx} + P_{yy} + P_{zz}) = -p$
‡ Stokes:	$t^f_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i}),$
$^{\odot}$ Poisson:	stated his reduction of the number of independent t_{ij} from 9 to 6 is due to Cauchy. (cf.§5.2)
$^{\flat}$Maxwell:	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$
$^{\sharp}$Kirchhoff[23]:	$t^f_{ij} = (-p - 2kv_{l,l})\delta_{ij} + k(v_{i,j} + v_{j,i})$
$^{\natural}$Boltzmann[2]:	$t^f_{ij} = (-p - \frac{2}{3}\mathcal{R}v_{k,k})\delta_{ij} + \mathcal{R}(v_{i,j} + v_{j,i})$

We cannot ascribe to Euler a definite form for the stress tensor; however, Voigt[77] has presented a version in 1905. [4] He begins by introducing an exterior subscript index of the vector as also interior indices to the product of elements.

$$[\mathcal{B}.T]_1 = \mathcal{B}_1.T_1, \cdots$$

Then he defines the derivative of the synthetic function as follows: [5]

$$\frac{d}{dt}[w.T] \equiv \mathcal{D} \quad \Rightarrow \qquad (37)_V \qquad \left[T.\frac{dw}{dt}\right] + \Big[w.[w.T]\Big] \equiv \mathcal{D}\ ;$$

Here, he defines two vectors as follows:

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}, \quad [w.T] = \begin{bmatrix} w_1T_1 \\ w_2T_2 \\ w_3T_3 \end{bmatrix}$$

---

[4]As an aside, W.Voigt [77] states Euler equations with his invented tensor in 1905 as follows : (we show his sketched contents)

> Auch hier sind die Ausdrücke für die Componenten nach den Richtungen der Tensoren T_1, T_2, T_3 - auf denen eine Seite hervorzuheben ist - von Interesse ; es gilt nämlich, wenn diese Richtungen wieder durch die Indices 1, 2, 3 characterisirt werden, höchst einfach
>
> $$(19)_V \qquad [\mathcal{B}.T]_1 = \mathcal{B}_1.T_1, \cdots$$
>
> Bei Benutzung dieses Resultates und bei Berücksichtigung der Constanz der Componenten von T nach den mit dem Körper bewegten Axen nimmt die Gleichung $(32)_V$ ($\frac{d}{dt}[w.T] = \mathcal{D}$) die Form an
>
> $$(37)_V \qquad \left[T.\frac{dw}{dt}\right] + \Big[w.[w.T]\Big] \ \#\ \mathcal{D}\ ;$$
>
> es ist dabei zu beachten, daß dieselbe über die Richtungen, nach denen die Componenten der in ihnen auftretenden Vectoren zu nehmen sind, noch weite Freiheit läßt.
>
> Der wichtigste Fall ist der, daß jene Richtungen in die eine Seite der Tensoren T_1, T_2, T_3 - die Hauptträgheitsaxen des Körpers - fallen. Hier reducieren sich nach $(19)_V$ die Componenten von $[w.T]$ auf w_1T_1, w_2T_2, w_3T_3, und es folgt, da die T_h von der Zeit unabhängig sind, aus $(37)_V$,
>
> $$\begin{cases} T_1\frac{dw_1}{dt} + w_2w_3\{T_3 - T_2\} = \mathcal{D}_1, \\ T_2\frac{dw_2}{dt} + w_3w_1\{T_1 - T_3\} = \mathcal{D}_2, \\ T_3\frac{dw_3}{dt} + w_1w_2\{T_2 - T_1\} = \mathcal{D}_3 \end{cases}$$
>
> Das sind die Eulerschen Gleichungen. [77, §11, pp.14-15.]

[5]($\Downarrow$) By $\#$, Voigt means $\equiv$, i.e. equality by definition.

then if T_n are independent of time, we can deduce the vectorial form of $(37)_V$:

$$\begin{cases} T_1 \frac{dw_1}{dt} + w_2 w_3 \{T_3 - T_2\} = \mathcal{D}_1, \\ T_2 \frac{dw_2}{dt} + w_3 w_1 \{T_1 - T_3\} = \mathcal{D}_2, \\ T_3 \frac{dw_3}{dt} + w_1 w_2 \{T_2 - T_1\} = \mathcal{D}_3 \end{cases}$$

He states that these are the Euler equations as expressed in tensor form.

5. Derivations of the two constants and tensor

Recently, Darrigol [11, p.121] has concluded[6]

> "it is called that *the two-constant theory* is the one now accepted for isotropic, linear elasticity,"

but Poisson [62, p.4] had stated already in 1831:

> L'equation qui résulte de cette considération et celle qui appartient à la surface entière sont le deux équations du problème ; elles renferment les deux constantes spéciales donc j'ai parlé tout à l'heure. [62, p.4].
>
> [(Engl.transl.) The equation that results from this consideration and that which belongs to the whole surface are the two equations of problem; they incorporate the *two special constants* of which I have just spoken];

and from these two equations Laplace had provided explanations of various phenomena observed by physicists. Therefore, we believe that Laplace was the first to establish a "two-constant" theory [37] in Table 3.

5.1. Navier's two constants and tensor. In his theory of elasticity in (2), Navier deduced the single constant ε.

The corresponding NS equations derived for incompressible fluids by Navier himself (1) are in their original form as follows:

$$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\Big(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\Big) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w \,; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\Big(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\Big) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w \,; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\Big) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w \,; \end{cases} \tag{2}$$

along with the equation of continuity: $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$.

Navier provided an evaluation of the two constants as follows:

$$(3\text{-}10)_{Nf} \quad \varepsilon \equiv \frac{8\pi}{30}\int_0^\infty d\rho\rho^4 f(\rho) = \frac{4\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho), \quad E \equiv \frac{4\pi}{6}\int_0^\infty d\rho\rho^2 F(\rho) = \frac{2\pi}{3}\int_0^\infty d\rho\rho^2 F(\rho). \tag{3}$$

In the case of fluids, Navier was well aware of the necessity for the equation of continuity, because from

(2) he obtained $\varepsilon\Delta$ by differentiating the equation of continuity with $(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz})$. For example, the ε-terms in (2), as well as (4), are reduced to $\varepsilon\Delta\mathbf{u}$ as for example in (5).

This is solely due to the mass conservation law, according to the explanation given by Navier.

---

[6]($\Downarrow$) Darrigol [11, p.121] uses such terminology, however, not explaining his definition or concreate meaning of the the two-constant theory. Here, we introduce his sentence cited from Darrigol, to whome our motivation owe largely, as follows :

> In the final version of his theory, Cauchy proposed the more general, two-constant relation
>
> $$\tau_{i,j} = K'(\partial_i u_j + \partial_j u_i) + K''\delta_{ij}\partial_k u_k$$
>
> between stress and deformation. This allowed him to retrieve Navier's equation of equilibrium as the particular case for which $K' = K''$. *The two-constant theory* is the one now accepted for isotropic elasticity. [11, p.121]

As an aside, Navier always used his often-used method involving a four-step procedure to solve three of the equations, such as the equilibrium equation for the fluid [47], the kinetic equation for the elastic solid [46], and the kinetic equation for the fluid [47] with the following general method:

(1) initially, deduce either one constant or two constants, including the incomputable function such as $f\rho$, $f(\rho)$ or $F(\rho)$ in Table 3,
(2) then construct the indeterminate equation,
(3) next expand it in a Taylor series and integrate it by parts, exchanging d and δ, and then pair up with the same integral operator, and finally
(4) solve the indeterminate equation from the two conditions of the interior and the boundary.

We present more details of this procedure by outlining Navier's analysis of fluid flow [47].

5.1.1. **Indeterminate equation.** As called then by Navier, the indeterminate equation is given as follows:

$$
\begin{aligned}
(3\text{-}24)_{Nf} \quad 0 \quad = \quad & \iiint dxdydz \begin{cases} \left[P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right)\right]\delta u \\ \left[Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right)\right]\delta v \\ \left[R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right)\right]\delta w \end{cases} \\
- \quad & \varepsilon \iiint dxdydz \begin{cases} \left(3\frac{du}{dx}\frac{\delta du}{dx} + \frac{du}{dy}\frac{\delta du}{dy} + \frac{du}{dz}\frac{\delta du}{dz}\right) + \left(\frac{dv}{dy}\frac{\delta du}{dx} + \frac{dv}{dx}\frac{\delta du}{dy}\right) + \left(\frac{dw}{dz}\frac{\delta du}{dx} + \frac{dw}{dx}\frac{\delta du}{dz}\right) \\ \left(\frac{du}{dx}\frac{\delta dv}{dy} + \frac{du}{dy}\frac{\delta dv}{dx}\right) + \left(\frac{dv}{dx}\frac{\delta dv}{dx} + 3\frac{dv}{dy}\frac{\delta dv}{dy} + \frac{dv}{dz}\frac{\delta dv}{dz}\right) + \left(\frac{dw}{dy}\frac{\delta dv}{dz} + \frac{dw}{dz}\frac{\delta dv}{dy}\right) \\ \left(\frac{du}{dx}\frac{\delta dw}{dz} + \frac{du}{dz}\frac{\delta dw}{dx}\right) + \left(\frac{dv}{dy}\frac{\delta dw}{dz} + \frac{dv}{dz}\frac{\delta dw}{dy}\right) + \left(\frac{dw}{dx}\frac{\delta dw}{dx} + \frac{dw}{dy}\frac{\delta dw}{dy} + 3\frac{dw}{dz}\frac{\delta dw}{dz}\right) \end{cases} \\
+ \quad & \mathbf{S}ds^2 E(u\delta u + v\delta v + w\delta w),
\end{aligned}
\tag{4}
$$

where (P, Q, R) are the components of the applied force, **S** shows an integration to be performed in the total area of the surface and ds^2 is its area, and with the quantity E, varying it according the the nature of the material with which the surface contacts.

5.1.2. **Determinate equation from a Taylor series expansion and integration by parts.** Putting $\mathbf{S}ds^2E(u\delta u+v\delta v+w\delta w) = 0$ in the indeterminate equation (4) and performing a Taylor series expansion to first-order and neglecting higher-order terms, we have the determinate equation as follows:

$$
(3\text{-}29)_{Nf} \quad 0 = \iiint dxdydz \begin{cases} \left[P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right) + \varepsilon\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)\right]\delta u \\ \left[Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right) + \varepsilon\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right)\right]\delta v \\ \left[R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right) + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right)\right]\delta w \end{cases} \tag{5}
$$

From (5) we obtain (2), i.e. the kinetic equation, which is equivalent to the first equation of (1).

5.1.3. **Determinate equation deduced from boundary condition.** As a boundary condition, Navier used two constants in one equation. In this respect, his method is unique within developments of the period. Navier had explained his method as follows:

regarding the conditions which apply at points on the surface of the fluid element, if we substitute

- $dydz \quad \rightarrow \quad ds^2 \cos l$, where l is the angle by which the tangent plane makes with the yz-plane of the surface frame,
- $dxdz \quad \rightarrow \quad ds^2 \cos m$, where similarly m is the angle with the xz-plane,
- $dxdy \quad \rightarrow \quad ds^2 \cos n$, where similarly n is the angle with the xy-plane,
- $\iint dydz, \iint dxdz, \iint dxdy \quad \rightarrow \quad \mathbf{S}ds^2$, where **S** is the sign of integral in respect to ds^2 on the surface,

then, because the factors multiplying $\delta u, \delta v$ and δw respectively reduce to zero, the following determinate equations should hold for any point on the surface of the fluid element:

$$
(3\text{-}32)_{Nf} \quad \begin{cases} Eu + \varepsilon\left[\cos l\, 2\frac{du}{dx} + \cos m\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \cos n\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right] = 0, \\ Ev + \varepsilon\left[\cos l\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \cos m\, 2\frac{dv}{dy} + \cos n\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\right] = 0, \\ Ew + \varepsilon\left[\cos l\left(\frac{dw}{dx} + \frac{du}{dz}\right) + \cos m\left(\frac{dw}{dy} + \frac{dv}{dz}\right) + \cos n\, 2\frac{dw}{dz}\right] = 0. \end{cases} \tag{6}
$$

Here the value of the constant E must vary in accordance with the nature of solid with which the fluid is in contact. The equations of (6) are an expression of conditions prevailing on the boundary of the surface and constitute the so-called boundary conditions. The first terms on the

left-hand-side of (6) are defined in (3) for the expression that we seek for the sum of the momentum of all interactions arising between molecules on the boundary and the fluid, while the second terms are the normal derivatives. Here, derivative terms on the left-hand-side of (6) are expressible as $v_{i,j} + v_{j,i}$. If we introduce the basis of the tensor as $\left[\ \cos l \quad \cos m \quad \cos n\ \right]^T$, then the tensor part of (6) is expressible as:

$$t_{ij} = \varepsilon[\{2v_{i,j} - (v_{i,j} + v_{j,i})\}\delta_{ij} + (v_{i,j} + v_{j,i})] = \varepsilon\{0\delta_{ij} + (v_{i,j} + v_{j,i})\} = \varepsilon(v_{i,j} + v_{j,i}).$$

5.2. **Cauchy's two constants and tensor.** In this section we adopt the following definitions:

- $a,\ b,\ c$: the coordinate values of a molecule m in the rectangular axes of $x,\ y,\ z$;
- $a + \Delta a,\ b + \Delta b,\ c + \Delta c$: the coordinates of another molecule m;
- $\xi,\ \eta,\ \zeta$: three functions of $a,\ b,\ c$ representing infinitesimal displacements parallel to the axes of molecule m;
- $(x,\ y,\ z),\ (x + \Delta x,\ y + \Delta y,\ z + \Delta z)$: the coordinates of molecules m and m in the new state of the system;
- $r(1+\varepsilon)$: the distance between molecules m and m;[7]
- ε: the dilatation of the length r in the path from the first state to the second, and then we have $x = a + \xi,\ y = b + \eta,\ z = c + \zeta$;
- $X,\ Y,\ Z$: the quantities of the algebraic projections such that :
  - of which a resultant are consisted ;
  - from this resultant divided by m, or, which return to itself, by the accelerated force which acts on the molecule m and which will be due to the actions of molecules $m, m', m'', \cdots$

Cauchy deduced the following three elements of material points of elasticity after calculating the interactions of molecules, the details of which are omitted for the sake of brevity. However, to begin we start with the following equation of elasticity

$$(40)_C \quad \begin{cases} X = (L+G)\frac{\partial^2\xi}{\partial a^2} + (R+H)\frac{\partial^2\xi}{\partial b^2} + (Q+I)\frac{\partial^2\xi}{\partial c^2} + 2R\frac{\partial^2\eta}{\partial a\partial b} + 2Q\frac{\partial^2\zeta}{\partial c\partial a}, \\ Y = (R+G)\frac{\partial^2\eta}{\partial a^2} + (M+H)\frac{\partial^2\eta}{\partial b^2} + (P+I)\frac{\partial^2\eta}{\partial c^2} + 2P\frac{\partial^2\zeta}{\partial b\partial c} + 2R\frac{\partial^2\xi}{\partial a\partial b}, \\ Z = (Q+G)\frac{\partial^2\zeta}{\partial a^2} + (P+H)\frac{\partial^2\zeta}{\partial b^2} + (N+I)\frac{\partial^2\zeta}{\partial c^2} + 2Q\frac{\partial^2\xi}{\partial c\partial a} + 2P\frac{\partial^2\eta}{\partial b\partial c} \end{cases}$$

which displays all nine components of the tensor. (The invariants of the tensor are represented by the two constants G and R.) Cauchy said of the elements of the tensor, i.e. the fixed values: $G, H, I, L, M, N, P, Q, R$:

> If we suppose that the molecules $m, m', m'', \cdots$ are originally allocated by the same way in relation to the three planes made by the molecule m in parallel with the plane coordinates, then the values of these quantities become remain invariable, even though a series of changes are made among the three angles: α, β, γ.

Cauchy then resulted in the case of symmetric tensors such that:

$$(41)_C \quad G = H = I, \quad L = M = N, \quad P = Q = R, \quad (45)_C \quad L = 3R.$$

which reduces the form of the equations $(40)_C$ to the equations consisted of the tensor function with the main axis (the normal stress) of Laplacian with $R+G$ and the term of gredient of the divergence with $2R$:

$$(46)_C \begin{cases} X = (R+G)\left(\frac{\partial^2\xi}{\partial a^2} + \frac{\partial^2\xi}{\partial b^2} + \frac{\partial^2\xi}{\partial c^2}\right) + 2R\frac{\partial\nu}{\partial a}, \\ Y = (R+G)\left(\frac{\partial^2\eta}{\partial a^2} + \frac{\partial^2\eta}{\partial b^2} + \frac{\partial^2\eta}{\partial c^2}\right) + 2R\frac{\partial\nu}{\partial b}, \\ Z = (R+G)\left(\frac{\partial^2\zeta}{\partial a^2} + \frac{\partial^2\zeta}{\partial b^2} + \frac{\partial^2\zeta}{\partial c^2}\right) + 2R\frac{\partial\nu}{\partial c}, \end{cases} \quad \textit{where,} \quad (47)_C \quad \nu = \frac{\partial\xi}{\partial a} + \frac{\partial\eta}{\partial b} + \frac{\partial\zeta}{\partial c}$$

[7]($\Downarrow$) This bold type m is different from m, the latter is top one of $m, m', m'', \cdots$.

($\Downarrow$) Cauchy may have been the inventor of the term [8] "tensor", and Poisson supported Cauchy's priority in the symmetry properties of the tensor when he reduced the number of independent components from nine to six elements, by the following quote:

> D'un autre côté, il faut, pour l'equilibre d'un parallélépipède rectangle d'une étendue insensible, que les neuf composantes des pressions appliquées à ses trois faces non-parallélles, se réduisent à six forces qui peuvent être inégales. Cette proposition est due à M.Cauchy, et se déduit de la considération des momens.[9] [60, §38, p.83]
>
> [(Engl. transl.) On the other hand, one needs for equilibrium of the rectangular parallelepiped of infinitesimal volume, that the nine components of the pressure applied to its three non-paralleled faces, reduce to six forces which may be unequal. This proposition is due to Mr. Cauchy, and deduced from the consideration of momentum.]

($\Uparrow$)
Continuing, we define the density of molecules as:
$(48)_C$ $\Delta = \frac{\mathcal{M}}{\mathcal{V}}$, where $\mathcal{M}$ is the sum of the masses of molecules contained in the sphere and $\mathcal{V}$ is the volume of the sphere. We then find expressions for the two constants, G and R:

$$(50)_C \quad \begin{cases} G = \pm\frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3\mathrm{f}(r)\cos^2\alpha\sin p\,drdqdp = \pm\frac{2\pi\Delta}{3}\int_0^\infty r^3\mathrm{f}(r)dr, \\ R = \frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3 f(r)\cos^2\alpha\cos^2\beta\sin p\,drdqdp \\ \quad = \frac{2\pi\Delta}{15}\int_0^\infty r^3 f(r)dr = \pm\frac{2\pi\Delta}{15}\int_0^\infty\Big[r^4\mathrm{f}'(r) - r^3\mathrm{f}(r)\Big]dr \end{cases} \tag{7}$$

where we have used:

$$(51)_C \quad \cos\alpha = \cos p, \quad \cos\beta = \sin p\cos q, \quad \cos\gamma = \sin p\sin q.$$

When we calculate these values in the general case [10] then (7) yields the following expressions:

$$(56)_C \quad \begin{cases} A \equiv \Big[(L+G)\frac{\partial\xi}{\partial a} + (R-G)\frac{\partial\eta}{\partial b} + (Q-G)\frac{\partial\zeta}{\partial c}\Big]\Delta, \\ B \equiv \Big[(R-H)\frac{\partial\xi}{\partial a} + (M+H)\frac{\partial\eta}{\partial b} + (P-H)\frac{\partial\zeta}{\partial c}\Big]\Delta, \\ C \equiv \Big[(Q-I)\frac{\partial\xi}{\partial a} + (P-I)\frac{\partial\eta}{\partial b} + (N+I)\frac{\partial\zeta}{\partial c}\Big]\Delta, \end{cases} \qquad (57)_C \quad \begin{cases} D \equiv \Big[(P+I)\frac{\partial\eta}{\partial c} + (P+H)\frac{\partial\zeta}{\partial b}\Big]\Delta, \\ E \equiv \Big[(Q+G)\frac{\partial\zeta}{\partial a} + (Q+I)\frac{\partial\xi}{\partial c}\Big]\Delta, \\ F \equiv \Big[(R+H)\frac{\partial\xi}{\partial b} + (R+G)\frac{\partial\eta}{\partial a}\Big]\Delta, \end{cases}$$

By $(41)_C$ and $(45)_C$, we obtain the following reduced form:

$$\frac{A}{\Delta} = 2(R+G)\frac{\partial\xi}{\partial a} + (R-G)v, \quad \frac{B}{\Delta} = 2(R+G)\frac{\partial\eta}{\partial b} + (R-G)v, \quad \frac{C}{\Delta} = 2(R+G)\frac{\partial\zeta}{\partial c} + (R-G)v,$$
$$\frac{D}{\Delta} = (R+G)\Big(\frac{\partial\eta}{\partial b} + \frac{\partial\zeta}{\partial c}\Big), \quad \frac{E}{\Delta} = (R+G)\Big(\frac{\partial\zeta}{\partial a} + \frac{\partial\xi}{\partial c}\Big), \quad \frac{F}{\Delta} = (R+G)\Big(\frac{\partial\xi}{\partial b} + \frac{\partial\eta}{\partial a}\Big).$$

For the sake of convenience, in the particular case when both $(41)_C$ and $(45)_C$ hold, it is sufficient to have :

$$(59)_C \quad (R+G)\Delta \equiv \frac{1}{2}k, \quad (R-G)\Delta \equiv K, \quad \Rightarrow \quad R = \frac{k+2K}{4\Delta} \quad G = \frac{k-2K}{4\Delta}.$$

---

[8]($\Downarrow$) The editors of Hamilton's papers [20, p.237, footnote] say, "The writer believes that what originally led him to use the terms 'modulus' and 'amplitude,' was a recollection of M. Cauchy's nomenclature respecting the usual imaginaries of algebra."

[9]($\Downarrow$) Poisson always writes "moments" as "momens".

[10]($\Downarrow$) We obtained the following intermediate results, which were needed:

$$\begin{cases} \int_0^{2\pi}\int_0^\pi \cos^2\alpha\sin p dqdp = 2\pi\int_0^\pi\cos^2 p\sin\ pdp = 2\pi\Big[-\frac{\cos^3 p}{3}\Big]_0^\pi = \frac{4\pi}{3}, \\ \int_0^{2\pi}\int_0^\pi\cos^2\alpha\cos^2\beta\sin\ pdp = \int_0^{2\pi}\cos^2 qdq\int_0^\pi\cos^2 p(1-\cos^2 p)\sin pdp = \Big[\frac{q}{2} + \frac{1}{4}\sin 2q\Big]_0^{2\pi}\Big[-\frac{\cos^5 p}{5}\Big]_0^\pi = (\frac{2\pi}{2} - 0)(\frac{2}{3} - \frac{2}{5}) = \frac{4\pi}{15} \\ C_3 = \frac{1}{2}\frac{4\pi}{15} = \frac{2\pi}{15}, \quad C_4 = \frac{1}{2}\frac{4\pi}{3} = \frac{2\pi}{3}, \end{cases}$$

Equations $(56)_C$ and $(57)_C$ can be displayed in a more convenient manner

$$(60)_C \quad \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} = \begin{bmatrix} k\frac{\partial\xi}{\partial a}+K\nu & \frac{1}{2}k\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & \frac{1}{2}k\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) \\ \frac{1}{2}k\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & k\frac{\partial\eta}{\partial b}+K\nu & \frac{1}{2}k\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) \\ \frac{1}{2}k\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) & \frac{1}{2}k\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) & k\frac{\partial\zeta}{\partial c}+K\nu \end{bmatrix}. \quad (8)$$

Here, we must remark that the layout of the symmetric tensor of $(58)_C$ or $(60)_C$ is Cauchy's invention. If, moreover, the condition $(54)_C$: $R=-G$ holds, then $k=0$ holds, thus yielding the following identities: $(61)_C \quad A=B=C=K\nu, \quad D=E=F=0.$

5.2.1. Equilibrium and kinetic equation of fluid by Cauchy.

In what follows, equations referring to Cauchy's work on fluids will be designated in the form $(\cdot)_{C^*}$ instead of by $(\cdot)_C$ to distinguish these from equations appearing in his work on elasticity above.

(Verification of equations for fluids.)

By replacing $(a,\ b,\ c)$ of $(56)_C$ and $(57)_C$ with $(x,\ y,\ z)$, we derive an equivalent set of equations for fluids as for elasticity. We omit for the sake of brevity the precise process in leading to the two constants or equations and present the final form

$$(76)_{C^*} \quad \begin{cases} \frac{\partial A}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial E}{\partial z}+X\Delta=0, \\ \frac{\partial F}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial D}{\partial z}+Y\Delta=0, \\ \frac{\partial E}{\partial x}+\frac{\partial D}{\partial y}+\frac{\partial C}{\partial z}+Z\Delta=0, \end{cases} \Rightarrow \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} + \Delta \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{0}$$

We follow the layout of Cauchy's symmetric tensor as presented originally in $(76)_{C^*}$. By replacing $R+G$ and $2R$ with Cauchy's usage $C_1^* \equiv R+G=\frac{k}{2\Delta}$, $\quad C_2^* \equiv 2R=\frac{k+2K}{2\Delta}$, we can reduce these equations for fluids both in motion and in equilibrium to the same form $(46)_C$ found for elasticity. However, here, we would like to adopt not Cauchy's C_1^* and C_2^* but $C_1=R$ and $C_2=G$, because it is more rational to do so, as can be seen by checking the reciprocal coincidence in Table 3. [11]

($\Downarrow$) Here, C_1^* is the constant to the tensor function with the main axis of Laplacian. C_2^* corresponds to the grad.div term. In today's NS equations, the ratio of coefficients : $\frac{C_1^*}{C_2^*}=\frac{\text{coefficient of tensor}}{\text{coefficient of grad.div}}=\frac{k}{k+2K}$. By Prandtl [64, p.259] in 1934, the ratio was fixed at 3. We had have to wait the formulation by Poisson in fluid equation. In this Cauchy's paper, we can not confirm explicitly the deduction of the value, except for the elasticity. cf. Poisson's equations $(7\text{-}9)_{Pf}$ or Table 7. ($\Uparrow$)

(Comparison with and comments on Navier's equation in elasticity.)

Cauchy states: for the reduction of equations $(79)_{C^*}$ and $(80)_{C^*}$ to Navier's equations([46]) to determine the law of equilibrium and elasticity, it is necessary to assume such as the condition which we have mentioned above: $k=2K$. If $G=0$ then we get the equations of equilibrium and the kinetic equations in elastic equilibrium, then Cauchy's tensor is equivalent not only to the tensor in elasticity but also to the tensor of ε's term in Navier's fluid equation (2) (c.f. Table 5).

5.3. Poisson's two constants and tensor.

5.3.1. Principles and equations in elastic solids.

Below, we deduce K and k according to Poisson [59, pp.368-405, §1-§16]. For simplicity, we introduce the following definitions:

$$\begin{cases} ax_1+by_1+c(z_1-\zeta_1)\equiv\phi, \\ a'x_1+b'y_1+c'(z_1-\zeta_1)\equiv\psi, \\ a''x_1+b''y_1+c''(z_1-\zeta_1)\equiv\theta, \end{cases} \qquad \begin{cases} \phi\frac{du}{dx}+\psi\frac{du}{dy}+\theta\frac{du}{dz}\equiv\phi', \\ \phi\frac{dv}{dx}+\psi\frac{dv}{dy}+\theta\frac{dv}{dz}\equiv\psi', \\ \phi\frac{dw}{dx}+\psi\frac{dw}{dy}+\theta\frac{dw}{dz}\equiv\theta' \end{cases} \quad (9)$$

We assume that α is the average molecular distance, ω represents a finite surface

[11]($\Downarrow$) Here, C_1 and C_2 are not the two-constant defined earlier by us but introduced temporarily by Cauchy himself.

area, and $\frac{\omega}{\alpha^2}$ is the average number of molecules on ω. We then get the pressure terms.

$$P = \sum \frac{(\phi + \phi')\zeta}{\alpha^3 r'} fr', \quad Q = \sum \frac{(\psi + \psi')\zeta}{\alpha^3 r'} fr' \quad R = \sum \frac{(\theta + \theta')\zeta}{\alpha^3 r'} fr'. \tag{10}$$

By using Poisson's so-called *effective transformation,* [12] we obtain from (10) the following:

$$\begin{cases} P = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[(g+g')\sum \frac{r^3}{\alpha^5} fr + (gg' + hh' + ll')g \sum \frac{r^5}{\alpha^5} \frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ Q = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[(h+h')\sum \frac{r^3}{\alpha^5} fr + (gg' + hh' + ll')h \sum \frac{r^5}{\alpha^5} \frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ R = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[(l+l')\sum \frac{r^3}{\alpha^5} fr + (gg' + hh' + ll')l \sum \frac{r^5}{\alpha^5} \frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \end{cases} \qquad \Delta := \cos\beta \cdot \sin\beta \; d\beta \; d\gamma, \tag{11}$$

Later, Poisson solved again this problem in another book [60], [13] in which he deduces the general principles behind elasticity and fluids, and hence derives the representative two-constant theory with K and k for both elasticity and fluids as follows:

$$\begin{cases} P = \left[K(1+\frac{du}{dx}) + k\left(3\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right]c + \left[K\frac{du}{dy} + k\left(\frac{du}{dy} + \frac{dv}{dx}\right)\right]c' + \left[K\frac{du}{dz} + k\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right]c'', \\ Q = \left[K(1+\frac{dv}{dy}) + k\left(\frac{du}{dx} + 3\frac{dv}{dy} + \frac{dw}{dz}\right)\right]c' + \left[K\frac{dv}{dx} + k\left(\frac{dv}{dx} + \frac{du}{dy}\right)\right]c + \left[K\frac{dv}{dz} + k\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\right]c'', \\ R = \left[K(1+\frac{dw}{dz}) + k\left(\frac{du}{dx} + \frac{dv}{dy} + 3\frac{dw}{dz}\right)\right]c'' + \left[K\frac{dw}{dy} + k\left(\frac{dw}{dy} + \frac{dv}{dz}\right)\right]c' + \left[K\frac{dw}{dx} + k\left(\frac{dw}{dx} + \frac{du}{dz}\right)\right]c, \end{cases} \tag{12}$$

where, for simplicity, he uses similarly K and k.

($\Downarrow$) By the way, from (12) we can express the pressure : $[P,\ Q,\ R]^T$ by the two tensor on the basis of $[c,\ c',\ c'']^T$ corresponding to two constants : K and k are given as follows :

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = K \begin{bmatrix} 1+\frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & 1+\frac{dv}{dy} & \frac{dw}{dx} \\ \frac{dw}{dy} & \frac{dw}{dx} & 1+\frac{dw}{dz} \end{bmatrix} + k \begin{bmatrix} 3\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} & \frac{du}{dz} + \frac{dw}{dx} & \frac{du}{dy} + \frac{dv}{dx} \\ \frac{dv}{dz} + \frac{dw}{dy} & \frac{du}{dx} + 3\frac{dv}{dy} + \frac{dw}{dz} & \frac{dv}{dx} + \frac{du}{dy} \\ \frac{dw}{dy} + \frac{dv}{dz} & \frac{dw}{dx} + \frac{du}{dz} & \frac{du}{dx} + \frac{dv}{dy} + 3\frac{dw}{dz} \end{bmatrix}$$

Moreover, instead of α in (10), he introduces ε as the average distance between molecules, and from the following considerations:

- "on voit que la pression N restera la même en tous sens autour de ce point: elle sera normale à ce plan et dirigée de dehors en dedans de A, ou de dedans en dehors, selon que sa value sera positive ou negative,

  [(Engl.transl.) we see that the pressure N remains the same in all directions around this point: A, and directed for outward to inward or from inward to outward, according to that the value will be positive or negative,]
  ($\Downarrow$) (then we ought to consider as $\frac{1}{2}$)($\Uparrow$) ;
- from the assumption of isotropy and homogeneity of space, $r^2 = x^2 + y^2 + z^2$, implying $\Sigma \frac{z^2}{r} fr = \Sigma \frac{1}{3} r fr$, (cf. Poisson [60], pp. 32-34):

$$(3\text{-}8)_{Pe} \qquad K \equiv \frac{1}{6\varepsilon^3}\sum rfr = \frac{2\pi}{3}\sum \frac{rfr}{4\pi\varepsilon^3}, \quad k \equiv \frac{1}{30\varepsilon^3}\sum r^3 \frac{d.\frac{1}{r}fr}{dr} = \frac{2\pi}{15}\sum \frac{1}{4\pi\varepsilon^3} r^3 \frac{d.\frac{1}{r}fr}{dr}, \tag{13}$$

"... et étendant les sommes Σ à tous les points matériels du corps qui sont compris dans la sphère d'activité de M. (cf. Poisson [60], p. 46):

[(Engl.transl.) "... and extending the summation Σ to all the material points of the bodies contained in the active sphere of M.]

12($\Downarrow$) $\frac{1}{r'} fr' = \frac{1}{r} fr + (\phi\phi' + \psi\psi' + \theta\theta')\frac{d.\frac{1}{r}fr}{rdr}$ ([60, p.42]).

13($\Downarrow$) In Poisson [60] Chapter 3 is titled "*Calcul des Pressions dans les Corps élastiques ; équations différentielles de l'équilibre et du mouvement de ces Corps.*"

5.3.2. **Fluid pressure in motion**. Poisson's tensor of the pressures in a fluid,[14] which he assumes compressible, reads as follows:

$$(7\text{-}7)_{Pf}\quad \begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix} = \begin{bmatrix} \beta\Big(\frac{du}{dz}+\frac{dw}{dx}\Big) & \beta\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{du}{dx} \\ \beta\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dv}{dy} & \beta\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) \\ p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dw}{dz} & \beta\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) & \beta\Big(\frac{du}{dz}+\frac{dw}{dx}\Big) \end{bmatrix},$$

$$(k+K)\alpha=\beta,\quad (k-K)\alpha=\beta',\quad p=\psi t=K,\quad \Rightarrow\quad \beta+\beta'=2k\alpha,$$

where χt is the density of the fluid around the point M, and ψt is the pressure. Here K and k are the same constants as in $(3\text{-}8)_{P^e}(=(13))$ for an elastic body. Velocity and pressure are defined as follows:

$$\mathbf{u}=(u,v,w)=(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}),\quad \varpi\equiv p-\alpha\frac{d\psi t}{dt}-\frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt},\quad (\varpi\equiv p,\quad \text{if incompressible.})$$

from which follows:

$$\begin{cases} \frac{d^2x}{dt^2}=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}, \\ \frac{d^2y}{dt^2}=\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}, \\ \frac{d^2z}{dt^2}=\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}. \end{cases}$$

$$(7\text{-}9)_{Pf}\quad \begin{cases} \rho(X-\frac{d^2x}{dt^2})=\frac{d\varpi}{dx}+\beta(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}), \\ \rho(Y-\frac{d^2y}{dt^2})=\frac{d\varpi}{dy}+\beta(\frac{d^2u}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2w}{dz^2}), \\ \rho(Z-\frac{d^2z}{dt^2})=\frac{d\varpi}{dz}+\beta(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}) \\ \text{where } \varpi=p+\frac{\alpha}{3}(K+k)(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}), \end{cases} \qquad (14)$$

$$\Rightarrow^*\quad \begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}+\alpha(K+k)\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)+\frac{1}{3}\alpha(K+k)\frac{d}{dx}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0, \\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}+\alpha(K+k)\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)+\frac{1}{3}\alpha(K+k)\frac{d}{dy}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0, \\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}+\alpha(K+k)\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)+\frac{1}{3}\alpha(K+k)\frac{d}{dz}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0, \end{cases}$$

($\Downarrow$) Here, $\alpha(K+k)$ is the constant to the tensor function with the main axis (the normal stress) of Laplacian. $\frac{1}{3}\alpha(K+k)$ corresponds to the coefficient of grad.div term. In today's NS equations, the ratio of coefficient attached to the term of the tensor function with the main axis (the normal stress) of Laplacian to that of grad div : $\frac{\text{coefficient of tensor}}{\text{coefficient of grad div}}=3$, like Poisson deduced in $(7\text{-}9)_{Pf}$ and Stokes' $(12)_S$ through the tensor (15) by Saint-Venant. By Prandtl [64, p.259] in 1934, we had have to wait by the time, when including this ratio of two coefficients, as what is called the NS equations were expressed in fluid equation. cf. Table 7. ($\Uparrow$)

5.4. Saint-Venant's tensor.

Saint-Venant [15] explained that the object of his paper [67] was to simplify the description and calculation of the molecular interactions without specifying the molecular function. We present Saint-Venant's tensor, which from the extract seems to anticipate that of Stokes [67].[16] For this section, we introduce the following parameters: ξ,η,ζ are the velocity components at the arbitrary point m of a fluid in motion in the coordinate directions x,y,z respectively,

P_{xx},P_{yy},P_{zz} are the normal pressures and P_{yz},P_{zx},P_{xy} are the tangential pressures with sub-index pair indicating the perpendicular plane and direction of decomposition.

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[14]($\Downarrow$) In Poisson [60], Chapter 7 is tilted "*Calcul des Pressions dans les Fluides en mouvement ; équations différentielles de ce mouvement.*"

[15]($\Downarrow$) Adhémar Jean-Claude Barré de Saint-Venant (1797-1886).

[16]($\Downarrow$) This is an extract from his main paper, however we can't get this main paper until now. Even in all the list of Saint-Venant's works by Boussinesq and A.Flamant [5] : *Notice sur la vie et les travaux de Barré de Saint-Venant*, it does not appear.

where $\frac{1}{3}\Big(P_{xx}+P_{yy}+P_{zz}\Big)-\frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)=\pi.$
From the last equation, we solve the normal pressure as follows:

$$(2)_{SV}\quad P_{xx}=\pi+2\varepsilon\frac{d\xi}{dx},\quad P_{yy}=\pi+2\varepsilon\frac{d\eta}{dy},\quad P_{zz}=\pi+2\varepsilon\frac{d\zeta}{dz}.$$

From $(1)_{SV}$, obtaining the tangential pressures: P_{yz}, P_{zx}, P_{xy}, the tensor reduces to symmetric form as follows :

$$\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} \pi+2\varepsilon\frac{d\xi}{dx}, & \varepsilon\Big(\frac{d\xi}{dy}+\frac{d\eta}{dx}\Big) & \varepsilon\Big(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\Big) \\ \varepsilon\Big(\frac{d\xi}{dy}+\frac{d\eta}{dx}\Big) & \pi+2\varepsilon\frac{d\eta}{dy} & \varepsilon\Big(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\Big) \\ \varepsilon\Big(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\Big) & \varepsilon\Big(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\Big) & \pi+2\varepsilon\frac{d\zeta}{dz} \end{bmatrix}. \tag{15}$$

Saint-Venant stated that by using his theory, we can obtain concordance with Navier, Cauchy and Poisson:

> Si l'on remplace π par $\varpi-\varepsilon(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz})$, et si l'on substitue les équations $(2)_{SV}$ et $(3)_{SV}$ dans les relations connues entre les pressions et les forces accélératrices, on obtient, en supposant ε le même en tous les points du fluide, les équations différentielles données le 18 mars 1822 par M.Navier (*Mémoires de l'Institut*, t.VI), en 1828 par M.Cauchy (*Exercices de Mathématiques*, p.187) [17], et le 12 octobre 1829 par M.Poisson (même *Mémoire*, p.152) [18]. La quantité variable ϖ ou π n'est autre chose, dans les liquides, que la *pression normale moyenne* en chaque point. [67, p.1243]
>
> [(Engl.transl.) If we replace π with $\varpi-\varepsilon(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz})$, and substitute the quations $(2)_{SV}$ and $(3)_{SV}$ in the known relation between the pressures and accelerated forces, by supposing ε is the same at all points of fluid, then we get the differential equations given by Navier [47] on 18 March, 1822, by Cauchy [7, p.187] in 1828, and by Poisson [60, p.152] on 12 October in 1829. The variable quantity ϖ or π is nothing but the normal average pressure at every point in liquids.]

Saint-Venant's paper [67] seems to provide Stokes with a clue to the notion of tensor (19) and his principle, because we can see the close correspondence by comparing [19] Saint-Venant's t_{ij}:

$$\begin{aligned} t_{ij} &= \big(\pi+2\varepsilon v_{i,j}-\gamma\big)\delta_{ij}+\gamma,\quad (\text{where}\quad \gamma\equiv\varepsilon(v_{i,j}+v_{j,i})),\\ &= \Big(\frac{1}{3}(P_{xx}+P_{yy}+P_{zz})-\frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)+2\varepsilon v_{i,j}-\gamma\Big)\delta_{ij}+\gamma\\ &= (\frac{1}{3}(P_{xx}+P_{yy}+P_{zz})-\frac{2\varepsilon}{3}v_{k,k})\delta_{ij}+\varepsilon(v_{i,j}+v_{j,i})\quad\Leftarrow\quad 2\varepsilon v_{i,j}\delta_{ij}=\varepsilon(v_{i,j}+v_{j,i})\delta_{ij}=\gamma\delta_{ij} \end{aligned} \tag{16}$$

with Stokes' t_{ij} (20). Here, using (16), we put $P_{xx}=P_{yy}=P_{zz}=-p$ by assuming isotropy and homogeneity, [20] which Stokes similarly considers his principle as follows:

> If the molecules of E were in a state of relative equilibrium, the pressure would be equal in all directions about P, as in the case of fluids at rest. Hence I shall assume the following principle:
>
> - that the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure that would exist in all directions about P if the fluid in its neighborhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P, and
> - that the relative motion due to any rotary motion may be eliminated without affecting the differences of the pressures above mentioned.

---

[17]($\Downarrow$) Cauchy [6, p.226]

[18]($\Downarrow$) Poisson [60, p.152] $(7\text{-}9)_{Pf}$.

[19]($\Downarrow$) Here, for the sources of the tensorial descriptions of t_{ij} of Poisson and Cauchy we cite C.Truesdell [75], of Navier in G.Darrigol [11], otherwise in Schlichting [69].

[20]($\Downarrow$) cf. I.Imai [22, p.185].

[74, p.80].

As a consequence, we think that (16) is equivalent to Stokes' t_{ij} as follows. For example, if we put $\varepsilon \equiv \mu$, and choose the t_{xx} component of Saint-Venant's tensor from (15):

$$\begin{aligned} \pi + 2\varepsilon\frac{d\xi}{dx} &= -p + \Big(2 - \frac{2}{3}\varepsilon\frac{d\xi}{dx}\Big) - \frac{2\varepsilon}{3}\Big(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big) = -p + 2\varepsilon\Big\{\frac{2}{3}\frac{d\xi}{dx} - \frac{1}{3}\Big(\frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big)\Big\} \\ &= -p + 2\varepsilon\Big\{\frac{d\xi}{dx} - \frac{1}{3}\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big)\Big\} = -p + 2\varepsilon\Big(\frac{d\xi}{dx} - \delta\Big) \quad (19), \end{aligned}$$

which recovers the P_1 component derived by Stokes. The other tensor components are likewise demonstrated but we omit the proof here for brevity. Moreover, Saint-Venant proposed that if put the following :

$$\pi = \varpi - \varepsilon\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\Big) =^{*} \varpi - \varepsilon v_{k,k}$$

then

$$t_{ij} = (\varpi - \varepsilon v_{k,k} + 2\varepsilon v_{i,j} - \gamma)\delta_{ij} + \gamma = (\varpi - \varepsilon v_{k,k})\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}).$$

This form of his tensor plays a key role in common with Navier's, Cauchy's and Poisson's constants.

5.5. Stokes' equations and tensor.

By expressing the fluid equations in the following form

$$(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\Big(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\Big(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \end{cases} \qquad (17)$$

Stokes pointed out the coincidence with Poisson using the correspondence:
$\varpi = p + \frac{\alpha}{3}(K+k)\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)$ which then gives $\nabla\varpi = \nabla p + \frac{\beta}{3}\nabla(\nabla\cdot\mathbf{u})$. Stokes also commented:

> The same equations have also been obtained by Navier in the case of an incompressible fluid (Mém. de l'Académie, t. VI. p.389) [21],
> but his principles differ from mine still more than do Poisson's. [74, p.77, footnote]

He further stated:

> Observing that $\alpha(K+k) \equiv \beta$, this value of ϖ reduces Poisson's equation $(7\text{-}9)_{Pf}$ (=(14) in our renumbering) to the equation $(12)_S$ of this paper.

Stokes proposed his approximate equations in [74, p.93] :

$$(13)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) = 0, \end{cases} \qquad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \qquad (18)$$

which are identical to $(7\text{-}9)_{Pf}$(=(14)), adding that:

> "These equations are applicable to the determination of the motion of water in pipes and canals, to the calculation of the effect of friction on the motions of tides and waves, and such questions. ([74, p.93]).

Here we shall trace his deduction with the Stokes tensor in the form:

$$\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} p - 2\mu\Big(\frac{du}{dx} - \delta\Big) & -\mu\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & -\mu\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) \\ -\mu\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & p - 2\mu\Big(\frac{dv}{dy} - \delta\Big) & -\mu\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) \\ -\mu\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) & -\mu\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) & p - 2\mu\Big(\frac{dw}{dz} - \delta\Big) \end{bmatrix} \qquad (19)$$

He then remarked about δ :

[21]($\Downarrow$) cf. Navier [47].

"Tt may also be very easily provided directly that the value of 3δ, the rate of cubical dilatation, satisfies the equation

$$3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$$

[74, p.90]

We find that Stokes' tensor can be described compactly in component form as follows:

$$\begin{aligned} -t_{ij} &= \{p - 2\mu(v_{i,j} - \delta) + \gamma\}\delta_{ij} - \gamma, && \Leftarrow \text{where}, \quad \gamma = \mu(v_{i,j} + v_{j,i}), \\ &= \{p - 2\mu v_{i,j}\}\delta_{ij} + \gamma(-\delta_{ij} + \delta_{ij} - 1) && \Leftarrow \text{where}, \quad 2\mu v_{i,j}\delta_{ij} = \mu(v_{i,j} + v_{j,i})\delta_{ij} = \gamma\delta_{ij}, \\ &= (p + 2\mu\gamma)\delta_{ij} - \gamma = (p + \frac{2}{3}\mu v_{k,k})\delta_{ij} - \mu(v_{i,j} + v_{j,i}) \end{aligned} \tag{20}$$

Therefore, the sign of $-t_{ij}$ depends on the location of the tensor in the equation.

[22] Now, in considering the coincidence of (15) with (17), we see that Stokes' tensor may have originated in Saint-Venant's. The article [52] by J.J. O'Connor and E.F. Robertson pointed out this resemblance. Moreover, Stokes reported on the then academic activities within hydromechanics [73], in which he cites Saint-Venant [67] saying:

"I shall therefore suppose that for water, and by analogy for other incompressible fluids." ([74, p.93]).

At any rate, we obtain $(13)_S$ (=(18)) with (19) and the following (21):

$$\begin{cases} \rho\left(\frac{Du}{Dt} - X\right) + \frac{dP_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} = \rho\left(\frac{Du}{Dt} - X\right) + P = 0, \\ \rho\left(\frac{Dv}{Dt} - Y\right) + \frac{dT_3}{dx} + \frac{dP_2}{dy} + \frac{dT_1}{dz} = \rho\left(\frac{Dv}{Dt} - Y\right) + Q = 0, \\ \rho\left(\frac{Dw}{Dt} - Z\right) + \frac{dT_2}{dx} + \frac{dT_1}{dy} + \frac{dP_3}{dz} = \rho\left(\frac{Dw}{Dt} - Z\right) + R = 0, \end{cases}$$
$$\text{where} \quad \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{bmatrix} \tag{21}$$

6. The rapidly decreasing functions including in the "two-constant"

In Table 3, we show the form of g_1 and g_2, which are kernel functions and with which the progenitors of the fluid equation developed their formulae. Here we refer to these functions as rapidly decreasing functions (RDFs). While formulating the equilibrium equations, we obtain the competing theories of "two-constant" in capillary action between Laplace and Gauss.

In 1830, after Laplace's death, Gauss [17] started publishing his studies on capillarity following his famous paper on curved surfaces [15]. In the paper, Gauss criticized Laplace's calculations of 1805-7 in which the "two-constant" in his calculation of capillary action were introduced. At about this time, Gauss had studied what became to be called Gaussian function or *Gaussian curve* and using this as his RDF Gauss criticised Laplace's example function e^{-if} as the equivalent function to $\varphi(f)$. Here, $\varphi(f)$ is the RDF, which depends on distance f. In that paper, Gauss [17] pointed out various deficiencies:

1. Laplace had mentioned only attractive action and without considering the repulsive action;
2. Laplace could not identify the correct example function as the equivalent function of the RDF; and
3. Laplace lacked any proof from say a geometrical point of view.

The following are Gauss' criticisms to Laplace in the preface of [17].

- Judging from the second dissertation : $\prec$ *Supplément à la théorie de l'action capillaire* $\succ$ ([35]), Mr. Laplace had scarcely investigated of φf, not only the complete attraction, but also a part, and tacitly understood incompletely the general attraction ; by the way, if we would refer the latter in comparison with our sensible modification, on the contrary, we can assert it to be more inferior to the bad experiments and be clearly visible.

---

[22]($\Downarrow$) Schlichting reverses the sign of Stokes' tensor as follows: $\sigma_{ij} = -p\delta_{ij} + \mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}$ [69, p.58, in the footnote].

> But it is not at all necessary to limit the generality by such a large quantity, this point is more clear than words, we would see easiest, only by investigating if these integrations would be able to be extended, not only at infinity but also at an arbitrary sensible distance, or if the occurring in experiment could be wider extended up to the finitely measurable distance. [17, p.33]

Here, we can consider these arguments on the RDFs as simple examples of today's distribution and hyperfunction of Schwartz in 1954/55, but which were popular in the 1830s, during the time the NS equations were being discussed in their microscopically-descriptive formulation.

However, Gauss' criticisms in 1830 naturally drew no rebuttal. We present a sketch of these assertions on the RDFs in Tables 8 and 9 in their original, cross-indexed narratives, where, we show the then family of RDF by using our notation $f \in \mathcal{RDF}$, and f is a function included in the two-constant belonging to the then rapidly decreasing function.

7. Conclusions

The "two-constant" theory is the currently-accepted theory for isotropic, homogeneous, linear elasticity. (Darrigol[11, p.121]); the terminology : "two-constant" theory is due to Prof. Darrigol. In our report, we provided a universal expression of this theory within a historical context and identified the following features:

(1) the "two-constant" were defined in terms of kernel functions of $RDFs$, describing the characteristics of dissipation or diffusion within isotropic and homogeneous fluids that were necessary for the interpretation of the nature of fluid or the formulation of the equations of the fluid mechanics including kinetics, equilibrium and capillarity. With their origin perhaps arising in the work of Laplace in 1805, these sorts of functions are simple examples of today's distribution and hyperfunction of Schwartz proposed in 1954/55.

(2) the genealogy of tensor as it pertains to the development of the NS equations in the original mathematical formulations;

(3) the tensors and the corresponding equations as developed historically by Navier (1822), Cauchy (1828), Poisson (1829), Saint-Venant (1843) and Stokes (1845) (sic. in order); and finally

(4) the appearance of the notion of tensors especially in the work of Saint-Venant.

(5) Gauss [17] also contributed to develop fundamental conception of RDF or $MDNS$ equations for fluid mechanics including capillary action, because he formulated the equations with two-function instead of two-constant and these were the superior method than other contemporaries with the progenitors of NS equations.

(6) According to Bolza [3], Gauss [17] had broken one of the neck of fundamental problems, such as *multiple integral* and *calculus of variations*, however, we must recognize that even he owed the latter to its progenitor Lagrange, and calculation of capillarity to its progenitor Laplace.

It is our contention that Saint-Venant's was an epoch-making contribution, by simplifying and identifying the concordance between the earlier pioneers of the $MDNS$ equations, in using only tensors without recourse to the microscopical descriptions, and providing context for the contribution of Stokes.

TABLE 1. C_1, C_2, C_3, C_4: definitions of constants and computation of total momentum of molecular actions by Navier, Cauchy, Poisson, Saint-Venant & Stokes

no	name/problem	elastic solid	elastic fluid	remark
1	Navier elastic:[46] fluid:[47]	(Navier[46] only:) $C_1 = \varepsilon \equiv \frac{2\pi}{15}\int_0^\infty d\rho \cdot \rho^4 f\rho$ $C_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} \cos\varphi d\varphi \ g_3 \Rightarrow \{\frac{16}{15}, \frac{4}{15}, \frac{2}{5}\}$ $\Rightarrow \frac{1}{2}\frac{\pi}{4}\frac{16}{15} = 2\pi \ over 15$	(Navier[47] only:) $C_1 = \varepsilon \equiv \frac{2\pi}{15}\int_0^\infty d\rho \cdot \rho^4 f(\rho)$ $C_2 = E \equiv \frac{2\pi}{3}\int_0^\infty d\rho \cdot \rho^2 F(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi \ g_3$ $\Rightarrow \{\frac{\pi}{10}, \frac{\pi}{30}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi \ g_4 \Rightarrow \frac{2\pi}{3}$	$\alpha = \rho\cos\psi\cos\varphi$, $\beta = \rho\cos\psi\sin\varphi$, $\gamma = \rho\sin\psi$
2	Cauchy elastic and fluid[7]	(Cauchy[7]) $C_1 = R = \frac{2\pi\Delta}{15}\int_0^\infty r^3 f(r)dr$ $= \pm\frac{2\pi\Delta}{15}\int_0^\infty [r^4 \mathrm{f}'(r) - r^3\mathrm{f}(r)]dr$ $C_2 = G = \pm\frac{2\pi\Delta}{3}\int_0^\infty r^3 \mathrm{f}(r)dr$ $C_3 = \frac{1}{2}\int_0^{2\pi}\cos^2 qdq\int_0^\pi \cos^2\alpha\cos^2\beta dp$, $= \frac{1}{2}\int_0^{2\pi}\cos^2 qdq\int_0^\pi \cos^2 p\sin^2 p\sin pdp = \frac{2\pi}{15}$, $C_4 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi \cos^2\alpha\sin pdqdp$ $= \pi\int_0^\pi \cos^2 p\sin \ pdp = \frac{2\pi}{3}$,	(Cauchy[7]) same as elastic solid	$\cos\alpha = \cos p$, $\cos\beta = \sin p\cos q$, $\cos\gamma = \sin p\sin q$ $\Delta = \frac{\mathcal{M}}{\mathcal{V}}$: mass of molecules per volume.
3	Poisson elastic: [59, 60] fluid:[60]	(Poisson[59] only:) $C_1 = k \equiv \frac{2\pi}{15}\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}$ $C_2 = K \equiv \frac{2\pi}{3}\sum\frac{r^3}{\alpha^5}fr$ $C_3 = \int_0^{2\pi} d\gamma\int_0^{\frac{\pi}{2}}\cos\beta\sin\beta d\beta \ g_3 \Rightarrow \{\frac{2\pi}{5}, \frac{2\pi}{15}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{2\pi} d\gamma\int_0^{\frac{\pi}{2}}\cos\beta\sin\beta d\beta \ g_4 \Rightarrow \frac{2\pi}{3}$ Remark: C_3 is chosen as the common factor of $\{\cdot,\cdot\}$	(Poisson[60] both elastic and fluid) $C_1 = -k \equiv -\frac{1}{30\varepsilon^3}\sum r^3\frac{d.\frac{1}{r}fr}{dr}$ $= -\frac{2\pi}{15}\sum\frac{r^3}{4\pi\varepsilon^3}\frac{d.\frac{1}{r}fr}{dr}$ $C_2 = -K \equiv -\frac{1}{6\varepsilon^3}\sum rfr$ $= -\frac{2\pi}{3}\sum\frac{r}{4\pi\varepsilon^3}fr$ $C_3: \begin{cases} G = \frac{1}{10}\sum r^3\frac{d.\frac{1}{r}fr}{dr}, \\ E = F = \frac{1}{30}\sum r^3\frac{d.\frac{1}{r}fr}{dr} \end{cases}$ $\Rightarrow \{\frac{1}{10}, \frac{1}{30}\} \Rightarrow \frac{1}{30}$ $C_4: (3\text{-}2)_{Pf} \quad N = \frac{1}{6\varepsilon^3}\sum rfr \Rightarrow \frac{1}{6}$	In Poisson[60], he treats both elastic and fluid the same. $x_1 = r\cos\beta\cos\gamma$, $y_1 = r\sin\beta\sin\gamma$, $\zeta = -r\cos\beta$
4	Saint-Venant [67]		$C_1 = \varepsilon, \quad C_2 = \frac{\varepsilon}{3}$	
5	Stokes[74]	$C_1 = A, \quad C_2 = B$	$C_1 = \mu, \quad C_2 = \frac{\mu}{3}$	

TABLE 2. C_1, C_2 and equation of equilibrium of fluid containing exact differential by Poisson & Navier

no	name	C_1, C_2 of equilibrium	equation of equilibrium with exact differential term
1	Poisson [60]	$C_1 = -q \equiv \frac{1}{4\varepsilon^3}\sum\frac{r_1^2 z' R}{r}$ $C_2 = p \equiv \frac{1}{6\varepsilon^3}\sum rR$	$N = p + q\left(\frac{1}{\lambda} + \frac{1}{\lambda'}\right)$ where N : the vertical force, λ, λ' : the radii of the principal curvature
2	Navier fluid [47]	$C_1 = p \equiv \frac{4\pi}{3}\int_0^\infty d\rho\rho^3 f(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\psi\int_0^{\frac{\pi}{2}} d\varphi g_3$ $\Rightarrow \{\frac{2}{3}, \frac{1}{3}, \frac{\pi}{4}\} \Rightarrow \frac{8\pi}{6} = \frac{4\pi}{3}$	$0 = \iiint dxdydz\left[p\left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz}\right) + P\delta x + Q\delta y + R\delta z\right]$. By integration by parts $0 = \iiint dxdydz\left[\left(P - \frac{dp}{dx}\right)\delta x + \left(Q - \frac{dp}{dy}\right)\delta y + \left(R - \frac{dp}{dz}\right)\delta z\right]$ $-\iint dydz\left(p'\delta x' - p''\delta x''\right) - \iint dxdz\left(p'\delta y' - p''\delta y''\right) - \iint dxdy\left(p'\delta z' - p''\delta z''\right)$, $\Rightarrow$. condition of inner point and exact differential $\frac{dp}{dx} = P, \quad \frac{dp}{dy} = Q, \quad \frac{dp}{dz} = R. \quad \Rightarrow \quad dp = Pdx + Qdy + Rdz$ $\Rightarrow$. boundary condition and relation of variation $\delta x, \delta y, \delta z$ $0 = Pdx + Qdy + Rdz \quad \Rightarrow \quad 0 = \delta x\cos l + \delta y\cos m + \delta z\cos n$

TABLE 3. The expression of the total momentum of molecular actions by Laplace, Navier, Cauchy, Poisson, Saint-Venant & Stokes. (Remark. 6-8 : capillarity, 9-10 : equilibrium, else : kinetic equation)

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Navier 1827 [46]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$		ρ : radius
2	Navier fluid 1827 [47]	motion of fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$		ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$	
3	Cauchy 1828 [7]	system of particles	R		$\frac{2\pi}{15}$		$\int_0^\infty dr$	r^3		$f(r)$		$f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)]$
				G		$\frac{2\pi}{3}$	$\int_0^\infty dr$		r^3		f(r)	$f(r) \neq \mathrm{f}(r)$
4	Poisson 1829 [59]	elastic solid	k		$\frac{2\pi}{15}$		$\sum \frac{1}{\alpha^5}$	r^5		$\frac{d.\frac{1}{r}fr}{dr}$		
				K		$\frac{2\pi}{3}$	$\sum \frac{1}{\alpha^5}$		r^3		fr	
5	Poisson 1831 [60]	motion of fluid	k		$\frac{1}{30}$		$\sum \frac{1}{\varepsilon^3}$	r^3		$\frac{d.\frac{1}{r}fr}{dr}$		$C_3 = \frac{1}{4\pi}\frac{2\pi}{15} = \frac{1}{30}$
				K		$\frac{1}{6}$	$\sum \frac{1}{\varepsilon^3}$		r		fr	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
6	Laplace 1806,7 [37]	capillary action	H		2π		$\int_0^\infty dz$	z		$\Psi(z)$		z : distance
				K		2π	$\int_0^\infty dz$				$\Psi(z)$	
6-2	Rewritten by Poisson 1831 [62]		H		$\frac{\pi}{4}\rho^2$		$\int_0^\infty dr$	r^4		φr		[62, pp.14-15]
				K		$\frac{2\pi}{3}\rho^2$	$\int_0^\infty dr$		r^3		φr	
7	Gauss 1830 [17]	capillary action										attraction : $-fx.dx = d\varphi x$, $\int fx.dx = -\varphi x$, repulsion : $-Fx.dx = d\Phi x$, $\int Fx.dx = -\Phi x$
8	Poisson 1831 [62]	capillary action	H		$\frac{\pi}{4}\rho^2$		$\int_0^\infty dr$	r^4		φr		[62, p.14]
				K		$\frac{2\pi}{3}\rho^2$	$\int_0^\infty dr$		r^3		φr	[62, p.12]
9	Navier fluid 1827 [47]	equilibrium of fluid	p		$\frac{4\pi}{3}$		$\int_0^\infty d\rho$	ρ^3		$f(\rho)$		ρ : radius
10	Poisson 1831 [60]	equilibrium of fluid	q		$\frac{1}{4}$		$\sum \frac{1}{\varepsilon^3}$	$\frac{1}{r}$		$r_i^2 z' R$		$C_3 = \frac{1}{4\pi}\pi = \frac{1}{4}$
				p		$\frac{1}{6}$	$\sum \frac{1}{\varepsilon^3}$		r		R	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
11	Saint-Venant 1843 [67]	fluid	ε	$\frac{\varepsilon}{3}$								
12	Stokes 1849 [74]	fluid	μ	$\frac{\mu}{3}$								
13	Stokes 1849 [74]	elastic solid	A	B								$A = 5B$

TABLE 4. S_1, S_2, g_3, g_4 : the trianglar functions for calculation of total momentum of molecular actions in unit sphere by Poisson, Navier, Cauchy & Stokes

<table>
<tr><th>no</th><th>name</th><th>S_1, S_2, g_3, g_4</th></tr>
<tr><td>1</td><td>Poisson</td><td>

$\cdot g_3$ and g_4 are in the following tensor :

$$\begin{cases} g = a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta, & g' = g\frac{du}{dx} + h\frac{du}{dy} + l\frac{du}{dz}, \\ h = a'\sin\beta\cos\gamma + b'\sin\beta\sin\gamma - c'\cos\beta, & h' = g\frac{dv}{dx} + h\frac{dv}{dy} + l\frac{dv}{dz}, \\ l = a''\sin\beta\cos\gamma + b''\sin\beta\sin\gamma - c''\cos\beta, & l' = g\frac{dw}{dx} + h\frac{dw}{dy} + l\frac{dw}{dz} \end{cases}$$

$$\begin{cases} P = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(g+g')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')g\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ Q = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(h+h')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')h\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ R = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(l+l')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')l\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \end{cases}$$

i.e.

$$\Rightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\Delta\Big(\begin{bmatrix} g+g' & (gg'+hh'+ll')g \\ h+h' & (gg'+hh'+ll')h \\ l+l' & (gg'+hh'+ll')l \end{bmatrix}\begin{bmatrix} \sum\frac{r^3 fr}{\alpha^5} \\ \sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr} \end{bmatrix}\Big)$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} d\beta\, d\gamma\Big(\begin{bmatrix} g_4 & g_3 \end{bmatrix}\begin{bmatrix} K' \\ k' \end{bmatrix}\Big),$$

where $\Delta := \cos\beta\cdot\sin\beta\, d\beta\, d\gamma$, $K' := \sum\frac{r^3 fr}{\alpha^5}$, $k' := \sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}$.

$\cdot S_1$ and S_2 are given above.

</td></tr>
<tr><td>2</td><td>Navier elastic solid (1827)[46]</td><td>

$\cdot g_3$:

$g_3 = \frac{1}{2}\delta f^2$

$$f \equiv \rho\Big[\frac{dx}{da}\cos^2\psi\cos^2\varphi + \big(\frac{dx}{db} + \frac{dy}{da}\big)\cos^2\psi\sin\varphi\cos\varphi + \big(\frac{dx}{dc} + \frac{dz}{da}\big)\cos\psi\sin\psi\cos\varphi + \frac{dy}{db}\cos^2\psi\sin^2\varphi + \big(\frac{dy}{dc} + \frac{dz}{db}\big)\sin\psi\cos\psi\sin\varphi + \frac{dz}{dc}\sin^2\psi\Big].$$

</td></tr>
<tr><td>3</td><td>Navier fluid (1827)[47]</td><td>

$\cdot g_3$:

$\alpha = \rho\cos\psi\cos\varphi, \quad \beta = \rho\cos\psi\sin\varphi, \quad \gamma = \rho\sin\psi$

$$g_3 = V\delta V = [\alpha\Big(\frac{du}{dx}\alpha + \frac{du}{dy}\beta + \frac{du}{dz}\gamma\Big) + \beta\Big(\frac{dv}{dx}\alpha + \frac{dv}{dy}\beta + \frac{dv}{dz}\gamma\Big) + \gamma\Big(\frac{dw}{dx}\alpha + \frac{dw}{dy}\beta + \frac{dw}{dz}\gamma\Big)\ ] \times$$
$$[\alpha\Big(\frac{\delta du}{dx}\alpha + \frac{\delta du}{dy}\beta + \frac{\delta du}{dz}\gamma\Big) + \beta\Big(\frac{\delta dv}{dx}\alpha + \frac{\delta dv}{dy}\beta + \frac{\delta dv}{dz}\gamma\Big) + \gamma\Big(\frac{\delta dw}{dx}\alpha + \frac{\delta dw}{dy}\beta + \frac{\delta dw}{dz}\gamma\Big)\ ],$$

$\cdot g_4$:

$$g_4 = V\delta\ V = \begin{cases} \alpha'^2 & \{\ (u\sin^2 r - v\sin r\cos r)\delta u, \\ & \quad (-u\sin r\cos r + v\cos^2 r)\delta v\ \}, \\ \beta'^2 & \{\ (u\cos^2 r\sin^2 s + v\sin r\cos r\sin^2 s + w\cos r\sin s\cos s)\delta u, \\ & \quad (u\sin r\cos r\sin^2 s + v\sin^2 r\sin^2 s + w\sin r\sin s\cos s)\delta v, \\ & \quad (u\cos r\sin s\cos s + v\sin r\sin s\cos s + w\cos^2 s\qquad)\delta w\ \}, \\ \gamma'^2 & \{\ (u\cos^2 r\cos^2 s + v\sin r\cos r\cos^2 s - w\cos r\sin s\cos s)\delta u, \\ & \quad (u\sin r\cos r\cos^2 s + v\sin^2 r\cos^2 s - w\sin r\sin s\cos s)\delta v, \\ & \quad (-u\cos r\sin s\cos s - v\sin r\sin s\cos s + w\sin^2 s\qquad)\delta w\ \} \end{cases}$$

where $\alpha' = \rho\cos\psi\cos\varphi$, $\beta' = \rho\cos\psi\sin\varphi$, $\gamma' = \rho\sin\psi$.

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<tr><td>3</td><td>Cauchy (1828)[7]</td><td>

$\cdot g_3 = g_4 = \frac{v}{2}$:

$$(44)_C \quad \begin{cases} G = G\big(\cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1\big) \equiv\ GA_1, \\ L = L\big(\cos^4\alpha_1 + \cos^4\beta_1 + \cos^4\gamma_1\big) \\ \quad +6R\big(\cos^2\beta_1\cos^2\gamma_1 + \cos^2\gamma_1\cos^2\alpha_1 + \cos^2\alpha_1\cos^2\beta_1\big) \equiv LB + 6RC, \\ R = R\big(\cos^2\beta_1\cos^2\gamma_2 + \cos^2\beta_2\cos^2\gamma_1 + \cos^2\gamma_1\cos^2\alpha_2 \\ \quad + \cos^2\gamma_2\cos^2\alpha_1 + \cos^2\alpha_1\cos^2\beta_2\cos^2\alpha_2\cos^2\beta_1\big) \\ \quad +4R\big(\cos\beta_1\cos\beta_1\cos\gamma_1\cos\gamma_2 + \cos\gamma_1\cos\gamma_2\cos\alpha_1\cos\alpha_2 \\ \quad + \cos\alpha_1\cos\alpha_2\cos\beta_1\cos\beta_2\big) \\ \quad +L\big(\cos^2\alpha_1\cos^2\alpha_2 + \cos^2\beta_1\cos^2\beta_2 + \cos^2\gamma_1\cos^2\gamma_2\big) \equiv RD + 4RE + LF \end{cases}$$

$$\text{where} \quad \begin{cases} \cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1 = 1, \quad \cos^2\alpha_2 + \cos^2\beta_2 + \cos^2\gamma_2 = 1, \\ \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0 \end{cases}$$

$\cdot$ From $(49)_C$ $\quad G = \frac{\Delta}{2}\mathbf{S}[\pm r\cos^2\alpha \mathrm{f}(r)v]$, $\quad R = \frac{\Delta}{2}\mathbf{S}[r\cos^2\alpha\cos^2\beta f(r)v]$

$$\text{and} \quad (50)_C \quad \begin{cases} G = \pm\frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3\mathrm{f}(r)\cos^2\alpha\sin p dr dq dp, \\ R = \frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3 f(r)\cos^2\alpha\cos^2\beta\sin p dr dq dp \end{cases}$$

$$\begin{cases} S_1 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi\cos^2\alpha\cos^2\beta\sin\ pdp = \frac{1}{2}\int_0^{2\pi}\cos^2 q dq\int_0^\pi cos^2 p(1-\cos^2 p)\sin p dp \\ \quad = \frac{1}{2}\pi(\frac{2}{3} - \frac{2}{5}) = \frac{2\pi}{15} \equiv C_3, \\ S_2 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi\cos^2\alpha\sin p dq dp = \frac{1}{2}2\pi\int_0^\pi\cos^2 p\sin\ pdp = \frac{2\pi}{3} \equiv C_4. \end{cases}$$

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TABLE 5. Concurrences and variations of tensors

1	name/ problem	tensor (3×3)	coefficient matrix (3 × 5) in equations
1-1	Navier elasticity	$t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(5\text{-}4)_{N^e}$ $-\varepsilon\begin{bmatrix} \left(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ \left(\frac{dw}{dx}+\frac{du}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \left(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\right) \end{bmatrix}$ $= -\varepsilon\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix},$ where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	We define the coefficient matrix in elasticity : C_T^e as follows: C_T^e : the coefficient of $\begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial z^2} & \frac{\partial^2 v}{\partial x\partial y} & \frac{\partial^2 w}{\partial z\partial x} \\ \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial y^2} & \frac{\partial^2 v}{\partial z^2} & \frac{\partial^2 w}{\partial y\partial z} & \frac{\partial^2 u}{\partial x\partial y} \\ \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial y^2} & \frac{\partial^2 w}{\partial z^2} & \frac{\partial^2 u}{\partial z\partial x} & \frac{\partial^2 v}{\partial y\partial z} \end{bmatrix},$ then $(6\text{-}1)_{N^e} \quad \Rightarrow \quad C_T^e = -\varepsilon\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix} \quad \Rightarrow \quad (23)$
1-2	Navier fluid	$t_{ij} = (p - \varepsilon u_{k,k})\delta_{ij} - \varepsilon(u_{i,j} + u_{j,i})$ (23) $\begin{bmatrix} \epsilon'-2\varepsilon\frac{du}{dx} & -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) \\ -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & \epsilon'-2\varepsilon\frac{dv}{dy} & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) \\ -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) & \epsilon'-2\varepsilon\frac{dw}{dz}) \end{bmatrix},$ where $\epsilon' = p - \varepsilon(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz})$	Similarly, we define the coefficient matrix in fluid : C_T^f, which contains p in (1,1)-, (2,2)- and (3,3)-element. $(23) \quad \Rightarrow \quad C_T^f = \begin{bmatrix} p-3\varepsilon & -\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & p-3\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & -\varepsilon & p-3\varepsilon & -2\varepsilon & -2\varepsilon \end{bmatrix}$
2	Cauchy system (contains both elastic body and fluid)	$t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(60)_C$ $\begin{bmatrix} k\frac{\partial\xi}{\partial a}+K\nu & \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) \\ \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & k\frac{\partial\eta}{\partial b}+K\nu & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) \\ \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) & k\frac{\partial\zeta}{\partial c}+K\nu \end{bmatrix},$ where $\nu = \frac{\partial\xi}{\partial a}+\frac{\partial\eta}{\partial b}+\frac{\partial\zeta}{\partial c}$	$(46)_C \Rightarrow C_T^e = \begin{bmatrix} L & R & Q & 2R & 2Q \\ R & M & P & 2P & 2R \\ Q & P & N & 2Q & 2P \end{bmatrix}$ $\Rightarrow R\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix},$ where $P = Q = R, \quad L = M = N, \quad L = 3R.$
3-1	Poisson elasticity	$t_{ij} = -\frac{a^2}{3}(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(6)_{P^e}$ $-\frac{a^2}{3}\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix},$ where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	$(6)_{P^e}$ $\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2}+\frac{2}{3}\frac{d^2v}{dydx}+\frac{2}{3}\frac{d^2w}{dzdx}+\frac{1}{3}\frac{d^2u}{dy^2}+\frac{1}{3}\frac{d^2u}{dz^2}\right) \\ = 0, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2}+\frac{2}{3}\frac{d^2u}{dxdy}+\frac{2}{3}\frac{d^2w}{dzdy}+\frac{1}{3}\frac{d^2v}{dx^2}+\frac{1}{3}\frac{d^2v}{dz^2}\right) \\ = 0, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2}+\frac{2}{3}\frac{d^2u}{dxdz}+\frac{2}{3}\frac{d^2v}{dydz}+\frac{1}{3}\frac{d^2w}{dx^2}+\frac{1}{3}\frac{d^2w}{dy^2}\right) \\ = 0, \end{cases}$ $\Rightarrow C_T^e = -\frac{a^2}{3}\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix}$
3-2	Poisson fluid	$t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(7\text{-}7)_{Pf}$ $\begin{bmatrix} \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) & \pi+2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \pi+2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) \\ \pi+2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) \end{bmatrix},$ where $\pi = p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt}$	$(7\text{-}9)_{Pf} \quad \Rightarrow \quad C_T^f = \begin{bmatrix} \varpi+\beta & \beta & \beta & 0 & 0 \\ \beta & \varpi+\beta & \beta & 0 & 0 \\ \beta & \beta & \varpi+\beta & 0 & 0 \end{bmatrix}.$ According to Stokes: if we put $\varpi = p + \frac{\alpha}{3}(K+k)\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)$ $\Rightarrow \quad C_T^f = \begin{bmatrix} p+\frac{4\beta}{3} & \beta & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & p+\frac{4\beta}{3} & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & \beta & p+\frac{4\beta}{3} & \frac{\beta}{3} & \frac{\beta}{3} \end{bmatrix}$ $\Rightarrow (12)_S$ (= (120)). Remark: $\alpha(K+k) = \beta$.
4	Saint-Venant fluid	$t_{ij} = (\frac{1}{3}(P_{xx}+P_{yy}+P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j}+v_{j,i})$ $= (-p - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j}+v_{j,i})$ $\begin{bmatrix} \pi+2\varepsilon\frac{d\xi}{dx}, & \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) \\ \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \pi+2\varepsilon\frac{d\eta}{dy} & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) \\ \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) & \pi+2\varepsilon\frac{d\zeta}{dz} \end{bmatrix},$ where $\pi = \frac{1}{3}\left(P_{xx}+P_{yy}+P_{zz}\right) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$ $\equiv -p - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right) \quad (117)$	No description in [67], however, we can see easily that, for example, in case of P_1, $P_1 = \pi + 2\varepsilon\frac{d\xi}{dx} = -p - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right) + 2\varepsilon\frac{d\xi}{dx}$ $= -p + \frac{4\varepsilon}{3}\frac{d\xi}{dx} - \frac{2\varepsilon}{3}\left(\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right),$ others are in the same way. $\Rightarrow \quad C_T^f = \begin{bmatrix} -p+\frac{4\varepsilon}{3} & \varepsilon & \varepsilon & \frac{\varepsilon}{3} & \frac{\varepsilon}{3} \\ \varepsilon & -p+\frac{4\varepsilon}{3} & \varepsilon & \frac{\varepsilon}{3} & \frac{\varepsilon}{3} \\ \varepsilon & \varepsilon & -p+\frac{4\varepsilon}{3} & \frac{\varepsilon}{3} & \frac{\varepsilon}{3} \end{bmatrix}$
5	Stokes fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j}+v_{j,i}),$ tensor $= -1 \times$ $\begin{bmatrix} p-2\mu\left(\frac{du}{dx}-\delta\right) & -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-2\mu\left(\frac{dv}{dy}-\delta\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-2\mu\left(\frac{dw}{dz}-\delta\right) \end{bmatrix}$ where $3\delta = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} \quad (124)$	$(12)_S \quad \Rightarrow \quad C_T^f = \begin{bmatrix} -p+\frac{4\mu}{3} & \mu & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & -p+\frac{4\mu}{3} & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & \mu & -p+\frac{4\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} \end{bmatrix}$ $\Rightarrow$ (124). Remark: $\frac{4}{3}\mu = 2\mu(1-\frac{1}{3})$

TABLE 6. Concurrences and variations of tensors (continued from Table 5)

1	name/ problem	tensor (3×3)
6	Maxwell (1865-66) [43]fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i}),$ $\begin{bmatrix} p - \frac{M}{9k\rho\Theta_2}p\left(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - \frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx} - 2\frac{dv}{dy} - \frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - \frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx} - \frac{dv}{dy} - 2\frac{dw}{dz}\right) \end{bmatrix}$
7	Kirchhoff (1876)[23] fluid	$t_{ij} = (-p - 2kv_{l,l})\delta_{ij} + k(v_{i,j} + v_{j,i}),$ $\begin{bmatrix} p - 2k\frac{\partial u}{\partial x} & -k\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -k\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ -k\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - 2k\frac{\partial v}{\partial y} & -k\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ -k\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -k\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - 2k\frac{\partial w}{\partial z} \end{bmatrix}$
8	Boltzmann (1895)[2] fluid	$t_{ij} = (-p - \frac{2}{3}\mathcal{R}v_{k,k})\delta_{ij} + \mathcal{R}(v_{i,j} + v_{j,i}),$ $\begin{bmatrix} p - 2\mathcal{R}\left\{\frac{\partial u}{\partial x} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - 2\mathcal{R}\left\{\frac{\partial v}{\partial y} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - 2\mathcal{R}\left\{\frac{\partial w}{\partial z} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} \end{bmatrix}$ where, $\mathcal{R} = \frac{M}{6k\rho\Theta_2}p.$

TABLE 7. The kinetic equations of the hydrodynamics until the "Navier-Stokes equations" were fixed. (Rem. HD : hydrodynamics, N under entry-no : non-linear, gr.dv : grad.div, E : $\frac{\Delta}{gr.dv}$ in elastic, F : $\frac{\Delta}{gr.dv}$ in fluid. The group of entry 6-14 show $F = 3$ in fluid.)

no	name/prob	the kinetic equations	Δ	gr.dv	E	F
1 N	Euler (1752-55) [12, p.127] fluid	$\begin{cases} X - \frac{1}{h}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ Y - \frac{1}{h}\frac{dp}{dy} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ Z - \frac{1}{h}\frac{dp}{dz} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}, \end{cases} \qquad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$				
2	Navier (1827)[46] elastic solid	$(6\text{-}1)_{N^e} \quad \begin{cases} \frac{\Pi}{g}\frac{d^2x}{dt^2} = \varepsilon\left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dbda} + 2\frac{d^2z}{dcda}\right), \\ \frac{\Pi}{g}\frac{d^2y}{dt^2} = \varepsilon\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dcdb}\right), \\ \frac{\Pi}{g}\frac{d^2z}{dt^2} = \varepsilon\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right) \end{cases}$ where Π is density of the solid, g is acceleration of gravity.	ε	2ε	$\frac{1}{2}$	
3 N	Navier (1827)[47] fluid	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\,; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\,; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\,; \end{cases}$	ε	2ε		$\frac{1}{2}$
4	Cauchy (1828)[7] system of particles in elastic solid and fluid	$\begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2}, \end{cases}$ $G = H = I, \quad L = M = N, \quad P = Q = R, \quad L = 3R$	$R+G$	$2R$	if $G=0$ $\frac{1}{2}$	if $G=0$ $\frac{1}{2}$
5	Poisson (1831)[60] elastic solid defined in general equations	$\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2u}{dx^2}, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2v}{dy^2}, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = \frac{\Pi}{\rho}\frac{d^2w}{dz^2}, \end{cases}$	$\frac{a^2}{3}$	$\frac{2a^2}{3}$	$\frac{1}{2}$	
6	Poisson (1831)[60] fluid defined in general equations	$\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K+k)\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K+k)\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K+k)\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \end{cases}$ $\begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) \end{cases}$ where $\varpi \equiv p - \alpha\frac{d\psi t}{dt} - \frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \quad \beta \equiv \alpha(K+k)$	β	$\frac{\beta}{3}$		3
7	Saint-Venant (1843)[67] fluid	He didn't describe the equations in [67], however his tensor is in Table 5 (4).	ε	$\frac{\varepsilon}{3}$		3
8	Stokes (1849)[74] fluid	$(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0. \end{cases}$	μ	$\frac{\mu}{3}$		3
9	Maxwell (1865-66) [43] HD	$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{dp}{dx} - C_M\left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{dp}{dy} - C_M\left[\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{dp}{dz} - C_M\left[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Z \end{cases} \quad \text{where, } C_M \equiv \frac{pM}{6k\rho\Theta_2}$	C_M	$\frac{C}{3}$		3
10	Kirchhoff (1876)[23] HD	$\begin{cases} \mu\frac{du}{dt} + \frac{\partial}{\partial x} - C_K\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu X, \\ \mu\frac{dv}{dt} + \frac{\partial p}{\partial y} - C_K\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Y, \\ \mu\frac{dw}{dt} + \frac{\partial p}{\partial z} - C_K\left[\Delta z + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Z, \end{cases} \quad \begin{cases} \frac{1}{\mu}\frac{d\mu}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ \text{where, } C_K \equiv \frac{1}{3\kappa}\frac{p}{\mu} \end{cases}$	C_K	$\frac{\Delta}{3}$		3
11 N	Rayleigh (1883)[65] HD	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = -\frac{du}{dt} + \nu\nabla^2 u - u\frac{du}{dx} - v\frac{du}{dy}, \\ \frac{1}{\rho}\frac{dp}{dy} = -\frac{dv}{dt} + \nu\nabla^2 v - u\frac{dv}{dx} - v\frac{dv}{dy} \end{cases} \quad , \quad \frac{du}{dx} + \frac{dv}{dy} = 0$	ν			
12	Boltzmann (1895)[2] HD	$(221)_B \quad \begin{cases} \rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \mathcal{R}\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} - \mathcal{R}\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} - \mathcal{R}\left[\Delta w + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Z \end{cases}$	$\mathcal{R}$	$\frac{\mathcal{R}}{3}$		3
13 N	Prandtl (1905)[63] HD	$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + \nabla(V+p) = k\nabla^2 v, \quad \text{div}\ \mathbf{v} = 0$	k			
14 N	Prandtl (1934)[64] HD	$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$ for incompressible, it is simplified as follows : div $\mathbf{w} = 0$, $\quad \frac{D\mathbf{w}}{dt} = g - \frac{1}{\rho}\text{grad}\ p + \nu\Delta\mathbf{w}$	ν	$\frac{\nu}{3}$		3

TABLE 8. Assertions on the *RDF*s by Laplace, Fourier, Poisson, Navier, Cauchy and Gauss

no	Name/Problem/Bibl./ Corresp. (with/against disputer)	Assertions
1	Laplace (1749-1827) capillary action : [37]	•(L_1) We shall put, as in $\Pi(f) = c' - \int_0^f df.\varphi(f)$, the integral $\int df.\varphi(f)$ being taken from $f = 0$, and c being its value when f is infinite. $\Pi(f)$ will be a positive quantity, decreasing with extreme rapidity; and we shall have, by taking the integrals from $f = 0$ to infinity ; $\int f^4 df.\varphi(f) = -f^4.\Pi(f) + 4\int f^3 df.\Pi(f)$. $\quad -f^4.\Pi(f) = 0$ when $f = \infty$; for although f^4 then becomes infinite, the extreme rapidity with which $\Pi(f)$ is supposed to decrease, renders $f^4.\Pi(f)$ nothing. (cf, Navier (N_4)). •(L_2) The functions $\varphi(f)$ and $\Pi(f)$ may be very well compared with exponentials like c^{-if}; c being the number whose hyperbolic logarithm is unity, and i being a very great positive and integral number, where, f is used for the distance between two molecules by Laplace. [37, pp.812-813]
2	Fourier (1768-1830) heat : [13, 14]	•(F_1) On désigne par $\varphi(p)$ une certain fonction de la distance p a une grandeur solides et dans les liquides, devient nulle lorsque p a une grandeur sensible. [13, §59] •(F_2) On voit par ce résultat que la température des différents points de l'axe décroit rapidement à mesure qu' on s'éloigne de l'origin. [13, §332]
3	Poisson (1781-1840) elastic:[56, 59, 60] fluid:[60] The origin disputed :[56] (with Navier[57, 58])	•(P_1) The function as an example of fr by Poisson : $ab^{-(\frac{r}{n\alpha})^m}$, which can not express both the modes of attraction and rejection and is not coincident with his $\sum r^3 fr$ accrding to Navier. •(P_2) Poisson must use summation instead of integral. On this point, Navier points out Poisson's logic for the necessity of summation. (cf. Navier (N_2)) •(P_3) Mais si l'on exprime avec lui les forces moléculaires par des intégrales, on peu voir par une simple intégration par partie, que le coëfficient k ou ε s'evanouit en même temps que K ; en sort que les équations d'équilibre ne renferment plus rien qui dépende de l'action des molécules ; résultat absurde que l'on ne peut éviter qu'en exprimant les résultantes de cette action, par des sommes non réductibles à des intégrales, ce qui empêche qu'on n'ait nécessairement $\varepsilon = 0$ par suite de $K = 0$. : Poisson [58, p.207, §2]. Where, ε cited by Poisson is Navier[46]'s one, which is equal to Poisson's k according to Poisson.
4	Navier (1785-1836) elastic : [46] fluid : [47] (with Poisson [48, 49, 50, 51], with Arago[51])	•(N_1) Si par exemple on prend pour cette fonction $e^{-k\rho}$, e étant le nombre dont le logarithme népérien est l'unité, et k un coefficient constant, on aura $\int_0^\infty d\rho \cdot \rho^4 e^{-k\rho} = \frac{4!}{k^5}, \quad \int_0^\infty d\rho \cdot \rho^6 e^{-k\rho} = \frac{6!}{k^7}, etc.$ Or pour que la quantité $e^{-k\rho}$ décroisse avec une trés-grand rapidité, quand ρ augmente, il faut supposer que le coefficient k est un trés-grand nombre. [46, p.383] (cf. Laplace (L_1), (L_2), Gauss (G_2)). •(N_2) Navier explains the use of integral against Poisson's necessity of summation : Donc la difficulté d'accorder l'état naturel du corps avec l'état varié, c'est-à-dire, de faire en sorte que k conserve une valeur, tandis que K est nul, n'existe véritablement pas ; ou du moins il n'est pas nécessaire, pour résoudre, de supposer que *les quantités* k, K *sont de sommes plutôt que des intégrales : ils suffit de supposer que* $r^4 fr$ *n'est pas nul quand* $r = 0$. [50, p.103, §7]. •(N_3) Navier points out the following operation : $[r^4 f(r)]_0^\infty = 0$, for $f(r) \to 0$ in $r \to 0$. "J'ai remarqué d'abord qu'il fallait lire : ≪ si l'on fait attention que $r^4 fr$ est nulle aux deux limites, etc.≫ J'ai remarqué ensuite que rien n'obligeait à admettre que $r^4 fr$ est nulle à la limite corresponante à $r = 0$; •(N_4) It must read " If one observes that $r^4 fr$ is null at both limits etc...." Moreover, the writer [Poisson] does not show the necessity that $r^4 fr$ be null at the limit corresponding to $r = 0$.
5	Cauchy (1789-1857) elastic and fluid : [7]	•(C_1) D'ailleurs, si, pour des valeurs croissantes de la distance r, la fonction f(r) décroit plus rapidment que la fonction que $\frac{1}{r^4}$, si de plus le produit r^4f(r) s'évanouit pour $r = 0$, on trouvera, en supposant la fonction f$'(r)$ continue, et en intégrant par parties, $\int_0^\infty r^4 \mathrm{f}'(r) dr = -4\int_0^\infty r^3 \mathrm{f}(r) dr$. [7, p.242]
6	Gauss (1777-1855) capillary action : [17] (to Laplace [17], to Bessel[18])	•(G_1) Judging from the second dissertation : ≺ *Supplément à la théorie de l'action capillaire* ≻, Mr. Laplace had scarcely investigated of φf, not only the complete attraction, but also a part, and tacitly understood incompletely the general attraction ; by the way, if we would refer the latter in comparison with our sensible modification, on the contrary, we can assert it to be more inferior to the bad experiments and be clearly visible. •(G_2) Laplace considers exponential e^{-if} as an example of equivalent function to φf, denoting the large quantity by i, or $\frac{1}{i}$ becomes infinitesimal. (cf. Laplace (L_2).) But it is not at all necessary to limit the generality by such a large quantity, this point is more clear than words, we would see easiest, only by investigating if these integrations would be able to be extended, not only at infinity but also at an arbitrary sensible distance, or if the occurring in experiment could be wider extended up to the finitely measurable distance. [17, p.33]

TABLE 9. Cross-indexed differences on the RDFs $f \in \mathcal{RDF}$ (Remark. entry 1,5,6 : on capillarity, P_n, N_n, C_n, L_n, G_n : in Table 8.)

		1	2	3	4	5
no	Name/Problem/ Bibl. (Year read) - Year published/	Laplace	Poisson	Navier	$f(r)$ at $r=0$	$f(r)$ at $r=\infty$
1	Laplace capillary action : [37] 1806-07	$L_1 : K, H$ L_2 :force attractive only and $f \simeq c^{-if}$, $f \in \mathcal{RDF}$			0	0
2	Poisson elastic : [56],(1828)-28; [59],1829;[60],(1829)-31 fluid : [60],(1829)-31 disputing origin: [56],1828 (with Navier : [57],1828;[58],1828)	Refer to Laplace's $f \in \mathcal{RDF}$	$k,\ K$	$P_1 \to N_1$: $f \simeq ab^{-(\frac{r}{n\alpha})^m}$ $P_2 \to N_2$: not by integral but by sum because $k = -K = 0$ at once. $P_3 \to N_3 : k = \varepsilon$ of Navier $P_4 \to N_4 : f \in \mathcal{RDF}$	0	0
3	Navier elastic:[46],(1821)-27 fluid:[47],(1822)-27 (with Poisson : [48],1828; [49],1829; [50],1829; [51],1829 with Arago[51],1829)	Refer to Laplace's integral	$N_1 \to P_1 : f \simeq e^{-k\rho}$ $N_2 \to P_2$: not by sum but by integral as Laplace does $N_3 \to P_3 : [r^4 f(r)]_0^\infty \neq 0$, $\varepsilon \neq k$ $N_4 \to P_4 : r^4 f(r)$ for $r=0$, $f \in \mathcal{RDF}$ but only in $r=\infty$, $f(r) \not\to 0$ as $r \to 0$	ε in elastic $\varepsilon,\ E$ in fluid	$\neq 0$	0
4	Cauchy elastic & fluid :[7]				0	0
5	Gauss capillary action : [17] (to Laplace [17],1830 to Bessel[18],1830)	$G_1 \to L_1$:Laplace's deduction is conspicuous. $G_2 \to L_2$:no necessary to limit i of c^{-if} to be very large.			-	
6	Poisson capillary action : [62],1831, (to Gauss[62])	Same K and H with Laplace			0	0

APPENDIX A. Detailed commentary of principles and deduction of equations or tensor

A.1. From Lagrange to Laplace.

Lagrange had completed "*Mécanique analytique*", and told it to Lalplace in the letter in 1782. [23] Lagrange discusses the dynamics of the planetary corps with the attractions and applies it to the general dynamics. However, his dates are a few earlier for him to apply the existence of the atom or molecule to his equations of fluid dynamics. He passed away in 1813. In the early of the 19th century, Gay-Lussac, Dalton and Avogadro had proposed the atomic-molecular theories.

Lagrange had proposed three classes, by which all the system of corps action each other. [24]

> On peut range en trois classes tous les systèmes de corps qui agissent les uns sur les autres, et dont on peut déterminer le mouvement par les lois de la Mécanique ; car leur action mutuelle ne peut s'exercer que de trois manières différentes qui nous soient connues :
> - ou par des forces <u>d'attraction</u>, lorsque les corps sont isolés,
> - ou par des <u>liens</u> qui les unissent, [25]
> - ou enfin par la <u>collision</u> immédiate.
>
> Notre système planètaire appartient à la première classe, et par cette raison les problèmes qui s'y rapportient doivent tenir le première rang parmi tous les problèmes de la Dynamique. Nous allons en faire l'objet de cette Section. [31, Vol. **12**, Part 2 (*La Dynamique*), §7, p.1].

Our planerary system belongs to the first calss (caused by the force of attraction), then he seeks the mechanics in it.

Lagrange described the hydrodynamic equations :

$$\begin{cases} \Delta\left[\left(\frac{d^2x}{dt^2}+X\right)\frac{\partial x}{\partial a}+\left(\frac{d^2y}{dt^2}+Y\right)\frac{\partial y}{\partial a}+\left(\frac{d^2z}{dt^2}+Z\right)\frac{\partial z}{\partial a}\right]-\frac{\partial \lambda}{\partial a}=0, \\ \Delta\left[\left(\frac{d^2x}{dt^2}+X\right)\frac{\partial x}{\partial b}+\left(\frac{d^2y}{dt^2}+Y\right)\frac{\partial y}{\partial b}+\left(\frac{d^2z}{dt^2}+Z\right)\frac{\partial z}{\partial b}\right]-\frac{\partial \lambda}{\partial b}=0, \\ \Delta\left[\left(\frac{d^2x}{dt^2}+X\right)\frac{\partial x}{\partial c}+\left(\frac{d^2y}{dt^2}+Y\right)\frac{\partial y}{\partial c}+\left(\frac{d^2z}{dt^2}+Z\right)\frac{\partial z}{\partial c}\right]-\frac{\partial \lambda}{\partial c}=0 \end{cases} \tag{22}$$

[26] where where, $\mathbf{a} = (a, b, c)$: position on $t = 0$, $\mathbf{X} = (x,\ y,\ z)$: position on $t = t$, $\mathbf{X} = (X,\ Y,\ Z)$: outer force, Δ : density, λ : pressure.

The Lagrange's hydrodynamic equations of today's vectorial descripsion coppreponding to (22) are :

$$\rho\sum_{j=1}^{3}\frac{\partial x_j}{\partial a_i}\Big(\frac{\partial^2 x_j}{\partial t^2}-K_j\Big)=-\frac{\partial p}{\partial a_i}, \quad (i=1,2,3)$$

where, $\mathbf{a} = (a_1, a_2, a_3)$: position on $t = 0$, $\mathbf{X} = (x_1, x_2, x_3)$: position on $t = t$, $\mathbf{K} = (K_1, K_2, K_3)$: outer force, p : pressure, ρ : density.

Lagrange communicated his "*Méchanique analitique*" to Laplace, however, from the hydrodynamic equation by Lagrange, we can scarecely find the $MDFD$ equations, for lack of the epochal background on atomic-molecular theories.

Laplace studied the capillary action (cf. [34, 35, 36, 37]), in which he treated the attractive forces. In the introduction of [35] following with [34], Laplace says, "So as to make clear more and more about the identity of attractive forces, upon which this actions depends, which produce the affinities of the bodies" (*Supplément* [35]). We would like to discuss Laplace with Gauss later.

A.2. Naviers' principle and equations.

[23]($\Downarrow$) Lagrange corresponds with Laplace saying, "J'ai presque achevé un Traité de Mécanique analytique, fondé uniquement sur le principe ou formule que j'expose dans la première Section du Mémoire ci-joint; mais, comme j'ignore encore quand et oú pourrai la faire imprimer, je ne m'empresse pas d'y mettre la dernière main. [32, Vol. **14**, §16, No. 20 *Lagrange à Laplace. Berlin, 15 septembre 1782.* p.116]

[24]($\Downarrow$) This paragraph doesn't exist in the first edition. The following content we refer is in the 4th edition was published after the 3rd edition published by Bertrand. The first edition uses the title page in 1788 as the published year, instead of the 4the edition in 1789. It reads that Quatrième Édition. D'apres la troisieme édition de 1853 publiée par Bertrand.

[25]($\Downarrow$) Combination in chemistry, etc.

[26]($\Downarrow$) Lagrange [31, Vol. **12**, §11, p.280] *De mouvement des fluides incompressibles*, or [31, Vol. **12**, §12, p.325] *De mouvement des fluides compressibles et élastiques.*

A.2.1. **From Euler to Navier.**

The corresponding Navier-Stokes equations on the incompressible fluid (1) by Navier are as follow :

$$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\Big(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\Big) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\Big(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\Big) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\Big) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\ ; \end{cases} \tag{23}$$

and the equation of continuity :

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \tag{24}$$

He explains ε from various concepts in [45, p.251] :

> ε is the constant which we mentioned above. Many experiments teach that this constant takes the various values for each fluids, and varies with the temperature for each fluid. It is considerd also as variant with the pression ; but we have observed as the known facts, on the contrary, that ε is almostly independ of the force which tends to compress the partial differences of the fluid.

Navier cites the Euler's equations of the ideal fluid ([47, p.399]) :

$$\begin{cases} P - \frac{dp}{dx} = \rho\Big(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\Big), \\ Q - \frac{dp}{dy} = \rho\Big(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\Big), \\ R - \frac{dp}{dz} = \rho\Big(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\Big), \end{cases} \tag{25}$$

with (24). Referring to (25) by Euler who passed away in 1783, Navier observed in 1822 that he hadn't sought for *nouvelles forces moléculaires* anywhere in the Euler's descriptions and that had motivated to formulate self-made MD fluid dynamics equations :

> Mais, d'après les notations exposées ci-dessus[27], il est nécessaire d'admettre *l'existence de nouvelles forces moléculaires*, qui sont développées par l'etat de mouvement du fluid. La recherche des expressions analytiques de ce forces est le principal object que l'on s'est proposé dans la composition de ce mémoire. [47, p.399]

A.2.2. **Principles and means of constant ε in elastic solid.**

From Navier [46, p.386], we cite his context about the computation of momentum of total forces by integral :

• ¶ 4.

$$(3\text{-}5)_{N^e} \quad \sqrt{\alpha^2+\beta^2+\gamma^2} + \frac{1}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\Big[\frac{dx}{da}\alpha^2 + (\frac{dx}{db}+\frac{dy}{da})\alpha\beta + (\frac{dx}{dc}+\frac{dz}{da})\alpha\gamma + \frac{dy}{db}\beta^2 + (\frac{dy}{dc}+\frac{dz}{db})\beta\gamma + \frac{dz}{dc}\gamma^2\Big].$$

> Le premier terme est la valeur primitive de la distance MM' des deux points que l'on considère, qui a été représentée ci-dessuus pa ρ. Le second terme représente donc la variation que cette distance a subie par suite du changement de figure du corps, et à laquelle la force qui agit de M' sur M est proportionnelle. Si on remplace α, β, γ par les valeurs
>
> $$\begin{cases} \alpha = \rho\cos\psi\cos\varphi, \\ \beta = \rho\cos\psi\sin\varphi, \\ \gamma = \rho\sin\psi, \end{cases}$$
>
> cette variation deviendra
>
> $$f \equiv \rho\Big[\frac{dx}{da}\cos^2\psi\cos^2\varphi + (\frac{dx}{db}+\frac{dy}{da})\cos^2\psi\sin\varphi\cos\varphi + (\frac{dx}{dc}+\frac{dz}{da})\cos\psi\sin\psi\cos\varphi + \frac{dy}{db}\cos^2\psi\sin^2\varphi + (\frac{dy}{dc}+\frac{dz}{db})\sin\psi\cos\psi\sin\varphi + \frac{dz}{dc}\sin^2\psi\Big]. \tag{26}$$

[27]($\Downarrow$) (25)

($\Downarrow$) Here, Navier immediately introduces harmonic function, although which may simplify elastic structure and we can consider the elastic structure as simple, however by this, we can not get generality. By the way, Cauchy begins with the general case, and finally apply polar system by harmonic function. Cauchy criticizes Navier' special case. cf. Cauchy $(51)_C$. ($\Uparrow$)

Représentons pour abréger, cette quantité par f. La force avec laquelle le point M' attire M sera donc proportionnelle à f. Le moment de cette force, cette expression étant prise dans le même sens que dans la *Mécanique analytique*,[28] sera évidement proportionnel à $f\delta f$, ou à $\frac{1}{2}\delta f^2$. Parconséquent

- si l'on multiplie $\frac{1}{2}\delta f^2$ par $d\rho d\psi d\varphi\rho^2 \cos\psi f\rho$;
- si l'on transporte le signe δ en avant des signes d'intégration relatifs à ρ, ψ and φ, ce qui est permis ;
- et si l'on intègre entre les mêmes limites qu'on l'a fait dans le no 3 :

on aura une quantité proportionnelle à la somme des moments de toutes les forces intérieures par lesquelles le point M est sollicité. Cette quantité est donc (continue below) [46, p.386]

$$
\begin{aligned}
(4\text{-}7)_{N^e} \quad & \int_0^{\infty} d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} d\varphi\rho^4 \cos\psi f\rho(\frac{1}{2}\delta f^2) \\
= \quad & \frac{1}{2}\delta \int_0^{\infty} d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} d\varphi\rho^4 \cos\psi f\rho \\
\times \quad & \Big[\frac{dx}{da}\cos^2\psi\cos^2\varphi + (\frac{dx}{db}+\frac{dy}{da})\cos^2\psi\sin\varphi\cos\varphi + (\frac{dx}{dc}+\frac{dz}{da})\cos\psi\sin\psi\cos\varphi \\
+ \quad & \frac{dy}{db}\cos^2\psi\sin^2\varphi + (\frac{dy}{dc}+\frac{dz}{db})\sin\psi\cos\psi\sin\varphi + \frac{dz}{dc}\sin^2\psi\Big]^2.
\end{aligned}
$$

$$
\begin{aligned}
f^2 = & (\frac{dx}{da})^2\cos^4\psi\cos^4\varphi + \Big\{(\frac{dx}{db})^2 + (\frac{dy}{da})^2 + 2\frac{dx}{db}\frac{dy}{da}\Big\}\cos^4\psi\sin^2\varphi\cos^2\varphi \\
+ & \Big\{(\frac{dx}{dc})^2 + (\frac{dz}{da})^2 + 2\frac{dx}{dc}\frac{dz}{da}\Big\}\cos^2\psi\sin^2\psi\cos^2\varphi + (\frac{dy}{db})^2\cos^4\psi\sin^4\varphi \\
+ & \Big\{(\frac{dy}{dc})^2 + (\frac{dz}{db})^2 + 2\frac{dy}{dc}\frac{dz}{db}\Big\}\sin^2\psi\cos^2\psi\sin^2\varphi + (\frac{dz}{dc})^2\sin^4\psi
\end{aligned}
$$

($\Downarrow$) Here, we would like to show Navier's mistake. At first we integrate above with respect to φ. By using the formulae below including (89) :

$$
\begin{cases}
\int\cos^n x dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int\cos^{n-2}x dx, \\
\int\sin^n x dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int\sin^{n-2}x dx,
\end{cases}
$$

then :

$$
\int_0^{2\pi}\cos^4\varphi d\varphi = \int_0^{2\pi}\sin^4\varphi d\varphi = \frac{3}{4}\pi, \ \int_0^{2\pi}\sin^2\varphi\cos^2\varphi d\varphi = \frac{\pi}{4}, \ \int_0^{2\pi}\cos^2\varphi d\varphi = \int_0^{2\pi}\sin^2\varphi d\varphi = \pi \quad (27)
$$

[28]($\Downarrow$) Lagrange [31].

Hence, it follows that : [29]

$$
\begin{aligned}
&\frac{1}{2}\delta\int_0^\infty d\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi\int_0^{2\pi} d\varphi\rho^4\cos\psi f\rho f^2 \\
=\ & \frac{1}{2}\delta\int_0^\infty d\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \\
\times\ & \frac{\pi}{4}\Big[3\frac{d^2x}{da^2}\cos^5\psi+\Big\{\Big(\frac{dx}{db}+\frac{dy}{da}\Big)^2+2\frac{dx}{db}\frac{dy}{da}\Big\}\cos^5\psi \\
+\ & 4\Big\{\Big(\frac{dx}{dc}+\frac{dz}{da}\Big)^2+2\frac{dx}{dc}\frac{dz}{da}\Big\}\cos^3\psi\sin^2\psi+3\frac{d^2y}{db^2}\cos^5\psi \\
+\ & 4\Big\{\Big(\frac{dy}{dc}+\frac{dz}{db}\Big)^2+2\frac{dy}{dc}\frac{dz}{db}\Big\}\sin^2\psi\cos^3\psi+8\frac{d^2z}{dc^2}\sin^4\psi\cos\psi\Big].
\end{aligned}
\tag{28}
$$

($\Downarrow$) Here, we would like to notice our correction of the last term of $[\cdots 8\cdots]$ in (28) from 3 to 8, however this correction will not give any effect to Navier's description below. Next we integrate above with respect to ψ. Then

$$
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5\psi d\psi=\frac{16}{15},\quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\psi\sin^2\psi d\psi=\frac{4}{15},\quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^4\psi\cos\psi d\psi=\frac{2}{5}
\tag{29}
$$

After representing the coefficient which is on the front of the integral with respect to ρ with ε, we get from (28) the following :

$$
\begin{aligned}
\frac{1}{2}\varepsilon\delta\Big[\ \ &\Big\{3\frac{d^2x}{da^2}+\Big\{\Big(\frac{dx}{db}+\frac{dy}{da}\Big)^2+2\frac{dx}{db}\frac{dy}{da}\Big\}+\Big\{\Big(\frac{dx}{dc}+\frac{dz}{da}\Big)^2+2\frac{dx}{dc}\frac{dz}{da}\Big\} \\
&+3\frac{d^2y}{db^2}+\Big\{\Big(\frac{dy}{dc}+\frac{dz}{db}\Big)^2+2\frac{dy}{dc}\frac{dz}{db}\Big\}+3\frac{d^2z}{dc^2}\ \ \Big].
\end{aligned}
\tag{30}
$$

Here, choosing as a common factor, $\frac{\pi}{4}$ by integral with respact to φ from (27) and $\frac{16}{15}$ by integral with respect to ψ from (29) respectively, we get

$$
(3\text{-}9)_{N^e}\quad \varepsilon\equiv\frac{1}{2}\frac{\pi}{4}\frac{16}{15}\int_0^\infty d\rho\rho^4 f\rho=\frac{2\pi}{15}\int_0^\infty d\rho\rho^4 f\rho
\tag{31}
$$

This ε of (31) should be multiplied by $\frac{1}{2}$, when the momentum of the total forces *in the solid* are computed, namely it becomes the same as (52).

• ¶ 5.

$$
\begin{aligned}
(5\text{-}1)_{N^e}\quad 0 =\ & \varepsilon\iiint da\,db\,dc\left\{\begin{array}{l}
3\frac{dx}{da}\frac{\delta dx}{da}+\frac{dx}{db}\frac{\delta dx}{db}+\frac{dx}{db}\frac{\delta dy}{da}+\frac{dy}{da}\frac{\delta dx}{db}+\frac{dy}{da}\frac{\delta dy}{da}+\frac{dx}{da}\frac{\delta dy}{db}+\frac{dy}{db}\frac{\delta dx}{da} \\
\frac{dx}{dc}\frac{\delta dx}{dc}+\frac{dx}{dc}\frac{\delta dz}{da}+\frac{dz}{da}\frac{\delta dx}{dc}+\frac{dz}{da}\frac{\delta dz}{da}+\frac{dx}{da}\frac{\delta dz}{dc}+\frac{dz}{dc}\frac{\delta dx}{da}+3\frac{dy}{db}\frac{\delta dy}{db} \\
\frac{dy}{dc}\frac{\delta dy}{dc}+\frac{dy}{dc}\frac{\delta dz}{db}+\frac{dz}{db}\frac{\delta dy}{dc}+\frac{dz}{db}\frac{\delta dz}{db}+\frac{dz}{dc}\frac{\delta dy}{db}+\frac{dy}{db}\frac{\delta dz}{dc}+3\frac{dz}{dc}\frac{\delta dz}{dc}
\end{array}\right. \\
-\ & \iiint da\,db\,dc\,(X\delta x+Y\delta y+Z\delta z)-\int ds\,(X'\delta x'+Y'\delta y'+Z'\delta z').
\end{aligned}
\tag{32}
$$

When the first term of ε in the right-hand side of (32) is arranged in respect to $\delta x, \delta y$ and δz then :

$$
\varepsilon\iiint da\,db\,dc\left\{\begin{array}{l}
3\frac{dx}{da}\frac{\delta dx}{da}+\frac{dx}{db}\frac{\delta dx}{db}+\frac{dx}{dc}\frac{\delta dx}{dc}+2\frac{dy}{da}\frac{\delta dx}{db}+2\frac{dz}{da}\frac{\delta dx}{dc} \\
\frac{dy}{da}\frac{\delta dy}{da}+3\frac{dy}{db}\frac{\delta dy}{db}+\frac{dy}{dc}\frac{\delta dy}{dc}+2\frac{dx}{da}\frac{\delta dy}{db}+2\frac{dz}{db}\frac{\delta dy}{dc} \\
\frac{dz}{da}\frac{\delta dz}{da}+\frac{dz}{db}\frac{\delta dz}{db}+3\frac{dz}{dc}\frac{\delta dz}{dc}+2\frac{dx}{da}\frac{\delta dz}{dc}+2\frac{dy}{db}\frac{\delta dz}{dc}
\end{array}\right.
\tag{33}
$$

---

[29]($\Downarrow$) Remark : $f\rho$ does not mean $f\times\rho$ but $f(\rho)$. We compute $(\bullet+\bullet)^2$ in (28) as usual, for example : $\left(\frac{dx}{dc}+\frac{dz}{da}\right)^2=\frac{d^2x}{dc^2}+2\frac{dx}{dc}\frac{dz}{da}+\frac{d^2z}{da^2}$.

Moreover, we rearrange (33) for differential : $\frac{\delta x'}{da'}, \frac{\delta x'}{db'}, \frac{\delta x'}{dc'}, \frac{\delta y'}{da'}, \frac{\delta y'}{db'}, \frac{\delta y'}{dc'}, \frac{\delta z'}{da'}, \frac{\delta z'}{db'}, \frac{\delta z'}{dc'}$ as follows :

$$\varepsilon \iiint da\ db\ dc \left\{ \begin{array}{l} 3\frac{dx}{da}\frac{\delta dx}{da} + \frac{dy}{db}\frac{\delta dx}{da} + \frac{dz}{dc}\frac{\delta dx}{da} + \frac{dx}{db}\frac{\delta dx}{db} + \frac{dy}{da}\frac{\delta dx}{db} + \frac{dx}{dc}\frac{\delta dx}{dc} + \frac{dz}{da}\frac{\delta dx}{dc} \\ \frac{dx}{db}\frac{\delta dy}{da} + \frac{dy}{da}\frac{\delta dy}{da} + \frac{dx}{da}\frac{\delta dy}{db} + 3\frac{dy}{db}\frac{\delta dy}{db} + \frac{dz}{dc}\frac{\delta dy}{db} + \frac{dy}{dc}\frac{\delta dy}{dc} + \frac{dz}{db}\frac{\delta dy}{dc} \\ \frac{dx}{dc}\frac{\delta dz}{da} + \frac{dz}{da}\frac{\delta dz}{da} + \frac{dy}{dc}\frac{\delta dz}{db} + \frac{dz}{db}\frac{\delta dz}{db} + \frac{dx}{da}\frac{\delta dz}{dc} + 3\frac{dz}{dc}\frac{\delta dz}{dc} + \frac{dy}{db}\frac{\delta dz}{dc} \end{array} \right. \tag{34}$$

Using (33) and integration by parts of $\delta x, \delta y$ and δz, we make the top term of (35), in which $-\varepsilon$ is leading. And using (34), we show only the first differential order : $\delta x', \delta y', \delta z'$ in the middle term of (35) as follows :

$$\begin{aligned}
&(5\text{-}2)_{N^e} \quad 0 \\
&= -\varepsilon \iiint da\ db\ dc \left\{ \begin{array}{l} \left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dadb} + 2\frac{d^2z}{dadc}\right)\delta x \\ \left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dbdc}\right)\delta y \\ \left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right)\delta z \end{array} \right. \\
&+ \varepsilon\Big[\iint db'\ dc'\Big(3\frac{dx'}{da'} + \frac{dy'}{db'} + \frac{dz'}{dc'}\Big) + \iint da'dc'\Big(\frac{dx'}{db'} + \frac{dy'}{da'}\Big) + \iint da'db'\Big(\frac{dx'}{dc} + \frac{dz'}{da'}\Big)\Big]\delta x' \\
&+ \varepsilon\Big[\iint db'dc'\Big(\frac{dx'}{db'} + \frac{dy'}{da'}\Big) + \iint da'dc'\Big(\frac{dx'}{da'} + 3\frac{dy'}{db'} + \frac{dz'}{dc'}\Big) + \iint da'db'\Big(\frac{dy'}{dc'} + \frac{dz'}{db'}\Big)\Big]\delta y' \\
&+ \varepsilon\Big[\iint db'dc'\Big(\frac{dx'}{dc'} + \frac{dz'}{da'}\Big) + \iint da'dc'\Big(\frac{dy'}{dc'} + \frac{dz'}{db'}\Big) + \iint da'db'\Big(\frac{dx'}{da'} + \frac{dy'}{db'} + 3\frac{dz'}{dc'}\Big)\Big]\delta z' \\
&- \iiint da\ db\ dc(X\delta x + Y\delta y + Z\delta z) - \int ds(X'\delta x' + Y'\delta y' + Z'\delta z').
\end{aligned} \tag{35}$$

We solve the indetermate equations (35) [30] of equilibrium in an elastic solid as follows. At first, we get the following two equations from (35) :

- The force inside the solid corps :

$$-\iiint da\ db\ dc(X\delta x + Y\delta y + Z\delta z) = \varepsilon \iiint da\ db\ dc \left\{ \begin{array}{l} \left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dadb} + 2\frac{d^2z}{dadc}\right)\delta x \\ \left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dbdc}\right)\delta y \\ \left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right)\delta z. \end{array} \right. \tag{36}$$

- The force on the boundary :

$$\begin{aligned}
&\int ds(X'\delta x' + Y'\delta y' + Z'\delta z') \\
&= \varepsilon\Big[\iint db'\ dc'\Big(3\frac{dx'}{da'} + \frac{dy'}{db'} + \frac{dz'}{dc'}\Big) + \iint da'dc'\Big(\frac{dx'}{db'} + \frac{dy'}{da'}\Big) + \iint da'db'\Big(\frac{dx'}{dc} + \frac{dz'}{da'}\Big)\Big]\delta x' \\
&+ \varepsilon\Big[\iint db'dc'\Big(\frac{dx'}{db'} + \frac{dy'}{da'}\Big) + \iint da'dc'\Big(\frac{dx'}{da'} + 3\frac{dy'}{db'} + \frac{dz'}{dc'}\Big) + \iint da'db'\Big(\frac{dy'}{dc'} + \frac{dz'}{db'}\Big)\Big]\delta y' \\
&+ \varepsilon\Big[\iint db'dc'\Big(\frac{dx'}{dc'} + \frac{dz'}{da'}\Big) + \iint da'dc'\Big(\frac{dy'}{dc'} + \frac{dz'}{db'}\Big) + \iint da'db'\Big(\frac{dx'}{da'} + \frac{dy'}{db'} + 3\frac{dz'}{dc'}\Big)\Big]\delta z'.
\end{aligned} \tag{37}$$

$$\Rightarrow \int ds \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \varepsilon \begin{bmatrix} 3\frac{dx'}{da'} + \frac{dy'}{db'} + \frac{dz'}{dc'} & \frac{dx'}{db'} + \frac{dy'}{da'} & \frac{dx'}{dc'} + \frac{dz'}{da'} \\ \frac{dx'}{db'} + \frac{dy'}{da'} & \frac{dx'}{da'} + 3\frac{dy'}{db'} + \frac{dz'}{dc'} & \frac{dy'}{dc'} + \frac{dz'}{db'}, \\ \frac{dx'}{dc'} + \frac{dz'}{da'} & \frac{dy'}{dc'} + \frac{dz'}{db'} & \frac{dx'}{da'} + \frac{dy'}{db'} + 3\frac{dz'}{dc'} \end{bmatrix} \begin{bmatrix} \iint db'\ dc' \\ \iint da'dc' \\ \iint da'db' \end{bmatrix}, \tag{38}$$

here this tensor is symmetric. From (36), we get the inside forces of the elastic solid as follows :

30 (⇓) Navier says that (35) is usually called "equations indéfinies". [46, p.384,389]

$$(5\text{-}3)_{N^e}\quad \begin{cases} -X=\varepsilon\Big(3\frac{d^2x}{da^2}+\frac{d^2x}{db^2}+\frac{d^2x}{dc^2}+2\frac{d^2y}{dadb}+2\frac{d^2z}{dadc}\Big),\\ -Y=\varepsilon\Big(\frac{d^2y}{da^2}+3\frac{d^2y}{db^2}+\frac{d^2y}{dc^2}+2\frac{d^2x}{dadb}+2\frac{d^2z}{dbdc}\Big),\\ -Z=\varepsilon\Big(\frac{d^2z}{da^2}+\frac{d^2z}{db^2}+3\frac{d^2z}{dc^2}+2\frac{d^2x}{dadc}+2\frac{d^2y}{dbdc}\Big),\end{cases} \tag{39}$$

where X, Y and Z are positive values.

Next, we get also X', Y' and Z' from the (37) : we suppose that :[31]

- $db'dc' \quad \to \quad ds\cos l,\quad l$: the angles by which the tangent plane makes on the surface frame with the plane bc,
- $da'dc' \quad \to \quad ds\cos m,\quad m$: similarly, the angles with the plane ac,
- $da'db' \quad \to \quad ds\cos n,\quad n$: similarly, the angles with the plane ab,
- $\iint db'dc', \iint da'dc', \iint da'db' \quad \to \quad \int ds$,

then from (37), we get the forces operation on the surface of the elastic solid as follows :

$$(5\text{-}4)_{N^e}\quad \begin{cases} X'=\varepsilon\Big[\cos\ l\Big(3\frac{dx'}{da'}+\frac{dy'}{db'}+\frac{dz'}{dc'}\Big)+\cos\ m\Big(\frac{dx'}{db'}+\frac{dy'}{da'}\Big)+\cos\ n\Big(\frac{dx'}{dc'}+\frac{dz'}{da'}\Big)\Big],\\ Y'=\varepsilon\Big[\cos\ l\Big(\frac{dx'}{db'}+\frac{dy'}{da'}\Big)+\cos\ m\Big(\frac{dx'}{da'}+3\frac{dy'}{db'}+\frac{dz'}{dc'}\Big)+\cos\ n\Big(\frac{dy'}{dc'}+\frac{dz'}{db'}\Big)\Big],\\ Z'=\varepsilon\Big[\cos\ l\Big(\frac{dx'}{dc'}+\frac{dz'}{da'}\Big)+\cos\ m\Big(\frac{dy'}{dc'}+\frac{dz'}{db'}\Big)+\cos\ n\Big(\frac{dx'}{da'}+\frac{dy'}{db'}+3\frac{dz'}{dc'}\Big)\Big].\end{cases} \tag{40}$$

$$\Rightarrow \begin{bmatrix} X'\\ Y'\\ Z'\end{bmatrix}=\varepsilon\begin{bmatrix} 3\frac{dx'}{da'}+\frac{dy'}{db'}+\frac{dz'}{dc'} & \frac{dx'}{db'}+\frac{dy'}{da'} & \frac{dx'}{dc'}+\frac{dz'}{da'}\\ \frac{dx'}{db'}+\frac{dy'}{da'} & \frac{dx'}{da'}+3\frac{dy'}{db'}+\frac{dz'}{dc'} & \frac{dy'}{dc'}+\frac{dz'}{db'},\\ \frac{dx'}{dc'}+\frac{dz'}{da'} & \frac{dy'}{dc'}+\frac{dz'}{db'} & \frac{dx'}{da'}+\frac{dy'}{db'}+3\frac{dz'}{dc'}\end{bmatrix}\begin{bmatrix}\cos l\\ \cos m\\ \cos n\end{bmatrix} \tag{41}$$

($\Downarrow$) By the way, when we rearrange (32) to compare with equations of equilibrium in fluid, then (32) becomes (42) as follows :

$$\begin{aligned} &(5\text{-}1)'_{N^e}\quad 0\\ &=\ \varepsilon\iiint dadbdc\begin{cases}\Big(3\frac{dx}{da}\frac{\delta dx}{da}+\frac{dx}{db}\frac{\delta dx}{db}+\frac{dx}{dc}\frac{\delta dx}{dc}\Big)+\Big(\frac{dy}{db}\frac{\delta dx}{da}+\frac{dy}{da}\frac{\delta dx}{db}\Big)+\Big(\frac{dz}{dc}\frac{\delta dx}{da}+\frac{dz}{da}\frac{\delta dx}{dc}\Big)\\ \Big(\frac{dx}{da}\frac{\delta dy}{db}+\frac{dx}{db}\frac{\delta dy}{da}\Big)+\Big(\frac{dy}{da}\frac{\delta dy}{da}+3\frac{dy}{db}\frac{\delta dy}{db}+\frac{dy}{dc}\frac{\delta dy}{dc}\Big)+\Big(\frac{dz}{db}\frac{\delta dy}{dc}+\frac{dz}{dc}\frac{\delta dy}{db}\Big)\\ \Big(\frac{dx}{da}\frac{\delta dz}{dc}+\frac{dx}{dc}\frac{\delta dz}{da}\Big)+\Big(\frac{dy}{db}\frac{\delta dz}{dc}+\frac{dy}{dc}\frac{\delta dz}{db}\Big)+\Big(\frac{dz}{da}\frac{\delta dz}{da}+\frac{dz}{db}\frac{\delta dz}{db}+3\frac{dz}{dc}\frac{\delta dz}{dc}\Big)\end{cases}\\ &-\ \iiint dadbdc(X\delta x+Y\delta y+Z\delta z)-\int ds(X'\delta x'+Y'\delta y'+Z'\delta z'). \end{aligned} \tag{42}$$

Navier deduces the equations of equilibrium in fluid as follows :

[31]($\Downarrow$) On this method Navier cites Lagrange ([31, pp.113-188,1 partie ,§ 5]), *Solution de différents problèmes de statique.* In fluid case, Navier rethinks this method afterward. c.f. (69).

$$
\begin{aligned}
&(3\text{-}24)_{Nf} \quad 0 \\
= &\iiint dxdydz \begin{cases} \left[P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right)\right]\delta u \\ \left[Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right)\right]\delta v \\ \left[R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right)\right]\delta w \end{cases} \\
- &\varepsilon \iiint dxdydz \begin{cases} \left(3\frac{du}{dx}\frac{\delta du}{dx} + \frac{du}{dy}\frac{\delta du}{dy} + \frac{du}{dz}\frac{\delta du}{dz}\right) + \left(\frac{dv}{dy}\frac{\delta du}{dx} + \frac{dv}{dx}\frac{\delta du}{dy}\right) + \left(\frac{dw}{dz}\frac{\delta du}{dx} + \frac{dw}{dx}\frac{\delta du}{dz}\right) \\ \left(\frac{du}{dx}\frac{\delta dv}{dy} + \frac{du}{dy}\frac{\delta dv}{dx}\right) + \left(\frac{dv}{dx}\frac{\delta dv}{dx} + 3\frac{dv}{dy}\frac{\delta dv}{dy} + \frac{dv}{dz}\frac{\delta dv}{dz}\right) + \left(\frac{dw}{dy}\frac{\delta dv}{dz} + \frac{dw}{dz}\frac{\delta dv}{dy}\right) \\ \left(\frac{du}{dx}\frac{\delta dw}{dz} + \frac{du}{dz}\frac{\delta dw}{dx}\right) + \left(\frac{dv}{dy}\frac{\delta dw}{dz} + \frac{dv}{dz}\frac{\delta dw}{dy}\right) + \left(\frac{dw}{dx}\frac{\delta dw}{dx} + \frac{dw}{dy}\frac{\delta dw}{dy} + 3\frac{dw}{dz}\frac{\delta dw}{dz}\right) \end{cases} \\
+ &\mathbf{S}ds^2 E(u\delta u + v\delta v + w\delta w). \qquad (43)
\end{aligned}
$$

When we compare only the terms of ε between (42) in elastic solid and (43) in fluid, the defference is none, and the both tensor are symmetric respectively.

• ¶ 6. Navier computes the acceleration around the point M. Π is *density* of the solid per volume, g is *acceleration of gravity*, then

$$
(6\text{-}1)_{Ne} \quad \begin{cases} \frac{\Pi}{g}\frac{d^2x}{dt^2} = \varepsilon\left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dbda} + 2\frac{d^2z}{dcda}\right), \\ \frac{\Pi}{g}\frac{d^2y}{dt^2} = \varepsilon\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dcdb}\right), \\ \frac{\Pi}{g}\frac{d^2z}{dt^2} = \varepsilon\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right) \end{cases} \qquad (44)
$$

Poisson comments that ε in (39) and (44) equals Poisson's corresponding parameter in $(6)_{Pe}$ (= (94)), namely Navier's ε is equivalent to Poison's $\frac{a^2}{2}$, however Navier denies it.

A.2.3. **Deduction of the expressions of forces of the molecular action which is under the state of motion.**

Navier deduces the expressions of forces of the molecular action which is under the state of motion as follows : [32]

We consider the two molecules M and M'. x, y, z are the values of the rectangular coordinates of M and $x+\alpha, y+\beta, z+\gamma$ are the values of the rectangular coordinates of M'. The length of a rayon emitting from M : $\rho = \sqrt{\alpha^2+\beta^2+\gamma^2}$. The velocities of the molecule M are u, v, w and that of the molecules M' are

$$
(3\text{-}3)_{Nf} \quad u + \frac{du}{dx}\alpha + \frac{du}{dy}\beta + \frac{du}{dz}\gamma, \quad v + \frac{dv}{dx}\alpha + \frac{dv}{dy}\beta + \frac{dv}{dz}\gamma, \quad w + \frac{dw}{dx}\alpha + \frac{dw}{dy}\beta + \frac{dw}{dz}\gamma \qquad (45)
$$

V is the quantity on which the proportional action depends as follows :

$$
(3\text{-}4)_{Nf} \quad V = \frac{\alpha}{\rho}\left(\frac{du}{dx}\alpha + \frac{du}{dy}\beta + \frac{du}{dz}\gamma\right) + \frac{\beta}{\rho}\left(\frac{dv}{dx}\alpha + \frac{dv}{dy}\beta + \frac{dv}{dz}\gamma\right) + \frac{\gamma}{\rho}\left(\frac{dw}{dx}\alpha + \frac{dw}{dy}\beta + \frac{dw}{dz}\gamma\right). \qquad (46)
$$

V represents the force which exists between two certain molecules of fluid. The increment of V is as follows :

$$
(3\text{-}5)_{Nf} \quad \delta V = \frac{\alpha}{\rho}\left(\frac{\delta du}{dx}\alpha + \frac{\delta du}{dy}\beta + \frac{\delta du}{dz}\gamma\right) + \frac{\beta}{\rho}\left(\frac{\delta dv}{dx}\alpha + \frac{\delta dv}{dy}\beta + \frac{\delta dv}{dz}\gamma\right) + \frac{\gamma}{\rho}\left(\frac{\delta dw}{dx}\alpha + \frac{\delta dw}{dy}\beta + \frac{\delta dw}{dz}\gamma\right). \quad (47)
$$

$f(\rho)$ is a function depends on the distance ρ between M and M'. We define that ψ is the angle of the rayon ρ with its projection on the $\alpha\beta$-plane and φ is the angle which this projection forms with the α axis, and then

$$
(3\text{-}9)_{Nf} \quad \alpha = \rho\cos\psi\cos\varphi, \quad \beta = \rho\cos\psi\sin\varphi, \quad \gamma = \rho\sin\psi \qquad (48)
$$

[32](⇓) Navier ([46, pp.399-405])

TABLE 10. Combination between V and δV

	α^2	$\alpha\beta$	$\alpha\gamma$	$\beta\alpha$	β^2	$\beta\gamma$	$\gamma\alpha$	$\gamma\beta$	γ^2
α^2	1				5				7
$\alpha\beta$		2		4					
$\alpha\gamma$			3				6		
$\beta\alpha$		9		10					
β^2	8				11				14
$\beta\gamma$						12		13	
$\gamma\alpha$			16				19		
$\gamma\beta$						18		20	
γ^2	15				17				21

We calculate $d\rho d\psi d\varphi \rho^2 \cos\varphi$ of the element of the volume in the new system of coordinates : (α, β, γ), and integrate with respect to φ, ψ from 0 to $\frac{\pi}{2}$ and with respect to ρ from 0 to ∞.

$$(3\text{-}6)_{Nf} \quad \frac{1}{\rho^4} f(\rho) V \delta V =$$
$$\frac{f(\rho)}{\rho^4} \quad [\ \alpha\Big(\frac{du}{dx}\alpha + \frac{du}{dy}\beta + \frac{du}{dz}\gamma\Big) + \beta\Big(\frac{dv}{dx}\alpha + \frac{dv}{dy}\beta + \frac{dv}{dz}\gamma\Big) + \gamma\Big(\frac{dw}{dx}\alpha + \frac{dw}{dy}\beta + \frac{dw}{dz}\gamma\Big)\ ] \times$$
$$[\ \alpha\Big(\frac{\delta du}{dx}\alpha + \frac{\delta du}{dy}\beta + \frac{\delta du}{dz}\gamma\Big) + \beta\Big(\frac{\delta dv}{dx}\alpha + \frac{\delta dv}{dy}\beta + \frac{\delta dv}{dz}\gamma\Big) + \gamma\Big(\frac{\delta dw}{dx}\alpha + \frac{\delta dw}{dy}\beta + \frac{\delta dw}{dz}\gamma\Big)\ ], \tag{49}$$

here, by the symmetry we supposed, we get the relations as follows :

$$\Big|\alpha\frac{du}{dy}\beta\Big| = \Big|\beta\frac{dv}{dx}\alpha\Big|, \quad \Big|\beta\frac{dv}{dz}\gamma\Big| = \Big|\gamma\frac{dw}{dy}\beta\Big|, \quad \Big|\alpha\frac{du}{dz}\gamma\Big| = \Big|\gamma\frac{dw}{dx}\alpha\Big|,$$
$$\Big|\alpha\frac{\delta du}{dy}\beta\Big| = \Big|\beta\frac{\delta dv}{dx}\alpha\Big|, \quad \Big|\beta\frac{\delta dv}{dz}\gamma\Big| = \Big|\gamma\frac{\delta dw}{dy}\beta\Big|, \quad \Big|\alpha\frac{\delta du}{dz}\gamma\Big| = \Big|\gamma\frac{\delta dw}{dx}\alpha\Big|.$$

Because we integrate only $\frac{1}{8}$ volume of the total sphere, total of the sphere is multiplied by 8.

$$(3\text{-}7)_{Nf}$$
$$8\frac{f(\rho)}{\rho^4}\Big\{\Big(\frac{du}{dx}\frac{\delta du}{dx}\alpha^4 + \frac{du}{dy}\frac{\delta du}{dy}\alpha^2\beta^2 + \frac{du}{dz}\frac{\delta du}{dz}\alpha^2\gamma^2\Big) + \Big(\frac{dv}{dx}\frac{\delta du}{dy} + \frac{dv}{dy}\frac{\delta du}{dx}\Big)\alpha^2\beta^2 + \Big(\frac{dw}{dx}\frac{\delta du}{dz} + \frac{dw}{dz}\frac{\delta du}{dx}\Big)\alpha^2\gamma^2$$
$$+ \quad \Big(\frac{du}{dx}\frac{\delta dv}{dy} + \frac{du}{dy}\frac{\delta dv}{dx}\Big)\alpha^2\beta^2 + \Big(\frac{dv}{dx}\frac{\delta dv}{dx}\alpha^2\beta^2 + \frac{dv}{dy}\frac{\delta dv}{dy}\beta^4 + \frac{dv}{dz}\frac{\delta dv}{dz}\beta^2\gamma^2\Big) + \Big(\frac{dw}{dy}\frac{\delta dv}{dz} + \frac{dw}{dz}\frac{\delta dv}{dy}\Big)\beta^2\gamma^2$$
$$+ \quad \Big(\frac{du}{dx}\frac{\delta dw}{dz} + \frac{du}{dz}\frac{\delta dw}{dx}\Big)\alpha^2\gamma^2 + \Big(\frac{dv}{dy}\frac{\delta dw}{dz} + \frac{dv}{dz}\frac{\delta dw}{dy}\Big)\beta^2\gamma^2 + \Big(\frac{dw}{dx}\frac{\delta dw}{dx}\alpha^2\gamma^2 + \frac{dw}{dy}\frac{\delta dw}{dy}\beta^2\gamma^2 + \frac{dw}{dz}\frac{\delta dw}{dz}\gamma^4\Big)\Big\} \tag{50}$$

We get 21 terms in (50) from (49). We show the combination between V and δV in Table 10, in which the row is V and the column is δV and the numbers are the order of the description of the 21 terms in (50). By the formulae of the original function of infinite integral :

$$\begin{cases} \int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x, \quad \int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\sin 2x, \\ \int \sin^3 x dx = -\frac{1}{3}\cos x(\sin^2 x + 2), \quad \int \cos^3 x dx = \frac{1}{3}\sin x(\cos^2 x + 2), \\ \int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx, \quad \int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx, \\ \int \sin x \cos^m x dx = -\frac{\cos^{m+1} x}{m+1}, \quad \int \sin^m x \cos x dx = \frac{\sin^{m+1} x}{m+1} \end{cases}$$

We get the result of the integration except for $\int_0^\infty d\rho$ as follows :,

$$\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \alpha^4 \cos\psi = \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos^5\psi \cos^4\varphi = \frac{\pi}{10},$$
$$\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \beta^4 \cos\psi = \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos^5\psi \sin^4\varphi = \frac{\pi}{10},$$
$$\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \gamma^4 \cos\psi = \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos\psi \sin^4\varphi = \frac{\pi}{10},$$
$$\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \alpha^2\beta^2 \cos\psi = \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos^5\psi \sin^2\varphi\cos^2\varphi = \frac{\pi}{30},$$
$$\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \alpha^2\gamma^2 \cos\psi = \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos^3\psi \sin^2\varphi\cos^2\varphi = \frac{\pi}{30},$$
$$\frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \beta^2\gamma^2 \cos\psi = \frac{1}{\rho^4}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi \cos^3\psi \sin^2\varphi\sin^2\varphi = \frac{\pi}{30}.$$

Total of the sphere is multiplied by 8 taking ε as the common factor :

$$(3\text{-}10)_{Nf} \quad \varepsilon \equiv \frac{8\pi}{30}\int_0^\infty d\rho\rho^4 f(\rho) = \frac{4\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho) \tag{51}$$

We get now ε of (23), and using the law of conservation of mass : (24), it turns out the term of $\frac{\varepsilon}{\rho}\Delta\mathbf{u}$ of the today's formulation : (1) from next :

$$\varepsilon\begin{cases} 3\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+2\frac{dudv}{dxdy}+2\frac{dwdu}{dzdx}, \\ \frac{d^2v}{dx^2}+3\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}+2\frac{dudv}{dxdy}+2\frac{dudw}{dydz}, \\ \frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+3\frac{d^2w}{dz^2}+2\frac{dvdw}{dydz}+2\frac{dwdu}{dzdx} \end{cases} \quad \Rightarrow \quad \varepsilon\Delta\mathbf{u}.$$

Exactly speaking, Navier ([46, p.405]) says this ε must be mupltiply by $\frac{1}{2}$, for double count, when we get the total momentum of the forces caused by the reciprocal actions of the molecules of a fluid in the following section, as follows :

$$\boxed{(3\text{-}9)_{N\epsilon} \qquad \varepsilon \equiv \frac{2\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho),} \tag{52}$$

For this reason, Darrigol cites Navier's tensor from this by using tensor notation.[33]

A.2.4. **Deduction of the expressions of the total momentum of the forces caused by the reciprocal actions of the molecules of a fluid.**

Navier uses the above results to seek the expression of the total momentum of the forces caused by the reciprocal actions of the molecules of a fluid as follows : [34] Here, we rotate the rectangular coordinates for γ' to coincide with the direction of a rayon MN of which M is the common origin of the both rectangular coodinates of α, β, γ and α', β', γ' satisfying $\varphi = r$ and $\psi = s$ and then we get the new relation of α', β' and γ' from (48) as follows :

$$\alpha' = \rho\cos\psi\cos\varphi = \rho\cos r\cos s, \quad \beta' = \rho\cos\psi\sin\varphi = \rho\cos r\sin s, \quad \gamma' = \rho\sin\psi = \rho\sin r \tag{53}$$

[33]($\Downarrow$) O.Darrigol [10, p.112] interprets that this is Navier's tensor as follows :

$$\frac{2\pi}{15}\int_0^\infty d\rho\rho^4 f(\rho) \equiv k, \quad M = \int \sigma_{ij}\partial_i w_j d\tau,$$
$$\sigma_{ij} = -kN^2(\delta_{ij}\partial_k u_k + \partial_i u_j + \partial_j u_i) \equiv -kN^2(\delta_{ij}u_{kk} + u_{ji} + u_{ij}),$$
$$\text{where } N = 1.$$

In analogy with Lagrange's reasoning, Navier integrated it by parts to get

$$M = \oint \sigma_{ij}\partial_i w_j dS_i - \int (\partial_i\sigma_{ij})w_j d\tau.$$

[34]($\Downarrow$) Navier ([46, pp.405-416])

In fig.1, we suppose that : the point P is the projected point on $\alpha\beta$-plane from N. The angle of PMN equals s. N, R and Q are on the common line on the $\beta'\gamma'$-plane. Plane MNR and plane MRQ are on the common $\beta'\gamma'$-plane. $MN \perp MQ$, and $MR \perp MP$. Therefore, the angle made by MQ and MR equals s.

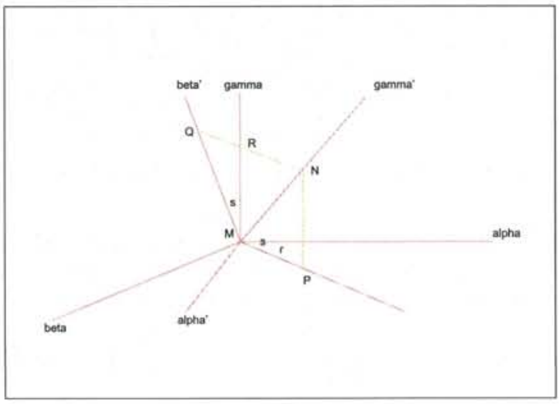

fig.1 Rotation of coordinates

From the above, we get the following :

$$(3\text{-}17)_{Nf} \quad \begin{cases} \alpha = -\alpha' \sin r + \beta' \cos r \sin s + \gamma' \cos r \cos s, \\ \beta = \quad \alpha' \cos r + \beta' \sin r \sin s + \gamma' \sin r \cos s, \\ \gamma = \qquad\qquad\quad \beta' \cos s \qquad\qquad\quad - \gamma' \sin s \end{cases}$$

$$or \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -\sin r & \cos r \sin s & \cos r \cos s \\ \cos r & \sin r \sin s & \sin r \cos s \\ 0 & \cos s & -\sin s \end{bmatrix} \begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix} \equiv A \begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix},$$

where last terms of the right hand-side of α, β, γ (or the values in the 3rd column of the 3×3 matrix for the transformation) are the original values of (48) except for the term of γ', and the rest terms are added by the rotation. ($\Downarrow$) By the way, if we call this rotation matrix A, we get $det(A) = 1$, so that $A^{-1} = \tilde{A} = A^T$, i.e.

$$\begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix} = A^{-1} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -\sin r & \cos r & 0 \\ \cos r \cos s & \sin r \sin s & \cos s \\ \cos r \cos s & \sin r \cos s & -\sin s \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}.$$

Using (45),(46) and (47), like (49), the expression : (54) is considered as the expression which must be integrated for all the value in respect to α' and β' and for the only positive value in respect to γ'. Then we get following :

$(3\text{-}18)_{Nf}$

$$\frac{1}{\rho^2} F(\rho) V \delta V = \frac{F(\rho)}{\rho^2} \times$$

$$[\quad \alpha'(-u \sin r + v \cos r) + \beta'(u \cos r \sin s + v \sin r \sin s + w \cos s) + \gamma'(u \cos r \cos s + v \sin r \cos s - w \sin s)\ ] \times$$

$$[\quad \alpha'(-\delta u \sin r + \delta v \cos r) + \beta'(\delta u \cos r \sin s + \delta v \sin r \sin s + \delta w \cos s) + \gamma'(\delta u \cos r \cos s + \delta v \sin r \cos s - \delta w \sin s)\ ] \tag{54}$$

We get the right-hand side of (54), except for $\frac{F(\rho)}{\rho^2}$ as follows :

$$[\alpha' a + \beta' b + \gamma' c][\alpha'(d\delta u + e\delta v) + \beta'(f\delta u + g\delta v + h\delta w) + \gamma'(i\delta u + j\delta v + k\delta w)]$$

or

$$\Big([\ a\ b\ c\ ] \begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix} \Big) \Big([\ \alpha'\ \beta'\ \gamma'\ ] \Big(\begin{bmatrix} d & e & 0 \\ f & g & h \\ i & j & k \end{bmatrix} \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix} \Big)\Big),$$

then we get

$$\begin{bmatrix} \alpha'\beta' \\ \beta'\gamma' \\ \gamma'\alpha' \end{bmatrix} = \begin{bmatrix} fa+db & ga+eb & ha \\ fc+ib & gc+jb & hc+kb \\ ia+cd & ja+ce & ka \end{bmatrix} \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix}.$$

We get effectively the following :

$$\begin{aligned} \alpha'\beta' = & \Big\{ -2u\sin r\cos r\sin s + v\sin s(\cos^2 r - \sin^2 r) - w\sin r\cos s \Big\}\delta u + \\ & \Big\{ \quad 2v\sin r\cos r\sin s + u\sin s(\cos^2 r - \sin^2 r) + w\cos r\cos s \Big\}\delta v + \\ & \Big\{ \cos s(v\cos r - u\sin r) \Big\}\delta w \end{aligned}$$

$$\begin{aligned} \beta'\gamma' = & \cos r\Big\{ 2u\cos r\sin s\cos s + 2v\sin r\sin s\cos s + w(\cos^2 s - \sin^2 s) \Big\}\delta u + \\ & \sin r\Big\{ 2u\cos r\sin s\cos s + 2v\sin r\sin s\cos s + w(\cos^2 s - \sin^2 s) \Big\}\delta v + \\ & \Big\{ u\cos r + v\sin r - 2w\sin s\cos s \Big\}\delta w \end{aligned}$$

$$\begin{aligned} \gamma'\alpha' = & \Big\{ -2u\sin r\cos r\cos s + v\cos s(\cos^2 r - \sin^2 r) + w\sin s\sin r \Big\}\delta u + \\ & \Big\{ \quad 2v\sin r\cos r\cos s + u\cos(\cos^2 r - \sin^2 r) - w\sin s\cos r \Big\}\delta v + \\ & \Big\{ \sin s(u\sin r - v\cos r) \Big\}\delta w \end{aligned}$$

On this point, Navier explains as follows :

> On the above expression, we must integrate for all the value with respect to α' and β', but with respect to γ', only positive value. This operation becomes symple by remarking :
>
> - that if we consider four points placed symmetrically, this sign for γ' is positive, but the other coordinates α' and β' differ from each other by sign of point two by two, and
> - that if we add the values which the above expression (54) takes in these four points, it rests, as the result of the addition, only the terms which are relative to the terms of α'^2 and the terms of β'^2, the terms gained are to be multiplied with 4.
>
> Hence performing the multi indexes, all is reduced to integrate the quality in the volume of $\frac{1}{8}$ of a sphere where α', β' and γ' take the positive values as follows :

$$(3\text{-}19)_{N'} \quad 4\frac{F(\rho)}{\rho^2}\left\{\begin{array}{lll} \alpha'^2 & \Big\{ & (u\sin^2 r - v\sin r\cos r)\delta u \\ & & (-u\sin r\cos r + v\cos^2 r)\delta v \quad \Big\} \\ \beta'^2 & \Big\{ & (u\cos^2 r\sin^2 s + v\sin r\cos r\sin^2 s + w\cos r\sin s\cos s)\delta u \\ & & (u\sin r\cos r\sin^2 s + v\sin^2 r\sin^2 s + w\sin r\sin s\cos s)\delta v \\ & & (u\cos r\sin s\cos s + v\sin r\sin s\cos s + w\cos^2 s \qquad)\delta w \quad \Big\} \\ \gamma'^2 & \Big\{ & (u\cos^2 r\cos^2 s + v\sin r\cos r\cos^2 s - w\cos r\sin s\cos s)\delta u \\ & & (u\sin r\cos r\cos^2 s + v\sin^2 r\cos^2 s - w\sin r\sin s\cos s)\delta v \\ & & (-u\cos r\sin s\cos s - v\sin r\sin s\cos s + w\sin^2 s \qquad)\delta w \quad \Big\} \end{array}\right. \tag{55}$$

$$\alpha' = \rho\cos\psi\cos\varphi, \quad \beta' = \rho\cos\psi\sin\varphi, \quad \gamma' = \rho\sin\psi$$

Making the calculation of the element of volume $d\rho d\psi d\varphi\rho^2\cos\psi$ with respect to ψ and φ from 0 to $\frac{\pi}{2}$, we get the following three results of the finite integrations :

$$\begin{aligned}
\frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\alpha'^2\cos\psi &= \frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\cos^3\psi\cos^2\varphi = \frac{\pi}{6},\\
\frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\beta'^2\cos\psi &= \frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\cos^3\psi\sin^2\varphi = \frac{\pi}{6},\\
\frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\gamma'^2\cos\psi &= \frac{1}{\rho^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\psi d\varphi\sin^2\psi\cos\psi = \frac{\pi}{6}
\end{aligned}$$

$F(\rho)$ is the same function as $f(\rho)$ in (49), which is a function which depends on the distance ρ between M and M'. Taking $\frac{\pi}{6}$ as the common factor, we put

$$\boxed{(3\text{-}22)_{Nf}\quad \frac{4\pi}{6}\int_0^{\infty} d\rho\rho^2F(\rho) = \frac{2\pi}{3}\int_0^{\infty} d\rho\rho^2F(\rho) \equiv E. \qquad (56)}$$

and define :

$$(3\text{-}23)_{Nf}\quad E(u\delta u + v\delta v + w\delta w)$$

for the expression which we seek for the sum of the momentum of the total actions caused between the molecules of the wall and the fluid, following the direction which passes the point M of the surface of the boundary of the fluid and the wall. E represents a constant of which the value are given by the experiment, according to the characteristic of the wall and the fluid, and which are able to be regarded as the measure of its reciprocal action. We get the following equilibrium of a fluid using ε of (51) and the above $E(u\delta u + v\delta v + w\delta w)$:

$$\begin{aligned}
(3\text{-}24)_{Nf}\quad 0 = &\iiint dxdydz\begin{cases}\left[P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right)\right]\delta u\\ \left[Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right)\right]\delta v\\ \left[R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right)\right]\delta w\end{cases}\\
- \ \varepsilon&\iiint dxdydz\begin{cases}3\frac{du}{dx}\frac{\delta du}{dx} + \frac{du}{dy}\frac{\delta du}{dy} + \frac{du}{dz}\frac{\delta du}{dz} + \frac{dv}{dy}\frac{\delta du}{dx} + \frac{dv}{dx}\frac{\delta du}{dy} + \frac{dw}{dz}\frac{\delta du}{dx} + \frac{dw}{dx}\frac{\delta du}{dz}\\ \frac{du}{dx}\frac{\delta dv}{dy} + \frac{du}{dy}\frac{\delta dv}{dx} + \frac{dv}{dx}\frac{\delta dv}{dx} + 3\frac{dv}{dy}\frac{\delta dv}{dy} + \frac{dv}{dz}\frac{\delta dv}{dz} + \frac{dw}{dy}\frac{\delta dv}{dz} + \frac{dw}{dz}\frac{\delta dv}{dy}\\ \frac{du}{dx}\frac{\delta dw}{dz} + \frac{du}{dz}\frac{\delta dw}{dx} + \frac{dv}{dy}\frac{\delta dw}{dz} + \frac{dv}{dz}\frac{\delta dw}{dy} + \frac{dw}{dx}\frac{\delta dw}{dx} + \frac{dw}{dy}\frac{\delta dw}{dy} + 3\frac{dw}{dz}\frac{\delta dw}{dz}\end{cases}\\
+ \ &\mathbf{S}ds^2E(u\delta u + v\delta v + w\delta w). \qquad (57)
\end{aligned}$$

Here, Here, $\mathbf{S}$ means the integration in the total surface of the fluid, E of (56) must vary in accordance with the nature of solid with which the fluid contacts. Shifting d to the front of δ of the middle term of the right hand-side of (57) and by Taylor expansion using the integration by parts

$$(3\text{-}25)_{Nf} \quad \varepsilon \iiint dxdydz \begin{cases} \left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dzdx}\right)\delta u \\ \left(2\frac{d^2u}{dxdy} + \frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2w}{dydz}\right)\delta v \\ \left(2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz} + \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2}\right)\delta w \end{cases} \tag{58}$$

$$+ \quad \varepsilon \iint dy'dz'\big[\Big(3\frac{du'}{dx'} + \frac{dv'}{dy'} + \frac{dw'}{dz'}\Big)\delta u' + \Big(\frac{du'}{dy'} + \frac{dv'}{dx'}\Big)\delta v' + \Big(\frac{du'}{dz'} + \frac{dw'}{dx'}\Big)\delta w'\big] \tag{59}$$

$$+ \quad \varepsilon \iint dx'dz'\big[\Big(\frac{du'}{dy'} + \frac{dv'}{dx'}\Big)\delta u' + \Big(\frac{du'}{dx'} + 3\frac{dv'}{dy'} + \frac{dw'}{dz'}\Big)\delta v' + \Big(\frac{dv'}{dz'} + \frac{dw'}{dy'}\Big)\delta w'\big] \tag{60}$$

$$+ \quad \varepsilon \iint dx'dy'\big[\Big(\frac{du'}{dz'} + \frac{dw'}{dx'}\Big)\delta u' + \Big(\frac{dv'}{dz'} + \frac{dw'}{dy'}\Big)\delta v' + \Big(\frac{du'}{dx'} + \frac{dv'}{dy'} + 3\frac{dw'}{dz'}\Big)\delta w'\big] \tag{61}$$

$$- \quad \varepsilon \iint dy''dz''\big[\Big(3\frac{du''}{dx''} + \frac{dv''}{dy''} + \frac{dw''}{dz''}\Big)\delta u'' + \Big(\frac{du''}{dy''} + \frac{dv''}{dx''}\Big)\delta v'' + \Big(\frac{du''}{dz''} + \frac{dw''}{dx''}\Big)\delta w''\big] \tag{62}$$

$$- \quad \varepsilon \iint dx''dz''\big[\Big(\frac{du''}{dy''} + \frac{dv''}{dx''}\Big)\delta u'' + \Big(\frac{du''}{dx''} + 3\frac{dv''}{dy''} + \frac{dw''}{dz''}\Big)\delta v'' + \Big(\frac{dv''}{dz''} + \frac{dw''}{dy''}\Big)\delta w''\big] \tag{63}$$

$$- \quad \varepsilon \iint dx''dy''\big[\Big(\frac{du''}{dz''} + \frac{dw''}{dx''}\Big)\delta u'' + \Big(\frac{dv''}{dz''} + \frac{dw''}{dy''}\Big)\delta v'' + \Big(\frac{du''}{dx''} + \frac{dv''}{dy''} + 3\frac{dw''}{dz''}\Big)\delta w''\big] \tag{64}$$

($\Downarrow$) By the way, we show again $(3\text{-}25)_{Nf}$ keeping the tensorial structure :

$$(3\text{-}25)_{Nf} \quad \varepsilon \iiint dxdydz \begin{cases} \left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dzdx}\right)\delta u \\ \left(2\frac{d^2u}{dxdy} + \frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2w}{dydz}\right)\delta v \\ \left(2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz} + \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2}\right)\delta w \end{cases}$$

$$+ \quad \varepsilon \iint \begin{cases} dy'dz'\big[\Big(3\frac{du'}{dx'} + \frac{dv'}{dy'} + \frac{dw'}{dz'}\Big)\delta u' + \Big(\frac{du'}{dy'} + \frac{dv'}{dx'}\Big)\delta v' + \Big(\frac{du'}{dz'} + \frac{dw'}{dx'}\Big)\delta w'\big] \\ dx'dz'\big[\Big(\frac{du'}{dy'} + \frac{dv'}{dx'}\Big)\delta u' + \Big(\frac{du'}{dx'} + 3\frac{dv'}{dy'} + \frac{dw'}{dz'}\Big)\delta v' + \Big(\frac{dv'}{dz'} + \frac{dw'}{dy'}\Big)\delta w'\big] \\ dx'dy'\big[\Big(\frac{du'}{dz'} + \frac{dw'}{dx'}\Big)\delta u' + \Big(\frac{dv'}{dz'} + \frac{dw'}{dy'}\Big)\delta v' + \Big(\frac{du'}{dx'} + \frac{dv'}{dy'} + 3\frac{dw'}{dz'}\Big)\delta w'\big] \end{cases} \quad \cdots \text{first order}$$

$$- \quad \varepsilon \iint \begin{cases} dy''dz''\big[\Big(3\frac{du''}{dx''} + \frac{dv''}{dy''} + \frac{dw''}{dz''}\Big)\delta u'' + \Big(\frac{du''}{dy''} + \frac{dv''}{dx''}\Big)\delta v'' + \Big(\frac{du''}{dz''} + \frac{dw''}{dx''}\Big)\delta w''\big] \\ dx''dz''\big[\Big(\frac{du''}{dy''} + \frac{dv''}{dx''}\Big)\delta u'' + \Big(\frac{du''}{dx''} + 3\frac{dv''}{dy''} + \frac{dw''}{dz''}\Big)\delta v'' + \Big(\frac{dv''}{dz''} + \frac{dw''}{dy''}\Big)\delta w''\big] \\ dx''dy''\big[\Big(\frac{du''}{dz''} + \frac{dw''}{dx''}\Big)\delta u'' + \Big(\frac{dv''}{dz''} + \frac{dw''}{dy''}\Big)\delta v'' + \Big(\frac{du''}{dx''} + \frac{dv''}{dy''} + 3\frac{dw''}{dz''}\Big)\delta w''\big] \end{cases} \quad \cdots \text{second order}$$

($\Downarrow$) By the way, if we check the ε terms of $\delta u', \delta v', \delta w'$, after replacing $\mathbf{u} = \{u, v, w\}$ of fluid $\Leftrightarrow$ $\{x, y, z\}$ of elastic solid, and the coordinate system : $\{x, y, z\}$ of fluid $\Leftrightarrow$ $\{a, b, c\}$ of elastic solid, then we can see the coincidence with the tensor between the equation (38) or (65) in elastic solid and (59)-(61) in the first order terms of fluid as follows [35]:

$$\int ds \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \varepsilon \begin{bmatrix} 3\frac{dx'}{da'} + \frac{dy'}{db'} + \frac{dz'}{dc'} & \frac{dx'}{db'} + \frac{dy'}{da'} & \frac{dx'}{dc'} + \frac{dz'}{da'} \\ \frac{dx'}{db'} + \frac{dy'}{da'} & \frac{dx'}{da'} + 3\frac{dy'}{db'} + \frac{dz'}{dc'} & \frac{dy'}{dc'} + \frac{dz'}{db'}, \\ \frac{dx'}{dc'} + \frac{dz'}{da'} & \frac{dy'}{dc'} + \frac{dz'}{db'} & \frac{dx'}{da'} + \frac{dy'}{db'} + 3\frac{dz'}{dc'} \end{bmatrix} \begin{bmatrix} \iint db'\,dc' \\ \iint da'dc' \\ \iint da'db' \end{bmatrix}, \tag{65}$$

here this tensor is symmetric.

Using the following equations deduced from the conservation law :

$$(3-26)_{Nf} \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

and

$$(3\text{-}27)_{Nf} \quad \begin{cases} \frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = \frac{d^2u}{dx^2} + \frac{d^2v}{dxdy} + \frac{d^2w}{dxdz} = 0, \\ \frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = \frac{d^2v}{dy^2} + \frac{d^2u}{dxdy} + \frac{d^2w}{dydz} = 0, \\ \frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = \frac{d^2w}{dz^2} + \frac{d^2u}{dxdz} + \frac{d^2v}{dydz} = 0, \end{cases}$$

[35]($\Downarrow$) Navier [46] neglected the terms below the second Taylor expansion in elastic case.

we get the short expression as follows :

$$
\begin{aligned}
(3\text{-}28)_{Nf} \quad & \varepsilon \iiint dxdydz\Big[\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)\delta u+\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)\delta v \\
+ \quad & \Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)\delta w\Big] \\
+ \quad & \varepsilon \iint dy'dz'[2\frac{du'}{dx'}\delta u'+\Big(\frac{du'}{dy'}+\frac{dv'}{dx'}\Big)\delta v'+\Big(\frac{du'}{dz'}+\frac{dw'}{dx'}\Big)\delta w'] \\
+ \quad & \varepsilon \iint dx'dz'[\Big(\frac{du'}{dy'}+\frac{dv'}{dx'}\Big)\delta u'+2\frac{dv'}{dy'}\delta v'+\Big(\frac{dv'}{dz'}+\frac{dw'}{dy'}\Big)\delta w'] \\
+ \quad & \varepsilon \iint dx'dy'[\Big(\frac{du'}{dz'}+\frac{dw'}{dx'}\Big)\delta u'+\Big(\frac{dv'}{dz'}+\frac{dw'}{dy'}\Big)\delta v'+2\frac{dw'}{dz'}\delta w'] \\
- \quad & \varepsilon \iint dy''dz''[2\frac{du''}{dx''}\delta u''+\Big(\frac{du''}{dy''}+\frac{dv''}{dx''}\Big)\delta v''+\Big(\frac{du''}{dz''}+\frac{dw''}{dx''}\Big)\delta w''] \\
- \quad & \varepsilon \iint dx''dz''[\Big(\frac{du''}{dy''}+\frac{dv''}{dx''}\Big)\delta u''+2\frac{dv''}{dy''}\delta v''+\Big(\frac{dv''}{dz''}+\frac{dw''}{dy''}\Big)\delta w''] \\
- \quad & \varepsilon \iint dx''dy''[\Big(\frac{du''}{dz''}+\frac{dw''}{dx''}\Big)\delta u''+\Big(\frac{dv''}{dz''}+\frac{dw''}{dy''}\Big)\delta v''+2\frac{dw''}{dz''}\delta w''] \qquad (66)
\end{aligned}
$$

We show again $(3\text{-}28)_{Nf}$ keeping the tensorial structure :

$$
\begin{aligned}
&(3\text{-}28)_{Nf} \\
&\quad \varepsilon \iiint dxdydz\Big[\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)\delta u+\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)\delta v+\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)\delta w\Big] \\
&+ \quad \varepsilon \iint \begin{cases} dy'dz'[2\frac{du'}{dx'}\delta u'+\Big(\frac{du'}{dy'}+\frac{dv'}{dx'}\Big)\delta v'+\Big(\frac{du'}{dz'}+\frac{dw'}{dx'}\Big)\delta w'] \\ dx'dz'[\Big(\frac{du'}{dy'}+\frac{dv'}{dx'}\Big)\delta u'+2\frac{dv'}{dy'}\delta v'+\Big(\frac{dv'}{dz'}+\frac{dw'}{dy'}\Big)\delta w'] & \cdots\text{first order} \\ dx'dy'[\Big(\frac{du'}{dz'}+\frac{dw'}{dx'}\Big)\delta u'+\Big(\frac{dv'}{dz'}+\frac{dw'}{dy'}\Big)\delta v'+2\frac{dw'}{dz'}\delta w'] \end{cases} \\
&- \quad \varepsilon \iint \begin{cases} dy''dz''[2\frac{du''}{dx''}\delta u''+\Big(\frac{du''}{dy''}+\frac{dv''}{dx''}\Big)\delta v''+\Big(\frac{du''}{dz''}+\frac{dw''}{dx''}\Big)\delta w''] \\ dx''dz''[\Big(\frac{du''}{dy''}+\frac{dv''}{dx''}\Big)\delta u''+2\frac{dv''}{dy''}\delta v''+\Big(\frac{dv''}{dz''}+\frac{dw''}{dy''}\Big)\delta w''] & \cdots\text{second order} \qquad (67) \\ dx''dy''[\Big(\frac{du''}{dz''}+\frac{dw''}{dx''}\Big)\delta u''+\Big(\frac{dv''}{dz''}+\frac{dw''}{dy''}\Big)\delta v''+2\frac{dw''}{dz''}\delta w''] \end{cases}
\end{aligned}
$$

Considering

- that $\mathbf{S}ds^2E(u\delta u+v\delta v+w\delta w)$ of (57) is zero,
- that all the remaining terms of the (66) are zero,

then, combining the first term of right-hand side of (57) with the first term of (66), we get the last expression as follows :

$$
(3\text{-}29)_{Nf} \quad 0 = \iiint dxdydz \begin{cases} [P-\frac{dp}{dx}-\rho\Big(\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\Big)+\varepsilon\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)]\delta u \\ [Q-\frac{dp}{dy}-\rho\Big(\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}\Big)+\varepsilon\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)]\delta v \\ [R-\frac{dp}{dz}-\rho\Big(\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}\Big)+\varepsilon\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)]\delta w \end{cases} \qquad (68)
$$

At last, solving what Navier called *the indeterminate equations*, we get (23) from (68), combining the terms under the symbol of integral of the right-hand side of (68) with zero of the left-hand side of (68).

On the other hand, to deduce (69) from (66), we transpose (59)-(61) as follows :

$$
(3\text{-}30)_{Nf} \begin{cases} Eu\delta u+\varepsilon\big[2\iint dy'dz'\frac{du'}{dx'}+\iint dx'dz'\Big(\frac{du'}{dy'}+\frac{dv'}{dx'}\Big)+\iint dx'dy'\Big(\frac{du'}{dz'}+\frac{dw'}{dx'}\Big)\big]\delta u'+\cdots\delta u''+\cdots=0 \\ Ev\delta v+\varepsilon\big[\iint dy'dz'\Big(\frac{du'}{dy'}+\frac{dv'}{dx'}\Big)+2\iint dx'dz'\frac{dv'}{dy'}+\iint dx'dy'\Big(\frac{dv'}{dz'}+\frac{dw'}{dy'}\Big)\big]\delta v'+\cdots\delta v''+\cdots=0 \\ Ew\delta w+\varepsilon\big[\iint dy'dz'\Big(\frac{du'}{dz'}+\frac{dw'}{dx'}\Big)+\iint dx'dz'\Big(\frac{dv'}{dz'}+\frac{dw'}{dy'}\Big)+2\iint dx'dy'\frac{dw'}{dz'}\big]\delta w'+\cdots\delta w''+\cdots=0. \end{cases}
$$

A.2.5. **Boundary condition.**

About the handling of $\mathbf{S}ds^2 E(u\delta u + v\delta v + w\delta w)$ of (57) and all the remaining terms of the (66), Navier explains as follows : regarding the conditions which react at the points of the surface of the fluid, if we substitute

- $dydz \quad \to \quad ds^2 \cos l, \quad l$: the angles by which the tangent plane makes on the surface frame with the plane yz,
- $dxdz \quad \to \quad ds^2 \cos m, \quad m$: similarly, the angles with the plane xz,
- $dxdy \quad \to \quad ds^2 \cos n, \quad n$: similarly, the angles with the plane xy,
- $\iint dydz, \iint dxdz, \iint dxdy \quad \to \quad \mathbf{S}ds^2$, where $\mathbf{S}$ is the sign of integral in respect to ds^2 on the surface,

then because the affected terms by the quantities $\delta u, \delta v$ and δw respectively are reduced to zero, the following determinate equations should hold for any points of the surface of the fluid :

$$(3\text{-}32)_{Nf} \quad \begin{cases} Eu + \varepsilon[\cos l 2\frac{du}{dx} + \cos m\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \cos n\left(\frac{du}{dz} + \frac{dw}{dx}\right)] = 0, \\ Ev + \varepsilon[\cos l\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \cos m 2\frac{dv}{dy} + \cos n\left(\frac{dv}{dz} + \frac{dw}{dy}\right)] = 0, \\ Ew + \varepsilon[\cos l\left(\frac{du}{dz} + \frac{dw}{dx}\right) + \cos m\left(\frac{dv}{dz} + \frac{dw}{dy}\right) + \cos n 2\frac{dw}{dz}] = 0, \end{cases} \tag{69}$$

here the value of the constant E which is varied according to the nature of the solid with which the fluid is in contact. (69) express the boundary condition. The first terms of the left-hand side of (69) are defined by (56) for the expression which we seek for the momentum of the total actions which caused between the molecules of the boundary and the fluid, and the second terms are the normal derivatives gained from (66). Here, (69) is put by :

$$E\begin{bmatrix} u \\ v \\ w \end{bmatrix} + \varepsilon \begin{bmatrix} 2\frac{du}{dx} & \frac{du}{dy} + \frac{dv}{dx} & \frac{du}{dz} + \frac{dw}{dx} \\ \frac{du}{dy} + \frac{dv}{dx} & 2\frac{dv}{dy} & \frac{dv}{dz} + \frac{dw}{dy} \\ \frac{du}{dz} + \frac{dw}{dx} & \frac{dv}{dz} + \frac{dw}{dy} & 2\frac{dw}{dz} \end{bmatrix} \begin{bmatrix} \cos l \\ \cos m \\ \cos n \end{bmatrix} = 0 \tag{70}$$

($\Downarrow$) If putting the basis of the tensor as $\begin{bmatrix} \cos l & \cos m & \cos n \end{bmatrix}^T$, then the tensor part of (70) is expressed as follows :

$$t_{ij} = \varepsilon[\{2v_{i,j} - (v_{i,j} + v_{j,i})\}\delta_{ij} + (v_{i,j} + v_{j,i})] = \varepsilon\{0\delta_{ij} + (v_{i,j} + v_{j,i})\} = \varepsilon(v_{i,j} + v_{j,i}).$$

Moreover, by using Darrigol's simple notation[36], we can express this condition as

$$E\mathbf{v} + \varepsilon\partial_\perp \mathbf{v}_\parallel = \mathbf{0},$$

where $\partial_\perp$ is the normal derivative, and $\mathbf{v}_\parallel$ is the component of the fluid velocity parallel to the surface.

($\Downarrow$) We have one question. Why Navier's E implies in the today's NS equations, in which E is not used ?

[36]($\Downarrow$) Darrigol [10, p.115]

A.3. Cauchy's deduction of tensor.

A.3.1. Deduction of the equations of accelerated force.

[37] We show the summary of Cauchy's twelve assumptions in Table 11 framed below, which are numbered #1 to #12 in the following.

TABLE 11. Assumptions of the system of particles by Cauchy

no	item	ref. equations
1	mouvement par des forces d'attraction ou de repulsion mutuelle	
2	La lettre **S** indiquant une somme de termes semblables, mais relatifs aux diverses molécules $m, m', \cdots$, et la signe $\pm$ devant être réduit au signe + ou au signe − suivant que la masse m sera attirée ou repoussée par molécule m.	$(3)_C$
3-1	L'etat du système de points matériels soit changé	$(4)_C - (6)_C$
3-2	Les molécules m, m, m', $\cdots$ se déplacent dans l'espace, mais de manière que la distance de deux molécules m et m varie dans un rapport peu différent de l'unité	$(7)_C$
4	ξ, η, ζ : des fonctions de a, b, c, qui représentent las déplacements très petits et parallèles aux axes d'une molécule quelconque m	$(8)_C - (11)_C$
5	Les déplacements ξ, η, ζ sont très petits, alors, en considérant ces déplacements comme infiniment petits du premier ordre, et négligeant les infiniment petits du second ordre.	$(12)_C - (31)_C$
6	Les équations qui expriment l'equilibre ou le mouvement du système des masses m, m, m', $\cdots$ soumises, non seulement à leurs attractions ou répulsions mutuelles, mais à de nouvelles forces accélératrice.	$(32)_C - (34)_C$
7	Les sommes comprises dans les formules $(26)_C$ et $(30)_C$ s'evanouissent. Les masses m, m', m'', $\cdots$ étant deux à deux égales entre elles, sont distribuées, symétriquement de part et d'autre de la molécule m	$(35)_C - (36)_C$
8	Parmi les sommes comprises dans les formules $(26)_C$, $(30)_C$ et $(31)_C$, toutes celles qui renferment des puissances impaires de $\cos\alpha$, de $\cos\beta$, ou de $\cos\gamma$ s'evanouissent.	$(37)_C - (40)_C$
8-1	Les molécules m, m', m'', $\cdots$ sont distribuées symétriquement par rapport à chacun des trois plans	
8-2	Deux molécules symétriquement placées à l'egard d'un des trois premiers plans offrent toujours des masses égales	
9	Les molécules $m, m', m'', \cdots$ primitivement distribuées de la même manière par rapport aux trois plans menés par la molécule m parallèlement aux plans coordonnés	$(41)_C - (42)_C$
10	Les molécules $m, m', m'', \cdots$ primitivement distribuées autour de la molécule m, de manière que les sommes comprises dans les équations (37), (38), (39) deviennent indépendantes des directions assignées aux axes des x, y, z	$(43)_C - (52)_C$

• ¶ 1. At first, we consider that the great number of molecules or material points are arbitrarily distributed in a certain potion of the space and its motion are brought about by the forces of mutual attraction or repulsion. Strictly speaking, we must cite Cauchy's assumptions as follows :

> #1. Considérons un très grand nombre de molécules ou points matériels distribués arbitrairement dans une portion de l'espace, et solicités au mouvement par des forces d'attraction ou de repulsion mutuelle. [7, p.227]

The definition of the various terms are :

- m (in roman style) : mass of this molecule ;
- m, m', m'' (in italic style) : masses of another molecules, of which the existing are assumed at a certain time ;
- a, b, c : the coordinate values of the molecule m on the rectangular coordinates : x, y, z ;
- $a+\Delta a$, $b+\Delta b$, $c+\Delta c$: the coordinate values of the molecule m ;
- r : the distance between m and m (with scalar value) ;

[37](⇓) For convenience' sake, we put "• ¶ (number)" as the paragraph number which is not in the text by Cauchy, but we count the number and show it, and moreover, we suppose the sections.

- $\alpha,\ \beta,\ \gamma$: the angles formed by the vector of ray : $\mathbf{r}$ with each half axis of the positive coordinates.

• ¶ 2. Cauchy's hypothesis of molecular activities are as follows :

> #2. la lettre **S** indiquant une somme de termes semblables, mais relatifs aux diverses molécules $m, m', \cdots$, et la signe $\pm$ devant être réduit au signe $+$ ou au signe $-$ suivant que la masse m sera attirée ou repoussée par molécule m. Ajoutons que les quantités $\Delta a,\ \Delta b,\ \Delta c$ pourront être exprimées en fonction de r et des angles $\alpha,\ \beta,\ \gamma$ par les formules : [7, p.228]

$$(3)_C \quad \Delta a = r\cos\alpha, \quad \Delta b = r\cos\beta, \quad \Delta c = r\cos\gamma.$$

• ¶ 3.

> #3. Supposons maintenant
> - que l'etat du système de points matériels soit changé, et
> - que les molécules m, m, $m', \cdots$ se déplacent dans l'espace, mais de manière que la distance de deux molécules m et m varie dans un rapport peu différent de l'unité.
>
> #4. Soient $\xi,\ \eta,\ \zeta$: des fonctions de $a,\ b,\ c$, qui représentent les déplacements très petits et parallèles aux axes d'une molécule quelconque m ; [7, p.228]

- $x,\ y,\ z$; $x+\Delta x,\ y+\Delta y,\ z+\Delta z$: les coordonnées des molécules m, m dans le nouvel état du système ;
- $r(1+\varepsilon)$: la distance des molécules m, m dans ce nouvel état ;
- ε : la dilatation très petite de la longueur r dans le passage du premier état au second ; et l'on aura évidement

$$(4)_C \quad x = a+\xi, \quad y = b+\eta, \quad z = c+\zeta.$$

$$(5)_C \quad \begin{cases} \Delta x = \Delta a + \Delta\xi = r\cos\alpha + \Delta\xi, \\ \Delta y = \Delta b + \Delta\eta = r\cos\beta + \Delta\eta, \\ \Delta z = \Delta c + \Delta\zeta = r\cos\gamma + \Delta\zeta. \end{cases}$$

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$$\begin{aligned}(6)_C \quad r^2(1+\varepsilon)^2 &= (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= r^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) + 2r(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta) + (\Delta\xi)^2 + (\Delta\eta)^2 + (\Delta\zeta)^2 \\ &= r^2 + 2r(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta) + (\Delta\xi)^2 + (\Delta\eta)^2 + (\Delta\zeta)^2.\end{aligned}$$

Here we used the following by $(8)_C$:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{(\Delta a)^2}{r^2} + \frac{(\Delta b)^2}{r^2} + \frac{(\Delta c)^2}{r^2} = 1.$$

$$(7)_C \quad 1+\varepsilon = \sqrt{1 + \frac{2}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta) + \frac{1}{r^2}(\Delta\xi)^2 + (\Delta\eta)^2 + (\Delta\zeta)^2)},$$

We can put the following with the equivalent expressions :[39]

$$(8)_C, (9)_C \quad \cos\alpha = \frac{\Delta a}{r} = \frac{\Delta x}{r(1+\varepsilon)}, \quad \cos\beta = \frac{\Delta b}{r} = \frac{\Delta y}{r(1+\varepsilon)}, \quad \cos\gamma = \frac{\Delta a}{r} = \frac{\Delta z}{r(1+\varepsilon)}.$$

• ¶ 4. After all, the algebraic projections of resultant forces of attractions and repulsions performed by the molecules $m, m', m'', \cdots$ on the molecule m come to be equal three products :

---

[38]($\Downarrow$) For misprints in Cauchy [7, p.228], we substituted "'+" by the second "=" in each line of $(5)_C$, for example, from $+r\cos\alpha$ into $= r\cos\alpha$ in sic. : $\Delta x = \Delta a + \Delta\xi + r\cos\alpha + \Delta\xi$.

[39]($\Downarrow$) In Cauchy, it reads :

les cosinus seront représentés, non plus par $(8)_C$ $\cos\alpha = \frac{\Delta a}{r}, \cdots$ main par $(9)_C$ $\frac{\Delta x}{r(1+\varepsilon)}, \cdots$. (sic).

Then we can state the expression combining each term of $(8)_C$ with the coressponding term of $(9)_C$.

$$(10)_C \quad \begin{cases} \mathbf{mS}\Big\{ \pm m\frac{\Delta x}{r(1+\varepsilon)}\mathrm{f}[r(1+\varepsilon)]\Big\}, \\ \mathbf{mS}\Big\{ \pm m\frac{\Delta y}{r(1+\varepsilon)}\mathrm{f}[r(1+\varepsilon)]\Big\}, \\ \mathbf{mS}\Big\{ \pm m\frac{\Delta z}{r(1+\varepsilon)}\mathrm{f}[r(1+\varepsilon)]\Big\}. \end{cases}$$

Here we put the accelerated force as follows :

$$(11)_C \quad \begin{cases} X = \mathbf{S}\Big\{ \pm m\frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta x\Big\}, \\ Y = \mathbf{S}\Big\{ \pm m\frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta y\Big\}, \\ Z = \mathbf{S}\Big\{ \pm m\frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta z\Big\}. \end{cases}$$

les trois products : mX, mY, mZ, et les trois quantités : X, Y, Z représenteront les projections algébriques :

- de la résultante dont il s'agit ;
- de cette résultante divisée par m, ou, qui revient au même, de la force accélératrice qui sollicitera la molécule m et qui sera due aux actions des molécules $m, m', m'', \cdots$

• ¶ 5. The deplacements : ξ, η, ζ are infinitesimal, then we can neglect these values of second order.

> #5. Dans l'hypothèse que nous avons admise, c'est-à-dire lorsque les déplacements ξ, η, ζ sont très petits, alors, en considérant ces déplacements comme infiniment petits du premier ordre, et négligeant les infiniment petits du second ordre, on tire de l'equation $(7)_C$: [7, p.230]

$$(7)_C \quad \Rightarrow \quad (12)_C \quad \varepsilon = \frac{1}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta).$$

$$(13)_C \quad \frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)} = (1-\varepsilon)\frac{\mathrm{f}(r)+\varepsilon r\mathrm{f}'(r)}{r} = \frac{\mathrm{f}(r)}{r} + \varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)-\varepsilon\mathrm{f}'(r)}{r} \simeq \frac{\mathrm{f}(r)}{r} + \varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}.$$

($\Downarrow$) Here, we introduce the method of simplified calculation by Cauchy : $(11)_C$ turns into from $(5)_C$ and $(13)_C$ as follows :

$$\begin{aligned} \frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta x &= \frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}(r\cos\alpha + \Delta\xi) \quad (\because \text{ from } (5)_C\ ) \\ &= \Big(1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\Big)\Big(\frac{\mathrm{f}(r)}{r}\Big)\Big(r\cos\alpha + \frac{\Delta\xi r\cos\alpha}{r\cos\alpha}\Big) \quad (\because \text{ from } (13)_C\ ) \\ &= \Big(1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\Big)\Big(1+\frac{\Delta\xi}{r\cos\alpha}\Big)\Big(\frac{\mathrm{f}(r)}{r}\Big)r\cos\alpha \\ &= \Big\{1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \frac{\Delta\xi}{r\cos\alpha} + \Big(\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\Big)\Big(\frac{\Delta\xi}{r\cos\alpha}\Big)\Big\}\mathrm{f}(r)\cos\alpha \\ &\simeq \Big(1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \frac{\Delta\xi}{r\cos\alpha}\Big)\mathrm{f}(r)\cos\alpha. \end{aligned}$$

Similarly, we can get the following :

$$\begin{aligned} \frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta y &= \Big\{1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \frac{\Delta\eta}{r\cos\beta} + \Big(\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\Big)\Big(\frac{\Delta\eta}{r\cos\beta}\Big)\Big\}\mathrm{f}(r)\cos\beta \\ &\simeq \Big(1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \frac{\Delta\eta}{r\cos\beta}\Big)\mathrm{f}(r)\cos\beta, \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{f}[r(1+\varepsilon)]}{r(1+\varepsilon)}\Delta z &= \Big\{1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \frac{\Delta\zeta}{r\cos\gamma} + \Big(\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\Big)\Big(\frac{\Delta\zeta}{r\cos\gamma}\Big)\Big\}\mathrm{f}(r)\cos\gamma \\ &\simeq \Big(1+\varepsilon\frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)} + \frac{\Delta\zeta}{r\cos\gamma}\Big)\mathrm{f}(r)\cos\gamma. \end{aligned}$$

(⇑)
According to Cauchy's assumption, we get the following $(14)_C$ from $(11)_C$ by combining $(5)_C$ with $(13)_C$.

$$(14)_C \quad \begin{cases} X = \mathbf{S}\Big\{ \pm m\big[1 + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\varepsilon + \frac{\Delta\xi}{r\cos\alpha}\big]\cos\alpha\mathrm{f}(r)\Big\}, \\ Y = \mathbf{S}\Big\{ \pm m\big[1 + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\varepsilon + \frac{\Delta\eta}{r\cos\beta}\big]\cos\beta\mathrm{f}(r)\Big\}, \\ Z = \mathbf{S}\Big\{ \pm m\big[1 + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{\mathrm{f}(r)}\varepsilon + \frac{\Delta\zeta}{r\cos\gamma}\big]\cos\gamma\mathrm{f}(r)\Big\}. \end{cases}$$

• ¶ 6. From the initial condition, by considering the equilibrium of X, Y, Z, we get some results.

lorsque le premier état du système des points matériels est état d'équibre, il suffit de remplacer ξ, η, ζ par zéro dans les formules $(14)_C$, pour faire évanouir X, Y, Z.

Then we get $(15)_C$ as follows :

$$(15)_C \quad \mathbf{S}[\pm m\cos\alpha\mathrm{f}(r)] = 0, \quad \mathbf{S}[\pm m\cos\beta\mathrm{f}(r)] = 0, \quad \mathbf{S}[\pm m\cos\gamma\mathrm{f}(r)] = 0.$$

$$(16)_C \quad \begin{cases} X = \mathbf{S}\Big\{ \pm m\big[\{r\mathrm{f}'(r) - \mathrm{f}(r)\}\varepsilon\cos\alpha + \frac{\mathrm{f}(r)}{r}\Delta\xi\big]\Big\}, \\ Y = \mathbf{S}\Big\{ \pm m\big[\{r\mathrm{f}'(r) - \mathrm{f}(r)\}\varepsilon\cos\beta + \frac{\mathrm{f}(r)}{r}\Delta\eta\big]\Big\}, \\ Z = \mathbf{S}\Big\{ \pm m\big[\{r\mathrm{f}'(r) - \mathrm{f}(r)\}\varepsilon\cos\gamma + \frac{\mathrm{f}(r)}{r}\Delta\zeta\big]\Big\}. \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{S}\Big\{ \pm m \begin{bmatrix} \varepsilon\cos\alpha & \Delta\xi \\ \varepsilon\cos\beta & \Delta\eta \\ \varepsilon\cos\gamma & \Delta\zeta \end{bmatrix} \begin{bmatrix} (r\mathrm{f}'(r) - \mathrm{f}(r))\varepsilon \\ \frac{\mathrm{f}(r)}{r} \end{bmatrix} \Big\},$$

From $(12)_C$

$$\Rightarrow \quad \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{S}\Big\{ \pm m \begin{bmatrix} \frac{1}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta)\cos\alpha & \Delta\xi \\ \frac{1}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta)\cos\beta & \Delta\eta \\ \frac{1}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta)\cos\gamma & \Delta\zeta \end{bmatrix} \begin{bmatrix} (r\mathrm{f}'(r) - \mathrm{f}(r))\varepsilon \\ \frac{\mathrm{f}(r)}{r} \end{bmatrix} \Big\},$$

$(16)_C \quad \Rightarrow$

$$(17)_C \quad \begin{cases} X = \mathbf{S}\Big\{ \pm m\Big[\Big(\frac{\mathrm{f}(r)}{r} + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\cos^2\alpha\Big)\Delta\xi + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\big(\cos\alpha\cos\beta\Delta\eta + \cos\alpha\cos\gamma\Delta\zeta\big)\Big]\Big\}, \\ Y = \mathbf{S}\Big\{ \pm m\Big[\Big(\frac{\mathrm{f}(r)}{r} + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\cos^2\beta\Big)\Delta\eta + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\big(\cos\beta\cos\gamma\Delta\zeta + \cos\beta\cos\alpha\Delta\xi\big)\Big]\Big\}, \\ Z = \mathbf{S}\Big\{ \pm m\Big[\Big(\frac{\mathrm{f}(r)}{r} + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\cos^2\gamma\Big)\Delta\zeta + \frac{r\mathrm{f}'(r)-\mathrm{f}(r)}{r}\big(\cos\gamma\cos\alpha\Delta\xi + \cos\gamma\cos\beta\Delta\eta\big)\Big]\Big\}. \end{cases}$$

• ¶ 7. The formulation of accelerated force.

$$(18)_C \quad \begin{cases} \Delta\xi = r\Big(\dfrac{\partial\xi}{\partial a}\cos\alpha + \dfrac{\partial\xi}{\partial b}\cos\beta + \dfrac{\partial\xi}{\partial c}\cos\gamma\Big) \\ +\frac{r^2}{1\cdot 2}\Big(\frac{\partial^2\xi}{\partial a^2}\cos^2\alpha + \frac{\partial^2\xi}{\partial b^2}\cos^2\beta + \frac{\partial^2\xi}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2\xi}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2\xi}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2\xi}{\partial a\partial b}\cos\alpha\cos\beta\Big) \\ +\cdots, \\ \Delta\eta = r\Big(\frac{\partial\eta}{\partial a}\cos\alpha + \frac{\partial\eta}{\partial b}\cos\beta + \frac{\partial\eta}{\partial c}\cos\gamma\Big) \\ +\frac{r^2}{1\cdot 2}\Big(\frac{\partial^2\eta}{\partial a^2}\cos^2\alpha + \frac{\partial^2\eta}{\partial b^2}\cos^2\beta + \frac{\partial^2\eta}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2\eta}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2\eta}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2\eta}{\partial a\partial b}\cos\alpha\cos\beta\Big) \\ +\cdots, \\ \Delta\zeta = r\Big(\frac{\partial\zeta}{\partial a}\cos\alpha + \frac{\partial\zeta}{\partial b}\cos\beta + \frac{\partial\zeta}{\partial c}\cos\gamma\Big) \\ +\frac{r^2}{1\cdot 2}\Big(\frac{\partial^2\zeta}{\partial a^2}\cos^2\alpha + \frac{\partial^2\zeta}{\partial b^2}\cos^2\beta + \frac{\partial^2\zeta}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2\zeta}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2\zeta}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2\zeta}{\partial a\partial b}\cos\alpha\cos\beta\Big) \\ +\cdots, \end{cases}$$

$$(19)_C \quad \frac{\partial\xi}{\partial a}, \quad \frac{\partial\xi}{\partial b}, \quad \frac{\partial\xi}{\partial c}, \quad \frac{\partial\eta}{\partial a}, \quad \frac{\partial\eta}{\partial b}, \quad \frac{\partial\eta}{\partial c}, \quad \frac{\partial\zeta}{\partial a}, \quad \frac{\partial\zeta}{\partial b}, \quad \frac{\partial\zeta}{\partial c},$$

$$(20)_C \quad \begin{cases} \frac{\partial^2\xi}{\partial a^2}, & \frac{\partial^2\xi}{\partial b^2}, & \frac{\partial^2\xi}{\partial c^2}, & \frac{\partial^2\xi}{\partial b\partial c}, & \frac{\partial^2\xi}{\partial c\partial a}, & \frac{\partial^2\xi}{\partial a\partial b}, \\ \frac{\partial^2\eta}{\partial a^2}, & \frac{\partial^2\eta}{\partial b^2}, & \frac{\partial^2\eta}{\partial c^2}, & \frac{\partial^2\eta}{\partial b\partial c}, & \frac{\partial^2\eta}{\partial c\partial a}, & \frac{\partial^2\eta}{\partial a\partial b}, \\ \frac{\partial^2\zeta}{\partial a^2}, & \frac{\partial^2\zeta}{\partial b^2}, & \frac{\partial^2\zeta}{\partial c^2}, & \frac{\partial^2\xi}{\partial b\partial c}, & \frac{\partial^2\zeta}{\partial c\partial a}, & \frac{\partial^2\zeta}{\partial a\partial b}, \end{cases}$$

We show ξ_1, η_1, ζ_1 with Jacobian :

$$(21)_C \quad \begin{cases} \xi_1 = \frac{\partial \xi}{\partial a}\cos\alpha + \frac{\partial \xi}{\partial b}\cos\beta + \frac{\partial \xi}{\partial c}\cos\gamma, \\ \eta_1 = \frac{\partial \eta}{\partial a}\cos\alpha + \frac{\partial \eta}{\partial b}\cos\beta + \frac{\partial \eta}{\partial c}\cos\gamma, \\ \zeta_1 = \frac{\partial \zeta}{\partial a}\cos\alpha + \frac{\partial \zeta}{\partial b}\cos\beta + \frac{\partial \zeta}{\partial c}\cos\gamma, \end{cases} \quad \Rightarrow^* \quad \begin{bmatrix} \xi_1 \\ \eta_1 \\ \zeta_1 \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial a} & \frac{\partial \xi}{\partial b} & \frac{\partial \xi}{\partial c} \\ \frac{\partial \eta}{\partial a} & \frac{\partial \eta}{\partial b} & \frac{\partial \eta}{\partial c} \\ \frac{\partial \zeta}{\partial a} & \frac{\partial \zeta}{\partial b} & \frac{\partial \zeta}{\partial c} \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}$$

$$(22)_C \quad \begin{cases} \xi_2 = \frac{\partial^2 \xi}{\partial a^2}\cos^2\alpha + \frac{\partial^2 \xi}{\partial b^2}\cos^2\beta + \frac{\partial^2 \xi}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2 \xi}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2 \xi}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2 \xi}{\partial a\partial b}\cos\alpha\cos\beta, \\ \eta_2 = \frac{\partial^2 \eta}{\partial a^2}\cos^2\alpha + \frac{\partial^2 \eta}{\partial b^2}\cos^2\beta + \frac{\partial^2 \eta}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2 \eta}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2 \eta}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2 \eta}{\partial a\partial b}\cos\alpha\cos\beta, \\ \zeta_2 = \frac{\partial^2 \zeta}{\partial a^2}\cos^2\alpha + \frac{\partial^2 \zeta}{\partial b^2}\cos^2\beta + \frac{\partial^2 \zeta}{\partial c^2}\cos^2\gamma + 2\frac{\partial^2 \xi}{\partial b\partial c}\cos\beta\cos\gamma + 2\frac{\partial^2 \zeta}{\partial c\partial a}\cos\gamma\cos\alpha + 2\frac{\partial^2 \zeta}{\partial a\partial b}\cos\alpha\cos\beta, \end{cases}$$

$$\Rightarrow^* \quad \begin{bmatrix} \xi_2 \\ \eta_2 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \xi}{\partial a^2} & \frac{\partial^2 \xi}{\partial b^2} & \frac{\partial^2 \xi}{\partial c^2} & \frac{\partial^2 \xi}{\partial b\partial c} & \frac{\partial^2 \xi}{\partial c\partial a} & \frac{\partial^2 \xi}{\partial a\partial b} \\ \frac{\partial^2 \eta}{\partial a^2} & \frac{\partial^2 \eta}{\partial b^2} & \frac{\partial^2 \eta}{\partial c^2} & \frac{\partial^2 \eta}{\partial b\partial c} & \frac{\partial^2 \eta}{\partial c\partial a} & \frac{\partial^2 \eta}{\partial a\partial b} \\ \frac{\partial^2 \zeta}{\partial a^2} & \frac{\partial^2 \zeta}{\partial b^2} & \frac{\partial^2 \zeta}{\partial c^2} & \frac{\partial^2 \xi}{\partial b\partial c} & \frac{\partial^2 \zeta}{\partial c\partial a} & \frac{\partial^2 \zeta}{\partial a\partial b} \end{bmatrix} \begin{bmatrix} \cos^2\alpha \\ \cos^2\beta \\ \cos^2\gamma \\ 2\cos\beta\cos\gamma \\ 2\cos\gamma\cos\alpha \\ 2\cos\alpha\cos\beta \end{bmatrix}$$

From $(18)_C$, we get the following :

$$(23)_C \quad \Delta\xi = r\Big(\xi_1 + \frac{r}{2}\xi_2\Big), \quad \Delta\eta = r\Big(\eta_1 + \frac{r}{2}\eta_2\Big), \quad \Delta\zeta = r\Big(\zeta_1 + \frac{r}{2}\zeta_2\Big),$$

and from $(12)_C$ and $(23)_C$, we get the following :

$$(24)_C \quad \varepsilon = \frac{1}{r}(\cos\alpha\Delta\xi + \cos\beta\Delta\eta + \cos\gamma\Delta\zeta) = \xi_1\cos\alpha + \eta_1\cos\beta + \zeta_1\cos\gamma + \frac{r}{2}\Big(\xi_2\cos\alpha + \eta_2\cos\beta + \zeta_2\cos\gamma\Big)$$

The equation $(14)_C$ turns into the following :

$$\boxed{(14)_C \quad \Rightarrow \quad (25)_C \quad X = X_0 + X_1 + X_2, \quad Y = Y_0 + Y_1 + Y_2, \quad Z = Z_0 + Z_1 + Z_2}$$

($\Downarrow$) Cauchy will calculate the following matrix :

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} + \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$$

($\Uparrow$)

$$\boxed{(26)_C \quad X_0 = \mathbf{S}[\pm m\cos\alpha \mathrm{f}(r)], \quad Y_0 = \mathbf{S}[\pm m\cos\beta \mathrm{f}(r)], \quad Z_0 = \mathbf{S}[\pm m\cos\gamma \mathrm{f}(r)]}$$

$$(27)_C \begin{cases} X_1 = \mathbf{S}[\pm m\xi_1 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm m(\xi_1\cos\alpha + \eta_1\cos\beta + \zeta_1\cos\gamma)\cos\alpha[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \\ Y_1 = \mathbf{S}[\pm m\eta_1 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm m(\xi_1\cos\alpha + \eta_1\cos\beta + \zeta_1\cos\gamma)\cos\beta[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \\ Z_1 = \mathbf{S}[\pm m\zeta_1 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm m(\xi_1\cos\alpha + \eta_1\cos\beta + \zeta_1\cos\gamma)\cos\gamma[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \end{cases}$$

$$(28)_C \begin{cases} X_2 = \mathbf{S}[\pm \frac{mr}{2}\xi_2 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm \frac{mr}{2}(\xi_2\cos\alpha + \eta_2\cos\beta + \zeta_2\cos\gamma)\cos\alpha[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \\ Y_2 = \mathbf{S}[\pm \frac{mr}{2}\eta_2 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm \frac{mr}{2}(\xi_2\cos\alpha + \eta_2\cos\beta + \zeta_2\cos\gamma)\cos\beta[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \\ Z_2 = \mathbf{S}[\pm \frac{mr}{2}\zeta_2 \mathrm{f}(r)] + \mathbf{S}\Big[\ \pm \frac{mr}{2}(\xi_2\cos\alpha + \eta_2\cos\beta + \zeta_2\cos\gamma)\cos\gamma[r\mathrm{f}'(r) - \mathrm{f}(r)]\ \Big], \end{cases}$$

We put $f(r)$ in brief as follows :

$$(29)_C \quad f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)] \qquad (71)$$

$$
(30)_C\begin{cases}
X_1 = X_0\frac{\partial\xi}{\partial a} + Y_0\frac{\partial\xi}{\partial b} + Z_0\frac{\partial\xi}{\partial c}\\
\quad + \frac{\partial\xi}{\partial a}\mathbf{S}[mf(r)\cos^3\alpha] + \frac{\partial\xi}{\partial b}\mathbf{S}[mf(r)\cos^2\alpha\cos\beta] + \frac{\partial\xi}{\partial c}\mathbf{S}[mf(r)\cos^2\alpha\cos\gamma]\\
\quad + \frac{\partial\eta}{\partial a}\mathbf{S}[mf(r)\cos^2\alpha\cos\beta] + \frac{\partial\eta}{\partial b}\mathbf{S}[mf(r)\cos\alpha\cos^2\beta] + \frac{\partial\eta}{\partial c}\mathbf{S}[mf(r)\cos\alpha\cos\beta\cos\gamma]\\
\quad + \frac{\partial\zeta}{\partial a}\mathbf{S}[mf(r)\cos^2\alpha\cos\gamma] + \frac{\partial\zeta}{\partial b}\mathbf{S}[mf(r)\cos\alpha\cos\beta\cos\gamma] + \frac{\partial\zeta}{\partial c}\mathbf{S}[mf(r)\cos\alpha\cos^2\gamma],\\
Y_1 = X_0\frac{\partial\eta}{\partial a} + Y_0\frac{\partial\eta}{\partial b} + Z_0\frac{\partial\eta}{\partial c}\\
\quad + \frac{\partial\xi}{\partial a}\mathbf{S}[mf(r)\cos^2\alpha\cos\beta] + \frac{\partial\xi}{\partial b}\mathbf{S}[mf(r)\cos\alpha\cos^2\beta] + \frac{\partial\xi}{\partial c}\mathbf{S}[mf(r)\cos\alpha\cos\beta\cos\gamma]\\
\quad + \frac{\partial\eta}{\partial a}\mathbf{S}[mf(r)\cos\alpha\cos^2\beta] + \frac{\partial\eta}{\partial b}\mathbf{S}[mf(r)\cos^3\beta] + \frac{\partial\eta}{\partial c}\mathbf{S}[mf(r)\cos^2\beta\cos\gamma]\\
\quad + \frac{\partial\zeta}{\partial a}\mathbf{S}[mf(r)\cos\alpha\cos\beta\cos\gamma] + \frac{\partial\zeta}{\partial b}\mathbf{S}[mf(r)\cos^2\beta\cos\gamma] + \frac{\partial\zeta}{\partial c}\mathbf{S}[mf(r)\cos\beta\cos^2\gamma],\\
Z_1 = X_0\frac{\partial\zeta}{\partial a} + Y_0\frac{\partial\zeta}{\partial b} + Z_0\frac{\partial\zeta}{\partial c} + \cdots
\end{cases}
$$

($\Downarrow$) The equations in $(30)_C$ have the $36(= 3\times 12)$ terms at maximum. ($\Uparrow$)

$$
\Rightarrow\quad\begin{cases}
X_1 = mf(r)\begin{bmatrix}
\frac{\partial\xi}{\partial a}\left(\frac{X_0}{mf(r)\cos\alpha} + \mathbf{S}\cos^2\alpha\right) & \frac{\partial\xi}{\partial b}\mathbf{S}\cos^2\alpha & \frac{\partial\xi}{\partial c}\mathbf{S}\cos^2\alpha\\
\frac{\partial\eta}{\partial a}\left(\frac{Y_0}{mf(r)\cos\alpha} + \mathbf{S}\cos\alpha\cos\beta\right) & \frac{\partial\eta}{\partial b}\mathbf{S}\cos\alpha\cos\beta & \frac{\partial\eta}{\partial c}\mathbf{S}\cos\alpha\cos\beta\\
\frac{\partial\zeta}{\partial a}\left(\frac{Z_0}{mf(r)\cos\alpha} + \mathbf{S}\cos\alpha\cos\gamma\right) & \frac{\partial\zeta}{\partial b}\mathbf{S}\cos\alpha\cos\gamma & \frac{\partial\zeta}{\partial c}\mathbf{S}\cos\alpha\cos\gamma
\end{bmatrix}\begin{bmatrix}\cos\alpha\\ \cos\beta\\ \cos\gamma\end{bmatrix}\\
Y_1 = mf(r)\begin{bmatrix}
\frac{\partial\xi}{\partial a}\mathbf{S}\cos\alpha\cos\beta & \frac{\partial\xi}{\partial b}\left(\frac{X_0}{mf(r)\cos\beta} + \mathbf{S}\cos\alpha\cos\beta\right) & \frac{\partial\xi}{\partial c}\mathbf{S}\cos\alpha\cos\beta\\
\frac{\partial\eta}{\partial a}\mathbf{S}\cos^2\beta & \frac{\partial\eta}{\partial b}\left(\frac{Y_0}{mf(r)\cos\beta} + \mathbf{S}\cos^2\beta\right) & \frac{\partial\eta}{\partial c}\mathbf{S}\cos^2\beta\\
\frac{\partial\zeta}{\partial a}\mathbf{S}\cos\beta\cos\gamma & \frac{\partial\zeta}{\partial b}\left(\frac{Z_0}{mf(r)\cos\beta} + \mathbf{S}\cos\beta\cos\gamma\right) & \frac{\partial\zeta}{\partial c}\mathbf{S}\cos\beta\cos\gamma
\end{bmatrix}\begin{bmatrix}\cos\alpha\\ \cos\beta\\ \cos\gamma\end{bmatrix}\\
Z_1 = mf(r)\begin{bmatrix}
\frac{\partial\xi}{\partial a}\mathbf{S}\cos\alpha\cos\gamma & \frac{\partial\xi}{\partial b}\mathbf{S}\cos\alpha\cos\gamma & \frac{\partial\xi}{\partial c}\left(\frac{X_0}{mf(r)\cos\gamma} + \mathbf{S}\cos\alpha\cos\gamma\right)\\
\frac{\partial\eta}{\partial a}\mathbf{S}\cos\beta\cos\gamma & \frac{\partial\eta}{\partial b}\mathbf{S}\cos\beta\cos\gamma & \frac{\partial\eta}{\partial c}\left(\frac{Y_0}{mf(r)\cos\gamma} + \mathbf{S}\cos\beta\cos\gamma\right)\\
\frac{\partial\zeta}{\partial a}\mathbf{S}\cos^2\gamma & \frac{\partial\zeta}{\partial b}\mathbf{S}\cos^2\gamma & \frac{\partial\zeta}{\partial c}\left(\frac{Z_0}{mf(r)\cos\gamma} + \mathbf{S}\cos^2\gamma\right)
\end{bmatrix}\begin{bmatrix}\cos\alpha\\ \cos\beta\\ \cos\gamma\end{bmatrix}
\end{cases}\qquad(72)
$$

We see X, Y and Z are computed according to $(25)_C$ by only X_2, Y_2, Z_2, because in $(26)_C, (27)_C$ and $(30)_C$, all the terms contain the terms of $\cos\alpha$ or $\cos\beta$ or $\cos\gamma$ in odd power, which become zero by summation under the symbol of $\not\mathbf{S}$.

$$
(31)_C\begin{cases}
X_2 = \frac{\partial^2\xi}{\partial a^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha\mathrm{f}(r)] + \frac{\partial^2\xi}{\partial b^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\beta\mathrm{f}(r)] + \frac{\partial^2\xi}{\partial c^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma\mathrm{f}(r)]\\
\quad + \ \frac{\partial^2\xi}{\partial b\partial c}\ \not\mathbf{S}[\pm mr\cos\beta\cos\gamma\mathrm{f}(r)] + \frac{\partial^2\xi}{\partial c\partial a}\ \not\mathbf{S}[\pm mr\cos\gamma\cos\alpha\mathrm{f}(r)] + \frac{\partial^2\xi}{\partial a\partial b}\ \not\mathbf{S}[\pm mr\cos\alpha\cos\beta\mathrm{f}(r)]\\
\quad + \ \frac{\partial^2\xi}{\partial a^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^4\alpha] + \frac{\partial^2\xi}{\partial b^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta] + \frac{\partial^2\xi}{\partial c^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\gamma]\\
\quad + \ \frac{\partial^2\xi}{\partial b\partial c}\ \not\mathbf{S}[mrf(r)\cos^2\alpha\cos\beta\cos\gamma] + \frac{\partial^2\xi}{\partial c\partial a}\ \not\mathbf{S}[mrf(r)\cos^3\alpha\cos\gamma] + \frac{\partial^2\xi}{\partial a\partial b}\ \not\mathbf{S}[mrf(r)\cos^3\alpha\cos\beta]\\
\quad + \ \frac{\partial^2\eta}{\partial a^2}\ \not\mathbf{S}[\frac{mr}{2}f(r)\cos^3\alpha\cos\beta] + \frac{\partial^2\eta}{\partial b^2}\ \not\mathbf{S}[\frac{mr}{2}f(r)\cos\alpha\cos^3\beta] + \frac{\partial^2\eta}{\partial c^2}\ \not\mathbf{S}[\frac{mr}{2}f(r)\cos\alpha\cos\beta\cos^2\gamma]\\
\quad + \ \frac{\partial^2\eta}{\partial b\partial c}\ \not\mathbf{S}[mrf(r)\cos\alpha\cos^2\beta\cos\gamma] + \ \frac{\partial^2\eta}{\partial a\partial b}\ \not\mathbf{S}[mrf(r)\cos^2\alpha\cos\beta\cos\gamma] + \ \frac{\partial^2\eta}{\partial a\partial b}\ \mathbf{S}[mrf(r)\cos^2\alpha\cos^2\beta]\\
\quad + \ \frac{\partial^2\zeta}{\partial a^2}\ \not\mathbf{S}[\frac{mr}{2}f(r)\cos^3\alpha\cos\gamma] + \frac{\partial^2\zeta}{\partial b^2}\ \not\mathbf{S}[\frac{mr}{2}f(r)\cos\alpha\cos^2\beta\cos\gamma] + \frac{\partial^2\zeta}{\partial c^2}\ \not\mathbf{S}[\frac{mr}{2}f(r)\cos\alpha\cos^3\gamma]\\
\quad + \ \frac{\partial^2\zeta}{\partial b\partial c}\ \not\mathbf{S}[mrf(r)\cos\alpha\cos\beta\cos^2\gamma] + \frac{\partial^2\zeta}{\partial c\partial a}\ \mathbf{S}[mrf(r)\cos^2\alpha\cos^2\gamma] + \frac{\partial^2\zeta}{\partial a\partial b}\ \not\mathbf{S}[mrf(r)\cos^2\alpha\cos\beta\cos\gamma],\\
Y_2 = \frac{\partial^2\eta}{\partial a^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha\mathrm{f}(r)] + \frac{\partial^2\eta}{\partial b^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\beta\mathrm{f}(r)] + \frac{\partial^2\eta}{\partial c^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma\mathrm{f}(r)]\\
\quad + \cdots\\
\quad + \frac{\partial^2\eta}{\partial a^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta] + \frac{\partial^2\eta}{\partial b^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^4\beta] + \frac{\partial^2\eta}{\partial c^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\beta\cos^2\gamma]\\
\quad + \cdots\\
\quad + \frac{\partial^2\xi}{\partial a\partial b}\ \mathbf{S}[mrf(r)\cos^2\alpha\cos^2\beta]\\
\quad + \cdots\\
\quad + \frac{\partial^2\zeta}{\partial b\partial c}\ \mathbf{S}[mrf(r)\cos^2\beta\cos^2\gamma],\\
Z_2 = \frac{\partial^2\zeta}{\partial a^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha\mathrm{f}(r)] + \frac{\partial^2\zeta}{\partial b^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\beta\mathrm{f}(r)] + \frac{\partial^2\zeta}{\partial c^2}\ \mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma\mathrm{f}(r)]\\
\quad + \cdots\\
\quad + \frac{\partial^2\zeta}{\partial a^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta] + \frac{\partial^2\xi}{\partial b^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^2\beta\cos^2\gamma] + \frac{\partial^2\zeta}{\partial c^2}\ \mathbf{S}[\frac{mr}{2}f(r)\cos^4\gamma]\\
\quad + \cdots\\
\quad + \frac{\partial^2\xi}{\partial c\partial c}\ \mathbf{S}[mrf(r)\cos^2\gamma\cos^2\alpha]\\
\quad + \cdots\\
\quad + \frac{\partial^2\eta}{\partial b\partial c}\ \mathbf{S}[mrf(r)\cos^2\beta\cos^2\gamma]
\end{cases}
$$

(⇓) The equations in $(31)_C$ have the 72 (= 3 × 24) terms at maximum. By $(25)_C$, we have to calculate totally 117 (= 9+36+72) terms at maximum, however, according to the following articles, we can reduce these difficulties of calculation.

- By $(35)_C$, all terms in $(26)_C$ are zero.
- The terms led by the symbol of $\cancel{\mathbf{S}}$ terms in $(31)_C$ are deleted.
- Finally, in #8, by introducing original idea of Cauchy on the symmetric tensor, we can reduce it.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} + \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$$

By the way, Poisson calculates 63 terms at maximum, which we will mention below. (⇑)

• ¶ 8. Remark : in the right-hand sides of $(25)_C$, X_2, Y_2, Z_2 are not only the largest valued terms in $(25)_C$, but in $(25)_C$ even non-zero terms too, owing to the same sign. Cauchy says :

- To gain the sum which is relative to the coefficients of $(20)_C$ in the second terms (which are only under the symbol of $\cancel{\mathbf{S}}$) in $(31)_C$,[40]
  - it is sufficient to multiply succeedingly the quantities of
    - the term under the symbol **S** in the right-hand side of $(26)_C$ [41]
    - the second terms under the symbol **S** in $(30)_C$

    by the three factors $r\cos\alpha,\ r\cos\beta,\ r\cos\gamma$, or $\frac{1}{2}$ of these, viz. $\frac{1}{2}r\cos\alpha$ etc. (as the basis like (72), where, $(30)_C$ are multiplied by only cos as the basis,) and
  - if each of these value differs infinitesimally to zero, even if it is due to a infinitesimal value of vector **r**, we can neglect X_2, Y_2 and Z_2 in $(25)_C$, in comparison with the quantities X_0, Y_0, Z_0, X_1, Y_1 and Z_1.
- However, we must consider that each of sum contained in $(31)_C$ is composed of the terms to which the sum has an effect with the same sign, while each sum is composed of the terms to which the sum has an effect with the contrary sign, when they correspond to the molecules situated in the part, and the other point with $(a,\ b,\ c)$ on the direction orienting to the same point.
- It turns out that the latter sums are to disapper in the most cases, however, they are not one with the same as the former.
- Hence, we can conclude that the terms X_2, Y_2 and Z_2 in the second term of $(25)_C$ are not only of having numerical large value, but also of just nonzero terms.

By the way, to be exact, we cite Cauchy's original as follows :

> Comme, pour obtenir les sommes qui servent de coefficients aux expressions $(20)_C$ dans les seconds membres des formules $(31)_C$, il suffit de multiplier successivement les quantités renfermées sous le signe **S** dans les seconds membres des formules $(26)_C$ et $(30)_C$ par les trois facteurs $r\cos\alpha,\ r\cos\beta,\ r\cos\gamma$ ou par les moitiés de ces facteurs, et que chacun de ceux-ci différe très peu de zéro quand on attribue au rayon vecteur **r** une valeur très petite, il semble, au premier abord, qu'on pourrait, dans les équations $(25)_C$, négliger $X_2,\ Y_2,\ Z_2$ vis-à-vis des quantités $X_0,\ Y_0,\ Z_0,\ X_1,\ Y_1,\ Z_1$.
>
> Mais on doit observer que chacun des sommes comprises dans les formules $(31)_C$ se compose de termes qui sont tous affectés du même signe, tandis que chacune des sommes compose de termes qui sont affectés de signes contraires quand ils correspondent à des molécules situées du part et d'autre du point $(a,\ b,\ c)$ sur une droite quelconque menée par ce même point. Il en résulte que les dernières sommes peuvent s'évanouir dans beaucoup de cas, mais qu'il n'en est pas de même des autres. Donc il peut arriver que, dans les seconnnds membres des équations $(25)_C$, les termes $X_2,\ Y_2,\ Z_2$ soient, non seulement ceux qui offrent les plus grandes valeurs numériques, mais encore les seuls qui différent de zéro. [7, p.236]

• ¶ 9. Remark : the equations of accelerated forces follow not only in the forces come from its mutual attraction or repulsion but also in the new accelerated forces. [42]

---

[40](⇓) Reamrk. Cauchy uses the symbol only by **S**, however, we use **S** in $(30)_C$, while in $(31)_C$, we invent $\cancel{\mathbf{S}}$ of ours to discriminate between both, where $\cancel{\mathbf{S}}$ means the **S** to be deleted defined like #2 in ¶ 2.

[41](⇓) According to his original below, there is the second term in $(26)_C$ as well, however, there is actually not.

[42](⇓) These analyses don't appear in Navier's papers.

Les valeurs de $X,\ Y,\ Z$ étant déterminées par les formules $(25)_C$, $(26)_C$, $(30)_C$ et $(31)_C$ en fonction des quantités $(19)_C$ et $(20)_C$, on établira sans peine les équations qui expriment l'equilibre ou le mouvement du système des masses m, $m,\ m',\cdots$ soumises, non seulement à leurs attractions ou répulsions mutuelles, mais à de nouvelles forces accélératrice. [7, p.236]

> #6 En effet, supposont que, au bout du temps t, l'état d'équilibre ou de mouvement du système coïncide avec l'état dans lequel les coordonnées de la molécule m se trouvent représentées par $x,\ y,\ z$; et soient à cette époque $\mathcal{X},\ \mathcal{Y},\ \mathcal{Z}$ le projections algébriques de la nouvelle force accélératrice φ appliquée à la molécule m sur les axes coordonnées. On aura évidemment, si le système est en équilibre, [7, p.236]

$$(32)_C \quad X+\mathcal{X}=0, \quad Y+\mathcal{Y}=0, \quad Z+\mathcal{Z}=0.$$

> Au contraire, si le système se meut, en désignant par ψ la force accélératrice qui serait capable de produire à elle seule le mouvement effectif de la molécule m, et par $\dot{X},\ \dot{Y},\ \dot{Z}$ les projections algébriques de cette force sur les axes coordonnées, on devra, dans les équations $(32)_C$, remlacer les quantités $\mathcal{X},\ \mathcal{Y},\ \mathcal{Z}$ par les différences $\mathcal{X}-\dot{X},\ \mathcal{Y}-\dot{Y},\ \mathcal{Z}-\dot{Z}$. Comme on trouvera, d'ailleurs, en prenant $a,\ b,\ c$ pour variables indépendantes, et ayant égard aux formules $(4)_C$, [7, pp.236-7]

$$(4)_C \quad x=a+\xi, \quad y=b+\eta, \quad z=c+\zeta.$$

$$(33)_C \quad \dot{X}=\frac{\partial^2 x}{\partial t^2}=\frac{\partial^2 \xi}{\partial t^2}, \quad \dot{Y}=\frac{\partial^2 y}{\partial t^2}=\frac{\partial^2 \eta}{\partial t^2}, \quad \dot{Z}=\frac{\partial^2 z}{\partial t^2}=\frac{\partial^2 \zeta}{\partial t^2},$$

il est clair que le mouvement d'un molécule quelconque m sera déterminé par les équations.

Replacing $\mathcal{X},\ \mathcal{Y},\ \mathcal{Z}$ of $(32)_C$ with $\mathcal{X}-\dot{X},\ \mathcal{Y}-\dot{Y},\ \mathcal{Z}-\dot{Z}$, and considering $(4)_C$, we get the new accelerated forces as follows :

$$(32)_C \quad \Rightarrow \quad \begin{cases} X+\mathcal{X}-\dot{X}=0, \\ Y+\mathcal{Y}-\dot{Y}=0, \\ Z+\mathcal{Z}-\dot{Z}=0, \end{cases} \quad \Rightarrow \quad (34)_C \quad \begin{cases} X+\mathcal{X}=\dot{X}=\frac{\partial^2\xi}{\partial t^2}, \\ Y+\mathcal{Y}=\dot{Y}=\frac{\partial^2\eta}{\partial t^2}, \\ Z+\mathcal{Z}=\dot{Z}=\frac{\partial^2\zeta}{\partial t^2}, \end{cases}$$

A.3.2. **Reduction of tensor.**

• ¶ 10. The values of X, Y and Z , determined by the statements $(25)_C$, $(26)_C$, $(30)_C$, $(31)_C$, are simplified with several hypotheses as follows :

• ¶ 11. $(26)_C$ and $(30)_C$ disappear and $X,\ Y,\ Z$ are reduced to only $X_2,\ Y_2,\ Z_2$ for the symmetric distribution of the molecules.

> #7. D'abord on peut supposer que les sommes comprises dans les formules $(26)_C$ et $(30)_C$ s'evanouissent. C'est ce qui arrivera en particulier si, dans l'etat primitif du système, les masses $m,\ m',\ m'',\cdots$ étant deux à deux égales entre elles, sont distribuées, symétriquement de part et d'autre de la molécule m, sur des droites menées par le point $(a,\ b,\ c)$ avec lequel cette molécule coincide. En effet, comme chacun des termes renfermés sous le signe **S** dans les formules $(26)_C$ et $(30)_C$, offrant un nombre impaire de facteurs égaux aux cosinus $\cos\alpha,\ \cos\beta,\ \cos\gamma$, change nécessairement de signe avec ces mêmes facteurs, ces termes, comparés deux à deux, seront évidemment, dans le cas dont il s'agit, équivalents au signe près, mais affectés de signes contraires. [7, p.237]

Alors les formules $(15)_C$ seront vérifiées, c'est-à-dire que l'etat primitif du système sera un état d'equilibre ; et, comm on aura d'ailleurs

$$(35)_C \quad X_1 = 0,\ Y_1 = 0,\ Z_1 = 0,$$

les valeurs de $X,\ Y,\ Z$ se réduiront à celles de $X_2,\ Y_2,\ Z_2$. [7, pp.237-8]

• ¶ 12. At last, we get X, Y and Z from $(26)_C$, $(30)_C$, $(31)_C$ after deleting the terms containing the terms of $\cos\alpha$ or $\cos\beta$ or $\cos\gamma$ in odd power, which becomes zero in summation, (ex. including $\cos\alpha,\ \cos^3\alpha,\ \cos\beta,\ \cos^3\beta,\ \cdots$.)

#8. On peut supposer encore que, parmi les sommes comprises dans les formules $(26)_C$, $(30)_C$ et $(31)_C$, toutes celles qui renferment des puissances impaires de $\cos\alpha$, de $\cos\beta$, ou de $\cos\gamma$ s'evanouissent. C'est ce qui arriva en particulier

- si, dans l'etat primitif du système, les molécules $m,\ m',\ m'',\cdots$ sont distribuées symétriquement par rapport à chacun des trois plans qui, renfermant le point $(a,\ b,\ c)$, sont parallèles aux plans coordonnés des y, z, des $z,\ x$ et des $x,\ y$, et
- si deux molécules symétriquement placées à l'egard d'un des trois premiers plans offrent toujours des masses égales.

Dans la supposition dont il s'agit, non seulement les formes $(15)_C$ et $(35)_C$ seront vérifiées, mais de plus les valeurs de $X,\ Y,\ Z$, équivalentes à celles de $X_2,\ Y_2,\ Z_2$, se réduiront à [7, p.238]

$$(36)_C \quad \begin{cases} X = \frac{\partial^2\xi}{\partial a^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha \mathrm{f}(r)] + \frac{\partial^2\xi}{\partial b^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta \mathrm{f}(r)] + \frac{\partial^2\xi}{\partial c^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma \mathrm{f}(r)] \\ \quad + \frac{\partial^2\xi}{\partial a^2}\mathbf{S}[\frac{mr}{2}\cos^4\alpha f(r)] + \frac{\partial^2\xi}{\partial b^2}\mathbf{S}[\frac{mr}{2}\cos^2\alpha\cos^2\beta f(r)] + \frac{\partial^2\xi}{\partial c^2}\mathbf{S}[\frac{mr}{2}\cos^2\alpha\cos^2\gamma f(r)] \\ \quad + \frac{\partial^2\eta}{\partial a\partial b}\mathbf{S}[mr\cos^2\alpha\cos^2\beta f(r)] + \frac{\partial^2\zeta}{\partial a\partial c}\mathbf{S}[mr\cos^2\alpha\cos^2\gamma f(r)], \\ Y = \frac{\partial^2\eta}{\partial a^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha \mathrm{f}(r)] + \frac{\partial^2\eta}{\partial b^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta \mathrm{f}(r)] + \frac{\partial^2\eta}{\partial c^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma \mathrm{f}(r)] \\ \quad + \frac{\partial^2\eta}{\partial a^2}\mathbf{S}[\frac{mr}{2}\cos^2\alpha\cos^2\beta f(r)] + \frac{\partial^2\eta}{\partial b^2}\mathbf{S}[\frac{mr}{2}\cos^4\beta f(r)] + \frac{\partial^2\eta}{\partial c^2}\mathbf{S}[\frac{mr}{2}\cos^2\beta\cos^2\gamma f(r)] \\ \quad + \frac{\partial^2\xi}{\partial a\partial b}\mathbf{S}[mr\cos^2\alpha\cos^2\beta f(r)] + \frac{\partial^2\zeta}{\partial b\partial c}\mathbf{S}[mr\cos^2\beta\cos^2\gamma f(r)], \\ Z = \frac{\partial^2\zeta}{\partial a^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha \mathrm{f}(r)] + \frac{\partial^2\zeta}{\partial b^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta \mathrm{f}(r)] + \frac{\partial^2\zeta}{\partial c^2}\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma \mathrm{f}(r)] \\ \quad + \frac{\partial^2\zeta}{\partial a^2}\mathbf{S}[\frac{mr}{2}\cos^2\alpha\cos^2\beta f(r)] + \frac{\partial^2\xi}{\partial b^2}\mathbf{S}[\frac{mr}{2}\cos^2\beta\cos^2\gamma f(r)] + \frac{\partial^2\zeta}{\partial c^2}\mathbf{S}[\frac{mr}{2}\cos^4\gamma f(r)] \\ \quad + \frac{\partial^2\xi}{\partial c\partial a}\mathbf{S}[mr\cos^2\gamma\cos^2\alpha f(r)] + \frac{\partial^2\eta}{\partial b\partial c}\mathbf{S}[mr\cos^2\beta\cos^2\gamma f(r)] \end{cases}$$

$\Rightarrow$

$$\begin{cases} X = \frac{\partial^2\xi}{\partial a^2}\Big\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha \mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^4\alpha]\Big\} \\ \quad + \frac{\partial^2\xi}{\partial b^2}\Big\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta \mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta]\Big\} \\ \quad + \frac{\partial^2\xi}{\partial c^2}\Big\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma \mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\gamma]\Big\} \\ \quad + \frac{\partial^2\eta}{\partial a\partial b}\mathbf{S}[mrf(r)\cos^2\alpha\cos^2\beta] + \frac{\partial^2\zeta}{\partial c\partial a}\mathbf{S}[mrf(r)\cos^2\alpha\cos^2\gamma], \\ Y = \frac{\partial^2\eta}{\partial a^2}\Big\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha \mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta]\Big\} \\ \quad + \frac{\partial^2\eta}{\partial b^2}\Big\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta \mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^4\beta]\Big\} \\ \quad + \frac{\partial^2\eta}{\partial c^2}\Big\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma \mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\beta\cos^2\gamma]\Big\} \\ \quad + \frac{\partial^2\zeta}{\partial b\partial c}\mathbf{S}[mrf(r)\cos^2\beta\cos^2\gamma] + \frac{\partial^2\xi}{\partial a\partial b}\mathbf{S}[mrf(r)\cos^2\alpha\cos^2\beta], \\ Z = \frac{\partial^2\zeta}{\partial a^2}\Big\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha \mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\alpha\cos^2\beta]\Big\} \\ \quad + \frac{\partial^2\zeta}{\partial b^2}\Big\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\beta \mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^2\beta\cos^2\gamma]\Big\} \\ \quad + \frac{\partial^2\zeta}{\partial c^2}\Big\{\mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma \mathrm{f}(r)] + \mathbf{S}[\frac{mr}{2}f(r)\cos^4\gamma]\Big\} \\ \quad + \frac{\partial^2\xi}{\partial c\partial a}\mathbf{S}[mrf(r)\cos^2\gamma\cos^2\alpha] + \frac{\partial^2\eta}{\partial b\partial c}\mathbf{S}[mrf(r)\cos^2\beta\cos^2\gamma] \end{cases} \qquad (73)$$

$$
\begin{aligned}
\Rightarrow\ & \begin{bmatrix} X & Y & Z \end{bmatrix} \\
= & \begin{bmatrix} \frac{\partial^2\xi}{\partial a^2} & \frac{\partial^2\xi}{\partial b^2} & \frac{\partial^2\xi}{\partial c^2} & \frac{\partial^2\eta}{\partial a\partial b} & \frac{\partial^2\zeta}{\partial c\partial a} \\ \frac{\partial^2\eta}{\partial a^2} & \frac{\partial^2\eta}{\partial b^2} & \frac{\partial^2\eta}{\partial c^2} & \frac{\partial^2\zeta}{\partial b\partial c} & \frac{\partial^2\xi}{\partial a\partial b} \\ \frac{\partial^2\zeta}{\partial c^2} & \frac{\partial^2\zeta}{\partial b^2} & \frac{\partial^2\zeta}{\partial c^2} & \frac{\partial^2\xi}{\partial c\partial a} & \frac{\partial^2\eta}{\partial b\partial c} \end{bmatrix} \\
\times\ & \frac{mr}{2} \begin{bmatrix} \mathbf{S}\Big\{\pm\cos^2\alpha \mathrm{f}(r) + f(r)\cos^4\alpha\Big\} & \mathbf{S}\Big\{\pm\cos^2\alpha \mathrm{f}(r) + f(r)\cos^2\alpha\cos^2\beta\Big\} & \mathbf{S}\Big\{\pm\cos^2\alpha \mathrm{f}(r) + f(r)\cos^2\alpha\cos^2\beta\Big\} \\ \mathbf{S}\Big\{\pm\cos^2\beta \mathrm{f}(r) + f(r)\cos^2\alpha\cos^2\beta\Big\} & \mathbf{S}\Big\{\pm\cos^2\beta \mathrm{f}(r) + f(r)\cos^4\beta\Big\} & \mathbf{S}\Big\{\pm\cos^2\beta \mathrm{f}(r) + f(r)\cos^2\beta\cos^2\gamma\Big\} \\ \mathbf{S}\Big\{\pm\cos^2\gamma \mathrm{f}(r) + f(r)\cos^2\alpha\cos^2\gamma\Big\} & \mathbf{S}\Big\{\pm\cos^2\gamma \mathrm{f}(r) + f(r)\cos^2\beta\cos^2\gamma\Big\} & \mathbf{S}\Big\{\pm\cos^2\gamma \mathrm{f}(r) + f(r)\cos^4\gamma\Big\} \\ 2\mathbf{S}f(r)\cos^2\alpha\cos^2\beta & 2\mathbf{S}f(r)\cos^2\beta\cos^2\gamma & 2\mathbf{S}f(r)\cos^2\gamma\cos^2\alpha \\ 2\mathbf{S}f(r)\cos^2\alpha\cos^2\gamma & 2\mathbf{S}f(r)\cos^2\alpha\cos^2\beta & 2\mathbf{S}f(r)\cos^2\beta\cos^2\gamma \end{bmatrix} \\
\equiv\ & \begin{bmatrix} \frac{\partial^2\xi}{\partial a^2} & \frac{\partial^2\xi}{\partial b^2} & \frac{\partial^2\xi}{\partial c^2} & \frac{\partial^2\eta}{\partial a\partial b} & \frac{\partial^2\zeta}{\partial c\partial a} \\ \frac{\partial^2\eta}{\partial a^2} & \frac{\partial^2\eta}{\partial b^2} & \frac{\partial^2\eta}{\partial c^2} & \frac{\partial^2\zeta}{\partial b\partial c} & \frac{\partial^2\xi}{\partial a\partial b} \\ \frac{\partial^2\zeta}{\partial c^2} & \frac{\partial^2\zeta}{\partial b^2} & \frac{\partial^2\zeta}{\partial c^2} & \frac{\partial^2\xi}{\partial c\partial a} & \frac{\partial^2\eta}{\partial b\partial c} \end{bmatrix} \begin{bmatrix} G+L & G+R & G+Q \\ H+P & H+M & H+P \\ I+Q & I+P & I+N \\ 2R & 2P & 2Q \\ 2Q & 2R & 2P \end{bmatrix},
\end{aligned}
$$

where, we define 9 parameters in (73) by G, H, I, L, M, N, P, Q and R as follows :

$$
\begin{cases}
(37)_C \quad G \equiv \mathbf{S}[\pm\frac{mr}{2}\cos^2\alpha\ \mathrm{f}(r)], \quad H \equiv \mathbf{S}[\pm\frac{mr}{2}\cos^2\beta\ \mathrm{f}(r)], \quad I \equiv \mathbf{S}[\pm\frac{mr}{2}\cos^2\gamma\ \mathrm{f}(r)], \\
(38)_C \quad L \equiv \mathbf{S}[\frac{mr}{2}\cos^4\alpha\ f(r)], \quad M \equiv \mathbf{S}[\frac{mr}{2}\cos^4\beta\ f(r)], \quad N \equiv \mathbf{S}[\frac{mr}{2}\cos^4\gamma\ f(r)], \\
(39)_C \quad P \equiv \mathbf{S}[\frac{mr}{2}\cos^2\beta\cos^2\gamma\ f(r)], \quad Q \equiv \mathbf{S}[\frac{mr}{2}\cos^2\gamma\cos^2\alpha\ f(r)], \quad R \equiv \mathbf{S}[\frac{mr}{2}\cos^2\alpha\cos^2\beta\ f(r)],
\end{cases}
$$

Then from (73) it turns into the following :

$$
(40)_C \quad \begin{cases}
X = (G+L)\frac{\partial^2\xi}{\partial a^2} + (H+R)\frac{\partial^2\xi}{\partial b^2} + (I+Q)\frac{\partial^2\xi}{\partial c^2} + 2R\frac{\partial^2\eta}{\partial a\partial b} + 2Q\frac{\partial^2\zeta}{\partial c\partial a}, \\
Y = (G+R)\frac{\partial^2\eta}{\partial a^2} + (H+M)\frac{\partial^2\eta}{\partial b^2} + (I+P)\frac{\partial^2\eta}{\partial c^2} + 2P\frac{\partial^2\zeta}{\partial b\partial c} + 2R\frac{\partial^2\xi}{\partial a\partial b}, \\
Z = (G+Q)\frac{\partial^2\zeta}{\partial a^2} + (H+P)\frac{\partial^2\zeta}{\partial b^2} + (I+N)\frac{\partial^2\zeta}{\partial c^2} + 2Q\frac{\partial^2\xi}{\partial c\partial a} + 2P\frac{\partial^2\eta}{\partial b\partial c}.
\end{cases}
$$

• ¶ 13. Invariable values : $G, H, I, L, M, N, P, Q, R$.

If we suppose that the molecules $m, m', m'', \cdots$ are originally distributed by the same way in relation to the three planes made by the molecule m in parallel with the plane coordinates, then the values of quantities: $G, H, I, L, M, N, P, Q, R$ should remain invariable, even though a series of changes are made among the three angles : α, β, γ.

> #9. Si l'on supposait les molécules $m, m', m'', \cdots$ primitivement distribuées de la même manière par rapport aux trois plans menés par la molécule m parallèlement aux plans coordonnés, les valeurs des quantités $G, H, I, L, M, N, P, Q, R$ devraient rester les mêmes après un ou plusieurs échanges opérés entre les trois angles α, β, γ; et l'on aurait par suite [7, p.239]

$$
(41)_C \quad G = H = I, \quad L = M = N, \quad P = Q = R.
$$

$$
(42)_C \quad \begin{cases}
X = (L+G)\frac{\partial^2\xi}{\partial a^2} + (R+G)\Big(\frac{\partial^2\xi}{\partial b^2} + \frac{\partial^2\xi}{\partial c^2}\Big) + 2R\Big(\frac{\partial^2\eta}{\partial a\partial b} + \frac{\partial^2\zeta}{\partial c\partial a}\Big), \\
Y = (L+G)\frac{\partial^2\eta}{\partial b^2} + (R+G)\Big(\frac{\partial^2\eta}{\partial c^2} + \frac{\partial^2\eta}{\partial a^2}\Big) + 2R\Big(\frac{\partial^2\zeta}{\partial b\partial c} + \frac{\partial^2\xi}{\partial a\partial b}\Big), \\
Z = (L+Q)\frac{\partial^2\zeta}{\partial c^2} + (R+G)\Big(\frac{\partial^2\zeta}{\partial a^2} + \frac{\partial^2\zeta}{\partial b^2}\Big) + 2R\Big(\frac{\partial^2\xi}{\partial c\partial a} + \frac{\partial^2\eta}{\partial b\partial c}\Big).
\end{cases} \tag{74}
$$

$$
\Rightarrow \quad \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \frac{\partial^2\xi}{\partial a^2} & \Big(\frac{\partial^2\xi}{\partial b^2} + \frac{\partial^2\xi}{\partial c^2}\Big) & \Big(\frac{\partial^2\eta}{\partial a\partial b} + \frac{\partial^2\zeta}{\partial c\partial a}\Big) \\ \frac{\partial^2\eta}{\partial a^2} & \Big(\frac{\partial^2\eta}{\partial b^2} + \frac{\partial^2\eta}{\partial c^2}\Big) & \Big(\frac{\partial^2\zeta}{\partial b\partial c} + \frac{\partial^2\xi}{\partial a\partial b}\Big) \\ \frac{\partial^2\zeta}{\partial c^2} & \Big(\frac{\partial^2\zeta}{\partial b^2} + \frac{\partial^2\zeta}{\partial c^2}\Big) & \Big(\frac{\partial^2\xi}{\partial c\partial a} + \frac{\partial^2\eta}{\partial b\partial c}\Big) \end{bmatrix} \begin{bmatrix} L+G \\ R+G \\ 2R \end{bmatrix}.
$$

• ¶ 14. For the angles : $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3$ are perpendicular among each planes, the values of sums : $G, H, I, L, M, N, P, Q, R$ do not alter even by replacing $\cos\alpha, \cos\beta, \cos\gamma$ with the trinomial :

#10. Supposons enfin les molécules $m, m', m'', \cdots$ primitivement distribuées autour de la molécule m, de manière que les valeurs des sommes comprises dans les équations (37), (38), (39) deviennent indépendantes des directions assignées aux axes des x, y, z. Alors, non seulement les conditions (41) devront être satisfaites, mais de plus, si l'on nomme $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3$ les angles formés par trois demi-axes perpendiculaires entre eux avec les demi-axes des x, y et z positives, on n'altérera pas valeurs des sommes $G, H, I, L, M, N, P, Q, R$ en y remplaçant les trois quantités $\cos\alpha, \cos\beta, \cos\gamma$ par les trinômes [7, p.239]

$$\begin{cases} \cos\alpha \;\Rightarrow\; \cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1, \\ \cos\beta \;\Rightarrow\; \cos\alpha\cos\alpha_2 + \cos\beta\cos\beta_2 + \cos\gamma\cos\gamma_2, \\ \cos\gamma \;\Rightarrow\; \cos\alpha\cos\alpha_3 + \cos\beta\cos\beta_3 + \cos\gamma\cos\gamma_3, \end{cases}$$

$$(43)_C \quad \begin{cases} G = \mathbf{S}\Big[\pm\frac{mr}{2}\big(\cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1\big)^2 \mathrm{f}(r)\Big], \\ L = \mathbf{S}\Big[\frac{mr}{2}\big(\cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1\big)^4 f(r)\Big], \\ R = \mathbf{S}\Big[\frac{mr}{2}\big(\cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1\big)^2\big(\cos\alpha\cos\alpha_2 + \cos\beta\cos\beta_2 + \cos\gamma\cos\gamma_2\big)^2 f(r)\Big] \end{cases} \tag{75}$$

$(44)_C$

$$\begin{cases} G = G\big(\cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1\big) \equiv\; GA_1, \\ L = L\big(\cos^4\alpha_1 + \cos^4\beta_1 + \cos^4\gamma_1\big) + 6R\big(\cos^2\beta_1\cos^2\gamma_1 + \cos^2\gamma_1\cos^2\alpha_1 + \cos^2\alpha_1\cos^2\beta_1\big) \equiv LB + 6RC, \\ R = R\big(\cos^2\beta_1\cos^2\gamma_2 + \cos^2\beta_2\cos^2\gamma_1 + \cos^2\gamma_1\cos^2\alpha_2 + \cos^2\gamma_2\cos^2\alpha_1 + \cos^2\alpha_1\cos^2\beta_2 + \cos^2\alpha_2\cos^2\beta_1\big), \\ \quad +4R\big(\cos\beta_1\cos\beta_2\cos\gamma_1\cos\gamma_2 + \cos\gamma_1\cos\gamma_2\cos\alpha_1\cos\alpha_2 + \cos\alpha_1\cos\alpha_2\cos\beta_1\cos\beta_2\big) \\ \quad +L\big(\cos^2\alpha_1\cos^2\alpha_2 + \cos^2\beta_1\cos^2\beta_2 + \cos^2\gamma_1\cos^2\gamma_2\big) \equiv RD + 4RE + LF, \end{cases} \tag{76}$$

where

$$\begin{cases} \cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1 = 1, \\ \cos^2\alpha_2 + \cos^2\beta_2 + \cos^2\gamma_2 = 1, \\ \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0 \end{cases}$$

($\Downarrow$) and

$$\begin{cases} A_1 \equiv \cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1, \\ A_2 \equiv \cos^2\alpha_2 + \cos^2\beta_2 + \cos^2\gamma_2, \\ B = \cos^4\alpha_1 + \cos^4\beta_1 + \cos^4\gamma_1, \\ C \equiv \cos^2\beta_1\cos^2\gamma_1 + \cos^2\gamma_1\cos^2\alpha_1 + \cos^2\alpha_1\cos^2\beta_1, \\ D \equiv \cos^2\beta_1\cos^2\gamma_2 + \cos^2\beta_2\cos^2\gamma_1 + \cos^2\gamma_1\cos^2\alpha_2 + \cos^2\gamma_2\cos^2\alpha_1 + \cos^2\alpha_1\cos^2\beta_2\cos^2\alpha_2\cos^2\beta_1, \\ E \equiv \cos\beta_1\cos\beta_2\cos\gamma_1\cos\gamma_2 + \cos\gamma_1\cos\gamma_2\cos\alpha_1\cos\alpha_2 + \cos\alpha_1\cos\alpha_2\cos\beta_1\cos\beta_2, \\ F \equiv \cos^2\alpha_1\cos^2\alpha_2 + \cos^2\beta_1\cos^2\beta_2 + \cos^2\gamma_1\cos^2\gamma_2 \end{cases}$$

then

$$\begin{cases} 1 - B = A_1^2 - B = 2C, \\ 1 - D = A_1A_2 - D = F = -2E, \end{cases} \tag{77}$$

($\Uparrow$) namely :

$$\begin{cases} 1-\left(\cos^4\alpha_1+\cos^4\beta_1+\cos^4\gamma_1\right)=\left(\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1\right)^2-\left(\cos^4\alpha_1+\cos^4\beta_1+\cos^4\gamma_1\right)\\ \quad =2\left(\cos^2\beta_1\cos^2\gamma_1+\cos^2\gamma_1\cos^2\alpha_1+\cos^2\alpha_1\cos^2\beta_1\right),\\ 1-\left(\cos^2\beta_1\cos^2\gamma_2+\cos^2\beta_2\cos^2\gamma_1+\cos^2\gamma_1\cos^2\alpha_2+\cos^2\gamma_2\cos^2\alpha_1+\cos^2\alpha_1\cos^2\beta_2\cos^2\alpha_2\cos^2\beta_1\right)\\ \quad =\left(\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1\right)\left(\cos^2\alpha_2+\cos^2\beta_2+\cos^2\gamma_2\right)\\ \qquad -\left(\cos^2\beta_1\cos^2\gamma_2+\cos^2\beta_2\cos^2\gamma_1+\cos^2\gamma_1\cos^2\alpha_2+\cos^2\gamma_2\cos^2\alpha_1+\cos^2\alpha_1\cos^2\beta_2\cos^2\alpha_2\cos^2\beta_1\right)\\ \quad =\left(\cos^2\alpha_1\cos^2\alpha_2+\cos^2\beta_1\cos^2\beta_2+\cos^2\gamma_1\cos^2\gamma_2\right)\\ \quad =\left(\cos\beta_1\cos\beta_2\cos\gamma_1\cos\gamma_2+\cos\gamma_1\cos\gamma_2\cos\alpha_1\cos\alpha_2+\cos\alpha_1\cos\alpha_2\cos\beta_1\cos\beta_2\right) \end{cases}$$

From the second equation of (76) (: $(44)_C$) by (77)

$$L(1-B)=2LC=6RC$$

$$(45)_C \quad L=3R,$$

or, from the third equation by (77)

$$R(1-D)=-2RE=4RE+LF \quad \Rightarrow 2LE=6RE$$

$$(45)_C \quad L=3R$$

From $(74)(=(42)_C)$ we get $(78)(=(46)_C)$ by $(45)_C$ as follows :

$$(46)_C \quad \begin{cases} X=(R+G)\left(\frac{\partial^2\xi}{\partial a^2}+\frac{\partial^2\xi}{\partial b^2}+\frac{\partial^2\xi}{\partial c^2}\right)+2R\frac{\partial\nu}{\partial a},\\ Y=(R+G)\left(\frac{\partial^2\eta}{\partial a^2}+\frac{\partial^2\eta}{\partial b^2}+\frac{\partial^2\eta}{\partial c^2}\right)+2R\frac{\partial\nu}{\partial b},\\ Z=(R+G)\left(\frac{\partial^2\zeta}{\partial a^2}+\frac{\partial^2\zeta}{\partial b^2}+\frac{\partial^2\zeta}{\partial c^2}\right)+2R\frac{\partial\nu}{\partial c}, \end{cases} \tag{78}$$

$$\text{where} \qquad (47)_C \quad \nu=\frac{\partial\xi}{\partial a}+\frac{\partial\eta}{\partial b}+\frac{\partial\zeta}{\partial c}.$$

(c.f. $(78)(=(46)_C)\Rightarrow(115)(=(7\text{-}9)_{Pf})\Rightarrow(120)(=(12)_S)$). Moreover, from $(41)_C$:

$$G=H=I, \quad L=M=N, \quad P=Q=R.$$

($\Downarrow$) By the way, Cauchy says, when we put $G=H=I=0$ in $(40)_C$, we can see the coincidence of Cauchy's R with Navier's ε, as follows :

$$\begin{bmatrix} L & R & Q & 2R & 2Q\\ R & M & P & 2P & 2R\\ Q & P & N & 2Q & 2P \end{bmatrix} = R\begin{bmatrix} 3 & 1 & 1 & 2 & 2\\ 1 & 3 & 1 & 2 & 2\\ 1 & 1 & 3 & 2 & 2 \end{bmatrix} \tag{79}$$

These coefficients of (79) equal (39) of Navier.

• ¶ 15. Density : Δ defined by mass of a sphere : $\mathcal{M}$ and the volume of a sphere : $\mathcal{V}$ as follows :

> Concevons maintenant que, dans l'etat primitif du système des molécules $m, m', m'', \cdots$, et, du point (a, b, c) comme centre avec un rayon l convenablement choisi, on décrive une sphère qui renferme toutes les molécules dont l'action sur la masse m à une valeur sensible. Divisions le volume $\mathcal{V}$ de cette sphère en éléments très petits $v, v', v'', \cdots$, mais dont chacun renferme encore un très grand nombre de molécules. Soient $\mathcal{M}$ la somme des masses des molécules comprises dans la sphère, et [7, p.241]

$$(48)_C \quad \Delta=\frac{\mathcal{M}}{\mathcal{V}}=\frac{\text{mass of system of particles}}{\text{volume of system of particles}}=\text{density}$$

#11. Enfin supposons que les sommes des masses comprises sous les volumes élémentaires $v,\ v',\ v'',\cdots$ soient proportionelles à ces même volumes, et représentées en conséquence par les produits $\Delta v,\ \Delta v',\ \Delta v'',\cdots$. Alors, si la fonction f(r) est telle que, sans altérer sensiblement les sommes désignées par G et par R, on puisse faire abstraction de celles des molécules $m,\ m',\ m'',\cdots$ qui sont les plus voisines de la molécule m, les valeurs de $G,\ R$ fournies par les équations $(37)_C$ et $(39)_C$ différeront très peu de celles que déterminent les formules

$$(49)_C \quad \begin{cases} G = \frac{\Delta}{2}\mathbf{S}[\pm r\cos^2\alpha \mathrm{f}(r) v], \\ R = \frac{\Delta}{2}\mathbf{S}[r\cos^2\alpha\cos^2\beta f(r) v] \end{cases}$$

quand on étend le signe **S**, non plus à tous les points matériels $m,\ m',\ m'',\cdots$, mais à tous les éléments $v,\ v',\ v'',\cdots$ du volume $\mathcal{V}$.

Or, dans cette dernière hypothèse, le second membre de chacune des expression $(49)_C$ pourra être remplacé par une intégrale triple relative à trois coordonées polaires dont l'une serait le rayon vecteur **r**, tandis que les deux autres représenteraient les angles formés :

- par le rayon vecteur **r** avec l'axe des x ;
- par le plan qui renferme le même rayon et l'axe des x avec le plan des $x,\ y$.

[7, p.241-2]

#12. Soient $p,\ q$ les deux angles dont il s'agit. Chaque intégrale triple devra être prise entre les limites $p=0,\ q=\pi,\ q=0,\ q=2\pi,\ r=0,\ r=l$; et l'on pourra même, sans erreur sensible, remplacer la seconde limite de r ou le rayon l par l'infini positif. [7, p.242]

$$(50)_C \quad \begin{cases} G = \pm\frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3\mathrm{f}(r)\cos^2\alpha\sin p dr dq dp, \\ R = \frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3 f(r)\cos^2\alpha\ \cos^2\beta\sin p dr dq dp \end{cases} \tag{80}$$

We compute in general case such that :

$$(51)_C \quad \begin{cases} \cos\alpha = \cos p, \\ \cos\beta = \sin p\cos q, \\ \cos\gamma = \sin p\sin q \end{cases}$$

(⇓) At this step after various considerations and calculations, Cauchy introduces his polar system $(51)_C$ for the first time in his paper. This means "Cauchy's rigorous calculus" based on his rigidity in mathematics. By the way, Navier uses it at first step of his calculation in (26). cf. Grabiner [19] (⇑)

$$\begin{cases} \int_0^{2\pi}\int_0^\pi \cos^2\alpha\ \sin p\ dq\ dp = 2\pi\int_0^\pi\cos^2 p\ \sin\ p\ dp = 2\pi\Big[-\frac{\cos^3 p}{3}\Big]_0^\pi = \frac{4\pi}{3}, \\ \int_0^{2\pi}\int_0^\pi\cos^2\ \alpha\ \cos^2\beta\sin\ p\ dp = \int_0^{2\pi}\cos^2 q\ dq\int_0^\pi\cos^2 p(1-\cos^2 p)\sin p\ dp \\ = \Big[\frac{q}{2}+\frac{1}{4}\sin 2q\Big]_0^{2\pi}\Big[-\frac{\cos^5 p}{5}\Big]_0^\pi = (\frac{2\pi}{2}-0)(\frac{2}{3}-\frac{2}{5}) = \frac{4\pi}{15} \end{cases} \tag{81}$$

$$C_3 = \frac{1}{2}\frac{4\pi}{15} = \frac{2\pi}{15}, \quad C_4 = \frac{1}{2}\frac{4\pi}{3} = \frac{2\pi}{3}$$

Then (80) turns out by (71) the following :

$$(52)_C \quad \begin{cases} G = \pm\frac{2\pi\Delta}{3}\int_0^\infty r^3 \mathrm{f}(r)dr, \\ R = \frac{2\pi\Delta}{15}\int_0^\infty r^3 f(r)dr = \pm\frac{2\pi\Delta}{15}\int_0^\infty \Big[r^4\mathrm{f}'(r) - r^3\mathrm{f}(r)\Big]dr \end{cases} \tag{82}$$

D'ailleurs, si, pour des valeurs croissantes de la distance r, la fonction $\mathrm{f}(r)$ décroit plus rapidement que la fraction $\frac{1}{r^4}$, si de plus le produit $r^4\mathrm{f}'(r)$ s'évanouit pour $r = 0$, on trouvera, en supposant la fonction $\mathrm{f}'(r)$ continue, et en intégrant par parties,

$$(53)_C \quad \int_0^\infty r^4\mathrm{f}'(r)dr = -4\int_0^\infty r^3\mathrm{f}(r)dr$$

On aura donc alors

$$(54)_C \quad R = -G,$$

et, par conséquent, on tirera des formules $(46)_C$

$$(55)_C \quad X = 2R\frac{\partial v}{\partial a}, \quad Y = 2R\frac{\partial v}{\partial b}, \quad Z = 2R\frac{\partial v}{\partial c}$$

($\Downarrow$) This interpretation is very important in the sense of RDF by Cauchy.

• ¶ 16.

Lorsque les quantités, désinées dans les formules $(40)_C$ et $(48)_C$ par les lettres $G, H, I, L, M, N, P, Q, R$ et Δ, deviennent constantes, c'est-à-dire, indépendantes des coordonées a, b, c, ou, ce qui revient au même, de la place qu'occupe la molécule m, alors, en faisant, pour plus de commodité,

$$(56)_C \quad \begin{cases} A = \Big[(L+G)\frac{\partial\xi}{\partial a} + (R-G)\frac{\partial\eta}{\partial b} + (Q-G)\frac{\partial\zeta}{\partial c}\Big]\Delta, \\ B = \Big[(R-H)\frac{\partial\xi}{\partial a} + (M+H)\frac{\partial\eta}{\partial b} + (P-H)\frac{\partial\zeta}{\partial c}\Big]\Delta, \\ C = \Big[(Q-I)\frac{\partial\xi}{\partial a} + (P-I)\frac{\partial\eta}{\partial b} + (N+I)\frac{\partial\zeta}{\partial c}\Big]\Delta, \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \Delta \begin{bmatrix} L+G & R-G & Q-G \\ R-H & M+H & P-H \\ Q-I & P-I & N+I \end{bmatrix} \begin{bmatrix} \frac{\partial\xi}{\partial a} \\ \frac{\partial\eta}{\partial b} \\ \frac{\partial\zeta}{\partial c} \end{bmatrix}$$

$$(57)_C \quad \begin{cases} D = \Big[(P+I)\frac{\partial\eta}{\partial c} + (P+H)\frac{\partial\zeta}{\partial b}\Big]\Delta, \\ E = \Big[(Q+G)\frac{\partial\zeta}{\partial a} + (Q+I)\frac{\partial\xi}{\partial c}\Big]\Delta, \\ F = \Big[(R+H)\frac{\partial\xi}{\partial b} + (R+G)\frac{\partial\eta}{\partial a}\Big]\Delta, \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} D \\ E \\ F \end{bmatrix} = \Delta \begin{bmatrix} 0 & P+I & P+H \\ Q+I & 0 & Q+G \\ R+H & R+G & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\partial\xi}{\partial c} & \frac{\partial\xi}{\partial b} \\ \frac{\partial\eta}{\partial c} & 0 & \frac{\partial\eta}{\partial a} \\ \frac{\partial\zeta}{\partial b} & \frac{\partial\zeta}{\partial a} & 0 \end{bmatrix}$$

Then $(40)_C$ is reduced to the following :

$$(58)_C \quad \begin{cases} X = \frac{1}{\Delta}\Big(\frac{\partial A}{\partial a} + \frac{\partial F}{\partial b} + \frac{\partial E}{\partial c}\Big), \\ Y = \frac{1}{\Delta}\Big(\frac{\partial F}{\partial a} + \frac{\partial B}{\partial b} + \frac{\partial D}{\partial c}\Big), \\ X = \frac{1}{\Delta}\Big(\frac{\partial E}{\partial a} + \frac{\partial D}{\partial b} + \frac{\partial C}{\partial c}\Big), \end{cases} \quad \Rightarrow \quad \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \Delta \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial a} \\ \frac{\partial}{\partial b} \\ \frac{\partial}{\partial c} \end{bmatrix}$$

By $(41)_C$ and $(45)_C$,

$$G = H = I, \quad L = M = N, \quad P = Q = R, \quad L = 3R.$$

$$\begin{aligned}
\frac{A}{\Delta} = & \ (L+G)\frac{\partial \xi}{\partial a} + (R-G)\Big(\frac{\partial \eta}{\partial b} + \frac{\partial \zeta}{\partial c}\Big) \\
= & \ (L+G)\frac{\partial \eta}{\partial b} + (R-G)(\nu - \frac{\partial \zeta}{\partial c}) \\
= & \ (3R+G-R+G)\frac{\partial \xi}{\partial a} + (R-G)\nu \\
= & \ 2(R+G)\frac{\partial \xi}{\partial a} + (R-G)\nu \\
\frac{B}{\Delta} = & \ (L+G)\frac{\partial \eta}{\partial b} + (R-G)\Big(\frac{\partial \xi}{\partial a} + \frac{\partial \zeta}{\partial c}\Big) \\
= & \ (L+G)\frac{\partial \eta}{\partial b} + (R-G)(\nu - \frac{\partial \eta}{\partial b}) \\
= & \ (3R+G-R+G)\frac{\partial \eta}{\partial b} + (R-G)\nu \\
= & \ 2(R+G)\frac{\partial \eta}{\partial b} + (R-G)\nu
\end{aligned}$$

where,

$$(47)_C \quad \nu = \frac{\partial \xi}{\partial a} + \frac{\partial \eta}{\partial b} + \frac{\partial \zeta}{\partial c},$$

By the same way,

$$\begin{cases}
\frac{C}{\Delta} = 2(R+G)\frac{\partial \zeta}{\partial c} + (R-G)\nu, \\
\frac{D}{\Delta} = (R+G)\Big(\frac{\partial \eta}{\partial b} + \frac{\partial \zeta}{\partial c}\Big), \\
\frac{E}{\Delta} = (R+G)\Big(\frac{\partial \zeta}{\partial a} + \frac{\partial \xi}{\partial c}\Big), \\
\frac{F}{\Delta} = (R+G)\Big(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\Big)
\end{cases}$$

For convenience's sake, in the particular case, for $(41)_C$ and $(45)_C$ to hold, it is sufficient to be as follows :

$$(59)_C \quad (R+G)\Delta \equiv \frac{1}{2}k, \quad (R-G)\Delta \equiv K$$

For the equations $(56)_C$ and $(57)_C$,

$$(60)_C \quad \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} = \begin{bmatrix} k\frac{\partial \xi}{\partial a} + K\nu & \frac{1}{2}k\Big(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\Big) & \frac{1}{2}k\Big(\frac{\partial \zeta}{\partial a} + \frac{\partial \xi}{\partial c}\Big) \\ \frac{1}{2}k\Big(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\Big) & k\frac{\partial \eta}{\partial b} + K\nu & \frac{1}{2}k\Big(\frac{\partial \eta}{\partial c} + \frac{\partial \zeta}{\partial b}\Big) \\ \frac{1}{2}k\Big(\frac{\partial \zeta}{\partial a} + \frac{\partial \xi}{\partial c}\Big) & \frac{1}{2}k\Big(\frac{\partial \eta}{\partial c} + \frac{\partial \zeta}{\partial b}\Big) & k\frac{\partial \zeta}{\partial c} + K\nu \end{bmatrix}$$

If, moreover, the condition $(54)_C$: $R = -G$ holds, then $k = 0$ holds , and the following hold :

$$(61)_C \quad A = B = C = K\nu, \quad D = E = F = 0.$$

A.3.3. **Consideration of Elastic Fluid by Cauchy.**

We show the equation number of fluid by Cauchy in below, with $(\cdot)_{C^*}$ instead by $(\cdot)_C$ for discrimination with the elastic equations as above.

• ¶ 17. Assumption of elastic fluid.

As the equations in equilibrium :

$$(67)_{C^*} \quad \begin{cases}
(L+G)\frac{\partial^2 \xi}{\partial x^2} + (R+H)\frac{\partial^2 \xi}{\partial y^2} + (Q+I)\frac{\partial^2 \xi}{\partial z^2} + 2R\frac{\partial^2 \eta}{\partial x \partial y} + 2Q\frac{\partial^2 \zeta}{\partial z \partial x} + X = 0, \\
(R+G)\frac{\partial^2 \eta}{\partial x^2} + (M+H)\frac{\partial^2 \eta}{\partial y^2} + (P+I)\frac{\partial^2 \eta}{\partial z^2} + 2P\frac{\partial^2 \zeta}{\partial y \partial z} + 2R\frac{\partial^2 \xi}{\partial x \partial y} + Y = 0, \\
(Q+G)\frac{\partial^2 \zeta}{\partial x^2} + (P+H)\frac{\partial^2 \zeta}{\partial y^2} + (N+I)\frac{\partial^2 \zeta}{\partial z^2} + 2Q\frac{\partial^2 \xi}{\partial z \partial x} + 2P\frac{\partial^2 \eta}{\partial y \partial z} + Z = 0,
\end{cases}$$

and as the equations in motion :

$$(68)_{C^*} \quad \begin{cases}
(L+G)\frac{\partial^2 \xi}{\partial x^2} + (R+H)\frac{\partial^2 \xi}{\partial y^2} + (Q+I)\frac{\partial^2 \xi}{\partial z^2} + 2R\frac{\partial^2 \eta}{\partial x \partial y} + 2Q\frac{\partial^2 \zeta}{\partial z \partial x} + X = \frac{\partial^2 \xi}{\partial t^2}, \\
(R+G)\frac{\partial^2 \eta}{\partial x^2} + (M+H)\frac{\partial^2 \eta}{\partial y^2} + (P+I)\frac{\partial^2 \eta}{\partial z^2} + 2P\frac{\partial^2 \zeta}{\partial y \partial z} + 2R\frac{\partial^2 \xi}{\partial x \partial y} + Y = \frac{\partial^2 \eta}{\partial t^2}, \\
(Q+G)\frac{\partial^2 \zeta}{\partial x^2} + (P+H)\frac{\partial^2 \zeta}{\partial y^2} + (N+I)\frac{\partial^2 \zeta}{\partial z^2} + 2Q\frac{\partial^2 \xi}{\partial z \partial x} + 2P\frac{\partial^2 \eta}{\partial y \partial z} + Z = \frac{\partial^2 \zeta}{\partial t^2}
\end{cases}$$

Si de plus les valeurs de $G,\ H,\ I,\ L,\ M,\ N,\ P,\ Q,\ R$ deviennent indépendantes en chaque point des directions assignées aux des $x,\ y$ et z, les conditions $(41)_C$ et $(45)_C$ seront vérifiées, et, en supposant la quantité v déterminée par l'equation $(47)_C$, ou, ce qui revient au même, par la suivante

$$(69)_{C^*}\quad \nu = \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z} = \nabla\cdot\mathbf{u} = div\ \mathbf{u},\quad \mathbf{u} = (\xi,\ \eta,\ \zeta).$$

As the equilibrium of fluid :

$$(70)_{C^*}\quad \begin{cases} (R+G)\Big(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\Big) + 2R\frac{\partial\nu}{\partial x} + X = 0, \\ (R+G)\Big(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} + \frac{\partial^2\eta}{\partial z^2}\Big) + 2R\frac{\partial\nu}{\partial y} + Y = 0, \\ (R+G)\Big(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2} + \frac{\partial^2\zeta}{\partial z^2}\Big) + 2R\frac{\partial\nu}{\partial z} + Z = 0, \end{cases}$$

and as the equations in motion :

$$(71)_{C^*}\quad \begin{cases} (R+G)\Big(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\Big) + 2R\frac{\partial\nu}{\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ (R+G)\Big(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} + \frac{\partial^2\eta}{\partial z^2}\Big) + 2R\frac{\partial\nu}{\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ (R+G)\Big(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2} + \frac{\partial^2\zeta}{\partial z^2}\Big) + 2R\frac{\partial\nu}{\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2}, \end{cases}$$

By $(54)_C$: $\quad R = -G$, we get $\Delta = 0$ of $(71)_{C^*}$, then

$$(72)_{C^*}\quad 2R\frac{\partial\nu}{\partial x} + X = 0,\quad 2R\frac{\partial\nu}{\partial y} + Y = 0,\quad 2R\frac{\partial\nu}{\partial z} + Z = 0$$

$$(73)_{C^*}\quad 2R\frac{\partial\nu}{\partial x} + X = \frac{\partial^2\xi}{\partial t^2},\quad 2R\frac{\partial\nu}{\partial y} + Y = \frac{\partial^2\eta}{\partial t^2},\quad 2R\frac{\partial\nu}{\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2}$$

On doit observer

- que la quantité v, déterminée par formule $(69)_{C^*}$, représente la dilatation qu'éprouve un volume très petit, mais choisi de manière à renfermer avec la molécule $\mathbf{m}$ un grand nombre de molécules voisines, tandis que ces molécules changent de position dans l'espace.
- Ajoutons que les formules $(72)_{C^*}$ et $(73)_{C^*}$, étant semblables aux formules $(63)_C$, $(72)_{C^*}$ et $(77)_{C^*}$ des pages 173, 175 and 176,[43] paraissent convenir à un système de molécules qui seraient disposées de manière à constituer un fluide élastique.

[7, p.248]

• ¶ 18. Verification of equations in elastic fluid.
By replacing $(a,\ b,\ c)$ of $(56)_C$ and $(57)_C$ with $(x,\ y,\ z)$, we get $(74)_{C^*}$, $(75)_{C^*}$ of the equivalence of $(56)_C$ and $(57)_C$.
• ¶ 19.

$$(67)_C \quad\Rightarrow\quad (76)_{C^*}\quad \begin{cases} \frac{\partial A}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial E}{\partial z} + X\Delta = 0, \\ \frac{\partial F}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial D}{\partial z} + Y\Delta = 0, \\ \frac{\partial E}{\partial x} + \frac{\partial D}{\partial y} + \frac{\partial C}{\partial z} + Z\Delta = 0, \end{cases} \quad\Rightarrow\quad \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} + \Delta \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{0}$$

[43]($\Downarrow$) Equations $(63)_C$, $(72)_C$ and $(77)_C$ of p.173, 175, 176 are included in [6], which are as follows :

$$(63)_C\quad \frac{\partial l(p)}{\partial x} = k(X - \frac{\partial u}{\partial t}),\quad \frac{\partial l(p)}{\partial y} = k(Y - \frac{\partial v}{\partial t}),\quad \frac{\partial l(p)}{\partial z} = k(Z - \frac{\partial w}{\partial t});$$

$$(72)_C\quad \frac{\partial l(\mathcal{P})}{\partial x} = kX,\quad \frac{\partial l(\mathcal{P})}{\partial y} = kY,\quad \frac{\partial l(\mathcal{P})}{\partial z} = kZ;$$

$$(77)_C\quad k\frac{\partial u}{\partial t} = \frac{\partial\infty}{\partial x},\quad k\frac{\partial v}{\partial t} = \frac{\partial\infty}{\partial y},\quad k\frac{\partial w}{\partial t} = \frac{\partial\infty}{\partial z};$$

• ¶ 20. Two-constant by Cauchy.

(⇓) In this article, Cauchy states two-constant in his fluid equations, in which the two-constant corresponds to the tensor function with the main axis (the normal stress) of Laplacian and grad.div term. (⇑)

We get the tensor from $(76)_{C^*}$ as follows :

$$(77)_{C^*} \quad \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix}$$

Then $(74)_C$ and $(75)_C$ are reduced to the following :

$$(60)_C \quad \Rightarrow \quad (78)_{C^*} \quad \begin{bmatrix} A & F & E \\ F & B & D \\ E & D & C \end{bmatrix} = \begin{bmatrix} k\frac{\partial \xi}{\partial x} + K\nu & \frac{1}{2}k\left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right) & \frac{1}{2}k\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right) \\ \frac{1}{2}k\left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right) & k\frac{\partial \eta}{\partial y} + K\nu & \frac{1}{2}k\left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right) \\ \frac{1}{2}k\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right) & \frac{1}{2}k\left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right) & k\frac{\partial \zeta}{\partial z} + K\nu \end{bmatrix}$$

By replacing $R+G$ and $2R$ in the equation $(70)_{C^*}$ and $(71)_{C^*}$ with the following : [44]

$$C_1^* \equiv R + G = \frac{k}{2\Delta}, \quad C_2^* \equiv 2R = \frac{k+2K}{2\Delta}$$

(⇓) Here, C_1^* is the constant to the tensor function with the main axis (the normal stress) of Laplacian. C_2^* corresponds to therms of the gradient of divergence of $\mathbf{u}$. In today's NS equations, the value of ratio of coefficients : $\frac{C_1^*}{C_2^*} = \frac{\text{coefficient of tensor}}{\text{coefficient of grad.div}} = \frac{k}{k+2K}$. By Prandtl [64, p.259] in 1934, the ratio was fixed at 3. We had have to wait by the time, when including this ratio of two coefficients, what is called the NS equations were expressed by Prandtl in fluid equation. cf. Table 7. (⇑)

As the equations in equilibrium of fluid :

$$(79)_{C^*} \quad \begin{cases} C_1^*\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}\right) + C_2^*\frac{\partial v}{\partial x} + X = 0, \\ C_1^*\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2}\right) + C_2^*\frac{\partial v}{\partial y} + Y = 0, \\ C_1^*\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2}\right) + C_2^*\frac{\partial v}{\partial z} + Z = 0, \end{cases}$$

and as the equations in motion of fluid :

$$(80)_{C^*} \quad \begin{cases} C_1^*\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}\right) + C_2\frac{\partial v}{\partial x} + X = \frac{\partial^2 \xi}{\partial t^2}, \\ C_1^*\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2}\right) + C_2^*\frac{\partial v}{\partial y} + Y = \frac{\partial^2 \eta}{\partial t^2}, \\ C_1^*\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2}\right) + C_2^*\frac{\partial v}{\partial z} + Z = \frac{\partial^2 \zeta}{\partial t^2}. \end{cases}$$

• ¶ 21. Comparison with Navier's equation in elasticity.

Cauchy says : for the reduction of the equations $(79)_{C^*}$ and $(80)_{C^*}$ to Navier's equations([46]) to determine the law of equilibrium and elasticity, it is necessary to assume such as the condition which we have mentioned above :

$$(81)_{C^*} \quad k = 2K$$

• ¶ 22. Comments on Navier's equations in elasticity.

> On voir au rest que, si l'on considère un corps élastique comme un système de points matériels qui agissent les uns sur les autres à de très petites distances, les lois de l'équilibre ou du mouvement intérieur de ce corps seront exprimées dans beaucoup de cas par des équations différentes de celles qu'a données M.Navier.
>
> • Les formules $(67)_{C^*}$ et $(68)_{C^*}$ paraissent spécialement applicables au cas où, l'élasticité n'étant pas la même dans les diverses directions, le corps offre trois axes d'élasticité rectangulaires entre eux, et parallèles aux axes des x, des y et des z.
>
> • Les formules $(70)_{C^*}$ et $(71)_{C^*}$, au contraire, semblent devoire s'appliquer au cas où le corps est également élastique dans tous les sens ; et alors on retrouvera les formules de M.Navier, si l'on attribue à la quantité G une valeur nulle.

[44](⇓) The following notation : C_1^* and C_2^* are not our two-constant but the two symbols by Cauchy. To avoiding confusion, we don't use C_1 and C_2 by Cauchy but C_1^* and C_2^*.

- Ajoutons que, si, dans les formules $(67)_{C^*}$ et $(68)_{C^*}$, on réduit à zéro, non seulement la quantité G, mais encore les quantités de même espèce H et I, ces formules deviendront respectivement $(83)_{C^*}$ et $(84)_{C^*}$ [7, pp.251-252]

If $G = 0$, then we get the equations of equiblium in equal elasticity :

$$(67)_{C^*} \quad \Rightarrow \quad (83)_{C^*} \quad \begin{cases} L\frac{\partial^2\xi}{\partial x^2} + R\frac{\partial^2\xi}{\partial y^2} + Q\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = 0, \\ R\frac{\partial^2\eta}{\partial x^2} + M\frac{\partial^2\eta}{\partial y^2} + P\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = 0, \\ Q\frac{\partial^2\zeta}{\partial x^2} + P\frac{\partial^2\zeta}{\partial y^2} + N\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = 0, \end{cases}$$

and as the equations of motion in equal elasticity:

$$(68)_{C^*} \quad \Rightarrow \quad (84)_{C^*} \quad \begin{cases} L\frac{\partial^2\xi}{\partial x^2} + R\frac{\partial^2\xi}{\partial y^2} + Q\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ R\frac{\partial^2\eta}{\partial x^2} + M\frac{\partial^2\eta}{\partial y^2} + P\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ Q\frac{\partial^2\zeta}{\partial x^2} + P\frac{\partial^2\zeta}{\partial y^2} + N\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2} \end{cases}$$

A.4. Poisson's equations deduced from his principle.

A.4.1. Principle for the equations in elastic solid.

We deduce K and k in accordance with Poisson [59, p.368-405, §1-§16] as follows.

- § 2. For abbreviation, we put the following :

$$\begin{cases} ax_1 + by_1 + c(z_1 - \zeta_1) \equiv \phi, \\ a'x_1 + b'y_1 + c'(z_1 - \zeta_1) \equiv \psi, \\ a''x_1 + b''y_1 + c''(z_1 - \zeta_1) \equiv \theta, \end{cases} \qquad \begin{cases} \phi\frac{du}{dx} + \psi\frac{du}{dy} + \theta\frac{du}{dz} \equiv \phi', \\ \phi\frac{dv}{dx} + \psi\frac{dv}{dy} + \theta\frac{dv}{dz} \equiv \psi', \\ \phi\frac{dw}{dx} + \psi\frac{dw}{dy} + \theta\frac{dw}{dz} \equiv \theta' \end{cases} \tag{83}$$

Namely,

$$\begin{bmatrix} \phi \\ \psi \\ \theta \end{bmatrix} \equiv \begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 - \zeta \end{bmatrix}, \quad \begin{bmatrix} \phi' \\ \psi' \\ \theta' \end{bmatrix} \equiv \begin{bmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ \theta \end{bmatrix} = \nabla\mathbf{u} \cdot \begin{bmatrix} \phi \\ \psi \\ \theta \end{bmatrix}$$

$$r^2 = \phi^2 + \psi^2 + \theta^2,$$

$$(r')^2 = (\phi + \phi')^2 + (\psi + \psi')^2 + (\theta + \theta')^2,$$

$$r^2 = x_1^2 + y_1^2 + (z_1 - \zeta_1)^2,$$

$$(r')^2 = r^2 + 2\phi\phi' + 2\psi\psi' + 2\theta\theta' + (\phi')^2 + (\psi')^2 + (\theta')^2$$

- § 3. We assume that α : the average molecular interval, ω : surface, $\frac{\omega}{\alpha^2}$: the number of molecules on ω.

$$P = \sum \frac{(\phi + \phi')\zeta}{\alpha^3 r'} fr', \quad Q = \sum \frac{(\psi + \psi')\zeta}{\alpha^3 r'} fr' \quad R = \sum \frac{(\theta + \theta')\zeta}{\alpha^3 r'} fr'. \tag{84}$$

- § 4.

$$r' = r + \frac{1}{r}(\phi\phi' + \psi\psi' + \theta\theta')$$

At the same degree of approximation, we get : [45]

$$\boxed{\frac{1}{r'} fr' = \frac{1}{r} fr + (\phi\phi' + \psi\psi' + \theta\theta') \frac{d.\frac{1}{r} fr}{rdr}}$$

We get the three elements of force $P,\ Q,\ R$ from (83) and (84) :[46]

$$\boxed{(1)_{Pe} \quad \begin{cases} P = \sum \frac{(\phi+\phi')\zeta}{\alpha^3 r} fr + \sum(\phi\phi' + \psi\psi' + \theta\theta') \frac{\phi\zeta}{\alpha^3 r} \frac{d.\frac{1}{r} fr}{dr}, \\ Q = \sum \frac{(\psi+\psi')\zeta}{\alpha^3 r} fr + \sum(\phi\phi' + \psi\psi' + \theta\theta') \frac{\psi\zeta}{\alpha^3 r} \frac{d.\frac{1}{r} fr}{dr}, \\ R = \sum \frac{(\theta+\theta')\zeta}{\alpha^3 r} fr + \sum(\phi\phi' + \psi\psi' + \theta\theta') \frac{\theta\zeta}{\alpha^3 r} \frac{d.\frac{1}{r} fr}{dr}, \end{cases}} \tag{85}$$

$$\Rightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \sum \left(\begin{bmatrix} (\phi + \phi') & (\phi\phi' + \psi\psi' + \theta\theta')\phi \\ (\psi + \psi') & (\phi\phi' + \psi\psi' + \theta\theta')\psi \\ (\theta + \theta') & (\phi\phi' + \psi\psi' + \theta\theta')\theta \end{bmatrix} \begin{bmatrix} \frac{\zeta fr}{\alpha^3 r} \\ \frac{\zeta}{\alpha^3 r} \frac{d.\frac{1}{r} fr}{dr} \end{bmatrix} \right)$$

---

[45]($\Downarrow$) We correct this equation. Poisson [60], the corresponding equation (100), there is $\frac{1}{rdr}$

[46]($\Downarrow$) We use $_{Pe}$ in the left-side equation number as Poisson's equation number in [59]. And $_{Pf}$ means Poisson's equation number in Poisson [60]

We donate :

β : the angle between the vectorial rayon of one of molecules : r and the axis of ζ, and

γ : the angle which the projection of the rayon on the $x-y$ plane makes with the axis of x. We have :

$$\begin{cases} x_1 = r\cos\beta\cos\gamma, \\ y_1 = r\sin\beta\sin\gamma, \\ \zeta = r\cos\beta, \end{cases}$$

The quantities which majored under the $\sum$ take the form : pFr, which is expressed by

p : an entire function with sines and cosines of β and γ,

Fr : a same function as fr, of which value are insensible for total sensible value of the variable, and moreover, which equals 0 for the particular value of $r = 0$.

We consider that the summation in question is composed by the parties of the form :

$$\sum_r [(\sum_\beta \sum_\gamma p)Fr],$$

here, the outer $\sum$ corresponds to r and can extend to $r = \infty$, and the inner double $\sum$s correspond to β and γ.

- § 5. The value : $\sum\sum p$ related to sr^2 is assumed the product of p and the number of molecules which contain in the surface of sr^2, and which is expressed by $\frac{sr^2}{\alpha^2}$. We consider a hemisphere with the radius : $r = 1$ on the x_1-y_1 plane as follows :

$$\frac{r^2}{\alpha^2}\sum_\beta\sum_\gamma ps,$$

This new summation extends to the all parties in the hemisphere of the unit for the radius. Because p doesn't decrease very rapidly, we can change s with the diffrential element of the above surface, and the sign : $\sum$ with the signs of integration, we can take the following :

$$s = \sin\beta d\beta d\gamma, \quad \sum_\beta\sum_\gamma ps = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} p\sin\beta d\beta d\gamma,$$

$$\sum_\beta\sum_\gamma p = \frac{r^2}{\alpha^2}\int_0^{\frac{\pi}{2}}\int_0^{2\pi} p\sin\beta d\beta d\gamma,$$

$$\sum_r [(\sum_\beta\sum_\gamma p)Fr] = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} p\sin\beta d\beta d\gamma \sum_r \frac{r^2}{\alpha^2}Fr.$$

- § 6.

$$\phi = gr, \quad \psi = hr, \quad \theta = lr, \quad \phi' = g'r, \quad \psi' = h'r, \quad \theta' = l'r,$$

$$\begin{cases} g = a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta, & g' = g\frac{du}{dx} + h\frac{du}{dy} + l\frac{du}{dz}, \\ h = a'\sin\beta\cos\gamma + b'\sin\beta\sin\gamma - c'\cos\beta, & h' = g\frac{dv}{dx} + h\frac{dv}{dy} + l\frac{dv}{dz}, \\ l = a''\sin\beta\cos\gamma + b''\sin\beta\sin\gamma - c''\cos\beta, & l' = g\frac{dw}{dx} + h\frac{dw}{dy} + l\frac{dw}{dz} \end{cases}$$

($\Downarrow$) In brief :

$$\begin{bmatrix} g \\ h \\ l \end{bmatrix} = \begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix} \begin{bmatrix} \sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ -\cos\beta \end{bmatrix}, \quad \begin{bmatrix} g' \\ h' \\ l' \end{bmatrix} = \begin{bmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{bmatrix} \begin{bmatrix} g \\ h \\ l \end{bmatrix} = \nabla\mathbf{u}\cdot\begin{bmatrix} g \\ h \\ l \end{bmatrix}$$

By using Poisson's so-called *effective transformation*, [47] we obtain from (84) the following :($\Uparrow$)

$$\begin{cases} P = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(g+g')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')g\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ Q = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(h+h')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')h\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ R = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(l+l')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')l\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \end{cases} \tag{86}$$

[47]($\Downarrow$) $\frac{1}{r'}fr' = \frac{1}{r}fr + (\phi\phi' + \psi\psi' + \theta\theta')\frac{d.\frac{1}{r}fr}{rdr}$ ([60, p.42]).

($\Downarrow$) (86) implies the following :

$$\begin{aligned}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \Delta\Big(\begin{bmatrix} g+g' & (gg'+hh'+ll')g \\ h+h' & (gg'+hh'+ll')h \\ l+l' & (gg'+hh'+ll')l \end{bmatrix} \begin{bmatrix} \sum \frac{r^3 fr}{\alpha^5} \\ \sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr} \end{bmatrix} \Big), \\ &\equiv \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \Delta\Big(\begin{bmatrix} g+g' & P' \\ h+h' & Q' \\ l+l' & R' \end{bmatrix} \begin{bmatrix} K \\ k \end{bmatrix} \Big), \end{aligned} \tag{87}$$

where

$$\begin{bmatrix} P' \\ Q' \\ R' \end{bmatrix} \equiv$$

$$\begin{bmatrix} (g^3\nabla_x u + g^2h\nabla_y u + g^2lg\nabla_z u) + (g^2h\nabla_x v + gh^2\nabla_y v + ghl\nabla_z v) + (g^2l\nabla_x w + ghl\nabla_y w + gl^2\nabla_z w) \\ (g^2h\nabla_x u + gh^2\nabla_y u + ghl\nabla_z u) + (gh^2\nabla_x v + h^3\nabla_y v + h^2l\nabla_z v) + (ghl\nabla_x w + h^2l\nabla_y w + hl^2\nabla_z w) \\ (g^2l\nabla_x u + ghl\nabla_y u + gl^2g\nabla_z u) + (ghl\nabla_x v + h^2l\nabla_y v + hl^2\nabla_z v) + (gl^2\nabla_x w + hl^2\nabla_y w + l^3\nabla_z w) \end{bmatrix} \tag{88}$$

$\Delta := \cos\beta\cdot\sin\beta\, d\beta\, d\gamma$, $\nabla_x u := \frac{du}{dx}$, etc , $K := \sum \frac{r^3 fr}{\alpha^5}$, $k := \sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}$.
Below, we use the following integral formulae :

$$\begin{cases} \int \sin^2 x dx = \frac{x}{2} - \frac{1}{4}\sin 2x, \\ \int \cos^2 x dx = \frac{x}{2} + \frac{1}{4}\sin 2x, \\ \int \sin x\cos x dx = \frac{1}{2}\sin^2 x, \\ \int \sin^2 x\cos^2 x dx = -\frac{1}{8}(\frac{1}{4}\sin 4x - x), \\ \int \sin x\cos^m x dx = -\frac{\cos^{m+1} x}{m+1}, \\ \int \sin^m x\cos x dx = \frac{\sin^{m+1} x}{m+1}, \\ \int \cos^m x\sin^n x dx = \frac{\cos^{m-1} x\sin^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2} x\sin^n x dx, \quad (m>2 \;\&\; n>1), \\ \int \cos^m x\sin^n x dx = -\frac{\sin^{n-1} x\cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x\sin^{n-2} x dx, \quad (m>1 \;\&\; n>2) \end{cases} \tag{89}$$

($\Uparrow$)

At first, we get the following :

$$\begin{cases} g+g' = (a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta)(1+\frac{du}{dx}) + h\frac{du}{dy} + l\frac{du}{dz}, \\ h+h' = g\frac{dv}{dx} + (a'\sin\beta\cos\gamma + b'\sin\beta\sin\gamma - c'\cos\beta)(1+\frac{dv}{dy}) + l\frac{dv}{dz}, \\ l+l' = g\frac{dw}{dx} + h\frac{dw}{dy} + (a''\sin\beta\cos\gamma + b''\sin\beta\sin\gamma - c''\cos\beta)(1+\frac{dw}{dz}) \end{cases}$$

For the integral of $g+g'$, we put : $A \equiv a\sin\beta\cos\gamma + b\sin\beta\sin\gamma$ and $B \equiv c\cos\beta$.

$$\begin{aligned} \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} g\Delta = & \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \big(a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta\big)\cos\beta\sin\beta d\beta \\ \equiv & \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} (A-B)\cos\beta\sin\beta d\beta. \end{aligned}$$

$$\begin{aligned} \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} A\cos\beta\sin\beta d\beta &= \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} d\beta\Big(a\sin^2\beta\cos\beta\cos\gamma + b\sin^2\beta\sin\gamma\cos\beta\Big) \\ &= a\Big[\frac{\sin^3\beta}{3}\Big]_0^{2\pi} \int_0^{2\pi} d\gamma\cos\gamma + b\Big[\frac{\sin^3\beta}{3}\Big]_0^{2\pi} \int_0^{2\pi} d\gamma\sin\gamma = 0 \end{aligned}$$

$$\int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} -B\cos\beta\sin\beta d\beta = c\int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} -\cos^2\beta\sin\beta d\beta = -c\int_0^{2\pi} d\gamma\Big[-\frac{\cos^3\beta}{3}\Big]_0^{\frac{\pi}{2}} = -\frac{2}{3}\pi c$$

We get the following summary of the first half of (87) by the same way as *above* :

$$\begin{cases} \int_0^{\frac{\pi}{2}}\int_0^{2\pi}(g+g')\Delta = -\frac{2\pi}{3}\Big(c + c\frac{du}{dx} + c'\frac{du}{dy} + c''\frac{du}{dz}\Big), \\ \int_0^{\frac{\pi}{2}}\int_0^{2\pi}(h+h')\Delta = -\frac{2\pi}{3}\Big(c' + c\frac{dv}{dx} + c'\frac{dv}{dy} + c''\frac{dv}{dz}\Big), \\ \int_0^{\frac{\pi}{2}}\int_0^{2\pi}(l+l')\Delta = -\frac{2\pi}{3}\Big(c'' + c\frac{dw}{dx} + c'\frac{dw}{dy} + c''\frac{dw}{dz}\Big), \end{cases} \quad \Rightarrow -\frac{2\pi}{3}(\mathbf{c} + \nabla\mathbf{u}\cdot\mathbf{c}),$$

where $\mathbf{c} = \begin{pmatrix} c & c' & c'' \end{pmatrix}^T$. Below, we use the relations as follows :

$$a^2 + b^2 + c^2 = 1, \quad aa' + bb' + cc' = 0, \quad a'a'' + b'b'' + c'c'' = 0, \quad a''a + b''b + c''c = 0.$$

A.4.2. **Summation of last half term.**

We show only the value of $\iint g^3 \Delta$ in (88) in detail.

$$\begin{aligned}
\iint g^3\Delta &= \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} g^3 \cos\beta \sin\beta d\beta \\
&= \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \big(a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta\big)^3 \cos\beta\sin\beta d\beta \\
&= \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} d\beta \Big[\big(a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta\big)\Big(a^2\sin^2\beta\cos^2\gamma + b^2\sin^2\beta\sin^2\gamma + c^2\cos^2\beta \\
&+ 2ab\sin^2\beta\cos\gamma\sin\gamma - 2bc\sin\beta\cos\beta\sin\gamma - 2ca\sin\beta\cos\beta\cos\gamma\Big)\cos\beta\sin\beta\Big]
\end{aligned}$$

When we arrange $\iint g^3\Delta$ with respect to c's terms, then we may compute only 5 terms : cA_1, cA_2, cB_1, cB_2 and cC_1 relative to c as follows :

$$\begin{aligned}
\iint g^3\Delta &= \Big[(a\sin\beta\cos\gamma) * (-2ca\sin\beta\cos\beta\cos\gamma) \\
&+ (b\sin\beta\sin\gamma) * (-2bc\sin\beta\cos\beta\sin\gamma) \\
&- c\cos\beta * \big(a^2\sin^2\beta\cos^2\gamma + b\sin^2\beta\sin^2\gamma + c^2\cos^2\beta\big)\Big]\cos\beta\sin\beta \\
&= -c\Big(-2a^2\sin^3\beta\cos^2\beta\cos^2\gamma \\
&+ b^2\sin^2\gamma\sin^3\beta\sin^2\beta \\
&+ a^2\sin^3\beta\cos^2\beta\cos^2\gamma + b^2\sin^3\beta\cos^2\beta\sin^2\gamma \\
&+ c^2\cos^4\beta\sin\beta\Big) \\
&\equiv -c(A_1 + B_1 + A_2 + B_2 + C)
\end{aligned}$$

We compute the first term : $-ca^2$ with the integral in the right-hand side above as follows :

- $-cA_1 : a * (-2ca)$:

$$-2a^2c\int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} d\beta \sin^3\beta\cos^2\beta\cos^2\gamma = -\frac{4}{15}a^2c\Big[\frac{\gamma}{2} + \frac{1}{4}\sin 2\gamma\Big]_0^{2\pi} = -\frac{4}{15}\pi ca^2$$

Similarly,

- $-cA_2 : -c * a^2$:

$$\begin{aligned}
&-ca^2\int_0^{2\pi}\cos^2\gamma d\gamma\int_0^{\frac{\pi}{2}}\sin^3\beta\cos^2\beta d\beta \\
&= -ca^2\int_0^{2\pi}\cos^2\gamma d\gamma\Big(\Big[-\frac{\sin^2\beta\cos^3\beta}{5}\Big]_0^{\frac{\pi}{2}} + \frac{2}{5}\int_0^{\frac{\pi}{2}}\cos^2\beta\sin\beta d\beta\Big) \\
&= -ca^2\int_0^{2\pi}\cos^2\gamma d\gamma\Big(\frac{2}{5}(-(-\frac{1}{3})) + \frac{2}{5}\Big[-\frac{\cos^3\beta}{3}\Big]_0^{\frac{\pi}{2}}\Big) \\
&= -ca^2\frac{2}{15}\int_0^{2\pi}\cos^2\gamma d\gamma = -ca^2\frac{2}{15}\Big[\frac{\gamma}{2} + \frac{1}{4}\sin 2\gamma\Big]_0^{2\pi} = -\frac{2\pi}{15}ca^2
\end{aligned}$$

- $-cB_1 : b * (-2bc)$:

$$-2b^2c\int_0^{2\pi} d\gamma\int_0^{\frac{\pi}{2}} d\beta\sin^3\beta\cos^2\beta\sin^2\gamma = -\frac{4}{15}b^2c\Big[\frac{\gamma}{2} - \frac{1}{4}\sin 2\gamma\Big]_0^{2\pi} = -\frac{4}{15}\pi cb^2$$

- $-cB_2 : -c * b^2$:

$$-cb^2\int_0^{2\pi}\sin^2\gamma d\gamma\int_0^{\frac{\pi}{2}}\sin^3\beta\sin^2\beta d\beta = -cb^2\frac{2}{15}\Big[\frac{\gamma}{2} - \frac{1}{4}\sin 2\gamma\Big]_0^{2\pi} = -\frac{2\pi}{15}cb^2$$

- $-cC : -cc^2$:

$$-c^3\int_0^{2\pi}d\gamma\int_0^{\frac{\pi}{2}}\cos^4\beta\sin\beta d\beta = -c^3\int_0^{2\pi}d\gamma\Big[-\frac{\cos^5\beta}{5}\Big]_0^{\frac{\pi}{2}} = -\frac{2\pi}{5}cc^2$$

The integral of the terms of $2ab$, $-2bc$ and $-2ca$ are all zero respectively. Therefore we get the following equation :

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}g^3\Delta &= (-\frac{2}{15}-\frac{4}{15})\pi ca^2+(-\frac{2}{15}-\frac{4}{15})\pi cb^2-\frac{2\pi}{5}cc^2\\ &= -\frac{2\pi c}{5}\Big(a^2+b^2+c^2\Big) = -\frac{2\pi c}{5}, \quad \text{where } a^2+b^2+c^2=1\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}g^2h\Delta &= -\frac{2\pi}{15}\Big(3c^2c'+2aa'c+2bb'c+a^2c'+b^2c'\Big)\\ &= -\frac{2\pi}{15}\Big\{2c(cc'+aa'+bb')+c'(c^2+a^2+b^2)\Big\} = -\frac{2\pi}{15}c',\end{aligned}$$

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\int_0^{2\pi}ghl\Delta\\ &= -\frac{2\pi}{15}\Big(3cc'c''+aa'c''+aa''c'+a'a''c+bb'c''+bb''c'+b'b''c\Big)\\ &= -\frac{2\pi}{15}\Big\{c''(cc'+aa'+bb')+c'(a''a+b''b+c''c)+c(a''a'+b''b'+c''c')\Big\} = 0.\end{aligned}$$

Therefore, in brief :

$$\iint g^3\Delta = -\frac{2\pi c}{5}, \quad \iint g^2h\Delta = -\frac{2\pi c'}{15}, \quad \iint ghl\Delta = 0.$$

We get the same as above.

$$\iint h^3\Delta = -\frac{2\pi c'}{5}, \quad \iint l^3\Delta = -\frac{2\pi c''}{5},$$

$$\iint gh^2\Delta = -\frac{2\pi c}{15}, \quad \iint gl^2\Delta = -\frac{2\pi c}{15}, \quad \iint g^2l\Delta = -\frac{2\pi c''}{15},$$

$$\iint h^2l\Delta = -\frac{2\pi c''}{15}, \quad \iint hl^2\Delta = -\frac{2\pi c'}{15}.$$

We show in brief :

$$\boxed{\frac{2\pi}{3}\sum\frac{r^3}{\alpha^5}fr \equiv K, \quad \frac{2\pi}{15}\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr} \equiv k.} \tag{90}$$

These coefficients were replaced later with (96), in Poisson[60], p.46, p.140.)

By using (90), we get the following from (86) :

$$\begin{cases}P = -K\Big(c+\frac{du}{dx}c+\frac{du}{dy}c'+\frac{du}{dz}c''\Big)-k\Big(3\frac{du}{dx}c+\frac{du}{dy}c'+\frac{du}{dz}c''+\frac{dv}{dx}c'+\frac{dv}{dy}c+\frac{dw}{dx}c''+\frac{dw}{dz}c\Big),\\ Q = -K\Big(c'+\frac{dv}{dx}c+\frac{dv}{dy}c'+\frac{dv}{dz}c''\Big)-k\Big(\frac{dv}{dx}c+3\frac{dv}{dy}c'+\frac{dv}{dz}c''+\frac{du}{dx}c'+\frac{du}{dy}c+\frac{dw}{dy}c''+\frac{dw}{dz}c'\Big),\\ R = -K\Big(c''+\frac{dw}{dx}c+\frac{dw}{dy}c'+\frac{dw}{dz}c''\Big)-k\Big(\frac{dw}{dx}c+\frac{dw}{dy}c'+3\frac{dw}{dz}c''+\frac{du}{dx}c''+\frac{du}{dz}c+\frac{dv}{dy}c''+\frac{dv}{dz}c'\Big).\end{cases} \tag{91}$$

($\Downarrow$) By the way, we can state the linear relation of $P,\ Q,\ R$, which made of two tensors on the basis of $[c,\ c',\ c'']^T$ are as follows :

$$\begin{bmatrix}P\\Q\\R\end{bmatrix} = -K\begin{bmatrix}1+\frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz}\\ \frac{dv}{dx} & 1+\frac{dv}{dy} & \frac{dw}{dz}\\ \frac{dw}{dx} & \frac{dw}{dy} & 1+\frac{dw}{dz}\end{bmatrix} -k\begin{bmatrix}3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} & \frac{du}{dy}+\frac{dv}{dx} & \frac{du}{dz}+\frac{dw}{dx}\\ \frac{dv}{dz}+\frac{dw}{dy} & \frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz} & \frac{dv}{dx}+\frac{du}{dy}\\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dw}{dy}+\frac{dv}{dz} & \frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\end{bmatrix}$$

- § 8.

$$(3)_{P^e}\begin{cases} X_\rho = \frac{dP_1}{dz} + \frac{dP_2}{dy} + \frac{dP_3}{dx}, \\ Y_\rho = \frac{dQ_1}{dz} + \frac{dQ_2}{dy} + \frac{dQ_3}{dx}, \\ Z_\rho = \frac{dR_1}{dz} + \frac{dR_2}{dy} + \frac{dR_3}{dx}. \end{cases}$$

- § 10.

$$(4)_{P^e}\begin{cases} X_1 + P_1c'' + P_2c' + P_3c = 0, \\ Y_1 + Q_1c'' + Q_2c' + Q_3c = 0, \\ Z_1 + R_1c'' + R_2c' + R_3c = 0. \end{cases}$$

where c, c' and c'' are cosines of the angles formed between the original coordinates x, y, z and the normal line to the surface of separation.

- § 14.

(⇓) This article is very important for discussion of disputes between Poisson and Navier or between Arago and Navier. Why Poisson uses $\sum$ instead of $\int$. According to Poisson, if we would compute K and k under the symbol $\int$ like in (93), then we end up the result : $k = -K = 0$ at the same time, so that Poisson uses the symbol $\sum$. (cf. Table 9.) [a] (⇑)

[a](⇓) cf. There are many referrnces on these topics : Navier with Poisson : [48] in 1828; [49] in 1829; [50]in 1829; [51] in 1829 and Navier with Arago [51] in 1829. Above all, in [51], Navier's everlasting assertions are repeated.

Les équations (3) et (4) conviennent aussi à l'état primitif du corps ; et pour les appliquer à ce cas particulier, il suffit d'y faire $u = 0$, $v = 0$, $w = 0$, et d'y supprimer toutes les forces données, extérieures ou intérieures. On a alors

$$R_1 = Q_2 = P_3 = -K;$$

les six autres quantités P_1, Q_1, etc., sont nulles, et les six équations (3) et (4) se réduisent à quatre, savoir :

$$\frac{dK}{dx} = 0, \quad \frac{dK}{dy} = 0, \quad \frac{dK}{dz} = 0, \quad K = 0.$$

D'après les trois premières, la quantité K est une constante qui est nulle en vertu de la dernière. En remettant donc pour K ce que cette lettre représente (no.6)[48], et supprimant le facteur constant $\frac{2\pi}{3\alpha^5}$, on aura

$$\sum r^3 fr = 0$$

Ainsi, dans l'état du corps qu'on peut regarder comme son état naturel, où il n'est soumis qu'à l'action mutuelle de ses molécules, due à leur attraction et à la chaleur, les intervalles qui les séparent doivent être tels que cette *é*quation ait lieu pour tous les points du corps. Si l'on y introduit une nouvelle quantité de chaleur, ce qui augumentera, pour la même distance, l'intensité de la force répulsive, sans changer celle de la force attractive, il faudra que les intervalles moléculaires augmentent de manière que cette équation continue de subsister; et de là vient la dilatation calorifique, différente dans les différentes matières, à cause que la fonction fr n'y est pas la même.

Cette équation donne lieu de faire une remarque importante ; c'est que les sommes $\sum$ du no.6, que représentent *les lettres K et k, ne peuvent être changées en des intégrales,* quoique la variable r croisse dans chacune d'elles par de très-petites différences égales à α ; car si cette transformation était possible, k serait zéro en même temps que K ; d'où il résulterait qu'après le changement de forme du corps, les forces P, Q, R, seraient nulles comme auparavant, et que des forces données qui agiraient sur le corps ne pourraient

[48](⇓) § 6.

se faire équilibre, ce qui est inadmissible. Pour faire voir que k s'évanouirait au même temps que K, observons qu'on aurait

$$K = \frac{2\pi}{3}\int_0^\infty \frac{r^3}{\alpha^6} frdr, \quad k = \frac{2\pi}{15}\int_0^\infty \frac{r^5}{\alpha^6} d.\frac{1}{r}fr, \tag{92}$$

en multipliant sous les signes $\sum$ par $\frac{dr}{\alpha}$, et remplaçant ces signes par ceux de l'intégration. Or, si l'on intègre par partie, et si l'on fait attention que fr est nulle aux deux limites, il en résultera

$$k = -\frac{2\pi}{3}\int_0^\infty \frac{r^3}{\alpha^6} frdr = -K \tag{93}$$

ce qui montre que la quantité K étant nulle, on aurait aussi $k = 0$. [59, p.398-399, § 14]

- § 16.

Je substitute, en outre, dans les équations $(3)_{Pe}$ à la place de P, Q, etc., leurs valeurs, et je suppose le corps homogène; en observant que $K = 0$, il vient

$$(6)_{Pe} \quad \begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = 0, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = 0, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = 0, \end{cases} \tag{94}$$

a^2 étant un coefficient, égal à $\frac{3k}{\rho}$. Ces équations ont la même forme que celles qui ont été données par M.Navier[49], et qu'il a obtenues en partant de l'hypothèse que les molécules du corps, après son changement de forme, s'attirent proportionnellement aux accroissements de leurs distances mutuelles; et en admettant, de plus, que les résultantes de ces forces peuvent s'exprimer *par des intégrales, ce qui rendrait nul le coefficient* a^2, ainsi qu'on l'a vu plus haut. Les équations relatives à la surface, formées de la même manière, se trouvent aussi dans le Mémoire de M.Navier. [60, p.403-4,§16]

We can see that $(6)_{Pe}$ (= (94)) is able to be modified to (44) as follows :

$$\begin{cases} X - \frac{d^2u}{dt^2} + \frac{a^2}{3}\left(3\frac{d^2u}{dx^2} + 2\frac{d^2v}{dydx} + 2\frac{d^2w}{dzdx} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) = 0, \\ Y - \frac{d^2v}{dt^2} + \frac{a^2}{3}\left(3\frac{d^2v}{dy^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dzdy} + \frac{d^2v}{dx^2} + \frac{d^2v}{dz^2}\right) = 0, \\ Z - \frac{d^2w}{dt^2} + \frac{a^2}{3}\left(3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz} + \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right) = 0, \end{cases} \tag{95}$$

A.4.3. **General principle and equations in elastic solid and fluid.**

Poisson proposed two constants k and K in his compressible fluid equations in 1829, and issued in 1831([60, p.46, p.140]) ,

$$(3\text{-}8)_{Pf} \qquad k \equiv \frac{1}{30\varepsilon^3}\sum r^3\frac{d.\frac{1}{r}fr}{dr} = \frac{2\pi}{15}\sum\frac{1}{4\pi\varepsilon^3}r^3\frac{d.\frac{1}{r}fr}{dr}, \quad K \equiv \frac{1}{6\varepsilon^3}\sum rfr = \frac{2\pi}{3}\sum\frac{rfr}{4\pi\varepsilon^3}, \tag{96}$$

ε : *la grandeur moyenne des intervalles moléculaires autour du point* M. (the mean value of the molecular intervals around the point M.)([60], p.141).

We summarize Poisson's deduction of k and K in [60], which is a little different from [59, p.368-405, § 1-§ 16].[50]

- § 15. Here, at first, we introduce the setting of situation by Poisson for strict description.

Soit ω de sa section horisontale; sur cette section élevons dans A une cylindre vertical, dont la hauteur soit au moins égale au rayon d'activité des molécules; appelons B ce cylindre : l'action des molécules de A' sur celles de B, divisée par

[49] By Poisson's footnote : Tome VII de ces Mémoires, which is Navier[46].

[50] (⇓) In Poisson [60], the title of the chapter 3 is "*Calcul des Pressions dans les Corps élastiques ; équations defférentielles de l'équiblibre et du mouvement de ces Corps.*"

> ω, sera la *pression* exercée par A sur A', rapportée à l'unité de surface et relative au point M. [60, p.29]

⇒ Namely,
- We put ω : the area of the holizontal section ; on this section in A, in which a vertical cylinder stands, at the height of which equals at least the radius of sphere of the molecular activity.
- We call B : the cylinder : the molecular action of A' on it of B, divided with ω, is the pressure activated by A' on A, related to the unit of the surface and relative to the point M.

Poisson continues :

> Nous la représenterons par $N\omega$, en sort que N soit la pression rapportée à l'unité de surface; et à cause que la composante verticale de la force fr aggisante au point m et dirigée de bas en haut, est $\frac{z}{r}fr$, nous aurons
>
> $$N\omega = \sum \frac{z}{r} fr$$
>
> la somme Σ s'etandant à tous les points m de B et m' de A'. [60, p.30]

⇒ We put $N\omega$ such that N : the pressure
- related to the unit of the surface ;
- and caused the vertical component of the force : fr acting at the point m and passing from below to above,

is $\frac{z}{r}fr$, then we get :

$$N\omega = \sum \frac{z}{r} fr,$$

where the sum $\sum$ covers all points m of B and m' of A'.

- § 16. We put N : the pressure, ε : the mean value of the molecular intervals around the point M as above. We put ν : a proportional number to the volume : ωz.

$$N\omega = \sum \frac{\nu z}{r} fr$$

$$\text{where} \quad \nu = \frac{\omega z}{\varepsilon^3}$$

then

$$(3\text{-}1)_{Pf} \quad N = \frac{1}{\varepsilon^3} \sum \frac{z^2}{r} fr$$

If we call μ the mass of a molecule, or its mean value, the mass of the cylinder : ωz turns equal to $\nu\mu$, and the ratio : $\frac{\nu\mu}{\omega z}$ expresses the density. Hence, we put it with ρ, and put its value for ν, we have :

$$\rho = \frac{\mu}{\varepsilon^3}$$

- § 17. We see also that the quantity : $\frac{z^2}{r}fr$ obeys under the sign $\sum$ being null for all the points of the plane moved by M, the sum which it makes, become $\frac{1}{2}$ of the same sum extended to all the points of A and of A'. Moreover, r^2, which is the square of the distance from M' to the three planes of the rectangle passing through M, and the sum $\sum \frac{z^2}{r}fr$ having the same sum which we replace successively z^2 with the two another squares : x^2 and y^2, then it turns that it equals $\frac{1}{3}\sum \frac{r^2}{r}fr$. After these considerations, we have the following :

$$(3\text{-}2)_{Pf} \quad N = \frac{1}{\varepsilon^3} \sum \frac{z^2}{r} fr = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{\varepsilon^3} \sum \frac{r^2}{r} fr = \frac{1}{6\varepsilon^3} \sum rfr; \qquad (97)$$

- § 20. (This section corresponds to the sections from § 2 to § 4 in [59] describing the elastic solid.)

$$(3\text{-}6)_{Pf} \quad \begin{cases} x' = ax_1 + by_1 - cz_1, \\ y' = a'x_1 + b'y_1 - c'z_1, \\ z' = a''x_1 + b''y_1 - c''z_1 \end{cases}$$

(⇓) Namely,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & -c \\ a' & b' & -c' \\ a'' & b'' & -c'' \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} a & a' & a'' \\ b & b' & b'' \\ -c & -c' & -c'' \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \quad (98)$$

(⇑) where, 9 coefficients $a,\ b,\ \cdots$ are the cosines of the angles which $x_1,\ y_1$ and the extension of the axis z_1, with the axis of $x,\ y,\ z$, and these cosines are given.

$$r_1^2 = (\varphi + \varphi')^2 + (\psi + \psi')^2 + (\theta + \theta')^2$$

Here, for abbreviation :

$$\begin{cases} ax_1 + by_1 - cz_1 \equiv \varphi, \\ a'x_1 + b'y_1 - c'z_1 \equiv \psi, \\ a''x_1 + b''y_1 - c''z_1 \equiv \theta, \end{cases} \qquad \begin{cases} \varphi\frac{du}{dx} + \psi\frac{du}{dy} + \theta\frac{du}{dz} \equiv \varphi', \\ \varphi\frac{dv}{dx} + \psi\frac{dv}{dy} + \theta\frac{dv}{dz} \equiv \psi', \\ \varphi\frac{dw}{dx} + \psi\frac{dw}{dy} + \theta\frac{dw}{dz} \equiv \theta' \end{cases} \quad (99)$$

(⇓) Namely,

$$\begin{bmatrix} \varphi \\ \psi \\ \theta \end{bmatrix} \equiv \begin{bmatrix} a & b & -c \\ a' & b' & -c' \\ a'' & b'' & -c'' \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \begin{bmatrix} \varphi' \\ \psi' \\ \theta' \end{bmatrix} \equiv \begin{bmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \\ \theta \end{bmatrix} = \nabla \mathbf{u} \cdot \begin{bmatrix} \varphi \\ \psi \\ \theta \end{bmatrix}$$

(⇑)

$$\omega P = -\sum \frac{\varphi + \varphi'}{r_1} fr_1, \quad \omega Q = -\sum \frac{\psi + \psi'}{r_1} fr_1 \quad \omega R = -\sum \frac{\theta + \theta'}{r_1} fr_1.$$

for the components of the total action of A' on B, in covering the summation $\sum$ to the all points m' of A' and to the all points m of B. Because the function fr_1 is regarded as positive or negative, in accordance with the distance : r_1, the force which it represents, becomes repulsive or attractive, the components act in the direction of x, y and z, positive or negative, with their values above turn into positive or negative.

$$P = -\frac{1}{\varepsilon^3} \sum \frac{(\varphi + \varphi')z_1}{r_1} fr_1, \quad Q = -\frac{1}{\varepsilon^3} \sum \frac{(\psi + \psi')z_1}{r_1} fr_1 \quad R = -\frac{1}{\varepsilon^3} \sum \frac{(\theta + \theta')z_1}{r_1} fr_1.$$

By observing that

$$r^2 = \varphi^2 + \psi^2 + \theta^2,$$

and by neglecting the quantities of the second order with respect to φ', ψ', θ', we get the following :

$$r_1 = r + \frac{1}{r}(\varphi\varphi' + \psi\psi' + \theta\theta')$$

At the same degree of approximation, we get the following :[51]

$$\boxed{\frac{1}{r_1} fr_1 = \frac{1}{r} fr + (\varphi\varphi' + \psi\psi' + \theta\theta') \frac{d.\frac{1}{r} fr}{rdr}} \quad (100)$$

[51](⇓) Because this equation (100) must equal (85) in [59] of elastic solid, we corrected here Poisson's misprint.

TABLE 12. The 63 coefficients of the components of $-H$

	$x_1^2 z_1^2$ of E	$y_1^2 z_1^2$ of F	$z_1^2 z_1^2 (= z_1^4)$ of G	number of term
$\frac{du}{dx}$	$a(ca+2ca) = ca^2 + 2caa = 3ca^2$	$b(cb+2cb) = cb^2 + 2cbb = 3cb^2$	$cc^2 = c^3$	7
$\frac{dv}{dy}$	$a'(ca' + 2c'a) = ca'^2 + 2c'aa'$	$b'(cb' + 2c'b) = cb'^2 + 2c'bb'$	cc'^2	7
$\frac{dw}{dz}$	$a''(ca'' + 2c''a) = ca''^2 + 2c''aa''$	$b''(cb'' + 2c''b) = cb''^2 + 2c''bb''$	cc''^2	7
$\frac{du}{dy} + \frac{dv}{dx}$	$a(c'a + 2ca') = c'a^2 + 2caa'$	$b(c'b + 2cb') = c'b^2 + 2cbb'$	$cc'c = c^2c'$	$7 \times 2 = 14$
$\frac{du}{dz} + \frac{dw}{dx}$	$a(c''a + 2ca'') = c''a^2 + 2caa''$	$b(c''b + 2cb'') = c''b^2 + 2cbb''$	$ccc'' = c^2c''$	$7 \times 2 = 14$
$\frac{dv}{dz} + \frac{dw}{dy}$	$ca'a'' + c'aa'' + c''aa'$	$cb'b'' + cbb'' + c''bb'$	$cc'c''$	$7 \times 2 = 14$
number of term	27	27	9	63

$$
(3\text{-}7)_{Pf} \quad \begin{cases} P = -\frac{1}{\varepsilon^3}\sum \frac{(\varphi+\varphi')z_1}{r} fr - \frac{1}{\varepsilon^3}\sum(\varphi\varphi' + \psi\psi' + \theta\theta')\varphi z_1 \frac{d.\frac{1}{r}fr}{rdr}, \\ Q = -\frac{1}{\varepsilon^3}\sum \frac{(\psi+\psi')z_1}{r} fr - \frac{1}{\varepsilon^3}\sum(\varphi\varphi' + \psi\psi' + \theta\theta')\psi z_1 \frac{d.\frac{1}{r}fr}{rdr}, \\ R = -\frac{1}{\varepsilon^3}\sum \frac{(\omega+\omega')z_1}{r} fr - \frac{1}{\varepsilon^3}\sum(\varphi\varphi' + \psi\psi' + \theta\theta')\omega z_1 \frac{d.\frac{1}{r}fr}{rdr}, \end{cases} \tag{101}
$$

$$
\Rightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = -\frac{1}{\varepsilon^3}\sum\left(\begin{bmatrix} (\varphi+\varphi')z_1 & (\varphi\varphi' + \psi\psi' + \theta\theta')\varphi z_1 \\ (\psi+\psi')z_1 & (\varphi\varphi' + \psi\psi' + \theta\theta')\psi z_1 \\ (\omega+\omega')z_1 & (\varphi\varphi' + \psi\psi' + \theta\theta')\omega z_1 \end{bmatrix} \begin{bmatrix} \frac{fr}{r} \\ \frac{d.\frac{1}{r}fr}{rdr} \end{bmatrix} \right)
$$

A.4.4. **The first coefficient : K in summation of P, Q, R in elastic solid.**

- § 21. (This section corresponds to the sections from § 5 to § 7 in [59] describing the elastic solid.)

$$
\sum \frac{z^2}{r} fr = \frac{1}{6}\sum rfr,
$$

$$
\begin{cases} \sum \frac{(\varphi+\varphi')z_1}{r} fr = -\frac{1}{6}\Big(c + c\frac{du}{dx} + c'\frac{du}{dy} + c''\frac{du}{dz}\Big)\sum rfr = -\frac{1}{6}\Big(c\big(1 + \frac{du}{dx}\big) + c'\frac{du}{dy} + c''\frac{du}{dz}\Big)\sum rfr, \\ \sum \frac{(\psi+\psi')z_1}{r} fr = -\frac{1}{6}\Big(c' + c\frac{dv}{dx} + c'\frac{dv}{dy} + c''\frac{dv}{dz}\Big)\sum rfr = -\frac{1}{6}\Big(c\frac{dv}{dx} + c'\big(1 + \frac{dv}{dy}\big) + c''\frac{dv}{dz}\Big)\sum rfr, \\ \sum \frac{(\omega+\omega')z_1}{r} fr = -\frac{1}{6}\Big(c'' + c\frac{dw}{dx} + c'\frac{dw}{dy} + c''\frac{dw}{dz}\Big)\sum rfr = -\frac{1}{6}\Big(c\frac{dw}{dx} + c'\frac{dw}{dy} + c''\big(1 + \frac{dw}{dz}\big)\Big)\sum rfr, \end{cases}
$$

A.4.5. **The second coefficient : k in summation of P, Q, R in elastic solid.**

We denote the second summation in P of (101)(= $(3\text{-}7)_{Pf}$) by H such as :

$$
H \equiv \sum (\varphi\varphi' + \psi\psi' + \theta\theta')\varphi z_1 \frac{d.\frac{1}{r}fr}{rdr} \tag{102}
$$

$$
\sum x_1^2 z_1^2 \frac{d.\frac{1}{r}fr}{rdr} \equiv E, \quad \sum y_1^2 z_1^2 \frac{d.\frac{1}{r}fr}{rdr} \equiv F, \quad \sum z_1^4 \frac{d.\frac{1}{r}fr}{rdr} \equiv G,
$$

We get the 63 coefficients of components of $-H$ as in Table 12.[52] The sums of E, F and G are equal for A and for A', because the terms related to the plane made of x_1 and y_1, become equal to zero by taking the differential : we can take the volume of the total body, and take successively the value as $\frac{1}{2}$ of it. When we regard the body as homogeneous in the sphere of the molecular activity, we get the following :

$$
\begin{cases} \sum z_1^4 \frac{d.\frac{1}{r}fr}{rdr} = \sum y_1^4 \frac{d.\frac{1}{r}fr}{rdr} = \sum x_1^4 \frac{d.\frac{1}{r}fr}{rdr}, \\ \sum y_1^2 z_1^2 \frac{d.\frac{1}{r}fr}{rdr} = \sum x_1^2 z_1^2 \frac{d.\frac{1}{r}fr}{rdr} = \sum x_1^2 y_1^2 \frac{d.\frac{1}{r}fr}{rdr}. \end{cases}
$$

[52]($\Downarrow$) Tables 12, 13 and 14 are made by us.

TABLE 13. The coefficients by the combination of the terms of $z_1^2 \times z_1^2$ in G

	$c^2x'^2$	$c'^2y'^2$	$c''^2z'^2$	$2cc'x'y'$	$2c'c''y'z'$	$2cc''x'z'$
$c^2x'^2$	$c^4x'^4$	$2c^2c'^2x'^2y'^2$	$2c^2c''^2x'^2z'^2$	$4c^3c'x'^3y'$	$4c^2c'c''x'^2y'z'$	$4c^3c''x'^3z'$
$c'^2y'^2$		$c'^4y'^4$	$2c'^2c''^2y'^2z'^2$	$4cc'^3x'y'^3$	$4c'^3c''y'^3z'$	$4cc'^2c''x'y'^2z'$
$c''^2z'^2$			$c''^4z'^4$	$4cc'c''^2x'y'z'^2$	$4c'c''^3y'z'^3$	$4cc''^3x'z'^3$
$2cc'x'y'$				$4c^2c'^2x'^2y'^2$	$8cc'^2c''x'y'^2z'$	$8c^2c'c''x'^2y'z'$
$2c'c''y'z'$					$4c'^2c''^2y'^2z'^2$	$8cc'c''^2x'y'z'^2$
$2cc''x'z'$						$4c^2c''^2x'^2z'^2$

TABLE 14. The 21 coefficients by the combination of the terms of $z_1^2 \times z_1^2$ in G

	$c^2x'^2$	$c'^2y'^2$	$c''^2z'^2$	$2cc'x'y'$	$2c'c''y'z'$	$2cc''x'z'$
$c^2x'^2$	$c^4x'^4$	$2c^2c'^2x'^2y'^2$	$2c^2c''^2x'^2z'^2$			
$c'^2y'^2$		$c'^4y'^4$	$2c'^2c''^2y'^2z'^2$			
$c''^2z'^2$			$c''^4z'^4$			
$2cc'x'y'$				$4c^2c'^2x'^2y'^2$		
$2c'c''y'z'$					$4c'^2c''^2y'^2z'^2$	
$2cc''x'z'$						$4c^2c''^2x'^2z'^2$

$$\begin{cases} \sum z'^4 \frac{d.\frac{1}{r}fr}{rdr} = \sum y'^4 \frac{d.\frac{1}{r}fr}{rdr} = \sum x'^4 \frac{d.\frac{1}{r}fr}{rdr} = 2G, \\ \sum y'^2 z'^2 \frac{d.\frac{1}{r}fr}{rdr} = \sum x'^2 z'^2 \frac{d.\frac{1}{r}fr}{rdr} = \sum x'^2 y'^2 \frac{d.\frac{1}{r}fr}{rdr} = 2E = 2F. \end{cases} \tag{103}$$

From (98) we get the following :

$$\begin{cases} x_1 = ax' + a'y' + a''z' \\ y_1 = bx' + b'y' + b''z' \\ z_1 = -cx' - c'y' - c''z' \end{cases}$$

In Table 13,

$$\begin{aligned} &4(c^3c'x'^3y' + c^2c'c''x'^2y'z' + c^3c''x'^3z' + cc'^3x'y'^3 + c'^3c''y'^3z' + cc'^2c''x'y'^2z' \\ &+cc'c''^2x'y'z'^2 + c'c''^3y'z'^3 + cc''^3x'z'^3 + 2cc'^2c''x'y'^2z' + 2c^2c'c''x'^2y'z' + 2cc'c''^2x'y'z'^2) \\ &= 4\Big[(c^2x'^2 + c'^2y'^2 + c''^2z'^2)(cc'x'y' + c'c''y'z' + cc''x'z') + 2(c^2c'c''x'y'z' + cc'^2c''x'y'^2z' + cc'c''x'y'z'^2)\Big] \end{aligned}$$

Hence, we can consider only the elements of Table 14. From (103) and the 21 elements in the upper-trianglar matrix including the diagonal of Table 14 , we get G as follows :

$$\begin{aligned} G &= \sum z_1^4 \frac{d.\frac{1}{r}fr}{rdr} \\ &= \frac{1}{2}\sum \frac{d.\frac{1}{r}fr}{rdr}\Big[\big(c^4x'^4 + c'^4y'^4 + c''^4z'^4\big) + 6\big(c^2c'^2x'^2y'^2 + c^2c''^2y'^2z'^2 + c'^2c''^2z'^2x'^2\big)\Big] \\ &= \frac{1}{2}\Big[2G\big(c^4 + c'^4 + c''^4\big) + 6 \cdot 2F\big(c^2c'^2 + c^2c''^2 + c'^2c''^2\big)\Big] \\ &= G\big(c^4 + c'^4 + c''^4\big) + 6F\big(c^2c'^2 + c^2c''^2 + c'^2c''^2\big). \end{aligned} \tag{104}$$

Here, we put the following for convenience' sake :

$$\alpha \equiv c^4 + c'^4 + c''^4, \qquad \beta \equiv c^2c'^2 + c^2c''^2 + c'^2c''^2.$$

Because of[53]

$$\begin{cases} c^2+c'^2+c''^2=1, \\ c^4+c'^4+c''^4+2c^2c'^2+2c^2c''^2+2c'^2c''^2=1, \end{cases}$$

$$\Rightarrow \quad \begin{cases} (c^2+c'^2+c''^2)^2=1, \\ c^4+c'^4+c''^4+2c^2c'^2+2c^2c''^2+2c'^2c''^2=1 \Rightarrow \alpha=1-2\beta, \end{cases}$$

then, from (104), we get $G=\alpha G+6\beta F$ and then :

$$G=(1-2\beta)G+6\beta F \qquad \Rightarrow \qquad 2\beta G=6\beta F,$$

it becomes at last :

$$G=3F.$$

Moreover, because of $r_1^2=x_1^2+y_1^2+z_1^2$, we get the following :

$$\begin{aligned} &\sum r^4\frac{d.\frac{1}{r}fr}{rdr} \\ =\ &\sum x_1^4\frac{d.\frac{1}{r}fr}{rdr}+\sum y_1^4\frac{d.\frac{1}{r}fr}{rdr}+\sum z_1^4\frac{d.\frac{1}{r}fr}{rdr}+2\sum x_1^2y_1^2\frac{d.\frac{1}{r}fr}{rdr}+2\sum x_1^2z_1^2\frac{d.\frac{1}{r}fr}{rdr}+2\sum y_1^2z_1^2\frac{d.\frac{1}{r}fr}{rdr} \end{aligned} \tag{105}$$

From (103), (105) we get :

$$\frac{1}{2}\sum r^3\frac{d.\frac{1}{r}fr}{dr}=3G+6F=5G=15E=15F.$$

$$G=\frac{1}{10}\sum r^3\frac{d.\frac{1}{r}fr}{dr}, \quad E=F=\frac{1}{30}\sum r^3\frac{d.\frac{1}{r}fr}{dr},$$

As the common factor, we take $\frac{1}{30}$, then finally H of (102) turns into :

$$H=-\frac{1}{30}\sum r^3\frac{d.\frac{1}{r}fr}{dr}\Big[c\Big(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)+c'\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)+c''\Big(\frac{du}{dz}+\frac{dw}{dx}\Big)\Big]. \tag{106}$$

The second summation contained in Q of (101) ($=$ (3-7)$_{Pf}$) is deduced from H with the cyclical permutation of u and v, x and y, c and c', and similarly in R with the cyclical permutation of u and w, x and z, c and c''. In this manner, the equations (101) ($=$ (3-7)$_{Pf}$) turn out as follows :

$$\begin{cases} P=\Big[K(1+\frac{du}{dx})+k\Big(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big]c+\Big[K\frac{du}{dy}+k\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)\Big]c'+\Big[K\frac{du}{dz}+k\Big(\frac{du}{dz}+\frac{dw}{dx}\Big)\Big]c'', \\ Q=\Big[K(1+\frac{dv}{dy})+k\Big(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big]c'+\Big[K\frac{dv}{dx}+k\Big(\frac{dv}{dx}+\frac{du}{dy}\Big)\Big]c+\Big[K\frac{dv}{dz}+k\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)\Big]c'', \\ R=\Big[K(1+\frac{dw}{dz})+k\Big(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\Big)\Big]c''+\Big[K\frac{dw}{dy}+k\Big(\frac{dw}{dy}+\frac{dv}{dz}\Big)\Big]c'+\Big[K\frac{dw}{dx}+k\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)\Big]c, \end{cases} \tag{107}$$

where, for abbreviation, Poisson uses :

$$(3\text{-}8)_{Pf} \qquad k\equiv\frac{1}{30\varepsilon^3}\sum r^3\frac{d.\frac{1}{r}fr}{dr}=\frac{2\pi}{15}\sum\frac{1}{4\pi\varepsilon^3}r^3\frac{d.\frac{1}{r}fr}{dr}, \quad K\equiv\frac{1}{6\varepsilon^3}\sum rfr=\frac{2\pi}{3}\sum\frac{rfr}{4\pi\varepsilon^3} \tag{108}$$

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[53](⇓) We corrected Poisson's mistake :

$$c^4+c'^4+c''^4+2c^2c'^2+2c^2c''^2+2c'^2c''^2=0 \quad \Rightarrow \quad =1.$$

Because if

$$c^2+c'^2+c''^2=1$$

then we get clealy

$$c^4+c'^4+c''^4+2c^2c'^2+2c^2c''^2+2c'^2c''^2=1$$

Inversely, if the equation equals 0, then

$$G=-2\beta G+6\beta F \qquad \Rightarrow \qquad (1+2\beta)G=6\beta F \qquad \Rightarrow \qquad G=\frac{6\beta}{1+2\beta}F.$$

Then we can't get $G=3F$.

(⇓) By the way, we can state the linear relation of $P,\ Q,\ R$, which made of two tensors on the basis of $[c,\ c',\ c'']^T$ of $L,\ M,\ N$ and $H,\ I,\ J$ respectively, are as follows :

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix} + \begin{bmatrix} H \\ I \\ J \end{bmatrix}, \quad \begin{bmatrix} L \\ M \\ N \end{bmatrix} = K \begin{bmatrix} 1+\frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & 1+\frac{dv}{dy} & \frac{dw}{dx} \\ \frac{dw}{dy} & \frac{dw}{dx} & 1+\frac{dw}{dz} \end{bmatrix}$$

$$\begin{bmatrix} H \\ I \\ J \end{bmatrix} = k \begin{bmatrix} 3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} & \frac{du}{dz}+\frac{dw}{dx} & \frac{du}{dy}+\frac{dv}{dx} \\ \frac{dv}{dz}+\frac{dw}{dy} & \frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz} & \frac{dv}{dx}+\frac{du}{dy} \\ \frac{dw}{dy}+\frac{dv}{dz} & \frac{dw}{dx}+\frac{du}{dz} & \frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz} \end{bmatrix}$$

- § 22. We get the general equations as follows :

$$(3\text{-}9)_{Pf} \quad \begin{cases} P = P_1c'' + P_2c' + P_3c, \\ Q = Q_1c'' + Q_2c' + Q_3c, \\ R = R_1c'' + R_2c' + R_3c \end{cases} \Rightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{bmatrix} \begin{bmatrix} c'' \\ c' \\ c \end{bmatrix},$$

then we get the tensor on the basis of $[c'',\ c',\ c]^T$ from (107) as follows :

$$\begin{bmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} K\frac{du}{dz}+k\Big(\frac{du}{dz}+\frac{dw}{dx}\Big) & K\frac{du}{dy}+k\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) & K(1+\frac{du}{dx})+k\Big(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big) \\ K\frac{dv}{dz}+k\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) & K(1+\frac{dv}{dy})+k\Big(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\Big) & K\frac{dv}{dx}+k\Big(\frac{dv}{dx}+\frac{du}{dy}\Big) \\ K(1+\frac{dw}{dz})+k\Big(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\Big) & K\frac{dw}{dy}+k\Big(\frac{dw}{dy}+\frac{dv}{dz}\Big) & K\frac{dw}{dx}+k\Big(\frac{dw}{dx}+\frac{du}{dz}\Big) \end{bmatrix}$$

- § 23.

When we suppose that the initial state of the elastic solid is natural, it turns $K=0$, so we get the following :

$$(3\text{-}11)_{Pf} \quad \begin{bmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} k\Big(\frac{du}{dz}+\frac{dw}{dx}\Big) & k\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) & k\Big(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big) \\ k\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) & k\Big(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\Big) & k\Big(\frac{dv}{dx}+\frac{du}{dy}\Big) \\ k\Big(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\Big) & k\Big(\frac{dw}{dy}+\frac{dv}{dz}\Big) & k\Big(\frac{dw}{dx}+\frac{du}{dz}\Big) \end{bmatrix}$$

- § 24.

$$(3\text{-}12)_{Pf} \quad \begin{cases} X\rho = \frac{dP_1}{dz}+\frac{dP_2}{dy}+\frac{dP_3}{dx}, \\ Y\rho = \frac{dQ_1}{dz}+\frac{dQ_2}{dy}+\frac{dQ_3}{dx}, \\ Z\rho = \frac{dR_1}{dz}+\frac{dR_2}{dy}+\frac{dR_3}{dx}, \end{cases} \Rightarrow \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \frac{1}{\rho} \begin{bmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{bmatrix} \begin{bmatrix} \frac{d}{dz} \\ \frac{d}{dy} \\ \frac{d}{dx} \end{bmatrix} \qquad (109)$$

- § 27.

In homogeneous case, δ means the difference of the contraction or dilatation :

$$\frac{r'-r}{r} \equiv -\delta$$

$$P = -5k\delta c, \quad Q = -5k\delta c', \quad R = -5k\delta c'';$$

$$K = -5k\delta.$$

Replacing ε and r of K in (108) ($=(3\text{-}8)_{Pf}$) with ε' and r',

$$K = \frac{1}{6\varepsilon'^3}\sum r'fr'$$

and r' and ε' with

$$r' = r - r\delta, \quad \varepsilon' = z - \varepsilon\delta.$$

[54] For δ is very small value, we can develope K into the convergent series, with the order followed the power of δ, and neglecting the bigger power than the first, then it turns out as follows :

$$K = \frac{1+\delta}{6\varepsilon^3}\sum rfr - \frac{\delta}{6\varepsilon^3}\sum r^3\frac{d.\frac{1}{r}fr}{dr},$$

then, because $\sum rfr = 0$, by the condition of natural state,

$$\boxed{K = -\frac{\delta}{6\varepsilon^3}\sum r^3\frac{d.\frac{1}{r}fr}{dr} = -5k\delta.}$$

- § 31. Finally, Poisson assumes isotropic elasticity in natural state and the perpendicular pressure on the surface of corps.

  Je substitute, en outre, dans les équations (12)[55], à la place de $P_1,\ Q_1,\cdots$, leurs valeurs. Je supose le corps homogène, et je prends alors pour son état naturel auquel répondent les coordonées $x,\ y,\ z$, du point quelconque M, un état dans lequel la surface du corps est soumise à une pression normale et la même en tous ses points. En la représentant par Π, on aura $K = \Pi$ (§ 27). La quantite k étant négative (même § 27) et indépendante de $x,\ y,\ z$, je fais, pour abréger

He puts Π the normal pressure on the corps, and for abbreviation, he uses :

$$-\frac{5k}{\rho} \equiv a^2$$

then the motional equations of elastic corps are as follows :

$$\boxed{\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2u}{dx^2}, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2v}{dy^2}, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = \frac{\Pi}{\rho}\frac{d^2w}{dz^2}, \end{cases} \qquad (110)}$$

A.4.6. **Fluid pressure in motion, the differential equation of motion.**

- § 60.[56]

$$\begin{cases} Ft = P_1'c'' + P_2'c' + P_3'c - Kc, \\ F't = Q_1'c'' + Q_2'c' + Q_3'c - Kc', \\ F''t = R_1'c'' + R_2'c' + R_3'c - Kc'' \end{cases} \Rightarrow \begin{bmatrix} Ft \\ F't \\ F''t \end{bmatrix} = \begin{bmatrix} P_1' & P_2' & P_3' \\ Q_1' & Q_2' & Q_3' \\ R_1' & R_2' & R_3' \end{bmatrix} - K\begin{bmatrix} c \\ c' \\ c'' \end{bmatrix} \qquad (111)$$

$$\begin{bmatrix} P_1' & P_2' & P_3' \\ Q_1' & Q_2' & Q_3' \\ R_1' & R_2' & R_3' \end{bmatrix} = \begin{bmatrix} (K+k)\left(\frac{du}{dz} + \frac{dw}{dx}\right) & (K+k)\left(\frac{du}{dy} + \frac{dv}{dx}\right) & K + 2(K+k)\frac{du}{dx} - (K+k)\text{div}\,\mathbf{u} \\ (K+k)\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & K + 2(K+k)\frac{dv}{dy} - (K+k)\text{div}\,\mathbf{u} & (K+k)\left(\frac{du}{dy} + \frac{dv}{dx}\right) \\ K + 2(K+k)\frac{dw}{dz} - (K+k)\text{div}\,\mathbf{u} & (K+k)\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & (K+k)\left(\frac{du}{dz} + \frac{dw}{dx}\right) \end{bmatrix},$$

where, div $\mathbf{u} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$.

- § 63. Deduction of compressible, fluid equations.

[54](⇓) Then we get :

$$K = \frac{1}{6(z-\varepsilon\delta)^3}\sum r(1-\delta)f(r - r\delta)$$

[55](⇓) §24, $(3\text{-}12)_{Pf}$ (= (109)).

[56](⇓) Below, we use $(\bullet)_{Pf}$, for example, $(7\text{-}9)_{Pf}$ means the equation numbered for the equation (9) in the chapter 7 described by Poisson [60], in which $_{Pf}$ means the equation in the fluid problem by Poisson, because he numbered them by the same number between chapters.

[57] Poisson's tensor of the pressures in fluid reads as follows :

$$(7\text{-}7)_{Pf}\qquad \begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix} = \begin{bmatrix} \beta\Big(\frac{du}{dz}+\frac{dw}{dx}\Big) & \beta\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{du}{dx} \\ \beta\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dv}{dy} & \beta\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) \\ p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dw}{dz} & \beta\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) & \beta\Big(\frac{du}{dz}+\frac{dw}{dx}\Big) \end{bmatrix},$$

$$(k+K)\alpha=\beta,\quad (k-K)\alpha=\beta',\quad p=\psi t=K,\quad \text{then}\quad \beta+\beta'=2k\alpha, \tag{112}$$

where χt is the density of the fluid around the point M, and ψt is the pressure, and both depend on t, so we mean χt and ψt as $\chi(t)$ and $\psi(t)$.

($\Downarrow$) By the way, here, we can show the conversion of tensor $(7\text{-}7)_{Pf}$, replacing the first column with the third one, then we see easily the conventional style of array as follows :

$$\begin{bmatrix} U_3 & U_2 & U_1 \\ V_3 & V_2 & V_1 \\ W_3 & W_2 & W_1 \end{bmatrix} = \begin{bmatrix} p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{du}{dx} & \beta\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) & \beta\Big(\frac{du}{dz}+\frac{dw}{dx}\Big) \\ \beta\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dv}{dy} & \beta\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) \\ \beta\Big(\frac{du}{dz}+\frac{dw}{dx}\Big) & \beta\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) & p-\alpha\frac{d\psi t}{dt}-\frac{\beta'}{\chi t}\frac{d\chi t}{dt}+2\beta\frac{dw}{dz} \end{bmatrix},$$

($\Uparrow$)

Poisson deduces his fluid equation by the following steps :

$$(7\text{-}8)_{Pf}\qquad \begin{cases} \rho(X-\frac{d^2x}{dt^2})=\frac{dU_1}{dz}+\frac{dU_2}{dy}+\frac{dU_3}{dx}, \\ \rho(Y-\frac{d^2y}{dt^2})=\frac{dV_1}{dz}+\frac{dV_2}{dy}+\frac{dV_3}{dx}, \\ \rho(Z-\frac{d^2z}{dt^2})=\frac{dW_1}{dz}+\frac{dW_2}{dy}+\frac{dW_3}{dx}. \end{cases} \quad\Rightarrow\quad \rho(\mathbf{f}-\mathbf{u}_t)=\begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix}\begin{bmatrix} \frac{d}{dz} \\ \frac{d}{dy} \\ \frac{d}{dx} \end{bmatrix}$$

where, $\mathbf{u}=(u,v,w)$, $\mathbf{f}=(X,Y,Z)$ and the elements of velocity $\mathbf{u}=(u,v,w)$ are :

$$\frac{dx}{dt}=u,\quad \frac{dy}{dt}=v,\quad \frac{dz}{dt}=w \tag{113}$$

From (113),

$$\begin{cases} \frac{d^2x}{dt^2}=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}, \\ \frac{d^2y}{dt^2}=\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}, \\ \frac{d^2z}{dt^2}=\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}, \end{cases} \qquad \begin{cases} \frac{\psi t}{dt}=\frac{dp}{dt}+u\frac{dp}{dx}+v\frac{dp}{dy}+u\frac{dp}{dz}, \\ \frac{d\chi t}{dt}=\frac{d\rho}{dt}+u\frac{d\rho}{dx}+v\frac{d\rho}{dy}+u\frac{d\rho}{dz}. \end{cases}$$

$$\varpi\equiv p-\alpha\frac{d\psi t}{dt}-\frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \tag{114}$$

Finally, we get the fluid equations in compressible condition :

$$(7\text{-}9)_{Pf}\qquad \begin{cases} \rho(X-\frac{d^2x}{dt^2})=\frac{d\varpi}{dx}+\beta(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}), \\ \rho(Y-\frac{d^2y}{dt^2})=\frac{d\varpi}{dy}+\beta(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}), \\ \rho(Z-\frac{d^2z}{dt^2})=\frac{d\varpi}{dz}+\beta(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}), \\ \text{where}\quad \varpi\equiv p-\alpha\frac{d\psi t}{dt}-\frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}. \end{cases} \tag{115}$$

($\Downarrow$) If we put $\mathbf{u}=(u,v,w)$ and $\mathbf{f}=(X,Y,Z)$, then (115) becomes as follows :

$$\frac{\partial\mathbf{u}}{\partial t}+\frac{\beta}{\rho}\Delta\mathbf{u}+\frac{1}{\rho}\nabla\Big(p-\alpha\frac{d\psi t}{dt}-\frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}\Big)=\mathbf{f}$$

[57]($\Downarrow$) In Poisson [60], the title of the chapter 7 is "*Calcul des Pressions dans les Fluides en mouvement ; équations défférentielles de ce mouvement.*"

A.4.7. **Stokes' comment on Poisson's fluid equations.**

($\Downarrow$) Stokes comments on Poisson's $(7\text{-}9)_{Pf}$ as follows :

> On this supposition we shall get the value of $\frac{d\Psi t}{dt}$ from that of $R_1' - K$ in the equations of page 140 by putting
> $$\frac{du}{dx} = \frac{dv}{dy} = \frac{dw}{dz} = -\frac{1}{3\chi t}\frac{d\chi t}{dt}.$$
> We have therefore
> $$\alpha\frac{d\chi t}{dt} = \frac{\alpha}{3}(K - 5k)\frac{d\chi t}{\chi t dt}.$$
> Putting now for $\beta + \beta'$ its value $2\alpha k$, and for $\frac{1}{\chi t}\frac{d\chi t}{dt}$ its value given by equation (116)[58], the expression for ϖ, page 152,[59] becomes
> $$\varpi = p + \frac{\alpha}{3}(K+k)(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}).$$
> Observing that $\alpha(K+k) = \beta$, this value of ϖ reduces Poisson's equation $(7\text{-}9)_{Pf}$[=(115)] to the equation $(12)_S$[=(120)] of this paper. ([74, p.119])

Namely, by using $\alpha(K+k) = \beta$ in (112) and the following :

$$\begin{cases} \frac{d\varpi}{dx} = \frac{dp}{dx} + \frac{\alpha}{3}(K+k)\frac{d}{dx}(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \\ \frac{d\varpi}{dy} = \frac{dp}{dy} + \frac{\alpha}{3}(K+k)\frac{d}{dy}(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \\ \frac{d\varpi}{dz} = \frac{dp}{dz} + \frac{\alpha}{3}(K+k)\frac{d}{dz}(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \end{cases}$$

then $(115)(= (7\text{-}9)_{Pf}$) turns out :

$$(7\text{-}9)_{Pf} \quad \begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2u}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2w}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) \\ \text{where } \varpi = p + \frac{\alpha}{3}(K+k)(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \end{cases}$$

$$\Rightarrow \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K+k)\Big(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\Big) + \frac{1}{3}\alpha(K+k)\frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K+k)\Big(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\Big) + \frac{1}{3}\alpha(K+k)\frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K+k)\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\Big) + \frac{1}{3}\alpha(K+k)\frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \end{cases}$$

$$\Rightarrow \quad (12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\Big(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\Big(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0. \end{cases}$$

Therefore, Poisson treats the matters on conditions of both compressible and incompressible fluid.

Here, $\alpha(K+k)$ is the constant to the tensor function with the main axis (the normal stress) of Laplacian. $\frac{1}{3}\alpha(K+k)$ corresponds to the coefficient of grad.div term. In today's NS equations, the ratio of coefficients : $\frac{\text{coefficient of tensor}}{\text{coefficient of grad div}} = 3$ as well as Poisson deduced in $(7\text{-}9)_{Pf}$ and Stokes' $(12)_S$ through the tensor by Saint-Venant. By Prandtl [64, p.259] in 1934, the ratio was fixed at 3. By then, we had have to wait the time of formulation by Prandtl in fluid equation. cf. Table 7. ($\Uparrow$)

---

[58]($\Downarrow$) Poisson [60, p.141],
$$(7\text{-}2)_{Pf} \qquad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = -\frac{1}{\chi t}\frac{d\chi t}{dt}. \qquad (116)$$

[59]($\Downarrow$) cf. (114)

A.5. **Saint-Venant's tensor.**

Saint-Venant[60] explains the object of his paper [67] to simplify the description and calculation of molecular relation without setting the molecular function. His method is an epoc-making method of tensor :

> Cette Note a pour objet de faciliter l'examen du Mémoire de 1834 et de ce qui y a été ajouté en 1837, en simplifiant, comme on va le dire, l'exposition du point principal, qui est la recherche des formules des pressions dans l'intérieur des fluides en mouvement, sans faire de supposition sur la grandeur des attractions et répulsions des molécules en founction, soit de leurs distances, soit de leurs vitesses relatives. [67, p.1240]

We show Saint-Venant's tensor, which seems to hint Stokes, from the extract [67]. ξ, η, ζ : velocities on the arbitrary point m of a fluid in motion of paralleled direction of the coordinate x, y, z respectively. P_{xx}, P_{yy}, P_{zz} : normal pressure and P_{yz}, P_{zx}, P_{xy} : tangential pressure with double sub-indices showing perpendicular plane and direction of decomposition, if strictly speaking, such as the following :

> P_{xx}, P_{yy}, P_{zz} les *pressions normales* supportées au même point par l'unité superficielle de petites faces perpendiculaires aux x, aux y, aux z, c'est-à-dire les composantes, dans un sens normale à ces faces fictives, des pressions qui s'exercent à travers ;
>
> P_{yz}, P_{zx}, P_{xy} les *pressions tangentielles* sur les mêmes faces et dans les trios sens, c'est-à-dire les composantes, parallèlement aux faces, des pressions dont nous venons de parler ;
>
> - la première sous-lettre désignant toujours la face, par la coordonnée qui lui est perpendiculaire, et
> - la deuxième spécifiant le sens de la décomposition. [67, p.1240]

$$(1)_{SV} \quad \frac{P_{xx}-P_{yy}}{2\left(\frac{d\xi}{dx}-\frac{d\eta}{dy}\right)} = \frac{P_{zz}-P_{xx}}{2\left(\frac{d\zeta}{dx}-\frac{d\eta}{dz}\right)} = \frac{P_{yy}-P_{zz}}{2\left(\frac{d\eta}{dy}-\frac{d\zeta}{dz}\right)} = \frac{P_{yz}}{\frac{d\eta}{dz}+\frac{d\zeta}{dy}} = \frac{P_{zx}}{\frac{d\zeta}{dx}+\frac{d\xi}{dz}} = \frac{P_{xy}}{\frac{d\xi}{dy}+\frac{d\eta}{dx}} = \varepsilon,$$

where, we put

$$\frac{1}{3}\Big(P_{xx}+P_{yy}+P_{zz}\Big) - \frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big) = \pi,$$

We put normal pressure respectively as follows :

$$(2)_{SV} \quad P_{xx} = \pi + 2\varepsilon\frac{d\xi}{dx}, \quad P_{yy} = \pi + 2\varepsilon\frac{d\eta}{dy}, \quad P_{zz} = \pi + 2\varepsilon\frac{d\zeta}{dz},$$

From $(1)_{SV}$, we get tangential pressure respectively as follows :

$$(3)_{SV} \quad P_{yz} = \varepsilon\Big(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\Big), \quad P_{zx} = \varepsilon\Big(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\Big), \quad P_{xy} = \varepsilon\Big(\frac{d\xi}{dy}+\frac{d\eta}{dx}\Big).$$

From $(2)_{SV}$, we get π as follows :

$$P_{xx}+P_{yy}+P_{zz} = 3\pi + 2\varepsilon\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big) \quad \Rightarrow \quad \pi = \frac{1}{3}\Big(P_{xx}+P_{yy}+P_{zz}\Big) - \frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big).$$

$$\boxed{\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} \pi+2\varepsilon\frac{d\xi}{dx}, & \varepsilon\Big(\frac{d\xi}{dy}+\frac{d\eta}{dx}\Big) & \varepsilon\Big(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\Big) \\ \varepsilon\Big(\frac{d\xi}{dy}+\frac{d\eta}{dx}\Big) & \pi+2\varepsilon\frac{d\eta}{dy} & \varepsilon\Big(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\Big) \\ \varepsilon\Big(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\Big) & \varepsilon\Big(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\Big) & \pi+2\varepsilon\frac{d\zeta}{dz} \end{bmatrix}, \quad \text{where} \quad \pi = \tfrac{1}{3}\Big(P_{xx}+P_{yy}+P_{zz}\Big) - \tfrac{2\varepsilon}{3}\Big(\tfrac{d\xi}{dx}+\tfrac{d\eta}{dy}+\tfrac{d\zeta}{dz}\Big). \qquad (117)}$$

Saint-Venant proposes the univarsal method that we can deduce the concurrence with Navier, Cauchy and Poisson as follows :

[60](⇓) Adhémar Jean Claude Barré de Saint-Venant (1797-1886).

Si l'on remplace π par $\varpi - \varepsilon(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz})$, et si l'on substitue les équations $(2)_{SV}$ et $(3)_{SV}$ dans les relations connues entre les pressions et les forces accélératrices, on obtient, en supposant ε le même en tous les points du fluide, les équations différentielles données le 18 mars 1822 par M.Navier (*Mémoires de l'Institut*, t.VI), en 1828 par M.Cauchy (*Exercices de Mathématiques*, p.187)[61], et le 12 octobre 1829 par M.Poisson (même *Mémoire*, p.152)[62].

La quantité variable ϖ ou π n'est autre chose, dans les liquides, que la *pression normale moyenne* en chaque point. [67, p.1243]

This paper [67] seems to give Stokes a hint of tensor (124), partly because Stokes reports on the Saint-Venant's paper [67] in the report [73] by Stokes before Stokes issues his paper [74]. And partly because we can see by comparing[63] t_{ij} with Stokes' t_{ij} (125) :

$$\begin{aligned} t_{ij} &= (\pi + 2\varepsilon v_{i,j} - \gamma)\delta_{ij} + \gamma \\ &= \Big(\frac{1}{3}(P_{xx}+P_{yy}+P_{zz}) - \frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big) + 2\varepsilon v_{i,j} - \gamma\Big)\delta_{ij} + \gamma \\ &= (\frac{1}{3}(P_{xx}+P_{yy}+P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j}+v_{j,i}) \quad \Leftarrow \quad 2\varepsilon v_{i,j}\delta_{ij} = \varepsilon(v_{i,j}+v_{j,i})\delta_{ij} = \gamma\delta_{ij} \end{aligned} \tag{118}$$

$$\text{where} \quad \gamma = \varepsilon(v_{i,j}+v_{j,i}), \quad v_{k,k} = \sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i} = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\cdots \text{Einstein's convention}$$

Here, using (118), if we put[64] $P_{xx} = P_{yy} = P_{zz} = -p$ by Stokes principle in § A.6, then (118) is equivalent to Stokes' t_{ij} as follows :

$$\begin{aligned} & t_{ij} \\ &= \Big\{(\frac{1}{3}(P_{xx}+P_{yy}+P_{zz}) - \frac{2\varepsilon}{3}v_{k,k}\Big\}\delta_{ij} + \varepsilon(v_{i,j}+v_{j,i}) = (-p - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j}+v_{j,i}) \\ &\Rightarrow \quad \text{Stokes'}: \ -t_{ij} = (p + \frac{2}{3}\mu v_{k,k})\delta_{ij} - \mu(v_{i,j}+v_{j,i}) \Rightarrow (125). \end{aligned}$$

Moreover Saint-Venant assumes that : if we put $\pi = \varpi - \varepsilon\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big) = \varpi - \varepsilon v_{k,k}$ then

$$t_{ij} = (\varpi - \varepsilon v_{k,k} + 2\varepsilon v_{i,j} - \gamma)\delta_{ij} + \gamma = (\varpi - \varepsilon v_{k,k})\delta_{ij} + \varepsilon(v_{i,j}+v_{j,i}) \tag{119}$$

($\Downarrow$) By the way, we check the coincidence of Saint-Venant's tensor with Stokes'(124) concerning only $(1,1)$ element or P_1.

$$\begin{aligned} & P_1 \text{ of } (117) \quad \Rightarrow \quad \pi + 2\varepsilon\frac{d\xi}{dx} \\ &= -p + (2-\frac{2}{3})\varepsilon\frac{d\xi}{dx} - \frac{2\varepsilon}{3}\Big(\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big) \\ &= -p + \varepsilon\Big\{\frac{4}{3}\frac{d\xi}{dx} - \frac{2}{3}\Big(\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)\Big\} \\ &= -p + 2\varepsilon\Big\{\frac{2}{3}\frac{d\xi}{dx} - \frac{1}{3}\Big(\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)\Big\} \\ &= -p + 2\varepsilon\Big\{\frac{d\xi}{dx} - \frac{1}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)\Big\} \\ &= -p + 2\varepsilon\Big(\frac{d\xi}{dx} - \delta\Big) \quad \Rightarrow \quad p - 2\mu\Big(\frac{du}{dx} - \delta\Big) \quad \Rightarrow \quad P_1 \text{ of Stokes (124)}. \end{aligned}$$

where,

$$\pi = \frac{1}{3}\Big(P_{xx}+P_{yy}+P_{zz}\Big) - \frac{2\varepsilon}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big) \equiv -p - 2\varepsilon\delta, \quad \delta = \frac{1}{3}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big).$$

[61] Cauchy [6, p.226]

[62] Poisson [60, p.152] $(7\text{-}9)_{Pf}$=(115).

[63] ($\Downarrow$) In our paper, we cite the description of t_{ij} of the tensor : of Poisson and Cauchy, from C.Truesdell [75], of Navier, from G.Darrigol [11]. in other case computed by ourselves or referred from Schlichting [69].

[64] ($\Downarrow$) cf.I.Imai [22, p.185].

Other elements are coincident with (124) in the same way.

From here, we get the t_{ij} of Poisson, Navier and Cauchy as follows :

- $t_{ij} = \varpi\delta_{ij} - \varepsilon v_{k,k}\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}), \varpi = -p,\ -\varepsilon = \lambda,\ \varepsilon = \mu$
  $\Rightarrow$ Poisson's $t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$,

- $t_{ij} = \varpi\delta_{ij} - \varepsilon v_{k,k}\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}), \varpi = 0,\ -\varepsilon = \lambda,\ \varepsilon = \mu$
  $\Rightarrow$ Cauchy's $t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$,

- $t_{ij} = \varpi\delta_{ij} - \varepsilon(v_{k,k}\delta_{ij} + v_{i,j} + v_{j,i}),\ \varpi = 0,$
  $\Rightarrow$ Navier's $t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$,

Moreover, we can add Stokes'

- $t_{ij} = \varpi\delta_{ij} - \varepsilon v_{k,k}\delta_{ij} + \varepsilon(v_{i,j} + v_{j,i}), \varpi = -p,\ -\varepsilon = -\frac{2}{3}\mu,\ \varepsilon = \mu$
  $\Rightarrow$ Stokes'$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$.

A.6. **Stokes' principle, equations and tensor.**
Stokes says in [74, p.80] :[65]

> If the molecules of E were in a state of relative equilibrium, the pressure would be equal in all directions about P, as in the case of fluids at rest. Hence I shall assume the following principle :
> - That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighborhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P ; and
> - that the relative motion due to any rotary motion may be eliminated without affecting the differences of the pressures above mentioned.

Stokes comments on Navier's equation :

> The same equations have also been obtained by Navier in the case of an incompressible fluid (Mém. de l'Académie, t. VI. p.389)[66], but his principles differ from mine still more than do Poisson's. [74, p.77, footnote]

$$(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\mu\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dx}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0,\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\mu\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dy}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0,\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\mu\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dz}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0. \end{cases} \tag{120}$$

where Stokes says the coincidence with Poisson :

$$\varpi = p+\frac{\alpha}{3}(K+k)\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big) \Rightarrow \nabla\varpi = \nabla p+\frac{\beta}{3}\nabla\cdot(\nabla\cdot\mathbf{u}). \tag{121}$$

Observing that $\alpha(K+k)\equiv\beta$, this value of ϖ reduces Poisson's equation $(9)_{Pf}$ (=(115) in our renumbering) to the equation $(12)_S$ of this paper.

($\Downarrow$) By the way, $(12)_S$ turns to :

$$\begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\mu\Big(\frac{4}{3}\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+\frac{1}{3}\frac{d^2v}{dxdy}+\frac{1}{3}\frac{d^2w}{dxdz}\Big),\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\mu\Big(\frac{d^2v}{dx^2}+\frac{4}{3}\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}+\frac{1}{3}\frac{d^2u}{dxdy}+\frac{1}{3}\frac{d^2w}{dydz}\Big),\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\mu\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{4}{3}\frac{d^2w}{dz^2}+\frac{1}{3}\frac{d^2u}{dxdz}+\frac{1}{3}\frac{d^2w}{dydz}\Big). \end{cases}$$

or

$$\begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\frac{\mu}{3}\Big(4\frac{d^2u}{dx^2}+3\frac{d^2u}{dy^2}+3\frac{d^2u}{dz^2}+\frac{d^2v}{dxdy}+\frac{d^2w}{dxdz}\Big),\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\frac{\mu}{3}\Big(3\frac{d^2v}{dx^2}+4\frac{d^2v}{dy^2}+3\frac{d^2v}{dz^2}+\frac{d^2u}{dxdy}+\frac{d^2w}{dydz}\Big),\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\frac{\mu}{3}\Big(3\frac{d^2w}{dx^2}+3\frac{d^2w}{dy^2}+4\frac{d^2w}{dz^2}+\frac{d^2u}{dxdz}+\frac{d^2w}{dydz}\Big), \end{cases} \tag{122}$$

moreover, when we use vectorial notation after replacing with $\mathbf{f}\equiv(X,Y,Z)$, we get :

$$\rho(\frac{D\mathbf{u}}{Dt}-\mathbf{f})+\nabla p-\mu\Big(\Delta\mathbf{u}+\frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\Big)=0 \quad \text{or} \quad \frac{D\mathbf{u}}{Dt}-\frac{\mu}{\rho}\Delta\mathbf{u}-\frac{1}{3\rho}\nabla(\nabla\cdot\mathbf{u})+\frac{1}{\rho}\nabla p=\mathbf{f}$$

Stokes proposes the Stokes' approximate equations in [74, p.93] :

$$(13)_S \quad \begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\mu(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2})=0,\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\mu(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2})=0,\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\mu(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2})=0, \end{cases} \qquad \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0. \tag{123}$$

Stokes proposes that :

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[65]($\Downarrow$) Stokes [74, pp.78-105] Section 1. Explanation of the Theory of Fluid Motion proposed. Formulation of the Differential Equations. Application of these Equations to a few simple cases.

[66]($\Downarrow$) Navier [47].

> These equations are applicable to the determination of the motion of water in pipes and canals, to the calculation of the effect of friction on the motions of tides and waves, and such questions.

Here we shall trace his deduction on Stokes' tensor :

$$\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} p-2\mu\Big(\frac{du}{dx}-\delta\Big) & -\mu\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) & -\mu\Big(\frac{dw}{dx}+\frac{du}{dz}\Big) \\ -\mu\Big(\frac{du}{dy}+\frac{dv}{dx}\Big) & p-2\mu\Big(\frac{dv}{dy}-\delta\Big) & -\mu\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) \\ -\mu\Big(\frac{dw}{dx}+\frac{du}{dz}\Big) & -\mu\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big) & p-2\mu\Big(\frac{dw}{dz}-\delta\Big) \end{bmatrix}, \quad \text{where} \quad 3\delta = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} \tag{124}$$

Here, he writes, "it may also be very easily provided directly that the value of 3δ, the rate of cubical dilatation".

($\Downarrow$) By the way, Stokes' tensor is described compactly as follows :

$$\begin{aligned} -t_{ij} &= \{p-2\mu(v_{i,j}-\delta)+\gamma\}\delta_{ij}-\gamma \\ &= \{p-2\mu v_{i,j}\}\delta_{ij}+\gamma(-\delta_{ij}+\delta_{ij}-1) \quad \Leftarrow \quad 2\mu\delta\delta_{ij}=\mu(v_{i,j}+v_{j,i})\delta_{ij}=\gamma\delta_{ij} \\ &= (p+2\mu\gamma)\delta_{ij}-\gamma \\ &= (p+\frac{2}{3}\mu v_{k,k})\delta_{ij}-\mu(v_{i,j}+v_{j,i}), \end{aligned} \tag{125}$$

Here, the sign of $-t_{ij}$ depends on the location of the tensor in the equation, and we consider the coincident with (120). [67] We see Stokes' tensor comes from Saint-Venant's tensor. From here, the article by J.J.O'Connor and E.F.Robertson points out this resemblance as well.[68]

By d'Alembert's principle [69],

$$\begin{cases} \rho\Big(\frac{Du}{Dt}-X\Big)+\frac{dP_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}=\rho\Big(\frac{Du}{Dt}-X\Big)+P=0, \\ \rho\Big(\frac{Dv}{Dt}-Y\Big)+\frac{dP_2}{dy}+\frac{dT_3}{dx}+\frac{dT_1}{dz}=\rho\Big(\frac{Dv}{Dt}-Y\Big)+Q=0, \\ \rho\Big(\frac{Dw}{Dt}-Z\Big)+\frac{dP_3}{dz}+\frac{dT_2}{dx}+\frac{dT_1}{dy}=\rho\Big(\frac{Dw}{Dt}-Z\Big)+R=0 \end{cases} \tag{126}$$

By (124) and (126), we get (123). We seek the tensor for t_{ij} such that :

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{bmatrix}$$

Using (124) and (126),

$$\begin{cases} \frac{d}{dx}\Big\{p-2\mu\Big(\frac{du}{dx}\Big)+2\mu\frac{1}{3}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}+\frac{d}{dy}\Big\{-\mu\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)\Big\}+\frac{d}{dz}\Big\{-\mu\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)\Big\}, \\ \frac{d}{dy}\Big\{p-2\mu\Big(\frac{dv}{dy}\Big)+2\mu\frac{1}{3}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}+\frac{d}{dx}\Big\{-\mu\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)\Big\}+\frac{d}{dz}\Big\{-\mu\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)\Big\}, \\ \frac{d}{dz}\Big\{p-2\mu\Big(\frac{dw}{dz}\Big)+2\mu\frac{1}{3}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}+\frac{d}{dx}\Big\{-\mu\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)\Big\}+\frac{d}{dy}\Big\{-\mu\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)\Big\} \end{cases}$$

$$= \begin{cases} \frac{d}{dx}p-\mu\Big\{\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)+\frac{1}{3}\frac{d}{dx}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}, \\ \frac{d}{dy}p-\mu\Big\{\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)+\frac{1}{3}\frac{d}{dy}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}, \\ \frac{d}{dz}p-\mu\Big\{\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)+\frac{1}{3}\frac{d}{dz}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)\Big\}, \end{cases} = \begin{cases} \frac{d}{dx}p-\mu\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big), \\ \frac{d}{dy}p-\mu\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big), \\ \frac{d}{dz}p-\mu\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big) \end{cases}$$

---

[67]($\Downarrow$) Schlichting writes Stokes' tensor with the minus sign as follows :

$$\sigma_{ij} = -p\delta_{ij}+\mu(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})-\frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}$$

[69, p.58, in footnote]

[68]($\Downarrow$) cf. J.J.O'Connor, E.F.Robertson,→ http://www-groups.dcs.st-and.ac.uk/ history/Printonly/Saint-Venant.html.[52]

[69]($\Downarrow$) In 1758, from the Newton's kinetic equation (the second law of motion) : $\mathbf{F}=m\mathbf{r}$, d'Alembert proposed $\mathbf{F}-m\mathbf{r}=0$, where, $\mathbf{F}$: the force, m : the gravity, $\mathbf{r}$: the acceleration. According to his assertion, the problem of kinetic dynamics turns into that of the static dynamics.

Therefore we get (123).

By the modern vectorial expression, if we take $\mathbf{f} = (X, Y, Z)$, $\nu \equiv \frac{\mu}{\rho}$, and if, as Stokes says, we put $Du/Dt = \partial\mathbf{u}/\partial t$, then (123) turns out as follows :

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{f}, \quad \mathrm{div}\ \mathbf{u} = 0.$$

($\Downarrow$) By the way, here we shall get the tensor of Stokes equations from Navier's (23). We put as the same as Stokes equations :

$$\begin{cases} \rho\Big(\frac{du}{dt} - X\Big) + \frac{dp}{dx} - \varepsilon\Big(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\Big) + \frac{du}{dx}\cdot u + \frac{du}{dy}\cdot v + \frac{du}{dz}\cdot w = 0\ ; \\ \rho\Big(\frac{dv}{dt} - Y\Big) + \frac{dp}{dy} - \varepsilon\Big(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\Big) + \frac{dv}{dx}\cdot u + \frac{dv}{dy}\cdot v + \frac{dv}{dz}\cdot w = 0\ ; \\ \rho\Big(\frac{dw}{dt} - Z\Big) + \frac{dp}{dz} - \varepsilon\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\Big) + \frac{dw}{dx}\cdot u + \frac{dw}{dy}\cdot v + \frac{dw}{dz}\cdot w = 0\ ; \end{cases}$$

Using d'Alembert principle (126), we transform the terms of the coefficient of 3 with $3 = 2 + 1$ and the last two terms of the coefficient of 2 with $2 = 1 + 1$, respectively. We show here the viscosity term as follows :

$$\begin{cases} -\varepsilon\Big((2\frac{d^2u}{dx^2} + \frac{d^2u}{dx^2}) + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\Big); \\ -\varepsilon\Big(\frac{d^2v}{dx^2} + (2\frac{d^2v}{dy^2} + \frac{d^2v}{dy^2}) + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\Big); \\ -\varepsilon\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + (2\frac{d^2w}{dz^2} + \frac{d^2w}{dz^2}) + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\Big); \end{cases}$$

$$= \begin{cases} -\varepsilon\Big\{2\frac{d^2u}{dx^2} + \frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big\} + \frac{d}{dy}\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) + \frac{d}{dz}\Big(\frac{du}{dz} + \frac{dw}{dx}\Big)\Big\}; \\ -\varepsilon\Big\{\frac{d}{dx}\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) + \Big\{2\frac{d^2v}{dy^2} + \frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big\} + \frac{d}{dz}\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big)\Big\}; \\ -\varepsilon\Big\{\frac{d}{dx}\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) + \frac{d}{dy}\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) + \Big\{2\frac{d^2w}{dz^2} + \frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big\}\Big\}; \end{cases}$$

We get the tensor t_{ij} :

$$\begin{bmatrix} p - \varepsilon\Big(2\frac{du}{dx} + \delta\Big) & -\varepsilon\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & -\varepsilon\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) \\ -\varepsilon\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & p - \varepsilon\Big(2\frac{dv}{dy} + \delta\Big) & -\varepsilon\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) \\ -\varepsilon\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) & -\varepsilon\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) & p - \varepsilon\Big(2\frac{dw}{dz} + \delta\Big) \end{bmatrix}, \quad \text{where} \quad \delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}, \tag{127}$$

or

$$\begin{bmatrix} p - 2\varepsilon\Big(\frac{du}{dx} + \delta\Big) & -\varepsilon\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & -\varepsilon\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) \\ -\varepsilon\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & p - 2\varepsilon\Big(\frac{dv}{dy} + \delta\Big) & -\varepsilon\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) \\ -\varepsilon\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) & -\varepsilon\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) & p - 2\varepsilon\Big(\frac{dw}{dz} + \delta\Big) \end{bmatrix}, \quad \text{where} \quad 2\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}. \tag{128}$$

Therefore we see (124), (127) and (128) are the invariant-tensors equivalent each other except for the sign of δ.

A.7. **The authorized expressions of two-constant and the NS equations by Prandtl.**

By Prandtl [64, p.259] in 1934, the ratio was fixed at 3. We had have to wait by the time, when including this ratio of two coefficients, what is called the NS equations were expressed by Prandtl in fluid equations :

$$(15\text{–}5)_{Pr} \quad \frac{D\mathbf{w}}{dt} = \mathbf{g} - \frac{1}{\rho}\text{grad } p + \frac{1}{3}\nu \text{ grad div } \Delta\mathbf{w} + \nu\Delta\mathbf{w} \tag{129}$$

where, $\frac{D\mathbf{w}}{dt} \equiv \frac{\partial \mathbf{w}}{\partial t} + \mathbf{w}\cdot\nabla\mathbf{w}$, $\nu = \frac{\mu}{\rho}$, $\mathbf{w} = (u, v, w)$, $\mathbf{g} = (X, Y, Z)$. Namely :

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{3}\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big) + \nu\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\Big), \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\nu}{3}\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big) + \nu\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\Big), \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{3}\frac{\partial}{\partial z}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big) + \nu\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\Big) \end{cases} \tag{130}$$

For incompressible, it is simplified as follows :

$$(15\text{–}6)_{Pr} \qquad \frac{D\mathbf{w}}{dt} = \mathbf{g} - \frac{1}{\rho}\text{grad } p + \nu\Delta\mathbf{w}, \text{ div } \mathbf{w} = 0$$

Prandtl shows NS equations deducing from the Newton's fundamental law of mechanics, Mass×acceleration=force.

$$(15\text{–}1)_{Pr} \qquad \rho\frac{D\mathbf{w}}{dt} = \mathbf{F} + \mathbf{G} \tag{131}$$

Prandtl says :

> where the total force has been decomposed in body forces **F** and surface forces **G**. Leaving out of the discussion systems in which centrifugal forces, Coriolis forces, etc., occur, the only body force is the force of gravity per unit volume : $\mathbf{F} = \rho\mathbf{g}$. [64, p.251]
>
> Now we have to come to the point where the total surface force **G** can be expressed as a function of the rate of change of deformation. [64, p.258]

$$G_x = \Big(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}\Big), \quad G_y = \Big(\frac{\partial\tau_y}{\partial y} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{yz}}{\partial z}\Big), \quad G_z = \Big(\frac{\partial\tau_z}{\partial z} + \frac{\partial\tau_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}\Big)$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \underbrace{\mu\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}}_{\mu \text{ div } \mathbf{w}(\Rightarrow\mu \text{ grad div } \mathbf{w} \text{ with } \nabla)} \underbrace{+\mu\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{bmatrix}}_{\mu\nabla\mathbf{w}(\Rightarrow\mu\Delta\mathbf{w} \text{ with } \nabla)} - \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} - \frac{2}{3}\mu\begin{bmatrix} \text{div } \mathbf{w} & 0 \\ 0 & \text{div } \mathbf{w} \\ 0 & 0 & \text{div} \end{bmatrix}$$

$$\Pi = \mu(\nabla\mathbf{w} + \mathbf{w}\nabla) - p - \frac{2}{3}\mu \text{ div } \mathbf{w} \tag{132}$$

$$\Rightarrow^* \quad \begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} = \begin{bmatrix} \frac{\partial\sigma_x}{\partial x} & \frac{\partial\tau_{xy}}{\partial y} & \frac{\partial\tau_{xz}}{\partial z} \\ \frac{\partial\tau_{yx}}{\partial y} & \frac{\partial\sigma_y}{\partial y} & \frac{\partial\tau_{yz}}{\partial z} \\ \frac{\partial\tau_{zx}}{\partial z} & \frac{\partial\tau_{zy}}{\partial y} & \frac{\partial\sigma_z}{\partial z} \end{bmatrix} = \begin{bmatrix} 2\mu\frac{\partial u}{\partial x} - p - \frac{2}{3}\mu \text{ div } \mathbf{w} & \mu\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) & \mu\Big(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\Big) \\ \mu\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) & 2\mu\frac{\partial v}{\partial y} - p - \frac{2}{3}\mu \text{ div } \mathbf{w} & \mu\Big(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\Big) \\ \mu\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\Big) & \mu\Big(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\Big) & 2\mu\frac{\partial w}{\partial z} - p - \frac{2}{3}\mu \text{ div } \mathbf{w} \end{bmatrix}$$

Then

$$\begin{cases} G_x = -\frac{\partial p}{\partial x} + \mu\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\Big) + \frac{\mu}{3}\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big), \\ G_y = -\frac{\partial p}{\partial y} + \mu\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\Big) + \frac{\mu}{3}\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big), \\ G_z = -\frac{\partial p}{\partial z} + \mu\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\Big) + \frac{\mu}{3}\frac{\partial}{\partial z}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big) \end{cases} \Rightarrow \mathbf{G} = -\text{grad } p + \frac{1}{3}\mu \text{ grad div } \Delta\mathbf{w} + \mu\Delta\mathbf{w} \tag{133}$$

Since, from (132)

$$\begin{aligned} \mathbf{G} = {} & \nabla\cdot\Pi \\ = {} & \mu\nabla(\nabla\mathbf{w} + \mathbf{w}\nabla) - \text{grad } p - \frac{2}{3}\mu \text{ grad div } \mathbf{w} \\ = {} & \mu\nabla\cdot\nabla\mathbf{w} + \mu \text{ grad div } \mathbf{w} - \text{grad } p - \frac{2}{3}\mu \text{ grad div } \mathbf{w} \\ = {} & -\text{grad } p + \frac{1}{3}\mu \text{ grad div } \mathbf{w} + \mu\Delta\mathbf{w} \end{aligned}$$

Substituting (133) into (131), we find (129) or (130).

APPENDIX B. The "two-constant" theory in capillarity

Gauss didn't mention the following fact, and Bowditch [70] also didn't comment on Gauss' work in Laplace's total works [37] except for only one comment of the name "Gauss" [37, p.686]. [71]

N.Bowditch comments as follows :

> This theory of capillary attraction was first published by La Place in 1806 ; and in 1807 he gave a supplement. In neither of these works is the repulsive force of the heat of fluid taken into consideration, because he supposed it to be unnecessary. But in 1819 he observed, that this action could be taken into account, by supposing the force $\varphi(f)$ to represent the difference between the attractive force of the particles of the fluid $A(f)$, and the repulsive force of the heat $R(f)$ so that the combined action would be expressed by, $\varphi(f) = A(f) - R(f)$; $\cdots$ [37, p.685].

We would like to pay attention to Bowditch's remark about the works of Gauss and Poisson as follows :

> In 1830, Gauss published a work on capillary attraction entitled "*Principia generalia theoriœ figurœ fluidorum in statu equilibrii, etc.,*" ("*General principle of theory of the figure of fluid in state equilibrium*"), where, by means of the principle of virtual velocities, he obtains the figure of the capillary surface, and other theorems as they are given by La Place in this volume, and he also gives a more complete demonstration of the constancy of the angle of contact of the fluid with the sides of the tube. Finally, M.Poisson, in 1831, published his "*Nouvelle théorie de l'action capillaire, etc.,*" ("*New theory of the capillary action*"), where he expressly introduces into the formulas the consideration of the change *of density of the fluid at its surface and near the sides of the tube* in consequence of the corpuscular attraction. [37, p.686]

In his historical descriptions about the study of capillary action, we would like to recognize that there is no counterattack to Gauss, but the correct valuation. Gauss [18] stated his conclusions about the papers by Laplace as follows :

> At hancce propositionnem cardinalem totius theoriae per calculum demonstrare ne suscepit quidem ill. Laplace ; quae enim in dissertatione priori p.5 huc spectantia afferuntur, argumentationem vagam tantummode exhibent et quad demonstrandum erat iam supponunt : calculi autem p.44 sq. suscepti effectu carent.
>
> (Engl.transl.) To this cardinal proposition of the total theory with calculation for demonstration, we can not accept the papers by Mr. Laplace ; in p.5, since not only he developed clearly incorrect argument but also showed even the false proofs : we consider that his calculations in the pages and the following after p.44 are the vain effects.[72] [18, pp.33-34]

---

[70](⇓) The present work is a reprint, in four volumes, of Nathaniel Bowditch's English translation of volumes I, II, III and IV of the French-language treatise *Traité de Mécanique Céleste* by P.S.Laplace. The translation was originally published in Boston in 1829, 1832, 1834, and 1839, under the French title, "*Mécanique Céleste*", which has now been changed to its English-language form, "*Celestial Mechanics.*"

[71](⇓) Bowditch's comment number [9173g].

[72](⇓) There are 35 pages of calculation between p.44 and p.78 in his *Supplément.*

APPENDIX C. Laplace and Gauss

C.1. Laplace's theory of the capillary action.

We show below the four available originals of the capillary action by Laplace, which we mention, in which the top halves are the original by Laplace, to which Gauss and Bowditch et al., referred, and last two are the translations by Bowditch, in the commentary of which he cited Gauss [17].

1 . [34] P.S.Laplace, *Traité de méchanique céleste. Supplément au dixième livre du traité de Mécanique céleste. Sur l'action capillaire*, Ruprat, Paris, 1798-1805, pp.1-66. (We use this original printed by Culture et Civilisation, 1967.)
2 . [35] P.S.Laplace, *Supplément à la théorie de l'action capillaire*, Tome Quatrième, Paris, 1805, pp.1-78. (op. cit.)
3 . [36] P.S.Laplace, *Supplément à la théorie de l'action capillaire*, translated by N. Bowditch, Vol. I §4 90-95, 1966. (This is the complete works of Laplace.)
4 . [37] P.S.Laplace, *On capillary attraction, Supplement to the tenth book of the Méchanique Céleste*, translated by N. Bowditch, same as above Vol. IV 685-1018, 1806,1807. 1966. (op. cit.)

C.1.1. Laplace's conclusions of theory of the capillary action.

Laplace stated his "*complete theory*" of attraction which have an effct on the capillary action in the introduction [34], as follows :

> J'ai cherché, il y a longtemps, à determiner les lois d'attraction qui représentent ces phénomènes : de nouvelles recherches m'ont enfin conduit à faire voir qu'ils sont tous représentés par les mêmes lois qui satisfont aux phénomès de la réfraction, c'est-à-dire par les lois dans lequelles l'attraction n'est sensible qu'à des distances insensibles; et il en résulte une théorie complète de l'action capillaire.[34, p.2]
>
> ...
>
> De ces résultats relatifs aux terminés par des segmens sensibles des surface sphérique, je conclus ce théorème général : ≪ Dans toutes les loi qui rendent l'attraction insensible á des distances sensibles, l'action d'un corps terminé par une surface courbe, sur un canal intérieur infiniment étroit, perpendicularire à cette surface dans un point quelconque, est égale à la demi-somme des actions sur le même canal, de deux sphères qui auraient pour rayons le plus grand et le plus petit des rayons osculateurs de la surface, à ce point ≫. [34, p.4]

From the translation by Bowditch [37], for brevity, we show the corresponding part with above as follows :

> A long while ago, I endevored in vain to determine the laws of attraction which would represent these phenomena ; but same late researches have rendered it evident that the whole may be represented by the same laws, which satisfy the phenomena of refraction ; that is, by laws in which the attraction is sensible only at insensible distances ; and from this principle we can deduce a complete theory of capillary attraction. [37, p.688]
>
> ...
>
> From these results, relative to bodies terminated by sensible segments of a spherical surface, I have deduced this general theorem. *"In all the laws which render the attraction insensible at sensible distance, the action of body terminated by a curve surface, upon an infinitely narrow interior canal, which is perpendicular to that surface, at any point whatever, is equal to the half sum of the actions upon the same canal, of two spheres which have the same radii as the greatest and the least radii of curvature of the surface at that point."* By means of this theorem, and of the laws of the equilibrium of fluids, we can determine the figure which a fluid must have, when it is included whithin a vessel of a given figure, and acted upon by gravity. [37, p.689]

The target of *Supplément*, Laplace says, is "so as to render more evident the identity of the attractive forces, upon which this action depends, with those which produce the affinities of bodies" (*Supplément* [35]) :

> L'objet de ce Supplément est de perfectionner la théorie que j'ai donnée, des phénomènes capillaires ; d'en étendre les applications ; de la confirme par de nouvelles comparisons de ses résultats avec l'expérience ; ce en présentant sous un nouveau point-de-vue les

effets de l'action capillaire, de mettre de plus en plus en évidence l'identité des forces attractives dont cette action dépend, avec celles qui produisent les affinités. [35, p.1]

(Engl. transl. by Bowditch) : ⇒ The object of this supplement are, to complete the theory which I have given of the capillary phenomina; to extend its application; to confirm its results by a comparison with experiment ; and to present, in a new point of view, the effects of the capillary action, so as to render more evident the identity of the attractive forces, upon which this action depends, with those which produce the affinities of bodies. [37, p.806]

C.1.2. **Laplace's theory of the capillary action.**
Laplace's theories of the capillary action are described in the 14 articles. We cite only the contents of *no* 1 ([35, pp.10-14]) of theory of [35] pointed out by Gauss:
¶ *no* 1 of the theory of capillary action :

Considérons vase $ABCD$ (*fig. 1*), [73] plein d'eau jusqu'en AB, et concevons un tube capillaire de verre, $NMEF$, ouvert par ses deux extrémités, et plongeant dans son extrémité inférieure; l'eau s'élevera dans le tube jusqu'en O, et sa surface prendra la figure concave MON, O étant le point le plus bas de cette surface. Imaginons par ce point et par l'axe du tube, un filet d'eau renfermé dans un canal infiniment étroit $OZRV$; il est clair, d'après le principe que nous venons d'exposer sur le peu d'étendue des attractions capillaires, que l'action de l'eau inférieure à l'horizontale IOK, sera la même sur la colonne OZ, que l'action du vase la colonne VR. Mais le ménisque $MIOKN$ agira sur la colonne OZ de bas en haut, et tendra parconséquent à soulever le fluide. Ainsi, dans l'état d'équilibre, l'eau du canal $OZRV$ devra être plus élevée dans le tube que dans le vase, pour compenser par son poids, cette action du ménisque.

Soit r la distance du point attiré, au centre d'une couche sphérique dont u est le rayon et du l'épaisseur. Soir encore θ l'angle que le rayon u fait avec la droit r, ϖ l'angle que la plan qui passe par les deux droites r et u fait avec un plan fixe passant par la droite r : l'élément de la couche sphérique sera $u^2du.d\varpi.d\theta.\sin.\theta$. Si l'on nomme ensuite f la distance de ce élément, au point attiré que nous supposerons extérieur à la couche; nous aurons

$$f^2 = r^2 - 2ru.\cos.\theta + u^2.$$

Représentons par $\varphi(f)$ la loi de l'attraction à la distance f, attraction qui, dans le cas présent, est insensible lorsque f a une valeur sensible; l'action de l'élément de la couche sur le point attiré, décomposée parallèlement à r, et dirigée vers le centre de la couche, sera

$$u^2du.d\varpi.d\theta.\sin.\theta.\frac{r-u.\cos.\theta}{f}.\varphi(f)$$

On a

$$\frac{r-u.\cos.\theta}{f} = \frac{df}{dr}$$

ce qui donne à la quantité précédente, cette forme

$$u^2du.d\varpi.d\theta.\sin.\theta.\frac{df}{dr}.\varphi(f)$$

Désignons par $c-\Pi(f)$, l'intégrale $fdf.\varphi(f)$, prise depuis $f=0$; c étant la valeur de cette intégrale, lorsque f est infini; $\Pi(f)$ sera une quantité positive décroî̂t avec une extrême rapidité; de manière à devenir insensible, lorsque f a une valeur sensible. [35, pp.10-11]

¶ *no* 4 ([35, p.18-23]) of the theory of capillary action :

Soit donc O(*fig. 3*) [74] le point le plus bas de la surface AOB de l'eau renfermée dans un tube. Nommonz z la coordonnée verticale OM; x et y, les deux coordonnées horizontales d'un point quelconque N de la surface. Soient R et R' le plus grand et le

[73](⇓) The original *fig. 1* by Laplace [35] is shown in the last page in the appendix § F.3 of our paper.

[74](⇓) The original *fig. 3* by Laplace [35] is shown in the last page in the appendix § F.3 of our paper.

plus petit des rayons osculateurs de la surface à ce point.

R et R' seront les deux racines de l'équation [75]

$$R^2.(rt-s^2)-R.\sqrt{(1+p^2+q^2)}.\{(1+q^2).r-2pqs+(1+p^2).t\}+(1+p^2+q^2)^2=0, \tag{134}$$

équation dans laquelle

$$p=\frac{dz}{dx};\quad q=\frac{dz}{dy};\quad r=\frac{d^2z}{dx^2};\quad s=\frac{d^2z}{dxdy}=^*\frac{dp}{dy}=^*\frac{dq}{dx};\quad t=\frac{d^2z}{dy^2}. \tag{135}$$

On aura donc

$$\frac{1}{R}+\frac{1}{R'}=\frac{(1+q^2).\frac{dp}{dx}-pq.\left(\frac{dp}{dy}+\frac{dq}{dx}\right)+(1+p^2).\frac{dq}{dy}}{(1+p^2+q^2)^{\frac{3}{2}}}=\frac{(1+q^2).r-2pqs+(1+p^2).t}{(1+p^2+q^2)^{\frac{3}{2}}} \tag{136}$$

Cela posé, si l'on conçoit un canal quelconque infiniment étroit NSO ; on doit avoir par la loi de l'équilibre du fluide renfermé dans ce canal,

$$K-\frac{H}{2}.\left(\frac{1}{R}+\frac{1}{R'}\right)+gz=K-\frac{H}{2}.\left(\frac{1}{b}+\frac{1}{b'}\right);\quad\Rightarrow\quad\left(\frac{1}{R}+\frac{1}{R'}\right)-\frac{2gz}{H}=\frac{1}{b}+\frac{1}{b'}; \tag{137}$$

b et b' étant le plus grand et le plus petit des rayons osculateurs de la surface au point O, et g étant la pesanteur. En effet, l'action du fluide sur le canal, au point N, est par ce qui précède, $K-\frac{H}{2}.\left(\frac{1}{R}+\frac{1}{R'}\right)$; et de plus, la hauteur du point N audessus du point O est z. L'équation précèdente donne, en y substituant pour $\frac{1}{R}+\frac{1}{R'}$, sa valeur, [76]

$$(a)\qquad\frac{(1+q^2).r-2pqs+(1+p^2).t}{(1+p^2+q^2)^{\frac{3}{2}}}-\frac{2gz}{H}=\frac{1}{b}+\frac{1}{b'}; \tag{138}$$

[35, p.19]

C.1.3. Laplace's *supplément* for theory of the capillary action.

Laplace stated *the supplément* under the title of *Nouvelle manière de considérer l'action capillaire* in [35, p.14]. We show the original contents of p.5 and p.18 of [35] pointed out by Gauss. These translations are available by Bowditch [37] [77].

¶ pages 5−6 of *Supplément* [35, pp. 5−6] : [78]

(1) L'intégrale relative à f peut être prise depuis $f=0$ jusqu'à f infini; ensorte qu'elle est indépendente des dimensions de la masse attirante. C'est là ce qui caractérise ce genre d'attractions qui n'étant sensibles qu'à des distances imperceptibles, permettent d'ajouter ou de négliger à volonté, les attractions des corps, à des distances plus grandes que le rayon de leur sphère d'activité sensible.

(2) Désignons comme dans le n^o 1 de ma Théorie de l'action capillaire, par $c-\Pi(f)$, l'intégrale $fdf.\varphi(f)$, prise depuis $f=0$; c étant la valeur de cette intégrale, lorsque f est infini. $\Pi(f)$ sera une quantité positive décroissante avec une extrême rapidité; et l'on aura, en prenant les intégrales depuis $f=0$,

$$\int f^4df.\varphi(f)=-f^4.\Pi(f)+4\int f^3df.\Pi(f).$$

$-f^4.\Pi(f)$ est nul, lorsque f est infini; car, quoique f^4 devienne alors infini, l'extrême rapidité avec laquelle $\Pi(f)$ est supposé décroître, rend $f^4.\Pi(f)$ nul.

(3) Les functions $\varphi(f)$ et $\Pi(f)$ ne peuvent être mieux comparées qu'à des exponentielles telles que c^{-if}, c étant le nombre dont le logarithme hyperbolique est l'unité, et i étant un très-grand nombre.

[75]($\Downarrow$) (134) is a quadratic equation with respect to R.

[76]($\Downarrow$) From (136) and (137) we get it.

[77]($\Downarrow$) In this translation by Bowditch[37], the relation with the original pages is not shown.

[78]($\Downarrow$) Remark. Here, the itemized style is not of Laplace but of ours, for convenience' sake.

(4) En effet, c^{-if} est fini lorsque f est nul, et devient nul lorsque f est infini; de plus, il décroît avec une extrême rapidité, et le produit $f^n.c^{-if}$ est toujours nul, quel que soit l'exposant n, lorsque f est infini.

(5) Soit encore, comme dans le n^o 1 de la Théorie citée,

$$\int f df.\Pi(f) = c' - \Psi(f); \tag{139}$$

c' étant la valeur de cette intégrale, lorsque f est infini. $\Psi(f)$ sera encore une quantité positive décroissante avec une extrême rapidité; et l'on aura

$$4.\int f^3 df.\Pi(f) = -4f^2.\Psi(f) + 8.\int f df.\Psi(f).$$

dans le cas de f infini, $f^2\Psi(f)$ devient nul; on a donc en prenant l'intégrale depuis $f = 0$, jusqu'à f infini,

$$4\int f^3 df.\Pi(f) = 8\int f df.\Psi(f).$$

(6) Einfin, si l'on désigne, comme dans le n^o cité, par $\frac{H}{2\pi}$ l'intégrale $\int f df.\Psi(f)$ prise depuis f nul, jusqu'à f infini; on aura

$$\int f^4 df\varphi(f) = 8\int f df.\Psi(f) = \frac{4H}{\pi}.$$

Les deux forces tangentielles précédentes parallèles aux axes des x et y deviendront ainsi :

$$(SC + E).H, \quad (3F + D).H.$$

[35, (*Supplément*) p.5]

(We show the translation by Bowditch as follows :)

(1) The integral relative to f may be taken from $f = 0$ to $f = \infty$, so that it is independent of the dimensions of the attracting mass. This is what characterizes this kind of attractions, which, being sensible only at insensible distance, allows us to notice or neglect, at pleasure, the attractions of the bodies situated beyond their sphere of sensible activity.

(2) We shall put, as in

$$\Pi(f) = c' - \int_0^f df.\varphi(f),$$

the integral $\int df.\varphi(f)$ being taken from $f = 0$, and c being its value when f is infinite. $\Pi(f)$ will be a positive quantity, decreasing with extreme rapidity; and we shall have, by taking the integrals from $f = 0$;

$$\int f^4 df.\varphi(f) = -f^4.\Pi(f) + 4\int f^3 df.\Pi(f). \tag{140}$$

$-f^4.\Pi(f)$ is nothing when $f = \infty$; for although f^4 then becomes infinite, the extreme rapidity with which $\Pi(f)$ is supposed to decrease, renders $f^4.\Pi(f)$ nothing.

(3) The functions $\varphi(f)$ and $\Pi(f)$ may be very well compared with exponentials like c^{-if}; c being the number whose hyperbolic logarithm is unity, and i being a very big positive number.

(4) For c^{-if} is finite when $f = 0$, and becomes nothing when f is finite; moreover it decreases with extreme rapidity, and in such a manner that the product $f^n.c^{-if}$ always vanishes when f is infinite, whatever be the value of exponent n.

(5) We shall now put, as in,

$$\int_0^f f df.\Pi(f) = c' - \Psi(f);$$

c' being the value of that integral when f is infinite. $\Psi(f)$ will also be a positive quantity decreasing with extreme rapidity; and we shall have

$$4\int f^3 df.\Pi(f) = -4f^2.\Psi(f) + 8\int f df.\Psi(f).$$

When f is infinite, $f^2.\Psi(f)$ becomes nothing; therefore we shall have, by taking the integral fron $f = 0$ to $f = \infty$

$$4\int_0^\infty f^3 df.\Pi(f) = 8\int_0^\infty f df.\Psi(f). \tag{141}$$

(6) Lastly if we put as in,

$$\frac{H}{2\pi} = \int_0^\infty f df.\Psi(f),$$

we shall have,

$$\int_0^\infty f^4 df \varphi(f) = 8\int_0^\infty f df.\Psi(f) = \frac{4H}{\pi}. \tag{142}$$

Thus the two preceding tangential force, parallel to the axes of x and y, will become

$$(SC + E).H, \quad (3F + D).H.$$

[37, pp.812-813]

Remark by us: above (142) tells us simply that we get its equation from (140) and (141),

$$\int_0^\infty f^4 df.\varphi(f) = -f^4.\cancel{\Pi}(f) \; + 4\int_0^\infty f^3 df.\Pi(f) \cdots (140), \quad 4\int_0^\infty f^3 df.\Pi(f) = 8\int_0^\infty f df.\Psi(f) \cdots (141).$$

¶ *p*.18 of *Supplément* :

Fixons à cette extrémité, l'origine des coordinées x, y, z d'un point quelconque du plan solide; l'axe des x étant sur la ligne a de la plus courte distance de l'extrémité de la droite au plan, et l'axe des y étant horizontal comme l'axe des x.

En désignant par z' l'abaissement au-dessous de l'origine des coordonées, d'un point quelconque de la ligne attirée; l'attraction vertical du plan solide sur ce point sera à la distance s, et s

$$\iiint dx.dy.dz.\frac{(z + z')}{s}.\varphi(s);$$

$\varphi(s)$ étant la loi de l'attraction à la distance d'un point attirant du plan, au point attiré de la ligne ; ensorte que l'on a

$$s^2 = x^2 + y^2 + (z + z')^2.$$

Pour avoir l'attraction verticale du plan solide, sur la ligne entière; il faut multiplier la triple intégrale précédente par dz', et l'intégrer par rapport à z' depuis $z' = 0$ jusqu'à z' infini.

En désignant donc comme dans le n^o 1 de ma Théorie de l'action capillaire, par $c - \Pi(s)$, l'intégrale $\int ds.\varphi(s)$ prise depuis $s = 0$, la constante c étant l'intégrale entière depuis s nul jusqu'à s infini; on aure

$$\int dz'.\frac{(z + z')}{s}.\varphi(s) = \Pi(s);$$

s étant dans la second membre de cette équation, ce que devient s, à l'origine des coordonnées, ou lorsque z' est nul.

L'attraction du plan solide sur la ligne entière sera donc

$$\iiint dx.dy.dz.\Pi(s).$$

[35, (*Supplément*) pp.18-19].

C.2. **Gauss' paper.**

C.2.1. **Gauss' papers of the capillary action.**

Gauss states common motivations with Laplace about MD equations. For example, in §10, §11, §12, which we mention below, he states the difficulties of integral $\int r^2\varphi r.dr$, in which he confesses that he also is included in the person who feels difficulties to calculate the MD integral.

C.2.2. **Gauss' letters corresponded with Bessel about Laplace's theory of the capillary action.**

Gauss corresponded with Bessel about Laplace's two papers [35].

> Allein in der ganzen ersten Abhandlung selbst finde ich kein Wort, was dienen kann diess zu beweisen. Es kann also wohl nichts gemeint sein als die Stelle in der Einleitung pag. 5, wo ich aber den Schluß, daß die ≫ plans (en question) sont également inclinés à leurs parois≪ keineswegs auf eine befriedigende Art begründet finde. Ich gestehe, daß mir dieser Hauptheil von Laplace's Theorie der praecisen mathematischen Begründung des übrigen keineswegs würdig zur Seite zu stehen, sondern mehr den Character der vaguen Aperçus, die man früher von dem granzen Phaenomene hatte, to tragen scheint.
>
> Freilich könnte man sagen, daß Laplace these Lücke einigermassen in der zeiten Abhandlung ausgefüllt hat. Das Rapprochement in der ersten Methode die Haarröhrchen zu behandeln mit der andern in der zweiten Abhandlung (die doch wohl im Grunde nichts weiter ist als die Ladande'sche) führt zu einer Bestimmung des Winkels quaestionis, pag. 18. (27. Januar 1829.) [18, pp.487-490].
>
> (Engl. transl.) Only in all the first paper, I can find no word to be useful for me. It is sufficient to be no meaning as the part of the introduction [79] in page 5, where I conclude that his phrase "the plane (in question) inclines equally to its wall" is not based on the admitted method. I can not help confessing that these main theory by Laplace's Theory is for me to be convinced which is never worth consulting it as the (concise) [80] mathematical ground.
>
> Although we can say, of course, that Laplace complemented these defects in the second paper, however, his approximation in the first method, dealt the capillar action with another one, in the second paper (which is fundamentally inferior to the writing by Ladande[81]), he deduces the doubtful formulae of angle. page 18.

C.2.3. **Bessel's reply to Gauss.**

> Gegen die Gleichung der Oberfläche habe ich nie ein Misstrauen empfinden, allein den Winkel habe auch ich nicht für erwiesenermaßen unabhängig von dem Durchmesser der Röhre u.s.w. gehalten, sondern diese vielmehr als der Erfahrung, welche mit dem Raisonnement Seite 5 zusammentrifft, entsprechend; denn das Aufsteigen der Flüssigkeit in eigen Röhren könnte nicht dem Durchmesser derselben umgekehrt proportional sein, wenn dieser Winkel nicht stets gleich bleibe. (10. Februar 1829) [18, pp.491-493].
>
> (Engl. transl.) To the equation of surface, I did not have any doubts, however, about that the angle is independent of the diameter of the tube, etc., I have not accepted as being beyond doubt, but also these, strictly speaking, in the experience, which considering with the assumption of the page 5, phenomena of fluid in the tube, it is impossible to be in inverse proportion to the diameter of the tube, because this angle is not always equal.

---

[79](⇓) The introduction takes 1-9 pages in [35] and 685-694 pages in [37].

[80](⇓) We do not know about the meaning "praecisen". We can consult the word "praecise" whose meaning is "in short, in few words, briefly, concisely " of only as adverb with the following dictionaries edited by C.T.Lewis, "Elementary Latin Dictionary Lexicon" [41], or "Lexicon Latino-Japonicum" by Kenkyusha. In this sentence by Gauss, it must be used as adjective, so that we use as "concise".

[81](⇓) Ladande, Joseph Jérôme Lafrançois de, (1732-1807), i.e. an astronomer who then was criticized for his astronomical writings by Gauss. cf. *Shogakukan Robert Dictionnaire Français-Japonais* by Shougakkan, 1988. p.1390

C.3. **Laplace's two-constant in the *Suppléments*.**

We show Laplace's usage of two-constants in calculating of the capillary action in *Supplément* [35, pp.9-14] as follows :

$$2\pi.\{1+(A+B).r\}.\Psi(r).$$

Maintenant, si l'on nomme R le rayon osculateur de la section de la surface, par un plan passant par les axes des x et des z, et si l'on nomme pareillement R' le rayon osculateur de la section de la surface, par un plan passant par les axes des y et des z ;

$$A=\frac{1}{2R},\quad B=\frac{1}{2R'}.$$

$$2\pi.\Big\{1+\frac{r}{2}.\Big(\frac{1}{R}+\frac{1}{R'}\Big)\Big\}.\Psi(r).$$

Laplace stated that :

> Pour avoir l'action entière du corps, sur un fluide renfermé dans un canal infiniment étroit per pendiculaire à la surface, et dont la base est prise pour unité; il faut multiplier l'expression précédente par dr, et l'intégrer depius $r=0$ jusqu'à r infini. Soit alors [82]
>
> $$2\pi\int\Psi f.df=K,\qquad 2\pi\int\Psi f.f.df=H, \tag{143}$$
>
> l'action du corps sur le canal, sera
>
> $$K+\frac{H}{2}.\Big(\frac{1}{R}+\frac{1}{R'}\Big);$$

(⇓)Here, (143) means that these K and H are the two-constant, which, we think, had appeard for the first time. These mean

$$2\pi\int\Psi(f)df=K,\qquad 2\pi\int f\Psi(f)df=H,$$

(⇑)

When we denote $h+z$ the height of the point on the sea level, g : mass gravity and D : density, then

$$gD.(h+z)=\frac{H}{2}.\Big(\frac{1}{R}+\frac{1}{R'}\Big).$$

However, if we denote by (135)

$$\frac{dz}{dx}\equiv p,\quad \frac{dz}{dy}\equiv q$$

and by the theory of curved surface :

$$\frac{1}{R}+\frac{1}{R'}=\frac{(1+q^2).\frac{dp}{dx}-pq.\Big(\frac{dp}{dy}+\frac{dq}{dx}\Big)+(1+p^2).\frac{dq}{dy}}{(1+p^2+q^2)^{\frac{3}{2}}} \tag{144}$$

$$\frac{1}{2}.H.\Big[\frac{(1+q^2).\frac{dp}{dx}-pq.\Big(\frac{dp}{dy}+\frac{dq}{dx}\Big)+(1+p^2).\frac{dq}{dy}}{(1+p^2+q^2)^{\frac{3}{2}}}\Big]=gD.(h+z)$$

équation qui est visiblement la mêmé que l'équation (a) [83] du n^o 4 de la Théorie citée.

> Maintenant, il est facile de s'assurer par la théorir des surfaces courbes, que si l'on nomme ϖ l'angle que la plan tangent à la surface du fluide intérieur au tube, forme avec les parois du tube toujours supposé vertical, à l'extrémité de sa sphère d'activité sensible ; on a
>
> $$\cos\,\varpi=\pm\frac{pdy-qdx}{ds.\sqrt{1+p^2+q^2}}$$

[82](⇓) cf. Gauss cites this Laplace's (143) in (176).

[83](⇓) cf. (138).

ds étant l'élément de la section ; on a donc en observant que l'angle ϖ est constant, comme je l'ai fait voir dans la théorie citée,

$$\pm \int \frac{pdy - qdx}{\sqrt{1+p^2+q^2}} = c.\cos\ \varpi$$

c étant le contour entier de la section; partant

$$\frac{1}{2}.H.\iint dxdy.\left\{\left(d.\frac{\frac{p}{\sqrt{1+p^2+q^2}}}{dx}\right) + \left(d.\frac{\frac{q}{\sqrt{1+p^2+q^2}}}{dy}\right)\right\} = \frac{1}{2}.H.c.\cos\varpi$$

ce qui donne

$$gD.V = \frac{1}{2}.H.c.\cos\varpi$$

ainsi le volume du fluide, élevé au-dessus du niveau par l'action capillaire, est proportionnel au contour de la section de la surface intérieure du tube. On peut parvenir à cette équation remarquable, en considérant sous le point de vue suivant, les effets de l'action capillaire.

APPENDIX D. *Disquisitiones generales circa superficies curvas. (General survey on the curved surface)*

We show the only relative and available articles : §8, §10, §11, §12, §21 and §22 of the deduction of theorem of curvature and the first and second fundamental forms to the next appendix E.

Remarks.

- The contents are not literal or word-for-word translation from Gauss, but our free translation commenting our interpretations.
- Throughout this paper, in citation of bibliographical sources, we show our own paragraph or sentences of commentaries by surrounding between ($\Downarrow$) and ($\Uparrow$). (($\Uparrow$) is used only when not following to next article/section). And by $=^*$, we detail the statement by original authors, because we would like to discriminate and to avoid confusion from the descriptions by original authors. The mark : $\Rightarrow$ mean transformation of the statements in brevity of ours. And all the frames surrounding the statements are inserted for important remark of ours. Of course, when explicitly without these marks, it is not the description in citation of bibliographical sources.
- Throughout both papers of appendix D and E, Gauss didn't use today's expression of array or determinant at all, so all the expressions of the sort of that are of ours.

D.8. Theorem of curvature.

Theorem D.8.1. *The curvature in surface point of fluid is expressed by the fraction, of which the numerator is a value and the denominator is, on the contrary, the product by the two radii of limit curvature in a sectioned normal plane.*

D.10. Deduction of the formula of curvature.

$$\begin{cases} \frac{d^2x}{dp^2} \equiv \alpha, & \frac{d^2x}{dpdq} \equiv \alpha', & \frac{d^2x}{dq^2} \equiv \alpha'', \\ \frac{d^2y}{dp^2} \equiv \beta, & \frac{d^2y}{dpdq} \equiv \beta', & \frac{d^2y}{dq^2} \equiv \beta'', \\ \frac{d^2z}{dp^2} \equiv \gamma, & \frac{d^2z}{dpdq} \equiv \gamma', & \frac{d^2z}{dq^2} \equiv \gamma'' \end{cases}$$

The letters $a,\ b,\ c$ are permuted cyclically.

$$\begin{cases} A \equiv bc' - cb', \\ B \equiv ca' - ac', \\ C \equiv ab' - ba' \end{cases} \Rightarrow^* \quad A = \begin{vmatrix} b & c \\ b' & c' \end{vmatrix}, \quad B = \begin{vmatrix} c & a \\ c' & a' \end{vmatrix}, \quad C = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \tag{145}$$

$Adx + Bdy + Cdz = 0$, namely $dz = -\frac{A}{C}dx - \frac{B}{C}dy$. We denote $\frac{dz}{dx} \equiv t = -\frac{A}{C}$ and $\frac{dz}{dy} \equiv u = -\frac{B}{C}$.

$$\begin{cases} Cdp = b'dx - a'dy, \\ Cdq = -bdx + ady \end{cases} \Rightarrow^* \quad C\begin{bmatrix} dp \\ dq \end{bmatrix} = \begin{bmatrix} b' & -a' \\ -b & a \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix} \tag{146}$$

We take the complete differential of (146) in respect to t and u.

$$\begin{cases} C^3dt = \left(A\frac{dC}{dp} - C\frac{dA}{dp}\right)(b'dx - a'dy) + \left(C\frac{dA}{dq} - A\frac{dC}{dq}\right)(bdx - ady), \\ C^3du = \left(B\frac{dC}{dp} - C\frac{dB}{dp}\right)(b'dx - a'dy) + \left(C\frac{dB}{dq} - B\frac{dC}{dq}\right)(bdx - ady) \end{cases} \tag{147}$$

$$\Rightarrow^* \quad C^3\begin{bmatrix} dt \\ du \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} A & C \\ \frac{dA}{dp} & \frac{dC}{dp} \end{vmatrix} & \begin{vmatrix} C & A \\ \frac{dC}{dq} & \frac{dA}{dq} \end{vmatrix} \\ \begin{vmatrix} B & C \\ \frac{dB}{dp} & \frac{dC}{dp} \end{vmatrix} & \begin{vmatrix} C & B \\ \frac{dC}{dq} & \frac{dB}{dq} \end{vmatrix} \end{bmatrix} \begin{bmatrix} \begin{vmatrix} b' & a' \\ dy & dx \end{vmatrix} \\ \begin{vmatrix} b & a \\ dy & dx \end{vmatrix} \end{bmatrix}$$

We substitute (147) to the followings :

$$\begin{cases} \frac{dA}{dp} = c'\beta + b\gamma' - c\beta' - b'\gamma, & \frac{dA}{dq} = c'\beta' + b\gamma'' - c\beta'' - b'\gamma', \\ \frac{dB}{dp} = a'\gamma + c\alpha' - a\gamma' - c'\alpha, & \frac{dB}{dq} = a'\gamma' + c\alpha'' - a\gamma'' - c'\alpha', \\ \frac{dC}{dp} = b'\alpha + a\beta' - b\alpha' - a'\beta, & \frac{dC}{dq} = b'\alpha' + a\beta'' - b\alpha'' - a'\beta' \end{cases}$$

$$\begin{cases} C^3T = (b')^2(\alpha A + \beta B + \gamma C) - 2bb'(\alpha' A + \beta' B + \gamma C) + b^2(\alpha'' A + \beta'' B + \gamma'' C), \\ C^3U = -a'b'(\alpha A + \beta B + \gamma C) + (ab' + ba')(\alpha' A + \beta' B + \gamma C) - ab(\alpha'' A + \beta'' B + \gamma'' C), \\ C^3V = (a')^2(\alpha A + \beta B + \gamma C) - 2aa'(\alpha' A + \beta' B + \gamma C) + a^2(\alpha'' A + \beta'' B + \gamma'' C) \end{cases} \tag{148}$$

$$\begin{cases} (1)_G \quad A\alpha + B\beta + C\gamma \equiv D, \\ (2)_G \quad A\alpha' + B\beta' + C\gamma' \equiv D', \\ (3)_G \quad A\alpha'' + B\beta'' + C\gamma'' \equiv D'' \end{cases} \tag{149}$$

$$\Rightarrow^* \begin{bmatrix} D \\ D' \\ D'' \end{bmatrix} \equiv \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{d^2x}{dp^2} & \frac{d^2y}{dp^2} & \frac{d^2z}{dp^2} \\ \frac{d^2x}{dpdq} & \frac{d^2y}{dpdq} & \frac{d^2z}{dpdq} \\ \frac{d^2x}{dq^2} & \frac{d^2y}{dq^2} & \frac{d^2z}{dq^2} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

Substituting (149) for (148), we get the following :

$$\begin{cases} C^3T = (b')^2D - 2bb'D' + b^2D'', \\ C^3U = -a'b'D + (ab' + ba')D' - abD'', \\ C^3V = (a')^2D - 2aa'D' + a^2D'' \end{cases} \Rightarrow^* \; C^3 \begin{bmatrix} T \\ U \\ V \end{bmatrix} = \begin{bmatrix} (b')^2 & -2bb & b^2 \\ -a'b' & (ab' + ba') & -ab, \\ (a')^2 & -2aa' & a^2 \end{bmatrix} \begin{bmatrix} D \\ D' \\ D'' \end{bmatrix}$$

$$C^6(TV - U^2) = \big(DD'' - (D')^2\big)(ab' - ba')^2 = \big(DD'' - (D')^2\big)C^2$$

Therefore, the formula for the curvature is as follows :

$$\boxed{k \equiv \frac{DD'' - (D')^2}{(A^2 + B^2 + C^2)^2}} \tag{150}$$

D.11. **Evolving the equation of curvature.**

Suppose that x, y and z are functions of two independent variables u and v, with all partial derivatives up to those of the third order. The letters a, b, c are permuted cyclically.

$$\begin{cases} dx = adu + a'dv, \\ dy = bdu + b'dv, \\ dz = cdu + c'dv \end{cases} \Rightarrow^* \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} a\; a' \\ b\; b' \\ c\; c' \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}$$

$$\begin{cases} a^2 + b^2 + c^2 = E, \\ aa' + bb' + cc' = F, \\ (a')^2 + (b')^2 + (c')^2 = G, \end{cases} \Rightarrow^* \begin{bmatrix} E \\ F \\ G \end{bmatrix} = \begin{bmatrix} a\; b\; c \\ a\; b\; c \\ a'\; b'\; c' \end{bmatrix} \begin{bmatrix} a\; a'\; a' \\ b\; b'\; b' \\ c\; c'\; c' \end{bmatrix} \tag{151}$$

Let us treat v as the independent variable, u as a function of it. For the square of the distance of arc we shall have

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

$$\begin{cases} (4)_G \quad a\alpha + b\beta + c\gamma = m, \\ (5)_G \quad a\alpha' + b\beta' + c\gamma' = m', \\ (6)_G \quad a\alpha'' + b\beta'' + c\gamma'' = m'', \end{cases} \tag{152}$$

$$\Rightarrow^* \begin{bmatrix} m \\ m' \\ m'' \end{bmatrix} \equiv \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{d^2x}{dp^2} & \frac{d^2y}{dp^2} & \frac{d^2z}{dp^2} \\ \frac{d^2x}{dpdq} & \frac{d^2y}{dpdq} & \frac{d^2z}{dpdq} \\ \frac{d^2x}{dq^2} & \frac{d^2y}{dq^2} & \frac{d^2z}{dq^2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{cases} (7)_G \quad a'\alpha + b'\beta + c'\gamma = n, \\ (8)_G \quad a'\alpha' + b'\beta' + c'\gamma' = n', \\ (9)_G \quad a'\alpha'' + b'\beta'' + c'\gamma'' = n'' \end{cases} \tag{153}$$

$$\Rightarrow^* \begin{bmatrix} n \\ n' \\ n'' \end{bmatrix} \equiv \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{bmatrix} \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} \frac{d^2x}{dp^2} & \frac{d^2y}{dp^2} & \frac{d^2z}{dp^2} \\ \frac{d^2x}{dpdq} & \frac{d^2y}{dpdq} & \frac{d^2z}{dpdq} \\ \frac{d^2x}{dq^2} & \frac{d^2y}{dq^2} & \frac{d^2z}{dq^2} \end{bmatrix} \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$$

From (145) and (151), we get the following expression :

$$A^2 + B^2 + C^2 = EG - F^2 =^* \begin{vmatrix} E & F \\ F & G \end{vmatrix} \equiv \Delta$$

The first expression $A^2 + B^2 + C^2 = EG - F^2$ is verified easily by extending the following :

$$\begin{aligned} A^2 + B^2 + C^2 &=^* \underbrace{(bc' - cb')^2}_{A^2} + \underbrace{(ca' - ac')^2}_{B^2} + \underbrace{(ab' - ba')^2}_{C^2} \\ &= \underbrace{(a^2 + b^2 + c^2)}_{E} \underbrace{\Big((a')^2 + (b')^2 + (c')^2\Big)}_{G} - \underbrace{(aa' + bb' + cc')^2}_{F^2} = EG - F^2 \end{aligned}$$

We deduce the relative formulae to curvature.

Step 1. At first, we should solve the following system of linear equations from the equations $(1)_G$ of (149), $(4)_G$ of (152) and $(7)_G$ of (153) :

$$\begin{cases} (1)_G & A\alpha + B\beta + C\gamma = D, \\ (4)_G & a\alpha + b\beta + c\gamma = m, \\ (7)_G & a'\alpha + b'\beta + c'\gamma = n, \end{cases}$$

$$\Rightarrow^* \begin{bmatrix} D \\ m \\ n \end{bmatrix} = \begin{bmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{bmatrix} \begin{bmatrix} \frac{d^2x}{dp^2} \\ \frac{d^2y}{dp^2} \\ \frac{d^2z}{dp^2} \end{bmatrix}$$

At the first step : **1-1**, eliminating β and γ, and multiplying these by $bc' - cb'$, $b'C - c'B$, $cB - bc$, and adding these expressions, we get the following expression :

$$\Big(A(bc' - cb') + a(b'C - c'B) + a'(cB - bC)\Big)\alpha = D(bc' - cb') + m(b'C - c'B) + n(cB - bC) \tag{154}$$

$$\Rightarrow^* \quad \alpha \begin{vmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{vmatrix} = \begin{vmatrix} D & B & C \\ m & b & c \\ n & b' & c' \end{vmatrix} \Rightarrow \quad AD = \begin{vmatrix} \alpha & m & n \\ a & E & F \\ a' & F & G \end{vmatrix} \tag{155}$$

($\Downarrow$) We see the expression (155) of AD, by extending the following determinant, which we substitute the defined values above mentioned for m, n, E, F and G of (155) :

$$\Rightarrow^* \quad AD = A(A\alpha + B\beta + C\gamma) = \begin{vmatrix} \alpha & a\alpha + b\beta + c\gamma & a'\alpha + b'\beta + c'\gamma \\ a & a^2 + b^2 + c^2 & aa' + bb' + cc' \\ a' & aa' + bb' + cc' & (a')^2 + (b')^2 + (c')^2 \end{vmatrix}$$

In fact, from (155), we can verify AD as follows :

$$\begin{aligned} AD &= \alpha \underbrace{(EG - F^2)}_{\Delta} + a(nF - mG) + a'(mF - nE) \\ &=^* \alpha \begin{vmatrix} E & F \\ F & G \end{vmatrix} + a \begin{vmatrix} n & m \\ G & F \end{vmatrix} + a' \begin{vmatrix} m & n \\ E & F \end{vmatrix} = \begin{vmatrix} \alpha & m & n \\ a & E & F \\ a' & F & G \end{vmatrix} \end{aligned}$$

Gauss deduces this relation without using the expression of array in this paper [15]] at all. ($\Uparrow$)

At the second step : **1-2**, eliminating similarly α and γ using the same equations $(1)_G$, $(4)_G$, $(7)_G$, and multiplying these by $ca' - ac'$, $c'A - a'C$, $aC - cA$, and adding these expressions, we get the following :

$$\Big(B(ca' - ac') + b(c'A - a'C) + b'(aC - cA)\Big)\beta = D(ca' - ac') + m(c'A - a'C) + n(aC - cA)$$

$$\Rightarrow^* \quad \beta \begin{vmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{vmatrix} = \begin{vmatrix} A & D & C \\ a & m & c \\ a' & n & c' \end{vmatrix} \Rightarrow^* \quad BD = \begin{vmatrix} \beta & m & n \\ b & E & F \\ b' & F & G \end{vmatrix}$$

At the final step : **1-3**, eliminating similarly α and β, and multiplying these by $ab' - ba'$, $b'A - a'B$, $bA - aB$, and adding these expressions, we get the following expression :

$$\Big(C(ab' - ba') + c(b'A - a'B) + c'(bA - aB)\Big)\gamma = D(ab' - ba') + m(b'A - c'B) + n(bA - aB)$$

$$\Rightarrow^* \quad \gamma \begin{vmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{vmatrix} = \begin{vmatrix} A & B & D \\ a & b & m \\ a' & b' & n \end{vmatrix} \quad \Rightarrow^* \quad CD = \begin{vmatrix} \gamma & m & n \\ c & E & F \\ c' & F & G \end{vmatrix}$$

Here, we get the three expressions :

$$\begin{cases} AD = \alpha\Delta + a(nF - mG) + a'(mF - nE), \\ BD = \beta\Delta + b(nF - mG) + b'(mF - nE), \\ CD = \gamma\Delta + c(nF - mG) + c'(mF - nE) \end{cases} \tag{156}$$

$$\Rightarrow^* \quad D \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \alpha & a & a' \\ \beta & b & b' \\ \gamma & c & c' \end{bmatrix} \begin{bmatrix} \begin{vmatrix} E & F \\ F & G \end{vmatrix} \\ \\ \begin{vmatrix} n & m \\ G & F \end{vmatrix} \\ \\ \begin{vmatrix} m & n \\ E & F \end{vmatrix} \end{bmatrix}$$

Multiplying (156) by α'', β'' and γ'' respectively, and adding the gained expressions in the each hand side, then we get the following expression :

$$\begin{aligned} DD'' &= (\alpha\alpha'' + \beta\beta'' + \gamma\gamma'')\Delta + m''(nF - mG) + n''(mF - nE) \\ &=^* \begin{vmatrix} \alpha\alpha'' + \beta\beta'' + \gamma\gamma'' & m & n \\ m'' & E & F \\ n'' & F & G \end{vmatrix} \end{aligned} \tag{157}$$

Step 2 . Similarly, from the equations $(2)_G$, $(5)_G$, $(8)_G$:

$$\begin{cases} (2)_G \quad A\alpha' + B\beta' + C\gamma' \equiv D', \\ (5)_G \quad a\alpha' + b\beta' + c\gamma' = m', \\ (8)_G \quad a'\alpha' + b'\beta' + c'\gamma' = n', \end{cases} \quad \Rightarrow^* \quad \begin{bmatrix} D' \\ m' \\ n' \end{bmatrix} = \begin{bmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{bmatrix} \begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \end{bmatrix}$$

We get the three expressions corresponding to (156) :

$$\begin{cases} AD' = \alpha'\Delta + a(n'F - m'G) + a'(m'F - n'E), \\ BD' = \beta'\Delta + b(n'F - m'G) + b'(m'F - n'E), \\ CD' = \gamma'\Delta + c(n'F - m'G) + c'(m'F - n'E) \end{cases} \tag{158}$$

$$\Rightarrow^* \quad D' \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \alpha' & a & a' \\ \beta' & b & b' \\ \gamma' & c & c' \end{bmatrix} \begin{bmatrix} \begin{vmatrix} E & F \\ F & G \end{vmatrix} \\ \\ \begin{vmatrix} n' & m' \\ G & F \end{vmatrix} \\ \\ \begin{vmatrix} m' & n' \\ E & F \end{vmatrix} \end{bmatrix}$$

Multiplying (158) by α', β' and γ' respectively, and adding the gained expressions in the each hand side, then we get the following expression corresponding to (157) :

$$\begin{aligned} (D')^2 &= ((\alpha')^2 + (\beta')^2 + (\gamma')^2)\Delta + m'(n'F - m'G) + n'(m'F - n'E) \\ &=^* \begin{vmatrix} (\alpha')^2 + (\beta')^2 + (\gamma')^2 & m' & n' \\ m' & E & F \\ n' & F & G \end{vmatrix} \end{aligned} \tag{159}$$

Step 3. From (157) and (159), we get the following expression :

$$\begin{aligned} DD''-(D')^2 &= \Big(\alpha\alpha''+\beta\beta''+\gamma\gamma''-\big((\alpha')^2+(\beta')^2+(\gamma')^2\big)\Big)\underbrace{(EG-F^2)}_{\Delta} \\ &+ E\big((n')^2-nn''\big)+F\big(nm''-2m'n'+mn''\big)+G\big((m')^2-mm''\big)\big) \end{aligned} \tag{160}$$

Here using the following relations :

$$\frac{dE}{dp}=2m,\quad \frac{dE}{dq}=2m',\quad \frac{dF}{dp}=m'+n,\quad \frac{dF}{dq}=m''+n',\quad \frac{dG}{dp}=2n',\quad \frac{dG}{dq}=2n''$$

$$m=\frac{1}{2}\frac{dE}{dp},\quad m'=\frac{1}{2}\frac{dE}{dq},\quad m''=\frac{dF}{dq}-\frac{1}{2}\frac{dG}{dp},\quad n=\frac{dF}{dp}-\frac{1}{2}\frac{dE}{dq},\quad n'=\frac{1}{2}\frac{dG}{dp},\quad n''=\frac{1}{2}\frac{dG}{dq},$$

then the first term in the right hand side of (160) except for Δ turns into :

$$\alpha\alpha''+\beta\beta''+\gamma\gamma''-\big((\alpha')^2+(\beta')^2+(\gamma')^2\big)=\frac{dn}{dq}-\frac{dn'}{dp}=\frac{dm''}{dp}-\frac{dm'}{dq}=-\frac{1}{2}\frac{ddE}{dq^2}+\frac{ddF}{dpdq}-\frac{1}{2}\frac{ddG}{dp^2}$$

From the equation of curvature (150) in the end of last article § D.10, we get the following expression :

$$(A^2+B^2+C^2)^2k=(EG-F^2)^2k=(EG-F^2)^2\frac{DD''-(D')^2}{(A^2+B^2+C^2)^2}=DD''-(D')^2 \tag{161}$$

Substituting (160) for the right hand side of (161), we get finally the following expression :

$$\boxed{\begin{aligned} 4(EG-F^2)^2k \;=\; 4\Delta^2k \;=\; 4(DD''-(D')^2) \;&= E\Big(\underbrace{\frac{dE}{dq}\frac{dG}{dq}-2\frac{dF}{dp}\frac{dG}{dq}}_{nn''}+\underbrace{\big(\frac{dG}{dp}\big)^2}_{(n')^2}\Big) \\ &+ F\Big(\underbrace{\frac{dE}{dp}\frac{dG}{dq}}_{4mn''}-\underbrace{\frac{dE}{dq}\frac{dG}{dp}-2\frac{dE}{dq}\frac{dF}{dq}+4\frac{dF}{dp}\frac{dF}{dq}-2\frac{dE}{dp}\frac{dG}{dp}}_{4(nm''-2m'n')}\Big) \\ &+ G\Big(\underbrace{\frac{dE}{dp}\frac{dG}{dp}-2\frac{dE}{dp}\frac{dF}{dq}}_{mm'}+\underbrace{\big(\frac{dE}{dq}\big)^2}_{(m')^2}\Big) \\ &- 2\underbrace{(EG-F^2)}_{\Delta}\quad \underbrace{\Big(\frac{ddE}{dq^2}-2\frac{ddF}{dpdq}+\frac{ddG}{dp^2}\Big)}_{2\left(\alpha\alpha''+\beta\beta''+\gamma\gamma''-\left((\alpha')^2+(\beta')^2+(\gamma')^2\right)\right)} \end{aligned}}$$

($\Downarrow$) This equation is a quadratic equation in respect to $EG-F^2$:

$$\begin{aligned} &4k(EG-F^2)^2+2\Big(\frac{ddE}{dq^2}-2\frac{ddF}{dpdq}+\frac{ddG}{dp^2}\Big)(EG-F^2) \\ -\;& E\big(\frac{dE}{dq}\frac{dG}{dq}-2\frac{dF}{dp}\frac{dG}{dq}+\big(\frac{dG}{dp}\big)^2\big) \\ -\;& F\Big(\frac{dE}{dq}\frac{dG}{dq}-\frac{dE}{dp}\frac{dG}{dp}-2\frac{dE}{dq}\frac{dF}{dq}+4\frac{dF}{dp}\frac{dF}{dq}-2\frac{dE}{dp}\frac{dG}{dp}\Big) \\ -\;& G\Big(\frac{dE}{dp}\frac{dG}{dp}-2\frac{dE}{dp}\frac{dF}{dq}+\big(\frac{dE}{dq}\big)^2\Big)=0 \end{aligned}$$

[84]

[84] This equation means that the curvature depends only on the first fundamental form : (174). cf. Kobayashi [28, p.200]

D.12. **Deduction of formulae of a line-segment on the curved surface.**

From the following expression :

$$dx^2 + dy^2 + dz^2 = Edp^2 + 2Fdpdq + Gdq^2,$$

the general line-segment on the curved surface is expressed as $\sqrt{Edp^2 + 2Fdpdq + Gdq^2}$. And we consider the following expression $\sqrt{E'dp^2 + 2F'dpdq + G'dq^2}$, as the line-segment $\sqrt{(dx')^2 + (dy')^2 + (dz')^2}$, denoting the functions : E', F', G' of p and q.

$$E = E', \quad F = F', \quad G = G'$$

From the formula in the article § D.11 we get the following theorem :

Theorem D.12.1. *(Invariability of curvature.) Even if the curved surface turns into another shape of surface, the curvature of surface stays invariable in each point.*

The following are clear : after the curved surface turns into another shape of surface, the shape of the surface again returns to the first shape. □

D.15. **Deduction of theorem of the shape.**

Theorem D.15.1. *The shape of the curved surface will reach the shortest length in the same oriented point of fluid length, taking the normal line at the limit.* □

D.21. **Deduction of formulae.**

We would like to restore the general meanings to the $\prec$ *characters* $p, q, E, F, G, \omega \succ$, which were accepted, additionally speaking, which are determined by dual alias variables p', q', where, a infinite line-segment is expressed by :

$$\sqrt{E'dp'^2 + 2F'dp'.dq' + G'dq'^2}$$

$$\begin{cases} dp' = \alpha dp + \beta dq, \\ dq' = \gamma dp + \delta dq \end{cases} \Rightarrow^* \begin{bmatrix} dp' \\ dq' \end{bmatrix} = \begin{bmatrix} \alpha\ \beta \\ \gamma\ \delta \end{bmatrix} \begin{bmatrix} dp \\ dq \end{bmatrix}$$

Now we would like to investigate the geometric meaning of these coefficients $\alpha, \beta, \gamma, \delta$.

Quatuor[85] is here the linear system considered in the curved surface, for these, they were constants such as q, p, q', p'. If we determine these by points, these respond to the variable values of q, p, q', p', the positive variations dq, dp, dq', dp' are responded

$$\sqrt{E}.dp, \quad \sqrt{G}.dq, \quad \sqrt{E'}.dp', \quad \sqrt{G'}.dq'$$

We denote the angles by M, N, M', N'

$$p + dp, \quad q + dq, \quad p' + dp', \quad q' + dp'$$

are independent of the values of variations dq, dp, dq', dp'

$$\sqrt{E}.dp.\sin M + \sqrt{G}.dq.\sin N = \sqrt{E'}.dp'.\sin M' + \sqrt{G'}.dq'.\sin N'$$

We, however, introduce these by notating

- $N - M = \omega$
- $N' - M' = \omega'$
- $N - M' = \psi$.

These equations of the invented methods are seen in the following formats

$$\sqrt{E}.dp.\sin\underbrace{(M' - \omega + \psi)}_{M} + \sqrt{G}.dq.\sin\underbrace{(M' + \psi)}_{N} = \sqrt{E'}.dp'.\sin M' + \sqrt{G'}.dq'.\sin\underbrace{(M' + \omega')}_{N'}, \tag{162}$$

or

$$\sqrt{E}.dp.\sin(\underbrace{N' - \omega}_{M'} - \omega' + \psi) + \sqrt{G}.dq.\sin\underbrace{(N' - \omega' + \psi)}_{M'+N-M'=N} = \sqrt{E'}.dp'.\sin\underbrace{(N' - \omega')}_{M'} + \sqrt{G'}.dq'.\sin N' \tag{163}$$

$$\sqrt{E'}.\sin\ \omega'.dp' = \sqrt{E}.\sin(\omega + \omega' - \psi).dp + \sqrt{G}.\sin(\omega' - \psi).dq \tag{164}$$

[85]($\Downarrow$) Here, we mean temporarily *Quatuor* as the *quaternion* named it. Hamilton [20]'s *Quatuor* is another one, in which Hamilton used his defined word "tensor". cf. the footnote (8) in Cauchy, § 8.

$$\sqrt{G'}.\sin\ \omega'.dq' = \sqrt{E}.\sin(\psi-\omega').dp + \sqrt{G}.\sin\ \psi.dq \tag{165}$$

We can construct the equations in combining the left hand-side of (164) with that made by substituting $N'=0$ in the left hand-side of (163). And also the left hand-side of (165) with that made by substituting $M'=0$ in the left hand-side of (162) then

$$\begin{cases}\sqrt{E'}.\sin\ \omega'.dp' = \sqrt{E}.dp.\sin(-\omega-\omega'+\psi) + \sqrt{G}.dq.\sin(-\omega'+\psi),\\ \sqrt{G'}.\sin\ \omega'.dq' = \sqrt{E}.dp.\sin(-\omega+\psi) + \sqrt{G}.dq.\sin(\psi)\end{cases}$$

($\Downarrow$) That is

$$\begin{bmatrix}\sqrt{E'}.\sin\ \omega'.dp'\\ \sqrt{G'}.\sin\ \omega'.dq'\end{bmatrix} = \begin{bmatrix}\sqrt{E}.\sin(\omega+\omega'-\psi) & \sqrt{G}.\sin(\omega'-\psi)\\ \sqrt{E}.\sin(\psi-\omega') & \sqrt{G}.\sin\ \psi\end{bmatrix}\begin{bmatrix}dp\\ dq\end{bmatrix} \Rightarrow \begin{cases}dp' = \alpha dp + \beta dq,\\ dq' = \gamma dp + \delta dq\end{cases}$$

($\Uparrow$)

$$\begin{cases}\alpha = \sqrt{\frac{E}{E'}}.\frac{\sin(\omega+\omega'-\psi)}{\sin\omega'},\\ \beta = \sqrt{\frac{G}{E'}}.\frac{\sin(\omega'-\psi)}{\sin\omega'},\\ \gamma = \sqrt{\frac{E}{G'}}.\frac{\sin(\psi-\omega')}{\sin\omega'}\\ \delta = \sqrt{\frac{G}{G'}}.\frac{\sin\ \psi}{\sin\ \omega'}\end{cases} \tag{166}$$

$$\begin{cases}\cos\ \omega = \frac{F}{\sqrt{EG}},\\ \cos\ \omega' = \frac{F'}{\sqrt{E'G'}},\\ \sin\ \omega = \sqrt{\frac{FG-F^2}{EG}},\\ \sin\ \omega' = \sqrt{\frac{F'G'-F'F'}{E'G'}}\end{cases}$$

$$\begin{cases}\alpha\sqrt{(E'G'-F'F')} = \sqrt{EG'}.\sin(\omega+\omega'-\psi),\\ \beta\sqrt{(E'G'-F'F')} = \sqrt{GG'}.\sin(\omega'-\psi),\\ \gamma\sqrt{(E'G'-F'F')} = \sqrt{EE'}.\sin(\psi-\omega),\\ \delta\sqrt{(E'G'-F'F')} = \sqrt{GE'}.\sin\ \psi\end{cases} \tag{167}$$

Substituting

$$\begin{cases}dp' = \alpha dp + \beta dq,\\ dq' = \gamma dp + \delta dq\end{cases} \Rightarrow^* \begin{bmatrix}dp'\\ dq'\end{bmatrix} = \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix}dp\\ dq\end{bmatrix} \Rightarrow^* \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}^{-1}\begin{bmatrix}dp'\\ dq'\end{bmatrix} = \begin{bmatrix}dp\\ dq\end{bmatrix} \tag{168}$$

for

$$Edp'^2 + 2F'dp'dq' + Gdq'^2 \tag{169}$$

and combining the value gained from (169) with the following value :

$$Edp^2 + 2Fdpdq + Gdq^2$$

then the following corresponding relation holds :

$$\begin{cases}E'\Big(\alpha dp+\beta dq\Big)^2 + 2F'\Big(\alpha dp+\beta dq\Big)\Big(\gamma dp+\delta dq\Big) + G'\Big(\gamma dp+\delta dq\Big)^2,\\ Edp^2 + 2Fdpdq + Gdq^2\end{cases}$$

Substituting the right hand side of each expression for zero, from these relation, we get the following expression using the relation between the coefficients and their roots of the quadratic equation :

$$EG - F^2 = (E'G' - F'F')(\alpha\delta - \beta\gamma)^2$$

From (168)

$$\begin{cases}(\alpha\delta-\beta\gamma)dp = \delta dp' - \beta dq',\\ (\alpha\delta-\beta\gamma)dq = -\gamma dp' + \alpha dq'\end{cases} \Rightarrow^* (\alpha\delta-\beta\gamma)\begin{bmatrix}dp\\ dq\end{bmatrix} = \begin{bmatrix}\delta & -\beta\\ -\gamma & \alpha\end{bmatrix}\begin{bmatrix}dp'\\ dq'\end{bmatrix}$$

$$\begin{cases} E\delta^2 - 2F\gamma\delta + G\gamma^2 = \frac{EG-F^2}{E'G'-F'F'}.E', \\ E\beta\delta - F(\alpha\delta + \beta\gamma) + G\alpha\gamma = -\frac{EG-F^2}{E'G'-F'F'}, \\ E\beta^2 - 2F\alpha\beta + G\alpha^2 = \frac{EG-F^2}{E'G'-F'F'}.G' \end{cases}$$

D.22. **First Fundamental Form and Second Fundamental Form.**

From the general survey in the previous article, we shift to the latest application, where p, q are put with the most general meaning, for p', q', adopted in the article 15, in which these $\prec$ *characters* $\succ$ were denoted by r, φ. We assume $E' = 1, F' = 0, \omega' = \frac{\pi}{2}, \sqrt{G'} = m$, then from (166) we get the following :

$$\begin{cases} \alpha = \sqrt{E}.\cos(\omega - \psi), \\ \beta = \sqrt{G}.\cos\psi, \\ m.\gamma = \sqrt{E}.\sin(\psi - \omega), \\ m.\delta = \sqrt{G}.\sin\ \psi \end{cases}$$

Here we show the *quaternion* equations (167) in the above article give the following, in replacing them with above values of $\alpha,\ \beta,\ \gamma,\ \delta,$

$$\sqrt{E}.\cos(\omega - \psi) = \frac{dr}{dp} \tag{170}$$

$$\sqrt{G}.\cos\ \psi = \frac{dr}{dq} \tag{171}$$

$$\sqrt{E}.\sin(\psi - \omega) = m.\frac{dr}{dp} \tag{172}$$

$$\sqrt{G}.\sin\ \psi = m.\frac{dr}{dq} \tag{173}$$

Moreover, the last two equations turn out as follows :

$$EG - F^2 = E\Big(\frac{dr}{dq}\Big)^2 - 2F.\frac{dr}{dp}.\frac{dr}{dq} + G\Big(\frac{dr}{dp}\Big)^2 \tag{174}$$

$$\Big(E.\frac{dr}{dq} - F.\frac{dr}{dp}\Big).\frac{d\varphi}{dq} = \Big(F.\frac{dr}{dq} - G.\frac{dr}{dp}\Big).\frac{d\varphi}{dp} \tag{175}$$

Here, the quantities $r,\ \varphi,\ \psi$ (and if one is necessary to get m, then even m) are determined from the gained equations above with respect to p and q : clearly, the integral of (174) gives r, and the integral of (175) gives φ, and another equations (170) and (171) give ψ itself, in addition to, (172) and (173) give m.

General integral equations (174) and (175) are necessary for two functions arbitrarily introduced, because it is very easily recognized that if it is perpendicular, then these equations are considered as unrestricted to this case.

APPENDIX E. ***Principia generalia theoriae figurae fluidrum in statu aequilibrii. (General principles of theory on fluid figure in equilibrium state)***

In this dissertation, Gauss treats many important topics to the modern mathematics such as the following :

- (E.0) Preface
- (E.1-E.5) Introduction
- (E.6-E.9) Reduction from the sextuple integral to the quadruple integral
- (E.10-E.12) Criticism of Laplace's molecular calculation of capillarity equations
- (E.13-E.17) Ideas by Gauss
- (E.18-E.19) Variation problem
- (E.20-E.24) Deduction of Gauss' integral formula
- (E.25-E.26) Geometric meaning of curvature ($\frac{d\xi}{dx} + \frac{d\eta}{dy}$ in V)
- (E.27-E.30) Application of geometrical method to meniscus
- (E.31-E.33) Attraction in condition by A, α, β
- (E.34) Summary

[86] In particular, in these contents, included with many important topics from the viewpoint of mathematics, such as

- Integral theory in §E.17 and §E.23 which he aims to be one of his proud points to publish this paper
- The unique "two-function", corresponding to "two-constant", which we show in Table 3 and in §E.2
- Idea of RDF, which we show in Table 8, 9, and in Preface ¶4
- Reduction of integral from sextuple to quadruple, in the articles §E.2, §E.16 and §E.17
- In and after §E.18, we show his calculus of variations in the capillarity against the RDF and calculation of it by Laplace.
- Finally, for the question to be solved by variational equation introduced in §E.18 and §E.19, we sketch his answers deduced from the previous work of theory in curved surface [15], to the height and angle in question in §E.28 and §E.29.

Throughout our paper, we show the process of formulation of *calculus of variations*[87] using the two functions characterized from the attraction and repulsion, and his criticism to Laplace imaging the *Gaussian function* as the rapidly decreasing function by Gauss in 1830. And we introduce a contribution to the hydromechanics, because he was a contemporary of the epoch of formulation of the NS equations, which are our main theme in our paper.

E.0. **Preface.**

¶ 2.

• Since Mr. Laplace, from here, presented conveniently the unique supposition about the inner, molecular activity, moreover, giving up diminution of law for the increasing distance, we have got the first result in the surface of the fluid figure based on the accurate calculation, and have established the general equation for the figure of equilibrium, not only the precise capillary phenomenon as described, but also try to explain the related problems.

• This investigation is discussed getting the consented with and confirmed in everywhere, by the exact experiment, among the first class of increasing natural philosophers, geometricians, and referred and criticized by some authorities from all the directions to the maximum part such as a minor or nonsense.

¶ 3. (Two RDF functions and two-constant defined by Laplace.)

• In the calculation by Mr. Laplace, we have at least a thing, which we can give evidence about it, and for which we would not absolutely consent with him.

• In the previous commentary : $\prec$ *Théorie de l'action capillaire* $\succ$, denoting by φf intensity of the

---

[86]($\Downarrow$) We entitled for explanation of contents in each article below, where, there was not at all name of title but only the number in Gauss' paper. The article number is the same as Gauss' numbering of article.

[87]($\Downarrow$) Lagrange [31, p.201]. Today's mathematical nomenclature is *calculus of variations* or *calcul des variations* by *The mathematical dictionary* (4th edition in 2007) edited by MSJ, 1954, p.432, (Japanese).

attraction in the distance f, the integrals read : [88]

$$\int_x^\infty \varphi f.df = \Pi x, \qquad \int_x^\infty \Pi f.f df = \Psi x,$$

Extending the integral interval by x to 0, Laplace defined the two constants : [89]

$$2\pi \int_0^\infty \Psi f.df = K, \qquad 2\pi \int_0^\infty \Psi f.f.df = H, \tag{176}$$

where denoting by π the $\frac{1}{2}$ of the circumference of the circle with radius = 1.

In a word, the $\prec$ *indoles* $\succ$ (characteristics) of the function φf reserves ineffective, as long as this f were insensible for all sensible value. Hence,

- from only this supposition, it is not deduced absolutely,
- moreover, Πf and Ψf are for the finite values, this function f needs to be infinitesimal, can not absolutely be true, $2\pi \int_0^{\text{finite}} \Psi f.df$ and $2\pi \int_0^{\text{finite}} \Psi f.f.df$ turn into another infinitesimal value of K and H as we read in the dissertation ;
- of course, the form of function φf may be considered to be infinite, although the fundamental supposition satisfies these would be erroneous conclusions for this.
- If it is supposed that φf is complete attraction, in fact, it will moreover conserve the fractional form $\frac{\alpha}{f^2}$, which depends on the general attraction ;
- but as long as we can not measure the attractive particle, even we know the occurrence in experiment, it is too infinitesimal in comparison with all the earth, then although, if we extend infinitely the second integral of (176), we should infer that the function Ψf is restricted to infinity.

¶ 4. (Criticism to Laplace by Gauss.)

88($\Downarrow$) cf. Laplace states the two-constant (176) in his original paper. Poisson cites these equations in (238).

89($\Downarrow$) Poisson rewrites these equivalent equations by using (176) by Laplace. cf. (239), (240).

• However, something similar to simple carelessness form the basis, such that Laplace discusses about the form than about the relating action with it.

• Judging from the second dissertation : ≺ *Supplément à la théorie de l'action capillaire* ≻ ([35]), Mr. Laplace had scarcely investigated of φf, not only the complete attraction, but also a part, and tacitly understood incompletely the general attraction ; by the way, if we would refer the latter in comparison with our sensible modification, on the contrary, we can assert it to be more inferior to the bad experiments and be clearly visible.

• He considers exponential e^{-if} as an example of equivalent function to φf, denoting the large quantity by i, namely $\frac{1}{i}$ becomes infinitesimal.

> But it is not at all necessary to limit the generality by such a large quantity, this point is more clear than words, we would see easiest, only by investigating if these integrations would be able to be extended, not only at infinity but also at an arbitrary sensible distance, or if the occurring in experiment could be wider extended up to the finitely measurable distance. [a]
>
> [a](⇓) We show the paragraph of his Latin original as follows :
>
> Sed ne opus quidem est, generalitatem tantopere limitare, quum is, qui rem potius quam verba intuetur, facillime videat, sufficere, si intergrationes illae non in infinitum, sed tantummode usque ad distantiam sensibilem arbitrariam, aut si mavis ad distantiam finitam dimensionibus in experimentis occurrentibus maiorem extendantur. [17, p.33]

¶ 5.

• On the other hand, a person studied this theory with more decisive mistakes, and to this theory, nobody criticize this sophist. Both are clearly to be criticized as a part owner of it.

• Here we established the general equation for fluid of liberal surface with differential by the partial coordinates : this equation depends on the force by molecular attraction, which the particles of the fluid are in motion, and additionally, this theory is absolute and is never rested essentially deficient in it.

• In addition to this equation between partial differential, (its integration, if it were postulated in analysis, an arbitrary function is induced) it is not sufficient for the figure of surface, determined from all aspects, unless the new conditions of the nature of the fluid in the defined boundary were accepted.

• Total condition is set up by another theory, which is, the angle of the plane to the surface of the liberal fluid in tangently contact with the vase (exactly speaking, in the boundary of the sensibly attractive force to the wall of vase) with the plane of the wall of vase, it is a tangential constant, we put with the relation with intensity of the molecular force determined between vase and fluid, so that, the continuity of figure at the neighborhood of the contacted with the liberal surface of the fluid is not interrupted.

• Hence, to the cardinal proposition of the total theory with calculation for demonstration, we can not accept the papers by Mr. Laplace ; in p.5, since not only he developed clearly incorrect argument but also showed the false proofs : we consider that calculations in the pages and the following after p.44 *are the vain effects.*

[18, p.33-34][90]

E.1. **Introduction.**

On the formulation of the equilibrium equation in the system of particles of a material, we would like to solve how much the motion confine the condition, provided that the principle of motion of force adapts

[90](⇓) There are 35 pages of calculation between p.44 and p.78 in his *Supplément.*

at maximum.

We would like to construct the system as follows :

- from the physical point $m,\ m',\ m'',\cdots$, in which we denote the mass of the concentrate [91] by this letter, we think, which is accepted,
- we figure that
  - P is an accelerated force which is active in point m, and these systems of motion, made by an arbitrary material, infinitesimally small, and accepts the condition of the affinity of system (motion of force),
  - dp is the motion of the point m in direction of the projection of force P, i.e. made by the angle of cosine, which face with the direction of the force P, multiplicated ;
- next, $\sum Pdp$ is the production of the sum of all similar one with respect to all force of the sole point m.
- As the same way, P' represents the indefinite force of the sole point m',
- in addition to, dp' is the motion of the point m' made with the projection of singular direction, similary with the other points.

From these idea, the condition of equilibrium of the system is consisted of that and the sum are

$$m\sum Pdp + m'\sum Pdp' + m''\sum Pdp'' + \cdots$$

Provided that the force of motion becomes $= 0$, we can explain precisely the principle of motion of the general force, and even in this case, for the sum of null motion, we can get the positive value. [92]

E.2. **Three basic forces and two kernel functions : f derived from φ and F derived from Φ.**

We consider the force reduced to three main forces.

- I. Gravity.
- II. The attractive force, which itself corresponds to the points $m, m', m'', \cdots$. The intensity of attraction of function is proportional with the distance if this function, the $\prec$ *characteristic* $\succ$ denoted by f in mass and supposed that the attraction is uniformly concentrated in the point.
- III. The forces, $m, m', m'', \cdots$ are attractive to the infinitesimal fixed points. For these forces, in the similar way, we will designate the $\prec$ *characteristic* $F \succ$ such that the inverse-directional distance is used, and with $M, M', M'', \cdots$, which are treated as a fixed point in one case, or a mass in the other case, which are supposed in these concentrate.

We get $\sum Pdp$ of the previous article as follows :

$$\begin{aligned}&-gdz\\ &-\quad m'f(m,m')d(m,m') - m''f(m,m'')d(m,m'') - m'''f(m,m''')d(m,m''') - \cdots\\ &-\quad MF(m,M)d(m,M) - M'F(m,M')d(m,M') - M''F(m,M'')d(m,M'') - \cdots \qquad (177)\end{aligned}$$

where, the difference $d(m,m'),\ d(m,m'')$ etc. are partial, relative to the only motion of the force of m. We denote :

$$\varphi \text{ such that : } -fx.dx = d\varphi x, \qquad \int fx.dx = -\varphi x, \qquad (178)$$

$$\Phi \text{ such that : } -Fx.dx = d\Phi x, \qquad \int Fx.dx \equiv -\Phi x \qquad (179)$$

where, $\varphi\infty = 0$, and in case of $\varphi t \quad \Rightarrow \int_t^\infty fx.dx = -\varphi t$.

(⇓) On the other hand, Gauss didn't describe explicitly $\varphi 0$. By the way, this method without taking of "two-constant" by Gauss corresponds to the expressions by Laplace, Poisson, Navier et al. Poisson [62, p.8] considers this method as one of Gauss' characteristic, however Poisson chose his own method like Laplace. cf. the entry no.8 in Table 3.(⇑)

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[91](⇓) In this paper, Gauss cites the concentrate in § $E.2, E.18$.

[92](⇓) Gauss didn't say "nonnegative" but "positive" value.

At any rate, we get the integral of it from (177) as follows :

$$\begin{aligned} &-gdz \\ &\quad + \ m'd\varphi(m,m') + m''d\varphi(m,m'') + m'''d\varphi(m,m''') + \cdots \\ &\quad + \ Md\Phi(m,M) + M'd\Phi(m,M') + M''d\Phi(m,M'') + \cdots \end{aligned}$$

$$\begin{aligned} \Omega = & -gmz - gm'z' - gm''z'' - gm'''z''' - \cdots \\ & + \ m\Big\{m'\varphi(m,m') + m''\varphi(m,m'') + m'''\varphi(m,m''') + \cdots\Big\} \\ & + \ m'\Big\{ \qquad m''\varphi(m,m'') + m'''\varphi(m,m''') + \cdots\Big\} \\ & + \ m''\Big\{ \qquad m'''\varphi(m,m''') + \cdots\Big\} \\ & + \ \cdots \\ & + \ m\Big\{M\Phi(m,M) + M'\Phi(m,M') + M''\Phi(m,M'') + \cdots\Big\} \\ & + \ m'\Big\{M\Phi(m',M) + M'\Phi(m',M') + M''\Phi(m',M'') + \cdots\Big\} \\ & + \ m''\Big\{M\Phi(m'',M) + M'\Phi(m'',M') + M''\Phi(m'',M'') + \cdots\Big\} \\ & + \ \cdots \end{aligned}$$

The function Ω is expressed by the following sequence :

$$\begin{aligned} \Omega = \sum m\{-gz + \frac{1}{2}m'\varphi(m,m') + \frac{1}{2}m''\varphi(m,m'') + \frac{1}{2}m'''\varphi(m,m''') + \cdots \\ +M\Phi(m,M) + M'\Phi(m,M') + M''\Phi(m,M'') + \cdots \end{aligned}$$

where, $\prec$ *characteristic* $\Sigma \succ$ represents the expression of sum, in which $m', m'', m''', \cdots$ follow by permuting cyclically after m's term.

E.3. The sum of force : Ω.

If we locate the discrete points $M, M', M'', \cdots$, and assume the continuous corpus extending in the space S, and C is the uniformalized density, then the sum

$$M\Phi(m,M) + M'\Phi(m,M') + M''\Phi(m,M'') + \cdots$$

is transformed into the integral

$$C\int dS.\Phi(m,dS)$$

in the total space S, in which we denote the second analogy with (m, dS), which means the distance from the point m to the arbitrary points in the space S, and we call its element dS.

In addition, if we locate the discrete points $m, m', m'', \cdots$, and assume the continuous corpus extending in the space s, and the density is uniformly c, then we get the sum :

$$-gz + \frac{1}{2}c\int ds.\varphi(\mu, ds) + C\int dS.\Phi(\mu, dS)$$

where, z is the altitude of the point μ in the hyperplane H, in addition, we integralate the first integral, over the total space s and the second integral, over the space S. By the following expression :

$$\Omega = c\int ds.[ds],$$

we integrate over the total space s. For brevity, we express :

$$\boxed{\Omega = -gc\int zds + \frac{1}{2}c^2\iint ds.ds'.\varphi(ds,ds') + cC\iint ds.dS.\Phi(ds,dS) \qquad (180)}$$

where, s, s' are specially denoted spaces (satisfied with the mobile material), however with the duplex integration[93], integrate twice with the element to resolve it.
($\Downarrow$) Here the integral (180) contains sextuple integral when using both (178) and (179). ($\Uparrow$)

Poisson cites Gauss' *minimum* denoted by Ω in (180) in his preface of [62] and states :

> Depuis que cet ouvrage est écrit, j'ai connaissance d'un Memoire de M.Gauss, qui paraît en ce moment sous le titre de *Principia generalia theoriæ figuræ fluidorum in statu oequilibri*[94] . Pour former les équations de cet équilibre, l'auteur a recours au principe des vitesses virtuelles, qu'il applique à la masse entière du liquide, et non pas, comme dans la *Mécanique analytique*[95], à une élement differentiel de cette masse. Il trouve, de cette maniére, qu'une certaine intégrale sextuple, étendue à toute cette masse, doit être un *minimum.* Dans le cas d'un liquide homogène et incompressible, il réduit d'abord cette quantité à une intégrale quadruple ; et en considérant spécialment le cas où les forces appliquées au liquide sont la pesanteur et l'attraction mutual de ses molécules, dont la sphère d'activité est insensible, il réduit *de nouveau* la quantité dont il s'agit, qui est ensuit composée de trois termes, savoir,
> (1) le produit du poids du liquide et de l'ordonnée verticale de son centre du gravité,
> (2) l'aire de sa surface libre multipliée par une constante[96] qui ne dépend que de la matière du liquide,
> (3) et l'aire des parois fixes contre lesquelles il s'appuie, multipliée par une seconde constante[97] de la matière du liquide et de celle de la partie solide du système.
>
> [62, pp.7-8]

E.4. **The** *characteristics, indoles* **of fluid.**

The $\prec$ *characteristics* $\succ$ ($\prec$ *indoles* $\succ$) of fluid consists of the perfect mobility, for example, in the minimum particles, however the figure were big, it can be induced to any size, or minimum potential, the mutual figure depends on each changing. In unexpansible fluid (the liquid), which we called in our discussion, the volume of this particle keep to be constant due to the all movable figure. Consider that the following motion of this fluid

- which is limited by the solid corpus (the vase),
- and which are obeyed by the attraction between the mutual particles,
- the attraction between the particles of fluid,
- and the attraction between the particle of fluid and that of the vase,
- the status of equilibrium,
- and value of this Ω, when Ω is maximum, etc.
- and without infinite transportation between the particle of fluid, this Ω can induce positive increment.

Why this Ω can get the value, as long as such as :

- how long the period the figure,
- what sort of fluid satisfy it,
- moved (only by the interior fluid),
- accepting the equilibrium,
- how many times Ω for zero bring up the infinitesimal motion with the figure of vase.

Therefore, here, we consider that, if we can assume the figure does not move at all, (the vase which the fluid is contained, is along and tangential in everywhere), the force can not move in the fluid the interior of the fluid, if the equilibrium is holds by itself.

E.5. **The expression of Ω : the fundamental theory of fluid equilibrium.**

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[93]($\Downarrow$) In below, Gauss uses "duplex" not only as both P and U, but also as two triangles.

[94]Gottingue, 1830. This is commented by Poisson [62].

[95]($\Downarrow$) cf. Lagrange [31].

[96]($\Downarrow$) It means c in the second term of(180).

[97]($\Downarrow$) It means C in the third term of (180).

We would like to proceed to precisely investigate the expression of Ω, which we must consider as if the fundamental of the theory of fluid equilibrium.

(Latin) $\Rightarrow$ Progredimur ad accuratiorem investigationem expressionis Ω, quae tamquam fundamentum theoriae aequilibrii floidorum considerari debet.

We would like to take up, at first,

- the term $\int zds$: the production made by the volume of the space s at the altitude of the central gravity of the surface plane H.
- In addition, $gc\int zds$: the production of mass at the altitude of the fluid.

Hence, thus fluid particles does not obey the other force except for the gravity, in the state of equilibrium, the center altitude of the gravity becomes minimum as possible as, and therefore, we get easy the liberal part or liberal parts of surface, the part of the horizontal plane in the one same place, it becomes the surface and boundary of fluid.

E.6. **Transformation of the expression and the definition of s, S, φ, Φ.**

We take the transformation as follows :

- of the second and third terms to two cases of the particular problem, where, proposition of the dual spaces whatever, single element of the first space with second element, we combine and product from the third factor, put from the element volume of the first space and the volume element of the second space, and the function data of the mutual distance, and then we can sum up to the last,
- the second term to the same way, where the both space is the same,
- the third to it, where all of a side of space is from the other side of space,

then, the problem is completed. The two different cases are completed, namely

- when one side of space is part of the other side of space,
- or when each side has the common part with the other part.

Although, moreover, the first case is sufficient to institute us, or we can easy return the rest to the other side, when the work evaluate, the problem in itself complete by accepting the general sign.

In this problem, we denote the spaces by s and S, the function on distance denoted with the $\prec$ *characteristic* $\varphi \succ$, as the same as in the application to the second located term S and s of (180), and to the third located term, we may replace Φ with φ. The integral is given as follows :

$$\iint ds.dS.\varphi(ds, dS) \tag{181}$$

We would like to show that the spatial elements, depending on the three variables, which imply that *the sextuple integral are to be reduced to the quadruple integral.*

($\Downarrow$) Here the integral (181) contains triple integral when using either (178) or (179), then (180) contains sextuple integral. [98]

E.7. **Preparation for evolving the equation.**

$$\int ds.\varphi(\mu, ds)$$

where μ is the fixed point in the exterior or interior of the space s. We consider the surface of sphere with radius $= 1$ of which the center is μ.

$$d\Pi = \pm\frac{dt'.\cos q'}{r'r'} = \pm\frac{dt''.\cos q''}{r''r''} = \pm\frac{dt'''.\cos q'''}{r'''r'''} \quad \text{etc.}$$

$$\int r^2\varphi r.dr = -\varphi r$$

[98]($\Downarrow$) Poisson recognized this Gauss' achievement in [62], however he investigated this problem by his own method.

We integrate :

$$\int ds.\varphi(\mu, ds)$$

where μ is the fixed point in the exterior or interior of the space s

$$\int ds.\varphi(\mu, ds) = d\Pi.(\psi r' - \psi r'' + \psi r''' + \text{ etc }) = \frac{dt'.\cos q'.\psi r'}{r'r'} + \frac{dt''.\cos q''.\psi r''}{r''r''} + \frac{dt'''.\cos q'''.\psi r'''}{r'''r'''} + \cdots$$

at the time when μ exists in the exterior of the space s :

$$\begin{aligned}
&\int ds.\varphi(\mu, ds) \\
&= d\Pi.(\psi 0 - \psi r' + \psi r'' - \psi r''' + \text{ etc }) \\
&= d\Pi.\psi 0 + \frac{dt'.\cos q'.\psi r'}{r'r'} + \frac{dt''.\cos q''.\psi r''}{r''r''} + \frac{dt'''.\cos q'''.\psi r'''}{r'''r'''} + \cdots
\end{aligned}$$

at the time when μ exists in the interior of the space s.

When we take the sum by the arbitrary surface of the spherical part, we get the integral $\int ds.\varphi(\mu, ds)$ is completed, then

$$\int ds.\varphi(\mu, ds) = \begin{cases} \frac{dt.\cos q.\psi r}{r^2} & \text{in the first case} \\ 4\pi\psi 0 + \frac{dt.\cos q.\psi r}{r^2} & \text{in the second case} \end{cases}$$

where

- dt : the infinite arbitrary elements on the surface of space s,
- q, r : these are the values underlined in the previous pages about the determinate expressions, with respect to the element of r,
- π : $\frac{1}{2}$ of the circumference of circle with its radius $= 1$.

We see easy the rest, if the point μ is neither interior, nor exterior of the space s, or in the surface of these, to satisfy the secondary formula, the factor will move 4π in 2π, even if the surface in the point μ were given neither as the cusp nor as the aciform [99] type ; however, by our proposition, it is not at all necessary to satisfy this case.

E.8. **Evolution of equation $\iint ds.dS.\varphi(ds, dS)$.**

By the discussion in the previous article, the evolution of equation $\iint ds.dS.\varphi(ds, dS)$ reduced to

$$4\pi\sigma\psi_0 + \iint dt.dS.\frac{\cos\ q.\psi(dt, dS)}{(dt, dS)^2}$$

where σ denotes volumes of these spaces, is common in both space s, S, if s, S alternate mutually, the first term $4\pi\sigma\psi_0$ vanishes. New integral seems duplex in appearance, however, it turns to quintuplex. When we reduce to the quadruple, we must consider the integral

$$\int dS.\frac{\cos\ q.\psi(\mu, dS)}{(\mu, dS)^2}$$

by the arbitrary elements of the space S are extended, denoting again μ fixed point, and q : angle between two straights ($0 \leq q \leq \pi$) emitting from this point. Others are easily perspective, if the point μ is only exterior or interior of the space s, evaluate the secondary formula, move the factor 4π to 2π, and then if our propositions are not useful for you, please read the following cases.

$$d\Pi = -dT''.\cos\chi' = dT'''.\cos\chi'' = -dT''''.\cos\chi''' \quad \text{etc.}$$

$$\int \frac{dr.\psi r}{r^2} = -\theta r$$

here, accepting arbitrary the integral constant, our integral of the interior space S of prism,

$$\begin{aligned}
&= d\Pi.(\theta R' - \theta R'' + \theta R''' - \quad \text{etc}) \\
&= -dT'.\cos\chi'\theta R' - dT''.\cos\chi''\theta R'' - dT'''.\cos\chi'''\theta R''' - \quad \text{etc}
\end{aligned}$$

$$\int dS.\frac{\cos q.\psi(\mu, dS)}{(\mu, dS)^2} = -\int dT.\cos\chi.\theta R$$

[99](⇓) E.g. A needle, a pin, a sting, etc.

$$4\pi\sigma\psi_0 - \iint dt.dT.\cos\chi.\theta(dt, dT)$$

where χ indicating the mutual inclination of the element dt, dT, by the normal-direction, which is measured by the outer direction to the space s, S, which the integral by the complete surface, of which the space can be extended.

E.9. The three cases of integral.

As the same as the previous method, the division of space S in the element of prism depending is, thus the second method is necessary for the same division of space S in the element of prism. We consider that :

- at first, the surface of the sphere of the radius = 1, and around the center μ, are described with the infinitesimally small elements divided ;
- next, toward points, these element $d\Pi$ draw the straight line to the point μ, and this surface of the space S are cut at the points $P', P'', P''', \cdots$;
- then, we denote the distances between these points $P', P'', P''', \cdots$ and μ by $R', R'', R''', \cdots$;
- finally, the straight line at μ toward all points on the peripheral elements $d\Pi$ in the form of pyramidal shape, and among $P', P'', P''', \cdots$ cut the elements from the surface space S, and we designate these elements with $dT', dT'', dT''', \cdots$.

Moreover, we assume Q' inner straight line $P'\mu$ then normal in the elements dT' extend exterior and $Q'', Q''', \cdots$ have the inclination of similar normal in the same way, drawn from the straight line toward μ. Therefore we put

$$d\Pi = \pm\frac{dT'.\cos Q'}{(R')^2} = \mp\frac{dT''.\cos Q''}{(R'')^2} = \pm\frac{dT'''.\cos Q'''}{(R''')^2} = \cdots$$

where the sign changes superior or inferior, according to that the line $\mu P'$ take interior or exterior of the space S. [100]

Then, it seems clear that for all partial spaces of S, inside of its pyramidal space, the angle q is constant, we deduce as if it were the same as in article 7, if we would set indefinitely,

$$\int \varphi r.dr = -\theta r$$

if we assume the integral constant as arbitrary, the integral

$$\int \frac{dS.\cos\ q.\psi(\mu, dS)}{(\mu, dS)^2}$$

(I) In the case of the point μ existing in the exterior of space S :

$$\int \frac{dT.\cos\ q.\cos Q.\theta R}{R^2}$$

(II) In the case of the point μ existing in the interior of space S :

$$\theta_0.\int d\Pi.\cos q$$

(III) In the case of the point μ existing on the surface of space S :

$$\theta_0.\int d\Pi.\cos q$$

$$\cos q = \cos k.\cos v + \sin k.\sin v.\cos w$$

Integral $\int d\Pi.\cos q$ becomes

$$\begin{aligned}&\int_{\frac{\pi}{2}}^{\pi} dv \int_0^{2\pi} dw\Big(\cos k.\cos v + \sin k.\sin v.\cos w\Big)\sin v \\ = &\int_{\frac{\pi}{2}}^{\pi} 2\pi\cos k.\cos v.\sin v.dv = -2\pi\cos k\Big[\frac{1}{2}\sin^2 v\Big]_{\frac{\pi}{2}}^{\pi} = -\pi\cos k\end{aligned}$$

Applied to our first integral $\iint ds.dS.\varphi(ds, dS)$ of (181), then

100(⇓) cf. (211).

- (I) If the surface space s, S do not have common part, then
$$4\pi\sigma\psi_0 + \iint \frac{dt.dT.\cos\ q.\cos Q.\theta(dt, dT)}{(dt, dT)^2}$$
- (II) If the surface space s, S have common part, which is $\mathcal{T}$, then
$$4\pi\sigma\psi_0 \mp \mathcal{T}\theta_0 + \iint \frac{dt.dT.\cos\ q.\cos Q.\theta(dt, dT)}{(dt, dT)^2}$$
- (III) If the surface space s, S have plural, finite and discrete common parts, then
$$4\pi\sigma\psi_0 + \pi(\mathcal{T}' - \mathcal{T})\theta_0 + \iint \frac{dt.dT.\cos\ q.\cos Q.\theta(dt, dT)}{(dt, dT)^2}$$

E.10. **Criticism of Laplace's molecular calculation of capillarity equations.**

• We are almost ready to introduce two transformations of the integral $\iint ds.dS.\varphi(ds, dS)$ in the articles 8 and 9, by praising ourselves, even the equations were evolved, we may apply each time our proposition to it.

• Here, the function φ is used originally as the function f, for the further study built on the hypothesis, on which Mr. Laplace studies, says that the force of molecular activity are more finite in the infinitesimal distance. This phrase when the liquid move adhering, how long keeps the uniformity, under everybody can observe it, the attractive activity fr, expressed by the function of distance r, and since he treats the gravity g as homogeneous, which is due to liquid mass ; this is a defect of his supposition. and denoting the liquid mass by M, whatever we can try in the experiment, and he says almost the same as nothing with respect to every part of media.

• Mfr in the infinitesimal distance is not only finite, but also even r can be decreased over an arbitrary boundaries.

• Without theory and the policy to investigate that the gravity comes from the hypothesis, in the other point, the law of the function fr, as the same as the unknown in general, which we can not help making a mistake about the mathematical $\prec$ *character* $\succ$, look like peculiar : namely, as long as even the fact, standing on the most precise mathematics, can not punish himself, if so, so much as the mathematical precision, more, even without the experiments, we can get the absolute level of value ; without an experiment (or proof), none is free from the amusement by oneself in seeking after absolute truth ; if you would success, withdraw your supposition itself.

101

E.11. **Function φr as the constant of integral $\int fr.dr$.**

($\Downarrow$) Remark. below, $\varphi r \equiv \varphi(r), \varphi 0 \equiv \varphi(0), fr \equiv f(r)$ like the function in (178) and (179).($\Uparrow$)

• Even if we suppose the function denoting by fr (or the function by Fr) of attraction, that the fact that the relation is proportional reciprocally with inverted r^2, had been proved in the astronomy, if the figure between the fluid and a vessel, in any infinitesimal particle, the gravity can also affect to the modification. r increasing in even infinitesimal, fr turns into, by itself, infinitesimal, but also more rapidly decreased rather than $\frac{1}{r^2}$.

• Hence, we can make a deduction from here as follows : even the integral $\int fr.dr$ in everywhere, it is finite, turns into infinitesimal, then that the constant of integral $\int fr.dr = -\varphi r$, is supposed to be acceptable and have $\varphi\infty = 0$, if φr this value of integral $\int_r^\infty fx.dx$ is extended.

• In any way, φr the distance denoting positive quality by r, not only infinitesimal, but also finite r ; continues to decrease with respect to the distance r, it can go beyond the arbitrary boundary, generally speaking, if there is non-obstacle, then $\varphi 0 = \infty$.

101($\Downarrow$) Navier cites the molecular theory by Laplace and chooses consistently repulsive force in Navier's papers [46, 47] as the function depending on the distance between molecules, however, N.Bowditch [102] points out that Laplace rethinks the repulsion theory and changes it, in 1819 : $\varphi(f) = A(f) - R(f)$, where $\varphi(f)$: a function depending on the distance f between the molecules, $A(f)$: attractive force, $R(f)$: repulsive force.

E.12. **The difficulty of calculating $\int r^2\varphi r.dr$.**

($\Downarrow$) Remark. below, $\varphi r \equiv \varphi(r)$ like the function in (178) and (179).($\Uparrow$)

• Hence, since the function φr, in everywhere, instead of the finite value of r it turns into infinitesimal, and increasing r continues to decrease, $\int r^2\varphi r.dr$ always allows to extend finitely to an arbitrary big value, and moreover keep infinite, then as long as the latter, whatever we are ambitious, even if any experiments can teach us, it is just that : about how to make the infinitesimal integral, even by the big interval, in the case which we were unsuccessful in integral.

> • The very calculations by Mr. Laplace show us all these situations, in which my supposition is included ; since nature of the unknown function φr is suggestive, and using it, we can supersede it or abstain from it to many suppositional hypotheses.

• This constants of integral $\int r^2\varphi r.dr = -\psi r$ are able to be determined as we choose it, to make $\psi r = 0$, for the value of fluid with the finite distance of r, moreover, by its experiment, we can afford to get the length of circumference of the body.

• Hence, ψr for all this sort of value will be always finite (positive for minimum, negative for maximum), speaking in general, if there is non-obstacle, then for the infinitesimal value of r, we can convert to the finite value : although ought to add, we give an explanation to the phenomenon, as the decreasing distance r in infinitesimal, the value ψr itself means always as finite, as long as $\psi 0$ depends on the finite quantity.

• Besides these, $\frac{c\psi r}{r}$ is the quantity when the gravity is homogeneous, $\frac{c\psi r}{g}$ is linear, especially, $\frac{c\psi\theta}{g}$ is already-known-linear (for natural body, in this case, the function fr is useful for the force of the attractive activity), of which the magnitude may be very suspect, however, in the known case, it is an almost-approximate value, except for suppositional hypothesis.

E.13. **Proof of that $\frac{\theta_0}{\psi_0}$ is linear in insensible magnitude and its avoidance.**

($\Downarrow$) Remark. below, $\psi r \equiv \psi(r), \psi 0 \equiv \psi(0), \theta r \equiv \theta(r)$ like the function in (178) and (179).($\Uparrow$)

We consider the completely equivalent integral :

$$\int \psi r = -\theta r \quad \Rightarrow^* \quad \int \psi(r) = -\theta(r) \tag{182}$$

Here, we suppose that :

- by choosing the constants, $\theta r = 0$ for an arbitrary value r, for an arbitrary sensible between its interval, on which the experiment tells us of the fact,[103] for this, we can set how we get the way insensible θr is for any sensible value r in everywhere, even if it evaluate sensible for the insensible value.
- We assume $\frac{c\theta r}{g}$ explains the area of two-dimensional figure, in particular, $\frac{\theta r}{\psi r}$ is linear.
- Naturally, another $\frac{\theta_0}{\psi_0}$ is linear in insensible magnitude, which we prove as follows.

When ψr continues decreasing from $r = 0$, and certainly, such as, insensible have gone, as soon as r get sensible value, for $\psi r = \frac{1}{2}\psi_0$, [104] must be insensible : denote this value of r by ρ. We would consider the integral $\int(\psi_0 - \psi r)dr$, which we integrate it from $r = 0$ to $r = R$, it becomes from (182),

$$\int_0^R (\psi_0 - \psi r)dr = \Big[\psi_0 r + \theta r\Big]_0^R = R\psi_0 - \theta_0 + \theta R. \tag{183}$$

Cleary, this integral more greater, when it is integrated from $r = \rho$ to $r = R$, the extension becomes at any times greater than the integral $\int(\psi_0 - \psi\rho)dr$ between the same limits. The last integral becomes

$$\int_\rho^R (\psi_0 - \psi\rho)dr = (\psi_0 - \psi\rho)(R - \rho) = \frac{1}{2}\psi_0.(R - \rho) \tag{184}$$

which is generalized for this value $R(> \rho)$ from (183) and (184),

$$R\psi_0 - \theta_0 + \theta R > \frac{1}{2}\psi_0.(R - \rho)$$

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[103]($\Downarrow$) Gauss cites frequently the word "experiment", for example, such as ¶4 in Preface, or §E.15, §E.18.

[104]($\Downarrow$) Which we say a half-life of the radiation.

If $R = \frac{\theta_0}{\psi_0}$, and moreover, if R is a sensible quantity, then

$$\frac{\theta_0}{\psi_0}\psi_0 - \theta_0 + \theta R = \theta R > \frac{1}{2}\psi_0.(R - \rho)$$

This expression becomes absurd and invites contradiction. ($\Downarrow$) $\lim_{R\to\infty} \theta(R) = 0.$($\Uparrow$) $\square$

> Solving method : If we can not avoid this tremendous magnitude of ψ_0, by cutting only zero, θ_0 is possible to be the usually sufficient quantity and to be comparable with the dimension of body in carrying out an experiment. (If so, we get the same situation as a usual condition of experiment)

E.14. **Integral (I) and (II).**

Moreover, that comes from this $\prec$ "*indole*" (characteristic) function : $\theta \succ$ with respect to the integral (I)

$$\text{integral (I)}: \quad \iint \frac{dt.dT \cos q.\cos Q.\theta(dt, dT)}{(dt, dT)^2},$$

($\Downarrow$)where, $(\bullet, \bullet)^2$ means the square of distance between them. ($\Uparrow$) We would like to investigate this integral (I), starting with the simplification of it, to be able to alternate the surface points μ, considering specially the integral (II)

$$\text{integral (II)}: \quad \int \frac{dt.\cos q.\cos Q.\theta(\mu, dt)}{(\mu, dt)^2}$$

by all the surface : t, we consider to extend it. We denote the following :

- Q the angle between two straight lines emitting from the point μ,
- the second toward the element dt,
- the second toward the fixed point;

similarly,

- q the angle between two straights emitting from the point dt
- the second toward the element μ,
- the second normal element toward the exterior direction

Then

- at first, we observe, if point μ is sensible in the distance on the surface : t, all value $\theta(\mu, dt)$ is insensible : in this case, total integral (II) are insensible. Here we can get sensible value in this integral, how long we can extend the surface t in insensible distance at point μ, clearly enough the integral (II) by this part, all neglected, that is sensible in distance.
- Next, instead of $\frac{dt.\cos q}{(\mu,dt)^2}$ of (II), we replace by $\pm d\Pi$, and denoting $d\Pi$ on the surface of the sphere with the radius $= 1$, with the center : μ, the description of element id, in which the element dt of the exterior or interior plane, direct the point μ.

Hence, we get the integral (II) as follows

$$\int \frac{dt.\cos q.\cos Q.\theta(\mu, dt)}{(\mu, dt)^2} =^* \int \pm d\Pi.\cos Q.\theta(\mu, dt), \quad \text{where,} \quad \pm\, d\Pi = \frac{dt.\cos q}{(\mu, dt)^2}$$

here it is clear that this integral refers to the value as long as it is sensible, in respect to all the elements $d\Pi$, at the insensible distance (μ, dt), then the sensible magnitude of space on the surface covers the sphere. We consider the following three cases.

(1) In which, the radius of curvature of the surface t is infinitesimal at the point μ.
(2) In which, the continuous curvature at the point μ which the inner distance is infinitesimal. (cf. [15, art.3]).
(3) In which, the radius : of curvature of the surface t is open at the point of μ.

We would like to treat other reserving problems in the following article.

E.15. **Integral (II).**

(⇓) Remark. below, $\theta r \equiv \theta(r)$, $\quad \theta' r \equiv \theta'(r)$ like the function in (178) and (179).(⇑)

It is the most clearest case that every point μ is not in the surface t, however, in the sensible distance from here : in this case of our integral, it is possible to have sensible value, which we would like to investigate precisely.

- At first, splitting a surface on the sphere, to a point μ made normal in surface t, moreover, we draw the fixed straights emitting from both the points G and H respectively;
- next, we assume the arc $GH = k$,
- then, as another arc made of G and an arbitrary point, assuming $G\bullet = v$ on the surface of the sphere ;
- finally, we assume the surface angle w made of arcs k and v.

Here, the method for the element $d\Pi$ is possible to admit the product $\sin v.dv.dw$, and we call the distance (μ, dt) r briefly. Then the integral (II) turns into :

$$\int dv \int dw \Big[\pm (\cos k.\cos v + \sin k.\sin v.\cos w)\theta r.\sin v \Big] \tag{185}$$

We denote this minimum distance with ρ (at this point G, it correspond to $v = 0$), $r = \frac{\rho}{\cos v}$, when $v = 0$, then $r = \rho$, if w is independent of it. When we integrate (185) with respect to w, from $w = 0$ to $w = 2\pi$, then

$$\begin{aligned} &\int dv \int_0^{2\pi} dw \Big[\pm (\cos k.\cos v + \sin k.\sin v.\cos w)\theta r.\sin v \Big] \\ &=^* \ \pm \int 2\pi\theta r.\cos k.\cos v.\sin v.dv \\ &=^* \ \pm \int 2\pi \cos k.\theta r.dr \Big(\frac{\rho}{r}\Big)\Big(\frac{\rho}{r^2}\Big).dr \\ &= \ \pm \int \frac{2\pi\cos k.\rho^2\theta r.dr}{r^3} \end{aligned}$$

(⇓) where, we used

$$r = \frac{\rho}{\cos v} \quad \Rightarrow \quad \cos v = \frac{\rho}{r} \quad \Rightarrow \quad \sin v.dv = \frac{\rho}{r^2}.dr \quad (⇑)$$

Here, we consider this integral as the interval from $r = \rho$ to an arbitrary sensible, however, small value, then

$$\pm \int \frac{2\pi\cos k.\rho^2\theta r.dr}{r^3} = \pm\pi\cos k\Big(2r^2 \int \frac{\theta r.dr}{r^3}\Big)$$

We consider generally :

$$2r^2 \int \frac{\theta r.dr}{r^3} = -\theta' r, \qquad \int \frac{\theta r.dr}{r^3} = 0, \tag{186}$$

for an arbitrary sensible as its interval, on which the experiment tells us of the fact, we neglect the insensible terms then the integral (II) :

$$\pm\pi\cos k\Big(2r^2 \int \frac{\theta r.dr}{r^3}\Big) = \pm\pi\cos k.\theta'\rho \tag{187}$$

If it seem to be doubtful, or to be right, we have the partial surface t inter insensible distance, to the point μ position for the plane, and consider this location of the sphere, and R the distance from the center of the sphere to the point μ taking as positive or negative, according to whether the center is in the direction toward G or in opposite direction.

Hence, we get the followings :

- $$\begin{cases} \cos v = \frac{\rho}{r}(1 - \frac{\rho}{2R}) + \frac{r}{2R} \\ \sin v.dv = [\frac{\rho}{r^2}(1 - \frac{\rho}{2R}) + \frac{1}{2R}]dr \end{cases}$$

  where, if the mode R is a sensible quantity, we can see easily that the integral for this case, is not different with the above-mentioned in (187), sensible quantity about the value, $\pm\pi\cos k.\theta'\rho$.

- Another is the curvature of the surface t in its part, come from this, as long as the radius of curvature is insensitive, always we can assign the dual surface of a sphere, surface t in this point μ, the nearest point by tangential angle, inter this t set,
- and these radii are sensible magnitudes, clearly, then our integral inter integral fall into the related surface,
- and therefore, we could explain without sensible error, by the same formula,
- which, not only above things, but also we would suffer from the exceptions, when the surface t in the insensible distance to the point μ, would offer even the curvature of insensible radius, or aciform type, or the cusp [105].

E.16. **Reduced integral from sextuple to quadruple.**

Therefore it is clear that the transform come out from the integral (II) to integral (I), here insensible occure not only in this case, but also when the sensible value is produced for null point of the surface T, but also when the complex element of the surface T, for which points the integral (II) becomes sensible, the area consists of also insensible magnitude. Which are considered rightly, the integral (I) will appear, how much is able to acquire the sensible value, how long be able to keep the partial surface T or partial sensible magnitude in the insensible distance to the positive surface t.

Our integral (I) neglecting the insensible factors :

$$= -\int \pi\theta'\rho.d\tau + \int \pi\theta'\rho.d\tau'$$

Clearly this is not important, either the parts τ and τ' or to the surface T to t is rather important. The value of (181) becomes

$$\underbrace{\iint ds.dS.\varphi(ds, dS)}_{\text{triple integral}} = 4\pi\sigma\psi 0 - \pi\mathcal{T}\theta 0 + \pi\mathcal{T}'\theta 0 - \pi\int d\tau.\theta'\rho + \pi\int d\tau'.\theta'\rho \tag{188}$$

($\Downarrow$) Just this transformation is boastful reducible method of integral from the sextuple to quadruple, what is called by Gauss in (181).

E.17. **Method of reduction of $\iint ds.dS.\varphi(ds, dS)$ from sextuple to quadruple.**

- Therefore, we can assume the primitive function θ' of (186), i.e.

$$2r^2\int \frac{\theta r.dr}{r^3} = -\theta' r \quad \Rightarrow^* \quad \frac{\theta' r}{r^2} = \int \frac{2\theta x.dx}{x^3} \tag{189}$$

- We consider the integral from $x = r$ to an arbitrary, sensible and constant value, denoted by R. Namely we integrate the following : [106]

$$\int_R^r \frac{2\theta x.dx}{x^3} = \frac{\theta r}{r^2} - \frac{\theta R}{R^2} \tag{190}$$

Clearly this integral is smaller than this $\int \frac{2\theta x.dx}{x^3}$ with the interval, this is $= \frac{\theta r}{r^2} - \frac{\theta R}{R^2}$. Moreover, it is smaller than $\frac{\theta r}{r^2}$. Otherwise, by infinite integral, it become as follows :

$$\int \frac{2\theta x.dx}{x^3} = -\frac{\theta x}{x^2} + \int \frac{d\theta x}{x^2} = -\frac{\theta x}{x^2} - \int \frac{\psi x.dx}{x^2} \tag{191}$$

Moreover, from (189), (190) and (191),

$$\frac{\theta' r}{r^2} = \int \frac{2\theta x.dx}{x^3} =^* \Big[-\frac{\theta x}{x^2} - \int \frac{\psi x.dx}{x^2}\Big]_{x=r} = \Big(\frac{\theta r}{r^2} - \frac{\theta R}{R^2}\Big) - \int \frac{\psi r.dx}{r^2} =^* \Big(\frac{\theta r}{r^2} - \frac{\theta R}{R^2}\Big) - \frac{\psi r}{r} \tag{192}$$

- Integrating with the smaller interval than the integral $\int \frac{\psi x.dx}{x^2}$. Moreover, from (192), this is smaller than $\frac{\psi r}{r}$; therefore, the value of $\frac{\psi' r}{r^2}$ is greater than the right-side expression of (193) [107]

$$\frac{\theta' r}{r^2} = \Big(\frac{\theta r}{r^2} - \frac{\theta R}{R^2}\Big) - \frac{\psi r}{r} \quad \Rightarrow \quad \theta' r = \theta r - \frac{r^2.\theta R}{R^2} - r\psi r \tag{193}$$

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[105]($\Downarrow$) cf. the footnote above in the last line of § E.7.

[106]($\Downarrow$) This function is rapidly decreasing function. Here, θr, θR mean $\theta(r)$, $\theta(R)$ and are assumed as $\theta(r) > \theta(R)$.

[107]($\Downarrow$) Multiplying by r^2, which is infinitesimal value. Today's description of (193) is $\theta'(r) = \theta(r) - \frac{r^2.\theta(R)}{R^2} - r\psi(r)$.

From (193), the interval of $\theta' r$:

$$\theta r \quad \text{and} \quad \theta r - r^2.\frac{\theta R}{R^2} - r\psi r =^* \theta' r$$

• If we differentiate this expression, by r decreasing infinitely, then we see clearly that we can evaluate this quantity to be infinitesimal, for example, when ψ_0 in (188) is the finite quantity. Thus we have concluded that it is due to $\theta_0' = \theta_0$. We see clearly that the formula (188) of previous art.16 (§E.16) turns into

- $-\pi \mathcal{T}\theta_0$ and for instance, under the interval $-\pi \int d\tau.\theta'\rho$
- $\pi \mathcal{T}'\theta_0$ and for instance, under the interval $\pi \int d\tau'.\theta'\rho$,

if the difference or the distance is insensible or considerable as null, to count respectively the part of $\mathcal{T}$, $\mathcal{T}'$ or τ, τ'.

> By using this method of solution, we can cultivate the elegant mathematical sense, however we must surpass to conserve the distinction of our proposition.

E.18. **Variation problem to be solved.**

In the application of previous survey to the evolution the second term of the expression Ω in §E.3 and §E.6 denote by S in § E.20 $\sigma, \mathcal{T}, \mathcal{T}'$ will be use as $s, t, 0$, if t is the total surface of the space s, in which the fluid is filled. Therefore whenever this space extensional sensible part however insensible concentration is kept, this sort of gap (crevice), the second part of the expression Ω of (180) in the art. § E.3 becomes

$$= \frac{1}{2}\pi c^2(s\phi 0 - t\theta 0)$$

We assume the exceptions as follows :

(1) the space s contains the insensible part of the thickness, and this surface offers the dual sensible part of the liquid,
   - in which we denote the alternative t',
   - thick space in the neighborhood of the infinite elements : dt' by ρ,
   - by accepting the expression above terminology,

$$\pi c^2 \int \theta'\rho.dt'$$

(2) We put the $\prec$ *characteristic* $f \succ$ for the force of molecular attraction and $\prec$ *characteristic* $\succ$ F. The relation with the vase ought to yield oneself to the attractive force, we denote the functions by the $\prec$ *characteristic* $\succ$ with $\phi, \psi, \theta, \theta'$ and similarly with $\Phi, \Psi, \Theta, \Theta'$ applying the same relation between F and f. The third part of the expression Ω becomes generally speaking :

$$\pi cCT\Theta 0$$

(3) If in the neighborhood of the sensible part T' of the surface T have the thick of fluid, we denote the next term, in which infinite thick of fluid by ρ, as we accept from the experiments

$$-\pi cCT\Theta'\rho.dT'$$

(4) If the surface of the vase is contiguous except for the part T, we offer T'' in the distance we denote the next term, in which by ρ indefinite distance for points in anywhere,

$$+\pi cCT\Theta'\rho.dT''$$

In static equilibrium it is due to the maximum value, this turns into

$$\boxed{-gc\int zds + \frac{1}{2}c^2 s\psi_0 - \frac{1}{2}\pi c^2 t\theta_0 + \pi cCT\Theta_0}$$

In an arbitrary fluid, of which the figure is yield oneself to the space s meaning invariant, of which the expression becomes as follows :

$$\int zds + \frac{\pi c\theta_0}{2g}.t - \frac{\pi CT\Theta_0}{g}.T$$

and in an equilibrium state which is due to *minimum*. Here, we denote

$$\frac{\pi c\theta_0}{2g} \equiv \alpha^2, \quad \frac{\pi C T\Theta_0}{2g} \equiv \beta^2, \quad t \equiv T + U \tag{194}$$

and denoting by W, then

$$\boxed{W \equiv \int z ds + (\alpha^2 - 2\beta^2)T + \alpha^2 U} \tag{195}$$

E.19. **Decomposition of variation of W.**

The first term of the variation of W by (195) is as follows :

$$ahdh + a'h'dh',$$

and T of the second term :

$$bdh + b'dh'.$$

The last term of the variation of W by (195)

$$dU \equiv 0$$

Then from (195) and above three conditions, we get dW as follows :

$$\boxed{dW = ahdh + a'h'dh' - (2\beta^2 - \alpha^2)(bdh + b'dh')}$$

Moreover, for the volume of the integral of fluid is invariant, then

$$adh + a'dh' = 0$$

$$dW = dh\Big[a(h - h') - (2\beta^2 - \alpha^2)(b - \frac{ab'}{a'})\Big]$$

$$h - h' = (2\beta^2 - \alpha^2)(\frac{b}{a} - \frac{b'}{a'})$$

We can assume $\frac{b}{a} >> \frac{b'}{a'}$ in comparison with $\frac{b}{a}$, then

$$h - h' = (2\beta^2 - \alpha^2)\frac{b}{a}$$

We get the maximum height h :

$$\boxed{h = (2\beta^2 - \alpha^2)\frac{b}{a}}$$

Then

$$h' = (2\beta^2 - \alpha^2)\frac{b'}{a'}, \quad h'' = (2\beta^2 - \alpha^2)\frac{b''}{a''}, \quad \cdots$$

E.20. **Geometric structure for analysis.**

Moreover, now, with theorem in §E.18, we would like to determine the $\prec$ "*indoles*" $\succ$ (characteristics) of the figure in equilibrium, these problems are changed in evolution of the general variation, expressed with W, if the motion of the figure of the space filled with a fluid occured in only infinitesimal. If when we variation calculation of the duplicated integral for case, then even the boundary as if the variable insufficiently investigated, we could approach this precise survey to a little profound.

We consider the following :

- the surface, denoted by s
- a part U, on which all the points is determined by the coordinate x, y, z, these three values are the distances to an arbitrary horizontal plane.

It is capable to recognize z is, for example, as the indetermate function by x, y, for these secondary partial differential with a conventional method, by omitting a bracket, we show it by

$$\frac{dz}{dx}.dx, \quad \frac{dz}{dy}.dy$$

($\Downarrow$) These descriptions by Gauss mean as follows :

$$\frac{dz}{dx}.dx \equiv \left(\frac{dz}{dx}\right)_x = \frac{d^2z}{dx^2}, \quad \frac{dz}{dy}.dy \equiv \left(\frac{dz}{dy}\right)_y = \frac{d^2z}{dy^2}, \quad \frac{dz}{dx}.d'x \equiv \left(\frac{dz}{dx}\right)_{x'} = \frac{d^2z}{dxx'}$$

($\Uparrow$)

The structure we are considering is the following :

(1) We define the points consisted of an arbitrary and every points on the surface, denoting s with respect to the rectangular surface, normal to the exterior direction of s, and in addition, we set an angle by cosine between this normal direction to the axis of rectangular coordinate x, y and z with parallel, which we denote by ξ, η and ζ. Thereby it will be :

$$\xi^2 + \eta^2 + \zeta^2 = 1, \quad \frac{dz}{dx} = -\frac{\xi}{\zeta}, \quad \frac{dz}{dy} = -\frac{\eta}{\zeta} \quad \Rightarrow^* \frac{\xi^2 + \eta^2 + \zeta^2}{\zeta^2} =^* 1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 =^* \frac{1}{\zeta^2} \tag{196}$$

(2) The boundary of surface U become linear in itself, as the same as denoted by P, and while the motion is supposed necessarily, this element dP (as the same way of dU as the surface) is treated as positive only.

(3) The angle by cosine, that directions of the element dP are expressed with the axis of coordinate of x, y, z, denoted by X, Y, Z : since we would avoid giving ambiguous sense about the direction, we define these angles as follows :

at first,

- we assume that the normal straight in the element dP to the surface U, and draw a tangent
- looking this line innerward, we draw the second side,
- at last, in the normal straight with respect to the surface, we put the third side in the space s to the exterior,

and constituting similarly the next system of three straights and the coordinate axis x, y, z.

Secondly, thus, we see easily the following expressions (cf. *Disquisitiones generales circa superficies curvas*), using the angle by cosine with the straights to the axis of the coordinates x, y, z are respectively

$$\eta^0 Z - \zeta^0 Y, \quad \zeta^0 X - \xi^0 Z, \quad \xi^0 Y - \zeta^0 X \quad \Rightarrow^* \quad \begin{bmatrix} \alpha & \beta & \gamma \\ X & Y & Z \\ \xi^0 & \eta^0 & \zeta^0 \end{bmatrix}, \tag{197}$$

here, we suppose that ξ^0, η^0, ζ^0 are the values of ξ, η, ζ for the points of the element dP.

($\Downarrow$) where, α, β, γ are temporarily used values of ours to correspond to (218). By the way, we see (197) is the same with the determinant to be mentioned again below (218).

E.21. Variation of a triangle dU of the surface U.

Here we would like to supplement the preliminary. We assume the surface U is the part by an arbitrary infinitesimal perturbation.

- If we consider sufficiently all the perturbation, for this boundary P always invariant, at any rate, it maintains, in this vertical surface, we can induce clearly the variation of only the third coordinate z, this problem is far easy to evaluate it ;
- moreover, the maximum problem in general, in the following investigating method, considering the variable boundary, in which ambiguity and difficulty combine elegantly, bring up perturbation ; how we can show, always from the start of all, three coordinates handle the variation.

We the force as we image it, and anywhere on the surface, in which the coordinates, which are x, y, z, had substituted in another, these coordinates are $x + \delta x$, $y + \delta y$, $z + \delta z$, where δx, δy, δz are able to regard as if these were the indeterminate functions of x, y, if these values stay infinitesimal. Now we would like to inquire into the variation of singular (individual) element, expressed with W and surely the initial are made of variation of these elements dU.

Now, we assume a triangle consisted of three points : $P_1,\ P_2,\ P_3$.[108] We put the element of U by a triangle dU consisted of these points, of which the coordinates are :

$$\begin{cases} P_1: & x & y & z \\ P_2: & x+dx & y+dy & z+\frac{dz}{dx}.dx+\frac{dz}{dy}.dy \\ P_3: & x+d'x & y+d'y & z+\frac{dz}{dx}.d'x+\frac{dz}{dy}.d'y \end{cases}$$

If we assume $dx.d'y - dy.d'x > 0$, then the twice area of this triangle is gained by our principle as follows :

$$(dx.d'y - dy.d'x)\sqrt{\left[1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right]} \tag{198}$$

⇓(198) becomes $\frac{(dx.d'y-dy.d'x)}{\zeta}$ from (196). (⇑)

- location value by perturbation of P_1 : $x+\delta x,\quad y+\delta y,\quad z+\delta z.$
- Location value by perturbation of P_2:

$$\begin{bmatrix} x+dx \\ y+dy \\ z+\frac{dz}{dx}.dx+\frac{dz}{dy}.dy \end{bmatrix}, \begin{bmatrix} \delta x+\frac{d\delta x}{dx}.dx+\frac{d\delta x}{dy}.dy \\ \delta y+\frac{d\delta y}{dx}.dx+\frac{d\delta y}{dy}.dy \\ \delta z+\frac{d\delta z}{dx}.dx+\frac{d\delta z}{dy}.dy \end{bmatrix}, \begin{bmatrix} (x+\delta x)+(1+\frac{d\delta x}{dx}).dx+\frac{d\delta x}{dy}.dy \\ (y+\delta y)+\frac{d\delta y}{dx}.dx+(1+\frac{d\delta y}{dy}).dy \\ (z+\delta z)+(\frac{dz}{dx}+\frac{d\delta z}{dx}).dx+(\frac{dz}{dy}+\frac{d\delta z}{dy}).dy \end{bmatrix}$$

- Location value by perturbation of P_3 :

$$\begin{bmatrix} x+d'x \\ y+d'y \\ z+\frac{dz}{dx}.d'x+\frac{dz}{dy}.d'y \end{bmatrix}, \begin{bmatrix} \delta x+\frac{d\delta x}{dx}.d'x+\frac{d\delta x}{dy}.d'y \\ \delta y+\frac{d\delta y}{dx}.d'x+\frac{d\delta y}{dy}.d'y \\ \delta z+\frac{d\delta z}{dx}.d'x+\frac{d\delta z}{dy}.d'y \end{bmatrix}, \begin{bmatrix} (x+\delta x)+(1+\frac{d\delta x}{dx}).d'x+\frac{d\delta x}{dy}.d'y, \\ (y+\delta y)+\frac{d\delta y}{dx}.d'x+(1+\frac{d\delta y}{dy}).d'y, \\ (z+\delta z)+(\frac{dz}{dx}+\frac{d\delta z}{dx}).d'x+(\frac{dz}{dy}+\frac{d\delta z}{dy}).d'y \end{bmatrix}$$

(⇓) Totally, we can see that the values of coordinate of each point are as follows :

$$\begin{cases} (P_1): x+\delta x,\quad y+\delta y,\quad z+\delta z, \\ (P_2): x+\delta x+(1+\frac{d\delta x}{dx}).dx+\frac{d\delta x}{dy}.dy,\quad y+\delta y+\frac{d\delta y}{dx}.dx+(1+\frac{d\delta y}{dy}).dy,\quad z+\delta z+(\frac{dz}{dx}+\frac{d\delta z}{dx}).dx+(\frac{dz}{dy}+\frac{d\delta z}{dy}).dy \\ (P_3): x+\delta x+(1+\frac{d\delta x}{dx}).d'x+\frac{d\delta x}{dy}.d'y,\quad y+\delta y+\frac{d\delta y}{dx}.d'x+(1+\frac{d\delta y}{dy}).d'y,\quad z+\delta z+(\frac{dz}{dx}+\frac{d\delta z}{dx}).d'x+(\frac{dz}{dy}+\frac{d\delta z}{dy}).d'y \end{cases}$$

We can also show the matrix with variation only as follows :

$$\begin{aligned} &\begin{bmatrix} \delta x & \delta y & \delta z \\ (1+\frac{d\delta x}{dx}).dx+\frac{d\delta x}{dy}.dy & \frac{d\delta y}{dx}.dx+(1+\frac{d\delta y}{dy}).dy & (\frac{dz}{dx}+\frac{d\delta z}{dx}).dx+(\frac{dz}{dy}+\frac{d\delta z}{dy}).dy \\ (1+\frac{d\delta x}{dx}).d'x+\frac{d\delta x}{dy}.d'y & \frac{d\delta y}{dx}.d'x+(1+\frac{d\delta y}{dy}).d'y & (\frac{dz}{dx}+\frac{d\delta z}{dx}).d'x+(\frac{dz}{dy}+\frac{d\delta z}{dy}).d'y \end{bmatrix} \\ &= \begin{bmatrix} \delta x & \delta y & \delta z \\ (1+\frac{d\delta x}{dx}).dx+\frac{d\delta x}{dy}.dy & \frac{d\delta y}{dx}.dx+(1+\frac{d\delta y}{dy}).dy & E.dx+D.dy \\ (1+\frac{d\delta x}{dx}).d'x+\frac{d\delta x}{dy}.d'y & \frac{d\delta y}{dx}.d'x+(1+\frac{d\delta y}{dy}).d'y & E.d'x+D.d'y \end{bmatrix} \\ &\qquad \text{where,}\quad E \equiv \frac{dz}{dx}+\frac{d\delta z}{dx},\quad D \equiv \frac{dz}{dy}+\frac{d\delta z}{dy} \end{aligned} \tag{199}$$

By the way, these principle comes from Lagrange [31, pp.189-236], [109] in which Lagrange states his *méthode des variations*[110] in hydrostatics. (⇑)

The duplex triangles [111] including these points, by the same method, for brevity, by denoting the sum by N, (198) is expressed as follows :

$$(dx.d'y - dy.d'x)\sqrt{N}$$

[108](⇓) The symbols : $P_1,\ P_2,\ P_3$ are of ours instead of "the first point", etc.

[109](⇓) Article 7. *De l'equilibre des fluids incompressibles*, §2. *Oú l'on déduit les dois générales de l'équibre des fluides incompressibles de la nature des particules qui les composent.* [31, pp.204-236]

[110](⇓) Lagrange [31, p.201]. Today's mathematical nomenclature is *calculus of variations* or *calcul des variations*.

[111](⇓) The duplex triangles construct a rectangle made of two adjoining triangles.

These values : $dxd'y - dyd'x$, $dzd'x - dxd'z$ and $dyd'z - dzd'y$ are calculated in permutation by *Jacobian* $|J|$ of the three determinants extracted from (199) :

$$(x,y): \begin{vmatrix} 1+\frac{d\delta x}{dx} & \frac{d\delta x}{dy} \\ \frac{d\delta y}{dx} & 1+\frac{d\delta y}{dy} \end{vmatrix}, \quad (x,z): \begin{vmatrix} 1+\frac{d\delta x}{dx} & \frac{d\delta x}{dy} \\ E & D \end{vmatrix}, \quad (y,z): \begin{vmatrix} 1+\frac{d\delta y}{dy} & \frac{d\delta y}{dx} \\ D & E \end{vmatrix}$$

($\Uparrow$)
We denote temporarily the following sum by N, then

$$\begin{aligned} N \quad = \quad & \underbrace{\Big[\Big(1+\frac{d\delta x}{dx}\Big)\Big(1+\frac{d\delta y}{dy}\Big) - \frac{d\delta x}{dy}.\frac{d\delta y}{dx}\Big]^2}_{\text{C. outer product of (x,y)}} + \underbrace{\Big[\Big(1+\frac{d\delta x}{dx}\Big)\underbrace{\Big(\frac{dz}{dy}+\frac{d\delta z}{dy}\Big)}_{D} - \frac{d\delta x}{dy}\underbrace{\Big(\frac{dz}{dx}+\frac{d\delta z}{dx}\Big)}_{E}\Big]^2}_{\text{outer product of (x,z)}} \\ + \quad & \underbrace{\Big[\Big(1+\frac{d\delta y}{dy}\Big)\underbrace{\Big(\frac{dz}{dx}+\frac{d\delta z}{dx}\Big)}_{E} - \frac{d\delta y}{dx}\underbrace{\Big(\frac{dz}{dy}+\frac{d\delta z}{dy}\Big)}_{D}\Big]^2}_{\text{outer product of (y,z)}} \\ =^* \quad & C^2 + \Big[\Big(1+\frac{d\delta x}{dx}\Big)D - \frac{d\delta x}{dy}E\Big]^2 + \Big[\Big(1+\frac{d\delta y}{dy}\Big)E - \frac{d\delta y}{dx}D\Big]^2 \\ =^* \quad & C^2 + \Big[\Big(1+\frac{d\delta x}{dx}\Big)^2 + \Big(\frac{d\delta y}{dx}\Big)^2\Big]D^2 + \Big[\Big(\frac{d\delta x}{dy}\Big)^2 + \Big(1+\frac{d\delta y}{dy}\Big)^2\Big]E^2 - 2\Big[\Big(1+\frac{d\delta x}{dx}\Big)\frac{d\delta x}{dy} + \Big(1+\frac{d\delta y}{dy}\Big)\frac{d\delta y}{dx}\Big]DE \\ =^* \quad & C^2 + \underbrace{\Big[D_1^2 + D_2^2\Big]}_{G' \text{ of } (202)} D^2 + \underbrace{\Big[E_1^2 + E_2^2\Big]}_{E' \text{ of } (202)} E^2 - 2\underbrace{\Big[D_1E_2 + E_1D_2\Big]}_{F' \text{ of } (202)} DE, \end{aligned} \tag{200}$$

where, $C \equiv \Big(1+\frac{d\delta x}{dx}\Big)\Big(1+\frac{d\delta y}{dy}\Big) - \frac{d\delta x}{dy}.\frac{d\delta y}{dx} = 1 + \frac{d\delta x}{dx} + \frac{d\delta y}{dy}$, $D \equiv \frac{dz}{dy} + \frac{d\delta z}{dy}$, $E \equiv \frac{dz}{dx} + \frac{d\delta z}{dx}$

and D_1, D_2, E_1, E_2 are the two terms consisting of D and E respectively, and these coefficients are correspond to the variables of the equation (201) showed in our footnote on the theory of curved surface by Gauss [15]. [112]

Extending (200) with neglecting the second order of δ, for example, $\frac{d\delta x}{dy}.\frac{d\delta y}{dx}$ or $(\frac{d\delta y}{dy})^2$, etc., and for brevity, denoting the sum by L, then
($\Downarrow$)

- $C^2 = \Big(1+\frac{d\delta x}{dx}+\frac{d\delta y}{dy}\Big)^2 \cong 1 + 2\Big(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}\Big)$

- $$\begin{aligned} \Big[\Big(1+\frac{d\delta x}{dx}\Big)^2 + \Big(\frac{d\delta y}{dx}\Big)^2\Big]D^2 \quad &= \quad \Big[\Big(1+2\frac{d\delta x}{dx}+\Big(\frac{d\delta x}{dx}\Big)^2\Big) + \Big(\frac{d\delta y}{dx}\Big)^2\Big]\Big(\frac{dz}{dy}+\frac{d\delta z}{dy}\Big)^2 \\ &\cong \quad \Big(1+2\frac{d\delta x}{dx}\Big)\Big(\Big(\frac{dz}{dy}\Big)^2 + 2\frac{dz}{dy}\frac{d\delta z}{dy} + \Big(\frac{\cancel{d\delta} z}{\cancel{d}y}\Big)^2\Big) \\ &\cong \quad \Big(1+2\frac{d\delta x}{dx}\Big)\Big(\Big(\frac{dz}{dy}\Big)^2 + 2\frac{dz}{dy}\frac{d\delta z}{dy}\Big) \\ &= \quad \Big(\frac{dz}{dy}\Big)^2 + 2\frac{d\delta x}{dx}\Big(\frac{dz}{dy}\Big)^2 + 2\frac{dz}{dy}\frac{d\delta z}{dy} + 4\frac{d\delta x}{dx}\frac{\cancel{d}z}{\cancel{d}y}\frac{d\delta z}{dy} \\ &\cong \quad \Big(\frac{dz}{dy}\Big)^2 + 2\frac{d\delta x}{dx}\Big(\frac{dz}{dy}\Big)^2 + 2\frac{dz}{dy}\frac{d\delta z}{dy} \end{aligned}$$

[112]($\Downarrow$) In *Disquisitiones generales circa superficies curvas*, Gauss deduces the following concluding equation (cf. [15]) :

$$EG - F^2 = E\Big(\frac{dr}{dq}\Big)^2 - 2F.\frac{dr}{dp}.\frac{dr}{dq} + G\Big(\frac{dr}{dp}\Big)^2 \tag{201}$$

We see (200) resembles one in [15].

$$N = C^2 + G'D^2 + E'E^2 - 2F'DE \tag{202}$$

If we assume that $\frac{dr}{dp} \equiv D$, $\frac{dr}{dq} \equiv E$, $E' = E_2^2 + E_1^2$, $F' = D_1E_2 + E_1D_2$ and $G' = D_1^2 + D_2^2$, then E', F' and G' correspond to E, F and G in [15].

Similarly changing x with y in corresponding expression,

$$\bullet \quad \Big[\Big(\frac{d\delta x}{dy}\Big)^2+\Big(1+\frac{d\delta y}{dy}\Big)^2\Big]E^2 = \Big[\Big(\frac{d\delta x}{dy}\Big)^2+\Big(1+2\frac{d\delta y}{dy}+\Big(\frac{d\delta y}{dy}\Big)^2\Big)\Big]\Big(\frac{dz}{dx}+\frac{d\delta z}{dx}\Big)^2$$
$$\cong \Big(1+2\frac{d\delta y}{dy}\Big)\Big(\big(\frac{dz}{dx}\big)^2+2\frac{dz}{dx}\frac{d\delta z}{dx}+\big(\frac{d\delta z}{dx}\big)^2\Big)$$
$$\cong \Big(1+2\frac{d\delta y}{dy}\Big)\Big(\Big(\frac{dz}{dx}\Big)^2+2\frac{dz}{dx}\frac{d\delta z}{dx}\Big)$$
$$= \Big(\frac{dz}{dx}\Big)^2+2\frac{d\delta y}{dy}\Big(\frac{dz}{dx}\Big)^2+2\frac{dz}{dx}\frac{d\delta z}{dx}+4\frac{d\delta y}{dy}\frac{dz}{dx}\frac{d\delta z}{dx}$$
$$\cong \Big(\frac{dz}{dx}\Big)^2+2\frac{d\delta y}{dy}\Big(\frac{dz}{dx}\Big)^2+2\frac{dz}{dx}\frac{d\delta z}{dx}$$

$$\bullet \quad -2\Big[\Big(1+\frac{d\delta x}{dx}\Big)\frac{d\delta x}{dy}+\Big(1+\frac{d\delta y}{dy}\Big)\frac{d\delta y}{dx}\Big]DE$$
$$= -2\Big[\Big(1+\frac{d\delta x}{dx}\Big)\frac{d\delta x}{dy}+\Big(1+\frac{d\delta y}{dy}\Big)\frac{d\delta y}{dx}\Big]\Big(\frac{dz}{dx}+\frac{d\delta z}{dx}\Big)\Big(\frac{dz}{dy}+\frac{d\delta z}{dy}\Big)$$
$$= -2\Big[\frac{d\delta x}{dy}+\frac{d\delta x}{dx}\frac{d\delta x}{dy}+\frac{d\delta y}{dx}+\frac{d\delta y}{dy}\frac{d\delta y}{dx}\Big]\Big(\frac{dz}{dx}\frac{dz}{dy}+2\frac{d\delta z}{dx}\frac{dz}{dy}+\frac{d\delta z}{dx}\frac{d\delta z}{dy}\Big)$$
$$\cong -2\Big(\frac{d\delta x}{dy}+\frac{d\delta y}{dx}\Big)\Big(\frac{dz}{dx}\frac{dz}{dy}+2\frac{d\delta z}{dx}\frac{dz}{dy}\Big)$$
$$= -2\Big(\frac{d\delta x}{dy}\frac{dz}{dx}\frac{dz}{dy}+\frac{d\delta y}{dx}\frac{dz}{dx}\frac{dz}{dy}+2\frac{d\delta x}{dy}\frac{d\delta z}{dx}\frac{dz}{dy}+2\frac{d\delta y}{dx}\frac{d\delta z}{dx}\frac{dz}{dy}\Big)$$
$$\cong -2\frac{dz}{dx}\frac{dz}{dy}\Big(\frac{d\delta x}{dy}+\frac{d\delta y}{dx}\Big)$$

(⇑)

$$\sqrt{N}=\Big(\Big[1+\big(\frac{dz}{dx}\big)^2+\big(\frac{dz}{dy}\big)^2\Big].\Big[1+\frac{L}{1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2}\Big]\Big)^{\frac{1}{2}}$$

where, L is gained by extracting only one order terms in the expanded terms from (200) :
(⇓) Here we can't solve a question : where the inconsistency by the coefficient 2 in front of L in (203) of
The calculations by Gauss are interpreted as follows :

$$N$$
$$=^* \; C^2+(\bullet)D^2+(\bullet)E^2+(\bullet)DE$$
$$=^* \; \underbrace{1+2\Big(\frac{d\delta x}{dx}+\frac{d\delta y}{dy}\Big)}_{\bullet C^2}+\underbrace{\Big(\frac{dz}{dy}\Big)^2+2\frac{d\delta x}{dx}\Big(\frac{dz}{dy}\Big)^2+2\frac{dz}{dy}\frac{d\delta z}{dy}}_{\bullet D^2}+\underbrace{\Big(\frac{dz}{dx}\Big)^2+2\frac{d\delta y}{dy}\Big(\frac{dz}{dx}\Big)^2+2\frac{dz}{dx}\frac{d\delta z}{dx}}_{\bullet E^2}\underbrace{-2\frac{dz}{dx}\frac{dz}{dy}\Big(\frac{d\delta x}{dy}+\frac{d\delta y}{dx}\Big)}_{\bullet DE}$$
$$=^* \; \underbrace{2\Big[\frac{d\delta x}{dx}\Big\{[1+\Big(\frac{dz}{dy}\Big)^2\Big\}-\frac{dz}{dx}\frac{dz}{dy}\Big(\frac{d\delta x}{dy}+\frac{d\delta y}{dx}\Big)+\frac{d\delta y}{dy}\Big\{[1+\Big(\frac{dz}{dx}\Big)^2\Big\}+\Big(\frac{dz}{dy}\frac{d\delta z}{dy}+\frac{dz}{dx}\frac{d\delta z}{dx}\Big)\Big]}_{L}+\Big[1+\Big(\frac{dz}{dx}\Big)^2+\Big(\frac{dz}{dy}\Big)^2\Big]$$
$$=^* \; 2L+\Big[1+\Big(\frac{dz}{dx}\Big)^2+\Big(\frac{dz}{dy}\Big)^2\Big] \tag{203}$$

(⇑) We continue from Gauss. From (203), a first triangle L is the following :

$$L =^* \frac{1}{2}\Big[N-\Big\{1+\Big(\frac{dz}{dx}\Big)^2+\Big(\frac{dz}{dy}\Big)^2\Big\}\Big]$$
$$= \frac{d\delta x}{dx}\Big[1+\Big(\frac{dz}{dy}\Big)^2\Big]-\frac{d\delta x}{dy}\frac{dz}{dx}\frac{dz}{dy}-\frac{d\delta y}{dx}\frac{dz}{dx}\frac{dz}{dy}+\frac{d\delta y}{dy}\Big[1+\Big(\frac{dz}{dx}\Big)^2\Big]+\frac{d\delta z}{dx}\frac{dz}{dx}+\frac{d\delta z}{dy}\frac{dz}{dy}$$
$$=^* \frac{d\delta x}{dx}\Big[1+\Big(\frac{dz}{dy}\Big)^2\Big]-\frac{dz}{dx}\frac{dz}{dy}\Big(\frac{d\delta x}{dy}+\frac{d\delta y}{dx}\Big)+\frac{d\delta y}{dy}\Big[1+\Big(\frac{dz}{dx}\Big)^2\Big]+\frac{d\delta z}{dx}\frac{dz}{dx}+\frac{d\delta z}{dy}\frac{dz}{dy} \tag{204}$$

Here we may recall (196), then the following holds :

$$\xi^2+\eta^2+\zeta^2=1,\quad \frac{dz}{dx}=-\frac{\xi}{\zeta},\quad \frac{dz}{dy}=-\frac{\eta}{\zeta} \quad\Rightarrow\quad 1+\Big(\frac{dz}{dx}\Big)^2+\Big(\frac{dz}{dy}\Big)^2=^*\frac{\xi^2+\eta^2+\zeta^2}{\zeta^2}=\frac{1}{\zeta^2} \tag{205}$$

The ratio of the first triangle to the second and plus 1 becomes,

$$1+\frac{L}{1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2} =^* 1+\frac{\text{1st triangle}}{\text{2nd triangle}} =^* 1+\zeta^2 L$$

(⇓) The two triangles of first and second are contiguous and construct a quadrilateral by two.(⇑)
Moreover, this is independent of the figure of triangle dU, then, it turns out,

$$\delta dU = \frac{LdU}{1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2} =^* \zeta^2 LdU \tag{206}$$

Moreover, this is independent of the figure of a triangle dU, then, it turns out,

$$\delta dU = \frac{LdU}{1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2} =^* \zeta^2 LdU \tag{207}$$

Expanding L in (207) using (196) and (204), then

$$\delta dU = dU\Big[\frac{d\delta x}{dx}\Big(\eta^2+\zeta^2\Big) - \Big(\frac{d\delta x}{dy}+\frac{d\delta y}{dx}\Big)\xi\eta + \frac{d\delta y}{dy}\Big(\xi^2+\zeta^2\Big) - \frac{d\delta z}{dx}\xi\zeta - \frac{d\delta z}{dy}\eta\zeta\Big], \tag{208}$$

(⇓) where, we used the following : $\zeta^2\Big(1+\frac{dz}{dx}\Big) = \zeta^2+\zeta^2\frac{\xi^2}{\zeta^2} = \xi^2+\zeta^2,\quad \zeta^2\Big(1+\frac{dz}{dy}\Big) = \zeta^2+\zeta^2\frac{\eta^2}{\zeta^2} = \zeta^2+\eta^2.$
Here, the coefficient of 2 in (204) is unnecessary, since dU is a triangle defined in §E.21, then according to Gauss' description, dU means a triangle in (207) or (208).

E.22. **Integral expression by decomposing dU into dQ and dU.**

From (208), all variation of the surface U is obtained by the following two integrals

$$\int dU\Big[\Big(\eta^2+\zeta^2\Big)\frac{d\delta x}{dx} - \xi\eta\Big(\frac{d\delta y}{dx}\Big) - \xi\zeta\frac{d\delta z}{dx}\Big] \equiv A, \quad (x\text{−differential part}) \tag{209}$$

$$\int dU\Big[-\xi\eta\frac{d\delta x}{dy} + \Big(\xi^2+\zeta^2\Big)\frac{d\delta y}{dy} - \eta\zeta\frac{d\delta z}{dy}\Big] \equiv B, \quad (y\text{−differential part}) \tag{210}$$

and these are separately treated. We consider the following :

- at first, we take a plane, normal to the coordinate axis y, and such as,
  - for the value of this y to be determinate suitably taking the exterior value to the peripheral,
  - for the last value of y to be in the surface U.
- next, for this plane, on the peripheral P, we split into two part, or four, or six, etc., for the points of which by the first coordinate, to be followed by $x^0, x', x'', \cdots$; namely, as if the indices are different each other, we should number suitably by the indices to these points ;
- then, by the same way, we split the surface with other plane, for this infinite neighborhood to be parallel, and to encounter with the point of the second coordinate $y+dy$;
- finally, between these planes, we could get the elements of peripheral $dP^0, dP', dP'', \cdots$,

then we could see easily the left-hand side being expressed as follows :

$$dy = -Y^0 dP^0 = +Y' dP' = -Y'' dP'' = +Y''' dP''' \text{ etc.} \tag{211}$$

(⇓) where dP^* means the various P not the derivative, and the sign changes superior or inferior, according to that the line μP^* from the center μ takes interior or exterior of the space S. cf. §E.9. (⇑)
If, in addition to, we consider the infinitely many planes, normal to the coordinate axis x, of which the element dx between x^0 and x', or between x'' and x''', or etc., it corresponds to the element : [113]

$$dU = \frac{dx.dy}{\zeta}, \tag{212}$$

(⇓) Namely, this correspondence comes from (208)

$$\begin{aligned}\int \delta dU = & \int\Big[dU\Big(\eta^2+\zeta^2\Big)\frac{d\delta x}{dx} - \frac{d\delta y}{dx}\xi\eta - \frac{d\delta z}{dx}\xi\zeta\Big] + \int dU\Big[\Big(\xi^2+\zeta^2\Big)\frac{d\delta y}{dy} - \frac{d\delta x}{dy}\xi\eta - \frac{d\delta z}{dy}\eta\zeta\Big] \\ = & \underbrace{dy\int dx\frac{1}{\zeta}\Big[\Big(\eta^2+\zeta^2\Big).\frac{d\delta x}{dx} - \frac{d\delta y}{dx}\xi\eta - \frac{d\delta z}{dx}\xi\zeta\Big]}_{A} + \underbrace{dx\int dy\frac{1}{\zeta}\Big[\Big(\xi^2+\zeta^2\Big).\frac{d\delta y}{dy} - \frac{d\delta x}{dy}\xi\eta - \frac{d\delta z}{dy}\eta\zeta\Big]}_{B}\end{aligned}$$

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[113](⇓) In fact, comparering the two expressions : (209) with (213) and (210) with (213), then this correspondence deduced.

(⇑)
Therefore, from here, it is clear that the part A, which corresponds to the part of the surface depending on between the interval : $y,\ y+dy$, turns into the following integral, i.e. substituting the right hand-side of (212) into A of (209), then

$$A = dy\int dx\Big(\frac{\eta^2+\zeta^2}{\zeta}.\frac{d\delta x}{dx} - \frac{\xi\eta}{\zeta}.\frac{d\delta y}{dx} - \xi d\delta z\Big)$$

extending from $x = x^0$ to $x = x'$, next, from $x = x''$ to $x = x'''$ etc. In fact, the limit of this integration by parts is expressed as follows :

$$A = \Big(\frac{\eta^2+\zeta^2}{\zeta}\delta x - \frac{\xi\zeta}{\zeta}\delta y - \xi\delta z\Big)dy - dy\int\Big(\delta x\frac{\frac{\eta^2+\zeta^2}{\zeta}}{dx} - \delta y\frac{d\frac{\xi\eta}{\zeta}}{dx} - \delta z\frac{d\xi}{dx}\Big)dx \tag{213}$$

Here, we construct A using (211) and (212), then

$$\begin{aligned}
&\Big(\frac{\eta^{02}+\zeta^{02}}{\zeta^0}\delta x^0 - \frac{\xi^0\eta^0}{\zeta^0}\delta y^0 - \xi^0\delta z^0\Big)Y^0dP^0\\
+\ &\Big(\frac{\eta'^2+\zeta'^2}{\zeta'}\delta x' - \frac{\xi'\eta'}{\zeta'}\delta y' - \xi'\delta z'\Big)Y'dP'\\
+\ &\Big(\frac{\eta''^2+\zeta''^2}{\zeta''}\delta x'' - \frac{\xi''\eta''}{\zeta''}\delta y'' - \xi''\delta z''\Big)Y''dP''\\
+\ &\text{etc.}\\
-\ &\int\zeta dU\Big(\delta x\frac{\frac{\eta^2+\zeta^2}{\zeta}}{dx} - \delta y\frac{d\frac{\xi\eta}{\zeta}}{dx} - \delta z\frac{d\xi}{dx}\Big)
\end{aligned}$$

or in sum,

$$\sum\Big(\frac{\eta^2+\zeta^2}{\zeta}\delta x - \frac{\xi\eta}{\zeta}\delta y - \xi\delta z\Big)YdP - \int\zeta dU\Big(\delta x\frac{\frac{\eta^2+\zeta^2}{\zeta}}{dx} - \delta y\frac{d\frac{\xi\eta}{\zeta}}{dx} - \delta z\frac{d\xi}{dx}\Big)$$

This total quantity A is expressed by

$$\boxed{A = \int\Big(\frac{\eta^2+\zeta^2}{\zeta}\delta x - \frac{\xi\eta}{\zeta}\delta y - \xi\delta z\Big)YdP - \int\zeta dU\Big(\delta x\frac{\frac{\eta^2+\zeta^2}{\zeta}}{dx} - \delta y\frac{d\frac{\xi\eta}{\zeta}}{dx} - \delta z\frac{d\xi}{dx}\Big)}$$

where, the first integral is extended to all the circumference of P, and the second is extended to all the surface of U.

E.23. **Analytic reduction of δU to two integrals of Q and V via A and B.**
By calculation from (209) as the same as (210), we get B similarly and immediately

$$A = \int\Big(\frac{\eta^2+\zeta^2}{\zeta}\delta x - \frac{\xi\eta}{\zeta}\delta y - \xi\delta z\Big)YdP - \int\zeta dU\Big(\delta x\frac{\frac{\eta^2+\zeta^2}{\zeta}}{dx} - \delta y\frac{d\frac{\xi\eta}{\zeta}}{dx} - \delta z\frac{d\xi}{dx}\Big) \tag{214}$$

$$B = \int\Big(\frac{\xi\eta}{\zeta}\delta x - \frac{\xi^2+\zeta^2}{\zeta}\delta y - \eta\delta z\Big)XdP + \int\zeta dU\Big(\delta x\frac{\frac{\xi\eta}{\zeta}}{dy} - \delta y\frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy} + \delta z\frac{d\eta}{dy}\Big) \tag{215}$$

Here we determine for all the circumference P, we get ζQ from the first terms of both (214) and (215),

$$\Big(\frac{\xi\eta}{\zeta}\delta x - \frac{\xi^2+\zeta^2}{\zeta}\delta y - \eta\delta z\Big)X + \Big(\frac{\eta^2+\zeta^2}{\zeta}\delta x - \frac{\xi\eta}{\zeta}\delta y - \xi\delta z\Big)Y \equiv Q,$$

$$\Big[X\xi\eta + Y\Big(\eta^2+\zeta^2\Big)\Big]\delta x - \Big[X\big(\xi^2+\zeta^2\big) + Y\xi\eta\Big]\delta y + \big(X\eta\zeta - Y\xi\zeta\big)\delta z = \zeta Q$$

Moreover, for every point of the surface U, we get V from the second terms of both (214) and (215),

$$\Big(\frac{d\frac{\xi\eta}{\zeta}}{dy}-\frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx}\Big)\zeta\delta x+\Big(\frac{d\frac{\xi\eta}{\zeta}}{dx}-\frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy}\Big)\zeta\delta y+\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}\Big)\zeta\delta z\equiv V \tag{216}$$

That is, we can put

$$\boxed{\delta U=\int QdP+\int VdU} \tag{217}$$

The first integral is to be extended along all the circumference P, and the second is on all surface U. (⇓) This is what is called the *Gaussian integral formula* in two dimensions.

E.24. **Geometric reduction of Q and V.**

Formulae for Q and V notably contradict $X\xi+Y\eta+Z\zeta=0$,[114],Q has always the symmetric form as follows :

$$Q=(Y\zeta-Z\eta)\delta x+(Z\xi-X\zeta)\delta y+(X\eta-Y\xi)\delta z \quad\Rightarrow\quad \text{the value of determinant : } \begin{vmatrix}\delta x&\delta y&\delta z\\ X&Y&Z\\ \xi&\eta&\zeta\end{vmatrix} \tag{218}$$

(⇓) Here, the expression by determinant is of ours not by Gauss.(⇑)

When we see the form of V, we can reduce from the formulae (196)

$$\frac{dz}{dx}=-\frac{\xi}{\zeta}\,,\quad \frac{dz}{dy}=-\frac{\eta}{\zeta},$$

the following as

$$\frac{d\frac{\xi}{\zeta}}{dy}=\frac{d\frac{\eta}{\zeta}}{dx} \tag{219}$$

therefore,

$$\frac{d\frac{\xi\eta}{\zeta}}{dy}=\frac{\xi}{\zeta}.\frac{d\eta}{dy}+\eta\frac{d\frac{\xi}{\zeta}}{dy}=\frac{\xi}{\zeta}.\frac{d\eta}{dy}+\eta\frac{d\frac{\eta}{\zeta}}{dx}$$

Moreover, for $\xi^2+\eta^2+\zeta^2=1$, we can deduce

$$\xi\frac{d\xi}{dx}+\eta\frac{d\eta}{dx}+\zeta\frac{d\zeta}{dx}=0 \tag{220}$$

by dividing the both side of hand of (220) with ζ,

$$\frac{\xi}{\zeta}\frac{d\xi}{dx}=-\Big(\frac{\eta}{\zeta}\frac{d\eta}{dx}+\frac{d\zeta}{dx}\Big) \tag{221}$$

and therefore by (221)

$$\frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx}=\eta\frac{d\frac{\eta}{\zeta}}{dx}+\Big(\frac{\eta}{\zeta}.\frac{d\eta}{dx}+\frac{d\zeta}{dx}\Big)=\eta\frac{d\frac{\eta}{\zeta}}{dx}-\frac{\xi}{\zeta}.\frac{d\xi}{dx} \tag{222}$$

We may replace the coefficient of $\zeta\delta x$ in V of (216), using (219) and (222),

$$\begin{aligned}&\frac{d\frac{\xi\eta}{\zeta}}{dy}-\frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx}\\ =&\ \frac{d\frac{\xi\eta}{\zeta}}{dy}-\eta\frac{d\frac{\eta}{\zeta}}{dx}+\frac{\xi}{\zeta}.\frac{d\xi}{dx}\quad (\text{, from (222), })\\ =&\ \Big(\frac{\xi}{\zeta}\frac{d\eta}{dy}+\eta\cancel{\frac{d\frac{\xi}{\zeta}}{dy}}\Big)-\eta\cancel{\frac{d\frac{\xi}{\zeta}}{dy}}+\frac{\xi}{\zeta}.\frac{d\xi}{dx}\quad (\text{, from (219), })\\ =&\ \frac{\xi}{\zeta}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}\Big)\end{aligned}$$

[114](⇓) This means $X\xi+Y\eta+Z\zeta\neq 0$.

Similarly for $\zeta\delta y$

$$\frac{d\frac{\xi\eta}{\zeta}}{dx}-\frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy}=\frac{\eta}{\zeta}\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}\Big)$$

Then V of (216) is reduced as follows :

$$\boxed{V=(\xi\delta x+\eta\delta y+\zeta\delta z)\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}\Big)}$$

E.25. **Geometric meaning of $\frac{d\xi}{dx}+\frac{d\eta}{dy}$ in V.**

Before going forward, we must illustrate conveniently the important geometrical expression. Here we restrict the various directions of methods, we would like to present the intuitionally simple method as follows, which we introduced in *Disquisitiones generales circa superficies curvas*. We consider the following layout of structure.

- At first, we consider a sphere, of which the radius $=1$ at the center of an arbitrary surface, and we denote the values of the axes of coordinates x,y and z by the points $(1),(2)$ and (3),
- next, taking exterior domain denoted by s, we number a point denoting by the point (4) toward the normal direction on surface ;
- then, from an arbitrary point on surface, we draw a straight line toward different point, which we denote by the point (5),
- finally, for the variation of itself, we suppose that for the quantity $\sqrt{\delta x^2+\delta y^2+\delta z^2}$ to be always positive, and we denote the quantity by δe for brevity, then

$$\begin{cases}\delta x=\delta e.\cos(1,5)\\ \delta y=\delta e.\cos(2,5)\\ \delta z=\delta e.\cos(3,5)\end{cases},\quad \text{where}\quad \delta e\equiv\sqrt{\delta x^2+\delta y^2+\delta z^2}.$$

($\Downarrow$) (Remark. If we assume each ($\bullet$) a unique point each other in both, then ($\bullet,\bullet$) means the angle between two points taking an intermediate of an origin.) By the way, before Gauss' method of description of angle, we can show the same method by Lagrange in 1788 as follows :

> Comme ces quatre systèmes de coordonnèes répondent aux quatre angles du nouveau quadrilatère dans lequel s'est changé le rectangle $dx\ dy$, il est clair qu'on aura les côtés de ce quadrilatère en prenant la racine carrée de la somme des carrés des différences des coordonnées pour deux angles adjacents à chaque côté. Ainsi, en marquant la droit qui joint deux angles par la réunion des deux numerérons qui répondent à ces angles, on aura $(1,2)=dx\sqrt{\cdots}$ Lagrange [31, pp.207-208]

(Trans.)

It is clear that by two adjacent angles made by each side or edge are By $\sqrt{\delta x^2+\delta y^2+\delta z^2}$: the square root of sum of the each square of differences. Therefore by marking the line joining the two angles, with the pair of two number corresponding to these angles, we have $(1,2)=dx\sqrt{\cdots}$ $\cdots$ ($\Uparrow$)

Here, we would like to express the every point on the surface. About this boundary, when we treat the periphery P, we can approach this from the two different directions. [115] Hence,

- at first, we denote the point corresponding to dP by the point (6),
- next, we draw a straight line of the inner normally-directed tangential to the surface, then we denote the point by (7),

[115]($\Downarrow$) Namely, clockwise and counterclockwise.

TABLE 15. Comparison of Q and V in $\delta U = \int QdP + \int VdU$ between analytic and geometric method

no	value	analytic method	geometric method
1	Q	$Q = \Big(\frac{\xi\eta}{\zeta}\delta x - \frac{\xi^2+\zeta^2}{\zeta}\delta y - \eta\delta z\Big)X + \Big(\frac{\eta^2+\zeta^2}{\zeta}\delta x - \frac{\xi\eta}{\zeta}\delta y - \xi\delta z\Big)Y$	$Q = -\delta e.\cos(5,7)$
2	V	$V = \Big(\frac{d\frac{\xi\eta}{\zeta}}{dy} - \frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx}\Big)\zeta\delta x + \Big(\frac{d\frac{\xi\eta}{\zeta}}{dx} - \frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy}\Big)\zeta\delta y + \Big(\frac{d\xi}{dx} + \frac{d\eta}{dy}\Big)\zeta\delta z$	$V = \delta e.\cos(4,5).\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy}\Big)$ $= \delta e.\cos(4,5).\Big(\frac{1}{R} + \frac{1}{R'}\Big)$

- then, by this hypothesis, these points $(6),(7)$ and (4) look toward the same direction, [116] using above-mentioned $(1),(2)$ and (3) then $(4,6),(4,7)$ and $(6,7)$ make a cube, [117] if we assume each angle as the rectangle. [118]

Thus, the equations (197) in the above-mentioned (§E.20) are transformed into

$$\begin{cases} \eta Z - \zeta Y = \cos(1,7) \\ \zeta X - \xi Z = \cos(2,7) \\ \xi Y - \eta X = \cos(3,7) \end{cases}$$

(⇓) Namely $\cos(1,7)$, $\cos(2,7)$, $\cos(3,7)$ are determined by the its cofactor of following matrix :

$$\begin{bmatrix} \cos(1,7) & \cos(2,7) & \cos(3,7) \\ X & Y & Z \\ \xi & \eta & \zeta \end{bmatrix}$$

(⇑)

The formulae in the previous article take forms as follows :

$$Q = -\delta e.\cos(5,7), \quad V = \delta e.\cos(4,5).\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy}\Big), \tag{223}$$

where,

- Q expresses the translation of this point along the periphery P, to which a plane of tangential surface U, taking as normal in the domain, positive to the opposite direction ;
- the factor V is, like $\cos(4,5)$ clearly indicates, the translation of this point on the surface U, taking as positive in the domain of the exterior space s.

Here we may summarize Q and V in $\delta U = \int QdP + \int VdU$ by the two methods between analytic and geometric in Table 15.

We may explain by replacing $\frac{d\xi}{dx} + \frac{d\eta}{dy}$ in V of (223), from the point of view in geometric meaning. In such case, it turns out that : from (196),

$$\xi = -\zeta.\frac{dz}{dx}, \quad \eta = -\zeta.\frac{dz}{dy} \tag{224}$$

$$\Rightarrow^* \quad \xi^2 + \eta^2 + \zeta^2 = \zeta^2 + \zeta^2\Big(\big(\frac{dz}{dx}\big)^2 + \big(\frac{dz}{dy}\big)^2\Big)$$

Then

$$\frac{1}{\zeta^2} = 1 + \big(\frac{dz}{dx}\big)^2 + \big(\frac{dz}{dy}\big)^2 \tag{225}$$

Taking derivative in both side of hand of (225)

$$-2\zeta^{-3} = 2\frac{dz}{dx}.\frac{d\frac{dz}{dx}}{d\zeta} + 2\frac{dz}{dy}.\frac{d\frac{dz}{dy}}{d\zeta}$$

[116](⇓) This image is considered that there are three directions emitting from a common point and making a certain angle with two directions (i.e. points.)

[117](⇓) $(4,6),(4,7)$ and $(6,7)$ make a plane consisting of a cube respectively.

[118](⇓) When $\cos(4,6) = \cos(4,7) = \cos(6,7) = \frac{\pi}{2}$.

$$1 = -\zeta\frac{dz}{dx}.\zeta^2\frac{d\frac{dz}{dx}}{d\zeta} - \zeta\frac{dz}{dy}.\zeta^2\frac{d\frac{dz}{dy}}{d\zeta} \tag{226}$$

and finally we get the following expression after replacing (226) with ξ and η from (224)

$$d\zeta = \xi\zeta^2 d\frac{dz}{dx} + \eta\zeta^2 d\frac{dz}{dy} \tag{227}$$

[119] Using (224) and (227),

$$\begin{aligned}\frac{d\xi}{dx} &= -\zeta\frac{d^2z}{dx^2} - \frac{dz}{dx}.\frac{d\zeta}{dx} = -\zeta\frac{d^2z}{dx^2}\underbrace{-\zeta\frac{dz}{dx}}_{=\xi}\xi\zeta\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} \\ &= -\zeta(1-\xi^2)\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} = -\zeta\big(\eta^2+\zeta^2\big)\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy}\end{aligned}$$

$$\frac{d\eta}{dy} = -\zeta\frac{d^2z}{dy^2} + \eta^2\zeta\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} = -\zeta(1-\eta^2)\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} = -\zeta\big(\xi^2+\zeta^2\big)\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy}$$

Therefore, again from (224)

$$\boxed{\begin{aligned}\frac{d\xi}{dx} + \frac{d\eta}{dy} &= -\zeta^3\Big[\frac{d^2z}{dx^2}\Big\{1+\Big(\frac{dz}{dy}\Big)^2\Big\} - \frac{2d^2z}{dx.dy}.\frac{dz}{dx}.\frac{dz}{dy} + \frac{d^2z}{dy^2}\Big\{1+\Big(\frac{dz}{dx}\Big)^2\Big\}\Big], \\ \text{where,}\quad \zeta^3 &= \Big[1+\big(\frac{dz}{dx}\big)^2+\big(\frac{dz}{dy}\big)^2\Big]^{-\frac{3}{2}}\end{aligned}} \tag{228}$$

This is equal to (174) in Gauss [15]. [120] This value turns into a constant such as [121]

$$\boxed{\frac{d\xi}{dx} + \frac{d\eta}{dy} = \frac{1}{R} + \frac{1}{R'},} \tag{229}$$

where R and R' are the radii of curvature respectively. [122]
($\Downarrow$) Together (228) with (229) are what Gauss called it *the first fundamental theorem* referred in §E.28.

---

[119]($\Downarrow$) The above expressions are to be used by ∂, that is

$$\partial\zeta = \xi\zeta^2\partial\frac{\partial z}{\partial x} + \eta\zeta^2\partial\frac{\partial z}{\partial y}$$

[120]($\Downarrow$) Kobayashi [28], p.138 (3.9), *the first fundamental form* :

$$\begin{aligned} I_\alpha &= E_\alpha du_\alpha du_\alpha + 2F_\alpha du_\alpha dv_\alpha + G_\alpha dv_\alpha dv_\alpha \\ &= (du_\alpha,\ dv_\alpha)\begin{bmatrix} E_\alpha & F_\alpha \\ F_\alpha & G_\alpha\end{bmatrix}\begin{bmatrix} du_\alpha \\ dv_\alpha\end{bmatrix}\end{aligned}$$

where,

$$E_\alpha = \frac{\partial\mathbf{p}}{\partial u_\alpha}.\frac{\partial\mathbf{p}}{\partial u_\alpha},\quad F_\alpha = \frac{\partial\mathbf{p}}{\partial u_\alpha}.\frac{\partial\mathbf{p}}{\partial v_\alpha},\quad G_\alpha = \frac{\partial\mathbf{p}}{\partial v_\alpha}.\frac{\partial\mathbf{p}}{\partial v_\alpha}$$

[121]($\Downarrow$) cf. Laplace, IV, p.826 [9853], the equation (136) :

$$\frac{1}{R} + \frac{1}{R'} = \frac{(1+q^2).\frac{dp}{dx} - pq.\Big(\frac{dp}{dy}+\frac{dq}{dx}\Big) + (1+p^2).\frac{dq}{dy}}{(1+p^2+q^2)^{\frac{3}{2}}}$$

[122]($\Downarrow$) cf. Poisson [60], p.105.

E.26. **Reduction of δU.**

From (217), (223) and (229)

$$(I) \quad \delta U = \int QdP + \int VdU = -\int \underbrace{\delta e.\cos(5,7)}_{-Q\ of\ (223)} .dP + \int \underbrace{\delta e.\cos(4,5).(\frac{1}{R} + \frac{1}{R'})}_{V\ of\ (223)} dU$$

Now, we consider Q of (223),

$$\int \delta e.\cos(5,7)dP' = \int \delta e.\cos(5,7)dP^{(3)} + \int \delta e.\cos(5,7)dP^{(4)} \tag{230}$$

$$\int \delta e.\cos(5,7)dP^{(2)} = \int \delta e.\cos(5,7)dP^{(3)} + \int \delta e.\cos(5,7)dP^{(5)} \tag{231}$$

If we add both hand sides of two equations (230) and (231) above, then

$$\int \delta e.\cos(5,7)dP' + \int \delta e.\cos(5,7)dP^{(2)} = \int \delta e.\cos(5,7)dP$$

For $\delta U = \delta U' + \delta U''$, the variational values of $\delta U'$, $\delta U''$ fit by substitution. Thus, we can see the truth of the formula (I).

> In short, it is observed that, the variational theorem (I) is to be deduced from the consideration of geometry, and moreover it is easier than by the analytic method to solve our problem, although we are managing to solve it, and ocasionally, by the variational calculation, for including double integral of the limit of the variable, so that we have sought for it insufficiently up to now.
>
> However, in some way, we would like to try to investigate it from the view-point of another geometrical method which we can challenge sufficently to lead the readers.

E.27. **Geometrical method. Deducing the parts of Q.**

Evolving further the variation, for the expression W is explained by the variation of figure of the space s, we would like to start to argue at first, from the variation of the space s. Recalling that we consider in §E.21, the prism with the equal sides and oriented to the solid body, then, on this point, we can see that this prism has the following :

- the size of basement : dU,
- the height : $\xi\delta x + \eta\delta y + \zeta\delta z = \delta e.\cos(4,5)$, where $\delta e = \sqrt{\delta x^2 + \delta y^2 + \delta z^2}$
- the sign ($\pm$) of height depends on the transposition of triangle, according to the location of whole solid lying whether interior or exterior of the space s.

Hence, we can get

$$(II) \quad \delta s = \int dU.\delta e.\cos(4,5)$$

Next, from (II), the variation of $\int zds$ (III) follows :

$$(III) \quad \delta \int zds = \int zdU.\delta e.\cos(4,5)$$

As long as the variational quantity T, we can see that P is the limit point having commonly the surface T and U, the transposing point of the circumference P satisfies owing to these condition, and newly keeps in the surface space S. By the transposing element dP, as the partial displacement of the surface T, we get easily $\pm dP.\delta e.\sin(5,6)$. In general, the choice of positive or negative sign depends on the sign of $\cos(4,5)$. We would like to explain it by introducing the new directions such that :

- the space S tangential in the surface plane,
- the normal-directional line P, and
- the exterior space s,

respectively. If denoting the responding direction with the point (8), then by the transposing element dP, we get the surface variation of T, from the definition, as $dP.\delta e.\cos(5,8)$, namely (IV) :

$$(IV) \quad \delta T = \int dP.\delta e.\cos(5,8),$$

where, the sign of factor $\cos(5,8)$ depends on the conditions of whether increment or decrement.

When we assume that :

- at first, the point (6) were the pole of the maximum circle passing through the two points : (7) and (8), then the point (5) is the highest point in the circle made by the two points (6) and (8) ;
- next, the points (5), (7) and (8) make a rectangular triangle, having the rectangle at the point (8) ;
- then, we can get the expression : $\cos(5,7) = \cos(5,8).\cos(7,8)$, where, the arc $(7,8)$ is the measure of angle between planes of the two surface spaces : s and S, which are tangential intersecting with the point P and the plane domain, including null space ;
- finally, we denote the angle making with $(7,8)$ by i, i.e. $i = (7,8)$ and by $2\pi - i$, the angle between plane domain, in which the space s is continue.

Then we can formulate (V) as follows :

$$(V) \quad \cos(5,7) = \cos(5,8).\cos i$$

E.28. **Result.1 : deduction of height from the first fundamental theorem.**

By the combination of above formulae I, $\cdots$, IV, we get the variational expression of W.

$$\delta W = \underbrace{\int dU.\delta e.\cos(4,5)}_{(II)\ \delta s}.\Big[z + \alpha^2\Big(\frac{1}{R} + \frac{1}{R'}\Big)\Big] - \underbrace{\int dP.\delta e.\cos(5,8)}_{(IV)\ \delta T}.(\alpha^2 \cos\ i - \alpha^2 + 2\beta^2)$$

where,

$$z + \alpha^2\Big(\frac{1}{R} + \frac{1}{R'}\Big) = \text{Const.}$$

The equation is constituted by $\prec$ *the first fundamental theorem* $\succ$, in the theory of fluid equilibrium, in which Mr. Laplace missed, however, it would come to be different if he had used our method.

If we set Const $= 0$, then

$$z = -\alpha^2\Big(\frac{1}{R} + \frac{1}{R'}\Big).$$

where, z is the height of capillary action, α and β are the values defined in (194). And moreover, the following corollaries follow :

Corollaries :

(1) If free surface U is not classified, in any point in a section, the surface must be concavo-convex, (i.e. concave curvature is greater than convex curvature,) in addition, convexing the maximum radius is equivalent to concaving with the maximum radius.

(2) For upper normal plane to surface, it becomes concavo-concave, (i.e. biconcave, which is concave in both sides,) or if there is in anywhere, convexo-concave, (i.e. convex curvature is greater than concave curvature,) concave curvature will be convex.

(3) It becomes convexo-convex, (i.e. biconvex, which is convex in both sides,) or if there is in anywhere, concavo-convex, convex curvature will be concave.

(4) Free surface U can not have partial finite plane if not horizontal and coincident with normal plane.

E.29. **Result.2 : deduction of angle from the second fundamental theorem.**

$$\delta W = -\int dP.\delta e.\cos(5,8).(\alpha^2 \cos i - \alpha^2 + 2\beta^2) = \alpha^2 \int dP.\delta e.\cos(5,8).\Big(1 - 2\Big(\frac{\beta}{\alpha}\Big)^2 - \cos\ i\Big)$$

Here, we assume A such that

$$\cos A = 1 - 2\sin^2(\frac{A}{2}) = 1 - 2\frac{\beta^2}{\alpha^2}. \tag{232}$$

$$\text{If} \quad \sin\frac{A}{2} = \frac{\beta}{\alpha}, \quad \text{then} \quad \delta W = \alpha^2 \int dP.\delta e.\cos(5,8).(\cos A - \cos i)$$

where, the integral is to be extended along the total line P. Remember that the factor $\cos(5,8)$ is equivalent with $\sin(5,6)$, [123] and the sign becomes plus or minus, according to fluid in motion in the neighborhood of element dP or moreover, it reaches to the end point of P, or it comes to disappear. Here, we conclude that as follows :

- in state of equilibrium, it becomes always $i = A$.
- If in every part of the line P, it becomes $i < A$, then initially generated momentum in this part keeps invariable in the line P, and W show negative variation.
- If in a part of the line P, it becomes $i > A$, then both cases of minimum condition and equilibrium confront.

This is $\prec$ *the second fundamental theorem* $\succ$, which Mr. Laplace has investigated almost without proof in the meaning of the principle of molecule.

E.30. **In case of the vase having the figure of cusp or aciform.**

- The theorem above of arrangement which lacks in singular case, we can not pass over it.
- On the surface of the vase near the ultimate limit P, there exists the *only* plane contact with the surface of vase.
- If the continuous curvature in this point P the singular line interrupted, it is considered easily that not only the cusp, but also the aciform [124] of line P sifts, we do not change our conclusions ;

$$\alpha^2 dP.\delta e.\sin(5,6).(\cos A - \cos i)$$

$$-\alpha^2 dP.\delta e.\sin(5,6).(\cos A + \cos k)$$

$$\begin{cases} i = A, & i > A \\ k = 2\pi - A, & k > 2\pi - A \end{cases}$$

In the state of equilibrium, therefore, it can not become $i + k < 2\pi$, if, that is equivalent to the following : *in the state of equilibrium, the limit of free surface of fluid can not become up to the finite extension, in the aciform, concave surface of vase.* [125] To the contrary, the quantities by this limit coincident with aciform convex, this is required and sufficient for equilibrium, where, a is the inclination.

- When the angle lies between fluid plane and tangent vase as follows :

$$\begin{cases} \text{between } A \text{ and } A + a \text{ (included)} & \Rightarrow^* A \le * \le A + a, & \text{exterior-measured fluid,} \\ \text{between } 2\pi - A \text{ and } 2\pi - A + a & \Rightarrow^* 2\pi - A < * < 2\pi - A + a, & \text{interior-measured fluid,} \end{cases}$$

where, * means the angle.

- When the angle lies between two surface planes of vase from both side to aciform tangent in this point indefinitely denoted with $2\pi - \alpha$, to what extent we can measure this angle of domain of vase.

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[123]($\Downarrow$) i.e. $\cos(5,8) = \sin(5,6)$, where the point (8) is the point of rectangle, the points (6), (8) and (5) make a straight line in the direction from left to right.

[124]($\Downarrow$) For example, a needle, a pin, a sting, etc. See the footnote above in the last line of § E.7.

[125]($\Downarrow$) This French is sic by Gauss.

in statu aequilibrii limes superficiei fluidi liberae U esse nequit, per extensionem finitam, in acie concava superficiei vasis.

E.31. **Relations of quantities of attractions between fluid and vase in respect to the angle** A.

The constant α^2 and β^2, which ratio of the angle A determined depending on the function f and F, and in a sense, we can consider as if the strength of molecular force, of the particle of fluid and using vase. If the function is compared with, fx and Fx are in ratio determination independent of the distance x, putting n and moreover N, we can clearly stated that $\alpha^2 : \beta^2 = cn : CN$, i.e. the constants α^2 and β^2 proportionate to the attraction, where each distance between two molecules of equal volume, one is fluid and the other is vase. In respect to the cases of A, we assume that it is acute, rectangular, obtuse and both are rectangular, as following : [126]

$$\begin{cases} \beta^2 < \frac{1}{2}\alpha^2, & \text{A is acute,} \\ \beta^2 = \frac{1}{2}\alpha^2, & \text{A is rectangular,} \\ \beta^2 > \frac{1}{2}\alpha^2 \text{ or } \beta^2 < \alpha^2 & \text{A is obtuse,} \\ \beta^2 = \alpha^2 & \text{both } \alpha \text{ and } \beta \text{ are rectangular} \end{cases}$$

: in a sense of such supposition (although there were no sufficient reasons, it looks like true, it does not contradict) it must be the following :

- in the first case, the double quantities of particulate attractions of fluid have mutually larger than the double attractions of particle of vase of fluid ;
- in the secondary case, the quantities of first attraction were equal to the double of another ;
- in the third case, the first quantities is minor than double attractions of the other, or the first quantities are larger than another ;
- finally, in the fourth case, the quantities of both attraction equal.

The first example explains the case of mercury in glass vase.

E.32. **In the case of** $\beta^2 > \alpha^2$.

- How much the value of angle A in this case, where the attraction of vase become the largest than the attraction of partial fluid mutually ?
- The imaginary value, which for $\beta^2 > \alpha^2$ the formula $\sin \frac{1}{2}A = \frac{\beta}{\alpha}$ the angle A assign, at the moment prove that the supposition in such case, non admissible.
- In fact the quality $\beta^2 > \alpha^2$, we can not consist the supposition of *limit* on the surface T with the minimal condition with respect to the function W.[127]
- It seems to be that, in everywhere, namely, if we consider infinitesimal expansion as the ultra limit of the fluid layer, as well as T, we take the argument T', and as well as U, to which this argument approximately equals, the value of function W assume the sensible variation equals negative quality $-(2\beta^2 - 2\alpha^2)T'$; this value W continues decreasing infinitesimally for a long time, would occupy total surface of vase up to T'.
- Variation $-(2\beta^2 - 2\alpha^2)T'$ the more it becomes exact, the more the thickness takes minor, and as long as we discuss the value of expression of W, nothing disturb, these thickness takes continuing to disappearance.
- However, this disappearing thickness (exactly distinguish with insensible) is exists except for the mathematical fiction, so that the minimum value for W is got in the case of $\beta^2 = \alpha^2$.
- However, we change the view into our problem of phyisics, when the following accessory of this thickness must be naturally pleasure, even if it is insensible, such that it can keep equilibrium.

126(⇓) cf. (194), (195).

127(⇓) By (194) and (195), we get

$$\int zds - \frac{1}{2g}cs\psi_0 + \frac{1}{2g}\pi ct\theta_0 - \frac{1}{g}\pi CT\Theta_0 \quad \Rightarrow \quad \int zds - \frac{1}{2g}cs\psi_0 + \alpha^2(T+U) - 2\beta^2 T$$

then we get (195) :

$$W \equiv \int zds + (\alpha^2 - 2\beta^2)T + \alpha^2 U$$

- Whenever this part approaches, the expression of W, such as we have mentioned in §E.18, it is incomplete, and we denote it the part of vase, which the layer covers by T', whose thickness in the point is indefine by ρ, the expression Ω[128] extends moreover the boundary

$$\pi c^2 \int \theta' \rho . dT' - \pi c C \int \Theta' \rho . dT'$$

Until this time, the value of this W,

$$\frac{\pi C}{g} \int \Theta' \rho . dT' - \frac{\pi c}{g} \int \theta' \rho . dT' = \int dT' \Big(\frac{2\beta^2}{\Theta_0} . \Theta' \rho - \frac{2\alpha^2}{\theta_0} . \theta' \rho \Big) \tag{233}$$

where, we substitute (233) by the terms as we had denoted in (194) and (195) as follows :

$$\alpha^2 \equiv \frac{\pi c \theta_0}{2g}, \quad \beta^2 \equiv \frac{\pi C T \Theta_0}{2g}, \quad t \equiv T + U,$$

- Therefore, the value of this W, by extension of such a layer, then accept the variation $2(\beta^2 - \alpha^2)T'$, the total variation, its value of W, which we have the situation of the layer omitted, then we have

$$-2 \int dT' \Big[\beta^2 \big(1 - \frac{\Theta' \rho}{\Theta_0}\big) - \alpha^2 \big(1 - \frac{\theta' \rho}{\theta_0}\big) \Big]$$

This variation, for $\theta'_0 = \theta_0$ and $\Theta'_0 = \Theta_0$, become zero for disappearance of thickness : $\theta' \rho$ and $\Theta' \rho$ reduce the density of ρ, the thickness decrease, and then for insensible value of this ρ, evaluated as insensible, the variation of thickness inverse the value $-2(\beta^2 - \alpha^2)T'$ converges, moreover for the equilibrium state of fluid, the expression W becomes never suitable correctly if ultra sensibly decrease, it turns equivalently into sensible.

$$\int zds - 2(\beta^2 - \alpha^2)(T + T') - \alpha^2 T + \alpha^2 U$$

If $\beta^2 - \alpha^2 = 0$ then

$$\int zds - \alpha^2 T + \alpha^2 U$$

i.e. which expression, in the minimum, become for the case $\beta^2 = \alpha^2$.

- Hence, we get the figure of equilibrium fluid in vase, as $\beta^2 > \alpha^2$, for brevity, as the figure of equilibrium fluid in vase $\beta^2 = \alpha^2$, here the difference is strict equilibrium results in the layer of the insensible thickness.
- Besides, Mr. Laplace then stated that, for this case of vase of fluid insensible thickness are covered equivalent to be strictly with such vase, whose particles, the attractive force of fluid particles exist mutually and uniformly.
- By itself, hence, the arrangement obeys the descriptions in §E.18 read as the vertical capillarity ascending fluid in tube : quantity clearly $\beta^2 > \alpha^2$, in which we proposed the formulae that can substitute β with α in this point.

E.33. In the case of $\beta^2 < \alpha^2$.

- In this case, where $\beta^2 < \alpha^2$, the wet vase with the insensible fluid layer can not have the point, even if law of function θ' and Θ' are, when for the value of the function

$$\alpha^2 \big(1 - \frac{\theta' \rho}{\theta'_0}\big) - \beta^2 \big(1 - \frac{\Theta' \rho}{\Theta'_0}\big)$$

for brevity, we describe as $Q\rho$, this value continues increasing, if ρ increases from the sensible value at the zero value : because, clearly from the characteristic of this function $Q\rho$ would contradict with minimal condition.

- By itself, this characteristic occurs the hypothesis, by that in the article 31, where we had stated that fx and Fx are determined independently in proportion of x, from this fact, we deduce that $\frac{\theta'\rho}{\theta'_0} = \frac{\Theta'\rho}{\Theta'_0}$, and namely, $Q\rho = (\alpha^2 - \beta^2)\big(1 - \frac{\theta'\rho}{\theta'_0}\big)$.

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[128] ($\Downarrow$) c.f. (180).

$$\Omega = -gc \int zds + \frac{1}{2} c^2 \iint ds.ds'.\varphi(ds, ds') + cC \iint ds.dS.\Phi(ds, dS)$$

- However, if the functions f and F will occur simultaneously as inverse, it is not at all impossible, that this value $\frac{\theta'\rho}{\theta'_0}$ rapidly decrease, as well as $\frac{\Theta'\rho}{\Theta'_0}$, the function $Q\rho$, in both insensible value of this ρ, at first negative, and after, their values reaches to minimum (i.e. at last, negative), while $\alpha^2 - \beta^2$ ascends by the value 0 of the inverse their positive limit.
- In this case equilibrium at least postulate with insensibility, this thickness in general, showing is stated such that $Q\rho$ contradict not at all sensibly with the least value.
- Although if we denote by $-(\beta')^2$, it turns to $(\beta')^2 < \beta^2$; the figure of other part of the substantial indeterminate fluid, moreover, if in vase, with respect to the situation, β^2 must substitute the quantity $(\beta')^2$, i.e. the angle between plane of the free surface of fluid in contacting substantial part tangent with the wall of vase turns into 2 arcsine$\frac{\beta'}{\alpha'}$. (cf. (232).)
- Moreover, doubts in such case existing in natural phenomena, seem to be filled with the more complicated phenomena.

E.34. **Summary.**

Another with our proposition we presented, the general principle of this sort of stability descending as a result of special phenomena, especially, essential principles fit the theory in this case, by Mr. Laplace and the contemporary with him rushed and succeeded, so many phenomena in fluid equilibrium were solved, the new and so many results were produced : however, even so, the reserved were remained. Inversely, from this, it is possible to indulge in giving out the new light of this argument, or to fall into incorrect interpretation.

¶ I.

- Our theory does not only arrogate by ourselves to determine the figure of fluid equilibrium in mathematical exactitude, but also we recognize that, of the determination of figure, such as, an equilibrium figure varies different only in sensible quantity.
- If we recognize that there are errors in theory something imperfect, then they were
  - to prove in total, or,
  - to prove how much it is possible, or,
  - to prove how long we ignore the molecular attraction.
- In state of equilibrium, the function Ω [129] becomes exactly maximum, so that, the function

$$\frac{2\pi cs\psi 0}{g} - \frac{\Omega}{gc}$$

becomes minimum, this, moreover, for the indole (characteristics) of the molecular attraction, not only the function W is the exact equation, nevertheless, but also insensible in this place different.
- Figure for this W fit minimum, not exact equilibrium figure, if differential become insensible, as long as everywhere move sensible, the function W becomes lowest in the value of figure.
- Clearly, sensible differential in surface curvature is not excluded, as long as it were limited by partially insensible surface :
  - because in equilibrium figure, exact constant-angle over A denotes impossible by considering it sufficient, that if there were immensurable distance between the vase, as Mr. Laplace then had thought correctly that, as if the inclination in limit of sphere of sensible attraction with vase is coincident with sensible value of A.

¶ II.

- We should clearly distinguish the equilibrium figure with quiet figure. Fluid equation in the state of equilibrium, it keeps. In the quiet figure of fluid have a little different equilibrium figure, nevertheless, may occur, and fluid in quiet permanent or if moving, accept the momentum in this moment, before reaching to the equilibrium of fluid, similarly, for example, cubic horizontal plane not only in equilibrium but also super plane.
- Clearly, the first fundamental equation (§28) independently of perturbative limit P, i.e. in addition to, not only minimum condition but also necessary condition, here, we suppose this invariable limit : why, how long this perfect fluid delights in flow, on the other hand, at the same time, another fluid is able to increase freedom, while we postulate the minimum force of motion, the fluid will accommodate inevitably itself to its condition.

---

[129] (⇓) c.f. (180).

- The second principal reason (§29) essentially depend on perfect limit of P on the surface of vase.
- Minimum condition in value W in itself we postulate the equation $i = A$: in fact, since surface fluid will accommodate itself to this first principle, the angle i does not yet reach the normal value, the value is not only W absolute minimum, but also in the equilibrium state, it can not become perfect without translocation of limit P if without fluid motion in contact with vase, what sort of motion can inevitably obstacle friction.
- From here, it is clear that, in an experiment, why each corps institutes this great differential would meet with the angular value i .
- Similarly, in the case, where, $\beta^2 > \alpha^2$, the fluid in vase, whose wall get wet at this time, above all, which is consisted of the law of equilibrium, next, in part, which is substantial fluid, become $i = 2\pi$:
  if this wall in vase were dry until now except for fluid, which is in the state of non equilibrium base of dry vase raise to be possible for equilibrium, after all, the value of angle i reaches to 2π.
- From here, on the other hand, the theory tells us that the capillary phenomena of fluid, such that including the wet wall,
  - in the dry tube, this shows many irregularities, ascending very frequently, small by far,
  - in the wet tube at this time, where the most beautiful harmony with theory is always seen.

¶ III.

The constant inequality made by α and β, from the phenomena it is deduced,

- when the inequality becomes $\beta > \alpha$: where, the figure whose fluid in vase forms equilibrium of various material by its case not defferent with respect to immensurable vase got wet.
- Another inequality $\beta < \alpha$: where, it determines the ratio inter the constant which is the aide of the angle i, therefore, when the mode of ratio that the force is scarcely estimated.
- On mercury in the glass vase, Mr. Laplace studied the angle to be $i = 43°12'$.
- In wide of large precision, by far, the constant α is able to be determined, especially if the wet vase can admit so.
- For water, at $8.5°\ C$ in temperature, we should determine according to the experience cited by Mr. Laplace. [130]
- These sorts of things were already studied by physicians Segner and Gay-Lussac :

E.35. **Conclusions of ours.**

(1) The "two-constant" were defined in terms of kernel functions of $RDFs$, describing the characteristics of dissipation or diffusion within isotropic and homogeneous fluids that were necessary for the interpretation of the nature of fluid or the formulation of the equations of the fluid mechanics including kinetics, equilibrium and capillarity. With their origin perhaps arising in the work of Laplace in 1805, these sorts of functions are simple examples of today's distribution and hyperfunction of Schwartz [70] proposed in 1954/55. Another evidence of the then backgrond is the atome theory by Galton, who suggested the existence of atom in 1808.

(2) Gauss [17] also contributed to develop his self-made RDF or $MDNS$ equations for fluid mechanics including capillary action, because he formulated the equations with two-function instead of two-constant and this is an exceptional case from other contemporaries of NS equations.

(3) According to Bolza [3], Gauss [17] had broken one of the neck of fundamental problems, such as *multiple integral* and *calculus of variations*, however, we must recognaize that even he owed the latter to its progenitor Lagrange, and calculation of capillarity to its progenitor Laplace.

130($\Downarrow$) Following is the footnote by Gauss : H denoted by Mr. Laplace corresponds to our $\pi c\theta 0$, since we denotes α in the author's expression (194), then the expression $\frac{g}{\pi c\theta 0}$ equals $\frac{1}{2\alpha^2}$.

APPENDIX F. **Poisson's paper of capillarity**

F.1. Poisson's comments on Gauss [17].

Poisson [62] commented in the preface about Gauss [17]:

- Gauss' success is due to the merit of his $\prec$ *characteristic* $\succ$
- even Gauss uses the same method as the given physics by Laplace.
- Gauss calculates by the condition only the same density and incompressibility

After all, Poisson insists that

- We can take even any method to solve the problem, and carefully check our own equations and conditions from every points.

The following is a paragraph of the preface by Poisson [62] :

> Par les règles connues du calcul des variations, on détermine la surface inconne du liquide qui rend cette somme un *minimum*, et, comme on sait, on trove à la fois l'équation gènèrale de cette surface et l'équation particulière de son contour, ce qui est l'avantage $\prec$ caractèristique $\succ$ de la méthode que M.Gauss a suive. Mais cet illustre géomètre étant parti des mêmes donnèes physiques que Laplace, et n'ayant pas non plus considéré la variation de densité aux extrèmités du liquide, qu'il a regardé, au contraire, comme incompressible dans tous ses parties, les objections qui s'élèvent contre la théorie de l'autre que par la manière de former les équations d'équilibre. On peut, á cet égard, employer différens moyens ; mais, sans craindre de compliquer le calcul et d'en augmenter les difficultés, il importe de ne négliger aucune des circonstances essentielles de la question, parmi lesquelles il faut compter surtout la dilatation du liquide près de sa surface libre et la condensation qui peut être produite par l'attraction du tube. [62, 8]

(Engl. transl.) By the method known as calculation of variations, we determine the unknown surface of fluid which this sum show minimum, and as we know, we get at once the general equation of the surface and the particular equation of the arbitrary height, these are due to the characteristic advantages of the method Mr. Gauss had approached. But even this great prodigious mathematician had based the similarly given phisics with Laplace, and not considering the variation of density at the extremity of liquid, where there is regard contrary, as the incompressible in all the particle, the objection which evolves to another theory than by the manner of formulation of the equilibrium equations. We can, in this point, use the different methods; but without being afraid to the calculation and the difficulties extended by it, it is important not to neglect any essential circumstances of the problems, among which, to challenge especially the dilatation of liquid in neighborhood of free surface and condensation producing by the attraction of tube.

F.2. Poisson's two constants : K and H in capillary action.

We cite Poisson's K and H from [62, 12-14].

$$K = 2\pi\rho^2 q \int_0^\infty r^3 \varphi r dr$$

where,

$$q \equiv \int_0^\infty \int_0^\infty \frac{(y+z)dydz}{[1+(y+z)^2]^{\frac{3}{2}}} = \frac{1}{3}\int_0^\infty \frac{dy}{(1+y^2)^{\frac{3}{2}}} = \frac{1}{3}$$

$$(1)_P \quad K = \frac{2}{3}\pi\rho^2 \int_0^\infty r^3 \varphi r \; dr \tag{234}$$

$$\eta = u\sin v, \quad \eta' = \cos v$$

$$\zeta = Q\eta^2 + Q'(\eta')^2 + Q''\eta\eta'$$

We denote λ and λ' radii of two principle curvatures.

$$\frac{1}{\lambda} = \frac{d\zeta}{d\eta^2} = 2Q, \quad \frac{1}{\lambda'} = \frac{d\zeta}{d(\eta')^2} = 2Q',$$

The average value

$$\mu = -H(Q+Q') = -\frac{1}{2}H\Big(\frac{1}{\lambda} + \frac{1}{\lambda'}\Big),$$

where, we denote H for convenience sake

$$H \equiv \pi\rho^2 \int_0^\infty \int_0^\infty \varphi r \frac{su^3}{r} duds$$

where,

$$s = ux, \quad ds = udx, \quad u = \frac{r}{\sqrt{1+x^2}}, \quad du = \frac{dr}{\sqrt{1+x^2}}$$

$$(2)_P \quad H = \pi\rho^2 \int_0^\infty r^4 \varphi r dr \int_0^\infty \frac{xdx}{\sqrt{1+x^2}} = \frac{1}{4}\pi\rho^2 \int_0^\infty r^4 \varphi r \ dr \tag{235}$$

The normal action on this point :

$$(3)_P \quad N = K - \frac{1}{2}H\Big(\frac{1}{\lambda} + \frac{1}{\lambda'}\Big) \tag{236}$$

F.3. Coincidence of Poisson's K and H with Laplace's K and H.

Poisson proved Laplace's formulae as follows :

> Les expressions des coefficiens K et H que cette formule renferme s'accordent avec celles que Laplace a trouvées, sous une autre forme, pour les mêmes quantités. En effet, on suppose, dans la *Mécanique céleste,*[131]
>
> $$\int \varphi r \ dr = c - \Pi r, \quad \int r \Pi r \ dr = c' - \Psi r \tag{237}$$
>
> les intégrales commençant avec r, c et c' étant leurs values quand r a une grandeur sensible, Πr et Ψr désignant des founctions qui s'évanouissent pour tout valeur sensible de r. D'apprés cela, on a
>
> $$K = 2\pi\rho^2 \int_0^h \Psi r \ dr, \quad H = 2\pi\rho^2 \int_0^h r\Psi r \ dr \tag{238}$$
>
> en rétablissant la densité ρ que Laplace a prise pour unité, et la limite h étant une quantité de grandeur sensible, qu'on pourra, si l'on vent, remplacer par l'infini. Or, si l'on intègre par partie, il vient
>
> $$K = 2\pi\rho^2 h\Psi h - 2\pi\rho^2 \int_0^h r\frac{d\Psi r}{dr} dr = 2\pi\rho^2 \int_0^h r^2 \Pi r \ dr,$$
>
> $$H = \pi\rho^2 h^2 \Psi h - \pi\rho^2 \int_0^h r^2 \frac{d\Psi r}{dr} dr = \pi\rho^2 \int_0^h r^3 \Pi r \ dr$$
>
> intégrant de nouveau, on a
>
> $$K = \frac{2\pi\rho^2}{3} h^3 \Pi h - \frac{2\pi\rho^2}{3} \int_0^h r^3 \frac{d\Pi r}{dr} dr = \frac{2\pi\rho^2}{3} \int_0^h r^3 \varphi r \ dr \tag{239}$$
>
> $$H = \frac{\pi\rho^2}{4} h^4 \Pi h - \frac{\pi\rho^2}{4} \int_0^h r^4 \frac{d\Pi r}{dr} dr = \frac{\pi\rho^2}{4} \int_0^h r^4 \varphi r \ dr \tag{240}$$
>
> ce qui coïncide avec les formes (234) et (235), en prenant $h = \infty$. [62, pp.14-15]

(Engl. transl.) The expressions with coefficients K and H which these formulae included are coincident with that which Laplace had found under another form of (238), for the same quantities. In fact, we see that in *Mécanique céleste,* as follows :

$\cdots$ (238)

the integrals of the right hand-side of (237) beginning with r, c, and c' in which values r were sensibly large, Πr and Ψr are designated as *the dissipating functions*, even if r were sensibly large value. For this reason, it turns : [132]

$\cdots$ (expressions)

Laplace set density by $\rho = 1$, and h the big value, then we substitute h with ∞. Or if integrate it by parts, it turns out

---

[131]($\Downarrow$) cf. (139).

[132]($\Downarrow$) We cite these two-constant (238) : K and H by Laplace replacing $h = \infty$ in Table 3. These equations are described above in the preface by Gauss. cf.(176).

$\cdots$ (239), (240)

where if we replace $h = \infty$, then we get a coincidence with our formulae (234) and (235). [133]

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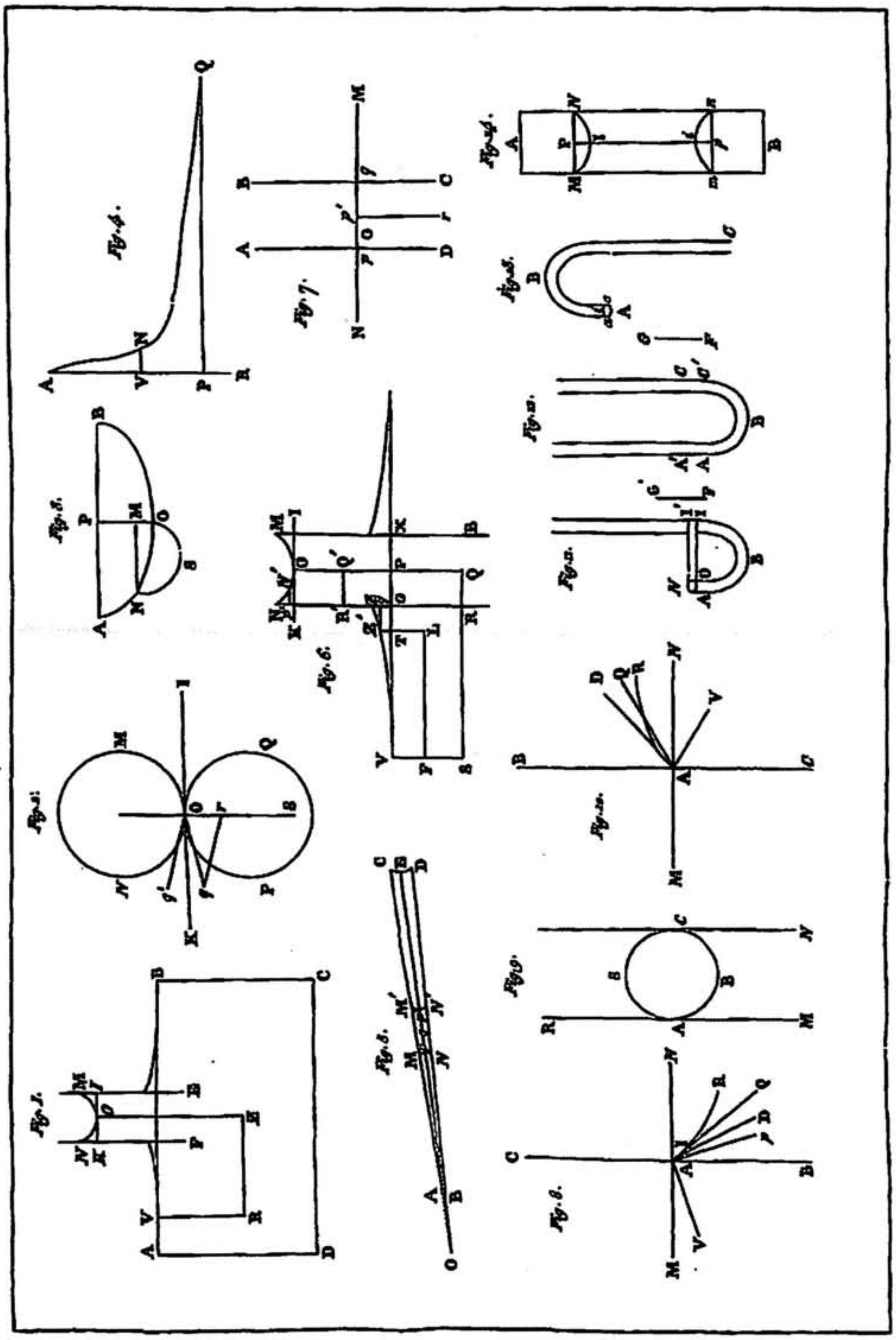
Remark: we use *Lu* (: in French) in the bibliography meaning "read" date by the referees of the journals, for example MAS. In citing the original paragraphs in our paper, the underscoring are of ours.

Appendix G. Figures

The following original figures were drawn by Laplace [35], in which we cited only *fig. 1* and *fig. 3* in the appendix C.1.2.



Remark : The *Fig.1* and *Fig.3* in these figures, correspond the above-mentioned figures which are identified with *fig.1* and *fig.3* in Appexdix C.1.2. citing from this original figures by Laplace.

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The microscopically-descriptive hydromechanics equations in the gas theory

ABSTRACT. The microscopically-description of hydromechanics equations are followed by the description of equations of gas theory by Maxwell, Kirchhoff and Boltzmann. Above all, in 1872, Boltzmann formulated the Boltzmann equations, expressed by the following today's formulation :

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = Q(f,g), \quad t>0, \quad \mathbf{x},\mathbf{v}\in\mathbb{R}^n (n\geq 3), \quad \mathbf{x}=(x,y,z), \quad \mathbf{v}=(\xi,\eta,\zeta), \tag{1}$$

$$Q(f,g)(t,x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{g(v'_*)f(v') - g(v_*)f(v)\}d\sigma dv_*, \quad g(v'_*) = g(t,x,v'_*), \text{ etc.} \tag{2}$$

These equations are able to be reduced for the general form of the hydrodynamic equations, after the formulations by Maxwell and Kirchhoff, and from which the Euler equations and the Navier-Stokes equations are reduced as the special case.

After Stokes' linear equations, the equations of gas theories were deduced by Maxwell in 1865, Kirchhoff in 1868 and Boltzmann in 1872, They contributed to formulate the fluid equations and to fix the Navier-Stokes equations, when Prandtl stated the today's formulation in using the nomenclature as the "so-called Navier-Stokes equations" in 1934, in which Prandtl included the three terms of nonlinear and two linear terms with the ratio of two coefficients as 3 : 1, which arose from Poisson in 1831, Saint-Venant in 1843, and Stokes in 1845.

Mathematics Subject Classification 2010 : 01Axx, 76A02, 76Mxx, 76-02, 76-03, 33A15, 35Qxx 35-xx.
Key words. The Boltzmann equations, the transport equations, gas theory, the Navier-Stokes equations, two-constant theory, tensor function, fluid dynamics, fluid mechanics, hydrostatics, hydrodynamics, hydromechanics, microscopically-descriptive equations, mathematical history.

Contents

1. Introduction

[1]

We have studied the original microscopically descriptive Navier-Stokes ($MDNS$) equations as the progenitors [2], Navier, Cauchy, Poisson, Saint-Venant and Stokes, and endeavor to ascertain their aims and conceptual thoughts in formulations these new equation. "The two-constant theory" was introduced first introduced in 1805 by Laplace [3] in regard to capillary action with constants denoted by H and K.

Thereafter, various pairs of constants have been proposed by their progenitors in formulating $MDNS$ equations or equations describing equilibrium or capillary situations. It is commonly accepted that this theory describes isotropic, linear elasticity. [4] We can find the "two-constant" in the equations of gas theories by Maxwell, Kirchhoff and Boltzmann, which were fixed into the common linear terms, and which originally takes its rise in Poisson and Stokes.

The gas theorists studied also the general equations of hydromechanics, which have the same proportion of coefficients as the equations deduced by Poisson and Stokes with only the linear term and the ratio of the coefficient of the tensor function with the main axis of Laplacian to that of gradient of divergence term is $3:1$. (cf Table 2.)

2. A universal method for the two-constant theory

In this section, we propose a universal method to describe the kinetic equations that arise in isotropic, linear elasticity. This method is outlined as follows:

- The partial differential equations describing waves in elastic solids or flows in elastic fluids are expressed by using one constant or a pair of constants C_1 and C_2 such that:

$$\text{for elastic solids:} \qquad \frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1 T_1 + C_2 T_2) = \mathbf{f},$$
$$\text{for elastic fluids:} \qquad \frac{\partial \mathbf{u}}{\partial t} - (C_1 T_1 + C_2 T_2) + \cdots = \mathbf{f},$$

  where $T_1,\ T_2, \cdots$ are the terms depending on tensor quantities constituting our equations. For example, the NS equations corresponding to incompressible fluids consist of the kinetic equation along with the continuity equation and are conventionally written, in modern vector notation, as follows:

$$\frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0. \tag{3}$$

  Here $\mathbf{u}$ is the velocity, $\mathbf{f}$ accounts for the body forces present, p the pressure and $\Delta \equiv \nabla \cdot \nabla$ the Laplacian operator.
- The two coefficients C_1 and C_2 associated with the tensor terms are the two constants of the theory, definitions of which depend on the contributing author. For example, ε and E were introduced by Navier, R and G by Cauchy, k and K in elastic and $(K+k)\alpha$ and $\frac{(K+k)\alpha}{3}$ in fluid by Poisson, ε and $\frac{\varepsilon}{3}$ by Saint-Venant, and μ and $\frac{\mu}{3}$ by Stokes. Since Poisson, the ratio of two coefficient in fluid was fixed at 3. Moreover, C_1 and C_2 can be expressed in the following form:

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[1] $(\Downarrow)$ Throughout this paper, in citation of bibliographical sources, we show our own paragraph or sentences of commentaries by surrounding between $(\Downarrow)$ and $(\Uparrow)$. $((\Uparrow)$ is used only when not following to next section,). And by $=^*$, we detail the statement by original authors, because we would like to discriminate and to avoid confusion from the descriptions by original authors. The mark : $\Rightarrow$ means transformation of the statements in brevity by ours. And all the frames surrounding the statements are inserted for important remark of ours. Of course, when the descriptions are explicitly distinct without these marks, these are not the descriptions in citation of bibliographical sources.

[2] $(\Downarrow)$ To establish a time line of these contributor, we list for easy reference the year of their birth and death: Sir I.Newton(1643-1727), D.Bernoulli(1700-1782), Euler(1707-1783), d'Alembert(1717-1783), Lagrange(1736-1813), Laplace(1749-1827), Fourier(1768-1830), Gauss(1777-1855), Navier(1785-1836), Poisson(1781-1840), Cauchy(1789-1857), Saint-Venant(1797-1886), Stokes(1819-1903). The order in our paper below is by date of proposal or publication.

[3] $(\Downarrow)$ Of capillary action, Laplace[11, V.4, Supplement p.2] achnowledges Clailaut and Clailaut cites Maupertuis.

[4] $(\Downarrow)$ Darrigol [6, p.121].

TABLE 1. The two constants in the kinetic equations

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Navier [14]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$		ρ : radius
2	Navier [15]	fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$		ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$	
3	Cauchy [3]	system of particles in elastic and fluid	R		$\frac{2\pi}{15}\Delta$		$\int_0^\infty dr$	r^3		$f(r)$		$f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)]$
				G		$\frac{2\pi}{3}\Delta$	$\int_0^\infty dr$		r^3		$\pm\mathrm{f}(r)$	$f(r) \neq \mathrm{f}(r)$, $\Delta = \frac{\mathcal{M}}{\mathcal{V}}$: mass of molecules per volume.
4	Poisson [16]	elastic solid	k		$\frac{2\pi}{15}$		$\sum \frac{1}{\alpha^5}$	r^5		$\frac{d.\frac{1}{r}fr}{dr}$		
				K		$\frac{2\pi}{3}$	$\sum \frac{1}{\alpha^5}$		r^3		fr	
5	Poisson [17]	elastic solid and fluid	k		$\frac{1}{30}$		$\sum \frac{1}{\varepsilon^3}$	r^3		$\frac{d.\frac{1}{r}fr}{dr}$		$C_3 = \frac{1}{4\pi}\frac{2\pi}{15} = \frac{1}{30}$
				K		$\frac{1}{6}$	$\sum \frac{1}{\varepsilon^3}$		r		fr	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
6	Saint-Venant [21]	fluid	ε	$\frac{\varepsilon}{3}$								
7	Stokes [22]	fluid	μ	$\frac{\mu}{3}$								
8	Stokes [22]	elastic solid	A	B								$A = 5B$

$$\begin{cases} C_1 \equiv \mathcal{L}r_1 g_1 S_1, \\ C_2 \equiv \mathcal{L}r_2 g_2 S_2, \end{cases} \quad \begin{cases} S_1 = \iint g_3 \to C_3, \\ S_2 = \iint g_4 \to C_4, \end{cases} \quad \Rightarrow \quad \begin{cases} C_1 = C_3\mathcal{L}r_1 g_1 = \frac{2\pi}{15}\mathcal{L}r_1 g_1, \\ C_2 = C_4\mathcal{L}r_2 g_2 = \frac{2\pi}{3}\mathcal{L}r_2 g_2. \end{cases}$$

Here $\mathcal{L}$ corresponds to either $\sum_0^\infty$ as argued for by Poisson or $\int_0^\infty$ as argued for by Navier. A heated debate had developed between the two over this point. It is a matter of personnel preference as to how the two constants should be expressed.

- The two constants depend on two radial functions r_1 and r_2 related to the radius of the active sphere of the molecules, raised to some power of n for Poisson's and Navier's cases; the relationship between these functions can be expressing by a logarithm with base r such that: $\log_r \frac{r_1}{r_2} = 2$.
- g_1 and g_2 are the kernel functions having both
  - the physical caracteristics come from the fluid dynamics described by the microscopically basic relations of the attraction and/or repulsion and
  - the mathematical requirements for the rapidly decreasing function.
- S_1 and S_2 are two expressions which determine the angular dependence on the surface of the active unit-sphere centered on a molecule through application of the double integral (or single sum in the case of Poisson's fluid).
- g_3 and g_4 are certain compound spherical harmonic functions determining the momentum over the unit sphere.
- C_3 and C_4 are indirectly determined as the common coefficients derived from the invariant tensor. With the exception of Poisson's fluid case, C_3 of C_1 is $\frac{2\pi}{3}$, and C_4 of C_2 is $\frac{2\pi}{15}$, which are evaluated over the unit spheres for each molecule, and which are independent of the preference in using integrals or summations. In Poisson's case, we obtain the same values as the above after multiplying by $\frac{1}{4\pi}$. The integrals are calculated from the total momentum of the active sphere surrounding the molecule.
- The ratio of C_3 to C_4 : $\frac{C_3}{C_4} = \frac{1}{5}$ including Poisson's case.

2.1. **Poisson's Fluid pressure in motion.**

TABLE 2. The kinetic equations of the hydrodynamics until the "Navier-Stokes equations" were fixed. (Rem. HD : hydrodynamics, N under entry-no : non-linear, gr.dv : grad.div, E : $\frac{\Delta}{gr.dv}$ in elastic, F : $\frac{\Delta}{gr.dv}$ in fluid. The group of entry 6-14 show $F = 3$ in fluid.)

no	name/prob	the kinetic equations	Δ	gr.dv	E	F
1 N	Euler (1752-55) [7, p.127] fluid	$\begin{cases} X - \frac{1}{h}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ Y - \frac{1}{h}\frac{dp}{dy} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ Z - \frac{1}{h}\frac{dp}{dz} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}, \end{cases} \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$				
2	Navier (1827)[14] elastic solid	$(6\text{-}1)_{Ne} \quad \begin{cases} \frac{\Pi}{g}\frac{d^2x}{dt^2} = \varepsilon\left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dbda} + 2\frac{d^2z}{dcda}\right), \\ \frac{\Pi}{g}\frac{d^2y}{dt^2} = \varepsilon\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dcdb}\right), \\ \frac{\Pi}{g}\frac{d^2z}{dt^2} = \varepsilon\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right) \end{cases}$ where Π is density of the solid, g is acceleration of gravity.	ε	2ε	$\frac{1}{2}$	
3 N	Navier (1827)[15] fluid	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\,; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\,; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\,; \end{cases}$	ε	2ε		$\frac{1}{2}$
4	Cauchy (1828)[3] system of particles in elastic solid and fluid	$\begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2}, \end{cases}$ $G = H = I, \quad L = M = N, \quad P = Q = R, \quad L = 3R$	$R+G$	$2R$	if $G = 0$ $\frac{1}{2}$	if $G = 0$ $\frac{1}{2}$
5	Poisson (1831)[17] elastic solid defined in general equations	$\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2u}{dx^2}, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2v}{dy^2}, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = \frac{\Pi}{\rho}\frac{d^2w}{dz^2}, \end{cases}$	$\frac{a^2}{3}$	$\frac{2a^2}{3}$	$\frac{1}{2}$	
6	Poisson (1831)[17] fluid defined in general equations	$\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K+k)\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K+k)\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K+k)\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \end{cases}$ $\begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) \end{cases}$ where $\varpi \equiv p - \alpha\frac{d\psi t}{dt} - \frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \quad \beta \equiv \alpha(K+k)$	β	$\frac{\beta}{3}$		3
7	Saint-Venant (1843)[21] fluid	He didn't describe the equations in [21], however his tensor is in Table 5 (4).	ε	$\frac{\varepsilon}{3}$		3
8	Stokes (1849)[22] fluid	$(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0. \end{cases}$	μ	$\frac{\mu}{3}$		3
9	Maxwell (1865-66) [12] HD	$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{dp}{dx} - C_M\left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{dp}{dy} - C_M\left[\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{dp}{dz} - C_M\left[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Z \end{cases} \quad \text{where, } C_M \equiv \frac{pM}{6k\rho\Theta_2}$	C_M	$\frac{C}{3}$		3
10	Kirchhoff (1876)[9] HD	$\begin{cases} \mu\frac{du}{dt} + \frac{\partial}{\partial x} - C_K\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu X, \\ \mu\frac{dv}{dt} + \frac{\partial p}{\partial y} - C_K\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Y, \\ \mu\frac{dw}{dt} + \frac{\partial p}{\partial z} - C_K\left[\Delta z + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Z, \end{cases} \quad \begin{cases} \frac{1}{\mu}\frac{d\mu}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ \text{where, } C_K \equiv \frac{1}{3\kappa}\frac{p}{\mu} \end{cases}$	C_K	$\frac{\Delta}{3}$		3
11 N	Rayleigh (1883)[20] HD	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = -\frac{du}{dt} + \nu\nabla^2 u - u\frac{du}{dx} - v\frac{du}{dy}, \\ \frac{1}{\rho}\frac{dp}{dy} = -\frac{dv}{dt} + \nu\nabla^2 v - u\frac{dv}{dx} - v\frac{dv}{dy} \end{cases} \quad , \quad \frac{du}{dx} + \frac{dv}{dy} = 0$	ν			
12	Boltzmann (1895)[2] HD	$(221)_B \quad \begin{cases} \rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \mathcal{R}\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} - \mathcal{R}\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} - \mathcal{R}\left[\Delta w + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Z \end{cases}$	$\mathcal{R}$	$\frac{\mathcal{R}}{3}$		3
13 N	Prandtl (1905)[18] HD	$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + \nabla(V+p) = k\nabla^2 v, \quad \text{div } \mathbf{v} = 0$	k			
14 N	Prandtl (1934)[19] HD	$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$ for incompressible, it is simplified as follows : div $\mathbf{w} = 0$, $\frac{D\mathbf{w}}{dt} = g - \frac{1}{\rho}\text{grad } p + \nu\Delta\mathbf{w}$	ν	$\frac{\nu}{3}$		3

TABLE 3. Geneology of tensors

1	name	tensor (3×3)	coefficient matrix (3 × 5) in equations
1-1	Navier elasticity	$t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(5\text{-}4)_{N^e}$ $-\varepsilon \begin{bmatrix} \left(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ \left(\frac{dw}{dx}+\frac{du}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \left(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\right) \end{bmatrix}$ $= -\varepsilon \begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix}$, where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	We define the coefficient matrix in elasticity : C_T^e : the coefficient of $\begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial z^2} & \frac{\partial^2 v}{\partial x\partial y} & \frac{\partial^2 w}{\partial z\partial x} \\ \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial y^2} & \frac{\partial^2 v}{\partial z^2} & \frac{\partial^2 w}{\partial y\partial z} & \frac{\partial^2 u}{\partial x\partial y} \\ \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial y^2} & \frac{\partial^2 w}{\partial z^2} & \frac{\partial^2 u}{\partial z\partial x} & \frac{\partial^2 v}{\partial y\partial z} \end{bmatrix}$, then $(6\text{-}1)_{N^e} \Rightarrow C_T^e = -\varepsilon \begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix}$
1-2	Navier fluid	$t_{ij} = (p - \varepsilon u_{k,k})\delta_{ij} - \varepsilon(u_{i,j} + u_{j,i})$ $\begin{bmatrix} \epsilon' - 2\varepsilon\frac{du}{dx} & -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) \\ -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & \epsilon' - 2\varepsilon\frac{dv}{dy} & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) \\ -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) & \epsilon' - 2\varepsilon\frac{dw}{dz} \end{bmatrix}$, where $\epsilon' = p - \varepsilon(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz})$	Similarly, we define the coefficient matrix in fluid : C_T^f , which contains p in (1,1)-, (2,2)- and (3,3)-element. $C_T^f = \begin{bmatrix} p-3\varepsilon & -\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & p-3\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & -\varepsilon & p-3\varepsilon & -2\varepsilon & -2\varepsilon \end{bmatrix}$
2	Cauchy system (contains both elasticity and fluid)	$t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(60)_C$ $\begin{bmatrix} k\frac{\partial\xi}{\partial a}+K\nu & \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) \\ \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & k\frac{\partial\eta}{\partial b}+K\nu & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) \\ \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) & k\frac{\partial\zeta}{\partial c}+K\nu \end{bmatrix}$, where $\nu = \frac{\partial\xi}{\partial a}+\frac{\partial\eta}{\partial b}+\frac{\partial\zeta}{\partial c}$	$(46)_C \Rightarrow C_T^e = \begin{bmatrix} L & R & Q & 2R & 2Q \\ R & M & P & 2P & 2R \\ Q & P & N & 2Q & 2P \end{bmatrix}$ $\Rightarrow R\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix}$, where $P = Q = R, \quad L = M = N, \quad L = 3R.$
3-1	Poisson elasticity	$t_{ij} = -\frac{a^2}{3}(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ $(6)_{P^e}$ $-\frac{a^2}{3}\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix}$, where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	$(6)_{P^e}$ $\begin{cases} X - \frac{d^2u}{dt^2} \\ +a^2\left(\frac{d^2u}{dx^2}+\frac{2}{3}\frac{d^2v}{dydx}+\frac{2}{3}\frac{d^2w}{dzdx}+\frac{1}{3}\frac{d^2u}{dy^2}+\frac{1}{3}\frac{d^2u}{dz^2}\right)=0, \\ Y - \frac{d^2v}{dt^2} \\ +a^2\left(\frac{d^2v}{dy^2}+\frac{2}{3}\frac{d^2u}{dxdy}+\frac{2}{3}\frac{d^2w}{dzdy}+\frac{1}{3}\frac{d^2v}{dx^2}+\frac{1}{3}\frac{d^2v}{dz^2}\right)=0, \\ Z - \frac{d^2w}{dt^2} \\ +a^2\left(\frac{d^2w}{dz^2}+\frac{2}{3}\frac{d^2u}{dxdz}+\frac{2}{3}\frac{d^2v}{dydz}+\frac{1}{3}\frac{d^2w}{dx^2}+\frac{1}{3}\frac{d^2w}{dy^2}\right)=0, \end{cases}$ $\Rightarrow C_T^e = -\frac{a^2}{3}\begin{bmatrix} 3 & 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 & 2 \end{bmatrix}$
3-2	Poisson fluid	$t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ $(7\text{-}7)_{Pf}$ $\begin{bmatrix} \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) & \pi+2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \pi+2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) \\ \pi+2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) \end{bmatrix}$, where $\pi = p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt}$	$(7\text{-}9)_{Pf} \Rightarrow C_T^f = \begin{bmatrix} \varpi+\beta & \beta & \beta & 0 & 0 \\ \beta & \varpi+\beta & \beta & 0 & 0 \\ \beta & \beta & \varpi+\beta & 0 & 0 \end{bmatrix}$. According to Stokes: if we put $\varpi = p + \frac{\alpha}{3}(K+k)\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)$ $\Rightarrow C_T^f = \begin{bmatrix} p+\frac{4\beta}{3} & \beta & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & p+\frac{4\beta}{3} & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & \beta & p+\frac{4\beta}{3} & \frac{\beta}{3} & \frac{\beta}{3} \end{bmatrix} \Rightarrow (12)_S.$ Remark: $\alpha(K+k) = \beta.$
4	Saint-Venant fluid	$t_{ij} = (\frac{1}{3}(P_{xx}+P_{yy}+P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j}+v_{j,i})$ $= (-p - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j}+v_{j,i})$ $\begin{bmatrix} \pi+2\varepsilon\frac{d\xi}{dx}, & \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) \\ \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \pi+2\varepsilon\frac{d\eta}{dy} & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) \\ \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) & \pi+2\varepsilon\frac{d\zeta}{dz} \end{bmatrix}$, where $\pi = \frac{1}{3}\left(P_{xx}+P_{yy}+P_{zz}\right) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$ $\equiv -p - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$	no description in [21].
5	Stokes fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j}+v_{j,i})$, tensor $= -1 \times$ $\begin{bmatrix} p-2\mu\left(\frac{du}{dx}-\delta\right) & -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-2\mu\left(\frac{dv}{dy}-\delta\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-2\mu\left(\frac{dw}{dz}-\delta\right) \end{bmatrix}$ where $3\delta = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	$(12)_S \Rightarrow C_T^f = \begin{bmatrix} -p+\frac{4\mu}{3} & \mu & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & -p+\frac{4\mu}{3} & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & \mu & -p+\frac{4\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} \end{bmatrix}$. Remark: $\frac{4}{3}\mu = 2\mu(1-\frac{1}{3})$

TABLE 4. Geneology of tensors (continued.)

1	name	tensor (3×3)
6	Maxwell fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$, $\begin{bmatrix} p - \frac{M}{9k\rho\Theta_2}p\left(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - \frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx} - 2\frac{dv}{dy} - \frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - \frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx} - \frac{dv}{dy} - 2\frac{dw}{dz}\right) \end{bmatrix}$
7	Kirchhoff fluid	$t_{ij} = (-p - 2kv_{l,l})\delta_{ij} + k(v_{i,j} + v_{j,i})$, $\begin{bmatrix} p - 2k\frac{\partial u}{\partial x} & -k\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -k\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ -k\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - 2k\frac{\partial v}{\partial y} & -k\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ -k\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -k\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - 2k\frac{\partial w}{\partial z} \end{bmatrix}$
8	Boltzmann fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$, $\begin{bmatrix} p - 2\mathcal{R}\left\{\frac{\partial u}{\partial x} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - 2\mathcal{R}\left\{\frac{\partial v}{\partial y} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - 2\mathcal{R}\left\{\frac{\partial w}{\partial z} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} \end{bmatrix}$ where, $\mathcal{R} = \frac{M}{6k\rho\Theta_2}p$.

- § 63.

[5] Poisson's tensor of the pressures in fluid reads as follows :

$(7\text{-}7)_{Pf}$

$$\begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix} = \begin{bmatrix} \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) \\ p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) \end{bmatrix},$$

$$(k+K)\alpha = \beta, \quad (k-K)\alpha = \beta', \quad p = \psi t = K, \quad \text{then} \quad \beta + \beta' = 2k\alpha, \tag{4}$$

where χt is the density of the fluid around the point M, and ψt is the pressure. Here we can replace the first column with the third one, then we see easily the conventional style of array as follows :

$$\begin{bmatrix} U_3 & U_2 & U_1 \\ V_3 & V_2 & V_1 \\ W_3 & W_2 & W_1 \end{bmatrix} = \begin{bmatrix} p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{du}{dx} & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) & \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) \\ \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dv}{dy} & \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) & \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dw}{dz} \end{bmatrix},$$

The elements of velocity $\mathbf{u} = (u, v, w)$ are :

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w$$

$$\begin{cases} \frac{d^2x}{dt^2} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ \frac{d^2y}{dt^2} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ \frac{d^2z}{dt^2} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz} \end{cases}$$

[5]($\Downarrow$) In Poisson [17], the title of the chaper 7 is "*Calcul des Pressions dans les Fluides en mouvement ; équations defferentielles de ce mouvement.*"

$$\varpi \equiv p - \alpha \frac{d\psi t}{dt} - \frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \tag{5}$$

a

$$(7\text{-}9)_{Pf} \qquad \begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}). \end{cases} \tag{6}$$

[a]($\Downarrow$) $(7\text{-}9)_{Pf}$ means the equation number with chapter of Poisson [17]

If we put $\mathbf{f} = (X, Y, Z)$ then (6) becomes as follows :

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\beta}{\rho}\Delta\mathbf{u} + \frac{1}{\rho}\nabla\varpi = \mathbf{f} \tag{7}$$

2.2. Stokes' comment on Poisson's fluid equations.

Stokes comments on Poisson's $(7\text{-}9)_{Pf}$ as follows :

> On this supposition we shall get the value of $\frac{d\Psi t}{dt}$ from that of $R_1' - K$ in the equations of page 140 by putting
>
> $$\frac{du}{dx} = \frac{dv}{dy} = \frac{dw}{dz} = -\frac{1}{3\chi t}\frac{d\chi t}{dt}.$$
>
> We have therefore
>
> $$\alpha\frac{d\chi t}{dt} = \frac{\alpha}{3}(K - 5k)\frac{d\chi t}{\chi t dt}.$$
>
> $$(7\text{-}2)_{Pf} \qquad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = -\frac{1}{\chi t}\frac{d\chi t}{dt}. \tag{8}$$
>
> Putting now for $\beta + \beta'$ its value $2\alpha k$, and for $\frac{1}{\chi t}\frac{d\chi t}{dt}$ its value given by equation (8) [6], the expression for ϖ, page 152, [7] becomes
>
> $$\varpi = p - \alpha\frac{d\psi t}{dt} - \frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt} = p - \Big(\frac{\alpha}{3}(K-5k) + 2\alpha k\Big)\frac{d\chi t}{\chi t dt} = p + \frac{\alpha}{3}(K+k)(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}).$$
>
> Observing that $\alpha(K + k) = \beta$, this value of ϖ reduces Poisson's equation $(7\text{-}9)_{Pf}$ [=(6)] to the equation $(12)_S$ of this paper. ([22, p.119]).

Namely, by using $\alpha(K + k) = \beta$ in (4), we get the following :

$$\begin{cases} \frac{d\varpi}{dx} = \frac{dp}{dx} + \frac{\beta}{3}\frac{d}{dx}(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \\ \frac{d\varpi}{dy} = \frac{dp}{dy} + \frac{\beta}{3}\frac{d}{dy}(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \\ \frac{d\varpi}{dz} = \frac{dp}{dz} + \frac{\beta}{3}\frac{d}{dz}(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \end{cases}$$

then (6)($= (7\text{-}9)_{Pf}$) turns out :

$$\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K+k)\Big(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\Big) + \frac{\alpha}{3}(K+k)\frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K+k)\Big(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\Big) + \frac{\alpha}{3}(K+k)\frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K+k)\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\Big) + \frac{\alpha}{3}(K+k)\frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \end{cases}$$

$$\Rightarrow (12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\Big(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\Big(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0. \end{cases}$$

[6]($\Downarrow$) Poisson[17, p.141]

[7]($\Downarrow$) cf. (5)

Therefore, Poisson contains both compressible and incompressible fluid.

3. *'Drafts of 'On the dynamical theory of Gases'* by Maxwell

3.1. A progenitor of gas theory after Poisson and Stokes.

($\Downarrow$) Even after Poisson, Saint-Venent and Stokes, we can cite the progenitors of microscopically descriptive, hydromechanical equations, which are specializes in gas theories, in which they describe the hydrodynamic equations, and they contribute to fix the tensor and equations of NS, so we have to trace them. cf. Table 2, 3, 4.

Maxwell [12] had presented between late 1865 and early 1866, the original equations calculating his original coefficient, with which his tensor coincides with Poisson and Stokes, and his gas theory prior to Kirchhoff [9] in 1876 and Boltzmann [2] in 1895 as follows: ($\Uparrow$)

if the motion is not very violent we may also neglect $\frac{\partial}{\partial t}(\rho\xi^2 - p)$ and then we have

$$\xi^2\rho = p - \frac{M}{9k\rho\Theta_2}p\Big(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\Big) \tag{9}$$

which similar expressions for $\eta^2\rho$ and $\zeta^2\rho$. By transformation of coordinates we can easily obtain the expressions for $\xi\eta\rho$, $\eta\zeta\rho$ and $\zeta\xi\rho$. They are of the form

$$\zeta\xi\rho = -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) \tag{10}$$

$$\begin{bmatrix} \rho\overline{\xi_0^2} & \rho\overline{\xi_0\eta_0} & \rho\overline{\xi_0\zeta_0} \\ \rho\overline{\xi_0\eta_0} & \rho\overline{\eta_0^2} & \rho\overline{\eta_0\zeta_0} \\ \rho\overline{\xi_0\zeta_0} & \rho\overline{\zeta_0\eta_0} & \rho\overline{\zeta_0^2} \end{bmatrix} = \begin{bmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix},$$

Having thus obtained the values of the pressures in different directions we may substitute them in the equation of motion

$$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{d}{dx}(\rho\xi^2) + \frac{d}{dy}(\rho\xi\eta) + \frac{d}{dz}(\rho\xi\zeta) = X\rho, \\ \rho\frac{\partial v}{\partial t} + \frac{d}{dx}(\rho\xi\eta) + \frac{d}{dy}(\rho\eta^2) + \frac{d}{dz}(\rho\eta\zeta) = Y\rho, \\ \rho\frac{\partial w}{\partial t} + \frac{d}{dx}(\rho\xi\zeta) + \frac{d}{dy}(\rho\zeta\eta) + \frac{d}{dz}(\rho\zeta^2) = Z\rho. \end{cases} \tag{11}$$

This becomes as follows :

$$\boxed{\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{dp}{dx} - \frac{pM}{6k\rho\Theta_2}\Big[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3}\frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{dp}{dy} - \frac{pM}{6k\rho\Theta_2}\Big[\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3}\frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{dp}{dz} - \frac{pM}{6k\rho\Theta_2}\Big[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} + \frac{1}{3}\frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big] = \rho Z. \end{cases}} \tag{12}$$

Maxwell states as follows:

> This is the equation of motion in the direction of x. The other equations may be written down by symmetry. The form of the equations is identical
> - with that deduced by Poisson [8] from the theory of elasticity by supposing the strain to be constantly relaxed at the given rate
> - and the ratio of the coefficients of ∇^2 to $\frac{d}{dx}\frac{1}{\rho}\frac{\partial\rho}{\partial t}$ agrees with that given by Professor Stokes, [9] which means (12) equals $(12)_S$.
>
> The quantity $\frac{pM}{6k\rho\Theta_2}$ is the coefficient of viscosity or of internal friction and is denoted by μ in the writings of Professor Stokes and in my paper on the Viscosity of Air and other Gases. [13, pp.261-262].

[8]($\Downarrow$) The Equation(9) in [17, p.139], which we cite as (6) $(7\text{-}9)_{Pf}$ above.

[9]($\Downarrow$) Stokes [22]

3.2. **Law of Volumes.**

In late 1865 or early 1866, Maxwell proposed this paper. It was likely that Boltzmann[10] had got his idea from this paper.

> $u,\ v,\ w$ are the components of the mean velocity of all the molecules which are at a given instant in a given element of volume, hence there is no motion of translation. $\xi,\ \eta,\ \zeta$ are the components of the relative velocity of one of these molecules with respect to the mean velocity, the 'velocity of agitation of molecules'.
>
> In the case of a single gas in motion let Q be the total energy of a single molecule then
>
> $$Q = \frac{1}{2}M\Big\{(u+\xi)^2 + (v+\eta)^2 + (w+\zeta)^2 + \beta(\xi^2+\eta^2+\zeta^2)\Big\}$$
>
> and
>
> $$\frac{\delta Q}{\delta t} = M(uX + vY + wZ).$$
>
> The general equation becomes

$$\begin{aligned}
&\frac{1}{2}\rho\frac{\partial}{\partial t}\Big\{u^2+v^2+w^2+(1+\beta)(\xi^2+\eta^2+\zeta^2)\Big\}\\
+&\ \frac{d}{dx}(u\rho\xi^2+v\rho\xi\eta+w\rho\xi\zeta)+\frac{d}{dy}(u\rho\xi\eta+v\rho\eta^2+w\rho\eta\zeta)+\frac{d}{dz}(u\rho\xi\zeta+v\rho\eta\zeta+w\rho\zeta^2)\\
+&\ \frac{1}{2}\frac{d}{dx}(1+\beta)\rho\xi(\xi^2+\eta^2+\zeta^2)+\frac{1}{2}\frac{d}{dy}(1+\beta)\rho\eta(\xi^2+\eta^2+\zeta^2)+\frac{1}{2}\frac{d}{dz}(1+\beta)\rho\zeta(\xi^2+\eta^2+\zeta^2)\\
=&\ \rho(uX+vY+wZ).
\end{aligned}$$

Substituting the values of $\rho X,\ \rho Y,\ \rho Z$

$$\begin{aligned}
&\frac{1}{2}\rho\frac{\partial}{\partial t}(1+\beta)(\xi^2+\eta^2+\zeta^2)\\
+&\ \rho\xi^2\frac{du}{dx}+\rho\eta^2\frac{dv}{dy}+\rho\zeta^2\frac{dw}{dz}+\rho\eta\zeta\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)+\rho\zeta\xi\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)+\rho\xi\eta\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)\\
+&\ \frac{1}{2}\rho(1+\beta)(\xi^2+\eta^2+\zeta^2)\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)\\
=&\ 0.
\end{aligned}$$

Deviding by ρ of both hand-side,

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(1+\beta)(\xi^2+\eta^2+\zeta^2)\\
+&\ \xi^2\frac{du}{dx}+\eta^2\frac{dv}{dy}+\zeta^2\frac{dw}{dz}+\eta\zeta\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)+\zeta\xi\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)+\xi\eta\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)\\
+&\ \frac{1}{2}(1+\beta)(\xi^2+\eta^2+\zeta^2)\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)\\
=&\ 0.
\end{aligned}$$

If we set $\mathcal{R} \equiv \frac{2}{(1+\beta)}$, then we get the second, linear term of the left hand-side by Maxwell is written by tensor

$$\begin{bmatrix} \rho\xi^2 & \rho\xi\eta & \rho\xi\zeta \\ \rho\xi\eta & \rho\eta^2 & \rho\eta\zeta \\ \rho\xi\zeta & \rho\zeta\eta & \rho\zeta^2 \end{bmatrix} = -\mathcal{R}\begin{bmatrix} \frac{\partial u}{\partial x} & \Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & \Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) \\ \Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & \frac{\partial v}{\partial y} & \Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) \\ \Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) & \Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) & \frac{\partial w}{\partial z} \end{bmatrix}$$

which is 'general tensor'.

---

[10](独) 1844-1906.

3.3. **Determination of the inequality of pressure in a medium.**

$$\xi^2\rho = p - \frac{M}{9k\rho\Theta_2}p\Big(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\Big),\ \eta^2\rho = p - \frac{M}{9k\rho\Theta_2}p\Big(\frac{du}{dx} - 2\frac{dv}{dy} - \frac{dw}{dz}\Big),\ \zeta^2\rho = p - \frac{M}{9k\rho\Theta_2}p\Big(\frac{du}{dx} - \frac{dv}{dy} - 2\frac{dw}{dz}\Big)$$

$$\eta\zeta\rho = -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big),\quad \xi\eta\rho = -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big),\quad \zeta\xi\rho = -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dw}{dx} + \frac{du}{dz}\Big)$$

Here, the relation of the coefficient between (13) and (14) is the relation between $\xi^2\rho$ ($= \eta^2\rho = \zeta^2\rho$) and $\eta\zeta\rho$ ($= \xi\eta\rho = \zeta\xi\rho$) become $\frac{2}{9}\frac{M}{k\rho\Theta_2} = \frac{1}{6}(1+\frac{1}{3})\frac{M}{k\rho\Theta_2}$. The left hand-side corresponds the coefficients of $\frac{du}{dx}$, $\frac{du}{dx}$, $\frac{du}{dx}$ on the diagonal of the right hand-side in (13). The right hand-side corresponds with the coefficients of $\frac{d^2u}{dx^2}$, $\frac{d^2v}{dy^2}$ and $\frac{d^2w}{dz^2}$ in (14).

Then we construct the tensor which is completely equal to (27) as follows :

$$\begin{bmatrix} \rho\xi^2 & \rho\xi\eta & \rho\xi\zeta \\ \rho\xi\eta & \rho\eta^2 & \rho\eta\zeta \\ \rho\xi\zeta & \rho\zeta\eta & \rho\zeta^2 \end{bmatrix} = \begin{bmatrix} p - \frac{M}{9k\rho\Theta_2}p\Big(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) \\ -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) & p - \frac{M}{9k\rho\Theta_2}p\Big(\frac{du}{dx} - 2\frac{dv}{dy} - \frac{dw}{dz}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) \\ -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\Big) & p - \frac{M}{9k\rho\Theta_2}p\Big(\frac{du}{dx} - \frac{dv}{dy} - 2\frac{dw}{dz}\Big) \end{bmatrix} \quad (13)$$

Having thus obtained the values of the pressures in different directions we may substitute them in the equation of motion.

$$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{d}{dx}(\rho\xi^2) + \frac{d}{dy}(\rho\xi\eta) + \frac{d}{dz}(\rho\xi\zeta) = X\rho, \\ \rho\frac{\partial v}{\partial t} + \frac{d}{dx}(\rho\xi\eta) + \frac{d}{dy}(\rho\eta^2) + \frac{d}{dz}(\rho\eta\zeta) = Y\rho, \\ \rho\frac{\partial w}{\partial t} + \frac{d}{dx}(\rho\xi\zeta) + \frac{d}{dy}(\rho\zeta\eta) + \frac{d}{dz}(\rho\zeta^2) = Z\rho, \end{cases}$$

which become the following equations that are completely equal to $(185)_B$

$$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{dp}{dx} - \frac{pM}{6k\rho\Theta_2}\Big\{\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3}\frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big\} = X\rho, \\ \rho\frac{\partial v}{\partial t} + \frac{dp}{dy} - \frac{pM}{6k\rho\Theta_2}\Big\{\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3}\frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big\} = Y\rho, \\ \rho\frac{\partial w}{\partial t} + \frac{dp}{dz} - \frac{pM}{6k\rho\Theta_2}\Big\{\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} + \frac{1}{3}\frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)\Big\} = Z\rho \end{cases} \quad (14)$$

If we set $\mathcal{R} \equiv \frac{Mp}{6k\rho\Theta_2}$, then these equations are completely equal to $(221)_B$. These facts state that Boltzmann had got his idea of special form of hydromechanics from Maxwell.

3.3.1. ***'Lectures on Gas Theory'* and *Lectures on Heat Theory* by Kirchhoff.**

We introduce *'Lectures on Gas Theory'* by Kirchhoff [9, pp.156-172]. He stated his theory citing only Maxwell in 1868 basing on Maxwell's theory as follows :

> Wir wenden uns jetzt zur Betrachtung eines Gases, das nicht in Ruhe ist, und folgen dabei der Maxwell'schen Darstellung.

He says, 'We turn here into the investigation of a gas, which is not stable, and follow the description by Maxwell." Afterward, Boltzmann referred many contents of gas theory from both Maxwell and Kirchhoff. For example, Kirchhoff states three assumptions of the number of molecule : we will investigate the change, which these integral operated in a time dt, where the time is infinitesimally small. We show the change by $\frac{\partial(N\overline{Q})}{\partial t}dt$. It consists of three parts :

- the value of Q enlarged by flowing into and flowing out a certain molecule in the parallelepiped in a time dt ;
- The outer force on the molecules, such as gravity operate, make change its velocity ;
- By the collision of each two molecules in the parallelepiped. [10, Lecture 15, p.157]

which Boltzmann cites almost assumptions. In Boltzmann's description about the condition no. 3,

(3) Those of our dn molecules that undergo a *collision* during the time dt will clearly have in general different velocity components after the *collision*.
- (Decrease :) Their velocity points will therefore be expected, as it were, from the parallelepiped by the *collision*, and thrown into a completely different parallelepiped. The number dn will thereby be *decreased*.
- (Increase :) On the other hand, the velocity points of m-molecules in other parallelepipeds will be throne into $d\omega$ by *collisions*, and dn will thereby *increase*.

- (Total increase by *collision* between m-molecules and m_1-molecules :) It is now a question of finding this total increase V_3 experienced by dn during time dt as a result of the *collisions* taking place between any m-molecules and any m_1-molecules.

In 1894, Kirchhoff, in *Lectures on Heat Theory* [10, p.194], stated hydrodynamic equations in incompressible fluid.

$$\begin{cases} \mu\frac{du}{dt} + \frac{\partial}{\partial x} - \frac{1}{3\kappa}\frac{p}{\mu}\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu X, \\ \mu\frac{dv}{dt} + \frac{\partial p}{\partial y} - \frac{1}{3\kappa}\frac{p}{\mu}\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Y, \\ \mu\frac{dw}{dt} + \frac{\partial p}{\partial z} - \frac{1}{3\kappa}\frac{p}{\mu}\left[\Delta z + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Z, \end{cases}$$

$$\frac{1}{\mu}\frac{d\mu}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Kirchhoff explains his viscosity term as follows :

> Als solche werden wir annehmen, daß $u,\ v,\ w$ in dem Gas dieselben Werthe haben, wie in dem festen Körper, also verschwinden, wenn dieser ruht; und daß die absolute Temperatur im Gas, die $\frac{p}{\mu}$ mal einer Constanten ist, gleich ist der Temperatur des fasten Körpers. $\cdots$
>
> Die mit $\frac{1}{\kappa}$ proportionalen Glieder, durch welch unsere Gleichungen sich unterscheiden von den in erster Annäherung geltenden, bedinden die Erscheinungen der *Reibung* und der *Wärmeleitung*. $\cdots$
>
> Die Grösse $\frac{1}{3\kappa}\frac{p}{\mu}$ heißt der *Reibungscoefficient*. [10, §3, pp.194-5]

[(transl.) We assume it as such that $u,\ v,\ w$ in the gas have each value in the solid, when these move, and that the absolute temperature in gas which is equal to the multiplied by $\frac{p}{\mu}$ of an constant, is equal to the temperature of solid. $\cdots$ The proportional terms with $\frac{1}{\kappa}$, by which our equations are distinguished with one in the first adaption, bring up as the phenomena of viscosity and the heat conduction. $\cdots$ The term $\frac{1}{3\kappa}\frac{p}{\mu}$ is called by viscosity coefficient. $\cdots$]

He introduces the real value of $\frac{1}{3\kappa}\frac{p}{\mu}$ in his following context, which we omit it for lack of space.

4. *'Lectures on Gas theory'* by Boltzmann

In general, according to Ukai [23], we can state the Boltzmann equations as follows: [11]

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = Q(f,g), \quad t>0, \quad \mathbf{x},\mathbf{v}\in\mathbb{R}^n(n\geq 3), \quad \mathbf{x}=(x,y,z), \quad \mathbf{v}=(\xi,\eta,\zeta), \tag{15}$$

$$Q(f,g)(t,x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{g(v'_*)f(v') - g(v_*)f(v)\}d\sigma dv_*, \quad g(v'_*) = g(t,x,v'_*), \tag{16}$$

$$v' = \frac{v+v_*}{2} + \frac{|v+v_*|}{2}\sigma, \quad v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad \sigma\in\mathbb{S}^{n-1} \tag{17}$$

where,

- $f = f(t,\ x,\ v)$ is interpretable as many meanings such as
  - density distribution of a molecule
  - number density of a molecule
  - probability density of a molecule

  at time : t, place : $\mathbf{x}$ and velocity : $\mathbf{v}$.
- $f(v)$ means $f(t,x,v)$ as abbreviating t and x in the same time and place with $f(v')$
- $Q(f,g)$ of the right-hand-side of (15) is the Boltzmann bilinear *collision operator.*
- $\mathbf{v}\cdot\nabla_{\mathbf{x}} f$ is the *transport operator,*
- $B(z,\sigma)$ of the right-hand-side in (16) is the non-negative function of *collision cross-section.*
- $Q(f,g)(t,x,v)$ is expressed in brief as $Q(f)$.
- $(v,\ v_*)$ and $(v',\ v'_*)$ are the velocities of a molecule before and after collision.
- According to Ukai [24], the *transport operators* are expressed with two sort of terms like Boltzmann's descriptions : $(114)_B$ and $(115)_B$ including the collision term $\nabla_{\mathbf{v}}\cdot(\mathbf{F}f)$ by exterior force $\mathbf{F}$ as follow : [12]

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}}\cdot(\mathbf{F}f) = Q(f) \tag{18}$$

$$Q(f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{f(v'_*)f(v') - f(v_*)f(v)\}d\sigma dv_* \tag{19}$$

where, $\mathbf{v}\cdot\nabla_{\mathbf{x}} f + \nabla_v\cdot(\mathbf{F}f)$ are *transport operators* operating under the exterior force : $\mathbf{F}(t,x,v) = (F_1,F_2,F_3)$. The right-hand side of (18) is expressed in brief as $Q(f)$ meaning $Q(f)(t,x,v)$.

4.1. Development of partial differential equations for f and F.

We show the Figure 6 in the last page of our paper, which defines the model of the *collision* between the molecule m_1 calling the point of it and the molecule m wich we call the point m. The instant when the molecule m passes vertically throught the disc of m_1 molecule, is defined as *collision.* We show Boltzmann's definitions as follow :

> We fix our attention on the parallelepiped representing all space points whose coordinates lie between the limits [13]
>
> $$(97)_B \quad [x,\ x+dx], \quad [y,\ y+dy], \quad [z,\ z+dz], \quad do = dxdydz$$
>
> We now construct a second rectangular parallelepiped, which include all points whose coordinates lie between the limits
>
> $$(98)_B \quad [\xi,\ \xi+d\xi], \quad [\eta,\ \eta+d\eta], \quad [\zeta,\ \zeta+d\zeta]$$
>
> We set its volume equal to

$$d\xi d\eta d\zeta = d\omega \tag{20}$$

---

[11]($\Downarrow$) We refer the *Lecture Note* by S.Ukai: *Boltzmann equations: New evolution of theory, Lecture Note of the Winter School in Kyushu of Non-linear Partial Differential Equations*, Kyushu University, 6-7, November, 2009.

[12]($\Downarrow$) In the Boltzmann' original equations, they are used with two terms like $(114)_B$, $(115)_B$. We can refer the *General lecture in the autumn meeting of MSJ* by S.Ukai [24] : *The study of Boltzmann equations: past and future*, MSJ, 23, September, 2010.

[13]($\Downarrow$) $(\ \cdot\ )_B$ in the top of the equation or expression means the number cited in Boltzmann[1] in below of our paper.

TABLE 5. The symbols and definitions

no	symbol	defined	content of conformation in modeling of collision. cf. The Fig. 6 in the last page.	cf.	m	m_1
1	$X,\ Y,\ Z$	(21)	The component of accelarating force of a molecule in a coödinate direction.			
2	mX, mY, mZ		The component of the external force acting on any m-molecule.		m	
3	$\xi,\ \eta,\ \zeta$	$(98)_B$	The component of velocity of any m-molecule in a coödinate direction.		m	
4	f	$(99)_B$	$f = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)$		m	
5	f_1	$(99)_B$	$f_1 = f(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)$, different only with velocity of f.		m	
6	F	$(100)_B$	$F = F(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)$			m_1
7	F_1	$(103)_B$	$F_1 = F(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)$, different only with velocity of F.			m_1
8	$\xi_1,\ \eta_1,\ \zeta_1$	$(102)_B$	The component of velocity of any m_1-molecule in a coödinate direction.			m_1
9	g	p.116	The moving direction (or velocity) of an m-molecule to an m_1-molecule.	Fig. 6	m	
10	gdt	p.116	The moving distance of an m-molecule to an m_1-molecule during dt.	Fig. 6	m	
11	b	$(104)_B$	The length of a line originated from m_1-molecule, where, b is the smallest possible distance of the two colliding molecules that could be attained if they moved without interaction in straight lines with the velocities they had before the collision. In other words, b is the line P_1P, where P_1 and P are the two points at which m_1 and m would be found at the moment of their closest approach if there were no interraction.	Fig. 6		m_1
12	σ		The limit of the length of a line. $[0,\ \sigma]$.	Fig. 6		m_1
13	ϵ	$(104)_B$	An angle formed between a line b and a line m_1H, where, ϵ is the angle between the two planes through the direction of relative motion, one parallel to P_1P along b, and the other to the abscissa axis.	Fig. 6		m_1
14	$\xi',\ \eta',\ \zeta'$	$(108)_B$	The component of velocity of a molecule after the *collision*.		m	
15	b'	$(109)_B$	The length of a line after the *collision*.	Fig. 6		m_1
16	ϵ'	$(109)_B$	An angle formed between a line b and a line m_1H after the *collision* .	Fig. 6		m_1
17	do : parallelepiped	$(97)_B$	We set $do = dxdydz$ in which the m-molecules lie, and we always call this parallelepiped the parallelepiped do.		m	
18	$d\omega$: parallelepiped of velocity point	(98_B) (20)	We set $d\omega = d\xi d\eta d\zeta$ in which velocity point of the m-molecules lie, and we always call this parallelepiped the parallelepiped $d\omega$.		m	
19	$d\omega_1$	(102_B) (24)	We set $d\omega_1 = d\xi_1 d\eta_1 d\zeta_1$ as well as $d\omega$, in which velocity point of the m_1-molecules lie, and we always call this parallelepiped the parallelepiped $d\omega_1$.			m_1
20	dn	$(99)_B$	The m-molecules that are in do at time t and whose velocity points lie in $d\omega$ at the same time will again be called the specified molecules, or the "dn molecules." $dn = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = fdod\omega$		m	
21	dn'	$(99)'_B$	The number of m-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time $t+dt$. $dn' = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t+dt)dod\omega$		m	
22	dN	$(100)_B$	The number of m_1-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time t. $dN = F(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = Fdod\omega$			m_1
23	dN_1	$(103)_B$	$dN_1 = F(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)dod\omega = F_1dod\omega_1$			m_1
24	ν_1	$(107)_B$	The number of all *collisions* of our dn molecules during dt with m_1-molecules.		m	m_1
25	ν_2	$(106)_B$	The number of m-points that pass an m_1-point at any distance less than σ during dt.		m	m_1
26	ν_3	$(105)_B$	The number of *collisions* between m-molecules and m_1-molecules.		m	m_1
27	V_1	(22)	The increase which dn experiences as a result of motion of the molecules during time dt, where all m-molecules whose velocity points lie in $d\omega$ move in the x-direction with velocity ξ, in the y-direction with velocity η, and in the z-direction with velocity ζ.	$A_2(\varphi)$	m	
28	V_2	(23)	As a result of the action of *external forces*, the velocity components of all the molecules change with time, and hence the velocity points of the molecules in do will move.	$A_3(\varphi)$	m	
28	i_1	$(111)_B$	The *total increase* experienced by dn as a result of *collisions* of m-molecules with m_1-molecules.		m	m_1
30	V_3	$(112)_B$	The *net increase* experienced by dn as a result of *collisions* of m-molecules with m_1-molecules. $V_3 = i_1 - \nu_1$.	$A_4(\varphi)$	m	m_1
31	V_4	$(113)_B$	The increment experienced by dn as a result of *collisions* of m or m_1-molecules with *each other*.	$A_5(\varphi)$	m	m_1
32	$\varphi,\ \sum_{d\omega,do}\varphi$	$(116)_B$	$\varphi = \varphi(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t),\quad \sum_{d\omega,do}\varphi \equiv \varphi f dod\omega$, multipling the number $fdod\omega$ by φ		m	
33	$\Phi,\ \sum_{d\omega,do}\Phi$	$(117)_B$	$\Phi = \Phi(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t),\quad \sum_{d\omega,do}\Phi \equiv \Phi F dod\omega$, multipling the number $Fdod\omega$ by Φ		m	
34	$\Phi_1,\ \sum_{d\omega,do}\Phi_1$	$(117)_B$	$\Phi_1 = \Phi(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)$, $\sum_{d\omega,do}\Phi_1 \equiv \Phi_1 F_1 dod\omega_1$, multipling the number $F_1dod\omega$ by Φ_1			m_1
35	$A_1(\varphi)$	$(121)_B$	The effect of explicit dependance of φ on t.			
36	$A_2(\varphi)$	$(122)_B$	The effect of *the motion* of the molecules.	V_1	m	
37	$A_3(\varphi)$	$(123)_B$	The effect of *external forces*.	V_2	m	
38	$A_4(\varphi)$	$(124)_B$	The effect of collisions of m-molecules with m_1-molecules.	V_3	m	m_1
39	$A_5(\varphi)$	$(125)_B$	The effect of collisions of m-molecules with *each other*.	V_4	m	
40	$B_1(\varphi)$	$(127)_B$	The total effect in ω of explicit dependance of φ on t.			
41	$B_2(\varphi)$	$(128)_B$	The effect in ω of *the motion* of the molecules.	V_1	m	
42	$B_3(\varphi)$	$(129)_B$	The effect in ω of *external forces*.	V_2	m	
43	$B_4(\varphi)$	$(134)_B$	The effect in ω of collisions of m-molecules with m_1-molecules.	V_3	m	m_1
44	$B_5(\varphi)$	$(139)_B$	The effect in ω of collisions of m-molecules with *each other*.	V_4	m	
45	$\{C_n(\varphi)\}_1^5$	$(125)_B$	The effect in ω and o as the same as $\{A_n(\varphi)\}_1^5$ or $\{B_n(\varphi)\}_1^5$			

and we call it the parallelepiped $d\omega$. The molecules that are in do at the time t and whose velocity points lie in $d\omega$ at the same time will again be called the specified molecules, or the "dn molecules." Their number is clearly proportional to the product $do \cdot d\omega$. Then all volume elements immediately adjacent to do find themselves subject to similar conditions, so that in a parallelepiped twice as large there will be twice as many molecules. We can therefore set this number equal to

$$(99)_B \quad dn = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = fdod\omega$$

Similarly the number of m_1-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time t will be :

$$(100)_B \quad dN = F(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = Fdod\omega$$

The two functions f and F completely characterize the state of motion, the mixing ratio, and the velocity distribution at all places in the gas mixture. We shall allow a very short time dt to elapse, and during this time we keep the size and position of do and $d\omega$ completely unchanged. The number of m-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time $t+dt$ is, according to Equation $(99)_B$,

$$dn' = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t+dt)dod\omega = fdod\omega$$

and the total increase experienced by dn during time dt is

$$(101)_B \quad dn' - dn = \frac{\partial f}{\partial t} do\ d\omega\ dt.$$

$\xi,\ \eta,\ \zeta$ are the rectangular coödinates of the velocity point. Although this is only an imaginary point, still it moves like the molecule itself in space. Since $X,\ Y,\ Z$ are the components of the accelerating force,[14] we have:

$$\frac{d\xi}{dt} = X, \quad \frac{d\eta}{dt} = Y, \quad \frac{d\zeta}{dt} = Z \tag{21}$$

4.2. Four different causes bringing up increase of dn.

Boltzmann explains an increase of dn as a result of the following *four different causes* of $V_1,\ V_2,\ V_3$ and V_4 :

- V_1 : increment by *transport* through do
- V_2 : increment by *transport* of external force
- V_3 : increment as a result of *collisions* of m-molecules with m_1-molecules
- V_4 : increment by *collision* of molecules with each other

We extract an outline by the Boltzmann [2] as follows :

The number dn experiences an increase as a result of *four different causes.*

(1) (V_1 : increase going out through do ;) All m-molecules whose velocity points lie in $d\omega$ move in the x-direction with velocity ξ, in the y-direction with velocity η, and in the z-direction with velocity ζ.

Hence through the left of the side of the parallelepiped do facing the negative abscissa direction there will enter during time dt as many molecules satisfying the condition (98_B) as may be found, at the beginning of dt, in a parallelepiped of base $dydz$ and height ξdt,[15] viz.

$$\xi \cdot f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dydzd\omega dt$$

molecules. Likewise, for the number of $m-$molecules that satisfying (98_B) and go out through the opposite face of do during time dt, the value:

$$\xi \cdot f(x+dx,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dydzd\omega dt$$

14(⇓) Da $X,\ Y,\ Z$ die Componenten der beschleunigenden Kraft sind, so ist: $\cdots$ Boltzmann [1, p.103].

15(⇓) ξ : the x-direction with velocity multiplied by dt becomes the length of a edge of which consists a parallelepiped with a base $dydz$.

By similar arguments for the four other sides of the parallelepiped, one finds that during time dt,

$$-\Big(\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}\Big)do\cdot d\omega\ dt$$

more molecules satisfying (98_B) enter do than leave it. This is therefore the increase V_1 which dn experiences as a result of motion of the molecules during time dt.

$$V_1=-\Big(\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}\Big)do\ d\omega\ dt \tag{22}$$

(2) (V_2 : increase by external force ;) As a result of the action of external forces, the velocity components of all the molecules change with time, and hence the velocity points of the molecules in do will move. Some velocity points will leave $d\omega$, others will come in, and since we always include in the number dn only those molecules whose velocity points lie in $d\omega$, dn likewise be changed for this reason.

$$V_2=-\Big(X\frac{\partial f}{d\xi}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}\Big)do\ d\omega\ dt \tag{23}$$

Boltzmann defines the effects of *collisions* as follows :

(3) (V_3 : increase as a result of *collisions* of m-molecules with m_1-molecules ;) Those of our dn molecules that undergo a *collision* during the time dt will clearly have in general different velocity components after the *collision*.

- (Decrease :) Their velocity points will therefore be expected, as it were, from the parallelepiped by the *collision*, and thrown into a completely different parallelepiped. The number dn will thereby be *decreased*.
- (Increase :) On the other hand, the velocity points of m-molecules in other parallelepipeds will be throne into $d\omega$ by *collisions*, and dn will thereby *increase*.
- (Total increase by *collision* between m-molecules and m_1-molecules :) It is now a question of finding this total increase V_3 experienced by dn during time dt as a result of the *collisions* taking place between any m-molecules and any m_1-molecules.

For this purpose we shall fix our attention on a very small fraction of the total number ν_1 of *collisions* undergone by our dn molecules during time dt with m_1-molecules. We construct a third parallelepiped which includes all points whose coordinates lie between the limits

$$(102)_B\quad [\xi_1,\ \xi_1+d\xi_1],\quad [\eta_1,\ \eta_1+d\eta_1],\quad [\zeta_1,\ \zeta_1+d\zeta_1]$$

Its volume is

$$d\omega_1=d\xi_1 d\eta_1 d\zeta_1 \tag{24}$$

It constitutes the parallelepiped $d\omega_1$. By analogy with Equation $(100)_B$, the number of m_1-molecules in do whose velocity points lie in $d\omega_1$ at time t is :

$$(103)_B\quad dN_1=F_1 do d\omega_1,$$

where F_1 is an abbreviation for $F(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1)$.

Boltzmann difines a passage of an m-point by an m_1-point as follows :

(a) (How to pass :) We define a passage of an m-point by an m_1-point as that instant of time when distance between the points has its smallest value ; thus m would pass through the plane through m_1 perpendicular to the direction g, if no interaction took place between the two molecules.

(b) (ν_2 : the number of passages of an m-point by an m_1-point :) Hence, ν_2 is equal to the number of passages of an m-point by an m_1-point that occurs during time dt, such that the smallest distance between the two molecules is less than σ.

(c) (A plane E :) In order to find this number, we draw through each m_1-point a plane E moving with m_1, perpendicular to the direction of g, and a line G, which parallel to this direction.

(d) (When a passage ends :) As soon as an m-point crosses E, a passage take place between it and the m_1-point.

(e) (A line m_1X :) We draw through each m_1-point a line m_1X parallel to the positive abscissa direction and similarly directed.

(f) (Half-plane :) The half-plane bounded by G, which contains the latter line, cuts E in the line m_1H, which of course again contains each m_1-point.

(g) (b and ϵ :) Furthermore, we draw from each m_1-point in each of the plane E a line of length b, which forms an angle ϵ with the line m_1H.

(h) (Rectangles of surface area R formed by b and ϵ :) All points of the plane E for which b and ϵ lie between the limits

$$(104)_B \quad [b,\ b+db], \quad [\epsilon,\ \epsilon+d\epsilon]$$

form a rectangle of surface area $R = bdbd\epsilon$.

In Figure 6 [16] the intersections of all these lines with a sphere circumscribed about m_1 are shown. The large circle (shown as an ellipse) lies in the plane E ; the circular arc GXH lies in the half-plane defined above. In each of planes E, an equal and identically situated rectangle will be found. We consider for the moment only those passages of an m-point by an m_1-point in which the first point penetrates one of the rectangles R.

$$\Pi = Rgdt = \underbrace{bdbd\epsilon}_{R}\, gdt, \quad \sum \Pi = dN_1\Pi = \underbrace{F_1 dod\omega_1}_{dN_1\ (103)_B}\ \underbrace{gbdbd\epsilon dt}_{\Pi}$$

Since these volumes are infinitesimal, and lie infinitely close to the point with coordinates $x,\ y,\ x$, then by analogy with Equation $(99)_B$ the number of m-points (i.e., m-molecules whose velocity points lie in $d\omega$) that are initially in the volumes $\sum \Pi$ is equal to :

$$(105)_B \quad \nu_3 = fd\omega \sum \Pi = fF_1 dod\omega d\omega_1 gbdbd\epsilon dt$$

This is at the same time the number of m-points that pass an m_1-point during time dt at a distance between b and $b+db$, in such a way that the angle ϵ lie between ϵ and $\epsilon + d\epsilon$.

By ν_2 we mean the number of m-points that pass an m_1-point at any distance less than σ during dt. We find ν_2 by integrating the differential expression ν_3 over ϵ from 0 to 2π, and over b from 0 to σ.

$$(106)_B \quad \nu_2 = \int_0^\sigma db \int_0^{2\pi} \nu_3 d\epsilon = dod\omega d\omega_1 dt \int_0^\sigma db \int_0^{2\pi} d\epsilon\ g \cdot b \cdot f \cdot F_1 \cdot$$

The number denoted by ν_1 of all *collisions* of our dn molecules during dt with m_1-molecules is therefore found by integrating over the three variable $\xi_1,\ \eta_1,\ \zeta_1$ whose differentials occur in $d\omega_1$, from $-\infty$ to $+\infty$; we indicate this a single integral sign :

$$(107)_B \quad \nu_1 = \int_{-\infty}^{\infty} \nu_2 d\omega_1 = do \cdot d\omega \cdot dt \int_{-\infty}^{\infty} d\omega_1 \int_0^\sigma db \int_0^{2\pi} fF_1 gbd\epsilon$$

We shall consider again those *collisions* between m-molecules and m_1-molecules, whose number was denoted by ν_3 and is given by Equation $(105)_B$.

[16]($\Downarrow$) We show this Figure 6 in the last page of our paper citing [1, p.107], which is equal to [2, p.117], however, we must correct the symbol R by H of [2, p.117].

These are the *collisions* that occur in unit time in the volume element do in such a way the following conditions are satisfied :

• The velocity components of the m-molecules and the m_1-molecules lie between the limits $(98)_B$ and $(102)_B$, respectively, before the interaction begins.

• We denote by b the closest distance of approach that would be attained if the molecules did not interact but retained the velocities they had before the *collision*.

The *total increment* i_1 experienced by dn as a result of *collisions* of m-molecules with m_1-molecules is founded by integrating over ϵ from 0 to 2π, over b from 0 to σ, and over ξ_1, η_1, ζ_1 from $-\infty$ to $+\infty$. We shall write the result of this integration in the form :

$$(111)_B \quad i_1 = dod\omega dt \iint_0^\sigma \int_0^{2\pi} f' F_1' gbd\omega_1 dbd\epsilon$$

Of course we cannot perform explicitly the integration with respect to b and ϵ since the variable ξ', η', ζ' and ξ_1', η_1', ζ_1' occurring in f' and F_1' are functions of (ξ, η, ζ, ξ_1', η_1', ζ_1', b and ϵ), which cannot be computed until the force law is given.[17]

The difference $i_1 - \nu_1$ expresses the *net increase* of dn during time dt as a result of *collisions* of m-molecules with m_1-molecules. It is therefore the *total increase* V_3 experienced by dn as a result of these *collisions*, and one has

$$(112)_B \quad V_3 = i_1 - \nu_1 = dod\omega dt \iint_0^\sigma \int_0^{2\pi} (f' F_1' - f F_1) d\omega_1 dbd\epsilon$$

(4) (V_4 : increment by collision of molecules with each other ;) The increment V_4 experienced by dn as a result of *collisions* of m-molecules with each other is found from Equation $(112)_B$ by a simple permutation. One now uses ξ_1, η_1, ζ_1 and ξ_1', η_1', ζ_1' for the velocity components of the other m-molecule *before and after the collision*, respectively, and one writes f_1 and f_1' for

$$f_1 = f(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t) \quad \text{and} \quad f_1' = f(x,\ y,\ z,\ \xi_1',\ \eta_1',\ \zeta_1',\ t)$$

Then :

$$(113)_B \quad V_4 = dod\omega dt \iint_0^\infty \int_0^{2\pi} (f' f_1' - f f_1) gbd\omega_1 dbd\epsilon$$

4.3. **Formulation of Boltzmann's transport equations.**

According to Boltzmann[1, pp.110-115], [18] his equations (so-called *transport equations*) are the following : [19]

Since now $V_1 + V_2 + V_3 + V_4$ is equal to the increment $dn' - dn$ of dn during time dt, and this according to Equation $(101)_B$ must be equal to $\frac{\partial f}{\partial t} dod\omega dt$, one obtains on substituting all the appropriate value and deviding by $dod\omega dt$ the following partial differential equation for the function f :

---

[17]($\Downarrow$) Hier kann die Integration nach b und ϵ natürlich nicht mehr sofort aus geführt werden, da die in f' and F_1' vorkommen den Variabeln ξ', η', ζ' und ξ_1', η_1', ζ_1' Function von ξ, η, ζ, ξ_1', η_1', ζ_1', b und ϵsind, welche nur berechnet werden können, wenn Virkungsgesetz der während eines Zusammenstosses wirksamen Kräfte gegeben ist. [1, p.112].

[18]($\Downarrow$) Boltzmann(1844-1906) had put the date in the foreword to part I as September in 1895, part II as August in 1898.

[19]($\Downarrow$) We mean the equation number in the left-hand side with $(\cdot)_B$ the citations from the Boltzmann[1] or [2]. We state only the symbol $\int$ instead of $\int_{-\infty}^{\infty}$. cf. $(107)_B$.

TABLE 6. Combination of function before and after collision

no	item	V_3 before	V_3 after	f of V_4 before	f of V_4 after	F of V_4 before	F of V_4 after
1	function of m_1	f	f'	f	f'	F	F'
2	function of m	F_1	F_1'	f_1	f_1'	F_1	F_1'
3	increment		$f'F_1' - fF_1$		$f'f_1' - ff_1$		$F'F_1' - FF_1$

$$(114)_B \quad \frac{\partial f}{\partial t} + \underbrace{\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}}_{V_1} + \underbrace{X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}}_{V_2}$$
$$= \underbrace{\iint_0^\infty \int_0^{2\pi} (f'F_1' - fF_1)gb\ d\omega_1\ db\ d\epsilon}_{V_3} + \underbrace{\iint_0^\infty \int_0^{2\pi} (f'f_1' - ff_1)gb\ d\omega_1\ db\ d\epsilon}_{V_4}$$
$$= \underbrace{\iint_0^\infty \int_0^{2\pi} \Big[(f'F_1' - fF_1) + (f'f_1' - ff_1)\Big]gb\ d\omega_1\ db\ d\epsilon}_{V_3+V_4}$$

Similarly we obtain the equation of F :

$$(115)_B \quad \frac{\partial F_1}{\partial t} + \underbrace{\xi_1\frac{\partial F_1}{\partial x} + \eta_1\frac{\partial F_1}{\partial y} + \zeta_1\frac{\partial F_1}{\partial z}}_{V_1} + \underbrace{X_1\frac{\partial F_1}{\partial x} + Y_1\frac{\partial F_1}{\partial y} + Z_1\frac{\partial F_1}{\partial z}}_{V_2}$$
$$= \underbrace{\iint_0^\infty \int_0^{2\pi} (f'F_1' - fF_1)gb\ d\omega_1\ db\ d\epsilon}_{V_3} + \underbrace{\iint_0^\infty \int_0^{2\pi} (F'F_1' - FF_1)gb\ d\omega_1\ db\ d\epsilon}_{V_4}$$
$$= \underbrace{\iint_0^\infty \int_0^{2\pi} \Big[(f'F_1' - fF_1) + (F'F_1' - FF_1)\Big]gb\ d\omega_1\ db\ d\epsilon}_{V_3+V_4}$$

where,

$$\begin{cases} f = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t), & f_1 = f(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t), & f_1' = f(x,\ y,\ z,\ \xi_1',\ \eta_1',\ \zeta_1',\ t), \\ F = F(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t), & F_1 = F(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t), & F_1' = F(x,\ y,\ z,\ \xi_1',\ \eta_1',\ \zeta_1',\ t) \end{cases} \tag{25}$$

Namely, we can verify $(114)_B$ for f :

$$\frac{V_1+V_2+V_3+V_4}{dod\omega dt} = \frac{\partial f}{\partial t} = -\underbrace{\Big(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}\Big)}_{V_1} - \underbrace{\Big(X\frac{\partial f}{d\xi} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}\Big)}_{V_2}$$
$$+ \underbrace{\iint_0^\infty \int_0^{2\pi} (f'F_1' - fF_1)gb\cdot d\omega_1 dbd\epsilon}_{V_3} + \underbrace{\iint_0^\infty \int_0^{2\pi} (f'f_1' - ff_1)gb\cdot d\omega_1 dbd\epsilon.}_{V_4}$$

Similarly we obtain $(115)_B$ for F.

$$\frac{V_1+V_2+V_3+V_4}{dod\omega dt} = \frac{\partial F_1}{\partial t} = -\Big(\xi\frac{\partial F_1}{\partial x} + \eta\frac{\partial F_1}{\partial y} + \zeta\frac{\partial F_1}{\partial z}\Big) - \Big(X\frac{\partial F_1}{d\xi} + Y\frac{\partial F_1}{\partial y} + Z\frac{\partial F_1}{\partial z}\Big)$$
$$+ \iint_0^\infty \int_0^{2\pi} (f'F_1' - fF_1)gb\cdot d\omega_1 dbd\epsilon + \iint_0^\infty \int_0^{2\pi} (F'F_1' - FF_1)gb\cdot d\omega_1 dbd\epsilon.$$

($\Downarrow$) Here, we can confirm the identity with the today's description of the Boltzmann equations (15) and (16) :

$$\partial_t f + \underbrace{\mathbf{v}\cdot\nabla_{\mathbf{x}} f}_{V_1} + \underbrace{\mathbf{w}\cdot\nabla_{\mathbf{v}} f}_{V_2} = \underbrace{Q(f,g)}_{V3,\ V4}, \quad \partial_t F + \underbrace{\mathbf{v}\cdot\nabla_{\mathbf{x}} F}_{V_1} + \underbrace{\mathbf{w}\cdot\nabla_{\mathbf{v}} F}_{V_2} = \underbrace{Q(F,G)}_{V3,\ V4},$$

$$Q(f,g)(t,x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{g(v_*')f(v') - g(v_*)f(v)\}d\sigma dv_*, \quad g(v_*') = g(t,x,v_*'), \text{ etc.}$$
$$t>0, \quad \mathbf{x},\mathbf{v},\mathbf{w}\in\mathbb{R}^n(n\geq 3), \quad \mathbf{x}=(x,y,z), \quad \mathbf{v}=(\xi,\eta,\zeta), \quad \mathbf{w}=(X,\ Y,\ Z).$$

In the case of (18) and (19)

$$\partial_t f + \underbrace{\mathbf{v}\cdot\nabla_{\mathbf{x}} f}_{V_1} + \underbrace{\nabla_{\mathbf{v}}\cdot(\vec{F}f)}_{V_2} = \underbrace{Q(f)}_{V_3, V_4}$$

$$Q(f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{f(v'_*)f(v') - f(v_*)f(v)\}d\sigma dv_*$$

4.4. Time-derivatives of sums over all molecules in a region.

Let φ be an arbitrary function of $x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t$. The value obtained by substituting therein the actual coödinates and velocity components of a particular molecule at time t will be called the value of φ corresponding to that molecule at time t. The sum of all values of φ corresponding to all the m-molecules that lie in the parallelepiped do and whose velocity points lie in the parallelepiped $d\omega$ at time t is obtained by multiplying φ by the number $f do d\omega$ of those molecules. We denote it by $(116)_B$.

Similarly we choose for the second kind of gas any other arbitrary function Φ of $x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t$ and denote by $(117)_B$. The sum of the values of Φ corresponding to all the m_1-molecules lying in do whose velocity points lie in $d\omega_1$. Φ_1 is the abbreviation for $\Phi(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)$. [2, §.17, pp.123-124].

4.5. General form of the hydrodynamic equations.

As the general expressions for fluid mechanics, he states that when we substitute for $\frac{\partial f}{\partial t}$ its value from Equation $(114)_B$, it turns into $(120)_B$, $(126)_B$, $(140)_B$, a sum of five terms, each of which has its own physical meaning, as follows:

$$\begin{cases} (116)_B\ \sum_{d\omega,do}\varphi \equiv \varphi f do d\omega, & (120)_B\ \frac{\partial}{\partial t}\sum_{d\omega,do}\varphi = \left(f\frac{\partial\varphi}{\partial t} + \varphi\frac{\partial f}{\partial t}\right)do d\omega = \left[\sum_{n=1}^{5} A_n(\varphi)\right]do d\omega, \\ (117)_B\ \sum_{d\omega,do}\Phi \equiv \Phi F do d\omega_1, & \sum_{d\omega,do}\Phi_1 = \Phi_1 F_1 do d\omega_1, \\ (118)_B\ \sum_{\omega,do}\varphi \equiv do\int\varphi f d\omega, & (126)_B\ \frac{\partial}{\partial t}\sum_{\omega,do}\varphi = do\int\left(f\frac{\partial\varphi}{\partial t} + \varphi\frac{\partial f}{\partial t}\right)d\omega = \left[\sum_{n=1}^{5} B_n(\varphi)\right]do, \\ (119)_B\ \sum_{\omega,o}\varphi \equiv \iint\varphi f do d\omega, & (140)_B\ \frac{d}{dt}\sum_{\omega,o}\varphi = \iint\left(f\frac{\partial\varphi}{\partial t} + \varphi\frac{\partial f}{\partial t}\right)do d\omega = \sum_{n=1}^{5} C_n(\varphi) \end{cases}$$

4.5.1. Conformation of $A_n(\varphi)$.

$$\begin{cases} (121)_B & A_1(\varphi) = \frac{\partial\varphi}{\partial t}f, \\ (122)_B & A_2(\varphi) = -\varphi\left(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}\right), \\ (123)_B & A_3(\varphi) = -\varphi\left(X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}\right), \\ (124)_B & A_4(\varphi) = \varphi\iint_0^\infty\int_0^{2\pi}(f'F'_1 - fF_1)gb\ d\omega_1\ db\ d\epsilon, \\ (125)_B & A_5(\varphi) = \varphi\iint_0^\infty\int_0^{2\pi}(f'f'_1 - ff_1)gb\ d\omega_1\ db\ d\epsilon, \end{cases}$$

where $\{A_n(\varphi)\}_{n=1}^5$ correspond to the effects such as

$$\begin{cases} A_1(\varphi): \text{the explicit dependence of } \varphi \text{ on } t; \\ A_2(\varphi): \text{the motion of the molecules;} \\ A_3(\varphi): \text{the external forces;} \\ A_4(\varphi): \textit{collisions} \text{ of } m\text{-molecules with } m_1\text{-molecules;} \\ A_5(\varphi): \textit{collisions} \text{ of } m\text{-molecules with each other;} \end{cases}$$

In order to find $\frac{\partial}{\partial t}\sum_{\omega,do}\varphi$, we have simply to integrate $\frac{\partial}{\partial t}\sum_{\omega,do}\varphi$ over all possible values of $d\omega$.

4.5.2. Conformation of $B_n(\varphi)$.

$$(126)_B \quad \frac{\partial}{\partial t}\sum_{\omega,do}\varphi = \left[\sum_{n=1}^{5} B_n(\varphi)\right]do.$$

One obtains each B by multiplying the corresponding A by $d\omega = d\xi d\eta d\zeta$ and integrating over all these variables from $-\infty$ to $+\infty$, which we indicate by a single integral sign. Thus :

$$(127)_B \quad B_1(\varphi) = \int A_1(\varphi)d\omega = \int \frac{\partial\varphi}{\partial t} f d\omega$$

$$(128)_B \quad B_2(\varphi) = \int A_2(\varphi)d\omega = -\int \varphi\Big(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}\Big)d\omega$$

$$(129)_B \quad B_3(\varphi) = \int A_3(\varphi)d\omega = -\int \varphi\Big(X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}\Big)d\omega$$

$$(134)_B \quad B_4(\varphi) = \int A_4(\varphi)d\omega = \frac{1}{2}\iiint_0^\infty \int_0^{2\pi} (\varphi - \varphi')(f'F_1' - fF_1)gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

$$(135)_B \quad B_{51}(\varphi) = \int A_5(\varphi' - \varphi)d\omega = \iiint_0^\infty \int_0^{2\pi} (\varphi' - \varphi)ff_1 gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

$$(136)_B \quad = \iiint_0^\infty \int_0^{2\pi} (\varphi - \varphi')f'f_1' gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

$$(135')_B \quad B_{51}'(\varphi) = \int A_5(\varphi_1' - \varphi_1)d\omega = \iiint_0^\infty \int_0^{2\pi} (\varphi_1' - \varphi_1)ff'gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

$$(136')_B \quad B_{52}'(\varphi) = \int A_5(\varphi_1 - \varphi_1')d\omega = \iiint_0^\infty \int_0^{2\pi} (\varphi_1 - \varphi_1')f'f_1' gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

From $(135)_B$,

$$(137)_B \quad B_{53}(\varphi) = \frac{1}{2}(B_{51} + B_{51}') = \frac{1}{2}\iiint_0^\infty \int_0^{2\pi} (\varphi' + \varphi_1' - \varphi - \varphi_1)ff'gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

From $(136)_B$,

$$(138)_B \quad B_{54}(\varphi) = \frac{1}{2}(B_{52} + B_{52}') = \frac{1}{2}\iiint_0^\infty \int_0^{2\pi} (\varphi + \varphi_1 - \varphi' - \varphi_1')f'f_1'gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

The arithmetic mean of $(137)_B$ and $(138)_B$,

$$(139)_B \quad B_5(\varphi) = \frac{1}{2}(B_{53} + B_{54}) = \frac{1}{4}\iiint_0^\infty \int_0^{2\pi} (\varphi + \varphi_1 - \varphi' - \varphi_1')(f'f_1' - ff_1)gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

4.5.3. **Conformation of $C_n(\varphi)$.**

$$(140)_B \quad \frac{d}{dt}\sum_{\omega,o}\varphi = \sum_{n=1}^{5} C_n(\varphi) = \underbrace{C_1(\varphi) + C_2(\varphi) + C_3(\varphi)}_{\text{increments except for those resulting from collisions}} + \underbrace{C_4(\varphi) + C_5(\varphi)}_{\text{increments of those resulting from collisions}}$$

Remark: since in $\sum_{\omega,o}\varphi$ of $(140)_B$ one has to integrate over all values of do and $d\omega$, this quantity is now a function only of time. Hence the use of symbol $\frac{\partial}{\partial t}$ is unnecessary, and we can express differentiation by the usual Latin letter d. Each C is obtained by multiplying the corresponding B by do and integrating over all volume elements, or else by multiplying the coresponding A by $do d\omega$ and integrating over all do and $d\omega$ as we show in $(119)_B$.

Integrating $\{B_n(\varphi)\}_{n=1}^3$ of $(127)_B$, $(128)_B$, $(129)_B$ by do from $-\infty$ to $+\infty$,

$$(141)_B \quad C_1(\varphi) + C_2(\varphi) + C_3(\varphi) = \iint f do d\omega\Big(\frac{\partial\varphi}{\partial t} + \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z} + X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}\Big)$$

Integrating $B_4(\varphi)$ of $(134)_B$ by do from $-\infty$ to $+\infty$,

$$(142_1)_B \quad C_4(\varphi) = \frac{1}{2}\iiiint_0^\infty \int_0^{2\pi} (\varphi - \varphi')(f'F_1' - fF_1)gb\ do\ d\omega\ d\omega_1\ db\ d\epsilon$$

Integrating $B_5(\varphi)$ of $(139)_B$ by do from $-\infty$ to $+\infty$,

$$(142_2)_B \quad C_5(\varphi) = \frac{1}{4}\iiiint_0^\infty \int_0^{2\pi} (\varphi + \varphi_1 - \varphi' - \varphi_1')(f'f_1' - ff_1)gb\; do\; d\omega\; d\omega_1\; db\; d\epsilon$$

4.5.4. **More general proof of the entropy theorem. Treatment of the equations corresponding to the stationary state.** Boltzmann assert the following conditions

$$(147)_B \quad ff_1 = f'f_1', \quad FF_1 = F'F_1', \quad fF_1 = f'F_1'.$$

4.5.5. **Linearity of $A_k,\ B_k,\ C_k$.**

Since $A,\ B,\ C$ are only the increments of definite quantities resulting from specified causes, most authors express them as derivatives of those quantities. Maxwell writes $\frac{\partial}{\partial t}\sum_{\omega,\ do}\varphi$, Kirchhoff $\frac{D}{Dt}\sum_{\omega,\ do}\varphi$ for $B_5(\varphi)$ etc. As with all differentials, the A for a sum of two functions is equal to the A's for the addends :

$$\begin{cases} A_k(\varphi+\psi) = A_k(\varphi) + A_k(\psi), \\ B_k(\varphi+\psi) = B_k(\varphi) + B_k(\psi), \\ C_k(\varphi+\psi) = C_k(\varphi) + C_k(\psi) \end{cases}$$

for any subscript k. These equations follows from the circumstance that φ occurs in all the integrals $A,\ B,\ C$ only linearly.

4.6. **Special form of the incompressible, hydrodynamic equations.**

$$(171)_B \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$(173)_B \quad \begin{cases} \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho X - \frac{\partial(\rho\overline{\xi_0^2})}{\partial x} - \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial y} - \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial z}, \\ \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho Y - \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial x} - \frac{\partial(\rho\overline{\eta_0^2})}{\partial y} - \frac{\partial(\rho\overline{\zeta_0\eta_0})}{\partial z}, \\ \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho Z - \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial x} - \frac{\partial(\rho\overline{\eta_0\zeta_0})}{\partial y} - \frac{\partial(\rho\overline{\zeta_0^2})}{\partial z} \end{cases}$$

Boltzmann says, "These equations as well as Equation $(171)_B$, are *only special cases of the general equation* $(126)_B$ and were derived from it by Maxwell and (following him) by Kirchhoff." Boltzmann concludes that if one collects all these terms, then Equation (126) reduces in this special case to:

$$(177)_B \quad \frac{\partial(\rho\overline{\varphi})}{\partial t} + \frac{\partial(\rho\overline{\xi\varphi})}{\partial x} + \frac{\partial(\rho\overline{\eta\varphi})}{\partial y} + \frac{\partial(\rho\overline{\zeta\varphi})}{\partial z} - \rho\Big[X\overline{\frac{\partial\varphi}{\partial\xi}} + Y\overline{\frac{\partial\varphi}{\partial\eta}} + Z\overline{\frac{\partial\varphi}{\partial\zeta}}\Big] = \underbrace{m\Big[B_4(\varphi) + B_5(\varphi)\Big]}_{\textit{collision terms}}$$

Boltzmann states about $(177)_B$:

From this equation Maxwell calculated the viscosity, diffusion, and heat conduction and Kirchhoff therefore calls it the basic equation of the theory. If one sets $\varphi = 1$, he obtains at once the continuity equation (171); for it follows from Equations (134) and (137) that $B_4(1) = B_5(1) = 0$. Subtraction of the continuity equation, multiplied by φ, from (177) gives (using the substitution [158]): [2, p.152].

where, (158) : $\xi = \xi_0 + u, \quad \eta = \eta_0 + v, \quad \zeta = \zeta_0 + w.$

$$\begin{aligned}(178)_B \quad \rho\Big(\frac{\partial\overline{\varphi}}{\partial t} + u\frac{\partial\overline{\varphi}}{\partial x} + v\frac{\partial\overline{\varphi}}{\partial y} + w\frac{\partial\overline{\varphi}}{\partial z}\Big) \quad &+ \quad \frac{\partial(\rho\overline{\xi_0\varphi})}{\partial x} + \frac{\partial(\rho\overline{\eta_0\varphi})}{\partial y} + \frac{\partial(\rho\overline{\zeta_0\varphi})}{\partial z} - \rho\Big[X\overline{\frac{\partial\varphi}{\partial\xi}} + Y\overline{\frac{\partial\varphi}{\partial\eta}} + Z\overline{\frac{\partial\varphi}{\partial\zeta}}\Big] \\ &= \quad \underbrace{m\Big[B_4(\varphi) + B_5(\varphi)\Big]}_{\textit{collision terms}}\end{aligned}$$

If one denotes the six quantities $(179)_B : \rho\overline{\xi_0^2}, \rho\overline{\eta_0^2}, \rho\overline{\zeta_0^2}, \rho\overline{\eta_0\zeta_0}, \rho\overline{\xi_0\zeta_0}, \rho\overline{\xi_0\eta_0}$ by $X_x,\ Y_y,\ Z_z,\ Y_z = Z_y,\ Z_x = X_z,\ X_y = Y_x$, namely, when we use the symmetric tensor, then we get the following :

$$\begin{bmatrix} \rho\overline{\xi_0^2} & \rho\overline{\xi_0\eta_0} & \rho\overline{\xi_0\zeta_0} \\ \rho\overline{\xi_0\eta_0} & \rho\overline{\eta_0^2} & \rho\overline{\eta_0\zeta_0} \\ \rho\overline{\xi_0\zeta_0} & \rho\overline{\zeta_0\eta_0} & \rho\overline{\zeta_0^2} \end{bmatrix} = \begin{bmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix}, \tag{26}$$

$$(180)_B \quad \begin{cases} \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = \rho X, \\ \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} = \rho Y, \\ \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = \rho Z \end{cases}$$

These are not NS equations for lack of the pressure term. Moreover $(181)_B$: $p = \overline{\rho\xi_0^2} = \overline{\rho\eta_0^2} = \overline{\rho\zeta_0^2}$, $\overline{\xi_0\eta_0} = \overline{\xi_0\zeta_0} = \overline{\eta_0\zeta_0} = 0$. Here, he assumes that from the supposition of isotropy and homogeneity, $p = \frac{1}{3}(X_x + Y_y + Z_z)$, which is the same as the principle by Saint-Venant or Stokes.

He deduces a special case of the hydrodynamic equations as follows:

> For the present, we assume as a fact of experience that in gases the normal pressure is always nearly equal in all directions, and that tangential elastic forces are very small; so that Equations (181) are approximately true. Substitution of the values given by this equation into Equation (173) yields:

$$(183)_B \quad \begin{cases} \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \frac{\partial p}{\partial x} - \rho X = 0, \\ \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + \frac{\partial p}{\partial y} - \rho Y = 0, \\ \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) + \frac{\partial p}{\partial z} - \rho Z = 0 \end{cases}$$

which are the so-called Euler equations in incompressible condition of $(171)_B$.

$$(185)_B \quad \begin{cases} \rho\frac{\partial u}{\partial t} + \frac{\partial(\rho\overline{\xi_0^2})}{\partial x} + \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial y} + \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial z} - \rho X = 0, \\ \rho\frac{\partial v}{\partial t} + \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial x} + \frac{\partial(\rho\overline{\eta_0^2})}{\partial y} + \frac{\partial(\rho\overline{\eta_0\zeta_0})}{\partial z} - \rho Y = 0, \\ \rho\frac{\partial w}{\partial t} + \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial x} + \frac{\partial(\rho\overline{\zeta_0\eta_0})}{\partial y} + \frac{\partial(\rho\overline{\zeta_0^2})}{\partial z} - \rho Z = 0 \end{cases}$$

We set the values of (26) as follows, which is the same tensor as Stokes :

$$(220)_B \quad \begin{bmatrix} \rho\overline{\xi_0^2} & \rho\overline{\xi_0\eta_0} & \rho\overline{\xi_0\zeta_0} \\ \rho\overline{\xi_0\eta_0} & \rho\overline{\eta_0^2} & \rho\overline{\eta_0\zeta_0} \\ \rho\overline{\xi_0\zeta_0} & \rho\overline{\zeta_0\eta_0} & \rho\overline{\zeta_0^2} \end{bmatrix} = \begin{bmatrix} p - 2\mathcal{R}\left\{\frac{\partial u}{\partial x} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - 2\mathcal{R}\left\{\frac{\partial v}{\partial y} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - 2\mathcal{R}\left\{\frac{\partial w}{\partial z} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} \end{bmatrix}$$

From $(220)_B$, we calculate the components of $(185)_B$ as follows:

$$\begin{bmatrix} \frac{\partial(\rho\overline{\xi_0^2})}{\partial x} & \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial y} & \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial z} \\ \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial x} & \frac{\partial(\rho\overline{\eta_0^2})}{\partial y} & \frac{\partial(\rho\overline{\eta_0\zeta_0})}{\partial z} \\ \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial x} & \frac{\partial(\rho\overline{\zeta_0\eta_0})}{\partial y} & \frac{\partial(\rho\overline{\zeta_0^2})}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} p - \mathcal{R}\left\{2\frac{\partial u}{\partial x} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - \mathcal{R}\left\{2\frac{\partial v}{\partial y} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - \mathcal{R}\left\{2\frac{\partial w}{\partial z} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Then, substitution of these values into the equations of motion $(185)_B$ yields:

$$(221)_B \quad \begin{cases} \rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \mathcal{R}\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] - \rho X = 0, \\ \rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} - \mathcal{R}\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] - \rho Y = 0, \\ \rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} - \mathcal{R}\left[\Delta w + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] - \rho Z = 0 \end{cases}$$

We can interpret that as the special cases, Boltzmann have deduced the NS equations after substituting the tensor $(220)_B$ to $(173)_B$, for lack of pressure terms.

We can construct the tensor with the Equations (13) and (14) as follows:

$$\begin{bmatrix} \rho\xi^2 & \rho\xi\eta & \rho\xi\zeta \\ \rho\xi\eta & \rho\eta^2 & \rho\eta\zeta \\ \rho\xi\zeta & \rho\zeta\eta & \rho\zeta^2 \end{bmatrix} = \begin{bmatrix} p-\frac{M}{9k\rho\Theta_2}p\left(2\frac{du}{dx}-\frac{dv}{dy}-\frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & p-\frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx}-2\frac{dv}{dy}-\frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) & p-\frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx}-\frac{dv}{dy}-2\frac{dw}{dz}\right) \end{bmatrix} \quad (27)$$

From $\mathcal{R} \equiv \frac{M}{6k\rho\Theta_2}p$, we get $(220)_B$. The equations (11) equals $(185)_B$ and (12) equals $(221)_B$ except for the coefficient.

4.7. **Entropy.**

The word entropy was deduced by Clausius [4] in 1865, and following his nomenclature, Boltzmann constructed his first version of equations in 1872, applying entropy to his gas theory. We show citing [4] Clausius' Greek nomenclature, meaning "conversion" of material as follows :

$$(60)_C \quad S = S_0 + \int \frac{dQ}{T}, \quad (65)_C \quad \int \frac{dQ}{T} = S - S_0,$$

> welch, nur etwas anders geordnet, dieselb ist, wie die unter (60) angeführt zur Bestimung von S dienendene Gleichung.
>
> Sucht man für S einen bezeichnenden Namen, so könnte man, ähnlich wie von der Grösse U gesagt ist, sie sey der *Wärme - und Wirkinhalt* des Körpers. Das ich es aber für besser halt, die Namen derartiger für dir Wissenschaft wichtiger Grössen aus den alten Sprachen zu entnehmen, damit sie unverändert in allen neuen Sprechen angewandt werden können, so schlage ich vor, die Grösse S nach dem griechischen Worte ἡ τροπὴ, die Verwandlung, die *Entropie* des Körpers zu nennen.
>
> Das Wort *Entropie* habe ich absichtlich dem Wort *Entropie* durch diese Worte banannte werden sollen, sind ihren physikalischen Bedeutung nach einander so nahe verwandt, dass eine gewisse Gleichartigkeit in der Benennung mir zweckmässig zu seyn scheint. [4, 389-390]
>
> (Transl.) $(60)_C$, $(65)_C$, which seemed to be like only reallocated expression, however, the usage cited in $(60)_C$, is useful equation.
>
> We sought some suitable name for the nomenclature for S, like the quantity U, such as the value of warm and value of work of a material. I considered that it seemed to be suitable to be adopted from the old Greek as the nomenclature for the important quantity, so I owed it to the quantity S from Greek word ἡ τροπὴ, which means "conversion", the *Entropy* of the material. $\cdots$

Boltzmann consider when the following conditions do not hold, where, the number of the two molecules f and f_1, F and F_1 and f and F_1 before and after collision, namely from $(147)_B$,

$$ff_1 \neq f'f_1', \quad FF_1 \neq F'F_1', \quad fF_1 \neq f'F_1'.$$

We construct the expression H for the gas contained in the volume element *do*. The value thus found will be multiplied by $-RM$ and divided by *do*. Let this quantity be

$$J = -RM \int f \ln f d\omega.$$

Jdo is then the "entropy" of the gas contained in *do*, if it had the same energy (heat) content and the same progressive motion in space, and obeyed the Maxwell velocity distribution law. It can be calculated just as in §19, and has the value

$$\frac{R\rho}{\mu} \ln(\frac{T^{\frac{3}{2}}}{\rho})$$

here, this value $\frac{R\rho}{\mu}$ is called *Boltzmann constant* and it was inscribed on his epitaph as

$$S = k \ln w$$

which is also

$$(\frac{T^{\frac{3}{2}}}{\rho})^k = \exp S$$

5. Conclusions

Maxwell in 1865, Boltzmann in 1895 and Prandtl[18, 19] in 1904 both used the "well-known hydrodynamic equations" and at latest in 1929, used the nomenclature of "Navier-Stokes equations", using the two-constant not of Navier, but of Saint-Venant, Stokes, and expanded by Maxwell, Kirchhoff and Boltzmann. These three persons verified the hydrodynamic equations without the name as Navier-Stokes equations.

In short, we can state that after formulating by Navier (1827) [15], Cauchy (1828) [3], Poisson (1831) [17], Saint-Venant (1843) [21] and Stokes (1849) [22], the topics of hydrodynamic history are rebuilt by Maxwell (1865) [12], Boltzmann (1895) [2] and Prandtl (1927) [19] in the cyclic interval of about 30 years or so.

As the two constants, Saint-Venant had used ε and $\frac{\varepsilon}{3}$, and Stokes μ and $\frac{\mu}{3}$, while Boltzmann used $\mathcal{R}$ and $\frac{\mathcal{R}}{3}$ after tracing Maxwell. According to Prandtl[18], we can suppose that the naming may be decided in "The third international mathematical Congress" in Heidelberg in 1904 or few years later than it. Boltzmann states hydrodynamic equations as well as the Euler equations of $(183)_B$:

> Die Gleichungen 221 sind die bekannten auf innere Reibung corrigirten hydrodynamischen Gleichungen. [1, p.169]
>
> (transl.) Equations (221) are the well-known hydrodynamic equations corrected for internal viscosity. [2, p.176]

According to Boltzmann's description, we can suppose the fact that the then academic society had not fixed yet the name of this equations, up to 1895 or 1898.

Basically, the NS equations were deduced from Newton's kinetic equation (the second law of motion) : $\mathbf{F} = m\mathbf{r}$, [20] however, the gas equations by Boltzmann were not deduced from it, but he based on and evolved the idea of gas theory by its progenitors Maxwell and Kirchhoff.

When we consider the contribution by Boltzmann to the NS equations, Boltzmann show the Euler equations and the NS equation as the special case of his general hydrodynamic equations. He verified the validity of the Euler equations and the NS equations, which were recognized in 1934 at latest by Prandtl [19, p.259], and at the epoch about one hundred years after Navier's paper [15] in 1821.

6. Epilogue. Boltzmann and Humanity

In 1898, Boltzmann had published *Vorlesungen über Gastheorie*, II Teil. (*The lecture of gas theory,* Part II), in which preface, he had expressed his fear that the theory of gases were temporarily thrown into oblibion as follows :

> Es wäre daher meines Erachtens ein Schaden für die Wissenschaft, wenn die Gastheorie durch die augenblicklich herrschende ihr feindselige Stimmung zeitweilig in Vergessenheit geriethe, wie z.B. einst die Undulationstheorie durch die Autorität Newton's. [1, Vorwort]
>
> In my opinion it would be a great tragedy for science if the theory of gases were temporarily thrown into oblibion because of a momentary hostile attitude toward it, as was for example the wave theory because of Newton's authority. Forward to Part II. [2, p.215]

After eight years, a newspaper in Wien '*Neue Freie Presse*', (New Free Press, Wien, Freitag, 07/September in 1906, Nr. 15102) reports Mach's consternation confronted by the news of Boltzmann who had taken his life. Here we cite our transcription from the Fraktur printing style of the newspaper in 1906, which is in Broda [5] [21], and we show it in our last page of our paper, thanking Saburo Ichii and

[20] (⇓) The Newton's kinetic equation (the second law of motion) : $\mathbf{F} = m\mathbf{r}$, where, $\mathbf{F}$: the force, m : the gravity, $\mathbf{r}$: the acceleration.

[21] (⇓) The original by Broda didn't cite this newspaper, however, the translators into Japanese [5] cites a photo of the then news stories in the Fraktur printing style. Here we cite our transcription from the Fraktur printing style into the today's German style for convenience' sake.

Toshihiko Tsuneto and the publishing company Misuzu Shobo. From here, we can see Boltzmann was having both the ardent passion to the learning and the pure humanity in his lifetime.

Remark. Mach had been the supervisor of Boltzmann and both were the then position of 'Hofrat', namely the advisor to Court of the Empire of Austria-Hungary, [22] so that the news reads 'Hofrat Mach' or 'Hofrat Boltzmann'.

Hofrat Professor Mach über den Tod Boltzmanns.

• Hofrat. Mach, der durch den Tod Boltzmanns zehr schmerzlich berührt worden ist, feilte und mit, daß das fraurige Ende der durch Selbstmord geräde jetzt nicht zu befürchten war, da sich sein geistiger Zustand in der lasten Zeit etwas gebessert hatte. Seit etwas zwei Jahren war zu er allerdings Unfällen von Irrwahn ausgefahrt, in denen sich bei ihm namentlich der Trieb zur Flucht fühlbar machte. Er mußte deshalb sorgfältig über macht werden. Doch traten wieder Momente ein, in denen er beruhigender Zusprache zugänglich war. Dies war auch der Fall, als er zur Erholung nach Duino gebracht wurde. Er versprach sich ruhig zu verhalten, und die Familie glaubte, daß die Besserung anhalten werde, so daß man nicht aus den Eintritt seiner verbürgten Gerüchten zufolge hat Boltzmann schon damals verführt, Hand an sich zu legen.

• Gelegentlich der Unwesenheit von Professor Dftmalb in Wien habe ich Boltzmann zum leztenmal in wirtlich froher Laune gesehen, in so guter Stimmung, wie selten vorher und nie wieder seither. Wir wohnten damals zusammen den Borirägen des Berliner Gastes im Ingenieur- und Architektenverein bei und zum Abschied war die Sachwelt bei einem Bankett vereinigt. Dftmalb saß auf den Ehrenplatz, Boltzmann zu seiner Rechten und ich zur Linken. Die "Glücksformel", die Dftmalb entwickelt hatte, gab Boltzmann Anlaß zu einer geistsprühenden den Tischrede. Lange saßen wir beisammen, und nach Mittelnacht geleitete ich ihn heim. Boltzmann war von einer kindlichen Reinheit des Geistes, von unerschöpflicher Liebenswürdigkeit und glücklich, wenn er jemanden gefällig sein konnte.

• Un Unerkennung als Gelehrter hat es ihm nie geschkt. Seine Bedeutung war je überagend, daß man sich ihr nicht entziehen konnt. Es war ihm auch beschieden, aus dem Kreise seiner Schüler große Männer hervorgehen zu sehen. Der Schwede Arrhenius, der Berliner Bernst, beide Koryphäen der Wissenschaft, waren Hörer Boltzmanns, und beide haben oft betont, wie unendlich viel sie ihrem Meister zu danken haben. Nach der Pensionierung von Professor Mach hat Hofrat Boltzmann auch philosophische Vorfräge gehalten, die sich außerordentlich guten Besuches zu erfreuen hatten.

• Es ist ein Jammer, daß ein Mensch von der gewartigen Bedeutung Boltzmanns vor der Zeit aus dem Leben geschieden ist. Er hat der Wissenscaft Immenses geleistet, aber es war immer noch Prozeß von ihm zu erhoffen.

Translated sketches of the news story :

• Mach was surprised at the news of Boltzmann's death. Mach had heard that Boltzmann was saying himself his recent steady calm, so the people of family had supposed that Boltzmann was recovering from being in the low spirits and had not been afraid of such an imminent state of mind.

• We lived then together with the gests from Berlin of the association of tecknology and architecture in Borirägen. He avoided the drinking party or banquet for his standard of value. Dftmalb took the seat of honor, to whom Boltzmann sat the right side and I the left side. Dftmalb proposes "the formula of happiness", Boltzmann gave the oppotunities for the speech. We were sitting together with him. At midnight, we went back to home. Boltzmann had a childish unalloyed genuine of mind and devoted endless kindness in perfect happyness to anybody, whom, when he could be kind to.

• His temperate obstinancy as a scholer didn't allow him to play his cards well. His idea was so noble that one should have not been easy to get along with him. Boltzmann kept away from the troubles with the scholars. Arrhenius of Swedish and Bernst of the Berliner were the authorities in each academic arena and colaborators of studies with Boltzmann and also the good listeners of Boltzmann's talks, and both have emphasized that how very frequently they had thanked their savant, Boltzmann. Boltzmann gave also the lectures on philosophy.

• The interviewee, Mach concludes his talk in the last paragraph with the following evaluation to Boltzmann : "It is greatly to be regretted that a promissing person upon his future, considering the

[22]($\Downarrow$) The Empire of Austria-Hungary : 1867-1918.

importance of Boltzmann, passed away his life. He had achieved the great tasks, however, it was still under the process of extending it eternally."

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Remark: we use *Lu* (: in French) in the bibliography meaning "read" date by the referees of the journals, for example MAS. In citing the original paragraphs in our paper, the underscoring are of ours.

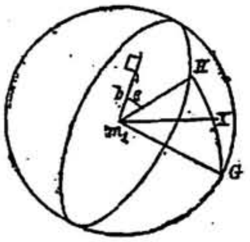

Fig. 6.

Hofrat Professor Mach über den Tod Boltzmanns.

Hofrat Mach, der durch den Tod Boltzmanns sehr schmerzlich berührt worden ist, teilte uns mit, daß das traurige Ende des Gelehrten durch Selbstmord gerade jetzt nicht zu befürchten war, da sich sein geistiger Zustand in der letzten Zeit etwas gebessert hatte. Seit etwa zwei Jahren war er allerdings Anfällen von Irrwahn ausgesetzt, in denen sich bei ihm namentlich der Trieb zur Flucht fühlbar machte. Er mußte deshalb sorgfältig überwacht werden. Doch traten wieder Momente ein, in denen er beruhigender Zusprache zugänglich war. Dies war auch der Fall, als er zur Erholung nach Duino gebracht wurde. Er versprach, sich ruhig zu verhalten, und die Familie glaubte, daß die Besserung anhalten werde, so daß man nicht auf den Eintritt einer verbürgten Gerüchten zufolge hat Boltzmann schon damals versucht, Hand an sich zu legen

Gelegentlich der Anwesenheit von Professor Ostwald in Wien habe ich Boltzmann zum letztenmal in wirklich froher Laune gesehen, in so guter Stimmung, wie selten vorher und nie wieder seither. Wir wohnten damals zusammen den Vorträgen des Berliner Gastes im Ingenieur- und Architektenverein bei und zum Abschied war die Fachwelt bei einem Bankett vereinigt. Ostwald saß auf dem Ehrenplatz, Boltzmann zu seiner Rechten und ich zur Linken. Die „Glücksformel“, die Ostwald entwickelt hatte, gab Boltzmann Anlaß zu einer geistsprühenden Tischrede. Lange saßen wir beisammen, und nach Mitternacht geleitete ich ihn heim. Boltzmann war von einer kindlichen Reinheit des Geistes, von unerschöpflicher Liebenswürdigkeit und glücklich, wenn er jemandem gefällig sein konnte.

An Anerkennung als Gelehrter hat es ihm nie gefehlt. Seine Bedeutung war so überragend, daß man sich ihr nicht entziehen konnte. Es war ihm auch beschieden, aus dem Kreise seiner Schüler große Männer hervorgehen zu sehen. Der Schwede Arrhenius, der Berliner Nernst, beide Koryphäen der Wissenschaft, waren Hörer Boltzmanns, und beide haben oft betont, wie unendlich viel sie ihrem Meister zu danken haben. Nach der Pensionierung von Professor Mach hat Hofrat Boltzmann auch philosophische Vorträge gehalten, die sich außerordentlich guten Besuches zu erfreuen hatten.

Es ist ein Jammer, daß ein Mensch von der gewaltigen Bedeutung Boltzmanns vor der Zeit aus dem Leben geschieden ist. Er hat der Wissenschaft Immenses geleistet, aber es war immer noch Großes von ihm zu erhoffen.

Doctoral Course in Mathematics, Graduate School of Tokyo Metropolitan Univ., **Part 4.** (2011/02/11), pp.193–235

The early studies of solutions of the Navier-Stokes equations

ABSTRACT. After the NS equations were fixed or so, many researchers of hydrodynamics studied the mathematical analyses, in particular, the functional analysis on the solutions of the NS equations. From the viewpoint of the mathematics, the full-scale studies have been begun to the weak solutions by Leray [12, 13, 14] in 1933/34 and by Hopf [4] in 1950/51. And soon after that, A.A.Kiselev [5, 7] in 1954/55, and Kiselev and Ladyzhenskaya [8] in 1957 and Ladyzhenskaya [11] in 1959 constructed the generalized solutions / the strong solutions. Prodi [23] and J.L.Lions [15] discussed the uniqueness of the solution of the NS equations in the three dimensions.

We sketch these historical facts at the beginning of the study on the solutions of the NS equations.

Finally, we show two sort of translations into English on the solutions of the NS equations, viz.

- from Hopf's German paper [4] only on the existence of a weak solution like Leray
- from Ladyzhenskaya's Russian paper [11] of a generalized / strong solution like Kiselev in the first time

We think that both are notable and full-scale studies not only of the NS equations or of the mathematical history, but also of the pure mathematics like functional analysis.

Mathematics Subject Classification 2010 : 01Axx, 76A02, 76Mxx, 76-02, 76-03, 33A15, 35Qxx 35-xx.

Key words. The Navier-Stokes equations, fluid dynamics, fluid mechanics, hydrostatics, hydrodynamics, hydromechanics, microscopically-descriptive equations, weak solution, strong solution, mathematical history.

Contents

1. Introduction

After Stokes' linear equations, the equations of gas theories were deduced by Maxwell in 1865, Kirchhoff in 1868 and Boltzmann in 1872, They contributed to formulate the fluid equations and to fix the Navier-Stokes equations, when Prandtl stated the today's formulation in using the nomenclature as the "so-called Navier-Stokes equations" in 1934, in which Prandtl included the three terms of nonlinear and two linear terms with the ratio of two coefficients as 3 : 1, which arose from Poisson in 1831, Saint-Venant in 1843, and Stokes in 1845.

In 1932, Hadamard published a book entitled *"Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques"*, (The Cauchy problem and the partial differential equations of the linear hyperbolic type.), in which he refers *théorème de Cauchy-Kowalewski* :

> Les trois questions suivantes se posent évidemment en ce qui concerne le problème de Cauchy :
> (1) Le problème de Cauchy a-t-il une solution ?
> (2) N'a-t-il qu'une seule solution (en d'autres termes, le problème est-il correctment posé ?);
> (3) Et enfin comment peut-on calculer cette solution ?
> Quoique les deux premières questions puissent être considérées simplement comme préliminaires, nous allons commencer par examiner comment on peut y répondre.
> On sait que Cauchy lui-meme, puis Sophie Kowalewski, et, au même moment, Darboux,[1] considéraient le cas dans lequel (2) ou (II) [2] peuvent être résolus par raport à r (ou r_m), savoir :
> $$(2')_{Ha} \quad r = f(u,\ x,\ y,\ p,\ q,\ r,\ s,\ t), \quad \text{ou}: \quad (\Pi')_{Ha} \quad r_m = f(u,\ x_1,\ \cdots),$$
> ce qui est le cas pour (2) ou (II) si :
> $$(3)_{Ha} \quad \frac{\partial\Phi}{\partial r} \neq 0, \quad \text{ou} \quad \frac{\partial\Phi}{\partial r_m} \neq 0;$$
> sous cette hypothèse, ils ons démontré (ou du moins sont considérés généralement comme ayant démontré) que le *problème de Cauchy*, par rapport à $x = 0$ (ou $x_m = 0$), admet toujours une solution et une seule. [3, pp.10-11, art. 7]

These questions correspond the following mathematical concepts :
(1) Existence of a solution of the Cauchy problem
(2) Uniqueness of the solution (well-posedness) of the problem
(3) Solvability of the solution on the problem

After the NS equations were fixed or so, many researchers of hydrodynamics studied step by step the mathematical analyses, in particular, the functional analysis on the solutions of the NS equations :

- At first, the full-scale studies have been begun with the weak solutions by Leray [12, 13, 14] in 1933/34 and by Hopf [4] in 1950/51.
- And soon after that, A.A.Kiselev [5, 7] in 1954/55, and Kiselev and Ladyzhenskaya [8] in 1957 and Ladyzhenskaya [11] in 1959 constructed the generalized solutions / the strong solutions.
- Prodi [23] and J.L.Lions [15] discussed the uniqueness of the solution of the NS equations in the three dimensions in 1959.

We sketch these historical facts and their assertions at the beginning of the solutions of the NS equations.

Finally, we show two sort of translations into English on the solutions of the NS equations, viz. :

[1] Cauchy, *C.R.Acad. Sc.*, vol. **14**, p.1020 ; vol. **15**, p.44, 85, 131(1842), Sophie Kowalewski, *Thesis*, Göttingen (1874) et *ournal für math.*, t. **80** (1875), pp.1-32 ; Darboux, *C.R. Acad. Sc.*, vol. **80** (1875), pp.101-104 et p.317.

($\Downarrow$) We can see that Cauchy is a progenitor of the analysis. He also published the two papers on the NS equations [1, 2] in 1828.

[2] ($\Downarrow$) The equations (2) and (II) defined by Hadamard are as follows :

$$(2)_{Ha} \quad \Phi(u,\ x,\ y,\ p,\ q,\ r,\ s,\ t) = 0, \quad (\Pi)_{Ha} \quad \Phi(u,\ x_i,\ p_i,\ r_i,\ s_{ik}) = 0, \quad i = 1, \cdots, m.$$

- From Hopf's German paper [4] only on the existence of a weak solution like Leray
- From Ladyzhenskaya's Russian paper [11] of a generalized / strong solution like Kiselev in the first time

We think that both are notable and full-scale studies not only of the NS equations or of the mathematical history, but also of the pure mathematics like functional analysis.

2. The studies of the weak solution of the NS equations

2.1. **Leray's introduction to construct the solution of turbulent flow.** Leray [3] [14, p.195] says : [4]

> If I should succeed to construct the solution of the equations of Navier[5] which become irregular, I shall have the right to insist that there exist effectivelly the solutions of turbulent flow marely no reducing, in the solutions of regular flow. Similarly, if this proposition should be false, *the notion of solution of turbulent flow* which will play no role any longer in the study of viscous liquid, will do no harm to its interest : it must well present the problems of mathematical physics for which physical cause of regularity is not sufficient to justify the hypothesis of the regularity made in setting of equation.

2.2. **Hopf's comment to Leray.** E.Hopf[4] comments on his **Lemma 5.1**, which we mention below, to the J.Leray[14] in [4] : [6]

> In the Rellich's theorem, the convergence of the x-integral on the quadratic of the derivation is presupposed. Our converging presupposition relates even to the (x,t)-integral and is therefore better adapted to the situation in our problem. Leray prove and use **Lemma** 2, which is even near to Rellich's lemma, operate like this theorem, only with (x)-integral. Our proof of convergence is more direct.

Hopf improved Leray's method described in [14] and proposed **Lemma** 2 in 1950/51. We show the English translation of Hopf [4] in § 5 below.

3. The study of the generalized solution / the strong solution

3.1. **Kiselev.** Kiselev [7][7], who published a paper titled *"Non-stationary flow of the viscous incompressible fluid on the smooth three-dimensional domain"* in 1956, is one of the progenitors of the **generalised solution** and the **strong solution** as follows :

$$L\mathbf{v} \equiv \frac{\partial \mathbf{v}}{\partial t} + \sum_{k=1}^{3} v_k \frac{\partial \mathbf{v}}{\partial x_k} - \nu \Delta \mathbf{v} = -\mathrm{grad}\ p + \mathbf{f}, \tag{1}$$

$$\mathrm{div}\ \mathbf{v} = 0, \tag{2}$$

$$\mathbf{v}|_{t=0} = \mathbf{a}, \tag{3}$$

$$\mathbf{v}|_S = 0 \tag{4}$$

where $\mathbf{f} = \mathbf{f}(x,t)$ and $\mathbf{a}(x)$ is the given vector, ν is the viscosity coefficient, which, for the brief description's sake, (we) deal as the constant. (We) call the vector $\mathbf{v}$ the **generalised solution** of the problem (1)-(4) on Q_t, if $\mathbf{v} \in L^2(Q_t)$, exists generally in the sense of S.L.Sobolev[25].

Theorem 1 (Uniqueness theorem). *The problem (1)-(4) have in Q_t not more than a generalised solution*

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[3](⇓) Leray, Jean. (1906-1998).

[4](⇓) This English version from French was made by the author of this paper. cf [14].

[5](⇓) Leray didn't use the NS equations but "the equations of Navier". Prandtl used the NS equations in his lecture in 1929 and in his text [22] published in 1934.

[6](⇓) This English version from Germany was made by the author of this paper.

[7](⇓) Kiselev, Andrei Alekseevich.

Theorem 2 (Existence theorem 1). *Supposing* $\mathbf{a} \in W_2^{(2)}$ *and satisfies the conditions (2) and (4),* $\mathbf{f} \in L_2(Q_t)$ *and* $\frac{\partial \mathbf{f}}{\partial t} \in L_2(Q_t)$ *and satisfies the condition* $\|\mathbf{a}\| \, \{ \, \|\mathbf{f} + L\mathbf{a}\| + \|\mathbf{f}\| \, \}_{t=0} < \frac{\nu^3}{\beta^2}$ *where* β *: a constant, depending on the domain* Ω*, and the symbol* $\| \cdot \|$ *means the norm in* $L_2(\Omega)$*. Then the problem (1)-(4) have the generalised solution, in any cases, for all* $t \in [0, T]$*, where* T *: an arbitrary number* $\leq l$*, satisfying* $\left(\|\mathbf{a}\| + \int_0^T \|\mathbf{f}\| dt\right)\left(\|\mathbf{f} - L\mathbf{a}\|_{t=0} + \max_{0 \leq t \leq T} \|\mathbf{f}\| + \int_0^T \|\frac{\partial \mathbf{f}}{\partial t}\| dt\right) < \frac{\nu^3}{\beta^2}$. $\square$

3.2. **Kiselev and Ladyzhenskaya.** They say in [8]:

In (our)[8] paper, (we) study the problems of the incompressible viscosity:

$$\frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + \sum_{k=1}^{3} v_k \frac{\partial \mathbf{v}}{\partial x_k} = -\text{grad } p + \mathbf{f}(x,t), \quad \text{div } \mathbf{v} = 0, \quad \mathbf{v}|_S = 0, \quad \mathbf{v}|_{t=0} = \mathbf{a} \tag{1}$$

Formulation 1. (We) shall call it a **generalized solution** of problem (1), that is the vector function $\mathbf{v}(x,t)$, having the generalized derivatives $\in L_2(Q_T)$ of the first order, summing to the power of 4 in a plane of $t = $ const for an arbitrary profile Q_T, $\int_\Omega \sum_i v_i^4(x,t)dx <$ const, and satisfying the conditions: div $\mathbf{v} = 0$, $\mathbf{v}|_S = 0$, $\mathbf{v}|_{t=0} = \mathbf{a}$ and the equality: $\int_0^T \int_\Omega \left[\frac{\partial \mathbf{v}}{\partial t}\mathbf{\Phi} + \nu \frac{\partial \mathbf{v}}{\partial x_k}\frac{\partial \mathbf{\Phi}}{\partial x_k} - v_k \mathbf{v} \frac{\partial \mathbf{\Phi}}{\partial x_k} - \mathbf{f}\mathbf{\Phi}\right] dxdt = 0 \cdots (3) \ {}^\forall \mathbf{\Phi} \in L_2(Q_T)$ such that $\frac{\partial \mathbf{\Phi}}{\partial x_k} \in L_2(Q_T)$, div $\mathbf{\Phi} = 0$, $\mathbf{\Phi}|_S = 0$. $\square$

Formulation 2. (We) shall call it a **generalized solution** of problem (1), that is the vector function $\mathbf{v}$(x,t), having the generalized derivatives $\in L_2(Q_T)$ in the form of $\frac{\partial^2 \mathbf{v}}{\partial t \partial x_i}$ and its all belongings satisfing the same condition as in **Formulation 1**. $\square$

(The Theorem 3,4 and 5 are new contents in [8] in comparison with [5, 7]. The following theorem is same as Kiselev[5, 7], about a strong solution which is already in [7].)

Theorem 6. *If* $\mathbf{a} \in J_{0,1}(\Omega) \cap W_2^2(\Omega)$*, and* $\mathbf{f}$ *and* $\mathbf{f}_t$ *are* $\in L_2(Q_t)$*,* $Q_t = \Omega \times [0, l]$*, then the problem (1) has the* **generalized solution** *in the sense of* **Formulation 2** *on the cylinder* $Q_T = \Omega \times [0, T]$*, such that* T *: no-smaller than an arbitrary number, depending on* ν*,* $\|\mathbf{a}\|_{W_2^2(\Omega)}$*,* $\|\mathbf{f}\|_{L_2(Q_l)}$*,* $\|\mathbf{f}_t\|_{L_2(Q_l)}$ *and the scale of the domain* Ω*.* $\square$[9]

We show the English translation of Ladyzhenskaya [11] in § 7 below.

4. Study of function space L^p for the uniqueness of the solution

4.1. **Prodi.** [10] Prodi [23] is one of the progenitors with J.L.Lions [15, 16][11] of the modern style combining with the function spaces, which J.L.Lions didn't described in [15]. Prodi's main theorem in 1959 is the following :

when B is a space of Banach, (we) put $u \in L^p(0, \tau; B)$. This means as follows : u is the function of t with the value in B, and integrable to the power of p within the interval : $(0, \tau)$. In special case, $L^p(0, \tau; L^p)$ is equivalent with $L^p(\Omega \times (0, \tau))$. By setting p and q as the number such that $p > 3$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. (We) have evidently $2 < q < 6$. $\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} - \mu \Delta_2 u_j = -\frac{\partial p}{\partial x_j} + f_j, \quad \frac{\partial u_j}{\partial x_j} = 0, \quad (j = 1, 2, 3).$

Theorem 7. *A function* u *which is a solution of the defined problem is unique if satisfies the following condition* $u \in L^{\frac{2p}{p-3}}(0, \tau; L^p(\Omega))$ *by the arbitrary value of* p*, with* $3 < p \leq +\infty$ *.* $\square$

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[8]($\Downarrow$) We refer the original [15] in using (we/our). This English version from Russian was made by the author of this paper. The first English version : Amer. Math. Soc., Transl(2) **24**(1963) by John Abramowich without corrections and comments. After conveying deep gratitude to him, we corrected the original misprints, amended phrases and words.

[9]($\Downarrow$) cf. Kiselev [7, p.27].

[10]($\Downarrow$) This English version from Italian with comments was made by the author of this paper.

[11]($\Downarrow$) This is not yet found in J.L.Lions [15].

5. **Hopf : *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. (On the initial-value problem for the hydrodynamic control-equations)***

5.1. **Introduction.** Hopf [12] [4] is one of the most important papers for the weak solution of the NS equations, however we have scarcely had the English translated version of Hopf [4] from German up to now, so we introduce our translation below. [13]

Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen.
(On the Initial-value Problem for the Hydrodynamic Control-Equations)
Eberhard Hopf (Commented by the author of this paper.)

1. Preliminaries

$f, g, h:$ *function*, $a, u, v, w:$ *field*.
$G : (x)$-domain,
$\hat{G} : (x,t)$-domain,
$x : x_1, x_2, \cdots, x_n$,
$u_i : u_i = \{u_i \in u(x,t) | i = 1, 2, \cdots n\}$,
$dx : dx_1 dx_2 \cdots dx_n$.

The divergence-free property of an (x,t)-domain: $\hat{G}$ on $u = \{u(x,t)|\ u \in C^1\}$ [14] is described by the differential equality:

$$\text{div } u = \frac{\partial u}{\partial x_\nu} = 0, \quad \nu = 1, 2, \cdots, n. \tag{1}$$

We shall not use the summation symbol: $\sum$ but conventional summation-description. We say that (x,t)-domain $\hat{G}$ define scalar or vector-valued function $v(x,t) \in N\ in\ \hat{G}$, if $v \equiv 0$ holds in the exterior of the compact subset $\subset$ this domain. The functions of the often used class below converge even in $\partial\hat{G}$. Its description is like this: The divergence-free on u:

$$u = \{u(x,t)|\ u \in C^1\ in\ \hat{G}\}$$

$$\iff \int\int_{\hat{G}} u_i \frac{\partial h}{\partial x_i}\ dxdt$$

$$(\equiv \int (u \cdot \nabla h)\ dt = -\int (\text{div } u \cdot h)\ dt) = 0, \ \ {}^\forall h(x,t) \in \ N\ in\ \hat{G}. \tag{2}$$

We define the scalar product of the two vector fields: $v(x,t)$ and $w(x,t)$ in $\hat{G}$ by

$$\int\int_{\hat{G}} v_i w_i\ dxdt,$$

so we can say:
the divergence-free on $u = \{u(x,t)|\ u \in C^1\ in\ \hat{G}\} \Rightarrow\ u \perp \frac{\partial h}{\partial x_i},\quad h \in N\ in\ \hat{G} : unique.$[15]

The following contrapositions to these facts is here of interest:

- The continuous field $h'(x,t)\ in\ \hat{G}$ with its component h_i' : the gradient-field(: $h_i' = \frac{\partial h}{\partial x_i}$) of the function $h(x,t)$ which is unique and moreover its x−derivative continue.
- It is necessary and sufficient that in $\hat{G}$ continuously x-differentiable and divergence-free field of the class N is orthogonal.

[12]($\Downarrow$) Eberhard Hopf (1902-83).

[13]($\Downarrow$) Except for 11 remarks by Hopf, which we mark with $(E.H)$, and the other footnotes marked with ($\Downarrow$) are by the author of this paper. The numbers of equations correspond to that in the original paper.

[14]($\Downarrow$) C^1 is used in meaning that u is continuously x−differentiable, which is abbreviated by the author of this paper.

[15]$(E.H)$ The formulation of the concept in the (x,t)-domain instead of (x) is effective for our problem. Application of Hilbert space theory on the problem of the potential theory and mathematic hydro-dynamic, we find in the following papers: O.Nikodym,"*Sur un théorem de M.S.Zaremba concernant les fonctions harmoniqes.*" J.Math.pur.appl., Paris,Sér.IX **12** (1933), 95-109; J.Leray,"*Sur le mouvement d'un liquide visqueux emplissant l'espace.*" Acta math., Uppsala **63** (1934), 193-248; H.Weyl,"*The method of orthogonal projection in potential theory.*" Duke math.J.**7** (1940), 411-444.

The necessity is further the result of the integral theorem. That the condition is sufficient holds from the following thought: by using $w(x,t) = \varphi(t)w(t)$ with the scalar φ, we can restrict on the corresponding proposition for x-domain:G. It holds also

$$\int_{\hat{G}} w_i h_i^{'}\, dx = 0, \quad {}^{\forall} w = \{w(x) |\ w \in N\ in\ G:\ smooth\ \&\ divergence{-}free\}.$$

The proposition holds if we can satisfy that the circulation of $h^{'}$:

$$\int_{\mathcal{C}} h_i^{'} w_i\, dx_i = \int_{\mathcal{C}} h_s^{'} ds$$

converges along the closed path in G. We see clearly that we have to prove only for the continuous curved-path without self-crossing. We get the vanishing by selecting the suitable value on w. To arbitrary small, known $\epsilon > 0$, there is always the smooth and divergence-free stream: $w(x)$ with the following properties:

w vanishes at 0 only in the closed cylinder with the thickness $< \epsilon$ around the path: $\mathcal{C}$. On each plane $\mathcal{C}$ vertically-passing through cylindrical section, the vector w makes an angle $< \epsilon$ to normal-direction (direction of $\mathcal{C}$ in the section) . The section-flow of w, which is independent of the special section on the divergence-free, is equivalent to 1. This fact is sufficient for the proof of the converge of the circulation along $\mathcal{C}$. We use such a $w(x)$, corresponding to a given, but small enough selected $\epsilon > 0$.

We put the hyper-plane-element on the cylindrical section with dF and we select the arc-length: s along $\mathcal{C}$ as the parameter across the section, so we can put in the cylinder the volume-element: dx in the form $\rho dF ds$, where ρ is in the neighborhood of $\mathcal{C}$ continuous and on $\mathcal{C}$ equal to 1. Then, we get

$$\int h_i^{'} w_i\, dx = \int \Big[\int h_w^{'} |w| \rho dF \Big]\, ds.$$

We replace here the component $h_w^{'}$ by the component $h_s^{'}$, gained in the section-point of $\mathcal{C}$ with the section, and we replace again $|w(x)|$ by the component w_s, in the normal-direction of dF, and replace ρ with 1, then from integral of the right-hand side

$$\int h_s^{'} \Big[\int w_s dF \Big] ds = \int h_s^{'} ds,$$

that is, the circulation. On the ground of the given properties of w, it become clear in itself that by this replacement, the error with ϵ is evaluated $\to 0$. Therefore the proposition is proved. □

The control-equation of Navier-Stokes[16]for the movement of a homogeneous and incompressible fluid is :

$$\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_\beta \partial x_\beta}. \tag{3}$$

where μ is a positive constant, the kinematic viscosity coefficient[17], and

$$\text{div}\ u = 0.$$

Each of $u(x,t)$ and $p(x,t)$ is a solution in the (x,t)-domain: $\hat{G}$, moreover the derivatives which appear in the equation, u_t, u_x and u_{xx}, are continuous.

We give now a new time-dependent and in $\hat{G}$ divergence-free vector field: $a = a(x,t)$. It must be $a \in N\ in\ \hat{G}$ and smooth enough: a and the differentials: a_t, a_x, a_{xx} are continuous in $\hat{G}$. On a is not imposed any further restriction. For $a \in N\ in\ \hat{G}$[18] and for $u_\alpha \frac{\partial u_i}{\partial x_\alpha} = \frac{\partial u_i u_\alpha}{\partial x_\alpha}$ the following holds:

$$\int\int_{\hat{G}} a_i \frac{\partial u_i}{\partial t}\, dxdt = -\int\int_{\hat{G}} \frac{\partial a_i}{\partial t} u_i\, dxdt$$

$$\int\int_{\hat{G}} a_i u_\alpha \frac{\partial u_i}{\partial x_\alpha}\, dxdt = -\int\int_{\hat{G}} \frac{\partial a_i}{\partial x_\alpha} u_\alpha u_i\, dxdt$$

16(⇓) L.M.H.Navier(1785-1836)'s in 1822, G.G.Stokes(1819-1903)'s in 1844, was proposed respectively.

17(⇓) Due to H. Kozono [10] , (3): the kinetic equation conventionally used to be described as follows:

$$\frac{\partial u}{\partial t} - \mu \Delta u + u \cdot \nabla u + \nabla p = 0, here\ owing\ to\ dyadics,$$

$$u \cdot \nabla u = (u \cdot \nabla) u = \sum_{k=1}^{n} u_k \frac{\partial u}{\partial x_k} = (\sum_{k=1}^{n} u_k \frac{\partial u_1}{\partial x_k}, \cdots, \sum_{k=1}^{n} u_k \frac{\partial u_n}{\partial x_k}).$$

18(⇓) In the original paper, there is G instead of $\hat{G}$, which is corrected by the author of this paper.

$$\int\int_{\hat{G}} a_i \frac{\partial^2 u_i}{\partial x_\beta \partial x_\beta}\, dxdt = -\int\int_{\hat{G}} \frac{\partial a_i}{\partial x_\beta}\frac{\partial u_i}{\partial x_\beta}\, dxdt = \int\int_{\hat{G}} \frac{\partial^2 a_i}{\partial x_\beta \partial x_\beta} u_i\, dxdt$$

[19] For div $a = 0$ and $a \in N$

$$\int\int_{\hat{G}} a_i \frac{\partial p}{\partial x_i}\, dxdt (\equiv \int (a, \nabla p) dt = -\int (\text{div } a, p) dt) = 0.$$

We find therefore, that the field $u(x,t)$ satisfies the following condition:

$$\int\int_{\hat{G}} \frac{\partial a_i}{\partial t} u_i\, dxdt + \int\int_{\hat{G}} \frac{\partial a_i}{\partial x_\alpha} u_\alpha u_i\, dxdt + \mu \int\int_{\hat{G}} \frac{\partial^2 a_i}{\partial x_\beta \partial x_\beta} u_i\, dxdt = 0. \tag{4}$$

for every smooth enough field $a(x,t)$ in $\hat{G}$ with the property

$$\text{div } a = 0 \ in\ \hat{G}, \quad a \in N \ in\ \hat{G}. \tag{5}$$

Moreover even the divergence-free of field u:

$$\int\int_{\hat{G}} \frac{\partial h}{\partial x_i} u_i\, dxdt (\equiv \int (\nabla h, u) dt = -\int (h, \text{div } u) dt) = 0, \quad h \in N \ in\ \hat{G}, \tag{6}$$

holds for every smooth enough function of the given class in $\hat{G}$. The control-equation is brought into the form of an equation between a linear functional operator of the arbitrary field and functions a, h. The important matter is that the unknown field $u(x,t)$, on which this operator depends, appears without derivatives in them.

We must make it consent that, on the equation (4) and (6) which we understood in just clear sense, we return again to the differential-form of the equation, when we restrict sufficiently smooth solution-field $u \in \hat{G}$. We see already that under this presupposition (6) return again to div $u = 0$ in $\hat{G}$. For sufficiently smooth u, we may cancel all the familiar partial integration. It follows then, that

$$\int\int_{\hat{G}} a_i \Big\{ \frac{\partial u_i}{\partial t} u_i + u_\alpha \frac{\partial u_i}{\partial x_\alpha} - \mu \frac{\partial^2 u_i}{\partial x_\beta \partial x_\beta} \Big\}\, dxdt$$

must hold for every smooth field $a(x,t)$ of the form (5). By the familiar theorem, it follows that the term in brackets above must be the derivative $\frac{\partial p}{\partial x_i}$ of a unique function $p(x,t)$ in $\hat{G}$. We see that the arbitrary integral form of equation is the physical formulation of uniqueness of pressure.

It comes from the general mathematical theory on the integral-form of the equation. It is however effective to free from the technical restriction on the smooth solution-field u. Two bilinear forms :

$$\int u_i u_i dx, \quad \int \frac{\partial u_i}{\partial x_\beta}\frac{\partial u_i}{\partial x_\beta} dx$$

in the energy-equation become the problem on Hilbert space of the vector field. It means that a methodical profit in wider space of the differentiable-property of the solution u become the theme of a formulation, which is able to be studied almost separately on the existence-problem.[20]

The arbitrary initial-value-problem of the hydro·dynamic control-equation is as follows. The solution $u(x,t)$ is to be on the above unstable domain $G(t)$, $t > 0$ of x-space, when $u(x,0)$ in $G(0)$ is as above, is given (with a suitable formulation condition of the continuous connection for $t \to 0$) and when the boundary condition of u in the boundary of $G(t)$, $t \geq 0$ (in suitable formulated sense of connection). J.Leray had devoted to three works in the early 30 years.[21] Leray had already solved this study by the aide of method of Hilbert space and by integral interpretation of equations in three dimensions.[22] Leray had solved this existence-problem for all $t > 0$ in his three papers in the following cases:

a) $G =$ total plane with the kinetic energy $< \infty$.
b) $G =$ fixed ellipsoid with the boundary value$= 0$.
c) $G =$ total three dimensional spaces with the kinetic energy $< \infty$.

[19] (⇓) In the original paper, u_i in last term is absent, and this is inserted by the author of this paper.

[20] $(E.H)$ cf. Additionally, "*The handling of the quadratic variational and linear differential-problem with the method of Hilbert space*" by R.Courant and D.Hilbert, *Methoden der mathematischen Physik*, Bd.2 Berlin 1937, Chap.VII.

[21] (⇓) J. Leray(1906-1998). a) "*Étude de diverses équations intégrales non linéaires et quelques problèmes que pose l'Hydrodynamique.*" J. Math.pur.appl., Paris,Sér.IX **12** (1933), 1-82; b)"*Essay sur les mouvements plans d'un liquide visqueux que limitent des parois.*" J.Math.pur.appl., Paris,Sér.IX **13**(1934), 331-418; c)l.c.f.n.

[22] $(E.H)$ C.W.Oseen based already long ago on his famous hydrodynamic study of a form of the controll equation, in which the secondary derivative is zero. He successed in proof on the existence only for sufficiently short time. cf. his work: Hydrodynamik(Leipzig 1927).

On the remarkable work of the differentiability problem, Leray points out a marked difference between the cases: $n = 2$ and $n > 2$. In the first case at least, if $G = total\ plane$, differentiability of the solution went well, but the method for $n \geq 3$ was gave up, to which we should evaluate as natural. Even by smoothness of all above data, the smoothness of solution in all time space was not proved. Strange to say, he gave up the proof on the uniqueness in 3D. These problems have never been explained enough even now. It is difficult to believe that the initial-value problem of the viscid fluid for $n = 3$ could have more than one solution, and we are to devote more in the approach on the uniqueness problem. It comes, however, only from another new studies, that, with the nonlinear partial differential problem, the number of the independent variable on the local properties of the solution has a fundamental impact.

We ignore the problem of the differentiability and even uniqueness in the proposing paper, at least initial-value problem, in which we start from the view on the integral of the equation as the primitive form. On this fact, we want to return to the (in our space not easy) proof of energy-equation in our following papers. The main purpose of this paper is **the construction of the approximate solution**,[23] which occupy a very wide space in Leray's work, here is treated by a simplified process, which can apply to wider class of the partial differential-problem. Also to that we want to return later. The method suggests the solution of the initial-value problem $^\forall t > 0$ in considerable generality, however, in this first paper, **the origin of the methodically basic idea** is more important for us than the generality of the result. We restrict here on the case, in which x-domain G is fixed in time, however completely arbitrary, and where, u is supposed to have the vanishing boundary value. The boundary condition is defined by the concept of Hilbert space, so wide enough the solvability is, and so close the uniqueness of the solution is, at least to hold in the two dimensions.[24] In the pure existence-theorem, the dimension number plays no role.

2. Function class H'. Solution of class H'

$f(x,t)$: *measurable function with the bounded norm of* $L^2(\hat{G})$ *on the class* $H \in \mathbf{R}$, H:Hilbert space.

$$\int\int_{\hat{G}} f^2\ dxdt < \infty.$$

$$s\text{-}\lim_{n\to\infty} f_n \to f^* \in H\ in\ \hat{G} \Rightarrow w\text{-}\lim_{n\to\infty} f_n \to f^* \in H\ in\ \hat{G}.$$

$$\int\int_{\hat{G}} fg\ dxdt \to \int\int_{\hat{G}} f^*g\ dxdt, \quad {}^\forall g \in H\ in\ \hat{G}.$$

$$\int\int_{\hat{G}} fg\ dxdt, \quad {}^\forall g:\ fixed\ function \in H,\ strong\ dense\ set$$

converges. There is then $^{\exists 1} f^*$: *weak admissible function in* G. Here instead of $\hat{G}$, we must use G. Then the norm

$$\int_G f^2\ dx$$

is fundamental. We recall the weak compactness of the function sequence: $\{f_n\}$ with uniformly bounded[25] norm (theorem of F. Riesz).[26]

Following criterion is often used by J. Leray for strong convergence, we also use it, namely:

$$w\text{-}\lim_{n\to\infty} f_n \to f^*(x,t) \Rightarrow \overline{\lim} \int\int_{\hat{G}} f^2\ dxdt \geq \int\int_{\hat{G}} f^{*2}\ dxdt,$$

here

$$s\text{-}\lim_{n\to\infty} f_n \to f^* \iff =\ holds\ in\ the\ above\ inequality.$$

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[23]($\Downarrow$) This solution is called "weak solution", a term which is not at all used in this paper.

[24]($\Downarrow$) If G is total x-space, in the condition of bounded kinetic energy and bounded dispersion-integral, it becomes the closed boundary condition. For understanding of the boundary condition is recommended by R.Courant and D.Hilbert, *Methoden der mathematischen Physik*, Bd.2 Berlin 1937, Chap.VII.

[25]($\Downarrow$) This original word is "*gleichmässig beschränkten*",cf J.Serrin,p.72, K.Masuda,p.644.

[26]($\Downarrow$) cf. Leray, I, §3. Forte convergence en moyenne. §4. Procédé diagonal de Cantor.

All this holds for the vector-fields u, v in $\hat{G}$, if the scalar product

$$\int\int_{\hat{G}} u_i v_i \; dxdt$$

and its corresponding norm are used.

Lemma 2.1.

$$w-\lim_{n\to\infty} u_n \to u^*(x,t) \Rightarrow \overline{\lim} \int\int_{\hat{G}} u_i u_i \; dxdt \geq \int\int_{\hat{G}} u_i^* u_i^* \; dxdt.$$

$$s-\lim_{n\to\infty} u_n \to u^*(x,t) \iff = \; holds \; in \; the \; above \; inequality. \quad \square$$

We use like Leray the concept of the generalized (total space) x-derivative[27] of function $f(x,t)$ and field $u(x,t)$.

Definition 2.1. $f(x,t)$ $defined\ in\ (x,t)\text{-}domain:\ {}^{\exists}\hat{G}\ should\ be\ f \in H^{'} in\ \hat{G} \iff it\ has\ the\ following\ properties$

$$f \in H \; in \; \hat{G}$$
$${}^{\exists}n, \; f_{'i} \in H \; in \; \hat{G} \; s.t.$$

$$\int\int_{\hat{G}} h f_{'i} \; dxdt = -\int\int_{\hat{G}} \frac{\partial h}{\partial x_i} f \; dxdt \;\; (h \in N \; in \; \hat{G}), \quad {}^{\forall}h(x,t), i = 1, 2, \cdots, n. \quad \square \tag{7}$$

$$f(x,t) \in H^{'}, \quad {}^{\forall}f \in C \; in \; \hat{G} \; s.t. \; f \; and \; all \; \tfrac{\partial f}{\partial x_i} \in \hat{G}.$$

For such a f,

$$\frac{\partial f}{\partial x_i} = f_{'i}.$$

This follows from the integral theorem and from the assumption that $h \in N$ i.e. h vanishes in the exterior of the given compact subset $\subset \hat{G}$. It is clear that $f \in H^{'}$ in $\hat{G}$
$\Rightarrow$ *generalized* x*-derivative* : $f_{'i}$ *in* G *uniquely determine until on the value in a* (x,t)*-null set.*

Lemma 2.2.

$$f \in H^{'} \; in \; \hat{G} \; and \; {}^{\forall}f, \; w-\lim_{n\to\infty} f_n \to f^*$$

$${}^{\forall}f, \quad \int\int_{\hat{G}} f^2 \; dxdt + \int\int_{\hat{G}} f_{'i} f_{'i} \; dxdt$$

$$uniformly \; bounded$$

$$\Rightarrow f^* \in \; H^{'} \; in \; \hat{G} \; and \; {}^{\forall}f_{'i}, \; w-\lim f_{'i_n} \to f_{'i}^* \quad \square$$

[28] <u>Proof</u>
${}^{\forall}f$ satisfies (7), where h is an arbitrary and admissible function,

$$w-\lim_{n\to\infty} \Big(-\int\int_{\hat{G}} \frac{\partial h}{\partial x_i} f_n \; dxdt \Big) = -\int\int_{\hat{G}} \frac{\partial h}{\partial x_i} f^* \; dxdt.$$

For fixed h, i along f_n

$$w-\lim_{n\to\infty} \int\int_{\hat{G}} h f_{'i_n} \; dxdt = \int\int_{\hat{G}} h f_{'i}^* \; dxdt.$$

The admissible function h in $\hat{G}$ in Hilbert space H strong dense and from the presupposition H-norm of $f_{'i}$ in $\hat{G}$ weak converges. We put f_i^* as limit function, so from (7) follows:

$$\int\int_{\hat{G}} h f_{'i}^* \; dxdt = -\int\int_{\hat{G}} \frac{\partial h}{\partial x_i} f^* \; dxdt \;\; {}^{\forall}h \; and \; {}^{\forall}i.$$

[27](⇓) cf. Leray, I, §7. Quasi-derivées., §9.Quelques lemmes concernant les quasi-derivées.
[28](⇓) $||f||_2 + ||\nabla f||_2 \equiv ||f||_{H^1}$

Definition2.1 $\Rightarrow f^* \in H^{'}\ in\ \hat{G}$ and from the uniqueness on x-derivation

$$f_i^* = f_{,i}^*. \quad \square$$

A field is called $\in H^{'}\ in\ \hat{G}$, when all components satisfy this. In the above integral-form of hydro·dynamic control-equation, no derivative of u appear. It is important in itself to set on the solution of weak differentiability-formation as belonging to class $H^{'}$. We may put therefore viscosity term in (4)

$$\mu \int\int_{\hat{G}} \frac{\partial^2 a_i}{\partial x_\beta \partial x_\beta} u_i\ dxdt = -\mu \int\int_{\hat{G}} \frac{\partial a_i}{\partial x_\beta} u_{i,\beta}\ dxdt \quad \square. \tag{8}$$

Definition 2.2. *u(x,t) is called* <u>*a solution of class* $H^{'}$</u> *of the hydro·dynamic equations in the domain:* (x,t) *of* $\hat{G}$*, if the following conditions are satisfied.*

a) $u \in H^{'}\ in\ \hat{G}$.

b) <u>*Divergence-free:*</u> $^{\forall}h \in N\ in\ \hat{G}$ *and* $h^{'} \in C$.

c) <u>*Kinetic equation:*</u> *the relation of (4) is satisfied by* $^{\forall}a(x,t) \in N\ in\ \hat{G}$, $div\ a = 0$ *and* a_t, a_x, a_{xx} *are continuous, namely* $a \in C^2$. $\square$

We consider that under the presupposition a) even that in u non-linear term in the control-equation(4) is a well-defined Lebesgue integral by the admissible field: a. If $u \in H\ in\ G$, this case already holds. For a) the condition of incompressibility b) is identified as follows:

$$\text{div}\ u \equiv u_{i,i} = 0, \quad for\ almost\ all\ x,\ t\ in\ \hat{G}$$

hold.[29] We think that in the control-equation(4) all integrands are zero in the exterior of $\hat{G}$. This is integrable if it looks at all over the (x,t)-space. With this arrangement the following theorem holds, which we shall prove, although we shall not use it in this paper.

Theorem 2.1.

$$\int_{t=\tau} a_i u_i dx = \int_{t<\tau}\int \frac{\partial a_i}{\partial t} u_i\ dxdt + \int_{t<\tau}\int \frac{\partial a_i}{\partial x_\alpha} u_\alpha u_i\ dxdt - \mu \int_{t<\tau}\int \frac{\partial a_i}{\partial x_\beta} u_{i,\beta}\ dxdt. \quad \square \tag{9}$$

Here $^{\forall}a(x,t)$*: admissible field under the* **Definition 2.2** *c).*

<u>Proof</u>

We think that with $a(x,t)$ even $h(t)a(x,t)$ is an admissible field, if $h(t)$: a total arbitrary $^{\forall}t$ differentiable function. We set in the bracket

$$\int\int K[a,u]\ dxdt = \int_{-\infty}^{\infty}\{\int_{t=\tau} K[a,u]dx\}d\tau = 0.$$

ha instead of a in the equation (4), so the equation also holds as follows

$$\int_{-\infty}^{\infty} h(\tau)\{\int_{t=\tau} K[a,u]dx\}d\tau = \int_{-\infty}^{\infty} h^{'}(\tau)\{\int_{t=\tau} a_i u_i dx\}d\tau = 0. \tag{10}$$

The brace in $-\infty < \tau < \infty$ by Lebesgue integral on τ, which for all $h(\tau)$ with continuous $h^{'}(\tau)$ is, as you know, is equivalent to

$$\int_{t=\tau} a_i u_i dx = \int_{-\infty}^{\tau}\{\int_{t\ fixed} K\ dx\}dt = \int_{t<\tau}\int K\ dxdt\ ^{\forall}\tau. \quad \square$$

3. The boundary condition for vanishing. The initial value problem

The cross section $t = const$ of (x,t)-domain $\hat{G}$ is x-domain $G(t)$. We must approximate to boundary of $G(t)$ nearest as could as possible with help on concept of Hilbert space of boundary condition of the vanishing of a function $g(x,t)\ ^{\forall}t$ and a field $u(x,t)\ ^{\forall}t$. This reach that we can get the function g from the function $\in N\ in\ \hat{G}$, by the suitable limit-procedure. Then it is necessary, sufficient effective restriction for the real space x-derivative of approximated function (not but the t-derivative) to make use of which the "vanishing" at the boundary of x-domain $G(t)$ is essentially preserved. We suggest the boundary condition with the belonging to the following functions class: $H'(N)$.

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[29] $(E.H)$ When we study that, we had not mentioned in this paper, the problem of limit-procedure: $\mu \to 0$ in the hydrodynamic fluid, we loss the function space H' of its condition which is essentially bound with the case $\mu > 0$. Obviously we must then be depend on the derivation-free definition of the divergence-free.

Definition 3.1.

$$w- \lim_{n \to \infty} \gamma_n(x,t) \to g(x,t), \quad \gamma \in \ N \ in \ \hat{G}, \quad \gamma_x \in C,$$

$$^{\forall}\gamma, \quad \int\int_{\hat{G}} \gamma^2 \ dxdt + \int\int_{\hat{G}} \gamma_{'i}\gamma_{'i} \ dxdt \tag{11}$$

is uniformly bounded,[30]

$$\Rightarrow g(x,t) \ is \ called \ g \in H^{'}(N) \ in \ \hat{G}. \quad \square$$

Lemma 3.1. $\hat{G}: cylindrical\ domain : x \ \subset \ G, \ 0 < t < T,$

$$s- \lim_{n \to \infty} \gamma_n(x,t) \Rightarrow g(x,t) \in \ N \ in \ \hat{G}$$

$\gamma(x,t) \in$ *the exterior of* $^{\exists}$*compact subset* $\subset G \ \Rightarrow \gamma \to 0;$ *uniformly bounded.*

$$\Longrightarrow g(x,t) \in H^{'}(N) \ in \ \hat{G}. \quad \square$$

Proof[31]
We put

$$\gamma \equiv \varphi(t)\gamma(x,t), \ \varphi \in C, \ ^{\forall}t \in < 0, T >,$$

$$\varphi \equiv \begin{cases} 0 & for \ 0 < t < \epsilon, \quad T - \epsilon < t < T, \\ 1 & for \ 2\epsilon < t < T - 2\epsilon \end{cases} \quad \Rightarrow 0 < \lim_{\epsilon \to 0} \varphi < 1.$$

In addition, we put $\tilde{\gamma} \equiv \varphi\gamma$,

$$\Rightarrow g \in H^{'}(N). \quad \square$$

Lemma 3.2. *By* $^{\forall}f \in H^{'} \ in \ \hat{G} \ and \ ^{\forall}g \in H^{'}(N) \ in \ \hat{G}$*, the following equation holds :*

$$\int\int_{\hat{G}} g_{'i}f \ dxdt = - \int\int_{\hat{G}} g f_{'i} \ dxdt, \quad (i = 1, 2, \cdots, n). \quad \square$$

Proof
Definition2.1 $\Rightarrow^{\forall} f,^{\forall} \gamma$: *continuously differentiable in* $\hat{G}$ *and* $\in N$. **Definition**3.1 $\Rightarrow$

$$w- \lim_{n \to \infty} \gamma_n \to g$$

with uniformly bounded integral by(11).
Lemma2.2 $\Rightarrow$ *instead* $(\gamma_{'i_n} \to g_{'i})$ *of* $(\gamma_n \to g)$ *in* $\hat{G}$.

$$^{\forall}f,^{\forall}\gamma \Rightarrow^{\forall} f,^{\forall} g. \quad \square$$

For effective formulation of the initial condition, we introduce now the class: $H(N)$. We restrict ourselves by it on the x-space and field $u(x)$, which in a x-domain G is cleared. When we regard only function $f(x)$, which belong to both class H and N in G, so it is clear that it is equal the strong convex hull of this function space with H. This holds on vector field in G. Otherwise it is however, when we are restricted on the divergence free field in G.

Definition 3.2. *A weakly limited field* $\in$ N *and* $\in$ C^2 *and divergence-free in* G
$\Rightarrow$ *(divergence-free field* $\in$ H *) is called* $\in$ $H(N)$ *in* G. $\square$

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We prove easily:
the field $u(x) \in H(N)\ in\ G$ is divergence-free and $\varphi(x) \in H^{'}\ in\ G \Rightarrow$

$$\int u_i \varphi_{'i} \ dx = 0.$$

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[30] $(E.H)$ $||\gamma||_2 + ||\nabla\gamma||_2 \equiv ||\gamma||_{H^1}$
[31] ($\Downarrow$) cf. Leray, III, §16.
[32] $(E.H)$ From the theorem by Saks, it is then even strong limit-field of just such a field.

[33] Belonging of the divergence-free field $\in H(N)$ compensate clearly the boundary condition of the vanishing of the normal component.

We can now formulate the existence theorem[34] for the hydro·dynamic initial value problem.

Existence Theorem
G : *x-domain*, $U(x)$: *divergence-free*, $^{\forall} U(x) \in H(N)$.
$\Rightarrow^{\forall} t$ *in* $G, ^{\exists} u(x,t)$ *defined with the following properties* :

A) *In every cylindrical domain* : $(x,t),\ x \subset G,\ 0 < t < T$,
u is a solution $\in H'$ *of hydro·dynamic control-equation*(**Definition 2.2**).

B) $^{\forall} t > 0$, *vanishing of boundary value* :
In every cylindrical domain, $u \in H'(N)$.

C) *Initial value condition*:

$$For\ t,\ s-\lim_{t\to 0} u_n(x,t) \to U(x)\ in\ G. \quad \square$$

4. Simplification of the problem. Approximation-process

For the construction of the solution u of the initial-value-problem for a fixed-time x-domain G, we get from the equation

$$\int_G a_i u_i dx\Big|_{t=\tau'} - \int_G a_i u_i dx\Big|_{t=\tau}$$
$$= \int_{\tau}^{\tau'} \int_G \frac{\partial a_i}{\partial t} u_i dx dt + \int_{\tau}^{\tau'} \int_G \frac{\partial a_i}{\partial x_\alpha} u_\alpha u_i dx dt + \mu \int_{\tau}^{\tau'} \int_G \frac{\partial^2 a_i}{\partial x_\beta \partial x_\beta} u_i dx dt. \tag{12}$$

[35]

Lemma 4.1. $^{\forall} t$ *in* G, *given*, $x \subset G,\ 0 < t < T,\ u(x,t) \in H$.
$\tau' >^{\forall} \tau > 0$ *and* $^{\forall} a(x) \in C^2$ *and* :

$$a = a(x), \quad \text{div } a = 0 \ in\ G, \quad a \in N\ in\ G, \tag{13}$$

namely, $a(x) \to 0, ^{\forall} a \in$ *the exterior of a suitable compact subset* $\subset G$.

$\Rightarrow$ *u holds* (12) *in the half-cylindrical domain* $\hat{G} : x \subset G,\ 0 < t$ *and* $^{\forall} a(x,t)$: *admissible field*, ($\rightarrow$ *See condition c) in* **Definition**2.2 *on the solution-definition*). $\square$

<u>Proof</u>
We describe (12) in the following abbreviated form:

$$f(\tau') - f(\tau) = \int_{\tau}^{\tau'} g(t)dt$$

$$\Rightarrow \int_0^\infty \varphi'(t) f(t) dt + \int_0^\infty \varphi(t) g(t) dt = 0 \quad ,^{\forall} \varphi \in C(0,\infty), \quad 0 <^{\forall} t < \infty.$$

We moreover, describe this equation fully, so we see that the equation (4) is satisfied in above half-cylinder by all fields $a = \varphi(t)a(t)$, where $a(x)$ is an arbitrary of above admissible field : (13) and $\varphi(t)$ is an arbitrary one of admissible function. We can approximate now, however, all the admissible field $a(x,t)$ in the condition c) of the solution-**Definition** 2.2, in the half-cylinder $\hat{G}$, so by the summation on fields with the special technique , with which we can exchange in the control-equation. We can always so arrange it, for example, that the convergence of field and its derivation until above-mentioned order in

[33] $(E.H)$

$$\int u_i \varphi'_i\ dx = -(\text{div } u \cdot \varphi) = 0.$$

[34] ($\Downarrow$) cf. Leray, III, §19, IV, §25.

[35] ($\Downarrow$) In the original paper, there is u instead of u_i, which is corrected by the author of this paper.

$\hat{G}$ is equivalent and that the approximated field vanishes in the exterior of a suitable fixed and compact subset of $\hat{G}$. $\square$

It is now clear, that a field $u(x,t)$, which is a solution of the equation (4) in the domain of lemma, which is further divergence-free in half-cylinder, and which satisfies the solution-**Definition** 2.2 $\in H'$ in all cylindrical sections.

We get an even more suitable form of the equation on the basis of the following facts.

Lemma 4.2. *There is a sequence:* $^{\exists}a^{\nu} \in C^2$ *in G and in G linear independent field in (12):*

$$a = a^{\nu}(x), div\ a^{\nu} = 0\ in\ G, \quad a^{\nu} \in\ N\ in\ G, \tag{14}$$

with the following properties:
$a \in C^2$ *in (12) is a uniformly-limit-field in G of a sequence of bounded linear-combinations:* $\{a^{\nu}\}$, *with uniformly converge of only the derivative until 2-order in G. By given a(x) in this approximation only such linear-combination is used, that is, have null-value in the exterior of the dependent compact subset* $\subset G$. $\square$

On the basis of this fact, it is clear that a field $u(x,t)$, which is in all cylindrical section $\in H$ and which satisfies the control-equation (12), $\tau' >^{\forall} \tau > 0$ and for all field : a of the above sequence, these effect automatically for all above admissible field (13). To sum up, we could say that the control-equation (4) for the present expression could be made up by the control-equation (12) with (14).

In the function space of the divergence-free vector-field: a, (12), (14) is an affine coordinated-description of hydro·dynamic equation. The affine system of the coordinated-vector(14) can be described simply by the linear transformation, that is, in the sense of bilinear form:

$$\int_G v_i w_i dx$$

is orthonormal. We may moreover presuppose that the sequence of (14): $\{a^{\nu}\}$ satisfies these conditions:

$$\int_G a_i^{\lambda} a_i^{\nu} dx = \delta_{\lambda\nu}. \tag{15}$$

Lemma 4.3. *The orthonormal system on the field* : a^{ν}
is complete in the field-space of divergence-free field : $U(x) \in H(N)$ *in G.* $\square$

Proof
Lemma 4.3 holds from **Definition** 3.2 *and* **Lemma** 4.2. $\square$

The approximation process

The k-th approximation step holds so that we think over only the first k of the unbounded, many control-equations: (12),(14),

$$a = a^{\nu}(x), \quad (\nu = 1, 2, \cdots, k), \tag{16}$$

and seek to solve these by the theorem

$$u = u^k(x,t) = \sum_{\nu=1}^{k} \lambda_{\nu}(t) a^{\nu}(x) \tag{17}$$

with to-be-given scalar factor $\lambda_{\nu} = \lambda_{\nu}^{k}$. This theorem satisfies, from (14) by itself, the condition of divergence-free and the boundary condition of the vanishing:

$$\text{div } u^k = 0\ in\ G, \quad u^k \in N\ in\ G. \tag{18}$$

Because only differentiable $\lambda(t)$ come into question and because the admissible field a is independent of t, the first k of equation (12) could be described in the form

$$\int_G a_i \frac{\partial u_i}{\partial t} dx = \int_G \frac{\partial a_i}{\partial x_{\alpha}} u_{\alpha} u_i dx + \mu \int_G \frac{\partial^2 a_i}{\partial x_{\beta} \partial x_{\beta}} u_i dx. \tag{19}$$

For (15), the k's equation (19),(16) with (17) become an ordinary differential system

$$\frac{d\lambda_{\nu}}{dt} = F_{\nu}(\lambda_1, \cdots, \lambda_k) \quad (\nu = 1, 2, \cdots, k), \tag{20}$$

where the right-hand side $F_\nu = F_\nu^k$ is the polynomial on λ with constant coefficient. (19),(16) and (17) i.e. equivalent meaning equation (20) share now with the strict hydro·dynamic equation, the important property such that the energy-equation[36]:

$$\frac{d}{dt}\frac{1}{2}\int_G u_i u_i dx = -\mu \int_G \frac{\partial u_i}{\partial x_\beta}\frac{\partial u_i}{\partial x_\beta} dx. \tag{21}$$

holds for the solution.[37] Since namely the equation (19) exists for all field of (16), so also for its linear combination (17): $u = u^k$ holds. The energy-equation follows in the usual way (and without difficulty on boundary) because by (18)

$$\int_G \frac{\partial u_i}{\partial x_\alpha} u_\alpha u_i dx = \int_G \frac{\partial K}{\partial x_\alpha} u_\alpha dx = 0, \quad (K = \frac{1}{2} u_i u_i)$$

[38] [39] and

$$\int_G \frac{\partial^2 u_i}{\partial x_\beta \partial x_\beta} u_i dx = -\int_G \frac{\partial u_i}{\partial x_\beta}\frac{\partial u_i}{\partial x_\beta} dx, \quad (u = u^k)$$

hold.[40] From (21),

$$\int_G u_i u_i dx = \lambda_1^2 + \cdots + \lambda_k^2, \quad (u = u^k)$$

by no means increase[41]. Hence, we decide that each of the initial solution of the system (20) at $t = 0$ exists $^\forall t = 0$.

The approximation process mean formally very simple as follows. We think by ourselves the both sides of the Navier-Stokes differential equation and the solution u formal from the orthonormal system of the field a^ν expanded : $u = \lambda_\nu a^\nu$. We make then purely formally first order for the unbounded many scalar Fourier coefficients λ of a system of the unbounded many differential equations . Our k-th step is simply, so that we use only the first k-th of this equations and evaluate in them all the unknown with index $\nu > k$ equal to 0. This method, by which we prove bellow, the existence theorem, moreover supply us a proposition on the convergence properties of this simplest and nearest approximation process.

As the initial value of $\lambda_\nu(t)$ *at* $t = 0$, we select the Fourier coefficients of the expansion of the given field $U(x)$ from the a^ν. While the solution $\lambda(t)$ in k-th step, in general, of the k-depend is this initial value of which is independent. From the presupposition $U \in H(N)$ *in* G and from the completeness lemma 4.3 holds

$$s\!-\lim_{k\to\infty} u_k(x,0) \to U(x) \ in \ G. \tag{22}$$

5. Proof of existence theorem

We summarize : the sequence field $\{u^k(x,t)\}$ has the following necessary properties:[42]

5a) $^\forall u^k(x,t) \in C^2, ^\forall x \subset G, ^\forall t \geq 0$.

5b) $^\forall u^k(x,t) \in exterior\ of\ compact\ x\text{-}domain \subset G,\ depend\ on\ only\ k$
$\Rightarrow u^k(x,t) \to 0$.

5c) $^\forall u^k(x,t)\ satisfy\ (19)\ \ ^\forall t \geq 0$,
$and\ (12)\ at\ \tau' > \tau \geq 0\ in\ the\ k\text{-}th\ order\ of\ (14), \quad (\nu = 1, 2, \cdots, k)$.

36($\Downarrow$) cf. Leray, II,Movements infinitement lents., §13,III.Movements reguliers §17.

37$(E.H)$ $\|u\|_2^2 + 2\mu \int_0^t \|\nabla u\|_2^2 \, d\tau = 0$.

38$(E.H)$

$$\int_G \frac{\partial K}{\partial x_\alpha} u_\alpha dx \equiv -(K \cdot \text{div } u) = 0.$$

39($\Downarrow$) cf. Leray,III,§17, V,§27.

40$(E.H)$

$$\int_G \frac{\partial^2 u_i}{\partial x_\beta \partial x_\beta} u_i dx = \frac{\partial u_i}{\partial x_\beta} u_i - \int_G \frac{\partial u_i}{\partial x_\beta}\frac{\partial u_i}{\partial x_\beta} dx \quad , \ here\ \frac{\partial u_i}{\partial x_\beta} u_i = (\text{div } u) \cdot u = 0.$$

cf. definition by (1)

41($\Downarrow$) cf. Leray, V. Solutions turbulents. §31.

42($\Downarrow$) cf. Leray, V. §28.

5d) *The integrations :*

$$\int_G u_i u_i dx, \quad \int_0^T \int_G \frac{\partial u_i}{\partial x_\beta}\frac{\partial u_i}{\partial x_\beta} dxdt, \quad (u = u^k(x,t)) \tag{23}$$

exist under the convergence as $k \to \infty$. *These are independent of* $({}^\forall k, {}^\forall t, {}^\forall T)$.

5e) *The initial value* :${}^\forall u^k(x,0)$ *satisfies* (22). □

5*d*) follows directly from the combination of (21) and (22).

1st step:

${}^\forall a^\nu(x)$ is continuous and $a^\nu(x) \neq 0$ only in a compact subset $\subset G$. We apply the first half of 5d) to $(a = a^\nu)$ of the left-hand side of (19), in which we evaluate
· the linear term in $u = u^k$ by using the Schwarz inequality,[43] and
· the quadratic term in u by using the absolute-convergence for the derivative on a,
so we get as follows: the right-hand side of (19): $(a = a^\nu, u = u^k, k \geq \nu)$ is uniformly bounded by the fixed-value $\nu, {}^\forall k$ *and* ${}^\forall t$.
Of the left-hand side, we consider similarly as follows :

$$\frac{d}{dt}\int_G a_i u_i dx$$

By the fixed-value ν, the time function:

$$\int_G a_i^\nu(x) u_i^k(x,t) dx \geq 0$$

satisfies a k-independent Lipschitz condition[44] ${}^\forall t \geq 0$. In addition, this is uniformly bound ${}^\forall t$, ${}^\forall k$. From the famous axiom of choice[45], there is also $k' \in \mathbf{Z}$ such that

$$^\exists \lim_{k' \to \infty} \int_G a_i^\nu(x) u_i^{k'}(x,t) dx, \quad {}^\forall t \geq 0, {}^\forall \nu : fixed \tag{24}$$

where, this is uniform for each bounded t-interval. The sequence of k' : $\{u^{k'}\}$ is dependent of index ν, but we can select it to index: $\nu+1$ as the subsequence of the preceding sequence. With the diagonal method[46], then a fixed sequence of $k' \in \mathbf{Z}$, which we put moreover with k', make just-made limit-proposition true for all fixed ν, $\nu = 1, 2, \cdots$. By these sequence, k' is operated as follows:

2nd step:

We prove here that the $\{u^{k'}(x,t)\}$ converges weakly in G, ${}^\forall t \geq 0$, t : fixed. For the proof, we fix t_0. From the first half of 5d), the sequence of these field at $(t = t_0)$ is weak compact in G. This proposition would be proved, if we show that each sequence in G can have only unique weak-limit-field. We put $u^*(x, t_0)$: limit-field, and $k''(< k')$: partial sequence, which is depend of t_0, such that

$$\lim_{k'' \to \infty} \int_G w_i(x) u_i^{k''}(x, t_0)\, dx = \int_G w_i(x) u_i^*(x, t_0)\, dx, \quad {}^\forall w(x) \in \ H \ in \ G.$$

In the case of $w = a^\nu$, the value of the right-hand side is however already fixed by the limit of (24). u^* and u^{**} [47] are two weak-limit-fields and v is its differential-field, so then

$$\int_G a_i^\nu v_i \, dx = 0, \quad {}^\forall \nu.$$

holds. From **Definition** 3.2 u^*, u^{**}, $v \in H(N)$ *in* G. From **Lemma** 4.3 a^ν span field-space by itself in G. Therefore

$$\int_G v_i v_i dx = 0$$

43(⇓) cf. Leray, I. Préliminaires. §2. Rappelons l'inégalité de Schwarz.

44(⇓) cf. Leray, I, §5.

45(⇓) $A_{\lambda \in \Lambda} \neq \emptyset \Rightarrow \Pi_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. K.Masuda use the theorem of Ascoli-Arzelà instead of axiom of choice.

46(⇓) cf. Leray, I, §4. Procédé diagonal de Cantor. V,§29.

47(⇓) This symbol: u^{**} is not at all used in the other formulation in this paper.

and the proposition holds. It holds also that in G $^\forall t > 0$, a value-given field $u^*(x,t)$ such that

$$\lim_{k'\to\infty} \int_G w_i(x) u_i^{k'}(x,t)\, dx = \int_G w_i(x) u_i^*(x,t)\, dx, \quad ^\forall t > 0, ^\forall w(x) \in\ H\ in\ G \tag{25}$$

u^* satisfy the condition B) of the existence theorem in §3. It follows from 5b) and 2nd half of 5d) and by using **Lemma** 3.1. We prove simply also as follows:

$$w\text{-}\lim_{k'\to\infty} u^{k'} \to u^*, \quad u^{k'} \in \hat{G},\ 0 < t < T.$$

3rd step:

We prove that $u^*(x,t)$ satisfies the condition A) of the existence theorem. $u^* \in$ *the* $^\forall$cylindrical domain, $x \subset G,\ 0 < t < T, u^* \in H'$, which is the super-class of $H'(N)$: $H'(N) \subset H'$.

From the description in the first half of §4, we are sufficient if only we show that as follows : u^* satisfy (12), $^\forall a = a^\nu, \tau' >^\forall \tau \geq 0$. From 5c) $u = u^*$ satisfy (12) for the same $^\forall \tau, ^\forall \tau'$ and for the first k' : a^ν. We fix τ, τ' and index: ν, and get limit with $k' \to \infty$. It is clear that we may replace u^* instead of u on the left-hand side of (12). Similarly, the third integral on the right-hand side of (12) holds. We consider that the inner integral as the following :

$$\int_\tau^{\tau'} \Big[\int_G w_i(x) u_i^{k'}(x,t)\, dx\Big] dt \tag{26}$$

for 5d) the first half is uniformly bounded function on t, k', and that we can apply the Lebesgue's convergence theorem on the outer t-integral. We may also exchange the order of both limit-procedures: $(k' \to \infty)$ and integration in 2nd integral of the right-hand side in (12), by more consideration, we use that on the 2nd half on 5d). We need now the following theorem which we shall prove in §6 later.

Lemma 5.1. $\{f^k\} = \{f^k(x,t)\} | f^k \in C^1\ in\ x \subset G,\ 0 < t < T\}$ *have the following properties* :
$^\forall t, f^k \in N\ in\ G,$

$$w\text{-}\lim_{k\to\infty} f^k(x,t) \to f^*(x,t),\ in\ x\ and\ t\ (0 < t < T),$$

the integrations :

$$\int_G f^2(x,t) dx, \quad \int_0^T \int_G f_{,i} f_{,i}\ dx dt, \quad (f = f^k)$$

are uniformly bounded $^\forall k, ^\forall t$.

$$\Rightarrow s\text{-}\lim_{k\to\infty} f^k(x,t) \to f^*(x,t), \quad x \subset QG,\ (0 < t < T),$$

where, $^\forall\ Q : (section\ of\ x\text{-}space) < \infty$.
In particular, if G degenerates, this deduction satisfies in itself. □

From 5a),5b), from the result of 2nd step, and from 5d) $^\forall(fixed\ T)$, by the components of $\{u^{k'}(x,t)\}$, the presupposition of the lemma is satisfied.
$^\forall\ Q(:\ section\ of\ x\text{-}space) < \infty$

$$\Rightarrow \lim_{k'\to\infty} \int_0^T \int_{QG} (u - u_i^*)(u - u_i^*)\ dx dt \to 0, \quad (u = u^{k'}).$$

Therefore, we can justify the limit-procedure: $k' \to \infty$ of the 2nd integral of the right-hand side in (12) $(a = a^\nu,\ \nu :\ fixed)$. We consider that a-factor of the integrant vanishes on the exterior of the fixed compact subset $C(C \subset G)$. By evaluation on $Q \supset C$ and $T > \tau'$, by this integral,

$$\int_\tau^{\tau'} \int_{QG} (a_{i,\alpha} u_\alpha)(u_i)\ dx dt, \quad (a = a^\nu, \ \ u = u^{k'}).$$

produce the following situation. The first factor of the integration converges weakly in the integration domain to $a_{i,\alpha} u_\alpha^*$, while the second factor converges strongly to u_i^*. This holds limit-process : $k' \to \infty$ under the integral symbol, as we know. Therefore, it is proved that u^* satisfy (12) $^\forall a^\nu(x)$ and $^\forall \tau >$

$0, \tau' > 0$. The condition A) of the existence theorem, therefore, is verified until on the divergence-free. The last property is satisfied in itself by a trivial way, moreover $^\forall t$: fixed.

For the completion of the proof of the existence theorem, it is further necessary to show that the initial value condition C) holds. From energy-equation (21) follows

$$\frac{1}{2}\int_G u_i u_i \; dx\Big|_0 = \frac{1}{2}\int_G u_i u_i \; dx\Big|_T + \mu \int_0^T \int_G \frac{\partial u_i}{\partial x_\beta}\frac{\partial u_i}{\partial x_\beta} \; dxdt, \quad ^\forall u \in \{u^{k'}\}. \tag{27}$$

The left-hand side of (27) converges by (22) as follows :

$$s\text{-} \lim_{k' \to \infty} \frac{1}{2}\int_G u_i^{k'} u_i^{k'} \; dx \to \frac{1}{2}\int_G U_i U_i \; dx.$$

$$\Rightarrow w\text{-} \lim_{k' \to \infty} \frac{1}{2}\int_G U_i^{k'} U_i^{k'} \; dx \to \frac{1}{2}\int_G u^* u^* \; dx, \quad t = T.$$

In the (x,t)-cylindrical domain, from **Lemma** 2.2 and 5d)

$$w\text{-} \lim_{k' \to \infty} u_{i,\beta}^{k'} = u_{i,\beta}^*$$

holds. By the application of **Lemma** 2.1 from(27), therefore, the inequality[48] follows:

$$\frac{1}{2}\int_G U_i U_i \; dx \geq \frac{1}{2}\int_G u_i^* u_i^* \; dx\Big|_T + \mu \int_0^T \int_G u_{i,\beta}^* u_{i,\beta}^* \; dxdt, \quad ^\forall T > 0.$$

[49] In particular,

$$\overline{\lim}_{t\to 0}\int_G u_i^* u_i^* \; dx \leq \int_G U_i U_i \; dx.$$

[50] To the last inequality, by using **Lemma** 2.1 again, we get that the initial value condition C) is satisfied. On the problem of the strong convergence at fixed t, we shall no further mention here.

6. Proof on Lemma 5.1

The lemma is narrowly applied by Rellich's axiom of choice and is proved.[51][52] In advance, we would like to remark that when G isn't restricted, even Rellich's axiom of choice does not hold for G by itself. One of the antitheses is the case, where G is total x-space and

$$f^k(x,t) = f(x_1 + k, x_2, \cdots, x_n), \quad f \in H' \cup N \; in \; G.$$

In this case, $f^* = 0$ holds, but it does not satisfy strong convergence to 0.[53] The proof of **Lemma** 5.1 comes from Friedrichs' inequality:

$Q(: section\ of\ x\text{-}space) < \infty, to\ given\ ^\forall\epsilon > 0$ exists a bounded number of fixed functions: $\omega_\nu(x) \in H \; in \; Q$ s.t. the equation:

$$\int_Q f^2 \; dx \leq \sum_\nu \Big[\int_Q f\omega_\nu \; dx\Big]^2 + \epsilon \int_Q f_{'i} f_{'i} \; dx, \quad ^\forall f(x) \in H' \; in \; Q$$

48(⇓) cf. Leray, IV, §24. Due to H.Kozono [10], the energy inequality is described by $\mu = 1$ as follow:

$$||u(t)||_2^2 + 2\int_0^t ||\nabla u(\tau)||_2^2 d\tau \leq ||a||_2^2, \quad ^\forall a \in L_\sigma^2, \; 0 \leq t < \infty,$$
$$||u(t) - a||_2 \to 0, \quad t \to +0$$

where $a \in L_\sigma^2$: total of vector-value-functions: $a \in L^2$, satisfing div $a = 0$, namely σ : symbol of the "solenoidal". This usage is due to P.L.Lion [17], K.Masuda [18], H.Kozono [10] et al.

49(⇓) cf. Leray, IV, §24

50(⇓) cf. Leray, III, §19., IV, §24.

51($E.H$) cf. Courant-Hilbert, l.c.f.n. In the Rellich's theorem, the convergence of the x-integral on the quadratic of the derivation is presupposed. Our converging presupposition is related rather to the (x,t)-integral and is therefore better adapted to the situation in our problem. Leray prove and use **Lemma** 2 in (l.c.f.n.), which is even similar to Rellich's lemma, it is true to operate like this theorem, but only with the (x)-integral. Our proof of convergence is more direct.

52(⇓) cf. Leray, V,§30.

53(⇓) From the lemma, we can therefore only induce the strong convergence of the approximate: $u(x,t) \to u^*(x,t)$ in the cylindrical section, if G is restricted. Meanwhile the strong convergence holds clearly by the arbitrary G. Leray deduce by his approximation in the case, where G is of the total x-space, with the help of the complex estimation of the energy-distribution in G. We want to return to the strong convergence-properties of our approximation later.

is held by means of $f(x)$.[54]

For the proof of **Lemma** 5.1, we would like to remark at first on the fact that by the fixed t, $f^k(x,t)$ of lemma in G is $f \in C^1$ *and* $f \in N$. If we clear this function in the exterior of G by null, this fact is useful, if this proposition is verified in the total x-space instead of on the G. In particular,

$$t \to fix \Rightarrow f : (\in Q(: \ section\ of\ x\text{-}space) < \infty) \in H^{'}.$$

The generalizing of these functions and the last setting is produced by the total presupposition $:\in N$. This presupposing is however used only in these order. We set arbitrarily a section: Q and a number: $\epsilon > 0$ and select various parameter-functions such ω_ν as satisfy the Friedrichs' inequality in Q. We use by the fixed t on the function

$$f(x,t) = f^k(x,t) - f^l(x,t), \tag{28}$$

which is in Q surely $f \in H'$. By integration on t, follows that total functions (28) of the inequality

$$\int_0^T \int_Q f^2 \, dxdt \le \sum_\nu \int_0^T \Big[\int_Q f\omega_\nu \, dx \Big]^2 \, dt + \epsilon \int_0^T \int_Q f_{'i} f_{'i} \, dxdt \tag{29}$$

holds. From the presupposition (weak convergence by fixed t)

$$\lim_{k,l\to\infty} \int_Q f\omega_\nu \, dx = 0$$

holds. The presupposition of bounded (first half) on the basis exist moreover, the function on t

$$\int_Q (f^k - f^l)\omega_\nu \, dx$$

uniformly bounded with both k and l. Therefore, the first term of the right-hand side of the (29) with both $k \to \infty$ and $l \to \infty$:

$$\lim_{k,l\to\infty} \sum_\nu \int_0^T \Big[\int_Q f\omega_\nu \, dx \Big]^2 \, dt \to 0 \tag{30}$$

holds. From the presupposition, exists also the factor with ϵ for (28) under one fixed convergence. From

$$\overline{\lim_{k,l\to\infty}} \int_0^T \int_Q (f^k - f^l)^2 \, dxdt \le c\epsilon, \quad {}^\forall\epsilon > 0$$

follows, however ϵ was arbitrary, the strong convergence of our sequence in (x,t)-domain, $x \subset Q$, $0 < t < T$. We see easy that the limit function in the context of lemma is the function $f^*(x,t)$, so **Lemma** 5.1 is proved. □

[54]($\Downarrow$) The ω_ν would become as the orthogonal in Q. The inequality is then described by an evaluation of the difference in the Bessel's inequality. On the proof of the inequality, we find by Courant-Hilbert, l.c.f.t. Chap.VII, §3, par. 1. We agree by ourselves to that the proof leads us in two dimensions, even the function in n dimensions. The Friedrich's inequality doesn't hold for the arbitrarily restricted domain.

6. Appendices to Hopf's paper (($\Downarrow$))

Theorem 1 by J.Leray and J.P. Schauder [8] is as follows.

Theorem 1. *Let*

$$x - F(x, k) = 0. \tag{1}$$

be the given equation. We apply three following groups of the hypotheses :

(H_1) *The unknown x and all the limits, determining F, belong to the fully normed linear space $\mathcal{E}$. The total of the limit of the parameter k fill the segment K of the axis of the substantial number.*
$F(x, k)$ defined for all the pairs (x, k), where x is the derivative element from $\mathcal{E}$ and k is the derivative element from K.
$F(x, k)$ turns into each bounded space of the point $x \in \mathcal{E}$ in the compact space.
$F(x, k)$ completely continuous with respect to k to each bounded subspace of the space $\mathcal{E}$.

(H_2) *In the arbitrary point k_n of the segment K all solutions of the continuous and their indexes can investigate the method of the chapter II ; we shall assume the total of the indexes suitable in zero.*

(H_3) *At last, we shall assume such a proved fact that the solutions of the problem (1) is bounded in the own group (a priori independent of k).*

We show the another simple definition[55] of same theorem as follows :
X : Banach space.
D : the bounded open set including zero.

$F(x, t) : \bar{D} \times [0, 1] \to X$: the compact map, where $F(x, 0) \equiv 0$. and $F(x, t) \neq x$, if $x \in \partial D$.

$\Rightarrow$ The compact map : $F(x, 1)$ has a fixed point in D.

6.1. The fundamental solutions of the Stokes hydrodynamic differential equation (($\Downarrow$) Extracted from Oseen and translated from German).

6.1.1. Fundamental solutions for the condition on the velocity components. We turn back to our problem, to determine the fundamental solutions of the Stokes differential equations. We said that we shall select these fundamental solutions so that the detail functions v depend only on the two points P and $P^{(0)}$, moreover, that the system of these functions in all themselves way of the coordinate depended, we also select the right hand direction system. It is easy to assume that these new functions of the components of the one than the transformation (10)[56] of the invariant tensor with the range there are 2. We have used from these underlying, deduced, a tensor which in an arbitrary right hand direction system of the following components :

$$t_{jk} = \delta_{jk}\Delta\Phi(r) - \frac{\partial^2\Phi(r)}{\partial x_j x_k}, \quad r^2 = (x_j - x_j^{(0)})^2, \quad r > 0, \tag{11}$$

δ_{jk} is here and bellow the jk-component of a tensor, these diagonal components of $(j = k)$ have the value 1 and the else components have the value 0. The three functions t_{1k}, t_{2k}, t_{3k} satisfy always, i.e., when k have the value $1, 2, 3$, the equation :

$$\frac{\partial t_{jk}}{\partial x_j} = 0. \tag{12}$$

When we define that Φ of the equation :

$$\Delta_x\Delta_x\Phi = 0 \quad \left(\Delta_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) \tag{13}$$

[57] should be satisfied and when we put :

$$-\mu\frac{\partial}{\partial x_k}\Delta_x\Phi = p_k$$

[55]($\Downarrow$) "The Iwanami mathematical dictionary. Revised 3", Iwanami, Tokyo, 1996, pp.933-934, (in Japanese)

[56]($\Downarrow$) $x'_j = a_j + l_{jk}x_k, \quad x_j^{(0)'} = a_j + l_{jk}x_k^{(0)}$.

[57]By a function, which depends on many points, it is useful and sometimes important for the operator Δ to operate by an index of the point to indicate with respect to this.

so we have for all admitted j- and k-value :

$$\mu\Delta_x t_{jk} - \frac{\partial p_k}{\partial x_j} = 0. \tag{14}$$

For these k-value ($1, 2$ and 3) and also the three functions t_{1k}, t_{2k}, t_{3k} and p_k are one solution of the Stokes equation i.e., we can put the equation (13), Φ depends only on r, to the familiar transformation of Δ in the polar coordinate in the following form :

$$\frac{d^4(r\Phi)}{dr^4} = 0.$$

These generalized solution is also $\Phi = ar^2 + br + c + \frac{d}{r}$, where a, b, c, d are the constants. We put from the basis, which we define soon, $\Phi(r) = r$. We have then :

$$\Delta\Phi = \frac{2}{r}, \quad t_{jk} = \frac{\delta_{jk}}{r} + \frac{(x_j - x_j^{(0)})(x_k - x_k^{(0)})}{r^3},$$

$$p_k = -2\mu\frac{\partial}{\partial x_k}\frac{1}{r} = 2\mu\frac{(x_k - x_k^{(0)})}{r^3}.$$

We observe now a domain B of (x_1, x_2, x_3)-space. F is its boundary surface. We assume that the Stokes differential equation has a regular solution in B. We show with $P^{(0)} = (x_1^0, x_2^{(0)}, x_3^{(0)})$ of an arbitrary point in the interior of B. We surround with a sphere with $r = \varepsilon$ and select ε so small that this sphere lies in the interior of the F. $B(\varepsilon)$ is a subspace of B, which includes the exterior of the sphere with $r = \varepsilon$. We use the formula (2)[58] on the domain $B(\varepsilon)$, and we put with $v_{jk} = t_{jk},\ \bar{p} = p_k$. The boundary is consist of the two subspaces of F and the sphere with $r = \varepsilon$. Because the value of rt_{jk} is over even in the point of $P^{(0)}$ is stable and because the boundary of the sphere with $r = \varepsilon$ is proportional with ε^2, we have :

$$\lim_{\varepsilon\to 0}\int_{r=\varepsilon} t_{jk}\Big(\mu\frac{du_j}{dn} - pn_j\Big)dS = 0.$$

Moreover

$$\lim_{\varepsilon\to 0}\int_{r=\varepsilon} u_j\Big(\mu\frac{dt_{jk}}{dn} - p_k n_j\Big)dS = \lim_{\varepsilon\to 0}\int_{r=\varepsilon}\Big\{u_k + 3u_j(x_j - x_j^{(0)})\frac{x_k - x_k^{(0)}}{r^2}\Big\}\frac{dS}{r^2}.$$

We put

$$u_j = u_j^{(0)} + r\varphi \quad \text{where } u_j^{(0)} = u_j(P^{(0)})$$

and because we put with φ as a bounded function of the point P in the neighborhood of $P^{(0)}$. We have then because

$$\int_{r=\varepsilon}\frac{dS}{r^2} = 4\pi, \quad \int_{r=\varepsilon}\frac{(x_j - x^{(0)})(x_k - x_k^{(0)})}{r^2}\frac{dS}{r^2} = \frac{4\pi}{3}\delta_{jk} :$$

$$\lim_{\varepsilon\to 0}\int_{r=\varepsilon} u_j\Big(\mu\frac{dt_{jk}}{dn} - p_k n_j\Big)dS = 8\pi\mu u_k(P^{(0)}).$$

Therefore :

$$u_k(P^{(0)}) = \frac{1}{8\pi\mu}\int_F\Big\{t_{jk}\Big(\mu\frac{du_j}{dn} - pn_j\Big) - u_j\Big(\mu\frac{dt_{jk}}{dn} - p_k n_j\Big)\Big\}dS. \tag{15}$$

When we get the 12 functions T_{jk} and P_k, which in the interior of the boundary F, we can put in the form of : $T_{jk} = t_{jk} + \tau_{jk}, P_k = p_k + \tau_k$, where τ_{jk}, p_k for all k-value ($k = 1, 2$ or 3) of the interior of F, the regular solution of the given Stokes equations, so we can deduce directly owing to the product of (15)

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[58](⇓)

$$\int_F\Big|v_j\Big(\mu\frac{du_j}{dn} - pn_j\Big) - u_j\Big(\mu\frac{dv_j}{dn} - \bar{p}n_j\Big)\Big|dS = 0. \quad (2)$$

t_{jk}, p_k by T_{jk}, P_k. When the new T_{jk} all disappear when the point P, included in the boundary F so we get :

$$u_k(P^{(0)}) = \frac{1}{8\pi\mu}\int_F u_j\Big(\mu\frac{dT_{jk}}{dn} - P_k n_j\Big)dS. \tag{16}$$

6.2. On the boundary value problem of the hydrodynamic viscous fluid. (($\Downarrow$) Extracted from Odqvist and translated from German).

6.2.1. **Definitions, Expressions, Green formulae.** We investigate, in the following, the space domain Q, which is restricted to the finite many, bounded continuous surface. The points of Q is spanned by the certain fixed, right oriented coordinate system $x_1,\ x_2,\ x_3$. The interior points of Q is put with the coordinate x_i $(i = 1, 2, 3)$ or y_i in brief, (x) or (y). The spatial element is put with dQ. The standard point is meant, if necessary, as $d_xQ,\ d_yQ$. We put each domain Q as the "interior domain" with $Q_{(i)}$, we can include an exterior domain $Q_{(e)}$ such that $Q_{(i)} + Q_{(e)}$ makes the total space.

The total surface of the body ("the boundary surface") of the domain Q is called and consist a continuous tangential plane. The points on T are put as $(\xi),\ (\eta), \cdots$ and the surface elements with dT or $d_\xi T, \cdots$. The normal line from the interior point is put with n_i $(i = 1, 2, 3)$.

The functions of the coordinate $x_i, \xi_i, \cdots$ of the points, we use, in brief, with $f(x),\ f(\eta),\ f(x, \eta), \cdots$, etc. It holds therefore, $n_i = n_i(\eta)$.

The distance of the two points x and x' is put with $r_{xx'}$.

A function $f(x)$ of the, we know, 2 points x and x' of a domain Q of the form

$$|f(x) - f(x')| < C^h_{r_{xx'}},$$

where with h of the real, positive and smaller the an arbitrary, and C : a positive constant depending only on h, which we call as H-continuous with the exponential h (*Hölder*).

An surface of T guarantees in the neighborhood of the point, which the expression of the form : $\xi_3 = (\xi_1,\ \xi_2)$, if the coordinate system is suitably selected. The function $f(\xi_1, \xi_2)$ have the first derivative which is H-continuous with the exponential h, so the boundary belongs to the class Ah. Therefore, from here, the 2 differential quotients of $f(\xi_1, \xi_2)$ exist and H-continuous, and so T belongs to the class Bh. The extended domain Q belongs to the class of both Ah and Bh[59]. In our last result, we shall use the class Bh in §6. Although overall with the small presupposition of the class Ah so wide operate than it can.

In the following, the double index in an arbitrary product of the differential quatations, etc., follows with respect to these indices that we should summarize, we write the equation (0.01) in brief

$$\begin{cases} \mu\Delta u_i - \rho u_k \frac{\partial u_i}{\partial x_k} = \frac{\partial p}{\partial x_i} - \rho X_i, \\ \frac{\partial u_k}{\partial x_k} = 0. \end{cases} \tag{1}$$

We study for the moment only its conditions of the viscous fluid, which obey the simple differential equations, which, by the conventional way, we eliminate the quadratic term :

$$\rho u_k \frac{\partial u_i}{\partial x_k}.$$

We have the Stokes equation

$$\begin{cases} \mu\Delta u_i = \frac{\partial p}{\partial x_i} - \rho X_i, \\ \frac{\partial u_k}{\partial x_k} = 0. \end{cases} \tag{2}$$

We call the following problem the **first Stokes boundary value problem** :

Determine in a space domain Q of the class Ah of the functions $u_i,\ p$, such that on the other hand, in Q satisfy :

$$\mu\Delta u_i = \frac{\partial p}{\partial x_i}, \quad \frac{\partial u_k}{\partial x_k} = 0 \tag{3}$$

[59]Compare with L.Lichtenstein : *New evolution of the potential theory. Conformal mapping*, The Encyclopedia of Mathematical Science, IIC3, Teubner, 1919.

and on the other hand, the functions $u_i(x)$ get by the approximation of the boundary surface : T of Q the given value $u_i(\xi)$.

The stress tensor operating in the fluid is described by

$$T_{ik} = -p\delta_{ik} + \mu\Big(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\Big)$$

where

$$\delta_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

The stress tensor operating on the surface element with the normal : n_k is put with

$$T_{ik}n_k.$$

The problem of the equation (1.03) in a domain Q then to investigate that the operating stress tensor (: $T_{ik}n_k$) on the cubic surface (: T) get the given value, we call the **second Stokes boundary value problem**. We call the equation (1.03) the **homogeneous Stokes equation**.

We now two arbitrary, twice continuously differentiable vector u_i and v_i and an arbitrary continuously differentiable scalar value p, then the "**Green identity**"

$$\begin{aligned} &\int_{Q_{(i)}} \frac{\partial}{\partial x_k}[T_{ik}v_{ik}]dQ \\ \equiv\ & \int_{Q_{(i)}} \Big\{\frac{\mu}{2}\Big(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\Big)\Big(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\Big) - p\frac{\partial v_k}{\partial x_k} + \Big[\mu\Delta u_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_i}\Big(\frac{\partial u_k}{\partial x_k}\Big)\Big]v_i\Big\}dQ \\ =\ & \int_T \Big[p\delta_{ik} - \mu\Big(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\Big)\Big]n_kv_idT. \end{aligned} \tag{4}$$

holds. The first 2 integrals are the space integral on the space element dQ of $Q_{(i)}$, and the last one is a surface integral on the surface element dT of T.

We exchange in (1.04) u_i and v_i and put q instead of p and substitute them, then it turns out from (1.04), here we call "**Green reciprocal formula**" :

$$\begin{aligned} \int_{Q_{(i)}} \Big[\mu\Delta u_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_i}\Big(\frac{\partial u_k}{\partial x_k}\Big)\Big]v_i - q\frac{\partial v_k}{\partial x_k} - \Big[\mu\Delta v_i - \frac{\partial q}{\partial x_i} + \mu\frac{\partial}{\partial x_i}\Big(\frac{\partial v_k}{\partial x_k}\Big)\Big]u_i - p\frac{\partial v_k}{\partial x_k}\Big\}dQ \\ = \int_T \Big\{T_{ik}(u)n_kv_i - T'_{ik}(v)n_ku_i\Big\}dT, \end{aligned} \tag{5}$$

where,

$$\begin{cases} T_{ik(u)} = -p\delta_{ik} + \mu\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big), \\ T'_{ik}(v) = \ \ q\delta_{ik} + \mu\Big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\Big) \end{cases} \tag{6}$$

Now, put down $u_i = v_i$ in (1.04) and select for the functions : u_i, p, such that they satisfy the equation (1.02), then it turns out the "**Green energy formula**":

$$\int_{Q_{(i)}} \frac{\partial}{\partial x_i}\Big[T_{ik}(u)u_k\Big]dQ = -\int_T T_{ik}(u)n_ku_idT = \int_{Q_{(i)}} \Big\{\frac{\mu}{2}\Big(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\Big)^2 - \rho X_ku_k\Big\}dQ, \tag{7}$$

hence the physical fact to the expression turns out that : the operating force of the surface force equals to the viscosity force reduced by the operation of the outer force. Because the author acknowledges the heuristic meaning of this, never occurring,[60] Green formulae, as the true reason, so that it is lately success to construct the same formulation of the potential assuming by the double boundary layer, which we can deduce from its integral equations with "**regular**" kernel, and which was impossible with the same method as known already so far[61].

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[60]cf. F.K.G.Odqvist, *The boundary value problem of the hydrodynamic viscous fluid*, Stockholm, 1928, P.A.Norstedt o.Söner. p. 49.

[61]Compare with C.W.Oseen, *Hydrodynamick*, Leipzig, 1927.

6.2.2. **Hydrodynamic potential.** To get a particular solution of (1.02), as you know, we can use the fundamental tensor. Under this, we see the results :

$$\begin{cases} v_{ik}(x,y) = \frac{1}{8\pi\mu}\left\{\frac{\delta_{ik}}{r_{xy}} + \frac{(y_i-x_i)(y_k-x_k)}{r_{xy}^3}\right\}, \\ q_k(x,y) = \frac{x_k-y_k}{4\pi r_{xy}^3}, \end{cases} \tag{8}$$

and the unknown solutions are [62]

$$\begin{cases} U_i(x) = \rho\int_Q v_{ik}(x,y)X_K(y)dQ, \\ P(x) = \rho\int_Q q_k(x,y)X_K(y)dQ. \end{cases} \tag{9}$$

The fundamental solutions correspond to, as you know, the turning out of the function : $\frac{1}{r_{xy}}$ from the potential theory and we can say the total concept of the functions U_i, P as the hydrodynamic space potential. It holds the important relations for $(x) \neq (y)$:

$$\mu\Delta_x v_{ik} = \frac{\partial q_k}{\partial x_i}, \quad \frac{\partial v_{ik}}{\partial x_i} = 0, \quad \Delta_x q_k = 0, \tag{10}$$

$$\mu\Delta_y v_{ik} = -\frac{\partial q_k}{\partial y_i}, \quad \frac{\partial v_{ik}}{\partial y_i} = 0, \quad \Delta_y q_k = 0, \tag{11}$$

$$v_{ik}(x,y) = v_{ki}(y,x). \tag{12}$$

[62]Here $X_k(y)$ is to satisfy the given conditions, cf. the following theorem 1 on page 337.

7. Ladyzhenskaya : *Investigation of the Navier-Stokes equations for the stationary motion of the incompressible fluid*

7.1. Introduction.

Ladyzhenskaya [63] [11] is one of the most important papers including Leray [12, 13, 14] and Hopf [4], from the viewpoint of mathematical history, of the early studies of the solution on the Navier-Stokes equations. However, this paper [11] was written in Russian, we introduce her paper by our translation into English. In addition, the first English version : Amer. Math. Soc., Transl(2) **24**(1963) by John Abramowich was published without corrections and comments. After conveying deep gratitude to him, we corrected the misprints by translator, amended phrases and words. We show the paper of hers below. [64]

Investigation of the Navier-Stokes equations for the stationary motion of the incompressible fluid
Ladyzhenskaya, Ol'ga Aleksandrovna (Commented by the author of this paper.)

The motion of the viscous incompressible fluid for the model of Navier-Stokes is described by the four functions :

$$\mathbf{u}(x) = (u_1(x), u_2(x), u_3(x)), \quad p(x),$$

satisfying the equations :

$$\Delta\mathbf{u} - \text{grad } p = \sum_{k=1}^{3} u_k \frac{\partial \mathbf{u}}{\partial x_k} + \mathbf{f}, \tag{1}$$

$$\text{div } \mathbf{u} = 0. \tag{2}$$

where $f(x) = (f_1(x), f_2(x), f_3(x))$: the vector of the mass force, $\mathbf{u}(x)$: the vector of the velocity of the flow of the fluid at the point of $\mathbf{x} = (x_1, x_2, x_3)$, and $p(x)$: the pressure at the point. For the brief, of the description of the coefficients of the viscosity and the density of the location, we put by regarding as 1. We shall study the motion in this domain Ω of the three-dimensional, Euclidian space E_3, having its fixed boundary S (S may consist of an arbitrary isolated, closed surface). The case of the moving boundary, we assume the similar investigation. To the boundary S, we assume the essential, incidental condition :

$$\mathbf{u}|_S = 0. \tag{3}$$

The problem in the definition of $\mathbf{u}$ and p in the domain Ω in the equations (1),(2), and the equation (3), and the condition in infinity : $u \to u_\infty$ with respect to $|x| \to \infty$, if Ω : the unbounded domain. To this problem, in many papers devoted their times. Out of them, we select the papers, such as Lichtenstein [1], Odqvist [2] and Leray [3], in some studies the solubility of the problem (1)-(3) for the domain Ω of the derivative form. In the paper [2], Odqvist studied the method of the potential of the linearized problem (1)-(3) : problem, equations (1), omitted nonlinear terms. (We put that exists the various methods of the linealized equations of (1) ; all of them come to the equations, varying each other in the terms of the lowest order for the comparison with the term $\Delta\mathbf{u}$: describing below the method of the solution of the problem (1)-(3), thus, we consider this terms without showing the efforts.)

Except for this, in its study and its own problem (1)-(3) and show its solubility $\ll$ in the large $\gg$. In the paper [3] by Leray give a priori estimate for the solution of the problem (1)-(3). The result of the paper [3] in the combination with the following results of Leray and Schauder[65], to the fixed point completely the continuous transformation of the Banach space, may claim the solubility of the problem (1)-(3) $\ll$ in the large $\gg$ in the case of the sufficiently smooth S and f of the problem (1)-(3), in fact, the solution of J.Leray. On this problem, sophisticatedly show giving the examples, the force of the achievement by Leray and Schauder of the method of the study of the nonlinear problem, which may investigate the existence of the solution of the problem and in these cases, when we stay in the condition

[63]($\Downarrow$) Ladyzhenskaya, Ol'ga Aleksandrovna (1922-2004.)

[64]($\Downarrow$) Except for four remarks by Ladyzhenskaya, which we mark with $(O.L)$, and the other footnotes marked with ($\Downarrow$) are by the author of this paper. The numbers of equations correspond to that in the original paper.

[65]($\Downarrow$) This method is so-called the Leray-Schauder's fixed point theorem.

of the non·uniqueness of the solution of the proof of the solubility $\ll$ in the general $\gg$ problem (1)-(3) in the bounded and continuous domain Ω (interior and exterior problem), fundamentally using the stated above (all is sophisticatedly made) the paper by Odqvist and discover the paper [3] by Leray. For this, we propose all the studies in the two differential functional, generalized quadratic integrable space of the order 1, and in the space $C^1(\Omega)$: continuously differentiable functions. The study in the space $W_2^1(\Omega)$ have such merits, that they may prove the solubility of the problem (1)-(3), because of the only general properties of the operators, compared with the current problem, investigate such a special analytic formation, corresponding to them, with respect to this of the assumption of f and the boundary, even the minimum. In them, we have defined the existence of the so-called generalized solutions of the problem (1)-(3). (of course, with respecting to the performance of the defined conditions of the smoothness for f and S.)

For convenience's sake of the reader of the paper, we chose separately the case of the homogeneous boundary conditions (on S and in infinity). Their study seems to be possible proposal of the very large and little large sufficiently. In the current problem, we limit the investigation of the stable flow in infinity : u_∞ = const, in addition, in the paper, the methods allow to learn also very general cases. In the paper, we show that the stationary problem of the hydrodynamic (interior and exterior) have, at least, the unique solution for all the limit value of the Reynolds number. All these generalized solutions are functions, twice continuously differentiable in the interior of a certain domain, having up to S, continuously differentiable of the order 1, only if the domains of the solid S smooth (have the twice derivatives, satisfying the Hölder's condition) and the mass forces f satisfy the Hölder's condition.

We investigated also establishing the motion in the tube of an arbitrary profile end, which are cylindrical tubes, extending to infinity. We see, with respect to an arbitrary Reynolds number, exists, at least, the unique laminar motion, which, in infinity, converge to the stationary situation, corresponding with the unbounded cylindrical tubes, having as well as the profiles that also end of our tubes.

1. The generalized solutions.

1.1. The homogeneous boundary conditions.

¶ 1. The basic spaces and the formulation of the problem

We put $L_2(\Omega)$: Hilbert space of the vector $\mathbf{u}(x)$, defined in Ω, with the quadratically integrable (in Ω) components. The scalar product in it, is defined by the equation

$$(\mathbf{u},\mathbf{v}) = \int_\Omega u_i v_i dx \equiv \int_\Omega \mathbf{u}\mathbf{v} dx.$$

Here and below, for the pair of indexes, we use the implicit summation symbol in the range from 1 to 3. On the boundary S of the domain Ω, we impose that it has not measure of volume. Take an example of the set $\mathcal{M}$ of all the functions in Ω, continuously differentiable, solenoidal vector $\mathbf{u}(x)$ and introduce in it, the scalar product

$$(\mathbf{u},\mathbf{v})_H = \int_\Omega \mathbf{u}_{x_k}\mathbf{v}_{x_k} dx \equiv \int_\Omega u_{i_{x_k}} v_{i_{x_k}} dx. \tag{4}$$

The solenoidal of $\mathbf{u}$ means that div $\mathbf{u} = 0$, and the finiteness in Ω : as well as $\mathbf{u}$ nicely to zero only in this bounded, strictly interior sub domain of the domain Ω. We show that the equation (4) is, in fact, may turn out in the capacity of the scalar product in $\mathcal{M}$. for this, we must investigate only this, that from the equation : $(\mathbf{u},\mathbf{v})_H = 0$, we deduce the equation : $\mathbf{u} = 0$. For the boundedness of the domain, this deduce from the inequality

$$\int_\Omega v^2(x)dx \le C_\Omega \int_\Omega \sum_{k=1}^{3} (v^2)_{x_i} dx,$$

and that also from the larger, strong inequality (cf. [4]).

$$\Big(\int_\Omega v^4(x)dx\Big)^{\frac{1}{4}} \le C_\Omega\Big(\int_\Omega \sum_{i=1}^{3}(v^2)_{x_i}\Big)^{\frac{1}{2}} dx, \tag{5}$$

strictly, for an arbitrary function v equals zero on S. For the unbounded, however, in the domain Ω these inequalities are not available in the ($C_\Omega = \infty$). However, for any functions in Ω the functions $v(x)$ and any point y, strictly, inequality :

$$\int_\Omega \frac{v^2(x)}{|x-y|^2}dx \le 4\int_\Omega \sum_{k=1}^3 (v^2)_{x_k} dx. \tag{6}$$

It turns out from the equation : [66]

$$2\int_\Omega \sum_{k=1}^3 v(x)\frac{\partial v}{\partial x_k}\frac{y_k - x_k}{|x-y|^2}dx = \int_\Omega \sum_{k=1}^3 \frac{\partial v^2}{\partial x_k}\frac{y_k - x_k}{|x-y|^2}dx = -\int_\Omega v^2 \sum_{k=1}^3 \frac{\partial}{\partial x_k}\frac{y_k - x_k}{|x-y|^2}dx = 3\int_\Omega \frac{v^2(x)}{|x-y|^2}dx.$$

and follows from its inequality : [67]

$$\int_\Omega \frac{v^2(x)}{|x-y|^2}dx \le 2\sqrt{\int_\Omega \frac{v^2(x)}{|x-y|^2}dx}\sqrt{\int_\Omega \sum_k (v^2)_{x_k}}.$$

Inequality (6) may conclude that from $(\mathbf{u}, \mathbf{u}) = 0$ deduce $\mathbf{u} = 0$ and in the case of the unbounded domain. The complement of the space $\mathcal{M}$, for the norm $\|\cdot\|_H$, corresponding the scalar product (4), turns out in the ample Hilbert space, which we show through $H(\Omega)$.

We explain, by this property, it has all the elements of such a constructed space H. The element $\mathbf{v}$ from H have the quadratically integrable on Ω, the generalized derivatives of the order 1 and

$$\text{div } \mathbf{u} = \sum_{k=1}^3 \frac{\partial v_i}{\partial x_i} = 0.$$

These components v_i belong to the inequality (6) for all y and the inequality (5) for all the bounded domain Ω. On the boundary S the vector $\mathbf{v}$ moves to zero (because this embedding theorem [4] instruct us.) In the case of the unbounded domain Ω, the element $\mathbf{v}$, in the defined mean, disperse to zero with respect to $|x| \equiv \sqrt{\sum_{k=1}^3 x_i^2} \to \infty$. This mean is instructed by the inequality (6). The summation of all these functions $\mathbf{v}$ also belong to H. Let's define now, that we can see the unknown solution of the problem. By the generalized solution of the problem (1)-(3) belonging to the class H, we call the vector function $\mathbf{u}(x)$, belong to H and satisfying the integral equation :

$$\int_\Omega \mathbf{u}_{x_k}\mathbf{\Phi}_{x_k}dx - \int_\Omega u_k \mathbf{u}\mathbf{\Phi}_{x_k}dx = -\int_\Omega \mathbf{f}\mathbf{\Phi}dx \tag{7}$$

for all $\mathbf{\Phi} \in \mathcal{M}$.

On the validity of the reason of this problem of extensively conceptual solutions, we deduce the following idea. First, if it turns out that the generalized solution $\mathbf{u}$ has the locally, quadratically integrable generalized derivatives of the order 2, then from (7) by the partial integral, it turns out the equation : [68]

$$\int_\Omega \Big(\Delta\mathbf{u} - u_k\frac{\partial u}{\partial x_k} - \mathbf{f}\Big)\mathbf{\Phi}dx \equiv 0,$$

from which we deduce (cf. [5]), that the expression exists in the parentheses, is the gradient vector, i.e., that $\mathbf{u}$ satisfy the equation : [69]

$$\Delta\mathbf{u} - u_k\frac{\partial u}{\partial x_k} - \mathbf{f} = +\text{grad } p$$

with the completely, certain function p) [70] (we see that p, such as this seems to be directly defined up to an arbitrary constant element), second, in the case of the linearized problem (when we cut off the nonlinear terms and when we have in the condition of the theorem of the uniqueness of the classical

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[66]($\Downarrow$) We correct the last hand side of the next equation : $-\int_\Omega \frac{v^2(x)}{|x-y|^2}dx \to 3\int_\Omega \frac{v^2(x)}{|x-y|^2}dx$, because of $\sum_{k=1}^3 \frac{\partial y_k - x_k}{\partial x_k} = -3$.

[67]($\Downarrow$) From (6), after raising to the power of $\frac{1}{2}$ of the both hand sides of (6) and multiplying the both hand sides of (6) by the just-gained left hand side.

[68]($\Downarrow$) By using the partial integral, from the left hand side of (7), we deduce as follows :

$$\int_\Omega \mathbf{u}_{x_k}\mathbf{\Phi}_{x_k}dx - \int_\Omega u_k\mathbf{u}\mathbf{\Phi}_{x_k}dx = -\int_\Omega \frac{\partial \mathbf{u}_{x_k}}{\partial x_k}\mathbf{\Phi}dx + \int_\Omega u_k\frac{\partial \mathbf{u}}{\partial x_k}\mathbf{\Phi} = -\int_\Omega \mathbf{f}\mathbf{\Phi}dx$$

[69]($\Downarrow$) From (1).

[70]($O.L$) We can prove that $p(x)$ is the unique function of x

solution) it turns out the uniqueness theorem of the generalized solution. Certainly, for the solubility of the both of the generalized solutions may have the identity $(u_{x_i}, \mathbf{\Phi}_{x_i}) = 0$ with respect to all $\mathbf{\Phi} \in \mathcal{M}$, where, just now proposed above of the investigation for the scalar product (4) deduce that $\mathbf{u} \equiv 0$.

We can prove that it turns out the uniqueness theorem of the generalized solution and the nonlinear problem (1)-(3), if the small domain Ω or if the small f. We put that in the case of the bounded domain Ω for the generalized solution $\mathbf{u}$, the identity (7) have naturally for an arbitrary element $\mathbf{\Phi}$ from H, only if $f \in L_2(\Omega)$ (or, even if $\mathbf{f}$ define the linear functional in H). This deduce from (5) and its, that $\mathcal{M}$ is dense in H.

We turn now into the proof of the existence theorem, moreover at first investigate the linear case.

¶ 2. Linealized problem

It is defined as the function from H, satisfying the identities :

$$\Delta \mathbf{u} - \text{grad } p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0, \quad \mathbf{u}|_S = 0. \tag{8}$$

This defines such functions from H, satisfying the identity

$$(\mathbf{u}, \mathbf{\Phi})_H = -(\mathbf{f}, \mathbf{\Phi}) \tag{9}$$

for any $\mathbf{\Phi}$ from $\mathcal{M}$. It turns out

Theorem 1. *The problem (8) has* $\mathbf{u}$ *and moreover unique generalized solution from* H *for any* $\mathbf{f}$, *being the linear functional in* H. □

The value $(\mathbf{f}, \mathbf{\Phi})$ in (9) teach such as limit the linear function $\mathbf{f}$ to the element $\mathbf{\Phi}$. The proof of the **Theorem 1** directly deduce from Riesz's theorem[71] on this, that an arbitrary linear functional, including $(\mathbf{f}, \mathbf{\Phi})$, can become in the way of unique, proposed in the form of the scalar product of the continuous element $\mathbf{u}$ also become the unknown solution. (We see that for this identity (9) have naturally for all $\mathbf{\Phi}$ from H and not only from $\mathcal{M}$).

We put some sufficiently conditions, as $\mathbf{f}$ becomes the linear functional in H.

Result 1. The problem (8) has the unique, generalized solution from H,

1) if Ω : the bounded domain, $\mathbf{f}$ of the summed in Ω to the power of $\frac{6}{5}$, and the functional $(\mathbf{f}, \mathbf{\Phi})$ in (9) calculate as $\int_\Omega \mathbf{f\Phi} dx$;
2) if $f_i(x) = \frac{\partial f_{ik}}{\partial x_k} \quad (i, k = 1, 2, 3), \quad f_{ik} \in L_2(\Omega)$, and the functional $(\mathbf{f}, \mathbf{\Phi})$ is estimated as $\int_\Omega f_{ik} \Phi_{i_{x_k}} dx$;
3) if for f, the smooth integral

$$\int_\Omega |x - y|^2 \sum_{k=1}^{3} f_i^2(x) dx$$

for such a point y and $(\mathbf{f}, \mathbf{\Phi}) = \int_\Omega \mathbf{f\Phi} dx$.

In the lines 2) and 3), Ω can become the unbounded domain. □

The validity for all these satisfied easily prove owing to the Cauchy's inequality by reason of the inequality (5) and (6). We see that the problem (8) in the bounded domain from a viewpoint of the theory of the extended selfadjoint operator investigated in the paper by S.G.Krein [6]. It turns out here the method of the proof (he knows the own origin from Friedrichs) considerably simple. (cf. moreover [7]).

¶ 3. Nonlinear problem (bounded domain)

We put Ω : the bounded domain, and $\mathbf{f}$ is the linear functional in H. The integral : $\int_\Omega \mathbf{f\Phi} dx$, fixing in the right hand side of (7), will exist as the problem of the linear functional $\mathbf{f}$ by Φ and will deal this as $(\mathbf{f}, \mathbf{\Phi})$. Owing to Riesz's theorem $(\mathbf{f}, \mathbf{\Phi}) = (\mathbf{F}, \mathbf{\Phi})_H$, where $\mathbf{F} \in H$, and so on, integral $\int_\Omega u_k \mathbf{u} \mathbf{\Phi}_{x_k} dx$ define the linear functional in H on Φ with respect to an arbitrary element $\mathbf{u}$ from H. This deduce from

[71](⇓) This is called the Riesz's representation theorem.

(5), just

$$\begin{aligned}
&\Big|\int_\Omega u_k \mathbf{u}\mathbf{\Phi}_{x_k}dx\Big| \\
\le\ & \sqrt{3}\Big(\int_\Omega \sum_{k=1}^3 u_k^4 dx\Big)^{\frac14}\Big(\int_\Omega \sum_{i=1}^3 u_i^4 dx\Big)^{\frac14}\Big(\int_\Omega \sum_{i,k=1}^3 \mathbf{\Phi}_{i_{x_k}}^2 dx\Big)^{\frac12} \\
=\ & \sqrt{3}\Big(\int_\Omega \sum_{k=1}^3 u_k^4 dx\Big)^{\frac12}\|\mathbf{\Phi}\|_H \\
\le\ & \sqrt{3}C_\Omega^2 \|u\|_H^2 \|\mathbf{\Phi}\|_H.
\end{aligned}$$

Owing to the Riesz's theorem exists such element $A\mathbf{u}$ in H, such that

$$\int_\Omega u_k \mathbf{u}\mathbf{\Phi}_{x_k}dx = (A\mathbf{u},\mathbf{\Phi})_H. \tag{10}$$

As a result of the identity (7), we can rewrite in the form of

$$(\mathbf{u} - A\mathbf{u} + \mathbf{F}, \mathbf{\Phi})_H = 0. \tag{11}$$

Therefore the problem define the generalized solutions simplified for the solution of the nonlinear equation :

$$\mathbf{u} - A\mathbf{u} + \mathbf{F} = 0 \tag{12}$$

in the space H. We show that the operator A completely continuous in H, i.e., that it is continuous and it transforms compactly an arbitrary bounded set in H. Because of H : Hilbert space, then sufficiently, it shows that A arbitrarily, weakly converges in H, the sequence $\{\mathbf{v}^m\}$ converges strongly. Therefore, $\mathbf{v}^m$ may converge weakly in H to $\mathbf{v}$. Owing to the embedding theorem (cf. [4]) $\mathbf{v}^m$ may converge strongly on $\mathbf{v}$ in $L_4(\Omega)$.

$$(A\mathbf{v}^m - A\mathbf{v}^n, \mathbf{\Phi})_H = \int_\Omega (v_k^m \mathbf{v}^m - v_k^n \mathbf{v}^n)\mathbf{\Phi}_{x_k}dx = \int_\Omega (v_k^m \mathbf{v}^m - v_k^n \mathbf{v}^n)\mathbf{\Phi}_{x_k}dx + \int_\Omega v_k^n(\mathbf{v}^m - \mathbf{v}^n)\mathbf{\Phi}_{x_k}dx.$$

Applying for the estimate of the right hand side, the Hölder's inequality and the inequality in (5), as well as in the above, we see

$$(A\mathbf{v}^m - A\mathbf{v}^n, \mathbf{\Phi})_H \le C\|\mathbf{v}^n - \mathbf{v}^m\|_{L_4(\Omega)}(\|\mathbf{v}^n\|_H + \|\mathbf{v}^m\|_H)\|\mathbf{\Phi}\|_H,$$

where, we assume $\mathbf{\Phi} = A\mathbf{v}^m - A\mathbf{v}^n$,

$$\|A\mathbf{v}^m - A\mathbf{v}^n\| \le C_1\|\mathbf{v}^n - \mathbf{v}^m\|_{L_4(\Omega)} \to 0, \quad n, m \to \infty.$$

Hence, we prove that A : completely continuous. Hence for the investigation of the solubility of the equation (12), we can apply the Leray-Schauder's method (the Russian translated version of their paper given in [8]). Let's contain in (12), such a substantial parameter λ :

$$\mathbf{u} - \lambda A\mathbf{u} + \mathbf{F} = 0. \qquad (12_\lambda)$$

for $\lambda = 0$, the transformation : $\mathbf{v} = \mathbf{u} + \mathbf{F}$ is each other unique transformation of H in H. Hence the power in an arbitrary point $\mathbf{v} \in H$ of this transformation, investigating at all H, equals to 1. In particular, it equals to 1 also in $\mathbf{v} = 0$. This means that the highest index of the solution of the equation : $\mathbf{u} + \mathbf{F} = 0$ equals to 1. For this, to be able to claim the constancy of this index for the solution of the equation (12_λ) for all $\lambda \in [0, 1]$, we must prove sufficiently that all the possible solutions of the equation (12_λ) does not go beyond the limit of a certain sphere of the space H. If it turns out this last one, then, by reason of the **Theorem 1**[72] in our footnote[73] on this paper [8] (pp. 84-85), equation (12_λ) will at least the unique solution with respect to all $\lambda \in [0, 1]$. Like this, for the proof of the solubility of the equation (12_λ) sufficiently investigate the a priori estimate in H for all the spaces of this solutions with respect to $\lambda \in [0, 1]$. For this, we recall that (12_λ) turns out, putted in the form (7), is

$$(\mathbf{u}, \mathbf{\Phi})_H - \lambda\int_\Omega u_k \mathbf{u}\mathbf{\Phi}_{x_k}dx = -(\mathbf{F}, \mathbf{\Phi})_H.$$

[72]($\Downarrow$) We show this **Theorem 1** in our appendix by the author of this paper.

[73]($\Downarrow$) Uspekhi Mat.Nauk **1**(1946), no 3/4(13/14), 71-95. (Russian)

here, in the capacity of the $\mathbf{\Phi}$, we can select an arbitrary element $\in \mathcal{M}$, and by just it, and from H. We put $\mathbf{\Phi} = \mathbf{u}$. This integral, standing with respect to λ, disperse because

$$\int_\Omega u_k \mathbf{u}\mathbf{u}_{x_k} dx = \frac{1}{2}\int_\Omega \mathbf{u}_k \frac{\partial \mathbf{u}^2}{\partial x_k} dx = -\frac{1}{2}\int_\Omega (\text{div } \mathbf{u})\mathbf{u}^2 dx = 0.$$

and therefore

$$\|\mathbf{u}\|_H^2 = -(\mathbf{F}, \mathbf{u})_H \leq \|\mathbf{F}\| \; \|\mathbf{u}\|_H = |\mathbf{f}| \; \|\mathbf{u}\|_H,$$

where $|\mathbf{f}|$ is the norm of the linear functional $\mathbf{f}$. Hence, the desiring a priori estimate :

$$\|\mathbf{u}\|_H \leq |\mathbf{f}|, \tag{13}$$

and from these and the existence theorem,

Theorem 2. *The problem (1)-(3) in the bounded domain Ω has, at least, the unique generalized solution from H for any linear functional* $\mathbf{f}$ *in H, in particular, for all* $\mathbf{f}$, *integrable to the power of $\frac{6}{5}$ in Ω.* □

¶ 4. Nonlinear problem (unbounded domain)

We put Ω : the unbounded domain[74] and $\mathbf{f}$ the linear functional in $H(\Omega)$. We put the symbol Ω_n ($n = 1, 2, \cdots$) the sequence extending of the domain covering in the range of all the Ω. We see easily that if each from the vector $\mathbf{v}$, belonging to $H(\Omega_n)$, extends to all Ω, containing this, equal to zero in the exterior of $H(\Omega_n)$, then it become belong to $H(\Omega)$. Therefore $\mathbf{f}$ can be estimated as the linear functional in all $\in H(\Omega)$, moreover, for $\mathbf{\Phi} \in H(\Omega_n)$,

$$|(\mathbf{f}, \mathbf{\Phi})| \leq |\mathbf{f}| \; \|\mathbf{\Phi}\|_{H(\Omega)} = |\mathbf{f}| \; \|\mathbf{\Phi}\|_{H(\Omega_n)},$$

where $|\mathbf{f}|$: the norm of the linear functional $\mathbf{v}$ in H. For each from the domain Ω_n of the problem (1)-(3) have, at least, the unique solution $\mathbf{u}^n$ and for all these solutions, the valid estimate (13), just the estimate

$$\|\mathbf{u}^n\|_H \leq |\mathbf{f}|. \qquad (13')$$

Therefore, the total of the solutions $\{\mathbf{u}^n\}$ weak compact in $\in H(\Omega)$. We show that all from its weak limit of $\mathbf{u}$ become the generalized solution $\in H(\Omega)$ of the problem (1)-(3). For this, sufficiently prove that $\mathbf{u}$ become the following identity (7), just this identity :

$$(\mathbf{u}, \mathbf{\Phi})_H - \int_\Omega u_k \mathbf{u} \mathbf{\Phi}_{x_k} dx = -(\mathbf{f}, \mathbf{\Phi}) \qquad (7')$$

${}^\forall\mathbf{\Phi} \in \mathcal{M}$ (however ${}^\forall\mathbf{\Phi} \notin H(\Omega)$!). We take an example such $\mathbf{\Phi} \in \mathcal{M}$. It is bounded. Therefore, for it and all $\mathbf{u}^n$ with sufficiently distant at the value n become exactly identity (7′). Moving for (7′) to the limit for the partial sequences n_k, for all n_{n_k} converge weakly to $\mathbf{u}$, assure, using (5), that $\mathbf{u}$ satisfy (7′) with $\mathbf{\Phi}$, took an example $\in \mathcal{M}$. *QED.*

Theorem 3. *The problem (1)-(3) with zero satisfy in infinity, have, at least, the unique generalized solution $\in H(\Omega)$ for the unbounded domain Ω, if all* $\mathbf{f}$ *define the linear functional in $H(\Omega)$ (the enough conditions of this given in (2)-(3) and Result 1).* □

1.2. **heterogeneous boundary conditions.** In this paragraph, we want to investigate generally the problem which we sketch the system n of the flow of $\mathbf{u}$, which, in infinity, is equal to the problematic, accustomed vector $\mathbf{u}_\infty$. For this, in the capacity of the auxiliary problem, we study at first, the problems of the Navier-Stokes system, in the bounded domain with the heterogeneous boundary conditions.

¶ 1. Flow in the bounded domain.

We shall find the generalized solutions of the system (1)-(2) in the bounded domain Ω with the boundary S (this, as the everywhere be able to consist of the separated surfaces : $S_1 + S_2 + \cdots + S_n$) satisfying the boundary equation :

$$\mathbf{u}|_S = \mathbf{a}|_S. \tag{14}$$

[74](*O.L*) We mean that the domain can be extend up to the infinity.

We assume that the vector $\mathbf{a}$ is by the boundary value of the vector $\mathbf{a}(x) = \text{ rot } \mathbf{b}(x)$, where $\mathbf{b}(x) \in W_2^2(\Omega)$, such that

$$\max_{x\in\Omega} |\mathbf{b}(x)| \leq \text{ const }, \quad \int_\Omega \sum_{i,k=1}^{3} b_{i_{x_k}}^4 \, dx \leq \text{ const }.$$

With no difficulty, we summarize for as the condition of the boundary with respect to also $\mathbf{a}|_S$ that the continuation of $\mathbf{a}|_S$ with S zero in Ω possibly (of course, that $\int(\mathbf{a},\mathbf{n})dS$ essentially equal zero !)

The generalized solution of the problem (1),(2),(14) named vector function $\mathbf{u}$ satisfying the integral identity :

$$\int_\Omega \mathbf{u}_{x_k} \mathbf{\Phi}_{x_k} dx - \int_\Omega u_k \mathbf{u} \mathbf{\Phi}_{x_k} dx = -(\mathbf{f}, \mathbf{\Phi}) \tag{15}$$

with respect to all $\mathbf{\Phi} \in \mathcal{M}$ and that $\mathbf{v} = \mathbf{u} - \mathbf{a} \in H(\Omega)$.

The assumptions on $\mathbf{f}$ are the same as the smooth, in the first order. For these conditions, it turns out

Theorem 4. *The problem of (1),(2) and (14) have at least, the unique generalized solution for all* $\mathbf{f}$, *being by the liniar functional in* $H(\Omega)$. □

This theorem prove the same as by the method of the **Theorem 2,3** in §1. The various papers of the solutions of the problem comes just to the solution $\mathbf{v}$ of the equation :

$$\mathbf{v} - A_1\mathbf{v} + \mathbf{F} = 0, \tag{16}$$

where, $\mathbf{F}$: the given element of H, and A_1 : completely continuous operator in $H(\Omega)$. The solution $\mathbf{u}$ of the problem is connected with the solution of the equation (16) by the equation : $\mathbf{u} = \mathbf{v} + \mathbf{a}$. For the proof of the solubility of the equation (16) is proved with no difficulty, that the norm in H all the possible solution of the equation :

$$\mathbf{v} - \lambda A_1\mathbf{v} + \mathbf{F} = 0, \tag{17}$$

with respect to $\lambda \in [0,1]$, bounded in total such a constant. We show this. Let's $\mathbf{v}$ be such a solution of the equation (17). Then $\mathbf{v} + \mathbf{a}$ satisfy the identity (15), if in it with respect to the nonlinear term define the factor λ, i.e.,

$$\int_\Omega (\mathbf{v}+\mathbf{a})_{x_k} \mathbf{\Phi}_{x_k} dx - \lambda \int_\Omega (v_k + a_k)(\mathbf{v}+\mathbf{a}) \mathbf{\Phi}_{x_k} dx = -(\mathbf{f}, \mathbf{\Phi}). \tag{18}$$

We substitute in this identity for $\mathbf{\Phi} = \mathbf{u}$ and use such that

$$\int_\Omega (v_k + a_k)\mathbf{v}\mathbf{v}_{x_k} dx = \frac{1}{2}\int_\Omega (v_k + a_k)\frac{\partial \mathbf{v}^2}{\partial x_k} dx = 0,$$

$$\int_\Omega \mathbf{a}_{x_k}\mathbf{v}_{x_k} dx \leq \|\mathbf{a}\|_H \|\mathbf{v}\|_H,$$

$$|(\mathbf{f},\mathbf{v})| \leq |\mathbf{f}| \; \|\mathbf{v}\|_H,$$

where $|\mathbf{f}|$ is the norm of the linear functional, the given $\mathbf{f}$ in H

$$\Big| \int_\Omega a_k \mathbf{a} \mathbf{v}_{x_k} dx \Big| \leq C\sqrt{\int_\Omega \sum_i a_i^4 dx} \|\mathbf{v}\|_H \leq C_\Omega \|\mathbf{a}\|_H^2 \|\mathbf{v}\|_H, \tag{19}$$

where the constant C_Ω depends on the volume of Ω. Therefore from (18) we deduce the inequality

$$\|\mathbf{v}\|_H^2 \leq \lambda \Big| \int_\Omega v_k \mathbf{a} \mathbf{v}_{x_k} dx \Big| + \|\mathbf{a}\|_H \|\mathbf{v}\|_H + \lambda C_\Omega \|\mathbf{a}\|_H^2 \|\mathbf{v}\|_H + |\mathbf{f}| \; \|\mathbf{v}\|_H. \tag{20}$$

We assume that $\|\mathbf{v}\|_H$ with respect to all $\lambda \in [0,1]$ unbounded in total. Then it exists such a sequence as $\lambda = \lambda_1, \lambda_2, \cdots \to \lambda_0$ and the corresponding its solutions $\mathbf{v}^n = \mathbf{v}(x, \lambda_n)$ of the equality (17), for all the value

$$N_n = \|\mathbf{v}^n\|_H$$

converges to the limit with respect to $n \to \infty$. For all $\mathbf{v}^n$, the valid inequality (20) with the same constant C_Ω. Divide the both hand sides of the inequality (20) by N_n^2 and put this as the inequality for the function $\mathbf{w}_n = \frac{\mathbf{v}^n}{N_n}$

$$1 \le \lambda_n \Big| \int_\Omega w_k^n \mathbf{a} \mathbf{w}_{x_k}^n dx \Big| + \frac{1}{N_n}\|\mathbf{a}\|_H + \frac{\lambda_n C_\Omega}{N_n}\|\mathbf{a}\|_H^2 + \frac{1}{N_n}|\mathbf{f}|. \tag{21}$$

The set of the function $\{\mathbf{w}^n\}$ continuously, bounded in $H : \|\mathbf{w}^n\|_H = 1$, and therefore, it strong, compact in $L_4(\Omega)$. Without the boundedness of the coincidence can assume that all the sequence $\mathbf{w}^n$ converge on an arbitrary function $\mathbf{w}$, strongly in $L_4(\Omega)$ and weakly in H. The limit function $\mathbf{w} \in H$. With no difficulty, we verify that the integral $\int_\Omega w_k^n \mathbf{a} \mathbf{w}_{x_k}^n dx$ converges on $\int_\Omega w_k \mathbf{a} \mathbf{w}_{x_k} dx$. (21) converges $n \to \infty$. As a result, it turns out

$$1 \le \lambda_0 \Big| \int_\Omega w_k \mathbf{a} \mathbf{w}_{x_k} dx \Big|. \tag{22}$$

The functions : $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ found for the fixed vector $\mathbf{a}(x) \in \Omega$. However, $\mathbf{u}(x)$ depends on only the role of $\mathbf{a}(x)$ on S (cf. the definition of the generalized solution of the problem (1),(2) and (14).) If we, instead of the original : $\mathbf{a}(x) =$ rot $\mathbf{b}(x)$, get arbitrarily from the vector $\mathbf{a}(x,\delta) =$ rot $[\mathbf{b}(x)\xi(x,\delta)]$, where $\xi(x,\delta)$ is such a $\ll$ cutting-off $\gg$ function, that is twice continuously differentiable, and equals to 1 in the neighborhood of the boundary S and zero at the point of Ω, separating from S at the distance of the larger $\delta(\delta > 0)$, then the solution $\mathbf{u}(x)$ of the problem (1),(2) and (14), found according with such $\mathbf{a}(x)$, becomes the solution of this problem and for all such $\mathbf{a}(x,\delta)$. We construct the sequence of the $\ll$ cutting-off $\gg$ functions : $\xi(x,0)$ with $\delta \to 0$ such that

$$|\xi(x,\delta)| \le c, \quad \Big|\frac{\partial \xi(x,\delta)}{\partial x_k}\Big| \le \frac{c}{\delta},$$

with the unique and the same constant c for all $\delta \in (0,\delta_1)$. The vector $\mathbf{v}^n \equiv \mathbf{v}(x,\lambda_n) = \mathbf{u}(x,\lambda_n) - \mathbf{a}(x,\delta)$ depends on δ, however, the limit value for $\mathbf{w}^n = \frac{\mathbf{v}^n}{N_n}$ of the vector $\mathbf{w}$ is independent of δ same as $N_n \to \infty$ with respect to $n \to \infty$, hence $\frac{\mathbf{a}(x,\delta)}{N_n} \to 0$ with respect to $n \to \infty$. Hence the inequality (22), validly for $\mathbf{w}$ with all the vector $\mathbf{a}$ in the form of rot $[\mathbf{b}(x)\xi(x,\delta)]$. We see easily that

$$|\mathbf{a}(x,\delta)| \le c_1 \Big(\frac{1}{\delta} + \sum_k |\mathbf{b}_{x_k}(x)|\Big), \tag{23}$$

Hence from (22), it turns out

$$\begin{aligned}
1 &\le \lambda_0 \Big| \int_{\Omega_\delta} w_k \mathbf{w}_{x_k} \mathbf{a}(x,\delta) dx \Big| \\
&\le \lambda_0 c_1 \int_{\Omega_\delta} \big| w_k \mathbf{w}_{x_k} \big| \Big(\frac{1}{\delta} + \sum_i |\mathbf{b}_{x_i}(x)|\Big) dx \\
&\le \frac{\lambda_0 c_2}{\delta} \Big(\int_{\Omega_\delta} \sum_k w_k^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\Omega_\delta} \sum_k \mathbf{w}_{x_k}^2 dx\Big)^{\frac{1}{2}} + \lambda_0 c_2 \Big(\int_{\Omega_\delta} \sum_k w_k^4 dx\Big)^{\frac{1}{4}} \Big(\int_{\Omega_\delta} \sum_k \mathbf{w}_{x_k}^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\Omega_\delta} \sum_{i,k} b_{i_{x_k}}^4 dx\Big)^{\frac{1}{4}}.
\end{aligned} \tag{24}$$

Here Ω_δ : the boundary zone of the width δ, and c_2 : the absolute constant defined an arbitrary domain Ω. For $\mathbf{w} \in H(\Omega)$, then for this equals to the inequality :

$$\Big(\int_{\Omega_\delta} \mathbf{w}^2(x) dx\Big)^{\frac{1}{2}} \le c_3 \delta \Big(\int_{\Omega_\delta} \sum_k \mathbf{w}_{x_k}^2 dx\Big)^{\frac{1}{2}}. \tag{25}$$

This inequality deduce easily with using the Cauchy's inequality from the expression

$$\mathbf{w}(x) = \mathbf{w}(y)|_{y \in S} + \int_y^x \frac{\partial \mathbf{w}}{\partial l} dl,$$

if we consider that $\mathbf{w}|_S = 0$. Owing to (25) and (5) from (24) we deduce

$$1 \le \lambda_0 c_4 \int_{\Omega_\delta} \sum_k \mathbf{w}_{x_k}^2 dx.$$

However this inequality, such as $\int_{\Omega_\delta} \sum_{i,k} \mathbf{w}_{i_{x_k}}^2 dx$ converges on zero with respect to $\delta \to 0$. The gained contradiction proves the boundedness of $\|\mathbf{v}(x,\lambda)\|_H$ for $\lambda \in [0,1]$. On this account, **Theorem 4** is proved. □

At the symposium of the differential equations held in November 1957, in Kharkov, a theorem was repoted by I.I.Vorovič, (proved by him with V.I.Judovič), which was on the existence of the generalized solution of the problem (1),(2),(14) with respect to the condition $(\mathbf{a},\mathbf{n})|_S = 0$. Under the performance of this condition, it may be possible to give the direct a priori estimate of $\|\mathbf{v}^n\|$ through the given problem[75]. According to what I.I.Vorovič teached me, their study of the differential properties of this problem hasn't progressed very far.

¶ 2. Flow in the unbounded domain.

We put the system n of the fixed, bounded such that we sketch the flow flax $\mathbf{u}(x)$ with the given value of this by the limit $\mathbf{u}_\infty = \text{const}$ in infty. We put by $\mathbf{a}(x)$ such that is solenoidal,[76] locally quadratically integrable vector with the quadratically integrable in Ω, generalized, arbitrary derivatives, equal to zero on S and $\mathbf{u}_\infty$ with respect to the larger $|x|$ ($|x| \geq R_0$). The generalized solution of the problem sketched, we call the function $\mathbf{u}$, satisfying the integral identity[77]

$$\int_\Omega \mathbf{u}_{x_k}\mathbf{\Phi}_{x_k}dx - \int_\Omega u_k\mathbf{u}\mathbf{\Phi}_{x_k}dx = -(\mathbf{f},\mathbf{\Phi}) \tag{26}$$

with respect to all $\mathbf{\Phi} \in \mathcal{M}$ and that $\mathbf{v} = \mathbf{u} - \mathbf{a}(x) \in H(\Omega)$. We remember that the condition of the tool : $\mathbf{u} - \mathbf{a}(x)$ in H produce such that

$$\int_\Omega \frac{(\mathbf{u}(x) - \mathbf{a}(x))^2}{|x-y|}dx \leq \text{const}, \quad \int_\Omega \sum_{k=1}^3 (\mathbf{u}_{x_k} - \mathbf{a}_{x_k})^2 dx \leq \text{const}. \tag{27}$$

The inequality (27) also assure that $\mathbf{u}(x)$ in the defined sence, converges to $\mathbf{u}_\infty$ with respect to $|x| \to \infty$.

On $\mathbf{f}$, we make these assumptions as well as in ¶ 4 in § 1. The boundedness on S comes merely to the possibility of the construction of the ≪ cut-off ≫ functions $\xi(x,\delta)$, i.e., of the functions, equal 1 in the neighborhood of S, zero in the point of Ω, separating at S with the distance of the larger than δ, and obeying the inequality $|\xi| \leq c_1$, $|\frac{\partial \xi}{\partial x_i}| \leq \frac{c_1}{\delta}$.[78] To these functions, except for the obvious coincidence, we can get the twice continuously differentiable (because we can always manage to get by the complementary, averaged value $|\xi|$). We put the vector $\mathbf{b} = (\alpha_2 x_3, \alpha_3 x_1, \alpha_1 x_2)$, where $\alpha = \mathbf{u}_\infty$. Undoubtedly, that the vector $\mathbf{e}(x,\delta) = \text{ rot } (\mathbf{b}(x)\xi(x,\delta))$ coincides with $\mathbf{u}_\infty$ in the neighborhood of S and equal zero in the exterior of the adjoining zone Ω_δ. In the capacity of the functions $\mathbf{a}(x)$ in the definition of the generalized solution, can get any from the functions $\mathbf{a}(x,\delta) = \mathbf{u}_\infty - \mathbf{e}(x,\delta)$. We use this below.

We have

> **Theorem 5.** *The problem of the sketching the system n of the solid, the flow equal in infinity : $\mathbf{u}_\infty =$ const , have always, at least, the unique, generalized solution with respect to all* $\mathbf{f}$, *satisfying the linear functional in H, in particular, with respect to* $\mathbf{f} \equiv 0$. □

The construction of the generalized solution may have the propagation such as in ¶ 4 in §1. We construct just the sequence of the domain Ω_n, converge on Ω. For each from Ω_n select the solutions $\mathbf{u}^n$ of the system (1),(2) satisfying the boundary conditions :

$$\mathbf{u}^n\big|_S = 0, \quad \mathbf{u}^n\big|_{\Gamma_n} = \mathbf{a}(x)\big|_{\Gamma_n}.$$

($S + \Gamma_n$: the domain Ω_n), and we show that the norms in $H(\Omega_n)$ of all $\mathbf{v} = \mathbf{u} - \mathbf{a}$ uniformly (for n) bounded

$$\|\mathbf{v}^n\|_{H(\Omega_n)} \leq C. \tag{28}$$

The estimate (28) can be selected from $\mathbf{v}^n$ of the sequence, converge on an arbitrary function $\mathbf{u} = \mathbf{v} + \mathbf{a}$. This equation is performed as well as in ¶ 4 § 1, and therefore we would not repeat it here.

Hence, we remain to prove the validity of (28). This makes, in general, as well as in the above part for the proof of the uniform, for λ, boundedness of $\|\mathbf{v}(x,\lambda)\|_H$. We assume just the induction, that $N_n = \|\mathbf{v}^n\|_{H(\Omega_n)} \to \infty$ with respect to $n \to \infty$. We put it and bellow in the capacity of $\mathbf{a}$ of the function $\mathbf{a}(x,\delta) = \mathbf{u}_\infty - \mathbf{e}(x,\delta)$. We substitute in (18) $\mathbf{v} = \mathbf{v}^n$, $\mathbf{\Phi} = \mathbf{v}^n$, $\lambda = 1$, and we estimate in the right hand

[75] $(O.L)$ This fact owed already to the paper [3] by Leray.

[76] (⇓) The divergence of $\mathbf{u}$ is zero.

[77] (⇓) This equation is the same as (15).

[78] $(O.L)$ We assume, for example, a cylinder as S.

side of the gotten equation as well as in the above, considering only that because $\mathbf{a}(x,\delta)$ equals to the constant vector $\mathbf{u}_\infty$ in the exterior of the adjoining, partial zone Ω_δ, then, instead of (19), we can get

$$\Big|\int_{\Omega_n} a_k \mathbf{a}\mathbf{v}^n_x dx\Big| = \Big|\int_{\Omega_n} a_k \mathbf{a}_{x_k}\mathbf{v}^n dx\Big| = \Big|\int_{\Omega_\delta} a_k \mathbf{a}_{x_k}\mathbf{v}^n dx\Big| \le c_1\|\mathbf{a}\|^2_{H(\Omega_\delta)}\|\mathbf{v}^n\|_{H(\Omega_n)}.$$

with the same as the constant c_1 for all n and $\delta\in(0,1)$. Under this, instead of (20) we get

$$\|\mathbf{v}^n\|^2_{H(\Omega_n)} \le \Big|\int_{\Omega_n} v^n_k \mathbf{a}\mathbf{v}^n_{x_k} dx\Big| + \|\mathbf{a}\|_{H(\Omega_\delta)}\|\mathbf{v}^n\|_{H(\Omega_n)} + c_1\|\mathbf{a}\|^2_{H(\Omega_\delta)}\|\mathbf{v}^n\|_{H(\Omega_n)} + |\mathbf{f}|\ \|\mathbf{v}^n\|_H. \tag{29}$$

We repeat each other belong to $\mathbf{v}^n(x)$ by zero in the exterior of Ω_n in all Ω and introduce the functions :

$$\mathbf{w}^n(x) = \frac{\mathbf{v}^n(x)}{N_n},\qquad \text{where } N_n = \|\mathbf{v}^n\|_{H(\Omega_n)}.$$

The functions $\mathbf{w}^n(x)$ can be assumed as the elements of $H(\Omega)$, bounded in total in $H(\Omega)$. For them, the exact or the same as (29)

$$1 \le \Big|\int_{\Omega_\delta} w^n_k \mathbf{e}\mathbf{w}^n_{x_k} dx\Big| + \frac{1}{N_n}\|\mathbf{a}\|_{H(\Omega_\delta)} + \frac{c_1}{N_n}\|\mathbf{a}\|^2_{H(\Omega_\delta)} + \frac{1}{N_n}|\mathbf{f}|,$$

if we consider that

$$\int_{\Omega_n} v^n_k \mathbf{a}\mathbf{v}^n_{x_k}\, dx = -\int_{\Omega_\delta} v^n_k \mathbf{e}(x,\delta)\mathbf{v}^n_{x_k}\, dx.$$

Judging moreover, sequentially as well as in the above, we reach the contradiction with our assumption, that $N_n\to\infty$ with respect to $n\to\infty$. Hence (28) is proved, and **Theorem 5** is just also proved. □

2. The classical solution.

2.1. **Preliminary comments.** In this chapter, we intend the boundary S of the smooth (having an arbitrary twice derivatives, satisfying the Hölder's condition), and by the exponential. With respect to the execution of this condition each from the generalized solution $\mathbf{u}(x)$, an arbitrary existence proved in the chapter I, give the classical solution, more precisely speaking, twice continuously differentiable in the interior of Ω functions, and once continuously differentiable in $\bar{\Omega}$ functions. satisfying the equation (1),(2) and uniformly bounded. In §2, we give the estimates of the brief proof of this some proof fact. In just the following paragraph independent of the chapter I, the existence of the classical solution of the hydrodynamic problems.

To put it briefly, we limited in this chapter to study the problem only with respect to homogeneous boundary condition and the function $f(x)$. Considering the inhomogeneity of the boundary condition is executed as the same as in §2 in this chapter I and it turns out fundamentally to just its result that and in the case of the homogeneous condition. The fundamental results of this chapter belong, in fact, to Leray [3].

Oseen [9] had constructed the fundamental singular solution for the linealized system (8). His expression is as follows : [79]

$$T_{ij}(x,y) = \frac{1}{8\pi}\Big[\frac{\delta^j_i}{|x-y|} + \frac{(y_i-x_i)(y_j-x_j)}{|x-y|^3}\Big],\quad P_i(x,y) = \frac{1}{4\pi}\frac{y_i-x_i}{|y-x|^3},$$

and have the following properties[80] :

$$\begin{cases} \sum_{k=1}^3 \frac{\partial^2}{\partial y_k^2}T_{ij} - \frac{\partial P_i}{\partial y_j} = -\delta^j_i\delta(x-y) \quad (i,j=1,2,3),\\ \frac{\partial T_{ij}}{\partial y_j} = 0,\quad x\ne y, \end{cases} \tag{30}$$

where $\delta(x-y)$ is the three-dimensional δ-functions, and δ^j_i is the Kronecker symbol. Using this solution, Odqvist proposed in the paper [2], the potential of volume and the potentials of the double- and triple-layered and proved that they have the properties, analogous to the properties of the ordinary electro·static potentials, constructed by using $\frac{1}{4\pi|x-y|}$. This could use it for the solution of the boundary problem for the linear system (8) and, in particular, use the Green function $G_{ij}(x,y),\ g_i(x,y)$ (more precisely speaking,

[79]($\Downarrow$) cf. Our appendix by the author of this paper and §2. Hydrodynamische Potential. (2.01) p.334, [2].

[80]($\Downarrow$) cf. §1. Difinitionen. Bezeichungen. Greensche Formeln. *Stokesschen Gleichungen* (1.02) p.332, [2].

the Green tensor). The functions $G_{ij}(x,y),\ g_i(x,y)$ satisfy the system (30) and moreover the boundary condition

$$G_{ij}(x,y)=0 \text{ with respect to } x\in\Omega \text{ and } y\in S.$$

Solutions of the problem (8) owing to it we put the style

$$u_i(x)=-\int_\Omega G_{ij}(x,y)f_j(y)dy \quad (i=1,2,3),\quad p(x)=\int g_i(x,y)f_i(y)dy. \tag{31}$$

In the same paper, Odqvist studied the differentiable properties of the potentials and the solutions of the problem (8) in the boundary of the domain Ω. In particular, he proved the following estimate of the Green function in the closed, bounded domain $\bar\Omega$:[81]

$$\begin{cases}\left|G_{ij}(x,y)\right|\le\frac{c}{|x-y|}, & \left|\frac{\partial G_{ij}(x,y)}{\partial x_k}\right|\le\frac{c}{|x-y|^2},\quad x,y\in\Omega,\\ \left|\frac{\partial G_{ij}(x,y)}{\partial x_k}-\frac{\partial G_{ij}(\tilde x,y)}{\partial\tilde x_k}\right|\le\frac{c|x-\tilde x|\ |\ln|x-\tilde x||^2}{R^3}, & \end{cases} \tag{32}$$

moreover the last estimate is for any $x,\ \tilde x\in\Omega$, separating from y with the distance of the non-smaller than R.

We put that the method of the proof of the estimate (32), except for an arbitrary change, is adaptive also with the case of the Green function for Laplace operator. We shall not prove the referred-here proofs on the linealized problem (1) and (2). The principle direction of the problem is completely clear and the eventual proposition of it was made very sophisticatedly in the paper [2].

2.2. **The proof of the classical, generalized solutions.** In this paragraph, we shall sketch the fundamental method of the proof on it, that each generalized solution, gotten in the chapter I, is by the classical if for f and S under the condition, designated in the above paragraph. With respect to this, we limit the interior problem and the homogeneous boundary conditions. For another problem, this is adaptive analogously.

Thus, we may have the generalized solution $\mathbf{u}(x)$ of the problem (1),(2) and (3), i.e., $\mathbf{u}(x)$ belongs to H, and $^\forall\Phi\in\mathcal{M}$ (and even for it $^\forall\Phi\in H$) satisfy the equation (7) or that

$$\int_\Omega \mathbf{u}_{x_k}\Phi_{x_k}dx=-\int_\Omega u_k\mathbf{u}_{x_k}\Phi dx-\int_\Omega \mathbf{f}\Phi dx. \tag{33}$$

We put in (33) as Φ the $\ll$ cutting-off $\gg$ Green function : $G^m_{ij}(y,x)$, considering y fixed in the interior point of Ω. The $\ll$ cut-off $\gg$ of $G_{ij}(x,y)$ is able to propose variously, for example, such as

$$G^m_{ij}(y,x)=T^m_{ij}(y,x)+g_{ij}(y,x),$$

where $g_{ij}(y,x)$ is a smooth part of Green function and[82]

$$T^m_{ij}(y,x)=\begin{cases}T_{ij}(y,x) & for \quad -m\le T_{ij}(y,x)\le m,\\ m & for \quad T_{ij}(y,x)\ge m,\\ -m & for \quad T_{ij}(y,x)\le -m,\end{cases}$$

With respect to the fixed y and sufficiently large m, of the vectror $(G^m_{i1},\ G^m_{i2},\ G^m_{i3})$ $(i=1,2,3)$ belong to H. Substitute in (33) instead of $\Phi_j(x)$ the function G^m_{ij} and afterward move to the limit with $m\to\infty$. Because of the strict estimate (32) for G_{ij} (and i.e. for g_{ij}), and the inequality (13) for u, then with no difficulty, we verify that in the both terms of the right hand side of (33), we may move to the limit with m under the symbol of $\int$, move the limit to y, it turns out for example, in the norm of the subspace $L_{\frac{3}{2}}(\Omega')$, here Ω' is an arbitrary, interior sub domain of the domain Ω. We also transform the left hand side of (33) using the partial integral to the form of

$$-\int_{\Omega_{ij}(y)}u_j(x)\Delta T^m_{ij}(y,x)dx-\int_{S_{ij}(y)}u_j(x)\frac{\partial T^m_{ij}(y,x)}{\partial n}dS-\int_\Omega u_j(x)\Delta g_{ij}(y,x)dx,$$

[81](⇓) cf. §5. Der Greensche Tensor und Seine Eigenschaften. pp.357-366, [2]

[82](⇓) The original top statement in the following conditions, $T_{ij}(y,x)\quad for\ -m\le|T_{ij}(y,x)|\le m$, but it seems to be incorrect.

where $\Omega_{ij}(y)$: the domain, in any $T_{ij}^m = T_{ij}$, $S_{ij}(y)$: its boundary, and put using (30), move the limit with $m \to \infty$ in the norm of $L_{\frac{3}{2}}(\Omega')$. As a result of this, we put as the conclusion, that for almost all $y \in \Omega$, for $\mathbf{u}(x)$, the valid equation follows :

$$u_i(y) = -\int_\Omega G_{ij}(y,x)u_k u_{j_{y_k}} dx - \int_\Omega G_{ij}(y,x)f_j(x)dx \quad (i = 1,2,3). \tag{34}$$

From this relations on $u_i(y)$, the estimates (32) on G_{ij} and inequality (13), we may conclude that $u_i(y)$ are twice continuously differentiable in the interior of Ω, continuously differentiable up to S and satisfy all the conditions of (1),(2) and (3). Or such a thing that the functions $\int_\Omega G_{ij}(y,x)f_j(x)dx$ have the just now mentioned properties of the differentiabilities are besed on the reason of the estimate (32) and the Hölder's function f_j, as well as this operates in the Newton's potential theorem. By them, for the proof of that just the functions :

$$u_i(y) = \int_\Omega G_{ij}(y,x)u_k u_{j_{x_k}} dx,$$

have the properties, it is sufficient to show that $\mathbf{u}(y)$ and $\mathbf{u}_{y_k}$ are continuous in $\bar{\Omega}$ and satisfy the Hölder's conditions in the interior of Ω. From the inequalities (32),(13) and (6) deduce the boundedness of $|v_i(y)|$ in Ω. In this case also, $u_k \mathbf{u}_{x_k}$ is integrable in Ω, by the order 2. From this and the estimate (32), with no difficulty, we convict that $v_i(y)$ satisfy the Hölder's conditions on $\bar{\Omega}$. The differentiability for $v_i(y)$ at y_k also again using (32) and only that the defined properties for $\mathbf{u}$, consistently, and convict that u_{y_k} is integrable on Ω at the order 6, (cf. [4]) and afterward and in it convict that they satisfy the Hölder's condition.

This is the generalized method of the proof of the claim, stated at the beginning of the paragraph. We ought to propose the local investigation of the differentiable properties of the generalized solutions. For the unbounded domain, this claim establishes in principle as well as by the reference of the integral expression of the type of (34), instead of Ω, only select such a bounded partial part Ω_n of the domain Ω , and i.e., instead of, G_{ij}, : the Green's tensor for the domain Ω_n.

2.3. **The nonlinear problem. (The bounded domain, homogeneous boundary conditions.)** We assume in the system (1), parameter λ for the nonlinear terms and study it such linear, assuming the right hand side of (1) as the free term. Then, the formula (31) is

$$u_i(x) = -\lambda \int_\Omega G_{ij}(x,y)u_k(y)u_{j_{y_k}} dy - \int_\Omega G_{ij}(x,y)f_j(y)dy. \tag{35}$$

The differential of this equation with respect to x_l become moreover, the following relation :

$$\frac{\partial u_i(x)}{\partial x_l} = -\lambda \int_\Omega \frac{\partial G_{ij}(x,y)}{\partial x_l} u_k(y)u_{j_{y_k}} dy - \int_\Omega \frac{\partial G_{ij}}{\partial x_l}(x,y)f_j(y)dy. \tag{36}$$

To all this system, we put as the form of one equation

$$v = \lambda D v + \varphi \tag{37}$$

and study this in Banach space $C(\Omega)$, each belonging to arbitrary one are continuous in $\bar{\Omega}$ functions. The norm in $C(\Omega)$ is defined as

$$\|v\|_C = \max_{x \in \Omega,\ i=1,2,\cdots,12} |v_i(x)|$$

Owing to the estimate (32) of the ordinary method prove that if f_i are the continuous in $\bar{\Omega}$, then the components $\int_\Omega \frac{\partial G_{ij}}{\partial x_l} f_j dy$ are continuous and even satisfy the Hölder's condition with an arbitrary constant α(for example, with $\alpha < \frac{1}{4}$). We show this, for example, for $\int_\Omega \frac{\partial G_{ij}(x,y)}{\partial x_k} f_j dy$.[83]

$$I \equiv \Big| \int_\Omega \Big(\frac{\partial G_{ij}(x,y)}{\partial x_k} - \frac{\partial G_{ij}(\tilde{x},y)}{\partial \tilde{x}_k} \Big) f_j(y)dy \Big| \le \Big| \int_{\Omega \cap K_\rho} \Big| + \Big| \int_{\Omega - K_\rho \cap \Omega} \Big|.$$

where K_ρ is the sphere of $|\frac{x+\tilde{x}}{2} - y| \le \rho$.

$$\Big| \int_{\Omega \cap K_\rho} \Big| \le c \ \max|f_i| \int_{K_\rho} \Big(\frac{1}{|x-y|^2} + \frac{1}{|\tilde{x}-y|^2} \Big) dy \le c_1 \rho,$$

[83]($\Downarrow$) Correcting the first term of the right hand side of the following equation in the original of [11] ; $\Big| \int_{\cap K_\rho} \Big|$, we put it as $\Big| \int_{\Omega \cap K_\rho} \Big|$.

$$\Big|\int_{\Omega-\Omega\cap K_\rho}\Big| \le c_2\frac{|x-\tilde{x}|\,|\ln|x-\tilde{x}||^2}{\rho^3}.$$

We put $\rho = |x-\tilde{x}|^{\frac{1}{4}}$, then

$$I \le c_3|x-\tilde{x}|^{\frac{1}{4}}|\ln|x-\tilde{x}||^2.$$

Hence, we see that the operator D is completely continuous operator in $C(\Omega)$.

We use now the Schauder-Leray's parameter extension method[84] for the proof of the solubility of the equation (37) in $C(\Omega)$. With respect to $\lambda = 0$, the transformation : $Bv = v - \varphi$, investigated in all H, has the order 1 at all the points. Thus for the solubility of (37) with respect to $\lambda = 1$, sufficiently, we investigate the a priori estimate for the solution of the equation (37), i.e., prove that for all the perturbed solutions of the equation (37) with respect to all $\lambda \in [0,1]$ strict inequality $\|v\|_C \le c_4$ with not only the unique but also the constant c_4.

Let v be an arbitrary solution of the equation (37). Maybe proved that this last one are 9 components of derivatives up to x_l at the first 3 and i.e., the equation (37) can put in the form (35),(36) with $v_i = u_i$, $(i = 1,2,3)$. Functions $v_i(x)$ and f_i of the continuous, i.e., with just now proved φ_i and $v_i(i = 1,2,\cdots,12)$ it will be satisfied the Hölder's condition up to x. Therefore, the multiplication $u_k(y)u_{i_{y_k}}$ of the potential of volume in (35) and (36) satisfy the Hölder's condition. Also to this we estimate the possibility and for f_i. Owing to this, we can claim that $v_i = u_i$ $(i = 1,2,3)$ have continuously twice derivatives with respect to x_k in the interior of Γ. (This is able to be proved by the familiar theorem on the Newton's potential method.) As a result of (31), they will satisfy the system (1) with the parameter λ for the nonlinear terms. Increasing its scalar by $\mathbf{u}$ and integrate afterward with respect to Ω, we assume that for $\mathbf{u}$ the identical equation :

$$(\mathbf{u},\mathbf{u})_H = -(\mathbf{f},\mathbf{u}). \tag{38}$$

Hence, as is stated above, follows the estimate

$$\sum_{i,k}\int_\Omega u^2_{i_{x_k}}\,dx \le c_5 \tag{39}$$

with respect to all λ. From the formula (35) for u_i and the inequality (6),(32) and (39), we deduce directly :

$$\max_{x\in\Omega}|u_i(x)| \le c\int_\Omega\sum_{k,j}\frac{1}{|x-y|}|u_k u_{j_{y_k}}|dy + c_6 \le c\sqrt{\sum_{k,j=1}^{3}\int_\Omega\frac{u_k^2}{|x-y|^2}dy}\sqrt{\sum_{k,j=1}^{3}\int_\Omega u^2_{j_{x_k}}dy} + c_6 \le c_6'. \tag{40}$$

Owing to the estimate of $\max|\frac{\partial u_i}{\partial x_l}|$ we get the equation (36). From it, and also from (32) and (40) follows

$$\Big|\frac{\partial u_i(x)}{\partial x_l}\Big| \le c_7\int_\Omega\frac{1}{|x-y|^2}\sum_{j,k}|u_{j_{y_k}}|dy + c_7'. \tag{41}$$

Multiplying both hand sides of this inequality with $\frac{1}{|x-z|^2}$ integrate totally with respect to x in the domain Ω and summing up totally with respect to i and l, then as a result, we get

$$\sum_{i,l}\int_\Omega\frac{1}{|x-z|^2}\Big|\frac{\partial u_i}{\partial x_l}\Big|dx \le 9c_7\int_\Omega\sum_{j,k}\Big|\frac{\partial u_j}{\partial y_k}\Big|dy\int_\Omega\frac{1}{|x-z|^2}\frac{1}{|x-y|^2}dx + c_8 \le c_9\int_\Omega\frac{1}{|y-z|}\sum_{j,k}|u_{j_{y_k}}|dy + c_8. \tag{42}$$

With respect to this, we used the familiar formula construction of the integral by the functional form $\frac{1}{r^\alpha}$. We estimate now the right hand side of (42) with the Cauchy's inequality and use the inequality (39). This assure us such that the right hand side of (42), and i.e., of (41) is not superior than an arbitrary constant c_{10} such that

$$\max_{x,i,l}\Big|\frac{\partial u_i(x)}{\partial x_l}\Big| \le c_{10}. \tag{43}$$

All the constants starting with c_5, are independent of λ and determine only by the size of the domain Ω, max $|f_i|$ and the constant with from the inequality (32). Inequality (40) and (43) prove that all the positive solutions v of the equation (37) are not over the sphere of the space $C(\Omega)$ with a radius of $\rho = \max(c_6, c_{10})$, and therefore, on the equation (37), we are applicable of the Schauder-Leray's theorem,

[84]($\Downarrow$) This method is so-called the Leray-Schauder's fixed point theorem. cf. Our appendix by the author of this paper.

which assures at least, the unique solution for all the values of the parameter λ, in the number and with respect to $\lambda = 1$. *QED.*

Theorem 6. *The problem(1)-(3) have, at least, the unique solution $u_i(x)$, continuous together with derivatives in the first order in Ω and having continuously differentiable in the second order in the interior of Ω. The pressure $p(x)$ has a continuous in Ω and continuously differentiable in the interior of Ω. With respect to $f_i(x)$, we seem such that they satisfy the Hölder's condition with an arbitrary positive constant.* $\square$

We can prove that in the conditions of **Theorem 6**, the second derivatives u_i satisfy the Hölder's condition. We can also see the next improvement of the properties of the differentiabilities on the solution with the improvement of the differentiabilities properties of f_i and S.

2.4. **The nonlinear problem. (The unbounded domain with the homogeneous boundary condition.)** Now we take Ω - unbounded domain. In brief, $f_i(x)$ equals to zero for the large $|x|$. We suppose to take the paragraph 4 in §1, of the chapter I. We take the continuity extending into Ω of the domain $\Omega_n (n = 1, 2, \dots)$ and corresponding them to the classical solution $u^n(x)$, satisfying the null boundary conditions. We prove that from them we can choose the subspace, uniformly approaching together with the own derivatives to the solution of the system (1),(2) in any bounded sub domain Ω' of the domain Ω. We fix Ω'. We mean $G'_{ij}(x, y)$ as the Green's function, corresponding to the domain Ω'. We take S'-mean the boundary of Ω', belonging S, and Γ'- remained part. For all $\mathbf{u}^n$ with the sufficiently large number n, strict equation

$$u_i^n(x) = -\int_\Omega G'_{ij}(x,y)u_k^n(y)u^n_{j_{y_k}}dy - \int_{\Gamma'}\frac{\partial G'_{ij}(x,y)}{\partial n_y}u_j^n(y)dS_y - \int_{\Omega'} G'_{ij}(x,y)f_j(y)dy. \tag{44}$$

This is proved, by the accustomed method. The unique, excellent equation (44) deduced from (35), shows the existence of the integral on Γ', which does not disppear in this case, because on Γ', the functions u_j^n are not necessary to change to zero. For $\mathbf{u}_j^n(x)$ with respect to all n the exact estimate (39) with not only the unique but also the constant c_5. Certainly, the integral $(f, \mathbf{u}^n)$, considering the estimation of the finiteness of f by us, are extended in fact, into a certain bounded domain, which we name the domain Ω_1 to make it clear. Hence, using (5) and the Cauchy inequality it turns out

$$|(\mathbf{f}, \mathbf{u}^n)| \le \sqrt{\int_{\Omega_1}\mathbf{f}^2dx}\sqrt{\int_{\Omega_1}(\mathbf{u}^n)^2dx} \le c_{11}\sqrt{\int_{\Omega_n}\sum_{i,k}((u_i^n)_{x_k})^2dx}$$

where, the constant c_{11} is general for all $\mathbf{u}^n$. Substituted this inequality into (38), we see on the validity (39), where the constant : $c_5 = c_{11}^2$. In addition, as the result of (6), it turns out a general and the estimate :

$$\int_{\Omega_n}\frac{\sum_j u_j^{n^2}(y)}{|x-y|^2}dy \le 4c_5. \tag{45}$$

Fixing sub domain Ω'' of the domain Ω', which is separated from the boundary Γ', with a certain positive distance δ. We prove that for $x \in \Omega''$, functions $|u_i^n(x)|$ are uniformly bounded. Certainly, the uniform boundedness of the module of the first term in the right hand side of (44) is deduced from (39) and (45), (ref (40)). The boundedness of the third term is clear. The boundness of the second term is deduced from it, that by $x \in \Omega''$ and $y \in \Gamma$, it turns out the inequality $\left|\frac{\partial G'_{ij}(x,y)}{\partial n_y}\right| \le \frac{c}{\delta^2}$, and the integral $\int_{\Gamma'}|u_j^n|dS_y$ is estimated by $\int_{\Omega'}(u_j^{n^2} + \sum_k u_{j_{y_k}}^{n^2})dy$. (The latter is the result of the Sobolev's embedding theorem; this can also show and directly, using the formula of the function with the integration from its derivatives to the direction of the integral method.)

Thus, the proposal of (44) together with the estimate (39) and (45) are able to claim uniform boundedness of $|u_i^n(x)|$ for $x \in \Omega'' \subset \Omega'$. Let it be : $\Omega'''' \subset \Omega''' \subset \Omega''$ in addition, their distance between the boundaries : $\Gamma'''', \Gamma''', \Gamma''$, are not smaller than δ each other. We put for $|u_i^n(x)|$ of the formula such as (44), for the domain Ω'' with G''_{ij} defferentiate totally this with respect to x_l and consider that $|u_i^n(x)|$ is already estimated in the Ω'', for $\frac{\partial G''_{ij}}{\partial x_l}$ equals to the estimate of (32) and $u^n_{i_{y_k}}(y) \in L_2(\Omega'')$. In its function of $\int_{\Omega''}\frac{\partial G''_{ij}}{\partial x_l}u_k^n u^n_{j_{y_k}}dy$ it turns out belonging to $L_6(\Omega'')$ and two different terms become uniform boundedness for $x \in \Omega''$. Then, for $u^n_{i_{x_l}}$, we can claim of uniform boundedness in $L_6(\Omega'')$. After this, we put the formula (44) for u^n_i in the domain Ω'' with the function G''_{ij} and differentiate totally this with respect to x_l. From this, the last relation is already proved that in Ω'''' function $u^n_{i_{x_l}}$ is the uniform boundedness together with our own Hölder's constant.

Thus, in some steps, we are certain of uniform boundedness on u^n_i, $u^n_{i_{x_l}}$ and our Hölder's constant in an arbitrary bounded sub domain Ω'''' of the domain Ω. From this, we may choose from $\mathbf{u}^n$ the subsequence approaching any functions arbitrarily together with any first order in the bounded sub domain Ω'''' of the domain Ω. Get for u^n_i of the formula (44) for Ω'''' and G''''_{ij} and changing in it into boundary, we are certain that for u_i equal to very this formula. From this, we have proved now by the famous method on it that $\mathbf{u}$ satisfies equation (1), (2) and conditions (3). To them it remains to solve moreover , in this point that $\mathbf{u}$ gets null conditions in infinity.

We see that, by analogy with the above, we can prove the uniform boundedness in an arbitrary strict null bounded sub domain of the domain Ω of an arbitrary with $\mathbf{u}^n$ of the second order, and of the Hölder's constant for them. This guarantees the capability of an arbitrary convergence of $\mathbf{u}^{k_n}$ together with an arbitrary, all orders up to 2.

2.5. **The behavior of the founded classical solutions with respect to $|x| \to \infty$.** We show that the classical solution $u(x)$, founded in the above paragraph, become to dissipate to zero with respect to $|x| \to \infty$, if the boundary S is allocated completely in the bounded part of the space x. For this, we introduce the study, except for the fundamental singular solutions T_{ij}, P_i of the linearized Navier-Stokes system :

$$T_{ij}(x,y) = \frac{1}{8\pi}\Big[\frac{\delta^j_i}{|x-y|} + \frac{(y_i-x_i)(y_j-x_j)}{|x-y|^3}\Big], \quad P_i(x,y) = \frac{1}{4\pi}\frac{y_i-x_i}{|y-x|^3}$$

Already the singular solutions : T_{ij} and P_i of this system is defined the following equations :

$$T'_{ij}(x,y,R) = \frac{1}{8\pi}\Big[\frac{\delta^j_i}{R^3} + (3R^2-2r^2) - \frac{(y_i-x_i)(y_j-x_j)}{R^3}\Big], \quad P'_i(x,y,R) = \frac{5}{4\pi}\frac{y_i-x_i}{|y-x|^3}.$$

The singular solutions :

$$T''_{ij}(x,y,R) = T_{ij}(x,y) - T'_{ij}(x,y,R), \quad P''_i(x,y,R) = P_i(x,y) - P'_i(x,y,R)$$

have the following properties : in the domain $\Omega_R(x)$, included in the interior of the sphere $S_R(x)$ in a radius R with the center of the point x, follows the equation :

$$\begin{cases} \sum_k \frac{\partial^2}{\partial y_k^2} T''_{ij}(x,y,R) - \frac{\partial P''_i(x,y,R)}{\partial y_j} = -\delta^j_i \delta(x-y) \quad (i,j=1,2,3), \\ \frac{\partial T''_{ij}}{\partial y_j} = 0, \quad T''_{ij}(x,y,R)|_{y\in S_R} = 0. \end{cases} \tag{46}$$

We put R an arbitrary value such that S exists in the interior of S_R. This is from (46) for the founded earlier classical solution $u(x)$ and $p(x)$ of the exterior problem by the ordinary method, follows the given expression :

$$\begin{aligned} &u_i(x) \qquad\qquad (47)\\ &= -\int_{\Omega_R(x)} T''_{ij} u_k(y) u_{j_{y_k}}\,dy - \int_{\Omega_{R(x)}} T''_{ij} f_j(x)dy \\ &+ \int_S T''_{ij}\Big[\frac{\partial u_j}{\partial y_k} - p\delta^j_k\Big]\cos(ny_k)dS - \int_{S_R} u_j\Big[\frac{\partial T''_{ij}}{\partial y_k} - P''_i\delta^k_j\Big]\cos(ny_k)dS. \end{aligned}$$

We extend R to ∞ in this equation and then it turns out that in infinity exists the following expression for $u_i(x)$:

$$u_i(x) = -\int_\Omega T_{ij}u_k(y)u_{j_{y_k}}\,dy - \int_\Omega T_{ij}f_j(x)dy + \int_S T_{ij}\Big[\frac{\partial u_j}{\partial y_k} - p\delta^j_k\Big]\cos(ny_k)dS. \tag{48}$$

Certainly, we simply see that with respect to $R \to \infty$, it turns out validly as following (cf. (39),(6) and (32)).[85]

$$\int_{\Omega_{R(x)}} T''_{ij} u_k(y) u_{j_{y_k}} dy \to \int_\Omega T_{ij} u_k u_{j_{y_k}} dy, \quad \int_{\Omega_{R(x)}} T''_{ij} f_j(x) dy \to \int_\Omega T_{ij} f_j(x) dy,$$

$$\int_S T''_{ij} \Big[\frac{\partial u_j}{\partial y_k} - p\delta^j_k\Big] \cos(ny_k) dS \to \int_S T_{ij} \Big[\frac{\partial u_j}{\partial y_k} - p\delta^j_k\Big] \cos(ny_k) dS,$$

$$\int_{\Omega_{R(x)}} T'_{ij} f_j(x) dy \to 0.$$

We see that with respect to $R \to \infty$

$$j_R(x) \equiv \int_{\Omega_{R(x)}} T'_{ij} u_k u_{j_{y_k}} dy + \int_{S_R} u_j \Big[\frac{\partial T''_{ij}}{\partial y_k} - P''_i \delta^k_j\Big] \cos(ny_k) dS \to 0.$$

Certainly[86]

$$|j_R(x)| \le c_1 \Big[\frac{1}{R} \int_{\Omega_{R(x)}} \sum_{j,k} |u_k||u_{j_{y_k}}| dy + \frac{1}{R^2} \int_{S_R} \sum_j |u_j| dS\Big].$$

We multiply the both hand sides of this inequality by $\frac{1}{c_1 R}$ and integrating totally in the range of $R_1 > 0$ up to ∞, and put $\mathbf{u} \equiv 0$ in the interior of S and we put this integrated result in the following form :

$$\frac{1}{c_1} \int_{R_1}^\infty \frac{1}{R} |j_R(x)| dR \le \int_{R_1}^\infty \frac{dR}{R^2} \int_0^R \int_{|\omega|=1} \sum_{j,k} |u_k||u_{j_{y_k}}| r^2 dr d\omega + \int_{R_1}^\infty \frac{dR}{R^3} \int_{S_R} \sum_j |u_j| dS.$$

After some computations, exchanging the order to be integrated in the first term of the right hand side, , we see in the following inequality :

$$\begin{aligned} & \frac{1}{c_1} \int_{R_1}^\infty \frac{1}{R} |j_R(x)| dR \\ \le\ & \int_{R_1}^\infty r \sum_{k,j} \int_{|\omega|=1} |u_k u_{j_{y_k}}| dr d\omega + \int_0^{R_1} \frac{r^2}{R_1} \sum_{k,j} \int_{|\omega|=1} |u_k u_{j_{y_k}}| d\omega dr + \int_{R_1}^\infty dR \int_{S_R} \frac{\sum_j |u_j|}{r^3} dS \\ \le\ & \int_\Omega \sum_{k,j} \frac{|u_k u_{j_{y_k}}|}{|x-y|} dy + \int_{\Omega - \Omega_{R_1}(x)} \frac{\sum_j |u_j|}{|x-y|^3} dy \\ \le\ & 2\sqrt{3} \int_\Omega \sum_{k,j} u^2_{j_{y_k}} dy + \sqrt{\int_{\Omega-\Omega_{R_1}(x)} \frac{1}{|x-y|^4} dy} \sqrt{\int_{\Omega-\Omega_{R_1}(x)} \frac{\sqrt{3}\sum_j u_j^2}{|x-y|^2} dy} < const. \end{aligned}$$

Thus, we have proved that $\int_{R_1}^\infty \frac{1}{R} |j_R(x)| dR$ converge, i.e. exists such sequence as $R_k \to \infty$, for all $j_{R_k}(x) \to 0$. For this sequence, R_k turns to a limit value in (47) and get (48). From the expression (48), with no difficulty already proved, that $u_j(x) \to 0$ with respect to $|x| \to \infty$. Certainly the second and third terms in it converge with respect to $|x| \to \infty$, to zero, for $\frac{1}{|x|}$. The very first term is just divided into the two terms.

$$\int_\Omega T_{ij} u_k u_{j_{y_k}} dy = \int_{\Omega_\rho} T_{ij} u_k u_{j_{y_k}} dy + \int_{\Omega - \Omega_\rho} T_{ij} u_k u_{j_{y_k}} dy$$

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[85]($\Downarrow$) In the original, $T_{ij} u_k(y) u_{j_{y_k}} dy \to \int_\Omega T_{ij} u_k u_{j_{y_k}} dy$, $T_{ij} f_j(x) dy \to \int_\Omega T_{ij} f_j(x) dy$, but we correct these terms, because these are no terms in (47).

[86]($\Downarrow$) In the original, there is no range on the integration. We correct the range from $\int \sum_{j,k} |u_k||u_{j_{y_k}}| dy$ to $\int_{\Omega_{R(x)}} \sum_{j,k} |u_k||u_{j_{y_k}}| dy$, by considering of the rested terms from (47).

where Ω_ρ is the common sphere of $|y| < \rho$ with Ω. To the second term of the above equation, from this integration of the fundamental inequalities (39),(32) and (6), we estimate thus :

$$\begin{aligned}
&\left|\int_{\Omega-\Omega_\rho} T_{ij} u_k u_{j_{y_k}} dy\right| \\
\leq\ & c \int_{\Omega-\Omega_\rho} \sum_k \frac{|u_k|}{|x-y|} |u_{j_{y_k}}| dy \\
\leq\ & c_1 \sqrt{\int_\Omega \sum_k \frac{u_k^2}{|x-y|^2} dy} \sqrt{\sum_{j,k} \int_{\Omega-\Omega_\rho} u_{j_{y_k}}^2 dy} \\
\leq\ & c_2 \sqrt{\int_\Omega \sum_{j,k} u_{j_{y_k}}^2 dy} \sqrt{\int_{\Omega-\Omega_\rho} \sum_{j,k} u_{j_{y_k}}^2 dy}.
\end{aligned}$$

Hence, it seems that, choosing ρ, sufficiently large, we can deduce the right hand side an smaller arbitrary ε being independent of this, where x exists. After this, fixed ρ, we put $|x|$ such a number that $|\int_{\Omega_\rho} T_{ij} u_k u_{j_{y_k}} dy|$ stayed smaller than ε.

Thus, we assure that any classical solution $\mathbf{u}(x)$ from the best case by us, uniformly converge to zero with respect to $|x| \to \infty$.

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8. Conclusions

We show merely the early studies of the solutions of the NS equations during 1933-59, in particular, the followings are the first versions in the analytic history of the NS equations :

- Leray in 1933/34 and Hopf in 1950/51 discussed the weak solutions
- Kiselev in 1955/56/57 and Ladyzhenskaya in 1957/59 discussed the generalized / strong solution
- Prodi in 1959 and J.L.Lions in 1959 discussed the uniqueness of the solution in L^p function space in the three dimensions

We show the two translated versions into English : Hopf [4] and Ladyzhenskaya [11], because in these papers, there are historically first full-scale discussions of the solutions of the NS equations such that :

- Hopf [4] asserts the existence of a weak solution like Leray, without uniqueness
- Ladyzhenskaya [11] discusses a generalized / strong solution like Kiselev in the first time.

[87](⇓) Sobolev says in the preface to the third edition : In this edition misprints and errors are corrected, certain definitions and formulations of the theorems are refined, clarifications are added, corrections are made in a number of proofs, bibliographical remarks and comments are given, and editorial changes are introduced.

Regrettably, we omitted the citing of Leray's papers [12, 13, 14] partly because of lack of space, and partly because of availability of an English version by Dr. Bob Terrell (cf. [14].)

We can see that there are another full-scale discussions on the mathematical and fluid dynamics or functional analysis, in Lelay, Hopf, Kiselev, Ladyzhenskaya, Prodi, J.L.Lions, etc., just during 1933-59, and many great studies follow after that such as [6]. We think that the solving the problems of fluid dynamics have made to find a clue to many mathematical studies and its developments.

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Remark: we use *Lu* (: in French) in the bibliography meaning "read" date by the referees of the journals, for example MAS. In citing the original paragraphs in our paper, the underscoring are of ours.

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