RATIONAL SEQUENCES CONVERGING TO LEFT-C.E. REALS OF POSITIVE EFFECTIVE HAUSDORFF DIMENSION

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Abstract. In our previous work, we characterized Solovay reducibility using Lipschitz condition, and introduced quasi Solovay reducibility (qS-reducibility, for short) as a Hölder condition counterpart. In this paper, we investigate effective dimensions and ideals closely related to quasi Solovay reducibility by means of the rate of convergence. We show that the qS-completeness among left-c.e. reals is equivalent to having a positive effective Hausdorff dimension. The Solovay degrees of qS-complete left-c.e. reals form a filter. On the other hand, the Solovay degrees of non-qS-complete left-c.e. reals do not form an ideal. Based on observations on the relationships between rational sequences and reducibility, we introduce a stronger version of qS-reducibility. Given a degree of this reducibility, the lower cone (including the given degree) forms an ideal. By developing these investigations, we characterize the effective dimensions by means of the rate of convergence. We give a variation of the first incompleteness theorem based on Solovay reducibility.

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1. Introduction

1.1. **Background.** Martin-Löf randomness, also called 1-randomness, has been the central notion in the theory of algorithmic randomness. Informally speaking, a real is random if its binary expansion is complicated in a certain algorithmic sense. In particular, left-c.e. random reals have many interesting properties. Here, a real number is called *left-c.e.* if the left set of its Dedekind cut is computably enumerable. For the general background on the study of randomness, see Downey and Hirschfeldt[6] or Nies[13].

The starting point of our discussion is the following theorem.

Theorem 1.1 (Demuth[5], Downey *et al.* [7].)**.** If the sum of two left-c.e. reals α, β is 1-random, then at least one of α or β is 1-random.

See also Corollary 9.5.9 of [6]. The theorem intuitively says that, in the unit interval of the real line, the addition of two non-random reals results in a non-random real.

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Our main goal of this paper is to reinvestigate the theorem above in terms of the rate of the convergence inspired by the recent result by Barmpalias and Lewis (Theorem 1.2) and our previous work on quasi Solovay reducibility.

Solovay reducibility is a preorder that compares two reals in terms of algorithmic complexity. A real α is *Solovay reducible* to a real β, $\alpha \leq_S \beta$ in symbols, if there exists a partial computable function f from $\mathbb{Q}$ to $\mathbb{Q}$ and a positive constant c such that for each rational $q < \beta$, $f(q) < \alpha$ is defined and $\alpha - f(q) < c(\beta - q)$. Informally speaking, if we have a good approximation q of β then we get a good approximation $f(q)$ of α. If we restrict ourselves to the left-c.e. reals, Solovay completeness coincides with 1-randomness. Miyabe *et al.* [12] studied effective dimensions and Solovay degrees. Among others, the sum of non-random reals is studied by means of the concept of ideals of the partially ordered set.

We may view Solovay reducibility from the perspective of analysis. In our previous work (Kumabe *et al.* [8]), we showed that for left-c.e. reals α and β, $\alpha \leq_S \beta$ if and only if there exists a Lipschitz continuous function from $(-\infty, \beta)$ to $(-\infty, \alpha)$ satisfying certain conditions. We also introduced the concept of *quasi Solovay reducibility*, which corresponds to the Hölder continuous functions. A real α is quasi Solovay reducible to a real β, $\alpha \leq_{qS} \beta$ in symbols, if there exists a partial computable function f from $\mathbb{Q}$ to $\mathbb{Q}$ and positive constants d, ℓ such that for each rational $q < \beta$, $f(q) < \alpha$ is defined and $(\alpha - f(q))^\ell < d(\beta - q)$. We showed that for left-c.e. reals α and β, $\alpha \leq_{qS} \beta$ if and only if there exists a Hölder continuous function from $(-\infty, \beta)$ to $(-\infty, \alpha)$ satisfying certain conditions.

In this paper, we investigate effective dimensions and ideals closely related to quasi Solovay reducibility. The most important concept in our method is the rate of convergence. Barmpalias and Lewis-Pye[2] investigate a quantity similar to the left-hand derivative.

Theorem 1.2. (Barmpalias and Lewis-Pye[2]. See also Miller [11].) Suppose $\langle a_n \rangle \nearrow \alpha, \langle b_n \rangle \nearrow \beta$. If β is random, then the following hold.

$$\lim_{n \to \infty} \frac{\alpha - a_n}{\beta - b_n} \text{ exists.}$$

Moreover,

- The limit value is independent of the choice of sequences.
- The limit value $= 0$ if and only if α is not 1-random.

1.2. **Overview.** In Section 2, we characterize qS-completeness among left-c.e. reals by means of Solovay reducibility and dimension. In particular, qS-completeness is equivalent to having a positive dimension. In Section 3, we investigate some ideals of left-c.e. reals. We show that the Solovay degrees of qS-complete reals form a filter. On the other hand, the Solovay degrees of non-qS-complete reals do not form an ideal. We investigate the relationships between rational sequences and reducibility, and by means of those investigations, we introduce a stronger version of qS-reducibility. In the stronger version, the lower cone below ($\leq$) a given degree forms an ideal. In Section 4, by developing Section 3, we characterize the effective dimensions by means of the rate of convergence. Section

5 is an appendix. As a by-product of our observation, we show a variation of the first incompleteness theorem by means of Solovay reducibility.

1.3. **Notation.** In this paper, $\langle a_n \rangle_{n\in\mathbb{N}}$, or $\langle a_n \rangle$ denotes a sequence of numbers. Unless otherwise specified, α and β denote left-c.e. reals.

Convention on sequences: Throughout the paper, unless otherwise specified, $\langle a_n \rangle \nearrow \alpha$ denotes that $\langle a_n \rangle$ is a strictly increasing computable sequence of rationals, and $\langle a_n \rangle$ converges to α.

We are mainly interested in Solovay reducibility and quasi Solovay reducibility and their stronger versions. In their analysis, K-reducibility and strong K-reducibility play an important role to connect our analysis to effective dimensions.

Definition 1.1. A real α is called *K-reducible* to a real β, $\alpha \leq_K \beta$ in symbols, if $K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + O(1)$. Here, K is the prefix-free Kolmogorov complexity , and $\alpha \upharpoonright n$ is the first n bits of the binary expansion of α.

Definition 1.2. A real α is *strongly K-reducible* to a real β, $\alpha \ll_K \beta$ in symbols, if $\lim_{n\to\infty}(K(\beta \upharpoonright n) - K(\alpha \upharpoonright n)) = \infty$.

Clearly, if $\alpha \ll_K \beta$, then $\alpha \leq_K \beta$. The converse does not hold; we will see stronger results later.

2. Characterization of qS-complete reals

In this section, we characterize qS-complete reals by means of Solovay reducibility and dimension. In the first subsection, we review previous work on Solovay reducibility and K-reducibility, and we show a slightly stronger result than the original. Throughout the paper, we sometimes use Lemma 2.1. In the second subsection, we show that qS-complete left-c.e. reals are characterized by means of a positive effective Hausdorff dimension.

2.1. **Solovay reducibility and K-reducibility.** Solovay reducibility implies K-reducibility. It is known that the inverse implication does not hold ([7]). On the other hand, strong K-reducibility implies (a property slightly stronger than) Solovay reducibility.

Proposition 2.1. (Prop. 2.3 of [12]) Let α, β be left-c.e. reals. If $\alpha \ll_K \beta$ then $\alpha <_S \beta$.

The proof of Prop. 2.1 in [12] is useful when we are interested in rational sequences converging to α and β under assumption of $\alpha \ll_K \beta$. For later convenience, we reconstitute the main part of the proof in a generalized form. See the notation section for the convention on sequences $\langle a_s \rangle$.

Lemma 2.1. Let α, β be left-c.e. reals. Suppose that $\langle a_s \rangle \nearrow \alpha$ and $\langle b_s \rangle \nearrow \beta$. Let ℓ, m be positive reals.

(1) If $\lim_n(K(\beta \upharpoonright \ell n) - K(\alpha \upharpoonright n)) = \infty$, then there exists a strictly increasing sequence $\langle s_n \rangle$ of natural numbers with the following properties: For almost all n and for each s such that $s_n \le s \le s_{n+1}$, the inequalities Eq. (1) and Eq. (2) below hold:

$$\alpha - a_s \le 2^{-n} \tag{1}$$

$$\beta - b_s \ge m2^{-\ell(n+1)} \tag{2}$$

(2) If $\limsup_n(K(\beta \upharpoonright \ell n) - K(\alpha \upharpoonright n)) = \infty$ then there exists a strictly increasing sequence $\langle s_n \rangle$ of natural numbers with the following properties: For infinitely many n, the above-mentioned inequalities Eq. (1) and Eq. (2) hold for all s with $s_n \le s \le s_{n+1}$.

Remarks (for both assertions). (a) In the case where β is rational, the assumptions on the limit do not hold. Thus we may assume that β is irrational.

(b) The sequence $\langle s_n \rangle$ need not be computable.

(c) For simplicity, we concentrate on the case where ℓ, m are positive integers. The following proof works for the general case with minor changes. In the general case, a real number expressing the length of a string should be replaced by a certain integer. For example, $\beta \upharpoonright \ell n$ should be replaced by $\beta \upharpoonright \lfloor \ell n \rfloor$.

Proof. (1) Given n, let s_n be the least s such that $a_s \upharpoonright n = \alpha \upharpoonright n$. Then we have $\alpha - a_s \le \alpha - a_{s_n} \le 2^{-n}$ for each s such that $s_n \le s \le s_{n+1}$. Therefore, we have Eq. (1).

If the string $a_{s_n} \upharpoonright n$ is given, we know n as its length, and we can find s_n by means of sequence $\langle a_s \rangle$. Then we can compute $b_{s_n} \upharpoonright \ell n$, which implies the following inequality:

$$K(b_{s_n} \upharpoonright \ell n) \le K(a_{s_n} \upharpoonright n) + O(1) = K(\alpha \upharpoonright n) + O(1) \tag{3}$$

We are going to show that for almost all n and for each nonnegative integer k such that $k \le m/2+1$, $\beta \upharpoonright \ell n$ is neither lexicographic kth successor of $b_{s_n} \upharpoonright \ell n$ nor lexicographic kth predecessor of $\beta \upharpoonright \ell n$, where the 0th successor (predecessor) denotes $b_{s_n} \upharpoonright \ell n$ itself. The proof is as follows. If the above-mentioned assertion fails, for infinitely many n, we have $K(\beta \upharpoonright \ell n) \le K(b_{s_n} \upharpoonright \ell n) + O(1)$. By Eq. (3), we have $K(\beta \upharpoonright \ell n) \le K(\alpha \upharpoonright n) + O(1)$, which contradicts to our assumption of $\liminf_n(K(\beta \upharpoonright \ell n) - K(\alpha \upharpoonright n)) = \infty$.

Hence, for almost all n, it holds that $\beta - b_{s_n} \ge m2^{-\ell n}$. Thus, for almost all n and for each s such that $s_n \le s \le s_{n+1}$, it holds that $\beta - b_s \ge \beta - b_{s_{n+1}} \ge m2^{-\ell(n+1)}$. Hence, for almost all n we have Eq. (2).

(2) We define $\langle s_n \rangle$ in the same way as above. Then, we have Eq. (1). In addition, for all n we have Eq. (3).

Then we can show that for infinitely many n and for each nonnegative integer k such that $k \le m/2+1$, $\beta \upharpoonright \ell n$ is neither lexicographic kth successor of $b_{s_n} \upharpoonright \ell n$ nor lexicographic kth predecessor of $\beta \upharpoonright \ell n$. The proof is given by means of our assmuption of $\limsup_n(K(\beta \upharpoonright \ell n) - K(\alpha \upharpoonright n)) = \infty$.

Therefore, for infinitely many n, it holds that $\beta - b_{s_n} \ge m2^{-\ell n}$. Thus, for infinitely many n and for each s such that $s_n \le s \le s_{n+1}$, it holds that $\beta - b_s \ge \beta - b_{s_{n+1}} \ge m2^{-\ell(n+1)}$. Hence, for infinitely many n we have Eq. (2). □

Proof. (of Proposition 2.1, sketch) Under the assumption of Lemma 2.1 assertion (1), let f be a partial function from $\mathbb{Q}$ to $\mathbb{Q}$ such that for each q, $f(q) = a_s$, where s is the least one such that $q < b_s$. Then it holds that $(\alpha - f(q))^\ell \leq (2^\ell/m)(\beta - q)$. We look at the case where $\ell = m = 1$. Thus we know: If α, β are left-c.e. and irrational then $\alpha \ll_K \beta \implies \alpha \leq_S \beta$. If $\beta \leq_S \alpha$, then $\beta \leq_K \alpha$, which contradicts $\alpha \ll_K \beta$. □

2.2. **Characterization of qS-complete reals by dimension.** Effective Hausdorff dimension and effective packing dimension are characterized by prefix-free Kolmogorov complexity.

Theorem 2.1. (Mayordomo[10]) Given $\alpha \in 2^\omega$, the effective Hausdorff dimension $\dim(\alpha)$ is characterized as follows.

$$\dim(\alpha) = \liminf_n \frac{K(\alpha \upharpoonright n)}{n} \tag{4}$$

Theorem 2.2. (Athreya, Hichcock, Lutz and Mayordomo[1]) Given $\alpha \in 2^\omega$, the effective packing dimension $\mathrm{Dim}(\alpha)$ is characterized as follows.

$$\mathrm{Dim}(\alpha) = \limsup_n \frac{K(\alpha \upharpoonright n)}{n} \tag{5}$$

A survey of algorithmic dimensions may be found in Chapter 13 of [6].

For a left-c.e. real, Solovay completeness and 1-randomness are equivalent. In this section, we show that qS-completeness and $\dim(\alpha) > 0$ are equivalent. We are interested in the structure of the Solovay degrees of qS-complete sets. We will see that the set of these degrees forms a filter.

Tadaki[15], in his study on partial randomness, introduced the generalized halting probability $\Omega^T = \sum_{p \in \mathrm{dom}(U)} 2^{-|p|/T}$ for each positive real number $T \leq 1$, where p runs over the domain of a fixed universal prefix-free machine. In our previous work[8], we introduced a variation of the generalized halting probability for T of the form 2^{-n}.

Definition 2.1. Ω_{2^0} denotes Ω. For each positive integer n, letting $0.a_0a_1a_2\ldots$ be the binary expansion of $\Omega_{2^{-n}}$, we define $\Omega_{2^{-(n+1)}}$ as $0.b_0b_1b_2\ldots$, where $b_{2n} = a_n$, and $b_{2n+1} = 1 - a_n$ for each n.

In our previous work[8], we showed that $\Omega_{2^{-n}}$ are qS-complete. Now we improve the result as follows.

Theorem 2.3. The following are equivalent for a left-c.e. real α.

(1) α is qS-complete.
(2) For some $n \in \mathbb{N}^+$, letting $T = 2^{-n}$, $\Omega_T \leq_S \alpha$.
(3) $\dim(\alpha) > 0$

Proof. (1) $\implies$ (3): Since $\Omega \leq_{qS} \alpha$, there are $\ell \in \mathbb{N}$ and sequences $\langle \omega_s \rangle \nearrow \Omega$ and $\langle a_s \rangle \nearrow \alpha$ such that $(\Omega - \omega_s)^\ell < d(\alpha - a_s)$. Let U be a universal prefix-free machine (see Section 3.5 of [6]). Take a string σ such that $U(\sigma) = \alpha \upharpoonright (n\ell)$ and $|\sigma| = K(\alpha \upharpoonright (n\ell))$. We can compute ω_s such that $\Omega - \omega_s < d^{1/\ell}2^{-n}$ by means of σ and constant bits. If $\Omega - \omega_s < 2^{-m}$ then

$\Omega \upharpoonright m$ is either $\omega_s \upharpoonright m$ itself or that plus 2^{-m}, and the latter is given by additional 1 bit information. To sum up, we can find $\Omega \upharpoonright n$ by means of σ and constant bit. Therefore, it holds that $K(\alpha \upharpoonright (n\ell)) \geq K(\Omega \upharpoonright n) - O(1) \geq n - O(1)$. Hence $\dim(\alpha) \geq 1/\ell > 0$.

(3) $\implies$ (2): For some T of the form $T = 2^{-n}$, it holds that $\mathrm{Dim}(\Omega_T) < \dim(\alpha)$. Therefore, $\Omega_T \ll_K \alpha$. By Proposition 2.1, we have $\Omega_T \leq_S \alpha$.

(2) $\implies$ (1): By [8], $\Omega \leq_{qS} \Omega_T$. □

3. Ideals on left-c.e. reals

The underlying intuition of this section is that the addition of two non-random left-c.e. reals results in a non-random real. It is known that this is the case if randomness means 1-randomness. This means the non-complete Solovay degrees form an ideal. We will investigate some ideals and filters of qS-degrees. In the first subsection, we show that the Solovay degrees of qS-complete reals form a filter. On the other hand, Solovay degrees of non-qS-complete reals do not form an ideal. In the second subsection, we observe the relationships between rational sequences and degrees. Based on these observations, in the third subsection, we introduce a stronger version of qS-reducibility that has a nice property with respect to addition.

3.1. The filter of the Solovay degrees of qS-complete reals.

A subset F of a partially ordered set $(P, \leq_P)$ is a *filter* if the following hold.

- $a \in F \wedge a \leq_P b \implies b \in F$
- $a, b \in F \implies \exists c \in F\ c \leq_P a \wedge c \leq_P b$

A subset X is an *ideal* if the following hold.

- $a \in X \wedge b \leq_P a \implies b \in X$
- $a, b \in X \implies \exists c \in X\ a \leq_P c \wedge b \leq_P c$

We consider the Solovay degrees of left-c.e. reals. Given a Solovay degree $\mathbf{a}$, we define $\dim \mathbf{a}$ as to be $\dim(a)$ for some $a \in \mathbf{a}$. Since Solovay reducibility implies K-reducibility, this definition is well-defined (See [12, Section 4]). The Solovay complete degree forms a filter obviously. For each rational $r \in (0,1)$, let F_r denote the family of all Solovay degrees $\mathbf{a}$ such that $\dim \mathbf{a} > r$. Then, for each rational $r \in (0,1)$, F_r is a filter (Miyabe, Nies and Stephan [12, Theorem 5.1]).

An analogous question is whether the Solovay degrees of qS-complete reals form a filter. The answer is affirmative.

Corollary 3.1. (to Theorem 2.3) We consider the Solovay degrees of left-c.e. reals. Then the family of Solovay degrees of all qS-complete left-c.e. reals is a filter.

Proof. The first requirement of a filter is obviously satisfied. As the degree c in the second requirement, by Theorem 2.3, we can take the Solovay degree of some Ω_T, where $T = 2^{-n}$ for a natural number n. □

We are going to investigate ideals. By Theorem 1.1, the Solovay degrees of non-ML-random left-c.e. reals form an ideal. Miyabe, Nies and Stephan [12, Proposition 5.7] showed the following: For each $r \in [0,1]$, the family of left-c.e. degrees $\mathbf{a}$ such that $\mathrm{Dim}(\mathbf{a}) < r$

is an ideal of left-c.e. Solovay degrees. The same thing holds for $\mathrm{Dim}(\mathbf{a}) \leq r$ in place of $\mathrm{Dim}(\mathbf{a}) < r$.

We ask whether the family of left-c.e. degrees of non-qS-complete left-c.e. degrees forms an ideal. The answer is negative.

Corollary 3.2. (to Theorem 2.3) There are non-qS-complete left-c.e. reals α and β such that $\alpha + \beta$ (real addition) is qS-complete.

Proof. The following was shown in [12, Theorem 4.1]. Suppose that g is a computable function such that $\sum_n 2^{-g(n)}$ is finite and a computable real. For a string σ, let $C(\sigma)$ denote its plain Kolmogorov complexity (see Section 3.1 of [6]). Let α be a left-c.e. real such that $C(\alpha \restriction n) \leq n - g(n)$ for all n. There exist left-c.e. reals β, γ such that $\alpha = \beta + \gamma$, $\dim(\beta) = \dim(\gamma) = 0$ (and both β, γ satisfy certain requirements).

For example, we look at the case of $\alpha = \Omega_{1/2}$ and $g(n) = n/2 - O(1)$. By Theorem 2.3, α is qS-complete, and neither β nor γ is qS-complete. □

3.2. **Rational sequences and reducibility.** By Corollary 3.2, the Solovay degrees of non-qS-complete left-c.e. reals do not form an ideal: Neither do the qS-degrees of them. Now we ask whether there is a stronger version $\underline{\ll}_{qS}$ of qS-reducibility such that $\alpha, \beta \underline{\ll}_{qS} \gamma \implies \alpha + \beta \underline{\ll}_{qS} \gamma$. In this subsection, we investigate the relationships between computable rational sequences and reducibility. Based on these observations, we will see an example of a stronger version of qS-reducibility with the above-mentioned property in the next subsection. Solovay reducibility has many equivalent assertions. Downey *et al.* characterized it via rational sequences.

Lemma 3.1. (Downey *et al.* [7]) Suppose $\langle b_n \rangle \nearrow \beta$. The following are equivalent.

(1) $\alpha \leq_S \beta$
(2) $\exists \langle a_n \rangle \nearrow \alpha \ \exists d > 0$ such that $\forall n \in \mathbb{N} \ a_n - a_{n-1} < d(b_n - b_{n-1})$.

In the case of quasi Solovay reducibility, the following holds.

Proposition 3.1. The following are equivalent.

(1) $\alpha \leq_{qS} \beta$
(2) $\exists \langle a_n \rangle \nearrow \alpha, \langle b_n \rangle \nearrow \beta \ \exists d, \ell > 0$ such that
$$\forall n, m \in \mathbb{N} \ (n < m \implies (a_m - a_n)^\ell < d(b_m - b_n)).$$

Proof. The direction of (2) $\implies$ (1) is given by taking limit of $m \to \infty$. By carefully examining our construction of Hölder continuous function in Theorem 2 of (the preprint version of) [8], we show the direction of (1) $\implies$ (2). □

In the case where β is 1-random, the rational sequences have more interesting properties. We are interested in the limit of $(\alpha - a_n)^\ell/(\beta - b_n)$ under the assumption that β is qS-complete.

Lemma 3.2. For left-c.e. α and β, the following are equivalent.

(1) $\alpha \leq_{qS} \beta$
(2) $\forall \langle b_s \rangle \nearrow \beta \ \exists \langle a_s \rangle \nearrow \alpha \ \exists d, \ell \in \mathbb{N}^+ \forall s \ (\alpha - a_s)^\ell \leq d(\beta - b_s)$

(3) $\forall\langle a_s\rangle \nearrow \alpha\ \exists\langle b_s\rangle \nearrow \beta\ \exists d, \ell \in \mathbb{N}^+ \forall s\ (\alpha - a_s)^\ell \leq d(\beta - b_s)$
(4) $\exists\langle b_s\rangle \nearrow \beta\ \exists\langle a_s\rangle \nearrow \alpha\ \exists d, \ell \in \mathbb{N}^+ \forall s\ (\alpha - a_s)^\ell \leq d(\beta - b_s)$

Proof. (1) $\Longrightarrow$ (2): Let $a_s = f(b_s)$.

(1) $\Longrightarrow$ (3): Given $\langle a_s\rangle$, take a temporary sequence $\langle\beta_s^*\rangle \nearrow \beta$. Let $N \in \mathbb{N}^+$ be large enough. For each $s \geq N$, take the largest t such that $f(\beta_t^*) \leq a_s$, and we define b_s as β_t^*. Then $(\alpha - a_s)^\ell \leq (\alpha - f(\beta_t^*))^\ell \leq d(\beta - b_s)$.

(2) $\Longrightarrow$ (4), (3) $\Longrightarrow$ (4): These are obvious.

(4) $\Longrightarrow$ (1): Given $q < \beta$, find s such that $q \leq b_s$, and let $f(q) = a_s$. □

The $\exists - \forall$ version of (2) is not equivalent to (1), because when both of α and β are rationals, we can chose arbitrarily slow $\{a_s\}$ afterword. Therefore, $\forall - \forall$ version of (2) is not equivalent to (1) in general. The situation is different with the hypothesis that β is qS-complete. We are going to see this in Lemma 3.3. The $\exists - \forall$ version of (3) is not equivalent to (1), because when both of α and β are rationals, we can chose arbitrarily fast $\{b_s\}$ afterword.

Lemma 3.3. Suppose $\langle a_s\rangle \nearrow \alpha, \langle b_s\rangle \nearrow \beta$. If β is qS-complete, then there exist positive integers d, ℓ such that:

$$\forall k\ (\alpha - a_k)^\ell \leq d(\beta - b_k)$$

Proof. Suppose that β is qS-complete. Then the statement of Lemma 3.2 (2) holds with $\alpha = \Omega$. That is, it holds that: $\exists\{\omega_s\} \nearrow \Omega\ \exists d, \ell \in \mathbb{N}^+ \forall s\ (\Omega - \omega_s)^\ell \leq d(\beta - b_s)$

On the other hand, by Theorem 1.2, $\dfrac{\alpha - a_s}{\Omega - \omega_s}$ has a limit. Therefore, there exists a positive integer e of the following property: $\forall s\ \alpha - a_s \leq e(\Omega - \omega_s)$. Hence, it holds that $(\alpha - a_s)^\ell \leq e^\ell(\Omega - \omega_s)^\ell \leq e^\ell d(\beta - b_s)$. □

Lemma 3.4. Suppose $\langle a_s\rangle \nearrow \alpha, \langle b_s\rangle \nearrow \beta$. Suppose that $x \geq 1$ is a real number, and the following limit > 0 exists.

$$\lim_{s\to\infty} \frac{(\alpha - a_s)^x}{\beta - b_s}$$

Then x is uniquely determined (depending on $\langle a_s\rangle$ and $\langle b_s\rangle$).

Proof. Suppose that the limit > 0 exists for x and that the same thing holds for $y > x$ in place of x, too. Then we have:

$$\frac{(\alpha - a_s)^y}{\beta - b_s} = (\alpha - a_s)^{y-x} \frac{(\alpha - a_s)^x}{\beta - b_s} \to 0\ (s \to \infty)$$

This contradicts the assumption. □

Theorem 3.1. It holds that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). If β is random, we also have (2) $\Leftarrow$ (3).

(1) $\alpha \ll_K \beta$. To be more precise: $\lim_{n\to\infty}(K(\beta \upharpoonright n) - K(\alpha \upharpoonright n)) = \infty$.
(2) $\alpha \ll_S \beta$. To be more precise: $\forall\langle a_n\rangle \nearrow \alpha\ \forall\langle b_n\rangle \nearrow \alpha\ \lim_{n\to\infty} \dfrac{\alpha - a_n}{\beta - b_n} = 0$.
(3) $\alpha <_S \beta$

We call $\ll_S$ above strong Solovay reducibility. This definition was clearly inspired by Theorem 1.2.

Proof. (1) $\Longrightarrow$ (2): Assume (1). Given $\langle a_n \rangle$ and $\langle b_n \rangle$, apply Lemma 2.1 (1) to the case where $\ell = 1$. Then for almost all n, we have $(\alpha - a_n)/(\beta - b_n) \leq 2/m$. Since m was arbitrary, we are done.

(2) $\Longrightarrow$ (3) is obvious.

(3) and β is random $\Longrightarrow$ (2): By Theorem 2.3 (b) of [11]. □

In particular, if $\beta = \Omega$ the assertions (2) and (3) of Lemma 3.1 are equivalent. In the proof of the following lemma, we will observe that given a left-c.e. real γ, the set of left-c.e. α such that $\alpha \ll_S \gamma$ forms an ideal. These two facts are important to see that the non-random left-c.e. reals form an ideal.

Lemma 3.5. In Lemma 3.1, the following hold.

(a) If β is random then (2) implies the following (1−).
  (1−) $\limsup_{n\to\infty} K(\beta \restriction n) - K(\alpha \restriction n) = \infty$

(b) (2) does not imply (1).

(c) (3) does not imply (2).

Proof. (a) Assume (2) of Lemma 3.1. By [2] (see also [11]), α is not 1-random. Therefore, $\forall c \neg \forall^\infty n \; n - c \leq K(\alpha \restriction n)$. In other words, for all positive integer c, there are infinitely many n_c such that the following holds.

$$n_c - c > K(\alpha \restriction n_c) \tag{6}$$

Thus, there is an increasing sequence $\{n_c\}_{c\geq 1}$ of positive integers such that Eq. (6) holds for each c. Since β is random, for some positive integer d, it holds that $\forall n \; K(\beta \restriction n) > n - d$. Thus for each positive integer k, we have the following.

$$K(\beta \restriction n_k) - K(\alpha \restriction n_k) > (n_k - d) - (n_k - k) = k - d. \tag{7}$$

Hence, $\limsup_{n\to\infty} K(\beta \restriction n) - K(\alpha \restriction n) = \infty$.

(b) There exists a non-random left-c.e. real α such that $\liminf_n (K(\Omega \restriction n) - K(\alpha \restriction n)) < \infty$. Thus, $\alpha \not\ll_K \Omega$. On the other hand, by [2] (see the paragraph just after Theorem 1.2), we have $\alpha \ll_S \Omega$.

(c) Every non-random left-c.e. Solovay degree can split into lesser left-c.e. Solovay degrees (Downey *et al.* [7]. See also section 9.5 of [6]). We are going to observe that this property of $<_S$ is not shared by $\ll_S$. Let α, β, and γ be left-c.e. reals such that $\alpha, \beta \ll_S \gamma$. Given $\langle a_s \rangle \nearrow \alpha$ and $\langle c_s \rangle \nearrow \gamma$, modify $\langle c_s \rangle$, if necessary, so that $\langle c_s - a_s \rangle$ is an increasing sequence of positive rationals. Let $b_s = c_s - a_s$. Then we have the following.

$$\frac{\alpha + \beta - (a_s + b_s)}{\gamma - c_s} = \frac{\alpha - a_s}{\gamma - c_s} + \frac{\beta - b_s}{\gamma - c_s} \to 0 \tag{8}$$

Thus, it holds that $\alpha + \beta \ll_S \gamma$. We have shown $\alpha, \beta \ll_S \gamma \implies \alpha + \beta \ll_S \gamma$, therefore $<_S$ and $\ll_S$ are not equivalent for left-c.e. reals. However, we know that (2) implies (3). Hence (3) does not imply (2). □

For any left-c.e. real α, the family of left-c.e. degrees $\leq_S \alpha$ forms an ideal, but the family of left-c.e. degrees $<_S \alpha$ does not form an ideal unless α is 1-random. By the proof of (c) above, we know that the family of left-c.e. degrees $\ll_S \alpha$ forms an ideal. The case $\alpha = \Omega$ corresponds to Theorem 1.1.

3.3. **Ideals and a stronger version of qS-reducibility.** Based on the observation in the previous subsection, we introduce a stronger version of qS-reducibility.

Definition 3.1. $\alpha \lll_{qS} \beta$ denotes the following assertion.

$$\exists \ell \in \mathbb{N}\ \forall \langle a_n \rangle \nearrow \alpha\ \forall \langle b_n \rangle \nearrow \beta\ \lim_{n\to\infty} \frac{(\alpha - a_n)^\ell}{\beta - b_n} = 0$$

The goal of this section is to show that $\alpha, \beta \lll_{qS} \gamma \implies \alpha + \beta \lll_{qS} \gamma$.

Theorem 3.2. Suppose that β is left-c.e. Then we have $(1q) \implies (2q)$, and $(2q) \implies (3q)$. In addition, if β is qS-complete then we have $(3q) \implies (1q)$.

$(1q)$ For some ℓ, $\lim_{n\to\infty}(K(\beta \restriction \ell n) - K(\alpha \restriction n)) = \infty$.

$(2q)$ $\alpha \lll_{qS} \beta$

$(3q)$ $\alpha \leq_{qS} \beta$

Proof. $(1q) \implies (2q)$: Assume $(1q)$. Apply Lemma 2.1 (1) to given $\langle a_n \rangle$ and $\langle b_n \rangle$. Then for almost all n, we have $(\alpha - a_n)^\ell/(\beta - b_n) \leq 2^\ell/m$. Since m was arbitrary, we are done.

$(2q) \implies (3q)$ is obvious.

$(3q)$ and β is qS-complete $\implies (1q)$: We have $K(\alpha \restriction n) \leq K(\beta \restriction \ell n) + O(1)$. It is enough to show that for some $c > 0$ we have $\lim_{n\to\infty}(K(\beta \restriction cn) - K(\beta \restriction \ell n)) = \infty$. Let $d = \liminf_{n\to\infty} K(\beta \restriction n)/n$, and let $D = \limsup_{n\to\infty} K(\beta \restriction n)/n$. Since β is qS-complete, by Theorem 2.3, d is positive. It is not hard to see that for any $\varepsilon > 0$, for sufficiently large n we have $d - \varepsilon < K(\beta \restriction n)/n < 1 + \varepsilon$. Now, suppose $\varepsilon > 0$ is small enough. We take a sufficiently large integer k depending on ε. Then the following holds for all sufficiently large n.

$$\frac{K(\beta \restriction k\ell n) - K(\beta \restriction \ell n)}{\ell n} = k\frac{K(\beta \restriction k\ell n)}{k\ell n} - \frac{K(\beta \restriction \ell n)}{\ell n} \geq k(d - \varepsilon) - 1 - \varepsilon > 0 \tag{9}$$

Therefore it holds that $\lim_{n\to\infty}(K(\beta \restriction k\ell n) - K(\beta \restriction \ell n)) = \infty$. □

Proposition 3.2. In Theorem 3.2, we have the following.

(a) We can not replace $(3q)$ by $\alpha <_{qS} \beta$.
(b) $(3q)$ does not imply $(2q)$.
(c) $(2q)$ does not imply $(1q)$.

Proof. (a) Consider the case where $\alpha = \beta = \Omega$, $\ell = 2$, $\langle a_n \rangle \nearrow \Omega$ and $\langle b_n \rangle \nearrow \Omega$. By [2] (see also [11]), $(\alpha - a_n)/(\beta - b_n) \to 0$ $(n \to \infty)$. Therefore, $(\alpha - a_n)^2/(\beta - b_n) \to 1$ $(n \to \infty)$. Thus, $(2q)$ holds under the current assumptions. On the other hand, we have $\beta \leq_{qs} \alpha$, thus $\alpha <_{qS} \beta$ does not hold.

(b) Consider the case where both of α and β are computable. Then $(3q)$ holds, and $(2q)$ does not hold.

(c) follows from the two lemmas below. □

Definition 3.2. A real α is a *strongly c.e. real* if there exists a c.e. set A of natural numbers and $\alpha = \sum_{n \in A} 2^{-n}$.

It is easy to see that any strongly c.e. real is left-c.e.

Lemma 3.6. Let α be a computable real and β be a strongly c.e. real. Then, for all $\ell \in \mathbb{N}$ we have $K(\beta \upharpoonright \ell n) - K(\alpha \upharpoonright n) < O(1)$ for infinitely many n. Thus, (1q) does not hold.

Proof. This fact immediately follows from the fact that any strongly c.e. real is infinitely often K-trivial by Proposition 2.2. in [3]. For the sake of completeness, we give details here again.

Let $B \subseteq \mathbb{N}$ be an infinite c.e. set and β is the strongly c.e. real defined by B, that is, $\beta = \sum_{n \in B} 2^{-n}$. Take a computable sequence $\langle B_s \rangle$ of finite sets of natural numbers such that $B_s \subsetneq B_{s+1}$ and $B = \bigcup_s B_s$. Define $b_s = \sum_{n \in B_s} 2^{-n}$, which is the corresponding approximation of β. For each $n \in B$, There are infinitely many $n \in B$ such that n is enumerated into B at stage s and no $m \leq n$ is enumerated into B after the stage s, that is,

$$\exists^\infty n \exists s [n \in B_s \wedge \forall m < n (m \notin B_s \rightarrow \forall t > s \; m \notin B_s)] \tag{10}$$

For such a pair n, s, we have $b_s \upharpoonright n = \beta \upharpoonright n$. Hence, $K(\beta \upharpoonright n) \leq^+ K(n)$ for infinitely many n. This implies $K(\beta \upharpoonright \ell n) \leq^+ K(\ell n) \leq^+ K(n)$ for infinitely many n.

On the other hand, since $K(n) \leq^+ K(\alpha \upharpoonright n)$, we have $\liminf_{n \to \infty} K(\beta \upharpoonright \ell n) - K(\alpha \upharpoonright n) < \infty$. Thus, (1$q$) does not hold for any computable real α and for any strongly c.e. real β. □

Lemma 3.7. There exists a strongly c.e. real β such that $\alpha \lll_{qS} \beta$ for every computable real. Thus, (2q) holds for this β and any computable real α.

Our proof idea is as follows. We shall take a variant of the halting problem as a set B and let β be the corresponding real. Since B knows the strings with which the machine halts as inputs, one can compute a fast-growing function f that dominates all computable functions.

Our goal is to show

$$\lim_{s \to \infty} \frac{\alpha - a_s}{\beta - b_s} = 0. \tag{11}$$

Otherwise, b_s is a good approximation of β for infinitely many s, and one can compute a sufficiently long initial segment of B, from which one can compute a function g such that $g(s) \geq f(s)$ for infinitely many s. This would contradict the property of f.

Proof. First, we construct the strongly c.e. real β. Let σ_n be the enumeration of $2^{<\omega}$ in the length-lexicographical order, that is, empty string, 0, 1, 00, 01, 10, 11, ... We define a c.e. set B by

$$B = \{n \in \mathbb{N} \; : \; U(\sigma_n) \downarrow\}$$

and let $\beta = \sum_{n \in B} 2^{-n}$.

We define a function $f : \mathbb{N} \to \mathbb{N}$ as follows.

$$f(n) := \max\{s \in \omega : \exists k \leq 2n[U(\sigma_k) \text{ halts at stage } s.]\ \} \tag{12}$$

Here U is a fixed universal plain machine. By saying $U(\sigma)$ halts at stage s, we mean $U(\sigma)[s]\downarrow$ and $U(\sigma)[s-1]\uparrow$.

Claim 1: f dominates any computable function g, that is, $f(n) \geq g(n)$ for almost all n.

Fix an index e of a computable function such that Φ_e is total. Since U is universal, U can simulate $\Phi_e(n)$ within $C(\langle e, n\rangle) + O(1)$, where C denotes the plain Kolmogorov complexity. Since $C(\langle e, n\rangle) + O(1) = O(\log n)$, there exists a program σ_k with $k \leq 2n$ such that $U(\sigma_k)$ simulate $\Phi_e(n)$ for all sufficiently large n. By the usual convention, the halting stage is larger than the output. Hence, f dominates Φ_e. This proves Claim 1.

We are going to show (2q) $\alpha \underline{\ll}_{qS} \beta$ with $\ell = 1$. Suppose $\langle a_s\rangle \nearrow \alpha$ and $\langle b_s\rangle \nearrow \beta$. Later we need to construct a *total* computable function g from $\langle b_s\rangle$. For infinitely many s, the term b_s is a good approximation, but for other s, the term b_s may not be a good approximation, which may prevent the totality of g. Thus, we first translate $\langle b_s\rangle$ into a well-behaved sequence $\langle d_n\rangle$.

Claim 2: From each computable sequence $\langle b_s\rangle$, one can compute a computable increasing sequence $\langle D_n\rangle$ of finite sets of natural numbers such that, letting $d_n = \sum_{m \in D_n} 2^{-m}$, we have $D_n \nearrow B$ and $\langle d_n\rangle \nearrow \beta$. Here, by $D_n \nearrow B$, we mean that D_n is increasing and $\lim_n D_n = B$.

Let $\langle B_s\rangle$ be an increasing computable sequence of finite sets of natural numbers converging to B. Let B'_s be the binary expansion of b_s, that is, $b_s = \sum_{n \in B'_s} 2^{-n}$. If there are two such sets, choose one of them as you wish. Given n, we can computably find $k \geq n$ and $s \geq n$ such that $B'_k \upharpoonright (2n+1) \subseteq B_s \upharpoonright (2n+1)$ because $B'_k \nearrow B$ and B is not computable. Then define $D_n := B'_k \upharpoonright (2n+1)$ for this k and define $\langle d_n\rangle$ as above accordingly. The resulting d_n may be smaller than b_k and b_n because we cut the B'_k to make D_n, but the difference is at most 2^{-2n-1}. Therefore we have $b_n \leq d_n + 2^{-2n-1}$ and

$$\beta - d_n \leq \beta - b_n + 2^{-2n-1} \tag{13}$$

Therefore, $\langle d_n\rangle \nearrow \beta$. This proves Claim 2.

Now we are ready to prove the lemma. For simplicity, we first observe the case where $\alpha - a_n < 2^{-2n}/n$. Assume for a contradiction that Eq. (11) fails:

$$\exists \varepsilon > 0\ \exists^\infty n\ \frac{\alpha - a_n}{\beta - b_n} > \varepsilon \tag{14}$$

We define a function g by

$$g(n) := \max\{s \in \omega : \exists m \in D_n[U(\sigma_m) \text{ halts at stage } s.]\ \} + 1 \tag{15}$$

Since $D_n \subseteq B$, $U(\sigma_m)$ halts for every $m \in D_n$. Hence, g is a total computable function.

If Eq. (14) holds for some n, then we have $\beta - b_n < 1/(\varepsilon n 2^{2n})$ and $\beta - d_n < 2^{-2n}$ by Eq. (13). Then, $D_n \upharpoonright 2n = B \upharpoonright 2n$ and $g(n) = f(n) + 1$. Since there are infinitely many such n, this contradicts Claim 1.

In the general case, we need one more trick. Take a computable subsequence $\langle n_k \rangle$ such that $\alpha - a_{n_k} < 2^{-2k-2}/k$ for all k. We may have

$$\beta - d_n \le \beta - b_n + 2^{-2k-1}$$

for $n_{k-1} < n \le n_k$ by a similar construction in Claim 2. Now we define a total computable function g by

$$g(k) := \max_{n \in (n_{k-1}, n_k]} \max\{s \in \omega : \exists m \in D_n[U(\sigma_m) \text{ halts at stage } s.]\ \} + 1$$

If Eq. (14) holds for some n, then, by taking k such that $n_{k-1} < n \le n_k$, we hae

$$\begin{aligned} \beta - d_n &\le \beta - b_n + 2^{-2k-1} \le \beta - b_{n_{k-1}} + 2^{-2k-1} \\ &< \frac{\alpha - a_{n_{k-1}}}{\varepsilon} + 2^{-2k-1} < \frac{1}{\varepsilon k 2^{2k}} + 2^{-2k-1} < 2^{-2k}. \end{aligned}$$

For this n, k, we have $D_n \upharpoonright 2k = B \upharpoonright 2k$. Thus, there are infinitely many k such that $g(k) = f(k) + 1$, which contradicts Claim 1. □

Lemma 3.8. $\alpha, \beta \lll_{qS} \gamma \Rightarrow \alpha + \beta \lll_{qS} \gamma$

Proof. Take appropreate $\langle a_n \rangle \nearrow \alpha, \langle b_n \rangle \nearrow \beta$ and ℓ. Then for any $\langle c_n \rangle \nearrow \gamma$, we have the following.

$$\lim_{n\to\infty} \frac{(\alpha - a_n)^\ell}{\gamma - c_n} = 0, \ \lim_{n\to\infty} \frac{(\beta - b_n)^\ell}{\gamma - c_n} = 0$$

Here, we have the following.

$$\begin{aligned} ((\alpha + \beta) - (a_n + b_n))^\ell &= \sum_{k=0}^{\ell} \binom{\ell}{k} (\alpha - a_n)^k (\beta - b_n)^{\ell - k} \\ &\le \sum_{k=0}^{\ell} \binom{\ell}{k} (\max\{\alpha - a_n, \beta - b_n\})^\ell \\ &\le O(1)(\max\{\alpha - a_n, \beta - b_n\})^\ell \end{aligned}$$

Therefore, $((\alpha + \beta) - (a_n + b_n))^\ell / (\gamma - c_n) \to 0 \ (n \to \infty)$. □

By these results above, we know that the family of left-c.e. degrees $\lll_{qS} \alpha$ forms an ideal. We also note that since $\le_{qS}$ is a standard reducibility, the family of left-c.e. degrees $\le_{qS} \alpha$ also forms an ideal.

4. Effective dimensions via the rate of convergence

By developing the previous section, we characterize the effective Hausdorff dimension of the left-c.e. reals by the rate of convergence of their computable approximations. In order to sketch the motive for the following theorem, let us observe the case where $(\alpha - a_s)^\ell/(\Omega - \omega_s)$ has a positive limit $c \leq 1$. For sufficiently large s, the approximate value of $\ell \log(\alpha - \alpha_s)$ is given by $\log(\Omega - \omega_s) + \log c$. Thus it is natural to look at the ratio $\log(\Omega - \omega_s)/\log(\alpha - \alpha_s)$ to investigate the role of ℓ in this context.

Theorem 4.1. Let α be a left-c.e. real. Suppose that $\langle \omega_s \rangle \nearrow \Omega$ and $\langle a_s \rangle \nearrow \alpha$.

(1)

$$\limsup_{s\to\infty} \frac{\log(\Omega - \omega_s)}{\log(\alpha - a_s)} = \mathrm{Dim}(\alpha) \tag{16}$$

(2)

$$\liminf_{s\to\infty} \frac{\log(\Omega - \omega_s)}{\log(\alpha - a_s)} = \dim(\alpha) \tag{17}$$

Remark: In the both assertions, the results are independent of the choice of $\langle \omega_s \rangle$ and $\langle a_s \rangle$.

Proof. (1) Let $d = \mathrm{Dim}(\alpha)$. For simplicity, we often omit the floor symbols $\lfloor\ \rfloor$. In the proof, a real number expressing the length of a string should be replaced by a certain integer.

Claim 1. $\limsup_s \log(\Omega - \omega_s)/\log(\alpha - a_s) \leq d$.

Proof of Claim 1: Suppose $\varepsilon > 0$. We are going to show $\log(\Omega - \omega_s)/\log(\alpha - a_s) \leq d + \varepsilon$ for sufficiently large s. For some constant c, for all n we have the following.

$$K(\Omega \upharpoonright (d+\varepsilon)n) > \lfloor (d+\varepsilon)n \rfloor - c \tag{18}$$

Given a positive integer N, all sufficiently large n satisfies the following.

$$\lfloor (d+\varepsilon)n \rfloor - c > \lfloor (d+\varepsilon/2)n \rfloor - N \tag{19}$$

Since $d = \mathrm{Dim}(\alpha)$, for almost all n, we have the following.

$$(d+\varepsilon/2)n > K(\alpha \upharpoonright n) \tag{20}$$

By Eq. (18), Eq. (19), and Eq. (20), for almost all n we have $K(\Omega \upharpoonright (d+\varepsilon)n) - K(\alpha \upharpoonright n) > N$. Since N was arbitarary, it holds that $\lim_{n\to\infty}(K(\Omega \upharpoonright (d+\varepsilon)n) - K(\alpha \upharpoonright n)) = \infty$.

Therefore, we can apply Lemma 2.1 (1) to this case with $\ell = d + \varepsilon$, and $m = \lfloor 2^{d+\varepsilon} \rfloor$. Let $\langle s_n \rangle$ be the sequence in Lemma 2.1 (1). Then for almost n and each s such that $s_n \leq s \leq s_{n+1}$, we have $\alpha - a_s \leq 2^{-n}$ and $\Omega - \omega_s \geq 2^{-(d+\varepsilon)n}$. Noting that the divisor $\log(\alpha - a_s)$ is negative, we get $\limsup_s \log(\Omega - \omega_s)/\log(\alpha - a_s) \leq d + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have shown Claim 1. Q.E.D. (Claim 1)

Claim 2. $\limsup_s \log(\Omega - \omega_s)/\log(\alpha - a_s) \geq d$.

Proof of Claim 2: Suppose $\varepsilon > 0$. Given a positive integer N, all sufficiently large n satisfies the following.

$$K(\Omega \upharpoonright (d-\varepsilon)n) + N < (d-\varepsilon/2)n \tag{21}$$

For infinitely many n, we have the following.

$$(d-\varepsilon/2)n < K(\alpha \upharpoonright n) \tag{22}$$

By Eq. (21) and Eq. (22), for infinitely many n, it holds that $N < K(\alpha \upharpoonright n) - K(\Omega \upharpoonright (d-\varepsilon)n)$. Therefore, for infinitely many n, we have the following.

$$N < K(\alpha \upharpoonright n/(d-\varepsilon)) - K(\Omega \upharpoonright n) \tag{23}$$

Therefore we can apply Lemma 2.1 (2) to this case. The roles of α, β in the lemma are performed by Ω and α, respectively. Let $\ell = 1/(d-\varepsilon)$, and $m = 2^{1/(d-\varepsilon)}$. Let $\langle s_n \rangle$ be the sequence in Lemma 2.1 (2). There are infinitely many n such that for each s such that $s_n \leq s \leq s_{n+1}$, $\Omega - \omega_s \leq 2^{-n}$ and $\alpha - a_s \geq 2^{-n/(d-\varepsilon)}$. Thus, we have $\log(\Omega - \omega_s)/\log(\alpha - a_s) \geq d - \varepsilon$ infinitely often. Hence it holds that $\limsup_s \log(\Omega - \omega_s)/\log(\alpha - a_s) \geq d - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have shown Claim 2. Q.E.D. (Claim 2)

By Claims 1 and 2, we have shown Eq. (16).

(2) Let $d = \dim(\alpha)$.

Claim 3. $\liminf_s \log(\Omega - \omega_s)/\log(\alpha - a_s) \leq d$.

Proof of Claim 3: Suppose $\varepsilon > 0$. For some constant c, for all n we have Eq. (18). Given a positive integer N, all sufficiently large n satisfies Eq. (19). Infinitely many n satisfies Eq. (20). Therefore, for infinitely many n we have $K(\Omega \upharpoonright (d+\varepsilon)n) - K(\alpha \upharpoonright n) > N$. Since N was arbitrary, it holds that $\limsup_{n\to\infty}(K(\Omega \upharpoonright (d+\varepsilon)n) - K(\alpha \upharpoonright n)) = \infty$.

Therefore, we can apply Lemma 2.1 (2) to this case. We get $\liminf_s \log(\Omega - \omega_s)/\log(\alpha - a_s) \leq d + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have shown Claim 3. Q.E.D. (Claim 3)

Claim 4. $\liminf_s \log(\Omega - \omega_s)/\log(\alpha - a_s) \geq d$.

Proof of Claim 4: Suppose $\varepsilon > 0$. Given a positive integer N, all sufficiently large n satisfies Eq. (21). For almost all n, we have Eq. (22). Therefore for almost all n, it holds that $N < K(\alpha \upharpoonright n) - K(\Omega \upharpoonright (d-\varepsilon)n)$. Thus for almost all n, we have Eq. (23).

Therefore we can apply Lemma 2.1 (1) to this case. Letting $\langle s_n \rangle$ be the sequence in Lemma 2.1 (1), for almost all n and for each s such that $s_n \leq s \leq s_{n+1}$, $\Omega - \omega_s \leq 2^{-n}$ and $\alpha - a_s \geq 2^{-n/(d-\varepsilon)}$. Thus, for almost all n we have $\log(\Omega - \omega_s)/\log(\alpha - a_s) \geq d - \varepsilon$. Hence it holds that $\liminf_s \log(\Omega - \omega_s)/\log(\alpha - a_s) \geq d - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have shown Claim 4. Q.E.D. (Claim 4)

By Claims 3 and 4, we have shown Eq. (17). □

5. Appendix: A variant of the first incompleteness theorem

Based on non-computability Ω, Chaitin[4] showed a variant of Gödel's first incompleteness theorem. Suppose that T is a consistent computable theory extending PA. Then there exists a natural number b such that for every natural number a, we have $T \nvdash b < K(a)$.

As a by-product of our observation, based on Solovay reducibility, we give a variant of Chaitin's incompleteness theorem. For simplicity, we use non-random left-c.e. real α and Ω. However, the proof works for any pair of left-c.e. reals α, β provided that β is not Solovay reducible to α. The absence of Solovay reduction plays the role of Gödel sentence. For each natural number m, we let $\overline{m}$ denote the numeral for m in a given formal system.

Proposition 5.1. Suppose that T is a consistent c.e. theory extending PA. Suppose that β is a left-c.e. real that is not Solovay complete (among the left-c.e. reals). Assume that $\langle \omega_s \rangle, \langle q_s \rangle$ are computable (strictly) increasing sequences of rationals converging to Ω, and that $\langle b_s \rangle, \langle r_s \rangle$ are computable (strictly) increasing sequences of rationals converging to β. Then the following holds.

For every positive integer L, there exists a natural number n such that for all natural numbers m, t,

$$T \nvdash \forall s \geq \overline{t} \ \big(|\omega_s - q_{\overline{m}}| < L|b_s - r_{\overline{n}}|\big) \tag{24}$$

Proof. We prove the proposition by contradiction. We assume that for some positive rational number L, for every natural number n there exist natural numbers m, t with the following property.

$$T \vdash \forall s \geq \overline{t} \ \big(|\omega_s - q_{\overline{m}}| < L|b_s - r_{\overline{n}}|\big) \tag{25}$$

We fix L for a while. Let $\varphi(x, y, z)$ denote the following formula.

$$\forall s \geq x \ [|\omega_s - q_y| < L|b_s - r_z|] \tag{26}$$

Since T is c.e., we can effectively perform the following procedure: Given n, enumerate all proofs in T; if we find a proof of $\varphi(\overline{t}, \overline{m}, \overline{n})$ for some (t, m), output (t, m). By Eq. (25), each input n has an output.

To be more precise, there exists computable function $f : \mathbb{N} \implies \mathbb{N} \times \mathbb{N}$ such that for all natural number n we have the following. Here, $f : n \mapsto (t, m)$ and $1st((t, m)) = t$ and $2nd((t, m)) = m$.

$$T \vdash \forall s \geq \overline{1st(f(n))} \ \big(|\omega_s - q_{\ \overline{2nd(f(n))}}| < L|b_s - r_{\overline{n}}|\big) \tag{27}$$

Since T is Σ_1-sound, it is Π_1-sound, too. Therefore, the following holds (in the standsard model).

$$\forall s \geq 1st(f(n)) \ \big(|\omega_s - q_{2nd(f(n))}| < L|b_s - r_n|\big) \tag{28}$$

By taking the limit $s \implies \infty$, we get the following.

$$\Omega - q_{2nd(f(n))} \leq L(\beta - r_n). \tag{29}$$

Now, we replace L by a slightly larger rational number. We abuse notation and denote this new rational number by the same symbol L. Note that function $g(n) := 2nd(f(n))$ is computable. Then for all n we have $\Omega - q_{g(n)} < L(\beta - r_n)$. Therefore, it holds that $\Omega \leq_S \beta$.

This contradicts that Ω is Solovay complete, and β is not Solovay complete. □

Proposition 5.2. Let $\varphi(x, y, z)$ be the formula defined in the proof of Theorem 5.1.

(1) For any n there exists (t, m) such that $\mathbb{N} \models \varphi(\bar{t}, \overline{m}, \overline{n})$.
(2) T is incomplete.

Proof. (1) We fix a positive rational L and take a natural number n as in the proof of Theorem 5.1. If we take m big enough, then we have $\Omega - q_m < L(\beta - r_n)$.. Therefore $L(\beta - r_n) - (\Omega - q_m) > 0$. Let $h(s)$ denote $L(b_s - r_n) - (\omega_s - q_m)$. Then we have $\lim_{s\to\infty} h(s) = L(\beta - r_n) - (\Omega - q_m) > 0$.

Hence there exists t such that the following holds.

$$\forall s \geq t \ \big(\omega_s - q_m < L(b_s - r_n)\big) \tag{30}$$

In summary, for any n there exist t and m such that we have $\mathbb{N} \models \varphi(\bar{t}, \overline{m}, \overline{n})$.

(2) Now, assume for a contradiction that $T \vdash \neg\varphi(\bar{t}, \overline{m}, \overline{n})$ holds for the above-mentioned t, m and n. Since $\varphi(\bar{t}, \overline{m}, \overline{n})$ is a Π_1-sentence, $\neg\varphi(\bar{t}, \overline{m}, \overline{n})$ is a Σ_1-sentence. Thus, by Σ_1-soundness of T, we have $\mathbb{N} \models \neg\varphi(\bar{t}, \overline{m}, \overline{n})$. This contradicts to (1). Hence it holds that $T \nvdash \neg\varphi(\bar{t}, \overline{m}, \overline{n})$. On the other hand, by Theorem 5.1, for any t, m, we have $T \nvdash \varphi(\bar{t}, \overline{m}, \overline{n})$. Hence T is incomplete. □

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