Fourier–Mukai partners of elliptic ruled surfaces over arbitrary characteristic fields

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Contents

1 Introduction **3**

2 Preliminaries **6**
2.1 Adjoint functors 6
2.2 Exact functors 8
2.3 Fourier–Mukai functors 11
2.4 Criterion of fully faithfulness 13

3 Relative moduli spaces of sheaves on elliptic fibrations **17**
3.1 Fourier–Mukai partners of elliptic surfaces 17
3.2 Weil–Châtelet group 21

4 Elliptic curves and automorphisms **24**

5 Admissible subcategories **32**
5.1 Basic properties 32
5.2 Pirozhkov's result and its application 39

6 Elliptic ruled surfaces **42**
6.1 Basic properties of ruled surfaces 42
6.2 Vector bundles over elliptic curves 43

7 Fourier–Mukai partners of elliptic ruled surfaces **44**
7.1 Elliptic ruled surfaces 44
7.2 Case (i-2). 46
7.3 Proof of Theorem 1.3. 50

1 Introduction

For a smooth projective variety X defined over an algebraically closed field k of characteristic $p \geq 0$, we denote by $D^b(X)$ the bounded derived categories of coherent sheaves on X. We say that a smooth projective variety Y is *derived equivalent* to X or a *Fourier–Mukai partner* of X if there exists an equivalence $D^b(X) \cong D^b(Y)$ as k-linear triangulated categories.

The derived category of coherent sheaves was introduced by Grothendieck and Verdier in 1960s as an appropriate framework for their theory of duality. Later, it has been observed that the derived category of coherent sheaves contains a lot of information about the variety, and that the derived category of coherent sheaves has connections with other subjects. An important one is *homological mirror symmetry* conjectured by Kontsevich [24]. Roughly speaking, it states a correspondence between the algebraic geometry and the symplectic geometry. Since the appearance of this conjecture, there is a growing interest in the derived category of coherent sheaves.

Of course, if X and Y are isomorphic, then X and Y are Fourier–Mukai partners each other. But in general, there exist pairs of varieties which are Fourier–Mukai partners and not isomorphic to each other. The first example of such pair was discovered by Mukai in [29], which states that an abelian variety A and its dual $\hat{A}$ are derived equivalent. Note that A and $\hat{A}$ are not even birationally equivalent in general. The idea of using moduli spaces was applied to K3 surfaces by Orlov [30] and he got the characterization of Fourier–Mukai partners of K3 surfaces. Moreover, Borisov and Căldăraru [9] discovered a pair of Calabi–Yau threefolds which are derived equivalent and not birationally equivalent. Moreover, Kuznetsov discovered other examples of such pairs in [25] using his theory called *homological projective duality*.

We let $\mathrm{FM}(X)$ denote the set of isomorphism classes of Fourier–Mukai partners of X. It is a fundamental question to describe the set $\mathrm{FM}(X)$ explicitly. It is known that $|\,\mathrm{FM}(C)| = 1$ for any smooth projective curve C (see [3, Theorem 7.16]). On the other hand, smooth projective surfaces S may have non-trivial Fourier–Mukai partners, namely $|\,\mathrm{FM}(S)| \neq 1$ may occur. Bridgeland, Maciocia and Kawamata showed in [12] and [23] that if a smooth projective surface S over $\mathbb{C}$ has a non-trivial Fourier–Mukai partner T, then both are abelian surfaces, K3 surfaces or elliptic surfaces with non-zero Kodaira dimension. For these surfaces, there are characterizations of their Fourier–Mukai partners.

Theorem 1.1 ([30])**.** *Let X and Y be K3 surfaces over $\mathbb{C}$. The following conditions are equivalent:*

1. *There exists an equivalence $D^b(X) \cong D^b(Y)$.*
2. *There exist $v \in \tilde{H}(X, \mathbb{Z})$ satisfying $v^2 = 0$ and an ample divisor ω on X such that there exists an isomorphism $Y \cong M_\omega(v)$, where $M_\omega(v)$ is a moduli space of ω-stable sheaves on X.*
3. *There exists a Hodge isometry $\tilde{H}(X, \mathbb{Z}) \cong \tilde{H}(Y, \mathbb{Z})$.*

Here, $\tilde{H}(X, \mathbb{Z})$ is a Mukai lattice, i.e., $H^*(X, \mathbb{Z})$ endowed with the Mukai pairing.

Theorem 1.2 ([12])**.** *Let $\pi\colon S \to C$ be an elliptic surface over $\mathbb{C}$ with non-zero Kodaira dimension. Then*

$$\mathrm{FM}(S) = \{J^i(S) \mid (i, \lambda_\pi) = 1\}/\cong .$$

where $J^i(S)$ is the relative moduli space of stable pure-dimension 1 sheaves on the fiber π, which parametrizes degree i line bundles. For more details, see §3.

In positive characteristic, it is known that similar results hold to the case $k = \mathbb{C}$. For example, Lieblich and Olsson gave a generalization of Theorem 1.1 to fields with $p \neq 2$ in [27]. It is also known that Enriques and bielliptic surfaces do not have any non-trivial Fourier–Mukai partners [**?**]. Moreover, we get a generalization of Theorem 1.2 to arbitrary characteristic fields. (See Theorem 3.3.)

In this thesis, we study the set $\mathrm{FM}(S)$ of elliptic ruled surfaces S defined over k with arbitrary p. Here, an *elliptic ruled surface* means a smooth projective surface with a $\mathbb{P}^1$-bundle structure over an elliptic curve. The author and Uehara obtain the following theorem, which is a generalization of the result for $k = \mathbb{C}$ in [40] to an arbitrary algebraically closed field k.

Theorem 1.3. *Let S be an elliptic ruled surface defined over k and $\pi : S \to E$ be a $\mathbb{P}^1$-bundle over an elliptic curve E. If $|\mathrm{FM}(S)| \neq 1$, then S is of the form*

$$S = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$$

for some $\mathcal{L} \in \mathrm{Pic}^0 E$ of order $m \geq 5$. Furthermore, we have

$$\mathrm{FM}(S) = \{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \mid i \in \mathbb{Z} \text{ with } (i, m) = 1 \text{ and } 1 \leq i < m\}/\cong, \quad (1)$$

and

$$|\mathrm{FM}(S)| = \varphi(m)/|H^{\mathcal{L}}_{\hat{E}}|.$$

Here, φ is the Euler function, and we define

$$H^{\mathcal{L}}_{\hat{E}} := \{i \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \phi \in \mathrm{Aut}_0(E) \text{ such that } \phi^*\mathcal{L} \cong \mathcal{L}^i\} \tag{2}$$

as a subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$. We also have $|H^{\mathcal{L}}_{\hat{E}}| = 2, 4$ or 6, depending on the choice of E and $\mathcal{L}$.

In the case $p \nmid m = \mathrm{ord}(\mathcal{L})$, it is known (cf. [40, Equation (3.4)]) that $S = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$ is a quotient of $F_0 \times \mathbb{P}^1$ by a cyclic group action, where F_0 is an elliptic curve, and Uehara uses this fact to describe the set $\mathrm{FM}(S)$ in [40]. On the other hand, in the case $p > 0$ and $p \mid m$, an elliptic ruled surface $S = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$ does not admit a similar construction (see [39, §5.1]). Therefore, we need more general treatment to show Theorem 1.3.

Moreover in the proof of Theorem 1.3, we study the structure of the group $H^{\mathcal{L}}_{\hat{E}}$, which heavily depends on the structure of the automorphism group $\mathrm{Aut}_0 E$ of E. In the case $k = \mathbb{C}$, it is a rather simple task, but in the case arbitrary k, especially, in the case $p = 2, 3$ and $j(E) = 0 = 1728$, we need a different approach from the one in [40] (see Lemma 4.7).

In the proof of Theorem 1.3, we obtain some evidence of the Popa–Schnell conjecture in [35], which states that for any Fourier–Mukai partners Y of a given smooth projective variety X, there exists an equivalence of derived categories of their albanese varieties, i.e., $D^b(\mathrm{Alb}(X)) \cong D^b(\mathrm{Alb}(Y))$.

Proposition 1.4 (=Corollary 5.22)**.** *Let $X \to A$ and $Y \to B$ be $\mathbb{P}^n$-bundles over abelian varieties A and B for $n = 1, 2$. If X and Y are Fourier–Mukai partners, then so are A and B. Furthermore, the Popa–Schnell conjecture holds true in this case.*

The plan of this thesis is as follows. In §2, we state several facts about derived categories of coherent sheaves . A criterion of fully faithfulness in [5] is extended to the case including positive characteristic.

In §3, we explain some results and notation of relative moduli spaces of stable sheaves on elliptic fibrations, a main tool for the study of Fourier–Mukai partners of elliptic surfaces. We obtain a characterization of Fourier–Mukai partners of elliptic surfaces with non-zero Kodaira dimensions in Theorem 3.3 for arbitrary $p = \mathrm{ch}\, k$, which was originally proved by Bridgeland in the case $p = 0$.

In §4, we show several results on automorphisms of elliptic curves.

In §5, we apply Pirozhkov's result [34] to show Proposition 1.4.

In §6, we explain several results of ruled surfaces on elliptic curves, which are based on [17].

Finally, in §7, we first narrow down the candidates of elliptic ruled surfaces with non-trivial Fourier–Mukai partners by Proposition 1.4 and the main result in [39], and then prove Theorem 1.3.

Notation and conventions All algebraic varieties X are defined over an algebraically closed field k of characteristic $p \geq 0$. A point $x \in X$ means a closed point unless otherwise specified.

For an elliptic curve E, $\mathrm{Aut}_0(E)$ is the group of automorphisms fixing the origin.

By an *elliptic surface*, we will always mean a smooth projective surface S together with a smooth projective curve C and a relatively minimal projective morphism $\pi\colon S \to C$ whose general fiber is an elliptic curve. An *elliptic ruled surface* means a smooth projective surface with a $\mathbb{P}^1$-bundle structure over an elliptic curve.

For a morphism $\pi\colon X \to Y$ between algebraic varieties, the symbol $\mathrm{Aut}(X/Y)$ stands for the group of automorphisms of X preserving π.

2 Preliminaries

2.1 Adjoint functors

For the study of the derived category of coherent sheaves, adjoint functors appear frequently, for example, see Lemma 2.15. In this subsection, we recall the definition and basic properties of the adjoint functors. Most of results are found in [20].

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary categories and $F\colon \mathcal{A} \to \mathcal{B}$ be a functor. A functor $H\colon \mathcal{B} \to \mathcal{A}$ is *right adjoint* to F, we write $F \dashv H$, if there exist isomorphisms

$$\mathrm{Hom}_{\mathcal{B}}(F(a), b) \cong \mathrm{Hom}_{\mathcal{A}}(a, H(b)) \tag{3}$$

for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$ which are functorial in a and b.

A functor $G\colon \mathcal{B} \to \mathcal{A}$ is *left adjoint* to F, we write $G \dashv F$, if there exist isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(G(b), a) \cong \mathrm{Hom}_{\mathcal{B}}(b, F(a)) \tag{4}$$

for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, which are functorial in a and b.

Clearly, H is right adjoint to F if and only if F is left adjoint to H.

A right (or left) adjoint functor is, if it exists, unique up to isomorphism. It can be verified by Yoneda's lemma.

Suppose $F \dashv H$, under the correspondence

$$\mathrm{Hom}_{\mathcal{B}}(F(a), F(a)) \cong \mathrm{Hom}_{\mathcal{A}}(a, H(F(a))), \tag{5}$$

the morphism $\mathrm{id}_{F(a)}$ induces a morphism $a \to H(F(a))$. By the functoriality of the isomorphism, we get a natural transformation

$$h\colon \mathrm{id}_{\mathcal{A}} \to H \circ F. \tag{6}$$

Similarly, suppose $G \dashv F$, then we get a natural transformation

$$g\colon G \circ F \to \mathrm{id}_{\mathcal{A}}. \tag{7}$$

For the induced natural transformation h described in (6), composing the functor H from the right, we can define the natural transformation

$$h_H\colon H \to (H \circ F) \circ H.$$

Similarly, for the induced natural transformation $g\colon F \circ H \to \mathrm{id}_{\mathcal{A}}$, composing the functor H from the left, we can define

$$H(g)\colon H \circ (F \circ H) \to H.$$

We can check that the compositions

$$H \xrightarrow{h_H} (H \circ F) \circ H = H \circ (F \circ H) \xrightarrow{H(g)} H \tag{8}$$

$$F \xrightarrow{F(h)} F \circ (H \circ F) = (F \circ H) \circ F \xrightarrow{g_F} F \tag{9}$$

are the identities.

Lemma 2.2 (Lemma 1.21 in [20])**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $H\colon \mathcal{B} \to \mathcal{A}$ be functors and suppose $F \dashv H$. Then the induced natural transformation $h\colon \mathrm{id}_{\mathcal{A}} \to H \circ F$ induces the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(a,b) & \xrightarrow{h(b)} & \mathrm{Hom}_{\mathcal{A}}(a, H(F(b))) \\ & \searrow^{F} & \downarrow \cong \\ & & \mathrm{Hom}_{\mathcal{B}}(F(a), F(b)) \end{array} \tag{10}$$

for any $a, b \in \mathcal{A}$. Here, the vertical isomorphism is given in (3)*.*

Similarly, for functors $G \dashv F$, the induced natural transformation $g\colon G \circ F \to \mathrm{id}_{\mathcal{A}}$ induces the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(a,b) & & \\ \downarrow g(a) & \searrow^{F} & \\ \mathrm{Hom}_{\mathcal{A}}(G(F(a)),b) & \xrightarrow[\cong]{} & \mathrm{Hom}_{\mathcal{B}}(F(a), F(b)). \end{array} \tag{11}$$

Recall that we say a functor F is fully faithful if the natural map

$$\mathrm{Hom}_{\mathcal{A}}(a,b) \to \mathrm{Hom}_{\mathcal{B}}(F(a), F(b))$$

is bijective for all objects $a, b \in \mathcal{A}$. By Lemma 2.2, we get the following corollary.

Corollary 2.3. *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $H\colon \mathcal{B} \to \mathcal{A}$ be functors and suppose $F \dashv H$. Then F is fully faithful if and only if the induced natural transformation $h\colon \mathrm{id}_{\mathcal{A}} \to H \circ F$ is an isomorphism.*

Similarly, if $G \dashv F$, then F is fully faithful if and only if the induced natural transformation $g\colon G \circ F \to \mathrm{id}_{\mathcal{A}}$ is an isomorphism.

2.2 Exact functors

Let $\mathcal{A}$ and $\mathcal{B}$ be triangulated categories and $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor. In this subsection, we describe a useful criterion for F to be fully faithful.

Let us start with the definition of a spanning class.

Definition 2.4. Let $\mathcal{A}$ be a triangulated category. A set Ω of the objects of $\mathcal{A}$ is called a *spanning class* for $\mathcal{A}$ if for any object a of $\mathcal{A}$,

$$\operatorname{Hom}^i_{\mathcal{A}}(\omega, a) = 0 \quad \forall \omega \in \Omega, \forall i \in \mathbb{Z} \Rightarrow a \cong 0,$$
$$\operatorname{Hom}^i_{\mathcal{A}}(a, \omega) = 0 \quad \forall \omega \in \Omega, \forall i \in \mathbb{Z} \Rightarrow a \cong 0.$$

Example 2.5. Let X be a smooth projective variety. Then the set

$$\Omega = \{\mathcal{O}_x \mid x \in X\}$$

is a spanning class for $D^b(X)$.

In the following theorem, spanning classes play an important role.

Theorem 2.6 (Proposition 1.49 in [20])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be triangulated categories and $F\colon \mathcal{A} \to \mathcal{B}$ an exact functor with a left and right adjoint. Suppose that $\mathcal{A}$ has a spanning class Ω. Then F is fully faithful if and only if for all elements $\omega_1, \omega_2 \in \Omega$ and for all $i \in \mathbb{Z}$, the homomorphism*

$$\operatorname{Hom}^i_{\mathcal{A}}(\omega_1, \omega_2) \to \operatorname{Hom}^i_{\mathcal{B}}(F(\omega_1), F(\omega_2))$$

is an isomorphism.

Next, we give a criterion for a fully faithful functor to be an equivalence. Let us start with the definition of a Serre functor.

Definition 2.7. Let $\mathcal{A}$ be a triangulated category and suppose that all Hom's in $\mathcal{A}$ are finite dimensional, i.e. for any $a, b \in \mathcal{A}$, $\sum_i \dim \operatorname{Hom}^i(a, b) < \infty$ holds. An autoequivalence

$$\mathcal{S}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}$$

is said to be a *Serre functor* if for any $a, b \in \mathcal{A}$, there exists an isomorphism

$$\operatorname{Hom}_{\mathcal{A}}(a, b) \cong \operatorname{Hom}_{\mathcal{A}}(b, \mathcal{S}(a))^{\vee}$$

which is functorial in each a and b.

Lemma 2.8 (Lemma 1.30 and Remark 1.31 in [20])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be triangulated categories with finite dimensional* Hom*'s and $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor. Suppose that both $\mathcal{A}$ and $\mathcal{B}$ have Serre functors $\mathcal{S}_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$ respectively.*

(1) *The functor F admits a left adjoint functor if and only if F admits a right adjoint functor.*

(2) *If F is an equivalence, then it commutes with Serre functors, i.e.*

$$F \circ \mathcal{S}_{\mathcal{A}} \cong \mathcal{S}_{\mathcal{B}} \circ F$$

holds.

In particular, applying Lemma 2.8 (2) to the identity functor, we see that a Serre functor is unique if it exists.

Here we give some remarks on indecomposable triangulated categories.

Definition 2.9. We say that a triangulated category $\mathcal{A}$ is decomposed into triangulated subcategories $\mathcal{A}_1$ and $\mathcal{A}_2$ if the following three conditions hold:

(1) Both categories $\mathcal{A}_1$ and $\mathcal{A}_2$ are non-trivial.

(2) For all $a \in \mathcal{A}$, there exists a distinguished triangle

$$b_1 \to a \to b_2 \to b_1[1]$$

with $b_i \in \mathcal{A}_i$ for $i = 1, 2$.

(3) $\mathrm{Hom}_{\mathcal{A}}(b_1, b_2) = \mathrm{Hom}_{\mathcal{A}}(b_2, b_1) = 0$ for all $b_i \in \mathcal{A}_i$ for $i = 1, 2$.

A triangulated category $\mathcal{A}$ is said to be *indecomposable* if it cannot be decomposed.

Proposition 2.10 (Proposition 3.10 in [20])**.** *For a Noetherian scheme X, $D^b(X)$ is indecomposable if and only if X is connected.*

The following criterion is useful.

Theorem 2.11 (Corollary 1.56 in [20])**.** *Let $F \colon \mathcal{A} \to \mathcal{B}$ be an exact functor with left adjoint and right adjoint. Assume that $\mathcal{A}$ is non-trivial, Ω is a spanning class of $\mathcal{A}$, $\mathcal{B}$ is indecomposable and F is fully faithful. If for any $\omega \in \Omega$, there exists an isomorphism*

$$F \circ \mathcal{S}_{\mathcal{A}}(\omega) \cong \mathcal{S}_{\mathcal{B}} \circ F(\omega)$$

then F is an equivalence.

2.3 Fourier–Mukai functors

First, we note a generalization of Serre duality, called Grothendieck–Verdier duality.

Let X and Y be smooth projective varieties and $f\colon X \to Y$ be a morphism. We define the *relative dualizing bundle*

$$\omega_f := \omega_X \otimes f^*\omega_Y^{-1}$$

where ω_X and ω_Y are the canonical bundle of X and Y respectively.

The following theorem is a special version of Grothendieck–Verdier duality. For more general version, see [16].

Theorem 2.12. *For any $\mathcal{E} \in D^b(X)$ and $\mathcal{F} \in D^b(Y)$, there exists a bifunctorial isomorphism*

$$\mathbb{R}f_* \, \mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathbb{L}f^*(\mathcal{F}) \otimes \omega_f[\dim X - \dim Y]) \cong \mathbb{R}\mathcal{H}om_Y(\mathbb{R}f_*\mathcal{E}, \mathcal{F}). \quad (12)$$

Corollary 2.13. *Let X be a smooth projective variety. Then*

$$\mathcal{S}_X(\text{-}) := (\text{-}) \otimes \omega_X[\dim X] : D^b(X) \to D^b(X)$$

is a Serre functor.

Proof. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be objects in $D^b(X)$. We write $\mathcal{E}_2^\vee := \mathbb{R}\mathcal{H}om_X(\mathcal{E}_2, \mathcal{O}_X)$ for the derived dual of $\mathcal{E}_2$. Applying Theorem 2.12 to

$$f : X \to \operatorname{Spec} k,\ \mathcal{E} := \mathcal{E}_1 \overset{\mathbb{L}}{\otimes} \mathcal{E}_2^\vee \text{ and } \mathcal{F} = k,$$

we get an isomorphism

$$\mathbb{R}\operatorname{Hom}_X(\mathcal{E}_1, \mathcal{E}_2 \otimes \omega_X[\dim X]) \cong \mathbb{R}\operatorname{Hom}_X(\mathcal{E}_2, \mathcal{E}_1)^\vee.$$

Taking 0-th cohomology, we get the assertion. □

Let us define

$$f^! \colon D^b(Y) \to D^b(X), \quad \mathcal{E} \mapsto \mathbb{L}f^*(\mathcal{E}) \otimes \omega_f[\dim X - \dim Y]. \quad (13)$$

Corollary 2.14. *The functor $f^!$ is right adjoint to $\mathbb{R}f_*$.*

Proof. Applying $\mathbb{R}\Gamma$ to both sides of (12) we get an isomorphism

$$\mathbb{R}\operatorname{Hom}_X(\mathcal{E}, \mathbb{L}f^*(\mathcal{F}) \otimes \omega_f[\dim X - \dim Y]) \cong \mathbb{R}\operatorname{Hom}_Y(\mathbb{R}f_*\mathcal{E}, \mathcal{F}).$$

We get the conclusion by taking cohomology in degree zero. $\square$

We know that the functor $\mathbb{R}f_*$ is right adjoint to $\mathbb{L}f^*$. In conclusion we get

$$\mathbb{L}f^* \dashv \mathbb{R}f_* \dashv f^!.$$

Now we introduce a fundamental notion. For smooth projective varieties X and Y, we denote by π_X and π_Y the projections of $X \times Y$ to X and to Y, respectively. For $\mathcal{P} \in D^b(X \times Y)$, we define a *Fourier–Mukai functor* $\Phi^{\mathcal{P}}$ as follows:

$$\Phi^{\mathcal{P}}(\text{-}) := \mathbb{R}\pi_{Y*}(\mathcal{P} \overset{\mathbb{L}}{\otimes} \pi_X^*(\text{-})). \tag{14}$$

By the definition, we immediately see that a Fourier–Mukai functor is exact. If a Fourier–Mukai functor gives an equivalence, we say that it is a *Fourier–Mukai transform.*

Let $\mathcal{E}$ and $\mathcal{F}$ be objects of $D^b(X)$. For an integer i, we define

$$\operatorname{Hom}^i_{D^b(X)}(\mathcal{E}, \mathcal{F}) := \operatorname{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{F}[i]).$$

If $\mathcal{E}$ and $\mathcal{F}$ are sheaves, then these spaces are just the Ext-groups, i.e.

$$\operatorname{Hom}^i_{D^b(X)}(\mathcal{E}, \mathcal{F}) = \operatorname{Ext}^i_X(\mathcal{E}, \mathcal{F}).$$

We note that for any equivalence $\Phi\colon D^b(X) \to D^b(Y)$, we get

$$\operatorname{Hom}^i_{D^b(Y)}(\Phi(\mathcal{E}), \Phi(\mathcal{F})) = \operatorname{Hom}^i_{D^b(X)}(\mathcal{E}, \mathcal{F}) \tag{15}$$

since Φ commutes with the translation functors. From (15), we get

$$\chi(\Phi(\mathcal{E}), \Phi(\mathcal{F})) = \chi(\mathcal{E}, \mathcal{F}),$$

where $\chi(\mathcal{E}, \mathcal{F})$ is the relative Euler characteristic

$$\chi(\mathcal{E}, \mathcal{F}) := \sum_i (-1)^i \dim \operatorname{Hom}^i_{D^b(X)}(\mathcal{E}, \mathcal{F}).$$

By Riemann–Roch theorem, the relative Euler characteristic is given in terms of the Chern characters of $\mathcal{E}$ and $\mathcal{F}$. In particular, if X is a surface, we get

$$\begin{aligned}\chi(\mathcal{E}, \mathcal{F}) =& r(\mathcal{E})ch_2(\mathcal{F}) - c_1(\mathcal{E}) \cdot c_1(\mathcal{F}) + r(\mathcal{F})ch_2(\mathcal{E}) \\ &+ \frac{1}{2}(r(\mathcal{F})c_1(\mathcal{E}) - r(\mathcal{E})c_1(\mathcal{F})) \cdot K_X + r(\mathcal{E})r(\mathcal{F})\chi(\mathcal{O}_X),\end{aligned}$$

where K_X is the first Chern class of the canonical bundle ω_X.

2.4 Criterion of fully faithfulness

Let X and Y be smooth projective varieties and $\Phi^{\mathcal{P}} \colon D^b(X) \to D^b(Y)$ be a Fourier–Mukai functor. The aim of this subsection is to give a criterion for $\Phi^{\mathcal{P}}$ to be fully faithful.

Lemma 2.15 ([29])**.** *For any $\mathcal{P} \in D^b(X \times Y)$, we let*

$$\mathcal{P}_L := \mathcal{P}^\vee \otimes \pi_Y^* \omega_Y[\dim Y] \quad \text{and} \quad \mathcal{P}_R := \mathcal{P}^\vee \otimes \pi_X^* \omega_X[\dim X],$$

where $\mathcal{P}^\vee = \mathbb{R}\mathcal{H}om(\mathcal{P}, \mathcal{O}_{X\times Y})$, the derived dual of $\mathcal{P}$. Then the induced Fourier–Mukai functors

$$\Phi^{\mathcal{P}_L} \colon D^b(Y) \to D^b(X) \quad \text{and} \quad \Phi^{\mathcal{P}_R} \colon D^b(Y) \to D^b(X)$$

are left, respectively right adjoint to $\Phi^{\mathcal{P}}$.

Proof. The proof is a straightforward application of Corollary 2.14. For any $\mathcal{E} \in D^b(X)$ and $\mathcal{F} \in D^b(Y)$, we get a sequence of functorial isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{D^b(X)}(\Phi^{\mathcal{P}_L}(\mathcal{F}), \mathcal{E}) &\cong \mathrm{Hom}_{D^b(X)}(\mathbb{R}\pi_{X*}(\mathcal{P}_L \overset{\mathbb{L}}{\otimes} \pi_Y^*\mathcal{F}), \mathcal{E}) \\
&\cong \mathrm{Hom}_{D^b(X\times Y)}(\mathcal{P}_L \overset{\mathbb{L}}{\otimes} \pi_Y^*\mathcal{F}, \pi_X^*\mathcal{E} \overset{\mathbb{L}}{\otimes} \pi_Y^*\omega_Y[\dim Y]) \\
&\cong \mathrm{Hom}_{D^b(X\times Y)}(\mathcal{P}^\vee \overset{\mathbb{L}}{\otimes} \pi_Y^*\mathcal{F}, \pi_X^*\mathcal{E}) \\
&\cong \mathrm{Hom}_{D^b(X\times Y)}(\pi_Y^*\mathcal{F}, \mathcal{P} \overset{\mathbb{L}}{\otimes} \pi_X^*\mathcal{E}) \\
&\cong \mathrm{Hom}_{D^b(Y)}(\mathcal{F}, \mathbb{R}\pi_{Y*}(\mathcal{P} \overset{\mathbb{L}}{\otimes} \pi_X^*\mathcal{E})) \\
&\cong \mathrm{Hom}_{D^b(Y)}(\mathcal{F}, \Phi^{\mathcal{P}}(\mathcal{E})).
\end{aligned}$$

This shows $\Phi^{\mathcal{P}_L} \dashv \Phi^{\mathcal{P}}$. For $\Phi^{\mathcal{P}} \dashv \Phi^{\mathcal{P}_R}$, the proof is similar. □

The following theorem is essential.

Theorem 2.16 ([30])**.** *Let F be an exact functor from $D^b(X)$ to $D^b(Y)$ where X and Y are smooth projective varieties. Assume that F is fully faithful and has the right or left adjoint functor. Then there exists an object $\mathcal{P} \in D^b(X \times Y)$ such that F is isomorphic to the Fourier–Mukai functor $\Phi^{\mathcal{P}}$ defined in* (14)*, and the object $\mathcal{P}$ is unique up to isomorphism.*

Here, we give a criterion for a Fourier–Mukai functor to be fully faithful. The following theorem is proved in [5, 11] in the case $k = \mathbb{C}$. Most of the proof is similar to the one in [5, 11]. See also Remark 2.18.

Theorem 2.17. (1) *The functor $\Phi^{\mathcal{P}} \colon D^b(X) \to D^b(Y)$ is fully faithful if and only if the following two conditions holds:*

(1-a) For any points $x, y \in X$,

$$\mathrm{Hom}^i_{D^b(Y)}(\Phi^{\mathcal{P}}(\mathcal{O}_x), \Phi^{\mathcal{P}}(\mathcal{O}_y)) = \begin{cases} k & \text{if } x = y \text{ and } i = 0 \\ 0 & \text{if } x \neq y \text{ or } i \notin [0, \dim X] \end{cases}$$

and

(1-b) the homomorphism

$$\mathrm{Hom}^1_{D^b(X)}(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Hom}^1_{D^b(Y)}(\Phi^{\mathcal{P}}(\mathcal{O}_x), \Phi^{\mathcal{P}}(\mathcal{O}_x))$$

is injective.

(2) *Suppose that $\Phi^{\mathcal{P}}$ is fully faithful. Then $\Phi^{\mathcal{P}}$ is an equivalence if and only if for any point $x \in X$,*

$$\Phi^{\mathcal{P}}(\mathcal{O}_x) \otimes \omega_Y \cong \Phi^{\mathcal{P}}(\mathcal{O}_x)$$

holds.

Proof. (1) If the functor $\Phi^{\mathcal{P}}$ is fully faithful, the conditions (1-a) and (1-b) hold obviously.

For the opposite direction, note that $\{\mathcal{O}_x \mid x \in X\}$ given in Example 2.5 is a spanning class for $D^b(X)$. By Theorem 2.6, it is enough to show that for any point $x \in X$ and for all $i \in \mathbb{Z}$, the homomorphism

$$\mathrm{Hom}^i_{D^b(X)}(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Hom}^i_{D^b(Y)}(\Phi^{\mathcal{P}}(\mathcal{O}_x), \Phi^{\mathcal{P}}(\mathcal{O}_x))$$

is an isomorphism.

We shall use the following commutative diagram given in (11) associated to the adjointness $\Phi^{\mathcal{P}_L} \dashv \Phi^{\mathcal{P}}$.

$$\begin{array}{ccc} \mathrm{Hom}^i_{D^b(X)}(\mathcal{O}_x, \mathcal{O}_x) & \xrightarrow{\Phi^{\mathcal{P}}} & \\ \downarrow g_{\mathcal{O}_x} & & \searrow \\ \mathrm{Hom}^i_{D^b(X)}(\Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x), \mathcal{O}_x) & \xrightarrow[\cong]{} & \mathrm{Hom}^i_{D^b(Y)}(\Phi^{\mathcal{P}}(\mathcal{O}_x), \Phi^{\mathcal{P}}(\mathcal{O}_x)) \end{array}$$

Therefore, the desired conclusion is equivalent to that the vertical morphism is an isomorphism. So it is enough to show that $\Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x) \cong \mathcal{O}_x$ via the natural transformation g.

First we check that $\Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x)$ is a sheaf supported at the point x. For any point $z \overset{\iota_z}{\hookrightarrow} X$, there are isomorphisms of vector spaces

$$\begin{aligned}\mathbb{L}_i\iota_z^*(\Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x))^\vee &\cong \mathrm{Hom}^i_{D^b(X)}(\Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x), \mathcal{O}_z) \\ &\cong \mathrm{Hom}^i_{D^b(Y)}(\Phi^{\mathcal{P}}(\mathcal{O}_x), \Phi^{\mathcal{P}}(\mathcal{O}_z))\end{aligned}$$

coming from the adjunctions $\mathbb{L}\iota_z^* \dashv \mathbb{R}\iota_{z*}$. Here, $\mathbb{L}_i\iota_z^*$ is the i-th cohomology of $\mathbb{L}\iota_z^*$. Hence we see that $\mathbb{L}_i\iota_z^*(\Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x)) = 0$ for $z \neq x$ or $i \notin [0, \dim X]$ by the assumption. Thus by [5, Proposition 1.5], $\Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x)$ is a sheaf supported at the point x. Furthermore, for the adjunction morphism $\Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x) \to \mathcal{O}_x$, let K be the kernel of its morphism, then we get a short exact sequence

$$0 \to K \to \Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x) \to \mathcal{O}_x \to 0 \tag{16}$$

in $\mathrm{Coh}(X)$.

Let us show that $K = 0$. Consider the natural transformation $h \colon \mathrm{id}_{D^b(Y)} \to \Phi^{\mathcal{P}} \circ \Phi^{\mathcal{P}_L}$ and $g \colon \Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}} \to \mathrm{id}_{D^b(X)}$ defined in (6) and (7), respectively. The composition

$$\Phi^{\mathcal{P}}(\mathcal{O}_x) \xrightarrow{h_{\Phi^{\mathcal{P}}}} \Phi^{\mathcal{P}} \circ \Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x) \xrightarrow{\Phi^{\mathcal{P}}(g)} \Phi^{\mathcal{P}}(\mathcal{O}_x)$$

yields the identity by (8). Since $\Phi^{\mathcal{P}}(\mathcal{O}_x) \neq 0$ by the assumption, we see that $\Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x) \to \mathcal{O}_x$ is not zero.

Applying the functor $\mathrm{Hom}_X(\text{-}, \mathcal{O}_x)$ to (16), we get $\mathrm{Hom}_X(K, \mathcal{O}_x) = 0$ by the assumption (1-b) and $\mathrm{Ext}^i_X(K, \mathcal{O}_x) = 0$ for $i \neq 0$ since $\Phi^{\mathcal{P}_L} \circ \Phi^{\mathcal{P}}(\mathcal{O}_x)$ and $\mathcal{O}_x$ are sheaves supported at a point. Therefore, we get $K = 0$, which completes the proof.

(2) The proof of [11, Theorem 5.4] works for arbitrary algebraically closed fields.

□

Remark 2.18. There does not exist the condition (1-b) in [5, 11] because in the case $\mathrm{ch}(k) = 0$, the condition (1-b) holds automatically. See [20, Step 5 in the proof of Proposition 7.1]. On the other hand, in the case $\mathrm{ch}(k) > 0$, there exists a functor satisfying the assumption (1-a) but not fully faithful. Here we describe it given in [18, Remark 1.25].

Let X be a smooth projective variety of dimension d over k with $p > 0$. It is well known that the relative Frobenius morphism $\mathrm{Fr}\colon X \to X^{(p)}$ is topologically a homeomorphism. Let us consider the direct image functor

$$\mathrm{Fr}_*\colon D^b(X) \to D^b(X^{(p)})$$

which can also be described as a Fourier–Mukai functor Φ^{Γ} where $\Gamma \subset X \times X^{(p)}$ is the graph of Fr. Since $\mathrm{Fr}_*\mathcal{O}_x \cong \mathcal{O}_{\mathrm{Fr}(x)}$ for any $x \in X$, we see that Φ^{Γ} satisfies the condition (1-a) in Theorem 2.17. However, $\mathrm{Fr}_*\mathcal{O}_X$ is a locally free of rank p^d. So, we see that

$$\mathrm{Hom}^0_{D^b(X^{(p)})}(\mathrm{Fr}_*\mathcal{O}_X, \mathrm{Fr}_*\mathcal{O}_x) \cong \mathrm{Hom}^0_{D^b(X^{(p)})}(\mathrm{Fr}_*\mathcal{O}_X, \mathcal{O}_{\mathrm{Fr}(x)}) \cong k^{p^d}.$$

Since $\mathrm{Hom}^0_{D^b(X^{(p)})}(\mathcal{O}_X, \mathcal{O}_x) \cong k$, the functor Fr_* is not fully faithful.

Lemma 2.19 ([12])**.** *Let X and Y be surfaces and $\Phi\colon D^b(X) \to D^b(Y)$ be a Fourier–Mukai transform. For any point $x \in X$, there exists an inequality*

$$\sum_i \dim \mathrm{Ext}^1_Y(H^i\Phi(\mathcal{O}_x), H^i\Phi(\mathcal{O}_x)) \leq 2,$$

and moreover, each of the sheaves $\Phi^i(\mathcal{O}_x)$ satisfies $\Phi^i(\mathcal{O}_x) \otimes \omega_Y \cong \Phi^i(\mathcal{O}_x)$.

Proof. Let us consider the spectral sequence

$$E_2^{p,q} = \bigoplus_i \mathrm{Ext}^p_Y(\Phi^i(\mathcal{O}_x), \Phi^{i+q}(\mathcal{O}_x)) \Rightarrow \mathrm{Hom}^{p+q}_{D^b(Y)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)).$$

Since Y is a surface, the $E_2^{1,0}$ term survives to infinity. On the other hand, by (15) we get

$$\mathrm{Hom}^1_{D^b(Y)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) \cong \mathrm{Hom}^1_{D^b(X)}(\mathcal{O}_x, \mathcal{O}_x) \cong k^2,$$

so, the result follows. □

Recall that a smooth projective variety Y is called a Fourier–Mukai partner of X if $D^b(Y) \cong D^b(X)$ as k-linear triangulated categories. We write $\mathrm{FM}(X)$ for the set of isomorphism classes of Fourier–Mukai partners of X:

$$\mathrm{FM}(X) := \{Y \text{ smooth projective variety} \mid D^b(X) \cong D^b(Y)\}/\cong.$$

It can happen that X does not have any non-trivial Fourier–Mukai partners, i.e. $|\mathrm{FM}(X)| = 1$. For example, let X be a smooth projective variety with ample canonical or anticanonical sheaf. Then $|\mathrm{FM}(X)| = 1$ holds by [6].

On the other hand, there exists a variety which has a non-trivial Fourier–Mukai partners. For example, as mentioned in §1, for an abelian variety A, $\mathrm{FM}(A)$ contains $\hat{A}$. In general $\hat{A}$ is not isomorphic to A, so in this case $|\mathrm{FM}(X)| \neq 1$.

For a study of Fourier–Mukai partners, the following is useful. For a proof, see e.g. [20, Corollary 6.14].

Proposition 2.20. *Let X and Y be smooth projective varieties and $\Phi\colon D^b(X) \to D^b(Y)$ be a Fourier–Mukai transform. Suppose there exists a closed point $x_0 \in X$ such that*

$$\Phi(\mathcal{O}_{x_0}) \cong \mathcal{O}_{y_0}$$

for some $y_0 \in Y$. Then there exists an open neighborhood U of x_0 and a morphism $f\colon U \to Y$ such that $f(x_0) = y_0$ and

$$\Phi(\mathcal{O}_x) \cong \mathcal{O}_{f(x)}$$

for any closed point $x \in U$.

3 Relative moduli spaces of sheaves on elliptic fibrations

3.1 Fourier–Mukai partners of elliptic surfaces

We study the set $\mathrm{FM}(S)$ for elliptic surfaces S. Let $\pi\colon S \to C$ be an elliptic surface and F_π be a general fiber of π. We define

$$\lambda_\pi := \min\{D \cdot F_\pi \mid D \text{ is a horizontal effective divisor on } S\}. \qquad (17)$$

Fix a polarization on S and consider the relative moduli scheme $\mathcal{M}(S/C) \to C$ of purely 1-dimensional stable sheaves[1] on the fibers π, whose existence

[1]Here we consider the Gieseker stability, equivalently the slope stability for 1-dimensional sheaves. Moreover, the stability does not depend on the choice of polarizations for such sheaves.

is assured by Simpson in the case $p = 0$ in [37], and by Langer in the case of arbitrary p in [26]. For integers $a > 0$ and i with i coprime to $a\lambda_\pi$, let $J_S(a, i)$ be the union of those components of $\mathcal{M}(S/C)$ which contains a point representing a rank a, degree i vector bundle on a smooth fiber of π. Bridgeland shows in [10] that $J_S(a, i)$ is actually a smooth projective surface and the natural morphism $J_S(a, i) \to C$ is a minimal elliptic fibration.

Lemma 3.1 (Lemma 4.2 in [12]). *For any integers a and b with b coprime to $a\lambda_\pi$, there is an isomorphism*

$$J_S(a, b) \cong J^b(S).$$

Put $J^i(S) := J_S(1, i)$. We can also define an elliptic surface $J^j(S) \to C$ for arbitrary $j \in \mathbb{Z}$, which is not necessarily fine but the coarse moduli space of a suitable functor (see [21, §11.4]). We have $J^0(S) \cong J(S)$, the Jacobian surface associated to S, $J^1(S) \cong S$ and

$$J^i(J^j(S)) \cong J^{ij}(S) \tag{18}$$

for $i, j \in \mathbb{Z}$. See the argument after (23) for the proof of (18).

It is well known that the following statement holds in the case $p = 0$ by [10, Theorem 1.2]. We state that it is also true for arbitrary p.

Proposition 3.2. *Elliptic surfaces S and $J^i(S)$ for some integer i with $(i, \lambda_\pi) = 1$ are derived equivalent via a Fourier–Mukai functor*

$$\Phi^{\mathcal{P}} \colon D^b(J^i(S)) \to D^b(S)$$

for a universal sheaf $\mathcal{P}$ on $J^i(S) \times S$.

Proof. First note that the coprimary assumption implies that $J^i(S)$ is a fine moduli space. Let us check the conditions (1-a), (1-b) and (2) in Theorem 2.17. The conditions (1-a) and (2) can be checked same as the proof in [10].

For (1-b), we will consider the map

$$\mathrm{Ext}^1_{J^i(S)}(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Ext}^1_S(\mathcal{P}_x, \mathcal{P}_x),$$

which is the Kodaira–Spencer map. Actually, the map is an isomorphism in our case because $\mathcal{P}$ is a universal family. This completes the proof. □

We have a nice characterization of Fourier–Mukai partners of elliptic surfaces with non-zero Kodaira dimensions.

Theorem 3.3. *Let $\pi\colon S \to C$ be an elliptic surface and T a smooth projective variety. Assume that the Kodaira dimension $\kappa(S)$ is non-zero. Then the following are equivalent.*

(i) T is a Fourier–Mukai partner of S.

(ii) T is isomorphic to $J^i(S)$ for some integer i with $(i, \lambda_\pi) = 1$.

Proof. It follows from Proposition 3.2 that (ii) implies (i). The opposite direction was proved in [12, Proposition 4.4] when $p = 0$ and S has no (-1)-curves. The most of proof there works even for $p > 0$.

Let $\Phi\colon D^b(T) \to D^b(S)$ be an equivalence. Take a point $x \in S$ lying on a smooth fiber F_x of π and take $t \in T$ such that the support of $\mathcal{E} := \Phi(\mathcal{O}_t)$ contains x. Since

$$\mathrm{Hom}_{D^b(S)}(\mathcal{E}, \mathcal{E}) \cong \mathrm{Hom}_{D^b(T)}(\mathcal{O}_t, \mathcal{O}_t) \cong k,$$

the support of $\mathcal{E}$ is connected, hence either $\mathrm{Supp}\,\mathcal{E} = F_x$ or $\mathrm{Supp}\,\mathcal{E}$ consists of a single closed point.

If $\mathcal{E}$ is a vector bundle on F_x, its Chern class is of the form $(0, aF_\pi, b)$ for some integers a and b. Now we know that

$$\chi(\mathcal{E}, \Phi(\mathcal{O}_T)) = \chi(\mathcal{O}_t, \mathcal{O}_T) = 1.$$

Using Riemann–Roch theorem we see that $a\lambda_\pi$ is coprime to b by the definition of λ_π. Since $\mathcal{E}$ is supported on an elliptic curve,

$$\mathrm{Ext}^1_S(\mathcal{H}^i(\mathcal{E}), \mathcal{H}^i(\mathcal{E})) \neq 0$$

for any $i \in \mathbb{Z}$. Hence Lemma 2.19 induces that $\mathcal{E}$ has only one non-zero cohomology sheaf. Thus, we see that $\mathcal{E}$ is a shift of a simple sheaf. In particular, $\mathcal{E}$ is stable.

Let us consider the equivalence

$$\Phi^{\mathcal{P}}\colon D^b(J_S(a,b)) \to D^b(S)$$

induced by the universal family $\mathcal{P}$. It takes $\mathcal{O}_z$ to E for some point $z \in J_S(a,b)$. Hence the composition

$$\Psi := \Phi^{-1} \circ \Phi^{\mathcal{P}}\colon D^b(J_S(a,b)) \to D^b(T)$$

satisfies $\Psi(\mathcal{O}_z) = \mathcal{O}_t$. By Proposition 2.20, there is a rational map $f\colon T \dashrightarrow J^i(S)$ such that $\Psi(\mathcal{O}_z) \cong \mathcal{O}_{f(z)}$ for any z in an open subset of T. Because $\Phi^{\mathcal{P}}$ is an equivalence, we can avoid the possibility that f is purely inseparable, and hence f is a birational map. Then the proof of [12, Proposition 4.4] works in the rest (including the case that S is not minimal). Finally, Lemma 3.1 gives the conclusion of this case.

If $\mathcal{E}$ is concentrated in a single point $z \in S$, then $\Phi(\mathcal{O}_t) = \mathcal{O}_z$. Then we get a birational map between $J^1(S)$ and T. So, the above discussion gives the conclusion.

□

As a consequence of Theorem 3.3, we obtain

$$\mathrm{FM}(S) = \{J^i(S) \mid i \in \mathbb{Z},\ (i, \lambda_\pi) = 1\}/\cong .$$

Moreover, we see that there exist natural isomorphisms

$$J^i(S) \cong J^{i+\lambda_\pi}(S) \cong J^{-i}(S). \tag{19}$$

Hence, in order to count the cardinality of the set $\mathrm{FM}(S)$, we often regard an integer i as an element of the unit group $(\mathbb{Z}/\lambda_\pi\mathbb{Z})^*$. It follows from the isomorphisms (18) and (19) that the set

$$H_\pi := \{i \in (\mathbb{Z}/\lambda_\pi\mathbb{Z})^* \mid J^i(S) \cong S\} \tag{20}$$

forms a subgroup of $(\mathbb{Z}/\lambda_\pi\mathbb{Z})^*$. Moreover, we see from (18) that $J^i(S) \cong J^j(S)$ for $i, j \in (\mathbb{Z}/\lambda_\pi\mathbb{Z})^*$ if and only if $(S \cong) J^1(S) \cong J^{i^{-1}j}(S)$. Combining all together, we have the following.

Lemma 3.4. *For an elliptic surface $\pi\colon S \to C$ with $\kappa(S) \neq 0$, the set $\mathrm{FM}(S)$ is naturally identified with the group $(\mathbb{Z}/\lambda_\pi\mathbb{Z})^*/H_\pi$.*

Since H_π contains the subgroup $\{\pm 1\}$ if $\lambda_\pi \geq 3$, we see

$$|\,\mathrm{FM}(S)| \leq \varphi(\lambda_\pi)/2, \tag{21}$$

where φ is the Euler function.

3.2 Weil–Châtelet group

In this subsection, we recall the definition of the Weil–Châtelet group. For more details, see [36, Ch.X.3] and [21, Ch.11.5].

Definition 3.5. Let E_0 be an elliptic curve over a field K. A *homogeneous space* for E_0 is a pair (E, μ) where E is a smooth curve over K and

$$\mu\colon E \times E_0 \to E$$

satisfies the following properties:

(1) $\mu(p, O) = p \quad$ for all $p \in E$.

(2) $\mu(\mu(p, P), Q) = \mu(p, P + Q) \quad$ for all $p \in E$ and $P, Q \in E_0$.

(3) For all $p, q \in E$, there is a unique $P \in E_0$ satisfying $\mu(p, P) = q$.

We say two homogeneous space (E, μ) and (E', μ') are *equivalent* if there exists an isomorphism $\theta\colon E \to E'$ defined over K which is compatible with the action of E_0, i.e. the following diagram is commutative:

$$\begin{array}{ccc} E \times E_0 & \xrightarrow{\theta \times \mathrm{id}_{E_0}} & E' \times E_0 \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu'} \\ E & \xrightarrow[\theta]{} & E' \end{array}$$

The collection $WC(E_0)$ of equivalence classes of homogeneous spaces for E_0 has a natural group structure (cf. [36, Theorem X.3.6], [21, Proposition 11.5.1]), and it is called the *Weil–Châtelet group*.

Note that the minus element of (E, μ) is given by $(E, \mu \circ (\mathrm{id} \times (-1)))$.

Proposition 3.6 (Proposition X.3.3 in [36])**.** *Let (E, μ) be a homogeneous space for E_0. Then (E, μ) is in the trivial class if and only if $E(K)$ is not empty.*

Let $\pi\colon S \to C$ be an elliptic surface (over an algebraically closed field k) and J^i_η denote the generic fiber of $\pi_i\colon J^i(S) \to C$ for $i \in \mathbb{Z}$. Then J^0_η is an elliptic curve over the function field of C, and we have a natural homogeneous space structure

$$\mu_i\colon J^i_\eta \times J^0_\eta \to J^i_\eta \quad (\mathcal{L}, \mathcal{M}) \mapsto \mathcal{L} \otimes \mathcal{M},$$

and hence we can regard $(J_\eta^i, \mu_i) \in WC(J_\eta^0)$. We define

$$\xi := (J_\eta^1, \mu_1) \in WC(J_\eta^0), \tag{22}$$

then, we have

$$i\xi = (J_\eta^i, \mu_i) \tag{23}$$

(cf. [21, Remark 11.5.2]) and thus

$$\operatorname{ord} \xi \mid \lambda_\pi. \tag{24}$$

It follows from (23) that the generic fibers of $J^i(J^j(S)) \to C$ and $J^{ij}(S) \to C$ are isomorphic to each other, and taking the relative smooth minimal models of compactifications of generic fibers, we obtain $J^i(J^j(S)) \cong J^{ij}(S)$ as in (18).

Take a closed point $x \in C$ and consider the henselization of the local ring $\mathcal{O}_{C,x}$ and denote it by $\mathcal{O}_{C,x}^h$. We also denote the base change of $\pi_0\colon J^0(S) \to C$ by the morphism $\mathrm{Spec}\mathcal{O}_{C,x}^h \to C$ by

$$J_x^0 \to \mathrm{Spec}\mathcal{O}_{C,x}^h.$$

Then it is known by [13, Proposition 5.4.3 in p.314, Theorem 5.4.3 in p.321] that there exists an exact sequence:

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Br}(J^0(S)) & \to & WC(J_\eta^0) & \to & \bigoplus_{x\in C} WC(J_x^0) \\ & & & & \cup & & \cup \\ & & & & \xi & \mapsto & (\xi_x)_{x\in C} \end{array} \tag{25}$$

Here, we denote the image of ξ (given in (22)) in $WC(J_x^0)$ by ξ_x. It follows from [13, Proposition 5.4.2] that $m_x = \operatorname{ord} \xi_x$, where m_x is the multiplicity of the fiber of π over the point $x \in C$. Define

$$\lambda'_\pi := \mathrm{l.c.m.}_{x\in C}(m_x) = \operatorname{ord}((\xi_x)_{x\in C}). \tag{26}$$

Since $\operatorname{ord} \xi$ is divided by $\operatorname{ord}((\xi_x)_{x\in C})$, we see from (24) that

$$\lambda'_\pi \mid \lambda_\pi.$$

In particular, if $i \in \mathbb{Z}$ is coprime to λ_π, then i is coprime to each m_x, and thus we have

$$\operatorname{ord}(i\xi)_x = \operatorname{ord} i(\xi_x) = \operatorname{ord}(\xi_x) = m_x. \tag{27}$$

Lemma 3.7. *Let $\pi\colon S \to C$ be an elliptic surface. Then we have the following statements.*

(1) *For $i \in \mathbb{Z}$ with $(i, \lambda_\pi) = 1$, consider the elliptic fibration $\pi_i\colon J^i(S) \to C$. The multiplicities of the fibers F_x and F'_x of π and π_i over a fixed point $x \in C$ coincide. Furthermore, if the fiber F_x is smooth, then it is isomorphic to F'_x.*

(2) *Let S be an elliptic ruled surface, and take $S' \in \mathrm{FM}(S)$. Then S' is also an elliptic ruled surface with an elliptic fibration.*

Proof. (i) Combining (27) with (23), we know that the multiplicity of the fiber of π_i over the point x is also m_x. This shows the first statement. By the property of the relative moduli scheme, the fiber F'_x is the fine moduli space of line bundles of degree i on a smooth elliptic curve F_x. Consequently, the second statement follows.

(ii) Theorem 3.3 implies that there exists an integer i with $(i, \lambda_\pi) = 1$ such that $J^i(S) \cong S'$, which implies that S' has an elliptic fibration π'. The Kodaira dimension is derived invariant by [38, Corollary 4.4], hence S' is a rational elliptic surface or an elliptic ruled surface. Then, [19, Theorem B] implies that S' is also an elliptic ruled surface. □

Define a subgroup H'_π of the group $H_\pi (:= \{i \in (\mathbb{Z}/\lambda_\pi\mathbb{Z})^* \mid J^i(S) \cong S\}$ given in (20)) to be

$$H'_\pi := \{i \in H_\pi \mid i \equiv 1 \pmod{\lambda'_\pi}\}. \tag{28}$$

We use the following lemma to obtain a lower bound of the cardinality of the set $\mathrm{FM}(S)$.

Lemma 3.8. *Let $\pi\colon S \to C$ be an elliptic surface with $\mathrm{Br}(J^0(S)) = 0$. Then we have*

$$\left|H_\pi/H'_\pi\right| \leq \left|\mathrm{Aut}_0(J^0_\eta)\right|.$$

Proof. For each $i \in H_\pi$, fix an isomorphism $\theta_i\colon J^1_\eta \to J^i_\eta$ over the generic point $\eta \in C$. Then we obtain a structure of a homogeneous space on J^1_η by the action

$$\mu'_i := \theta_i^{-1} \circ \mu_i \circ (\theta_i \times \mathrm{id}_{J^0_\eta})\colon J^1_\eta \times J^0_\eta \to J^1_\eta$$

such that $(J_\eta^i, \mu_i) = (J_\eta^1, \mu_i')$ holds in $WC(J_\eta^0)$ by the definition. On the other hand, by [36, Exercise 10.4], $(J_\eta^1, \mu_i') = (J_\eta^1, \mu_1 \circ (\mathrm{id}_{J_\eta^1} \times \phi))$ for some $\phi \in \mathrm{Aut}_0(J_\eta^0)$. We define an equivalence relation $\sim$ of $\mathrm{Aut}_0(J_\eta^0)$ such that

$$\phi_1 \sim \phi_2$$

for $\phi_i \in \mathrm{Aut}_0(J_\eta^0)$ when

$$(J_\eta^1, \mu_1 \circ (\mathrm{id}_{J_\eta^1} \times \phi_1)) = (J_\eta^1, \mu_1 \circ (\mathrm{id}_{J_\eta^1} \times \phi_2)).$$

Then we can define a map

$$f \colon H_\pi \to \mathrm{Aut}_0(J_\eta^0)/\sim \quad i \mapsto \phi.$$

We see that $ij^{-1} \in H'_\pi$ if and only if $f(i) = f(j)$ as follows. First note that we have an injection

$$WC(J_\eta^0) \hookrightarrow \bigoplus_{x \in C} WC(J_x^0) \qquad \xi = (J_\eta^1, \mu_1) \mapsto (\xi_x)_{x \in C}$$

by the vanishing of the Brauer group $\mathrm{Br}(J^0(S))$ and (25), and hence

$$\mathrm{ord}\,\xi = \lambda'_\pi (:= \mathrm{ord}((\xi_x)_{x \in C})). \tag{29}$$

We observe that for $i, j \in H_\pi$, the condition $f(i) = f(j)$ is equivalent to the equality $i\xi = j\xi$ by (23), which is also equivalent to $i^{-1}j \in H'_\pi$ by (29).

Consequently, we obtain an inclusion

$$H_\pi / H'_\pi \hookrightarrow \mathrm{Aut}_0(J_\eta^0)/\sim$$

and the conclusion. □

4 Elliptic curves and automorphisms

Let F be an elliptic curve over an algebraically closed field k with $p = \mathrm{ch}\, k \geq 0$. The explicit description of the automorphism group $\mathrm{Aut}_0(F)$ fixing the origin O is well-known, and is given as follows.

Theorem 4.1 (cf. Appendix A in [36]). *The automorphism group* $\mathrm{Aut}_0(F)$ *is*

$$
\begin{array}{ll}
\mathbb{Z}/2\mathbb{Z} & \text{if } j(F) \neq 0, 1728, \\
\mathbb{Z}/4\mathbb{Z} & \text{if } j(F) = 1728 \text{ and } p \neq 2, 3, \\
\mathbb{Z}/6\mathbb{Z} & \text{if } j(F) = 0 \text{ and } p \neq 2, 3, \\
\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z} & \text{if } j(F) = 0 = 1728 \text{ and } p = 3, \\
Q \rtimes \mathbb{Z}/3\mathbb{Z} & \text{if } j(F) = 0 = 1728 \text{ and } p = 2.
\end{array}
$$

Note that in the last second case, $\mathbb{Z}/4\mathbb{Z}$ *acts on* $\mathbb{Z}/3\mathbb{Z}$ *in the unique non-trivial way, and in the last case, the group is so called a binary tetrahedral group, and* Q *is the quaternion group. In the last two cases* F *is necessarily supersingular.*

For points $x_1, x_2 \in F$, to distinguish the summation of divisors and of elements in the group scheme F, we write $x_1 \oplus x_2$ for the sum of them by the operation of F, and

$$i \cdot x_1 := x_1 \oplus \cdots \oplus x_1 \quad (i \text{ times}).$$

Furthermore, we use the symbol T_a to stand for the translation by $a \in F$:

$$T_a \colon F \to F \quad x \to a \oplus x.$$

We also write

$$ix_1 := x_1 + \cdots + x_1 \quad (i \text{ times})$$

for the divisors on F of degree i. We denote the dual abelian variety $\mathrm{Pic}^0 F$ of F by $\hat{F}$. It is well-known that there exists a group scheme isomorphism

$$F \to \hat{F} \quad x \mapsto \mathcal{O}_F(x - O), \tag{30}$$

where O is the origin of F.

We will use the following lemma several times.

Lemma 4.2. *Take a point* $a \in F$ *with* $\mathrm{ord}(a) \geq 4$*, and* $\phi \in \mathrm{Aut}_0(F)$*. If* $\phi(a) = a$*, then* $\phi = \mathrm{id}_F$*.*

Proof. In any of the cases in Theorem 4.1, we have $\mathrm{ord}(\phi) \in \{1, 2, 3, 4, 6\}$. Let us first consider the case $\mathrm{ord}(\phi) = 2, 4$ or 6. In this case, $\phi^i = -\mathrm{id}_F$ for some $i \in \mathbb{Z}$, and hence we get $-1 \cdot a = a$. The condition $\operatorname{ord}(a) \geq 4$ yields a contradiction. Next, consider the case $\operatorname{ord}(\phi) = 3$. Then we have

$$(\phi - \mathrm{id}_F)(\phi^2 + \phi + \mathrm{id}_F) = 0$$

in the domain $\operatorname{End}(F)$, which implies that $\phi^2 + \phi + \mathrm{id}_F = 0$, and hence $\phi^2(a) \oplus \phi(a) \oplus a = O$. By the assumption $\phi(a) = a$, we see that $3 \cdot a = O$. This is absurd by $\operatorname{ord}(a) \geq 4$. □

For a non-zero integer m, we define the m-torsion subgroup of F to be

$$F[m] := \{a \in F \mid m \cdot a = O\}.$$

Equivalently, $F[m]$ is the kernel of the multiplication map by m. Recall that

$$F[m] = \begin{cases} \mathbb{Z}/p^e\mathbb{Z} & \text{if } F \text{ is ordinary, } m = p^e,\ e > 0 \\ \{O\} & \text{if } F \text{ is supersingular, } m = p^e,\ e > 0 \\ \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} & \text{if } p \nmid m. \end{cases}$$

(See [36, Corollary III.6.4].) Note that these 3 cases do not exhaust all possibilities (i.e., cases where m is divisible by p but is not power of p is not covered.)

Take $a \in F$ with $\operatorname{ord}(a) = m$. In order to count Fourier–Mukai partners of elliptic ruled surfaces, we need to study the subgroup

$$H_F^a := \{i \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \phi \in \operatorname{Aut}_0(F) \text{ such that } \phi(a) = i \cdot a\} \tag{31}$$

of $(\mathbb{Z}/m\mathbb{Z})^*$. Note that the definition of $H_{\hat{E}}^{\mathcal{L}}$ given in (2) is compatible with (31). We obtain the following result as a direct consequence of Lemma 4.2.

Lemma 4.3. *Take $a \in F$ with $\operatorname{ord}(a) \geq 4$.*

(1) *We have an injective group homomorphism*

$$\iota \colon H_F^a \hookrightarrow \operatorname{Aut}_0(F). \tag{32}$$

Furthermore, we have $|H_F^a| = 2, 4$ or 6.

(2) *Suppose that $p > 0$ and $\operatorname{ord}(a) = p^e$. Then (32) is an isomorphism.*

Proof. (i) Take $i \in H_F^a$. Then there exists $\phi \in \mathrm{Aut}_0(F)$ such that $\phi(a) = i \cdot a$, and define $\iota(i)$ to be ϕ. The well-definedness of ι follows from Lemma 4.2, and ι is injective by the definition. Since H_F^a is regarded as an abelian subgroup of $\mathrm{Aut}_0(F)$ described in Theorem 4.1, and H_F^a contains $\{\pm 1\}$ as a subgroup, we obtain the second assertion.

(ii) The existence of an order p^e element in F implies that F is ordinary. Since $F[p^e] = \mathbb{Z}/p^e\mathbb{Z} = \langle a \rangle$, for any $\phi \in \mathrm{Aut}_0(F)$ we see that $\phi(a) = i \cdot a$ for some $i \in (\mathbb{Z}/p^e\mathbb{Z})^*$. Hence the injective homomorphism in (32) is surjective, and then we can confirm the statement. □

Remark 4.4. If $\mathrm{ord}(a) \leq 3$, the map (32) may not be well-defined in the same way as in the proof of Lemma 4.3. For example, let F be a complex torus $\mathbb{C}/(\mathbb{Z} + \frac{-1+\sqrt{-3}}{2}\mathbb{Z})$. In this case $\mathrm{Aut}_0(F)$ contains a complex multiplication by $\frac{-1+\sqrt{-3}}{2}$. Consider the element $a = \frac{1}{3} + \frac{2}{3} \cdot \frac{-1+\sqrt{-3}}{2}$. Then $\frac{-1+\sqrt{-3}}{2} \cdot a = a$, so the map (32) is not well-defined.

From now on, by (32) we often regard H_F^a as a subgroup of $\mathrm{Aut}_0(F)$ when $\mathrm{ord}\, a \geq 4$. Note that we immediately see that $H_F^a = \{1\}$ if $m = 2$ and $H_F^a = \{\pm 1\}$ if $m = 3$.

We need the following to show Lemma 4.7,

Lemma 4.5. *Let F be an elliptic curve, and take $g \in \mathrm{Aut}_0(F)$ with $\mathrm{ord}(g) \geq 3$ (note that this condition implies $j(F) = 0$ or 1728). Let m be a positive integer satisfying $p \nmid m$. Then there exists $\alpha \in F[m]$ with $\mathrm{ord}(\alpha) = m$ such that α and $g\alpha$ generate $F[m]$.*

Proof. For $k = \mathbb{C}$, F is a complex torus. Since $j(F) = 0$ or 1728, we can put

$$F = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \quad \tau = \frac{-1+\sqrt{-3}}{2} \text{ or } \sqrt{-1}.$$

Then we see that

$$F[m] = \left\{ \frac{n}{m} + \frac{n'}{m}\tau \mid m, n, n' \in \mathbb{Z} \right\}.$$

In each case, complex multiplication by τ is an element of $\mathrm{Aut}_0(F)$ and the elements $\frac{1}{m}$ and $\frac{1}{m}\tau$ generate $F[m]$.

Let us consider another field k with $p = 0$. The defining equations of F is

$$\begin{aligned} y^2 + y &= x^3 \quad \text{if } j(F) = 0 \\ y^2 &= x^3 + x \quad \text{if } j(F) = 1728, \end{aligned}$$

so F can be defined over $\mathbb{Q}$. Take the smallest finite extension K of $\mathbb{Q}$ such that $F[m] \subset F(K)$. Then $F[m] = \langle \alpha, g\alpha \rangle$ for some $\alpha \in F(K)$ by the result of the case $k = \mathbb{C}$. Note that the automorphism g and the base change of the field extension commutes. Since algebraically closed field k with $\operatorname{ch} k = 0$ contains K, the element α can be taken as an element in $F(k)$, which gives the assertion.

Let us consider the case $p > 0$ and $p \nmid m$. Regard F as an elliptic curve over K as above. By [36, Proposition VII.3.1], the reduction map

$$F[m] \to \tilde{F}[m]$$

is bijective, where $\tilde{F}$ is a mod p reduction of F. Moreover, observing the action of $\operatorname{Aut}_0(F)$ and $\operatorname{Aut}_0(\tilde{F})$, the above map commutes with the action of their automorphisms. Hence the assertion follows from the case $p = 0$. □

Note that in this case, we easily see that α and $g\alpha$ are linearly independent i.e. for any $x, y \in \mathbb{Z}/m\mathbb{Z}$, the equality $x \cdot \alpha \oplus y \cdot g\alpha = 0$ implies $x = y = 0$.

Remark 4.6. Set $L := \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} (\cong F[m])$. In Lemma 4.5, the automorphism g really need to come from an automorphism group of F. In fact, take $m = 5$ and the automorphism

$$h := \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

of L with $\operatorname{ord}(h) = 4$. Since we have $h\alpha = 2\alpha$ for any $\alpha \in L$, it turns out that α and $h\alpha$ cannot generate L.

A part of the following lemma is shown in [40, Lemma 2.3] in the case $k = \mathbb{C}$.

Lemma 4.7. *Let F be an elliptic curve, and take $a \in F$ with $\operatorname{ord}(a) = m \geq 4$.*

(i) One of the following cases occurs:

(a) There exists $n \in \mathbb{Z}$ such that m divides n^2+1 and $H_F^a = \{\pm 1, \pm n\}$.

(b) There exists $n \in \mathbb{Z}$ such that m divides $n^2 + n + 1$ and $H_F^a = \{\pm 1, \pm n, \pm n^2\}$.

(c) $H_F^a = \{\pm 1\}$.

(ii) Assume that p does not divide m.

(a) If $j(F) \neq 0, 1728$, then $H_F^a = \{\pm 1\}$.

(b) Suppose $j(F) = 1728$ and $p \neq 2, 3$. If there exists $n \in \mathbb{Z}$ such that m divides $n^2 + 1$ and $a \in \langle n \cdot \alpha \oplus g\alpha \rangle$ the subgroup of $F[m]$, for some $\alpha \in F[m]$ and $g \in \mathrm{Aut}_0(F)$ with $\mathrm{ord}(g) = 4$ satisfying the condition in Lemma 4.5. Then $H_F^a = \{\pm 1, \pm n\}$. Otherwise, $H_F^a = \{\pm 1\}$.

(c) Suppose $j(F) = 0$ and $p \neq 2, 3$. If there exists $n \in \mathbb{Z}$ such that m divides n^2+n+1 and $a \in \langle (n+1) \cdot \alpha \oplus g\alpha \rangle$ the subgroup of $F[m]$, for some $\alpha \in F[m]$ and $g \in \mathrm{Aut}_0(F)$ with $\mathrm{ord}(g) \geq 3$ satisfying the condition in Lemma 4.5. Then $H_F^a = \{\pm 1, \pm n, \pm n^2\}$. Otherwise, $H_F^a = \{\pm 1\}$.

Proof. (i) We freely use Theorem 4.1 below. In the case F is ordinary, the assertion directly follows from the description of automorphism groups in Theorem 4.1. Suppose that F is supersingular. In the case $p = 2$, H_F^a is an abelian subgroup of the binary tetrahedral group. This condition implies that H_F^a is isomorphic to one of $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$. Hence, we get the assertion. In the case $p = 3$, then $\mathrm{Aut}_0(F)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$, and then the rest of the proof is similar to the previous case.

(ii) Note that the cardinality $|H_F^a|$ is even, since H_F^a always contains $\{\pm 1\}$ as its subgroup. (a) is obvious by $|\mathrm{Aut}_0(F)| = 2$. Hence, let us consider the case (b). Take a generator g of the cyclic group $\mathrm{Aut}_0(F) \cong \mathbb{Z}/4\mathbb{Z}$. It follows from Lemma 4.5 that we can set

$$a = x \cdot \alpha \oplus y \cdot g\alpha$$

for some $x, y \in \mathbb{Z}$ and suppose that $ga = n \cdot a$ holds for some $n \in \mathbb{Z}$. Then we have

$$nx \equiv -y, \quad ny \equiv x \pmod{m}$$

since $g^2 = -1$. Hence, we deduce that $a \in \langle n \cdot \alpha \oplus g\alpha \rangle$ and m divides $n^2 + 1$.

In the case (c), the proof is similar to the case (b). $\square$

Assume that $m = np^e$ with $e \geq 0$ and $p \nmid n$. We easily see that the natural projection

$$(\mathbb{Z}/m\mathbb{Z})^* \to (\mathbb{Z}/n\mathbb{Z})^*.$$

induces a group homomorphism

$$H_F^a \to H_F^{p^e \cdot a}. \tag{33}$$

Lemma 4.8. *The homomorphism* (33) *is surjective.*

Proof. Assume that there exists $\phi \in \mathrm{Aut}_0(F)$ such that $\phi(p^e \cdot a) = i_1 p^e \cdot a$ for some $i_1 \in (\mathbb{Z}/n\mathbb{Z})^*$. Since

$$F[m] \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/p^e\mathbb{Z}$$

the element $a \in F[m]$ can be decomposed by $a = a_1 + a_2$ where $\mathrm{ord}(a_1) = m$ and $\mathrm{ord}(a_2) = p^e$. Note that by the assumption we get $\phi(a_1) = i_1 \cdot a_1$. Moreover, since a_2 generates $F[p^e](\cong \mathbb{Z}/p^e\mathbb{Z})$, there exists $i_2 \in (\mathbb{Z}/p^e\mathbb{Z})^*$ such that $\phi(a_2) = i_2 \cdot a_2$. Let us consider the isomorphism

$$(\mathbb{Z}/n\mathbb{Z})^* \times (\mathbb{Z}/p^e\mathbb{Z})^* \to (\mathbb{Z}/m\mathbb{Z})^*$$

and let $i \in (\mathbb{Z}/m\mathbb{Z})^*$ be the image of (i_1, i_2). Then we see that

$$\begin{aligned} i \cdot a &= i \cdot (a_1 + a_2) \\ &= i_1 \cdot a_1 + i_2 \cdot a_2 \\ &= \phi(a_1) + \phi(a_2) \\ &= \phi(a) \end{aligned}$$

so the homomorphism (33) is surjective. □

Lemma 4.9. *If $n \geq 3$, then the homomorphism* (33) *is an isomorphism.*

Proof. By Lemma 4.8, we see that if $|H_F^a| = 2$ then $|H_F^{p^e \cdot a}| = 2$. Hence it is enough to show that $|H_F^a| > 2$ implies $|H_F^{p^e \cdot a}| > 2$ because the groups H_F^a and $H_F^{p^e \cdot a}$ are isomorphic to one of $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$. In the case $H_F^a \cong \mathbb{Z}/4\mathbb{Z}$. Then there exists $i \in (\mathbb{Z}/m\mathbb{Z})$ such that $i^2 = -1$ and

$$\phi(a) = i \cdot a$$

for some $\phi \in \mathrm{Aut}_0(F)$. Denote by $\bar{i}$ the image of i by the projection $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, which induces $H_F^a \to H_F^{p^e \cdot a}$. Since we have $\phi(p^e \cdot a) = i \cdot p^e \cdot a$, we see that $\bar{i} \in H_F^{p^e \cdot a}$. Since $\bar{i}^2 = -1$, we get $H_F^{p^e \cdot a} \cong \mathbb{Z}/4\mathbb{Z}$. In the case $H_F^a \cong \mathbb{Z}/6\mathbb{Z}$, the proof is similar. □

Take $a \in F$ and put $m := \operatorname{ord}(a)$. Assume furthermore that $p \nmid m$. Let us define an elliptic curve E to be $F/\langle a \rangle$, and consider the quotient morphism

$$q \colon F \to E = F/\langle a \rangle.$$

Then we can find $\mathcal{L} \in \operatorname{Pic}^0 E = \hat{E}$ with $m = \operatorname{ord}(\mathcal{L})$ such that $\hat{F} \cong \hat{E}/\langle \mathcal{L} \rangle$ and the dual morphism

$$\hat{q} = q^* \colon \hat{E} \to \hat{F}$$

of q coincides with the quotient morphism

$$\hat{E} \to \hat{E}/\langle \mathcal{L} \rangle.$$

The following lemma for $k = \mathbb{C}$ is shown by case analysis in [40, Lemma 2.4]. We give a more direct proof for an arbitrary algebraically closed field k.

Lemma 4.10. *In the above notation, the equality* $H_F^a = H_{\hat{E}}^{\mathcal{L}}$ *holds.*

Proof. If $m = 2$ or 3, then $H_F^a = \{1\}$ or $\{\pm 1\}$, respectively, without depending the choice of $a \in F$. So the assertion is clear.

In the case $m \geq 4$, suppose that we are given $i \in H_F^a$. Then there exists $\phi \in \operatorname{Aut}_0(F)$ such that $\phi(a) = i \cdot a$, which means that ϕ preserves the subgroup $\langle a \rangle$ of F. Then we see that ϕ induces the automorphism ϕ_E of E which makes the diagram

$$\begin{array}{ccc} F & \xrightarrow{q} & E \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi_E} \\ F & \xrightarrow[q]{} & E \end{array}$$

commutative. Taking the dual of it, we obtain the commutative diagram:

$$\begin{array}{ccc} \hat{F} & \xleftarrow{\hat{q}} & \hat{E} \\ {\scriptstyle \hat{\phi}}\uparrow & & \uparrow{\scriptstyle \widehat{\phi_E}} \\ \hat{F} & \xleftarrow[\hat{q}]{} & \hat{E} \end{array}$$

Since $\operatorname{Ker} \hat{q} = \langle \mathcal{L} \rangle$, we see that $\widehat{\phi_E} = \phi_E^*$ preserves the subgroup $\langle \mathcal{L} \rangle \subset \hat{E}$. Hence, we get

$$\phi_E^* \mathcal{L} \cong \mathcal{L}^j$$

for some $j \in H^{\mathcal{L}}_{\hat{E}}$. It follows from Lemma 4.2 that the equality $j = 1$ holds only if $\phi^*_E = \mathrm{id}_{\hat{E}}$, in particular $\phi = \mathrm{id}_F$. Hence, we have an injection

$$H^a_F \hookrightarrow H^{\mathcal{L}}_{\hat{E}} \quad i \mapsto j.$$

Because the conditions on $a \in F$ and $\mathcal{L} \in \hat{E}$ are symmetry, we conclude the equality $H^a_F = H^{\mathcal{L}}_{\hat{E}}$. □

Remark 4.11. Suppose that $j(F) \neq 0, 1728$. Then Lemma 4.7 (ii-a) tells us that $|H^a_F| = 2$, and hence $|H^{\mathcal{L}}_{\hat{E}}| = 2$ by Lemma 4.10. A similar statement holds if we relace $a \in F$ with $\mathcal{L} \in \hat{E}$.

5 Admissible subcategories

In this section, we summarize some definitions and results in [34], and give their application to Popa–Schnell conjecture in [35]. Throughout this section, let $\mathcal{D}$ be a k-linear triangulated category. We also refer to [32] for fundamental notions of ∞-categories.

5.1 Basic properties

In this subsection, we give basic properties of admissible subcategories. For more details, see [20].

Definition 5.1. Let $\mathcal{A} \subset \mathcal{D}$ be a full triangulated subcategory. If the inclusion functor $\iota\colon \mathcal{A} \hookrightarrow \mathcal{D}$ has the left (resp. right) adjoint functor, we say $\mathcal{A}$ is a *left (resp. right) admissible subcategory* of $\mathcal{D}$. A left and right admissible subcategory is called an *admissible subcategory*.

Remark 5.2. Let $\mathcal{B}$ be a left (resp. right) admissible subcategory of $\mathcal{D}$ and $\mathcal{A}$ be a left (resp. right) admissible subcategory of $\mathcal{B}$. Then $\mathcal{A}$ is a left (resp. right) admissible subcategory of $\mathcal{D}$. Indeed, the inclusion functor

$$\mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{\theta} \mathcal{D}$$

has a left adjoint functor $\theta_L \circ \iota_L$, where ι_L and θ_L are left adjoint functor of ι and θ, respectively.

Remark 5.3. Let X be a smooth projective variety. In this case $\mathcal{A} \subset D^b(X)$ is left admissible if and only if right admissible. For the proof, see [8, Propositions 2.6, 2.8 and Theorem 2.14].

Example 5.4. Let $X := \mathbb{P}(\mathcal{E}) \to Y$ be a $\mathbb{P}^n$-bundle. Then $\pi^* D^b(Y)$ is an admissible subcategory of $D^b(X)$.

To see this, it is enough to show that $\pi^* \colon D^b(Y) \to D^b(X)$ is fully faithful since π^* has a right adjoint functor $\mathbb{R}\pi_*$. Because of Corollary 2.3, we will check that $\mathbb{R}\pi_*\pi^*\mathcal{F} \cong \mathcal{F}$ for any $\mathcal{F} \in D^b(Y)$. By the projection formula, we only have to check that $\mathbb{R}\pi_*\pi^*\mathcal{O}_Y \cong \mathcal{O}_Y$. Since a fiber of π is $\mathbb{P}^n$, $H^i(X_y, \mathcal{O}_{X_y}) = 0$ for any $i \neq 0$, where $X_y = \pi^{-1}(y)$. On the other hand, since $H^i(X_y, \mathcal{O}_{X_y}) \cong (\mathbb{R}^i\pi_*\mathcal{O}_X)|_y$, we see that $\mathbb{R}^i\pi_*\mathcal{O}_X = 0$ for $i \neq 0$. Hence, we get $\mathbb{R}\pi_*\pi^*\mathcal{O}_Y \cong \mathcal{O}_Y$.

Now let us recall some basic notions.

Definition 5.5. Let $\mathcal{D}$ be a k-linear triangulated category and $\mathcal{C}$ be its full triangulated subcategory.

(1) The *right orthogonal subcategory* to $\mathcal{C}$ is a full subcategory of $\mathcal{D}$

$$\mathcal{C}^{\perp} := \{E \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(F, E) = 0 \text{ for any } F \in \mathcal{C}\}.$$

Similarly, the *left orthogonal subcategory* to $\mathcal{C}$ is

$${}^{\perp}\mathcal{C} := \{E \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(E, F) = 0 \text{ for any } F \in \mathcal{C}\}.$$

(2) A triangulated subcategory $\mathcal{C}$ is called *thick* if it is closed under taking direct summands.

Definition 5.6. (1) An object $E \in \mathcal{D}$ is *exceptional* if $\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}}(E, E) = k$ holds, or equivalently,

$$\mathrm{Hom}^i_{\mathcal{D}}(a, a) = \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

holds.

(2) A collection of exceptional objects

$$(E_1, \dots, E_n),$$

where $E_1, \dots, E_n \in \mathcal{D}$ is an *exceptional collection* of length n if $\mathbb{R}\mathrm{Hom}_{\mathcal{D}}(E_j, E_i) = 0$ for any $1 \leq i < j \leq n$.

(3) An *exceptional pair* is an exceptional collection of length 2.

Lemma 5.7. *Let X be a smooth projective variety and $E \in D^b(X)$ be an exceptional object. Then $\langle E \rangle$ is an admissible subcategory of $D^b(X)$, where $\langle E \rangle$ is the smallest thick subcategory of $D^b(X)$ which contains E.*

Proof. Let us consider the functor

$$\text{-} \overset{\mathbb{L}}{\otimes} E \colon D^b(\operatorname{Spec} k) \to D^b(X).$$

By the definition of exceptional objects, any object in $\langle E \rangle$ is of the form $\bigoplus E[i]^{\oplus n_i}$. Hence, we see that the essential image of the functor $\text{-} \overset{\mathbb{L}}{\otimes} E$ is $\langle E \rangle$. We know that the functor

$$\mathbb{R}\operatorname{Hom}_X(E, \text{-}) \colon D^b(X) \to D^b(\operatorname{Spec} k)$$

is right adjoint to $\text{-} \overset{\mathbb{L}}{\otimes} E$. Since E is exceptional, we have isomorphisms of functors

$$\mathbb{R}\operatorname{Hom}_X(E, (\text{-}) \overset{\mathbb{L}}{\otimes} E) \cong \mathbb{R}\operatorname{Hom}_X(E, E) \overset{\mathbb{L}}{\otimes} (\text{-}) \cong \operatorname{id}_{D^b(\operatorname{Spec} k)}.$$

This completes the proof. □

Definition 5.8. A collection of full triangulated subcategories $\mathcal{D}_1, \ldots \mathcal{D}_n$ is called a semiorthogonal decomposition of $\mathcal{D}$ if the following two conditions hold.

(1) For all $1 \leq j < i \leq n$ and any objects $E_i \in \mathcal{D}_i$, $E_j \in \mathcal{D}_j$ we have $\operatorname{Hom}_{\mathcal{D}}(E_i, E_j) = 0$.

(2) The smallest triangulated subcategory of $\mathcal{D}$ containing $\mathcal{D}_1, \ldots, \mathcal{D}_n$ coincides with $\mathcal{D}$.

In this case we denote it by

$$\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$$

a semiorthogonal decomposition.

Let $(E_1, \ldots, E_n)$ be an exceptional collection in $\mathcal{D}$. If the smallest triangulated subcategory of $\mathcal{D}$ containing $E_1, \ldots, E_n$ coincides with $\mathcal{D}$, then we get a semiorthogonal decomposition

$$\mathcal{D} = \langle \langle E_1 \rangle, \ldots, \langle E_n \rangle \rangle.$$

In this case, we say $(E_1, \ldots, E_n)$ is a *full exceptional collection* and we denote it by

$$\mathcal{D} = \langle E_1, \ldots, E_n \rangle.$$

Theorem 5.9 ([4])**.** *There is a full exceptional collection*

$$D^b(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}(i), \mathcal{O}_{\mathbb{P}^n}(i+1), \ldots, \mathcal{O}_{\mathbb{P}^n}(i+n) \rangle$$

for any $i \in \mathbb{Z}$.

Proposition 5.10. *Let* $\mathcal{C}$ *be a full subcategory of* $\mathcal{D}$ *and* $\iota \colon \mathcal{C} \hookrightarrow \mathcal{D}$ *be the fully faithful functor. Then the following statements are equivalent.*

(1) *The functor* ι *has a right (resp. left) adjoint functor, i.e. the category* $\mathcal{C}$ *is a right (resp. left) admissible subcategory.*

(2) *There exists a semiorthogonal decomposition*

$$\mathcal{D} = \langle \mathcal{C}^{\perp}, \mathcal{C} \rangle \quad (\textit{resp. } \mathcal{D} = \langle \mathcal{C}, {}^{\perp}\mathcal{C} \rangle).$$

Proof. We denote by ι_R the right adjoint functor of ι. By the natural transformation given in (7) we get a distinguished triangle

$$\iota \circ \iota_R(E) \xrightarrow{g_E} E \to G \to \iota \circ \iota_R(E)[1]$$

for any object $E \in \mathcal{D}$. For any object $C \in \mathcal{C}$ apply the functor $\mathrm{Hom}_{\mathcal{D}}(\iota(C), \text{-})$ to the distinguished triangle, we get a long exact sequence

$$\begin{aligned} &\to \mathrm{Hom}^l_{\mathcal{D}}(\iota(C), \iota \circ \iota_R(E)) \xrightarrow{\phi_l} \mathrm{Hom}^l_{\mathcal{D}}(\iota(C), E) \to \mathrm{Hom}^l_{\mathcal{D}}(\iota(C), G) \\ &\to \mathrm{Hom}^{l+1}_{\mathcal{D}}(\iota(C), \iota \circ \iota_R(E)) \xrightarrow{\phi_{l+1}} \cdots . \end{aligned}$$

By the construction, we see that each ϕ_l is an isomorphism. Hence, we get

$$\mathbb{R}\mathrm{Hom}_{\mathcal{D}}(\iota(C), G) = 0$$

for any object $C \in \mathcal{C}$. So we get $G \in \mathcal{C}^\perp$, which gives the proof of (1) $\Rightarrow$ (2).

For the converse, suppose that there exists a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{C}^\perp, \mathcal{C} \rangle$. Then for any object $E \in \mathcal{D}$, we get a distinguished triangle

$$\iota(F) \to E \to G \to \iota(F)[1]$$

for some $F \in \mathcal{C}$ and $G \in \iota(\mathcal{C})^\perp$. Let us define $\pi(E) := F$, and we will show that π is a functor from $\mathcal{D}$ to $\mathcal{C}$. Let $\iota(F_i) \to E_i \to G_i \to \iota(F_i)[1]$ be distinguished triangles for $i = 1, 2$ and $f\colon E_1 \to E_2$ be a morphism in $\mathcal{D}$. Consider the following diagram

$$\begin{array}{ccccccc} \iota(F_1) & \longrightarrow & E_1 & \longrightarrow & G_1 & \longrightarrow & \iota(F_1)[1] \\ & & \downarrow{\scriptstyle f} & & & & \\ \iota(F_2) & \longrightarrow & E_2 & \longrightarrow & G_2 & \longrightarrow & \iota(F_2)[1]. \end{array}$$

Since $G_2 \in \iota(\mathcal{C})^\perp$, a composition of morphisms

$$\iota(F_1) \to E_1 \xrightarrow{f} E_2 \to G_2$$

is zero. So, we get a morphism $\iota(F_1) \to \iota(F_2)$. Since ι is fully faithful, we get a morphism $F_1 \to F_2$ in the category $\mathcal{C}$. Hence, π is a functor.

For each $C \in \mathcal{C}$, applying the functor $\mathrm{Hom}_{\mathcal{D}}(\iota(C), \text{-})$ we get an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(C, \pi(E)) \cong \mathrm{Hom}_{\mathcal{D}}(\iota(C), E)$$

because $G \in \iota(\mathcal{C})^\perp$. Hence, we see that π is right adjoint to ι. $\square$

Lemma 5.11. *Let $\mathcal{D}$ be a triangulated category. Then any left or right admissible subcategory of $\mathcal{D}$ is a thick subcategory.*

Proof. Let $\mathcal{C} \subset \mathcal{D}$ be a right admissible subcategory, $\iota\colon \mathcal{C} \hookrightarrow \mathcal{D}$ be an inclusion and $\iota_R\colon \mathcal{D} \to \mathcal{C}$ be its right adjoint. For an object $E \in \mathcal{C}$, suppose that there is a decomposition $\iota(E) = A \oplus B$. Since $\iota_R \circ \iota = \mathrm{id}_{\mathcal{C}}$, we get a decomposition

$$E = \iota_R(A) \oplus \iota_R(B)$$

in the category $\mathcal{C}$. On the other hand, since $E \in \mathcal{C}$ we get

$$\mathrm{Hom}_{\mathcal{D}}(A, G) \oplus \mathrm{Hom}_{\mathcal{D}}(B, G) = \mathrm{Hom}_{\mathcal{D}}(E, G) = 0$$

for any object $G \in \mathcal{C}^\perp$. Since ${}^\perp(\mathcal{C}^\perp) = \mathcal{C}$, we see that $A, B \in \mathcal{C}$.

In the case $\mathcal{C} \subset \mathcal{D}$ is a left admissible subcategory, the proof is similar. $\square$

For a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$, we see that

$$\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_{k-1}, \langle \mathcal{D}_k, \ldots, \mathcal{D}_l \rangle, \mathcal{D}_{l+1}, \ldots, \mathcal{D}_n \rangle$$

is also a semiorthogonal decomposition of $\mathcal{D}$ where $1 \leq k \leq l \leq n$.

Proposition 5.12. *Let X be a smooth projective variety. If there exists a semiorthogonal decomposition*

$$D^b(X) = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle,$$

then all $\mathcal{D}_i$ are admissible subcategories.

Proof. Since we have a decomposition

$$D^b(X) = \langle \mathcal{D}_1, \langle \mathcal{D}_2, \ldots, \mathcal{D}_n \rangle \rangle$$

we see that $\mathcal{D}_1$ is a left admissible subcategory by Proposition 5.10. Note that by Remark 5.3, $\mathcal{D}_1$ is also a right admissible subcategory. Similarly, we see that $\mathcal{D}_2$ is an admissible subcategory of $\langle \mathcal{D}_2, \ldots, \mathcal{D}_n \rangle$. By Remark 5.2 $\mathcal{D}_2$ is also an admissible subcategory of $\mathcal{D}$. Repeating this procedure, we see that $\mathcal{D}_i$ is admissible for each $1 \leq i \leq n$. □

For an exceptional pair $\mathcal{E}, \mathcal{F}$, the left mutation $L_{\mathcal{E}}\mathcal{F}$ of $\mathcal{F}$ through $\mathcal{E}$ and the right mutation $R_{\mathcal{F}}\mathcal{E}$ of $\mathcal{E}$ through $\mathcal{F}$ are defined by the following distinguished triangles:

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \xrightarrow{\varepsilon} \mathcal{F} \to L_{\mathcal{E}}\mathcal{F}$$
$$R_{\mathcal{F}}\mathcal{E} \to \mathcal{E} \xrightarrow{\eta} \mathbb{R}\operatorname{Hom}_{\mathcal{D}}(\mathcal{E}, \mathcal{F})^{\vee} \otimes \mathcal{F}$$

The following lemma is well-known. See e.g. [20, Corollary 3.15].

Lemma 5.13. *Let C be a smooth projective curve.*

(1) *For any object $E \in D^b(C)$, there is a decomposition*

$$E \cong \bigoplus_i \mathcal{H}^i(E)[-i]$$

into a direct sum of shifts of cohomology sheaves.

(2) *For any coherent sheaf $\mathcal{F}$ on C, there is a decomposition*

$$\mathcal{F} \cong \mathcal{T} \oplus \mathcal{E}$$

into a direct sum of a torsion sheaf $\mathcal{T}$ and a locally free sheaf $\mathcal{E}$.

The complete description of admissible subcategories of $D^b(\mathbb{P}^1)$ is easy to be seen. Here we give a proof of this.

Proposition 5.14. *Any non-trivial admissible subcategory in $D^b(\mathbb{P}^1)$ is of the form $\langle \mathcal{O}(j) \rangle$ for some $j \in \mathbb{Z}$.*

Proof. Let $\mathcal{A} \subset D^b(\mathbb{P}^1)$ be a non-trivial admissible subcategory and $E \in \mathcal{A}$ be an object. By Lemma 5.13 (1), there is a decomposition

$$E \cong \bigoplus_i \mathcal{H}^i(E)[-i]$$

and by Lemma 5.13 (2), each $\mathcal{H}^i(E)$ is decomposed into a direct sum of torsion sheaves and locally free sheaves. Note that any locally free sheaf on $\mathbb{P}^1$ is decomposed by a direct sum of line bundles $\bigoplus_l \mathcal{O}(j_l)$ for some $j_l \in \mathbb{Z}$.

Assume that there exists an object $E \in \mathcal{A}$ which contains a (shift of) line bundle $\mathcal{O}(j)$ as a direct summand. By Lemma 5.11, we see that $\mathcal{O}(j)$ is in $\mathcal{A}$, hence $\langle \mathcal{O}(j) \rangle \subset \mathcal{A}$. Since we know that ${}^\perp\langle \mathcal{O}(j) \rangle = \langle \mathcal{O}(j+1) \rangle$ by Theorem 5.9, ${}^\perp\mathcal{A} \subset \langle \mathcal{O}(j+1) \rangle$. Hence we see ${}^\perp\mathcal{A} = \langle \mathcal{O}(j+1) \rangle$ or 0. For the first case we get $\mathcal{A} = \langle \mathcal{O}(j) \rangle$ and for the latter case we get $\mathcal{A} = D^b(\mathbb{P}^1)$, which is a trivial one.

It remains the case that for any $E \in \mathcal{A}$, the object E contains no line bundles as direct summands. Since any torsion sheaf on $\mathbb{P}^1$ is a filtration of structure sheaves of finite points, so $\mathcal{A}$ contains $\mathcal{O}_x$ for some $x \in \mathbb{P}^1$. If we take a semiorthogonal decomposition

$$D^b(\mathbb{P}^1) = \langle \mathcal{A}, {}^\perp \mathcal{A} \rangle$$

then ${}^\perp\mathcal{A}$ also has no line bundles by a similar argument to the previous case. Since both $\mathcal{A}$ and ${}^\perp\mathcal{A}$ have only direct sums of shift of torsion sheaves and

$$\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_y, \mathcal{O}_x) = \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_x, \mathcal{O}_y) = 0$$

for $y \neq x$, we see that $D^b(\mathbb{P}^1)$ is decomposed into $\mathcal{A}$ and ${}^\perp\mathcal{A}$ in the sense of Definition 2.9. This contradicts Proposition 2.10. □

Theorem 5.15 (Theorem 4.2 in [33])**.** *Any admissible subcategory in $D^b(\mathbb{P}^2)$ is generated by a subcollection of a mutation of the standard exceptional collection $D^b(\mathbb{P}^2) = \langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2) \rangle$.*

5.2 Pirozhkov's result and its application

For a Noetherian scheme S over k, we denote by $\mathrm{Perf}(S)$ the full subcategory of $D^b(S)$ consisting of perfect complexes. Note that for a smooth projective variety S, we know that $\mathrm{Perf}(S) = D^b(S)$. A stable k-linear ∞-category $\mathcal{D}$ is said to be *S-linear* if there exists an action functor

$$a_{\mathcal{D}} \colon \mathcal{D} \times \mathrm{Perf}(S) \to \mathcal{D}$$

together with associativity data.

For a morphism $f \colon X \to S$ between smooth projective varieties X and S over k, the category $D^b(X)$ has a natural S-linear structure via the functor

$$D^b(X) \times D^b(S) \to D^b(X) \quad (\mathcal{E}, \mathcal{F}) \mapsto \mathcal{E} \overset{\mathbb{L}}{\otimes}_X \mathbb{L}f^*\mathcal{F}.$$

Definition 5.16 ([34]). Let S be a Noetherian scheme over a field k. We say that S is *noncommutatively stably semiorthogonally indecomposable*, or *NSSI* for brevity, if for arbitrary choices of

(1) $\mathcal{D}$, a S-linear category which is proper[2] over S and has a classical generator, and

(2) $\mathcal{A}$, a left admissible subcategory of $\mathcal{D}$ which is linear over k,

the subcategory $\mathcal{A}$ is closed under the action of $\mathrm{Perf}(S)$ on $\mathcal{D}$.

Remark 5.17. For a definition of a classical generator, see [7]. For a quasi-compact and quasi-separated scheme S, the category $\mathrm{Perf}(S)$ has a classical generator [7, Corollary 3.1.2]. In particular, for a smooth projective variety S, the category $D^b(S)$ has a classical generator.

Theorem 5.18 (Lemma 6.1 in [34]). *Let $\pi \colon X \to S$ be a smooth projective morphism which is an étale-locally trivial fibration with fiber X_0. Assume that S is a connected excellent scheme[3]. Then for any point $s \in S$ the base change map*

$$\left\{ \begin{matrix} S\text{-linear admissible} \\ \text{subcategories} \\ \mathcal{A} \subset D^b(X) \end{matrix} \right\} \xrightarrow{\text{restriction to } X_s \cong X_0} \left\{ \begin{matrix} \text{admissible subcategories} \\ \mathcal{A}_0 \subset D^b(X_0) \end{matrix} \right\}$$

is an injection.

[2]See [32] for this notion.

[3]In [34, Lemma 6.1], Pirozhkov assumes that S is a scheme over $\mathbb{Q}$, but it is not needed in its proof.

Definition 5.19. Let $\pi\colon X \to S$ be a smooth projective morphism of Noetherian schemes.

(1) An object $\mathcal{E} \in \mathrm{Perf}(X)$ is π-*exceptional* if $\mathbb{R}\pi_* \mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{E}) \cong \mathcal{O}_S$.

(2) A collection of π-exceptional objects $\mathcal{E}_1, \ldots, \mathcal{E}_N \in \mathrm{Perf}(X)$ is a π-*exceptional collection* if $\mathbb{R}\pi_* \mathbb{R}\mathcal{H}om_X(\mathcal{E}_j, \mathcal{E}_i) = 0$ for any $1 \leq i < j \leq N$.

(3) A π-*exceptional pair* is a π-exceptional collection of length 2.

For a π-exceptional pair $\mathcal{E}, \mathcal{F}$, the left π-mutation $L_{\mathcal{E}}\mathcal{F}$ of $\mathcal{F}$ through $\mathcal{E}$ and the right π-mutation $R_{\mathcal{F}}\mathcal{E}$ of $\mathcal{E}$ through $\mathcal{F}$ are defined by the following distinguished triangles:

$$\pi^*\mathbb{R}\pi_* \mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\varepsilon} \mathcal{F} \to L_{\mathcal{E}}\mathcal{F},$$
$$R_{\mathcal{F}}\mathcal{E} \to \mathcal{E} \xrightarrow{\eta} \pi^*\mathbb{R}\pi_* \mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{F})^\vee \otimes_{\mathcal{O}_X} \mathcal{F}.$$

We see that mutations commute with base change.

Lemma 5.20 (Lemma 2.22 in [22])**.** *Consider the following Cartesian square of finite dimensional Noetherian schemes, where π is smooth projective.*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\pi} \\ T & \xrightarrow[g]{} & S \end{array}$$

For any π-exceptional pair $(\mathcal{E}, \mathcal{F})$, it follows that $(f^*\mathcal{E}, f^*\mathcal{F})$ is an φ-exceptional pair and we have the following isomorphisms:

$$L_{f^*\mathcal{E}}\,(f^*\mathcal{F}) \simeq f^*(L_{\mathcal{E}}\mathcal{F})$$
$$R_{f^*\mathcal{F}}\,(f^*\mathcal{E}) \simeq f^*(R_{\mathcal{F}}\mathcal{E})$$

We apply Lemmas 5.18 and 5.20 to obtain the following.

Proposition 5.21. *Let $\pi\colon X \to S$ be a $\mathbb{P}^n$-bundle ($n = 1, 2$) over a smooth projective variety S. Then any non-trivial S-linear admissible subcategory of $D^b(X)$ is of the following form:*

(1) *(Case $n = 1$)*
$$D^b(S)(i)(:= \pi^* D^b(S) \otimes_{\mathcal{O}_X} \mathcal{O}_X(i))$$
for some $i \in \mathbb{Z}$.

(2) *(Case $n = 2$)*
$$\pi^* D^b(S) \otimes_{\mathcal{O}_X} \langle \mathcal{E}_1, \dots, \mathcal{E}_l \rangle ,$$
where $\mathcal{E}_1, \dots, \mathcal{E}_l$ ($1 \leq l \leq n+1$) is a π-exceptional collection.

Proof. (i) Any non-trivial admissible subcategory in $D^b(\mathbb{P}^1)$ is of the form $\langle \mathcal{O}_{\mathbb{P}^1}(i) \rangle$ for some $i \in \mathbb{Z}$ by Proposition 5.14. Since the restriction of the admissible category $D^b(S)(i)$ to a fiber is $\langle \mathcal{O}_{\mathbb{P}^1}(i) \rangle$, the injective base change map in Theorem 5.18 is surjective. Hence the result follows.

(ii) Theorem 5.15 states that any non-trivial admissible subcategory $\mathcal{A}$ in $D^b(\mathbb{P}^2)$ is generated by a subcollection of successive mutations of the standard exceptional collection $\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)$. Lemma 5.20 yields an S-linear admissible subcategory $\mathcal{A}_X$ of $D^b(X)$, which is generated by a π-exceptional subcollection obtained by successive π-mutations of the π-exceptional collection $\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)$, and its derived restriction on a fiber is $\mathcal{A}$. This means that the injective base change map in Theorem 5.18 is surjective, hence, we obtain the result. □

Popa and Schnell showed in [35] that for smooth projective varieties X and Y over $\mathbb{C}$, if $D^b(X) \cong D^b(Y)$ then $\mathrm{Alb}(X)$ and $\mathrm{Alb}(Y)$ are isogenous. On the other hand we know that derived equivalence between abelian varieties induces their isogeny by Orlov [31].

The Popa–Schnell conjecture in [35] states that for any Fourier–Mukai partners Y of a given smooth projective variety X, there exists an equivalence $D^b(\mathrm{Alb}(Y)) \cong D^b(\mathrm{Alb}(X))$ of derived categories.

From Proposition 5.21, we deduce that the Popa–Schnell conjecture holds true in certain situations.

Corollary 5.22. *Let $X \to A$ and $Y \to B'$ be $\mathbb{P}^n$-bundles over abelian varieties A and B for $n = 1, 2$. If X and Y are Fourier–Mukai partners, then so are A and B. Furthermore, the Popa–Schnell conjecture holds true in this case.*

Proof. Put $D^b(A)(i) = \pi^* D^b(A) \otimes \mathcal{O}_X(i)$, where π is the $\mathbb{P}^1$-bundle $X \to A$. Since abelian varieties are NSSI by [34, Theorem 1.4], any admissible category of $D^b(X)$ is A-linear. Proposition 5.21 implies that any non-zero indecomposable admissible subcategory of $D^b(X)$ is equivalent to $D^b(A)$. This completes the proof of the first assertion. We see that $A \cong \mathrm{Alb}(X)$ and $B \cong \mathrm{Alb}(Y)$, and hence obtain the second. □

If X is an elliptic ruled surface over $\mathbb{C}$, namely $n = 1$ and $k = \mathbb{C}$, in Corollary 5.22, the statement follows from [40, Theorem 1.1]. The proof given above for $n = 1, 2$ and arbitrary k is more direct and natural.

Remark 5.23. Let $X \to E$ and $X' \to E'$ be $\mathbb{P}^n$-bundles over elliptic curves E and E' for $n = 1, 2$. As a consequence of Corollary 5.22, if X and X' are Fourier–Mukai partners, then $D^b(E) \cong D^b(E')$, and hence $E \cong E'$ by [20, Corollary 5.46].

6 Elliptic ruled surfaces

6.1 Basic properties of ruled surfaces

In this subsection, we give some basic results of ruled surfaces. Most of results are found in [17, Ch. 5 §2].

Let $\pi \colon S \to C$ be a ruled surface i.e. π is a surjective morphism to a curve C such that the fiber S_p is isomorphic to $\mathbb{P}^1$ for any $p \in C$. The following characterization is well-known.

Proposition 6.1. *If $\pi \colon S \to C$ is a ruled surface, then there exists a locally free sheaf $\mathcal{E}$ of rank 2 such that $S \cong \mathbb{P}(\mathcal{E})$ over C. Conversely, every such $\mathbb{P}(\mathcal{E})$ is a ruled surface over C.*

Proposition 6.2. *If $S \to C$ is a ruled surface, then it is possible to write $S \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf on C with $H^0(\mathcal{E}) \neq 0$ but for any invertible sheaf on $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$, we have $H^0(\mathcal{E}) = 0$. Moreover, the integer $\deg \mathcal{E}$ depends only on S.*

If a locally free sheaf $\mathcal{E}$ satisfies the conditions in Proposition 6.2, we say $\mathcal{E}$ is *normalized*. Let us put $e := -\deg \mathcal{E}$, which is an invariant of S.

Proposition 6.3. *Let $S := \mathbb{P}(\mathcal{E}) \to C$ be a ruled surface over the curve C of genus g, where $\mathcal{E}$ is a normalized locally free sheaf.*

(1) *If $\mathcal{E}$ is decomposable, then $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ for some $\mathcal{L} \in \operatorname{Pic} C$ with $\deg \mathcal{L} \leq 0$. So, we see that $e \geq 0$. Moreover, all values of $e \geq 0$ are possible.*

(2) *If $\mathcal{E}$ is indecomposable, then we get $-2g \leq e \leq 2g - 2$.*

6.2 Vector bundles over elliptic curves

Atiyah classified indecomposable vector bundles on elliptic curves [1]. We summarize his result we need below.

Let $\mathcal{M}_E(r,d) = \mathcal{M}(r,d)$ be the set of isomorphism classes of indecomposable vector bundles of rank r and degree d on an elliptic curve E.

Theorem 6.4 (Theorem 5 in [1]). (1) *There exists a vector bundle $\mathcal{E}_{r,0}$ on E, unique up to isomorphism, with $h^0(\mathcal{E}_{r,0}) \neq 0$.*

(2) *If $\mathcal{E} \in \mathcal{M}(r,0)$, then there exist a line bundle $\mathcal{L} \in \operatorname{Pic}^0(E)$ such that $\mathcal{E} \cong \mathcal{E}_{r,0} \otimes \mathcal{L}$.*

Theorem 6.4 says that for $\mathcal{E} \in \mathcal{M}(r,0)$, we have

$$h^0(\mathcal{E}) = h^1(\mathcal{E}) = 0 \quad \text{when} \quad \mathcal{E} \neq \mathcal{E}_{r,0}$$

and

$$h^0(\mathcal{E}_{r,0}) = h^1(\mathcal{E}_{r,0}) = 1.$$

Hence we see that the vector bundle $\mathcal{E}_{r,0}$ is the only normalized vector bundle in $\mathcal{M}(r,0)$.

Actually we can define $\mathcal{E}_{r,0}$ by putting $\mathcal{E}_{1,0} = \mathcal{O}_E$ and the unique non-trivial extension

$$0 \to \mathcal{E}_{r,0} \to \mathcal{E}_{r+1,0} \to \mathcal{O}_E \to 0$$

inductively. We can also see that $\mathcal{E}_{r,0} \cong (\mathcal{E}_{r,0})^\vee$.

Corollary 6.5. *Let E be an elliptic curve. For any $\mathcal{L}_1, \mathcal{L}_2 \in \operatorname{Pic}^0 E$ we get*

$$\operatorname{Hom}_E(\mathcal{L}_1, \mathcal{L}_2) = \begin{cases} k & \text{if } \mathcal{L}_1 \cong \mathcal{L}_2 \\ 0 & \text{if } \mathcal{L}_1 \not\cong \mathcal{L}_2. \end{cases}$$

Next we describe indecomposable vector bundle of rank 2 and degree 1.

Proposition 6.6. *Let S be a ruled surface over an elliptic curve E, corresponding to an indecomposable vector bundle $\mathcal{E}$. Then $e = 0$ or -1 and there is exactly one such ruled surface over E for each two values of e.*

Indeed, we can construct indecomposable vector bundles with $e = -1$ explicitly. For a closed point $P \in E$, we define a vector bundle $\mathcal{E}_P$ by the following non-split exact sequence

$$0 \to \mathcal{O}_E \to \mathcal{E}_P \to \mathcal{O}_E(P) \to 0.$$

Although $\mathcal{E}_P \not\cong \mathcal{E}_Q$ for distinct points $P, Q \in E$, we have $\mathbb{P}(\mathcal{E}_P) \cong \mathbb{P}(\mathcal{E}_Q)$ by Proposition 6.6. We can also see that

$$\mathcal{M}(2,1) = \{\mathcal{E}_P \mid P \in E\}.$$

Let $f : S \to E$ be a ruled surface over an elliptic curve E, corresponding to a normalized vector bundle $\mathcal{E}$. Put a general fiber of f by F_f and take a section C_0 with $C_0^2 = -e$. Let us define a divisor D which satisfies $\mathcal{O}_S(D) \cong \det\mathcal{E}$. Then we have

$$K_S \equiv -2C_0 - eF_f$$

by [17, Corollary 2.11]. Note that if S has an elliptic fibration, then $-K_S$ is nef. Moreover, we can see that $-K_S$ is nef if and only if $e = 0, -1$. We can also deduce from the above discussion that

$$\mathcal{E} = \begin{cases} \mathcal{O}_E \oplus \mathcal{L} \text{ or } \mathcal{E}_{2,0} & \text{if } e = 0 \\ \mathcal{E}_P & \text{if } e = -1. \end{cases} \tag{34}$$

for some $\mathcal{E} \in \operatorname{Pic}^0 E$ and $P \in E$.

7 Fourier–Mukai partners of elliptic ruled surfaces

7.1 Elliptic ruled surfaces

In this subsection, we recall a result in [39].

Let S be an elliptic surface and $\pi\colon S \to C$ be a relatively minimal elliptic fibration. We denote by

$$\pi^*(p_i) = m_i D_i$$

the multiple fibers of π, where m_i is the multiplicity and $p_i \in C$ for $i = 1, 2, \dots, \lambda$. We know that $\mathbb{R}^1\pi_*\mathcal{O}_S \cong \mathcal{L}_\pi \oplus \mathcal{T}_\pi$, where $\mathcal{L}_\pi$ is an invertible sheaf and $\mathcal{T}_\pi$ is a torsion sheaf. Note that $\mathcal{T}_\pi = 0$ in the case $p = 0$. The multiple fiber is said to be *tame* for a point $p_i \notin \operatorname{Supp}\mathcal{T}_\pi$ and *wild* for a point $p_i \in \operatorname{Supp}\mathcal{T}_\pi$

The canonical bundles on S and C are related by the canonical bundle formula:

$$\omega_S \cong \pi^*(\omega_C \otimes \mathcal{L}_\pi^{-1}) \otimes \mathcal{O}_S(\sum_{i=1}^{\lambda} a_i D_i) \tag{35}$$

for some integer a_i with $0 \leq a_i \leq m_i - 1$. If $a_i \neq m_i - 1$, then $\pi^*(p_i)$ is known to be a wild fiber.

Let $\mathcal{E}$ be a normalized rank 2 vector bundle on an elliptic curve E and

$$f\colon S = \mathbb{P}(\mathcal{E}) \to E$$

be a $\mathbb{P}^1$-bundle on E defined by $\mathcal{E}$. If S has an elliptic fibration, then $e = 0$ or -1 holds by the discussion around (34).

The following theorem gives a classification of ruled surfaces over elliptic curves from the point of view of elliptic fibrations. In the tables contained therein, the symbol * stands for a wild fiber. Moreover, as mentioned above, if a multiple fiber $m_i D_i$ is tame, then $a_i = m_i - 1$ where a_i and m_i are given in (35). Hence, we omit the value of a in the list. For example, $(2, 0/2^*)$ in the case (ii-3) stands for one tame fiber of multiplicity 2 with $a_1 = 1$ and one wild fiber of multiplicity 2 with $a_2 = 0$.

Theorem 7.1 (Theorem 1.1 in [39]). *Let us consider the above situation.*

(1) *For e = 0, we have the following:*

	$\mathcal{E}$	$\exists$ *an elliptic fibration on* S*?*	p
(i-1)	$\mathcal{O}_E \oplus \mathcal{O}_E$	*no multiple fibers*	$p \geq 0$
(i-2)	$\mathcal{O}_E \oplus \mathcal{L}$, $\operatorname{ord} \mathcal{L} = m > 1$	(m, m)	$p \geq 0$
(i-3)	$\mathcal{O}_E \oplus \mathcal{L}$, $\operatorname{ord} \mathcal{L} = \infty$	*no elliptic fibrations*	$p \geq 0$
(i-4)	*indecomposable*	*no elliptic fibrations*	$p = 0$
(i-5)	*indecomposable*	$(p - 2/p^*)$	$p > 0$

Here $\mathcal{L}$ *is an element of* $\operatorname{Pic}^0 E$.

(2) *Suppose that e = −1. Then the isomorphism class of such vector bundle* $\mathcal{E}$ *on* E *is unique, and* S *has an elliptic fibration. The list of singular fibers are as follows:*

	(a_i/m_i)	E	p
(ii-1)	$(2, 2, 2)$		$p \neq 2$
(ii-2)	$(1/2^*)$	*supersingular*	$p = 2$
(ii-3)	$(2, 0/2^*)$	*ordinary*	$p = 2$

By [12] and [23], we know that if S has non-trivial Fourier–Mukai partners, S has an elliptic fibration. Hence, from now on, we suppose that S has an elliptic fibration $\pi\colon S \to \mathbb{P}^1$. Theorem 7.1 says that the multiplicities of all multiple fibers of π are the same number m.

When $e = 0$ (resp. $e = -1$), we see

$$F_\pi \cdot F_f = mC_0 \cdot F_f = m \quad \text{(resp. } F_\pi \cdot C_0 = m(2C_0 - F_f) \cdot C_0 = m) \tag{36}$$

by [39, Remark 4.2], and hence

$$\lambda_\pi = m = \lambda'_\pi \tag{37}$$

for both cases (recall the definitions of λ_π and λ'_π in (17) and (26) respectively). Here F_π (resp. F_f) is a fiber of π (resp. f), and C_0 stands for a section of f satisfying $C_0^2 = -e$.

Consider the case $|\operatorname{FM}(S)| \neq 1$. Then the inequality (21) yields $m = \lambda_\pi \geq 5$. Hence, S fits into either (i-2), $m \geq 5$ or (i-5), $p \geq 5$ in Theorem 7.1. Then $S' \in \operatorname{FM}(S)$ is also an elliptic ruled surface admitting an elliptic fibration π' fitting into the same case as S by Lemma 3.7.

Lemma 7.2. *Suppose that $|\operatorname{FM}(S)| \neq 1$. Then S fits into the case (i-2).*

Proof. It suffices to show that $|\operatorname{FM}(S)| = 1$ in the case (i-5). Suppose that S fits into the case (i-5). As we explained above, $S' \in \operatorname{FM}(S)$ is also an elliptic ruled surface in the case (i-5). In other words, S' has a $\mathbb{P}^1$-bundle structure $f'\colon \mathbb{P}(\mathcal{E}') \to E'$, where $\mathcal{E}'$ is the indecomposable vector bundle of rank 2, degree 0 on an elliptic curve E'. By Corollary 5.22, we have $E \cong E'$. Then, we see $S \cong S'$ by [17, Theorem V.2.15], in other words, $|\operatorname{FM}(S)| = 1$. □

The purpose of this paper is to describe the set $\operatorname{FM}(S)$ for elliptic ruled surfaces. Hence in the sequel, we will concentrate on the case (i-2), the unique candidate of S admitting non-trivial Fourier–Mukai partners.

7.2 Case (i-2).

Take $\mathcal{L} \in \operatorname{Pic}^0 E$ with $1 < m := \operatorname{ord} \mathcal{L} < \infty$, and set

$$S := \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}).$$

The following lemma is elementary and useful.

Lemma 7.3. (1) *There exists an isomorphism $S \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{M})$ over E if and only if $\mathcal{L} \cong \mathcal{M}^{\pm 1}$.*

(2) *For $\phi_E \in \operatorname{Aut}(E)$, we have an isomorphism $f^*\phi_E$ in the fiber product diagram:*

$$\begin{array}{ccc} \mathbb{P}(\mathcal{O}_E \oplus \phi_E^*\mathcal{L}) & \xrightarrow{f^*\phi_E} & S \\ \downarrow & \square & \downarrow f \\ E & \xrightarrow[\phi_E]{} & E \end{array} \tag{38}$$

(3) *For some $\mathcal{M} \in \operatorname{Pic}^0 E$, let $f_T \colon T := \mathbb{P}(\mathcal{O}_E \oplus \mathcal{M}) \to E$ be the $\mathbb{P}^1$-bundle over E. Suppose that we are given an isomorphism $\phi \colon T \to S$. Then, if we replace ϕ appropriately, we can take $\phi_E \in \operatorname{Aut}_0(E)$, which makes the diagram*

$$\begin{array}{ccc} T & \xrightarrow{\phi} & S \\ f_T \downarrow & & \downarrow f \\ E & \xrightarrow[\phi_E]{} & E \end{array} \tag{39}$$

commutative. Moreover, we have an isomorphism

$$T \cong \mathbb{P}(\mathcal{O}_E \oplus \phi_E^*\mathcal{L}) \tag{40}$$

over E, and an isomorphism

$$\mathcal{M} \cong \phi_E^*\mathcal{L}. \tag{41}$$

Proof. (i) This fact directly follows from [17, Exercise II.7.9(b)].

(ii) This assertion must be well-known. We leave the proof to readers. (For example, use [17, Proposition II.7.12].)

(iii) Since S has a unique $\mathbb{P}^1$-bundle structure, the existence of $\phi_E \in \operatorname{Aut}(E)$ fitting in (39) is assured. Next, write $\phi_E = T_a \circ \phi_E^0$ for some $\phi_E^0 \in \operatorname{Aut}_0(E)$ and $a \in E$. Since $T_a^*\mathcal{L} \cong \mathcal{L}$, the isomorphism f^*T_a (given as $f^*\phi_E$ in (38)) gives an automorphism of S. Then, if necessary, replace ϕ with $(f^*T_a)^{-1} \circ \phi$, we may assume that $\phi_E \in \operatorname{Aut}_0(E)$. By the universal property of the fiber product in (38), we obtain an isomorphism (40) over E. Then by (i) there exists an isomorphism $\mathcal{M}^{\pm 1} \cong \phi_E^*\mathcal{L}$. Since $(-\mathrm{id}_E)^*\mathcal{L} \cong \mathcal{L}^{-1}$, $f^*(-\mathrm{id}_E)$ also gives an automorphism of S. Thus, replace ϕ with $f^*(-\mathrm{id}_E) \circ \phi$ if necessary, we may assume that $\phi_E \in \operatorname{Aut}_0(E)$ and (41) holds simultaneously. $\square$

Lemma 7.4. *For $i \in (\mathbb{Z}/m\mathbb{Z})^*$, $S \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i)$ if and only if there exists an automorphism $\phi_E \in \mathrm{Aut}_0(E)$ such that $\phi_E^*\mathcal{L} \cong \mathcal{L}^i$. Consequently, the set*

$$\{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong$$

is naturally identified with the group

$$(\mathbb{Z}/m\mathbb{Z})^*/H^{\mathcal{L}}_{\hat{E}}.$$

Here, recall that $H^{\mathcal{L}}_{\hat{E}} := \{i \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \phi \in \mathrm{Aut}_0(E) \text{ such that } \phi^*\mathcal{L} \cong \mathcal{L}^i\}$.

Proof. "If" part follows from Lemma 7.3 (ii). "Only if" part follows from Lemma 7.3 (iii). □

Consider the dual morphism

$$q_1 \colon F_0 := \widehat{\hat{E}/\langle\mathcal{L}\rangle} \to E \tag{42}$$

of the quotient morphism $\hat{E} \to \hat{E}/\langle\mathcal{L}\rangle$. Then it follows from the definition of q_1 that $q_1^*\mathcal{L} \cong \mathcal{O}_{F_0}$ holds. Thus we have a diagram

$$\begin{array}{ccccc} F_0 & \xleftarrow{p_1} & F_0 \times \mathbb{P}^1 & \xrightarrow{p_2} & \mathbb{P}^1 \\ {\scriptstyle q_1}\downarrow & \square & \downarrow{\scriptstyle q_S} & & \downarrow{\scriptstyle q_2} \\ E & \xleftarrow[f]{} & S & \xrightarrow[\pi]{} & \mathbb{P}^1, \end{array} \tag{43}$$

where the left square diagram is a fiber product, and the right one is obtained by the Stein factorization of $\pi \circ q_S$. The reason why $\pi \circ q_S$ factors through p_2 is as follows. First, we have $q_S^*\omega_S \cong \omega_{F_0 \times \mathbb{P}^1}$ by [39, Lemma 2.14]. On the other hand, the elliptic fibration p_2 (resp. π) are defined by the linear system of some multiple of $-K_{F_0 \times \mathbb{P}^1}$ (resp. $-K_S$). Therefore $\pi \circ q_S$ factors through p_2.

Recall that the elliptic fibration π has exactly two multiple fibers.

Convention. By the action of $\mathrm{PGL}(1, k)$ on $\mathbb{P}^1$, we always assume below that in the case (i-2), the elliptic fibration π has multiple fibers over the points 0 and ∞ in $\mathbb{P}^1$. Furthermore, we also assume that $q_2(0) = 0$ and $q_2(\infty) = \infty$.

For $y_0 \in \mathbb{P}^1$ with $y := q_2(y_0) \in \mathbb{P}^1 \backslash \{0, \infty\}$, we denote by F_y the non-multiple fiber of π over the point y. Then it follows from $f \circ q_S = q_1 \circ p_1$ that the restriction of q_S induces the isomorphism

$$q_S|_{F_0 \times y_0} \colon F_0 \times y_0 \cong F_y, \tag{44}$$

since we see from (36) that $f|_{F_y}$ is finite morphism of degree m. We tacitly identify F_0 and F_y by this isomorphism.

Take $x_0 \in F_0$ and set $x := q_1(x_0) \in E$. Then in a similar way to (44), we have an isomorphism

$$q_S|_{x_0 \times \mathbb{P}^1} \colon x_0 \times \mathbb{P}^1 \cong F_x, \tag{45}$$

where F_x is the fiber of f over the point x. We identify $\mathbb{P}^1$ and F_x by (45). By our convention above, we see that the two multiple fibers of π intersect with each fiber $\mathbb{P}^1$ of f at 0 and ∞ respectively.

Recall that f has two minimal sections, let's say C_0 and C_1, corresponding to the projections

$$\mathcal{O}_E \oplus \mathcal{L} \to \mathcal{O}_E \quad \text{and} \quad \mathcal{O}_E \oplus \mathcal{L} \to \mathcal{L}. \tag{46}$$

Then the multiple fibers of π are given exactly mC_0 and mC_1 (see [39, Remark 4.2]).

We use the following lemma to show Claim 7.7.

Lemma 7.5. *Let us regard the multiplicative group $\mathbb{G}_m$ as a subgroup of $\mathrm{Aut}(\mathcal{O}_E \oplus \mathcal{L})(\cong \mathbb{G}_m \times \mathbb{G}_m)$ by the diagonal embedding. Then there exists an injective homomorphism*

$$\iota \colon \mathbb{G}_m \cong \mathrm{Aut}(\mathcal{O}_E \oplus \mathcal{L})/\mathbb{G}_m \hookrightarrow \mathrm{Aut}(S/E).$$

Here, for $\lambda \in \mathbb{G}_m$, the automorphism $\iota(\lambda)$ of S induces the action on each fiber $\mathbb{P}^1$ of f fixing the points 0 and ∞.

Proof. The existence of the injection ι is assured in [15, p.202].[4] Note that since any elements of $\mathrm{Aut}(\mathcal{O}_E \oplus \mathcal{L})$ preserve the projections in (46), any $\beta \in \mathrm{Im}\, \iota$ preserves the minimal sections C_0 and C_1, and hence it gives an automorphism on each fiber $\mathbb{P}^1$ of f fixing the points 0 and ∞. □

[4]See also [28, Lemma 3]). Because Δ in ibid. is trivial, we actually see that ι gives an isomorphism.

7.3 Proof of Theorem 1.3.

Let S be an elliptic ruled surface and suppose $|\operatorname{FM}(S)| \neq 1$. Lemma 7.2 implies that

$$S \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$$

for some $\mathcal{L} \in \operatorname{Pic}^0 E$ with $\operatorname{ord} \mathcal{L} = m \geq 5$. Now if $S' \in \operatorname{FM}(S)$, by the same reason we get $S' \cong \mathbb{P}(\mathcal{O}_{E'} \oplus \mathcal{L}')$ for some $\mathcal{L}' \in \operatorname{Pic}^0 E'$ with

$$m = \lambda_\pi = \operatorname{ord} \mathcal{L} = \operatorname{ord} \mathcal{L}'.$$

Moreover, by Corollary 5.22, we see that $E \cong E'$.

We divide the proof of Theorem 1.3 into two cases: The case $m = p^e \geq 5$ for some $e > 0$, and the case arbitrary $m \geq 5$ with $m \neq p^e$ for any $e > 0$. In both cases, first we define an injective map

$$\{J^i(S) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong \hookrightarrow \{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong, \tag{47}$$

and secondly, we shall see

$$|H_\pi| \leq |H^{\mathcal{L}}_{\hat{E}}|. \tag{48}$$

The cardinality of the L.H.S in (47) is $\varphi(m)/|H_\pi|$ by Lemma 3.4, and the cardinality of the R.H.S. in (47) is $\varphi(m)/|H^{\mathcal{L}}_{\hat{E}}|$ by Lemma 7.4. Therefore, combining (47) with (48), we can conclude that (47) is a bijection, and hence Theorem 3.3 yields

$$\operatorname{FM}(S) = \{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong$$

as required in Theorem 1.3.

Case: $m = p^e \geq 5$ for some $e > 0$. Theorem 7.1 implies that $J^i(S) \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}_i)$ for some $\mathcal{L}_i \in \operatorname{Pic}^0 E$ with $\operatorname{ord} \mathcal{L}_i = p^e$. But in this case, E is necessarily ordinary, and hence $\hat{E}[p^e]$ is a cyclic group generated by $\mathcal{L}$. So in this case, $\mathcal{L}_i \cong \mathcal{L}^{\beta(i)}$ for some $\beta(i) \in (\mathbb{Z}/m\mathbb{Z})^*$, and thus we can define an injective map (47) by $J^i(S) \mapsto \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\beta(i)})$.

Denote by F_0 the elliptic curve satisfying $\hat{F}_0 = \hat{E}/\langle \mathcal{L} \rangle$ as in §7.2. Then by (44), a general fiber of the elliptic fibration $\pi\colon S \to \mathbb{P}^1$ is isomorphic to F_0.

Claim 7.6. The inequality (48) holds (if $m = p^e \geq 5$).

Proof. [13, Propositions 5.3.3, 5.3.6] implies that $\kappa(J^0(S)) = -\infty$. Combining this fact with [13, Corollary 5.3.5], we see that $J^0(S)$ is an elliptic ruled surface with a section. Therefore, by the classification in Theorem 7.1 and [13, Theorem 5.3.1 (i)], we have $J^0(S) \cong F_0 \times \mathbb{P}^1$. Then we have $\mathrm{Br}(J^0(S)) = 0$ by [14, Proposition 2.1]. Moreover we have $\lambda_\pi = p^e = \lambda'_\pi$ by (37), and hence the group H'_π in Lemma 3.8 is trivial. Therefore Lemma 3.8 yields

$$\left|H_\pi\right| \leq \left|\mathrm{Aut}_0(J^0_\eta)\right|.$$

Recall that $H^{\mathcal{L}}_{\hat{E}} = \mathrm{Aut}_0(E)$ by Lemma 4.3 (ii) in the case $m = p^e \geq 5$. Hence, to obtain the conclusion, it suffices to check that $|\mathrm{Aut}_0(J^0_\eta)| \leq |\mathrm{Aut}_0(E)|$. Thus we may assume $2 < |\,\mathrm{Aut}_0(J^0_\eta)|$. Note that we have a surjective homomorphism

$$\mathrm{Aut}_0(J^0(S)/\mathbb{P}^1) \to \mathrm{Aut}_0(J^0_\eta),$$

where $\mathrm{Aut}_0(J^0(S)/\mathbb{P}^1)$ means the automorphism group of $J^0(S)(\cong F_0 \times \mathbb{P}^1)$ over $\mathbb{P}^1$, fixing the 0-section. Thus, we have an isomorphism $\mathrm{Aut}_0(J^0(S)/\mathbb{P}^1) \cong \mathrm{Aut}_0(F_0)$, and moreover obtain

$$2 < |\,\mathrm{Aut}_0(J^0_\eta)| = |\,\mathrm{Aut}_0(J^0(S)/\mathbb{P}^1)| = |\,\mathrm{Aut}_0(F_0)|.$$

This yields $j(F_0) = 0$ or 1728. Since the morphism $q_1 \colon F_0 \to E$ obtained in (42) is a composition of relative Frobenius morphisms (cf. [36, Theorem V.3.1]), [17, Exercise IV.4.20(a)] produces the isomorphism $E \cong F_0$, which completes the proof. □

Claim 7.6 completes the proof of Theorem 1.3 in the case $m = p^e \geq 5$.

Case: Arbitrary $m \geq 5$ with $m \neq p^e$ for any $e > 0$. We may put $m = np^e$ with $e \geq 0$, $n > 1$, $p \nmid n$. We generalize the method of [40] below.

Recall that $S \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$, and define elliptic curves F_0 and F as $\hat{F}_0 := \hat{E}/\langle \mathcal{L} \rangle$ and $\hat{F} := \hat{E}/\left\langle \mathcal{L}^{p^e} \right\rangle$. Denote by

$$q_E \colon F \to E$$

the dual morphism of the quotient morphism $\hat{E} \to \hat{F} = \hat{E}/\left\langle \mathcal{L}^{p^e} \right\rangle$. Set

$$\mathcal{M} := q_E^*\mathcal{L} \ \text{ and } \ T := \mathbb{P}(\mathcal{O}_F \oplus \mathcal{M}).$$

Then we see $\hat{F}_0 = \hat{F}/\langle\mathcal{M}\rangle$ and $\operatorname{ord}\mathcal{M} = p^e$. Moreover if $e > 0$, the existence of a non-zero element $\mathcal{M}$ of $\hat{F}[p^e]$ implies that F is ordinary, and the dual morphism of the quotient morphism

$$\hat{F} \to \hat{F}_0 = \hat{F}/\langle\mathcal{M}\rangle .$$

is the e-th iteration of the relative Frobenius morphisms (cf. [36, Theorem V.3.1]). Then we obtain the following commutative diagram:

$$\begin{array}{ccccc} F_0 & \xleftarrow{p_1} & F_0 \times \mathbb{P}^1 & \xrightarrow{p_2} & \mathbb{P}^1 \\ \downarrow{\scriptstyle \mathrm{Fr}^e} & \square & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle \mathrm{Fr}^e_{\mathbb{P}^1}} \\ F & \xleftarrow[f_1]{} & T & \xrightarrow[\pi_1]{} & \mathbb{P}^1 \\ \downarrow{\scriptstyle q_E} & \square & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_{\mathbb{P}^1}} \\ E & \xleftarrow[f]{} & S & \xrightarrow[\pi]{} & \mathbb{P}^1 \end{array} \tag{49}$$

Both of the left squares are fiber product diagrams, and the right squares are obtained by the Stein factorizations of $\pi_1 \circ h_1$ and $\pi \circ q$ respectively. Moreover, we have

$$\deg q_E = \deg q = \deg q_{\mathbb{P}^1} = n.$$

Take

$$i \in \mathbb{Z} \text{ with } 1 \leq i < m, \quad (i, m) = 1. \tag{50}$$

Note that this condition implies that $(i, p^e) = (i, n) = 1$, and hence we sometimes regard $i \in (\mathbb{Z}/p^e\mathbb{Z})^*$ or $i \in (\mathbb{Z}/n\mathbb{Z})^*$ below.

Recall that we have already proved Theorem 1.3 for line bundles whose order is p-th power. By applying it to $\mathcal{M}$, we obtain

$$J^i(T) \cong \mathbb{P}(\mathcal{O}_F \oplus \mathcal{M}^{\beta(i)}) \tag{51}$$

for some $\beta(i) \in (\mathbb{Z}/p^e\mathbb{Z})^*$. Moreover, since $(\mathrm{Fr}^e)^*\mathcal{M} \cong \mathcal{O}_{F_0}$, we have a diagram

$$\begin{array}{ccccc} F_0 & \xleftarrow{p_1} & F_0 \times \mathbb{P}^1 & \xrightarrow{p_2} & \mathbb{P}^1 \\ \downarrow{\scriptstyle \mathrm{Fr}^e} & \square & \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle \mathrm{Fr}^e_{\mathbb{P}^1}} \\ F & \xleftarrow[f_i]{} & J^i(T) & \xrightarrow[\pi_i]{} & \mathbb{P}^1 \end{array} \tag{52}$$

as in (43). Here f_i is a $\mathbb{P}^1$-bundle defined by using the $\mathbb{P}^1$-bundle structure on $\mathbb{P}(\mathcal{O}_F \oplus \mathcal{M}^{\beta(i)})$ and the isomorphism (51).

Fix an n-th primitive root of unity ζ. Consider the multiplication on $\mathbb{G}_m$ by ζ, and extend it to the automorphism of $\mathbb{P}^1$. Denote it by $g_{\mathbb{P}^1}$. Because we see that $q_{\mathbb{P}^1}$ in (49) fixes points 0 and ∞ in $\mathbb{P}^1$, it turns out that the morphism $q_{\mathbb{P}^1}$ is the quotient morphism by the action of the group $\langle g_{\mathbb{P}^1} \rangle \cong \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{P}^1$.

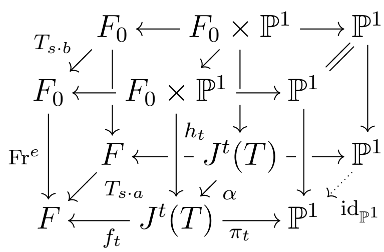
Take $a \in F$ such that $E \cong F/\langle a \rangle$ and $\operatorname{ord} a (= \operatorname{ord} \mathcal{L}^{p^e}) = n$. Then we can construct an action of the group $G := \mathbb{Z}/n\mathbb{Z}$ on $J^i(T)$ as follows.

Claim 7.7. For each $s \in (\mathbb{Z}/n\mathbb{Z})^*$ and $t \in (\mathbb{Z}/p^e\mathbb{Z})^*$, there exists an automorphism g_s of $J^t(T)$ which induces the translation $T_{s\cdot a}$ of F and the automorphism $g_{\mathbb{P}^1}$ of $\mathbb{P}^1$.

Proof. Since $T_{s\cdot a}^*\mathcal{M} \cong \mathcal{M}$, there exists an automorphism

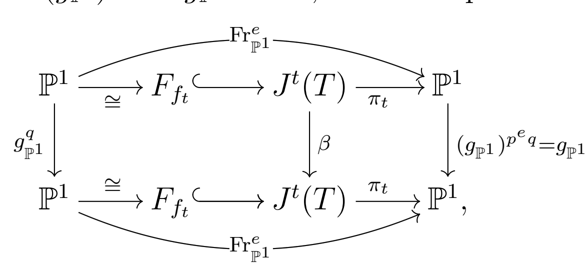
$$\alpha \in \operatorname{Aut}(J^t(T)) (\overset{(51)}{\cong} \operatorname{Aut}(\mathbb{P}(\mathcal{O}_F \oplus \mathcal{M}^{\beta(t)})))$$

compatible with $T_{s\cdot a}$ on F. Note that $T_{s\cdot a}$ lifts a translation $T_{s\cdot b}$ on F_0 for some $b \in F_0$ with $\mathrm{Fr}^e(b) = a$, and hence α lifts to $T_{s\cdot b} \times \mathrm{id}_{\mathbb{P}^1}$ on $F_0 \times \mathbb{P}^1$.



Therefore, α respects the elliptic fibration π_t, i.e. $\alpha \in \operatorname{Aut}(J^t(T)/\mathbb{P}^1)$.

Next take an integer q with $p^e q = 1$ in $(\mathbb{Z}/n\mathbb{Z})^*$. It follows from Lemma 7.5 that there exists an automorphism $\beta \in \operatorname{Aut}(J^t(T)/F)$ which induces the automorphism $g_{\mathbb{P}^1}^q$ on each fiber F_{f_t} (which we identify with $\mathbb{P}^1$ by (45)) of the $\mathbb{P}^1$-bundle f_t. Combining (45) with the commutativity of the right square in (52), we see that $\pi_t|_{F_{f_t}} \colon F_{f_t} \to \mathbb{P}^1$ coincides with $\mathrm{Fr}_{\mathbb{P}^1}^e$, and then β induces the automorphism $(g_{\mathbb{P}^1})^{p^e q} = g_{\mathbb{P}^1}$ on $\mathbb{P}^1$, the base space of π_t.



Hence, the automorphism $g_s := \alpha \circ \beta$ has the desired property. □

Denote by g a generator of the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$, and define the action of G on $J^t(T)$ by

$$\rho_{s,t} \colon G \to \mathrm{Aut}(J^t(T)) \quad g \mapsto g_s. \tag{53}$$

For the integer i given in (50), regard $i \in (\mathbb{Z}/n\mathbb{Z})^*$ and $i \in (\mathbb{Z}/p^e\mathbb{Z})^*$, and set $\rho_i := \rho_{i,i}$. We define the quotient variety to be

$$S_i := J^i(T)/_{\rho_i}G \tag{54}$$

by the action ρ_i, and denote the quotient morphism by

$$q_i \colon J^i(T) \to S_i.$$

It is easy to see that S is the quotient of $T = J^1(T)$ by the action $\rho_{s,1}$ for some s. Replace $a \in F$ with $s \cdot a$, and redefine g_s and $\rho_{s,t}$ by this new a, so that $S = S_1$ holds. After this replacement, we consider only the action ρ_i, but not general $\rho_{s,t}$.

We set

$$g_i^0 := T_{i\cdot b} \times g_{\mathbb{P}^1}^q \in \mathrm{Aut}(F_0 \times \mathbb{P}^1).$$

Then we see that $\operatorname{ord} g_i^0 = \operatorname{ord} T_{i\cdot b} = \operatorname{ord} g_{\mathbb{P}^1}^q = n$ and it is compatible with $g_i \in \mathrm{Aut}(J^i(T))$ defined in Claim 7.7:

$$h_i \circ g_i^0 = g_i \circ h_i. \tag{55}$$

We also define the action on $F_0 \times \mathbb{P}^1$ by

$$\rho_i^0 \colon G \to \mathrm{Aut}(F_0 \times \mathbb{P}^1) \quad g \mapsto g_i^0 \tag{56}$$

for each i.

Take an integer j with $1 \le j < m$, $(j, m) = 1$ and $ij = 1$ in $(\mathbb{Z}/m\mathbb{Z})^*$. For the projection

$$p_{13} \colon F_0 \times \Delta_{\mathbb{P}^1} \times F_0 \to F_0 \times F_0,$$

define a line bundle

$$\mathcal{U}_0 := p_{13}^* \mathcal{O}_{F_0 \times F_0}(\Delta_{F_0} + (j-1)F_0 \times O + (i-1)O \times F_0)$$

on

$$F_0 \times \Delta_{\mathbb{P}^1} \times F_0 (\cong (F_0 \times \mathbb{P}^1) \times_{\mathbb{P}^1} (F_0 \times \mathbb{P}^1)).$$

Then $F_0 \times \mathbb{P}^1$ in the second factor in R.H.S. serves as $J^i(F_0 \times \mathbb{P}^1)$ where $\mathcal{U}_0$ plays the role of a universal sheaf, and moreover it is shown in [40, page 3229] that it satisfies

$$(\rho_1^0(g) \times \rho_i^0(g))^* \mathcal{U}_0 \cong \mathcal{U}_0. \tag{57}$$

On the other hand, it follows from [10, Theorem 5.3] that we can take a universal sheaf $\mathcal{U}'$ on $T \times_{\mathbb{P}^1} J^i(T)$, which satisfies that $\mathcal{U}'|_{z \times J^i(T)}$ is a line bundle of degree j on F_0 for general $z \in T$. For a point $(x, y) \in F_0 \times (\mathbb{P}^1 \backslash \{0, \infty\})$, there exists an isomorphism

$$((h_1 \times h_i)^* \mathcal{U}')|_{(F_0 \times \mathbb{P}^1) \times_{\mathbb{P}^1} (x,y)} \cong \mathcal{U}'|_{T \times_{\mathbb{P}^1} h_i((x,y))}, \tag{58}$$

since the restriction of $h_1 \times h_i$ gives

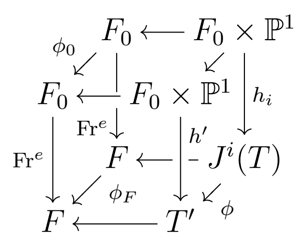
$$(F_0 \times \mathbb{P}^1) \times_{\mathbb{P}^1} (x, y) \cong F_0 \times y \cong F_y \cong T \times_{\mathbb{P}^1} h_i((x, y)),$$

where the second isomorphism comes from (44). Hence, we see that the L.H.S. in (58) is a line bundle of degree i on F_0. Then, by the universal property of $\mathcal{U}_0$, there exists an automorphism $\phi_0 \in \operatorname{Aut}(F_0)$ such that

$$(\mathrm{id}_{F_0 \times \Delta_{\mathbb{P}^1}} \times \phi_0)^* \mathcal{U}_0 \cong (h_1 \times h_i)^* \mathcal{U}' \otimes p_3^* \mathcal{N}_0$$

for some $\mathcal{N}_0 \in \operatorname{Pic}^0 F_0$.

We shall construct an elliptic ruled surface T' and (iso)morphisms ϕ_F, ϕ, h' which make the following diagrams commutative:



(59)

First, ϕ_0 descends to $\phi_F \in \operatorname{Aut}(F)$ via $\mathrm{Fr}^e \colon F_0 \to F$ by [36, Corollary II.2.12], and ϕ_F induces an isomorphism

$$\phi \colon J^i(T) \cong \mathbb{P}(\mathcal{O}_F \oplus \mathcal{M}^{\beta(i)}) \to T' := \mathbb{P}(\mathcal{O}_F \oplus \phi_{F*} \mathcal{M}^{\beta(i)}).$$

Note that $\phi_{F*} \in \operatorname{Aut}_0(\hat{F})$ preserves the subgroup $\ker \widehat{\mathrm{Fr}^e} = \hat{F}[p^e] = \langle \mathcal{M} \rangle$ of $\hat{F}$, and thus $\phi_{F*} \mathcal{M}^{\beta(i)} \in \langle \mathcal{M} \rangle$. Hence, we obtain a morphism

$$h' \colon F_0 \times \mathbb{P}^1 \cong \mathbb{P}(\mathcal{O}_{F_0} \oplus \mathcal{O}_{F_0}) \to T' \cong \mathbb{P}(\mathcal{O}_F \oplus \phi_{F*} \mathcal{M}^{\beta(i)}),$$

which fits into the diagram in (59). Moreover we have the following commutative diagram:

$$
\begin{array}{ccccc}
F_0 \times \Delta_{\mathbb{P}^1} \times F_0 & \xleftarrow{(\mathrm{id}_{F_0 \times \Delta_{\mathbb{P}^1}}) \times \phi_0} & F_0 \times \Delta_{\mathbb{P}^1} \times F_0 & \xrightarrow{p_3} & F_0 \\
{\scriptstyle h_1 \times h'} \downarrow & & \downarrow {\scriptstyle h_1 \times h_i} & & \downarrow {\scriptstyle \mathrm{Fr}^e} \\
T \times_{\mathbb{P}^1} T' & \xleftarrow[\mathrm{id}_T \times \phi]{} & T \times_{\mathbb{P}^1} J^i(T) & \xrightarrow[f_i \circ p_2]{} & F
\end{array}
$$

Take $\mathcal{N} \in \mathrm{Pic}^0 F$ such that $(\mathrm{Fr}^e)^*\mathcal{N} = \mathcal{N}_0$, and define a line bundle

$$\mathcal{U} := (\mathrm{id}_T \times \phi)_*(\mathcal{U}' \otimes (f_i \circ p_2)^*\mathcal{N})$$

on $T \times_{\mathbb{P}^1} T'$ so that

$$\mathcal{U}_0 \cong (h_1 \times h')^*\mathcal{U} \tag{60}$$

holds. The pair $(T', \mathcal{U})$ serves as $J^i(T)$ and its universal sheaf, ane thus we redefine T' to be $J^i(T)$.

Claim 7.8. The universal sheaf $\mathcal{U}$ on $T \times_{\mathbb{P}^1} J^i(T)$ satisfies

$$(\rho_1(g) \times \rho_i(g))^*\mathcal{U} \cong \mathcal{U}.$$

Proof. Take $y_0 \in \mathbb{P}^1 \backslash \{0, \infty\}$ with $y := \mathrm{Fr}^e(y_0) \in \mathbb{P}^1 \backslash \{0, \infty\}$. Denote by $F_y \times F'_y$ the fiber of $\pi_1 \times \pi_i \colon T \times_{\mathbb{P}^1} J^i(T) \to \mathbb{P}^1$ over the point y. Pull back the isomorphism (60) to the subscheme $F_0 \times y_0 \times F_0$, which is isomorphic to $F_y \times F'_y$ by (44), and combine (55) and (57) with it, then we have isomorphisms

$$((\rho_1(g) \times \rho_i(g))^*\mathcal{U})|_{F_y \times F_y} \cong ((\rho_1^0(g) \times \rho_i^0(g))^*\mathcal{U}_0)|_{F_0 \times y_0 \times F_0} \cong \mathcal{U}_0|_{F_0 \times y_0 \times F_0} \cong \mathcal{U}|_{F_y \times F_y}.$$

$$
\begin{array}{ccccc}
F_0 \times y_0 \times F_0 & \hookrightarrow & F_0 \times \Delta_{\mathbb{P}^1} \times F_0 & \xrightarrow{p_2} & \mathbb{P}^1 \ni y_0 \\
{\scriptstyle \cong} \downarrow & & {\scriptstyle h_1 \times h_i} \downarrow & & \downarrow {\scriptstyle \mathrm{Fr}^e_{\mathbb{P}^1}} \\
F_y \times F'_y & \hookrightarrow & T \times_{\mathbb{P}^1} J^i(T) & \xrightarrow[\pi_1 \times \pi_i]{} & \mathbb{P}^1 \ni y
\end{array}
$$

This yields that the line bundle $L := (\rho_1(g) \times \rho_i(g))^*\mathcal{U} \otimes \mathcal{U}^{-1}$ is trivial over the open set $(\pi_1 \times \pi_i)^{-1}(\mathbb{P}^1 \backslash \{0, \infty\})$ by [17, Exercise III.12.4]. We also see by (55), (57) and (60) that $(h_1 \times h_i)^*L$ is trivial over $\mathbb{P}^1 \backslash \{0, \infty\}$, and thus

$$L \cong \mathcal{O}_{T \times_{\mathbb{P}^1} J^i(T)}(b(D_0 \times D'_0 - D_\infty \times D'_\infty)) \tag{61}$$

for some $b \in \mathbb{Z}$, where $p^e D_0$ and $p^e D_0'$ (resp. $p^e D_\infty$ and $p^e D_\infty'$) are the multiple fibers over $0 \in \mathbb{P}^1$ (resp. ∞) of π_1 and π_i. Note that $\operatorname{ord} L$ divides p^e, the multiplicity of the multiple fibers. Since $\operatorname{ord}(\rho_1(g) \times \rho_i(g)) = n$ and the R.H.S. in (61) is $(\rho_1(g) \times \rho_i(g))$-invariant, we see that

$$\mathcal{U} \cong (\rho_1(g) \times \rho_i(g))^{n*}\mathcal{U} \cong (\rho_1(g) \times \rho_i(g))^{(n-1)*}\mathcal{U} \otimes L \cong \cdots \cong \mathcal{U} \otimes L^{\otimes n},$$

and hence $\operatorname{ord} L \mid n$. Since $p \nmid n$, we have $\operatorname{ord} L = 1$, as it is required. □

Recall that we have the following commutative diagram by the definition of S_i in (54):

$$\begin{array}{ccccc} F & \xleftarrow{f_i} & J^i(T) & \xrightarrow{\pi_i} & \mathbb{P}^1 \\ {\scriptstyle q_E}\downarrow & \square & \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle q_{\mathbb{P}^1}} \\ E & \longleftarrow & S_i & \xrightarrow[\pi_{S_i}]{} & \mathbb{P}^1 \end{array}$$

Here, q_E and $q_{\mathbb{P}^1}$ are the same one appeared in (49), and π_{S_i} is an elliptic fibration.

Claim 7.9. For each i, there exists $\alpha(i) \in (\mathbb{Z}/m\mathbb{Z})^*$ such that we have an isomorphism

$$S_i \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\alpha(i)}).$$

over E.

Proof. First of all, we know by Theorem 7.1 that there exists an isomorphism $S_i \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}_i)$ over E for some $\mathcal{L}_i \in \operatorname{Pic}^0 E$ with $\operatorname{ord} \mathcal{L}_i = m$. Then the result follows from

$$\mathcal{L}_i \in \ker(\widehat{\mathrm{Fr}^e} \circ \widehat{q_E}) = \langle \mathcal{L} \rangle \cong \mathbb{Z}/m\mathbb{Z}.$$

□

Recall that $S = S_1$ below.

Claim 7.10. There exists an isomorphism $J^i(S) \cong S_i$.

Proof. First, we shall show that there exists a coherent sheaf $\mathcal{U}_i$ on $S \times S_i$ such that

$$(q_1 \times \mathrm{id}_{J^i(T)})_*\mathcal{U} \cong (\mathrm{id}_S \times q_i)^*\mathcal{U}_i \tag{62}$$

for the morphisms

$$T \times J^i(T) \overset{q_1 \times \mathrm{id}_{J^i(T)}}{\to} S \times J^i(T) \overset{\mathrm{id}_S \times q_i}{\to} S \times S_i.$$

Claim 7.8 implies that

$$(\rho_1(g) \times \mathrm{id}_{J^i(T)})^* \mathcal{U} \cong (\mathrm{id}_T \times \rho_i(g)^{-1})^* \mathcal{U}.$$

Push forward the both sides by the morphism $q_1 \times \mathrm{id}_{J^i(T)}$. Then we obtain

$$(q_1 \times \mathrm{id}_{J^i(T)})_* \mathcal{U} \cong (\mathrm{id}_S \times \rho_i(g)^{-1})^* (q_1 \times \mathrm{id}_{J^i(T)})_* \mathcal{U},$$

that is, the sheaf $(q_1 \times \mathrm{id}_{J^i(T)})_* \mathcal{U}$ is G-invariant with respect to the diagonal action of G on $S \times J^i(T)$, where G acts on S trivially. Since $G = \langle g \rangle$ is a finite cyclic group, the G-invariance of coherent sheaves is equivalent to the G-equivariance, and hence there exists a coherent sheaf $\mathcal{U}_i$ on $S \times S_i$ satisfying (62).

For $z \in J^i(T)$, we have

$$\mathcal{U}_i|_{S \times q_i(z)} \cong ((q_1 \times \mathrm{id}_{J^i(T)})_* \mathcal{U})|_{S \times z} \cong q_{1*}(\mathcal{U}|_{T \times z}).$$

Here, the second isomorphism follows from [5, Lemma 1.3] and the smoothness of q_1. Suppose that z is not contained in multiple fibers of π_i, that is, $y := \pi_i(z) \in \mathbb{P}^1 \backslash \{0, \infty\}$ by the convention stated in §7.2. Then $\mathcal{U}|_{T \times z}$ is actually a sheaf on $F_y \times z$, and the restriction $q_1|_{F_y \times z}$ is an isomorphism by (44). It turns out that $\mathcal{U}_i|_{S \times q_i(z)}$ is also a line bundle of degree i on $F_{q_{\mathbb{P}^1}(y)} \times q_i(z)$.

Then, by the universal property of $J^i(S)$, there exists a morphism from

$$\pi_{S_i}^{-1}(\mathbb{P}^1 \backslash \{0, \infty\})(\subset S_i) \to \pi_{J^i(S)}^{-1}(\mathbb{P}^1 \backslash \{0, \infty\})(\subset J^i(S))$$

over $\mathbb{P}^1 \backslash \{0, \infty\}$, where π_{S_i} and $\pi_{J^i(S)}$ are the elliptic fibrations on S_i and $J^i(S)$ respectively. Since $\mathcal{U}_i|_{S \times q_i(z_1)} \ncong \mathcal{U}_i|_{S \times q_i(z_2)}$ on F_y for $z_1 \neq z_2 \in J^i(T)$, this morphism is injective, and hence S_i and $J^i(S)$ are birational over $\mathbb{P}^1$. Then, [2, Proposition III.8.4] implies that $S_i \cong J^i(S)$. $\square$

Combining Claims 7.9 and 7.10, we obtain the inclusion (47) by the map

$$J^i(S) \mapsto \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\alpha(i)}).$$

The next aim is to show (48).

Claim 7.11. There exists an injective group homomorphism

$$\overline{\alpha}\colon H_\pi/\{\pm 1\} \to H^{\mathcal{L}}_{\hat{E}}/\{\pm 1\}.$$

Proof. Take $i \in H_\pi (:= \{i \in (\mathbb{Z}/m\mathbb{Z})^* \mid J^i(S) \cong S\})$. We have $\alpha(i) \in (\mathbb{Z}/m\mathbb{Z})^*$ so that there exists an isomorphism

$$\psi\colon \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\alpha(i)}) \overset{\cong}{\to} S_i \overset{\cong}{\to} J^i(S)$$

by Claims 7.9 and 7.10. We use ψ and the $\mathbb{P}^1$-bundle structure on $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\alpha(i)})$ to fix a $\mathbb{P}^1$-bundle structure on $J^i(S)$:

$$f_{J^i(S)}\colon J^i(S) \to E$$

Then Lemma 7.3 (iii) implies that there exist an isomorphism φ and an automorphism $\varphi_E \in \mathrm{Aut}_0(E)$ fitting in the commutative diagram

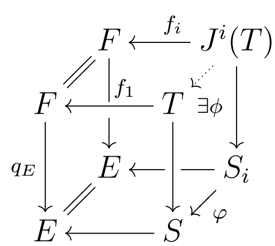
$$\begin{array}{ccccc} \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\alpha(i)}) & \xrightarrow{\psi} & J^i(S) & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow{\scriptstyle f_{J^i(S)}} & & \downarrow{\scriptstyle f} \\ E & = & E & \xrightarrow[\varphi_E]{} & E \end{array} \tag{63}$$

and $\varphi_E^*\mathcal{L} \cong \mathcal{L}^{\alpha(i)}$ is satisfied.

Take another isomorphism $\varphi'\colon J^i(S) \to S$. Then since $\varphi' \circ \varphi^{-1}$ is an automorphism of $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$, we have $(\varphi'_E \circ \varphi_E^{-1})^*\mathcal{L} \cong \mathcal{L}^{\pm 1}$ by Lemma 7.3 (i) and (ii). Thus, we obtain the group homomorphism

$$\alpha\colon H_\pi \to H^{\mathcal{L}}_{\hat{E}}/\{\pm 1\} (:= \{i \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \phi \in \mathrm{Aut}_0(E)\ \text{s.t.}\ \phi^*\mathcal{L} \cong \mathcal{L}^i\}/\{\pm 1\}.)$$

Thus it suffices to prove $\mathrm{Ker}\,\alpha = \{\pm 1\}$. Suppose $i \in \mathrm{Ker}\,\alpha$. Since $\varphi_E^*\mathcal{L} \cong \mathcal{L}^{\pm 1}$ holds in this case, Lemma 4.2 implies that φ_E fitting in the diagram (63) is either id_E or $-\mathrm{id}_E$. Replace φ with $f^*(-\mathrm{id}_E) \circ \varphi$ (see the notation in Lemma 7.3 (ii) and the proof of ibid. (iii)) if necessary, then we may assume that $\varphi_E = \mathrm{id}_E$. We have the following commutative diagram [5]:



(64)

[5]Here, we identify S_i and $J^i(S)$ by Claim 7.10.

Because the front and the back squares in (64) are the fiber product diagrams, there exists an isomorphism $\phi\colon J^i(T) \to T$ which makes the right square the fiber product.

Since ϕ descends to $\varphi\colon S_i = J^i(T)/_{\rho_i}G \to S = T/_{\rho_1}G$ for $G = \mathbb{Z}/n\mathbb{Z} = \langle g \rangle$, we have

$$\rho_1(g) \circ \phi = \phi \circ \rho_i(g)^l$$

for some l. Recall that both of $\rho_1(g)$ and $\rho_i(g)$ induce the same automorphism $g_{\mathbb{P}^1}$ on the base curve $\mathbb{P}^1$ of the elliptic fibrations on T and $J^i(T)$ (see Claim 7.7 and (53)), then we see $l = \pm 1$. Next recall $\rho_1(g)$ (resp. $\rho_i(g)$) induces the automorphism T_a (resp. $T_{i\cdot a}$) on F, the base curve of the $\mathbb{P}^1$-bundle f_1 (resp. f_i). Then we know that

$$T_a = (T_{i\cdot a})^l = T_{li\cdot a},$$

and hence, $1 = il$ in $(\mathbb{Z}/n\mathbb{Z})^*$. Therefore, we have $i = \pm 1$, and hence $\operatorname{Ker}\alpha \subset \{\pm 1\}$. The other direction is obvious. □

By Claim 7.11, we conclude that $|H_\pi| \leq |H_E^{\mathcal{L}}|$ as is required in (48).

Therefore, we complete the proof of the first statement in Theorem 1.3 for arbitrary $m \geq 5$. The second follows from Lemma 4.3 (ii).

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