Free products of coarsely convex spaces and the coarse Baum–Connes conjecture

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Contents

Chapter 1

Introduction

Fukaya and Oguni [9] introduced a wide class of metric spaces, called coarsely convex spaces. Coarsely convex spaces can be regarded as the counterpart of simply connected, complete, Riemannian manifolds with non-positive sectional curvature in coarse geometry. This class includes many classes of metric spaces. In particular, a class of geodesic coarsely convex spaces is an important subclass of coarsely convex spaces.

Let (X, d_X) be a geodesic metric space. Let $E \geq 1$ and $C \geq 0$ be constants. Let $\mathcal{L}$ be a family of geodesic segments. The metric space X is a geodesic $(E, C, \mathcal{L})$-coarsely convex space, if E, C, and $\mathcal{L}$ satisfy the following conditions:

i). For all $u, v \in X$, there exists a geodesic segment $\gamma \in \mathcal{L}$ with $\gamma : [0, a] \to X$ such that $\gamma(0) = u$ and $\gamma(a) = v$.

ii). Let $\gamma, \eta \in \mathcal{L}$ be geodesic segments with $\gamma : [0, a] \to X$ and $\eta : [0, b] \to X$. For all $t \in [0, a]$, $s \in [0, b]$, and $c \in [0, 1]$, we have

$$d_X(\gamma(ct), \eta(cs)) \leq (1 - c)Ed_X(\gamma(0), \eta(0)) + cEd_X(\gamma(t), \eta(s)) + C.$$

We say that X is a geodesic coarsely convex space if there exist E, C, and a family of geodesic segments $\mathcal{L}$ such that X is a geodesic $(E, C, \mathcal{L})$-coarsely convex space. A coarsely convex group is a group G acting properly and cocompactly on a coarsely convex space. The classes of metric spaces listed in Table 1.1 are examples of geodesic coarsely convex spaces.

- Geodesic Gromov hyperbolic metric spaces.
- CAT(0) spaces.
- Systolic complexes [15].
- Proper injective metric spaces [5], especially, the injective hulls of locally finite coarsely Helly graphs [3].

Table 1.1: Examples of geodesic coarsely convex spaces.

Pisanski and Tucker [16] introduced free products of Cayley graphs, and Bridson and Haefliger [2, Theorem II.11.18] constructed metric spaces on which free products of groups act properly and cocompactly. By slightly modifying their construction, we define free products of metric spaces with nets, and we obtain the following result.

THEOREM 1.0.1. *Let X and Y be metric spaces with nets. If X and Y are geodesic coarsely convex spaces, then the free product $X * Y$ is a geodesic coarsely convex space.*

Let X be a proper metric space. The coarse assembly map is a homomorphism from the coarse K-homology of X to the K-theory of the Roe algebra of X. The coarse Baum–Connes conjecture states that for "nice" proper metric spaces, the coarse assembly maps are isomorphisms. Fukaya and Oguni [9, Theorem 1.3] showed that for proper coarsely convex spaces, the coarse Baum–Connes conjecture holds. Combining this result and Theorem 1.0.1, we obtain the following

THEOREM 1.0.2. *Let X and Y be proper metric spaces with nets. If X and Y are geodesic coarsely convex spaces, then the free product $X*Y$ satisfies the coarse Baum–Connes conjecture.*

Let G and H be groups acting properly and cocompactly on X and Y, respectively. We can construct the free product X and Y with respect to their actions. Moreover, $G * H$ acts properly and cocompactly on it. Therefore, combining Theorem 1.0.2 and the Švarc–Milnor Lemma, we obtain the following.

THEOREM 1.0.3. *Let X and Y be proper metric spaces with nets. We suppose that X and Y are geodesic coarsely convex spaces. Let G and H be groups acting properly and cocompactly on X and Y, respectively. Then $G * H$ satisfies the coarse Baum–Connes conjecture.*

This doctoral thesis includes the content of the paper [6] to appear in Kyoto Journal of Mathematics.

Chapter 2

Preliminaries

2.1 Free products of metric spaces

To prepare for the definition of free products of metric spaces, we introduce several notations.

Definition 2.1.1. Let (X, d_X) be a metric space and let X_0 be a set. An *index map* is a map $i_X : X_0 \to X$ such that for any compact subset $K \subset X$, the preimage $i_X^{-1}(K)$ is a finite set. We choose a *base point* $e_X \in i_X(X_0)$. We call (X_0, i_X, e_X) a *net* of X. For $x_0 \in X_0$, we denote by $\overline{x_0}$ the image $i_X(x_0)$.

Example 2.1.2. Let (X, d_X, e_X) be a metric space with a base point e_X. Let G be a group acting on X by isometries. We say that G *acts properly* on X if for any compact subset $B \subset X$, the set

$$\{g \in G \mid g(B) \cap B \neq \emptyset\}$$

is a finite set. We say that G *acts cocompactly* on X if there exists a compact subset $K \subset X$ such that

$$\bigcup_{g \in G} g(K) = X.$$

When a group G acts properly and cocompactly on X, the orbit map $o(e_X) : G \to X, g \mapsto g(e_X)$ is an index map of X. Then $(G, o(e_X), e_X)$ is a net of X. We call $(G, o(e_X), e_X)$ the *G-net*.

Remark 2.1.3. In the definition of the net in Definition 2.1.1, we do not require that the index map is injective. Indeed, let G be a group acting on a metric space X with base point e_X as in Example 2.1.2. Then the orbit map $o(e_X)\colon G \to X$ is injective if and only if the stabilizer of e_X is trivial.

Let (X, d_X) and (Y, d_Y) be metric spaces with nets and let (X_0, i_X, e_X) and (Y_0, i_Y, e_Y) be nets of X and Y, respectively. We choose ϵ_X and ϵ_Y such that $i_X(\epsilon_X) = e_X$ and $i_Y(\epsilon_Y) = e_Y$, respectively. Let $X_0^* = X_0 \setminus \{\epsilon_X\}$ and $Y_0^* = Y_0 \setminus \{\epsilon_Y\}$. A *normal word* on $X_0^* \sqcup Y_0^*$ is a finite sequence

$$w_0 w_1 \cdots w_n \quad (n \geq 0, w_i \in X_0^* \sqcup Y_0^*)$$

such that for all i, we have

$$\begin{aligned} w_i \in X_0^* &\Rightarrow w_{i+1} \in Y_0^*, \\ w_i \in Y_0^* &\Rightarrow w_{i+1} \in X_0^*. \end{aligned}$$

Let $\omega = w_0 w_1 \cdots w_n$ be a normal word. We define the *length* of ω to be $n+1$. We denote by ϵ the *empty word*. We define that the length of the empty word is zero.

We define W to be the set consisting of the empty word ϵ and all normal words. We define W_X to be the subset of W consisting of ϵ and normal words whose last letter does not belong to X_0^*, that is,

$$W_X := \{w_0 w_1 \cdots w_n \in W \mid w_n \notin X_0^*\} \bigsqcup \{\epsilon\},$$

and we similarly define W_Y, that is,

$$W_Y := \{w_0 w_1 \cdots w_n \in W \mid w_n \notin Y_0^*\} \bigsqcup \{\epsilon\}.$$

We use the following disjoint union $\widetilde{X * Y}$,

$$\begin{aligned} \widetilde{X * Y} := &(W_X \times X) \bigsqcup (W_Y \times Y) \\ &\bigsqcup (W_X \times X_0^* \times [0,1]) \bigsqcup (W_Y \times Y_0^* \times [0,1]) \bigsqcup (\epsilon \times [0,1]). \end{aligned}$$

We define an equivalence relation $\sim$ on $\widetilde{X * Y}$ as follows:

- $(\epsilon, e_X) \sim (\epsilon, 0)$ and $(\epsilon, 1) \sim (\epsilon, e_Y)$.

- Let $\omega \in W_X$ and $x_0 \in X_0^*$.

$$\{\omega\} \times X \ni (\omega, \overline{x_0}) \sim (\omega, x_0, 0) \in \{\omega\} \times \{x_0\} \times [0,1],$$
$$\{\omega x_0\} \times Y \ni (\omega x_0, e_Y) \sim (\omega, x_0, 1) \in \{\omega\} \times \{x_0\} \times [0,1].$$

- Let $\tau \in W_Y$ and $y_0 \in Y_0^*$. we define

$$\{\tau\} \times Y \ni (\tau, \overline{y_0}) \sim (\tau, y_0, 0) \in \{\tau\} \times \{y_0\} \times [0,1],$$
$$\{\tau y_0\} \times X \ni (\tau y_0, e_X) \sim (\tau, y_0, 1) \in \{\tau\} \times \{y_0\} \times [0,1].$$

DEFINITION 2.1.4. *The free product* $X * Y$ is the quotient of $\widetilde{X * Y}$ by the equivalent relation $\sim$.

The free product $X * Y$ consists of the following two types of components (see Figure 2.1).

- The *sheets* consists of $\{\omega\} \times X$ and $\{\tau\} \times Y$, where $\omega \in W_X$ and $\tau \in W_Y$. For simplicity, we write $\{\omega\} \times X$ and $\{\tau\} \times Y$ as ωX and τY, respectively. We identify X and respectively Y with ϵX and respectively ϵY. We call ω and τ *index words.* For each sheet, the *height of the sheet* is the length of the index word.

- The *edges* consists of the following three:
  - There exists an edge $\{\epsilon\} \times [0,1]$ connecting $(\epsilon, e_X) \in \{\epsilon\} \times X$ and $(\epsilon, e_Y) \in \{\epsilon\} \times Y$
  - Let $\omega \in W_X$ and $x_0 \in X_0^*$. Then there exists an edge $\{\omega\} \times \{x_0\} \times [0,1]$ connecting $(\omega, \overline{x_0}) \in \{\omega\} \times X$ and $(\omega x_0, e_Y) \in \{\omega x_0\} \times Y$.
  - Let $\tau \in W_Y$ and $y_0 \in Y_0^*$. Then there exists an edge $\{\tau\} \times \{y_0\} \times [0,1]$ connecting $(\tau, \overline{y_0}) \in \{\tau\} \times Y$ and $(\tau y_0, e_X) \in \{\tau y_0\} \times X$.

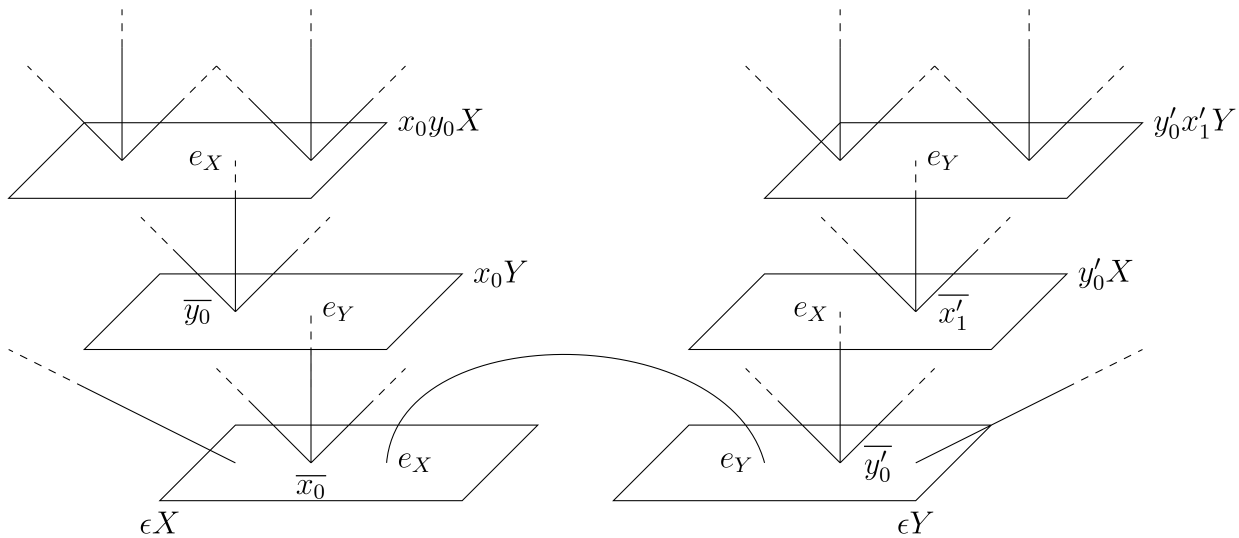

Figure 2.1: The free product $X * Y$. Here, $x_0, x'_1 \in X_0^*$ and $y_0, y'_0 \in Y_0^*$.

NOTATION 2.1.5. Each point $p \in X * Y$ can naturally identified with the triplet (ω, z, t) where $\omega \in W$, $z \in X \sqcup Y$ and $t \in [0, 1)$ satisfying:

- If $\omega \in W_X$, then $z \in X$,
- If $\omega \in W_Y$, then $z \in Y$,
- If $z \notin i_X(X_0) \sqcup i_Y(Y_0)$, then $t = 0$.

We call (ω, z, t) the *coordinate* of p. We say that p *belongs to sheets* if $t = 0$, and say that p *belongs to edges* if $t > 0$. When p belongs to a sheet, we abbreviate $(\omega, z, 0)$ as (ω, z).

NOTATION 2.1.6. We use the following notations.

- For $z \in X \sqcup Y$, set

$$\|z\| := \begin{cases} d_X(e_X, z) & \text{if } z \in X, \\ d_Y(e_Y, z) & \text{if } z \in Y. \end{cases}$$

- For $u, v \in X \sqcup Y$, set

$$d_{X\sqcup Y}(u,v) \coloneqq \begin{cases} d_X(u,v) & \text{if } \{u,v\} \subset X, \\ d_Y(u,v) & \text{if } \{u,v\} \subset Y, \\ \infty & \text{else.} \end{cases}$$

We will construct the metric d_* on the free product $X * Y$. First, we define a function $D_*\colon W \times W \to \mathbb{R}$. Let $\omega = uw_0 \cdots w_n \in W$ and $\omega' = uw'_0 \cdots w'_m \in W$, where $u = u_0 \dots u_k \in W$ is the maximal common prefix, that is, we have $w_0 \neq w'_0$. We define

$$D_*(\omega, \omega') \coloneqq d_{X\sqcup Y}(\overline{w_0}, \overline{w'_0}) + \sum_{i=1}^{n} \|\overline{w_i}\| + \sum_{j=1}^{m} \left\|\overline{w'_i}\right\| + n + m + 2.$$

For $\omega = w_0 \cdots w_n \in W$, we define

$$D_*(\epsilon, \omega) \coloneqq \sum_{i=1}^{n} \|\overline{w_i}\| + n + 1.$$

Finally we define $D_*(\epsilon, \epsilon) = 0$.

Now we will define the metric d_* on $X * Y$. First, we define d_* on sheets. Let $p, q \in X * Y$. We suppose that p and q belong to sheets. Let $(\omega, u, 0)$ and $(\tau, v, 0)$ be the coordinate of p and q, respectively, Here, $\omega, \tau \in W$ and $u, v \in X \sqcup Y$, as in Notation 2.1.5. We consider the following three cases.

(I) ω and τ are equal.

In this case, first, we suppose that $\omega = \tau \neq \epsilon$. Then we define

$$d_*(p,q) \coloneqq d_{X\sqcup Y}(u,v).$$

Next, we suppose that $\omega = \tau = \epsilon$. Then we define

$$d_*(p,q) \coloneqq \begin{cases} d_{X\sqcup Y}(u,v) & \text{if } \{u,v\} \subset X \text{ or } \{u,v\} \subset Y, \\ \|u\| + \|v\| + 1 & \text{else} \quad \text{(see Figure 2.2).} \end{cases}$$

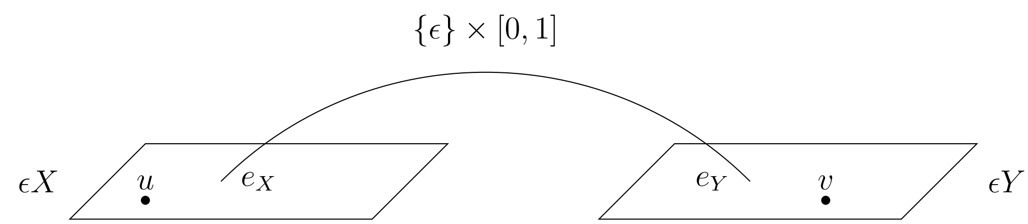

Figure 2.2: An example of case (I). Let $p = (\epsilon, u)$ and $q = (\epsilon, v)$, where $u \in X$, and $v \in Y$.

(II) ω is a proper subword of τ.

In this case, there exists $\tau' \in W \setminus \{\epsilon\}$ such that $\tau = \omega\tau'$. Let z be the initial letter of τ'. First, we assume that ω is not the empty word. See Figure 2.3. In this figure, $z = x_0'$. Then we define

$$d_*(p, q) := d_{X \sqcup Y}(u, \overline{z}) + D_*(\epsilon, \tau') + \|v\|.$$

Next, we assume that ω is the empty word. If $\{u, \overline{z}\} \subset X$ or $\{u, \overline{z}\} \subset Y$ holds, then we define

$$d_*(p, q) := d_{X \sqcup Y}(u, \overline{z}) + D_*(\epsilon, \tau') + \|v\|.$$

Otherwise, we define

$$d_*(p, q) := \|u\| + \|\overline{z}\| + D_*(\epsilon, \tau') + \|v\| + 1.$$

(III) Neither (I) nor (II).

In this case, there exist the maximal common prefix $\rho \in W$ (possibly the empty word) and $\omega', \tau' \in W \setminus \{\epsilon\}$ such that $\omega = \rho\omega'$ and $\tau = \rho\tau'$. Let z_0 and w_0 be the initial letter of ω' and τ', respectively. See Figure 2.4. In this figure, $z_0 = x_0$ and $w_0 = x_0'$. Then we define

$$d_*(p, q) := d_*((\rho, \overline{z_0}), (\rho, \overline{w_0})) + D_*(\epsilon, \omega') + D_*(\epsilon, \tau') + \|u\| + \|v\|.$$

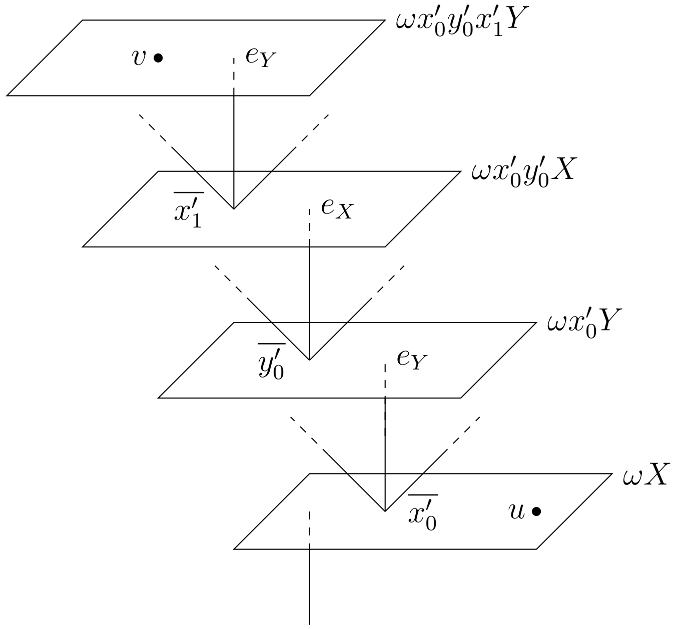

Figure 2.3: An example of case (II). Let $p = (\omega, u)$ and $q = (\omega x_0' y_0' x_1', v)$, where $\omega \in W_X \setminus \{\epsilon\}$, $x_i' \in X_0^*$, $y_0' \in Y_0^*$, $u \in X$, and $v \in Y$.

Finally, we extend d_* on edges in an obvious way. From the construction, it is clear that d_* is , non-degenerate, and satisfies the triangle inequality. This completes the construction of the metric d_* on the free product $X * Y$.

EXAMPLE 2.1.7. Let G and H be finitely generated groups and let S_G and S_H be finite generating sets of G and H, respectively. Let $\Gamma(G, S_G)$ and $\Gamma(H, S_H)$ be Cayley graphs of G and H. We can construct the free product $\Gamma(G, S_G)$ and $\Gamma(H, S_H)$ with respect to (G, ι_G, e_G) and (H, ι_H, e_H), where ι_G and ι_H are the inclusion maps. The resulting space $\Gamma(G, S_G) * \Gamma(H, S_H)$ coincides with the free product of Cayley graphs in the sense of Pisanski and Tucker [16].

The following example shows that the coarse geometry of a free product depends on the choice of nets.

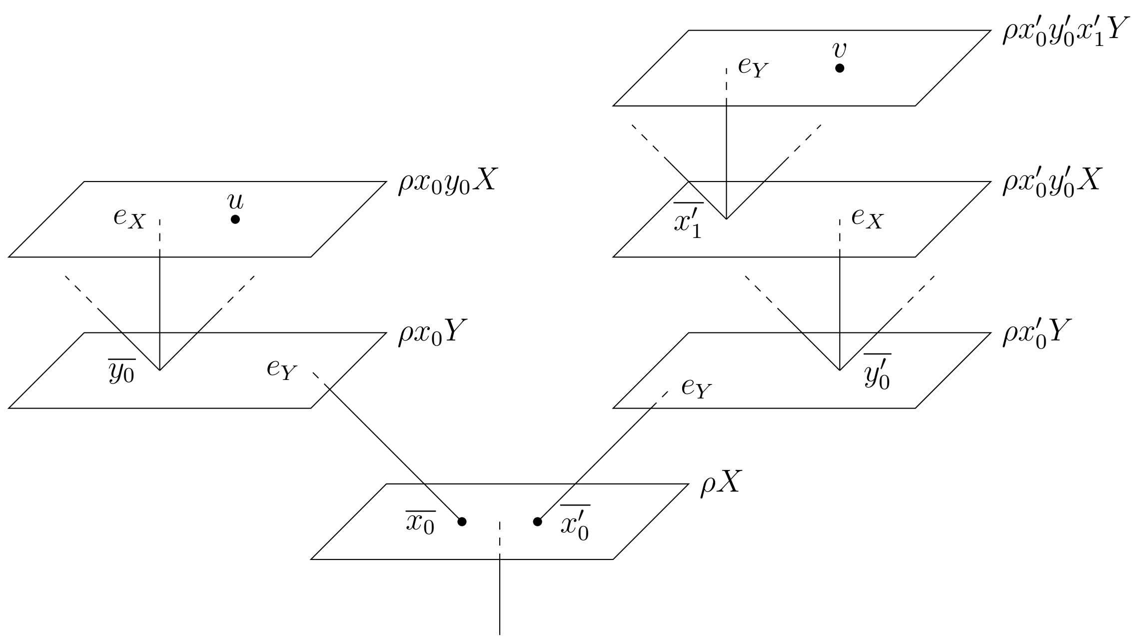

Figure 2.4: An example of (III). Let $p = (\rho x_0 y_0, u)$ and $q = (\rho x'_0 y'_0 x'_1, v)$, where $x_i, x'_i \in X_0^*$, $y_0, y'_0 \in Y_0^*$, $u \in X$, and $v \in Y$. Here, $\rho \in W_X$ and $x_0 \neq x'_0$.

Example 2.1.8. For $m \in \mathbb{N}$, set $G_m = \mathbb{Z}/m\mathbb{Z}$. We assume that G_m acts on $\mathbb{R}^2$ by rotations, that is,

$$[l] \cdot (r\cos\theta, r\sin\theta) = \left(r\cos\left(\theta + \frac{2\pi l}{m}\right), r\sin\left(\theta + \frac{2\pi l}{m}\right)\right).$$

Let $o((1,0))$ be an orbit map of $(1,0)$ by this action. The triplet $(G_m, o((1,0)), (1,0))$ is a net of $\mathbb{R}^2$. For $m, n \in \mathbb{N}$, we consider the free product $\mathbb{R}^2 * \mathbb{R}^2$ with respect to $(G_m, o((1,0)), (1,0))$ and $(G_n, o((1,0)), (1,0))$. We will see how the growth type of $\mathbb{R}^2 * \mathbb{R}^2$ depends on the choice of nets.

We denote by $V(R)$ the number of the points of the form $(\omega, (p,q))$ in the closed ball of radius R centered at $(\epsilon, (0,0))$, where ω is a word and p, q are integers.

Set $G_m^* = G_m \setminus \{0\}$ and let W be the set of normal words on $G_n^* \sqcup G_m^*$, and the empty word ϵ.

(1) Suppose $m = 1$. In this case, G_m^* is empty, so normal words consist only of single letters, namely, $W = G_n^* \sqcup \{\epsilon\}$. Then, the number of sheets is $n + 1$. We illustrate the shape of $\mathbb{R}^2 * \mathbb{R}^2$ in Figure 2.5. We can estimate $V(R)$ as follows:

$$V(R) \leq (n + 1) \cdot 2\pi R^2.$$

Therefore, in this case, $\mathbb{R}^2 * \mathbb{R}^2$ has a polynomial growth.

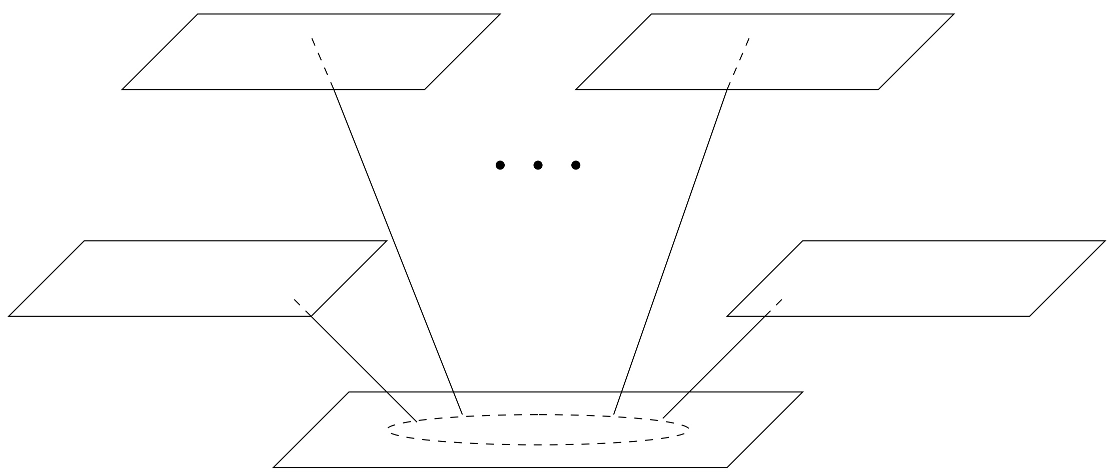

Figure 2.5: $m = 1$.

(2) Suppose $m = 2$ and $n = 2$. In this case the set of letters $G_2^* \sqcup G_2^* = \{1\} \sqcup \{1\}$ contains two elements. We illustrate the shape of $\mathbb{R}^2 * \mathbb{R}^2$ in Figure 2.6. We can estimate $V(R)$:

$$V(R) \leq R \cdot 2\pi R^2.$$

Therefore, in this case, $\mathbb{R}^2 * \mathbb{R}^2$ has a polynomial growth.

(3) Suppose $m \geq 3$ and $n \geq 2$. In this case G_m^* has at least two elements. Then the free product has a tree-like structure. Since the growth of the free product of the groups $G_m * G_n$ is exponential, the number of sheets grows exponentially. Therefore, in this case, $\mathbb{R}^2 * \mathbb{R}^2$ has a exponential growth.

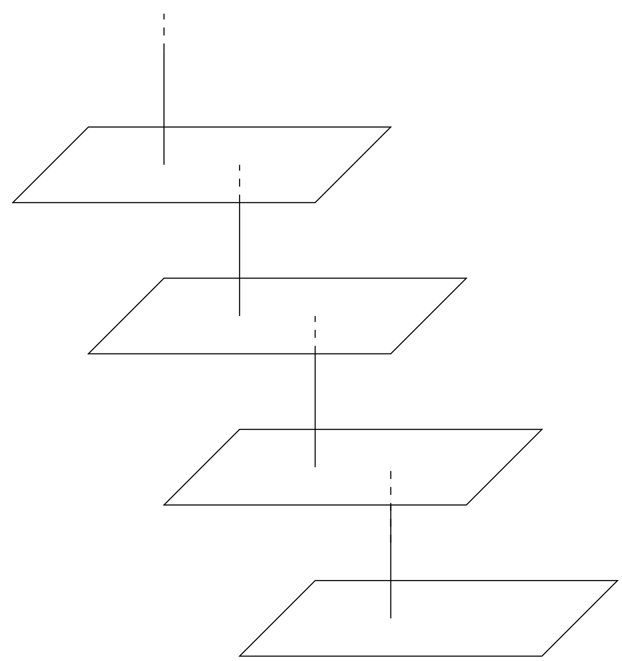

Figure 2.6: $m = 2$ and $n = 2$.

REMARK 2.1.9. Let X, X', and Y be metric spaces. We suppose that X and X' are quasi-isometric. In general, $X * Y$ and $X' * Y$ are not necessarily quasi-isometric. For example, let Γ_2 be a Cayley graph of $\mathbb{Z}/2\mathbb{Z}$ for some generating set, and Γ_3 be that of $\mathbb{Z}/3\mathbb{Z}$. It is clear that Γ_2 is quasi-isometric to Γ_3. By Proposition 3.2.1, $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ acts properly and cocompactly on $\Gamma_2 * \Gamma_2$, and so does $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ on $\Gamma_3 * \Gamma_2$. Since $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ has two ends and $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ has infinitely many ends, $\Gamma_2 * \Gamma_2$ and $\Gamma_3 * \Gamma_2$ are not quasi-isometric.

2.2 Coarsely convex spaces

In this section, we briefly review coarse geometry and coarsely convex spaces.

2.2.1 Geodesic metric spaces and proper metric spaces

A metric space (X, d_X) is a *geodesic metric space* if for any $x, x' \in X$, there exists a map $\gamma : [0, a] \to X$ such that $\gamma(0) = x$, $\gamma(a) = x'$, and $d_X(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, a]$.

We say that a metric space (X, d_X) is a *proper metric space* if every bounded closed subset in X is compact.

2.2.2 Coarse equivalence and quasi-isometry

Let (X, d_X) and (Y, d_Y) be metric spaces. We say that a map $f : X \to Y$ is a *coarse map* if there exist a non-decreasing function $\rho_+ : [0, \infty) \to [0, \infty)$ such that the inequality

$$d_Y(f(x), f(x')) \leq \rho_+(d_X(x, x'))$$

holds for any $x, x' \in X$, and for any bounded subset $B \subset Y$, the preimage $f^{-1}(B)$ is bounded.

We also say that a map $f : X \to Y$ is a *coarse embedding* if there exist non-decreasing functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ such that

$$\lim_{t \to \infty} \rho_-(t) = \infty,$$

and the inequality

$$\rho_-(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_+(d_X(x, x'))$$

holds for any $x, x' \in X$. When we can choose ρ_- and ρ_+ to be affine maps, we say that the map f is a *quasi-isometric embedding*.

Let $X' \subset X$. For $M \geq 0$, we say that X' is M*-dense* in X if $X = B_M(X')$, where $B_M(X')$ is the closed M-neighborhood of X'. We say that X and Y are *coarsely equivalent* if there exist a coarse embedding map $f : X \to Y$ and $M \geq 0$ such that $f(X)$ is M-dense in Y. We say that X and Y are *quasi-isometric* if there exist a quasi-isometric embedding map $f : X \to Y$ and $M \geq 0$ such that $f(X)$ is M-dense in Y.

Let $\lambda \geq 1$ and $k \geq 0$. A (λ, k)-*quasi-geodesic segment* is a (λ, k)-quasi-isometric embedding $\gamma : [0, a] \to X$, that is, the inequality

$$\lambda^{-1}|t - t'| - k \leq d_X(\gamma(t), \gamma(t')) \leq \lambda|t - t'| + k$$

holds for all $t, t' \in [0, a]$.

2.2.3 Coarsely convex spaces

Definition 2.2.1. Let (X, d_X) be a metric space. Let $\lambda \geq 1, k \geq 0, E \geq 1$, and $C \geq 0$ be constants. Let $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a non-decreasing function. Let $\mathcal{L}$ be a family of (λ, k)-quasi-geodesic segments. The metric space X is $(\lambda, k, E, C, \theta, \mathcal{L})$-*coarsely convex*, if $\mathcal{L}$ satisfies the following:

(CC1) For any $v, w \in X$, there exists a quasi-geodesic segment $\gamma \in \mathcal{L}$ with $\gamma : [0, a] \to X$, $\gamma(0) = v$ and $\gamma(a) = w$.

(CC2) Let $\gamma, \eta \in \mathcal{L}$ be quasi-geodesic segments with $\gamma : [0, a] \to X$ and $\eta : [0, b] \to X$. Then for all $t \in [0, a], s \in [0, b]$, and $c \in [0, 1]$, we have that

$$d_X(\gamma(ct), \eta(cs)) \leq (1 - c)E d_X(\gamma(0), \eta(0)) + cE d_X(\gamma(t), \eta(s)) + C.$$

We call this inequality *the coarsely convex inequality*.

(CC3) Let $\gamma, \eta \in \mathcal{L}$ be quasi-geodesic segments with $\gamma : [0, a] \to X$ and $\eta : [0, b] \to X$. Then for all $t \in [0, a]$ and $s \in [0, b]$, we have that

$$|t - s| \leq \theta(d_X(\gamma(0), \eta(0)) + d_X(\gamma(t), \eta(s))).$$

The family $\mathcal{L}$ satisfying (CC1), (CC2), and (CC3) is called a *system of good quasi-geodesic segments*, and elements $\gamma \in \mathcal{L}$ are called *good quasi-geodesic segments*.

We say that a metric space X is *a coarsely convex space* if there exist constants λ, k, E, C, a non-decreasing function $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a family of (λ, k)-quasi-geodesic segments $\mathcal{L}$ such that X is $(\lambda, k, E, C, \theta, \mathcal{L})$-coarsely convex.

If the family $\mathcal{L}$ consists of only geodesic segments, then $\mathcal{L}$ always satisfies (CC3) by the triangle inequality.

LEMMA 2.2.2. *Let (X, d_X) be a metric space. Let $\mathcal{L}$ be a family of geodesic segments. Then $\mathcal{L}$ satisfies* (CC3). *In particular, we can take a non-decreasing function satisfying* (CC3) *to the identity map* $\mathrm{id}_{\mathbb{R}_{\geq 0}}$.

PROOF. Let (X, d_X) be a metric space and let $\mathcal{L}$ be a family of geodesic segments. Let $\gamma, \eta \in \mathcal{L}$ be geodesic segments with $\gamma : [0, a] \to X$ and $\eta : [0, b] \to X$. Set $t \in [0, a]$ and $s \in [0, b]$. We suppose that $t > s$. Since γ and η are geodesic segments, we have

$$|t - s| = d_X(\gamma(0), \gamma(t)) - d_X(\eta(0), \eta(s)).$$

By the triangle inequality, we have

$$\begin{aligned}
&d_X(\gamma(0), \gamma(t)) - d_X(\eta(0), \eta(s)) \\
&\leq d_X(\gamma(0), \eta(0)) + d_X(\eta(0), \eta(s)) + d_X(\eta(s), \gamma(t)) - d_X(\eta(0), \eta(s)) \\
&= d_X(\gamma(0), \eta(0)) + d_X(\gamma(t), \eta(s)).
\end{aligned}$$

Therefore the family $\mathcal{L}$ satisfies (CC3). □

Let X be a metric space. For a map $\gamma : [a, b] \to X$, we denote by γ^{-1}, the map $\gamma^{-1} : [a, b] \to X$ defined by $\gamma^{-1}(t) := \gamma(b - (t - a))$ for $t \in [a, b]$. For $c \in [a, b]$, we denote by $\gamma|_{[a,c]}$ the restriction of γ to $[a, c]$. Let $\mathcal{L}$ be a family of quasi-geodesic segments in X. The family $\mathcal{L}$ is *symmetric* if $\gamma^{-1} \in \mathcal{L}$ for all $\gamma \in \mathcal{L}$, and $\mathcal{L}$ is *prefix-closed* if $\gamma|_{[a,c]} \in \mathcal{L}$ for all $\gamma \in \mathcal{L}$ with $\gamma : [a, b] \to X$ and for all $c \in [a, b]$.

The following Proposition 2.2.3 plays an important role in the proof of the main result.

PROPOSITION 2.2.3. *Let (X, d_X) be a metric space. Let $E \geq 1$ and $C \geq 0$ be constants. Let $\mathcal{L}$ be a family of geodesic segments. Suppose that $\mathcal{L}$ satisfies* *the following* (gCC1), (gCC2), *and* (gCC3), *then* X *is* $(1, 0, E, 2C, \mathrm{id}_{\mathbb{R}_{\geq 0}}, \mathcal{L})$-*coarsely convex:*

(gCC1) *For any* $v, w \in X$*, there exists a geodesic segment* $\gamma \in \mathcal{L}$ *with* $\gamma : [0, a] \to X$*,* $\gamma(0) = v$ *and* $\gamma(a) = w$*.*

Let $\gamma, \eta \in \mathcal{L}$ *be geodesic segments with* $\gamma : [0, a] \to X$ *and* $\eta : [0, b] \to X$*.*

(gCC2) *Suppose* $\gamma(0) = \eta(0)$*. Then for all* $t \in [0, a], s \in [0, b]$*, and* $c \in [0, 1]$*, we have that*

$$d_X(\gamma(ct), \eta(cs)) \leq cEd_X(\gamma(t), \eta(s)) + C.$$

(gCC3) *For* $t \in [0, a]$*, let* $\gamma' \in \mathcal{L}$ *with* $\gamma' : [0, d] \to X$*,* $\gamma'(0) = \eta(0)$*, and* $\gamma'(d) = \gamma(t)$*, we have*

$$d_X(\gamma(ct), \gamma'(cd)) \leq (1 - c)Ed_X(\gamma(0), \eta(0)) + C.$$

Proof. Since $\mathcal{L}$ is a family of geodesic segment, by Lemma 2.2.2, (CC3) holds. (gCC1) implies (CC1).

Let $\gamma, \eta \in \mathcal{L}$ be geodesic segments with $\gamma : [0, a] \to X$ and $\eta : [0, b] \to X$. Let $t \in [0, a]$, $s \in [0, b]$, and $c \in [0, 1]$. Let $\gamma' \in \mathcal{L}$ with $\gamma' : [0, d] \to X$, $\gamma'(0) = \eta(0)$, and $\gamma'(d) = \gamma(t)$. By (gCC3), we have

$$\begin{aligned} d_X(\gamma(ct), \gamma'(cd)) &\leq (1 - c)Ed_X(\gamma(0), \gamma'(0)) + C \\ &= (1 - c)Ed_X(\gamma(0), \eta(0)) + C. \end{aligned} \tag{2.1}$$

By (gCC2), we have

$$\begin{aligned} d_X(\gamma'(cd), \eta(cs)) &\leq cEd_X(\gamma'(d), \eta(s)) + C \\ &= cEd_X(\gamma(t), \eta(s)) + C \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2) yields

$$\begin{aligned} d_X(\gamma(ct), \eta(cs)) &\leq d_X(\gamma(ct), \gamma'(cd)) + d_X(\gamma'(cd), \eta(cs)) \\ &\leq (1 - c)Ed_X(\gamma(0), \eta(0)) + cEd_X(\gamma(t), \eta(s)) + 2C \end{aligned}$$

Therefore, $\mathcal{L}$ satisfies (CC2). □

We say that X is a *geodesic coarsely convex space* if there exist constants E, C, and a family of geodesic segments $\mathcal{L}$ such that $\mathcal{L}$ satisfies (gCC1), (gCC2), and (gCC3).

Chapter 3

Main results

In this section, we consider the free products of geodesic coarsely convex spaces.

3.1 Free products of geodesic coarsely convex spaces

THEOREM 3.1.1. *Let X and Y be metric spaces with nets. If X and Y are geodesic coarsely convex spaces, then the free product $X * Y$ is a geodesic coarsely convex space.*

In the rest of this section, we give the proof of Theorem 3.1.1. First, we construct the family of geodesic segments $\mathcal{L}_*$ in $X * Y$. Let (X_0, i_X, e_X) and (Y_0, i_Y, e_Y) be nets of X and Y, respectively, and let $\mathcal{S}$ be the sheets of $X * Y$. Since X and Y are geodesic coarsely convex, we can suppose that X and Y are $(1, 0, E_X, C_X, \mathrm{id}_{\mathbb{R}_{\geq 0}}, \mathcal{L}_X)$-coarsely convex and $(1, 0, E_Y, C_Y, \mathrm{id}_{\mathbb{R}_{\geq 0}}, \mathcal{L}_Y)$-coarsely convex, respectively. We define

$$E \coloneqq \max\{E_X, E_Y\},$$
$$C \coloneqq \max\{C_X, C_Y\}.$$

For any $S \in \mathcal{S}$ and for any $x, y \in S$, we choose a good geodesic in S from x to y, denoted by $\gamma_S(x,y)$. Let E be the set of edges. For $e \in E$ and $v, w \in e$, let $\gamma_e(v,w)$ be the geodesic segment on e from v to w.

Note that $e \in E$ is a unit interval. Then, for $e \in E$ and $v, w \in e$, we have $\gamma_e(w,v) = \gamma_e(v,w)^{-1}$. We define

$$\mathcal{L} \coloneqq \{\gamma_S(x,y) \mid S \in \mathcal{S},\ x, y \in S\} \bigsqcup \{\gamma_e(v,w) \mid e \in E, v, w \in e\}$$

For $a, b \in X * Y$, we define a geodesic segment from a to b by connecting $\gamma \in \mathcal{L}$ at the identified points, denoted by $\Gamma(a,b)$. Firstly, we suppose that a and b belong to sheets S_1 and S_2, respectively. Let e_{S_1} and e_{S_2} be base points of S_1 and S_2 and let (ω, u) and (τ, v) be coordinate of a and b, respectively. Here, $\omega, \tau \in W$ and $u, v \in X \sqcup Y$, as in Notation 2.1.5. We consider the following three cases.

(I) ω and τ are equal.

In this case, first, we suppose that $\omega = \tau \neq \epsilon$ or $S_1 = S_2$. Then we define

$$\Gamma(a,b)(t) \coloneqq \gamma_{S_1}(u,v)(t).$$

Next, we suppose that $\omega = \tau = \epsilon$ and $S_1 \neq S_2$. Let $e_\epsilon = \{\epsilon\} \times [0,1]$. Then we define

$$\Gamma(a,b)(t) \coloneqq \begin{cases} \gamma_{S_1}(u, e_{S_1})(t) & \text{for } 0 \leq t \leq \|u\|, \\ \gamma_{e_\epsilon}(e_{S_1}, e_{S_2})(t - t_1) & \text{for } t_1 \leq t \leq t_1 + 1, \\ \gamma_{S_2}(e_{S_2}, v)(t - t_2) & \text{for } t_2 \leq t \leq t_2 + \|v\|, \end{cases}$$

where $t_1 = \|u\|$ and $t_2 = t_1 + 1$.

(II) ω is a proper subword of τ.

In this case, there exists $\tau' \in W \setminus \{\epsilon\}$ such that $\tau = \omega\tau'$. Let z be the initial letter of τ'. We define the geodesic segment from a to b inductively for the length of τ'. First, we assume that ω is not the empty word and the height of τ' is equal to 1, that is, $\tau' = z$. Let $e = \{\omega\} \times \{z\} \times [0,1]$ Then we define

$$\Gamma(a,b)(t) \coloneqq \begin{cases} \gamma_{S_1}(u,\overline{z})(t) & \text{for } 0 \le t \le d_*(u,\overline{z}), \\ \gamma_{e}(\overline{z}, e_{S_2})(t-t_1) & \text{for } t_1 \le t \le t_1+1, \\ \gamma_{S_2}(e_{S_2}, v)(t-t_2) & \text{for } t_2 \le t \le t_2 + \|v\|, \end{cases}$$

where $t_1 = d_*(u,\overline{z})$ and $t_2 = t_1 + 1$. Next, we assume that ω is the empty word and $\tau' = z$. Let $e' = \{\epsilon\} \times \{z\} \times [0,1]$. If $\{u,\overline{z}\} \subset X$ or $\{u,\overline{z}\} \subset Y$ holds, then we define

$$\Gamma(a,b)(t) \coloneqq \begin{cases} \gamma_{S_1}(u,\overline{z})(t) & \text{for } 0 \le t \le d_*(u,\overline{z}), \\ \gamma_{e'}(\overline{z}, e_{S_2})(t-t_1) & \text{for } t_1 \le t \le t_1+1, \\ \gamma_{S_2}(e_{S_2}, v)(t-t_2) & \text{for } t_2 \le t \le t_2 + \|v\|, \end{cases}$$

where $t_1 = d_*(u,\overline{z})$ and $t_2 = t_1 + 1$. Otherwise, then we define

$$\Gamma(a,b)(t) \coloneqq \begin{cases} \gamma_{S_1}(u,e_{S_1})(t) & \text{for } 0 \le t \le \|u\|, \\ \gamma_{e_\epsilon}(e_{S_1}, e_{S_0})(t-t_1) & \text{for } t_1 \le t \le t_1+1, \\ \gamma_{S_0}(e_{S_0}, \overline{z})(t-t_2) & \text{for } t_2 \le t \le t_2 + \|\overline{z}\|, \\ \gamma_{e'}(\overline{z}, e_{S_2})(t-t_3) & \text{for } t_3 \le t \le t_3+1, \\ \gamma_{S_2}(e_{S_2}, v)(t-t_4) & \text{for } t_4 \le t \le t_4 + \|v\|, \end{cases}$$

where $t_1 = \|u\|$, $t_2 = t_1 + 1$, $t_3 = t_2 + \|\overline{z}\|$, and $t_4 = t_3 + 1$. Here, $S_0 = \epsilon X$ or $S_0 = \epsilon Y$. We denote by e_{S_0}, the base point of S_0. From the construction, it is clear that $\Gamma(a,b)$ is a geodesic segment.

Finally, we suppose that the length of τ' is greater than 1. Let $\tau' = l_0 l_1 l_2 \dots l_n$ and let $a' = (\omega l_0 l_1 \dots l_{n-1}, \overline{l_n})$. Let $\hat{e} = \{\omega l_0 l_1 \dots l_{n-1}\} \times \{l_n\} \times [0,1]$. By the assumption of induction, $\Gamma(a,a')$ is defined. Then we define $\Gamma(a,b)$ as follows:

$$\Gamma(a,b) \coloneqq \begin{cases} \Gamma(a,a')(t) & \text{for } 0 \leq t \leq d_*(a,a'), \\ \gamma_{\hat{e}}(\overline{l_n}, e_{S_2})(t-t_1) & \text{for } t_1 \leq t \leq t_1+1, \\ \gamma_{S_2}(e_{S_2}, v)(t-t_2) & \text{for } t_2 \leq t \leq t_2 + \|v\|, \end{cases}$$

where $t_1 = d_*(a,a')$, $t_2 = t_1 + 1$.

(III) Neither (I) nor (II).

In this case, there exist the maximal common prefix $\rho \in W$ (possibly the empty word) and $\omega', \tau' \in W \setminus \{\epsilon\}$ such that $\omega = \rho\omega'$ and $\tau = \rho\tau'$. Let z_0 and w_0 be the initial letter of ω' and τ', respectively. Set $a' = (\rho, \overline{z_0})$ and $b' = (\rho, \overline{w_0})$. Then we define

$$\Gamma(a,b) \coloneqq \begin{cases} \Gamma(a,a')(t) & \text{for } 0 \leq t \leq d_*(a,a'), \\ \Gamma(a',b')(t-t_1) & \text{for } t_1 \leq t \leq t_1 + d_*(\overline{z_0}, \overline{w_0}), \\ \Gamma(b',b)(t-t_2) & \text{for } t_2 \leq t \leq t_2 + d_*(b',b), \end{cases}$$

where $t_1 = d_*(a,a')$ and $t_2 = t_1 + d_*(\overline{z_0}, \overline{w_0})$.

Finally, we extend $\Gamma(a,b)$ on edges in an obvious way. We denote by $[a,b]$, the image of the geodesic segment $\Gamma(a,b)$. Define

$$\mathcal{L}_* \coloneqq \{\Gamma(a,b) \mid a,b \in X * Y\}.$$

EXAMPLE 3.1.2. Let $a = (ux_0y_0, x)$ and $b = (ux_0', y)$, where $u \in W_X$ is the maximal common prefix, $x_0, x_0' \in X_0^*$, $y_0 \in Y_0^*$, $x \in X$, and $y \in Y$. Let $e_1 = \{ux_0\} \times \{y_0\} \times [0,1]$, $e_2 = \{u\} \times \{x_0\} \times [0,1]$, and $e_3 = \{u\} \times \{x_0'\} \times [0,1]$. Then the path $\Gamma(a,b) : [0, d_*(a,b)] \to X * Y$ is given as follows (see Figure

3.1):

$$\Gamma(a,b)(t) = \begin{cases} \gamma_{ux_0y_0X}(x, e_X)(t) & 0 \le t \le d_X(e_X, x), \\ \gamma_{e_1}(e_X, \overline{y_0})(t - t_1) & t_1 \le t \le t_1 + 1, \\ \gamma_{ux_0Y}(\overline{y_0}, e_Y)(t - (t_1+1)) & t_1 + 1 \le t \le t_1 + d_Y(e_Y, \overline{y_0}) + 1, \\ \gamma_{e_2}(e_Y, \overline{x_0})(t - t_2) & t_2 \le t \le t_2 + 1, \\ \gamma_{uX}(\overline{x_0}, \overline{x_0'})(t - (t_2+1)) & t_2 + 1 \le t \le t_2 + d_X(\overline{x_0}, \overline{x_0'}) + 1, \\ \gamma_{e_3}(\overline{x_0'}, e_Y)(t - t_3) & t_3 \le t \le t_3 + 1, \\ \gamma_{ux_0'Y}(e_Y, y)(t - (t_3+1)) & t_3 + 1 \le t \le t_3 + d_Y(e_Y, y) + 1, \end{cases}$$

where $t_1 = d_X(e_X, x)$, $t_2 = t_1 + d_Y(e_Y, \overline{y_0}) + 1$, and $t_3 = t_2 + d_X(\overline{x_0}, \overline{x_0'}) + 1$.

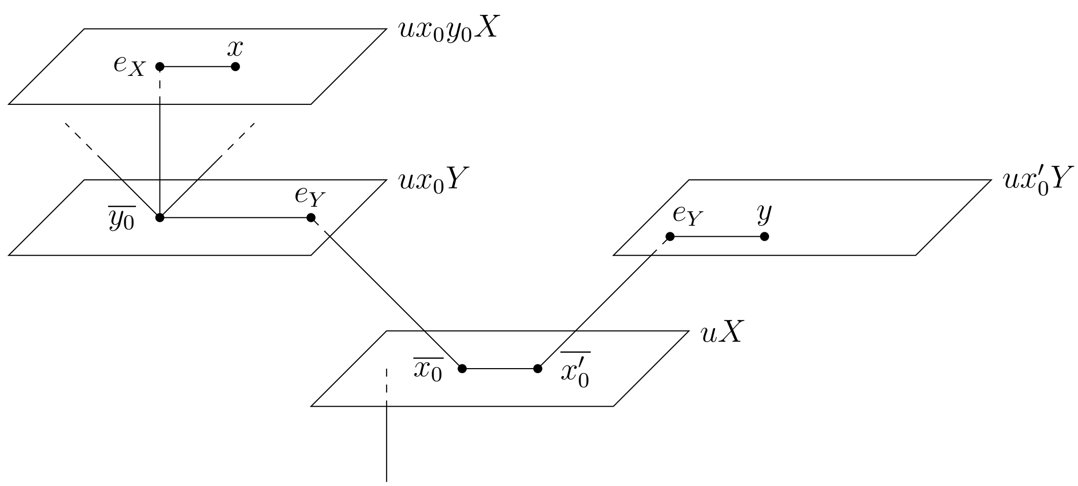

Figure 3.1: An example of $\Gamma(a,b) \in \mathcal{L}_*$

LEMMA 3.1.3. *We suppose that X and Y are (E,C)-geodesic coarsely convex spaces. For $v, w \in X * Y$ and $t' \in [0, d_*(v,w)]$, let $w' = \Gamma(v,w)(t')$. Then,*

$$d_*(\Gamma(v,w)(ct'), \Gamma(v,w')(ct')) \le C$$

holds for $c \in [0,1]$.

Proof. Set $\Gamma(v, w) = \Gamma_1$ and $\Gamma(v, w') = \Gamma_2$. Let $S' \in \mathcal{S}$ such that $w' \in S'$. Since Γ_1 and Γ_2 are geodesic segments, we can define Γ_2 to be $\Gamma_2 : [0, t'] \to X * Y$. Let

$$T \coloneqq \max\{t \in [0, t'] \mid \Gamma_1(s) = \Gamma_2(s) \text{ for any } s \in [0, t]\}$$

We can put $\Gamma_1(T) = \Gamma_1(c_0t') = \Gamma_2(c_0t')$ for some $c_0 \in [0, 1]$. Let $p = \Gamma_1(T)$. By the definition of $\mathcal{L}_*$, we obtain that p is in S'. Moreover, there exist $p' \in S'$ satisfying the following conditions: Let $\gamma_{S'}(p, p') \in \mathcal{L}$ be the geodesic segment with $\gamma_{S'}(p, p') : [0, u] \to S'$ and let $\gamma_{S'}(p, w') \in \mathcal{L}$ be the geodesic segment with $\gamma_{S'}(p, w') : [0, v] \to S'$. Such that

$$\begin{aligned}\Gamma_1([c_0t', t']) &\subseteq \gamma_{S'}(p, p')([0, u]),\\ \Gamma_2([c_0t', t']) &= \gamma_{S'}(p, w')([0, v]).\end{aligned}$$

Note that $\Gamma_1([0, c_0t']) = \Gamma_2([0, c_0t'])$, $u \geq v$, and $w' = \Gamma_1(t') = \gamma_{S'}(p, p')(v)$. Then for all $c \leq c_0$, we have that $\Gamma_1(ct') = \Gamma_2(ct')$. We suppose that $c > c_0$. Then there exist $c_1 \in [0, 1]$ such that

$$\begin{aligned}\Gamma_1(ct') &= \gamma_{S'}(p, p')(c_1v),\\ \Gamma_2(ct') &= \gamma_{S'}(p, w')(c_1v).\end{aligned}$$

Since X and Y are geodesic coarsely convex spaces, by the coarsely convex inequality, we have that

$$\begin{aligned}d_*(\Gamma_1(ct'), \Gamma_2(ct')) &= d_*(\gamma_{S'}(p, p')(c_1v), \gamma_{S'}(p, w')(c_1v))\\ &\leq c_1Ed_*(\gamma_{S'}(p, p')(v), \gamma_{S'}(p, w')(v)) + C\\ &= c_1Ed_*(w', w') + C = C.\end{aligned}$$

This complete the proof. □

Proposition 3.1.4. *The family of geodesic segments $\mathcal{L}_*$ satisfies* (gCC1), (gCC2), *and* (gCC3).

Proof. We remark that X and Y are geodesic spaces. Then, by the definition of $\mathcal{L}_*$, it is clear that $\mathcal{L}_*$ satisfies (gCC1).

We will prove that $\mathcal{L}_*$ satisfies (gCC2). Let $\Gamma_1 \in \mathcal{L}_*$ with $\Gamma_1 : [0,t] \to X * Y$ and let $\Gamma_2 \in \mathcal{L}_*$ with $\Gamma_2 : [0,s] \to X * Y$. We suppose that $\Gamma_1(0) = \Gamma_2(0)$. The geodesic triangle $\Delta(\Gamma_1(0), \Gamma_1(t), \Gamma_2(s))$ has the following form (see Figure 3.2): There exist $S \in \mathcal{S}$ and $p, a, b \in S$ such that

$$
\begin{aligned}
[\Gamma_1(0), \Gamma_1(t)] &= [\Gamma_1(0), p] \cup [p, a] \cup [a, \Gamma_1(t)], \\
[\Gamma_1(0), \Gamma_2(s)] &= [\Gamma_1(0), p] \cup [p, b] \cup [b, \Gamma_2(s)], \\
d_*(\Gamma_1(t), \Gamma_2(s)) &= d_*(\Gamma_1(t), a) + d_*(a, b) + d_*(b, \Gamma_2(s)).
\end{aligned}
$$

We set

$$
\begin{aligned}
p &\coloneqq \Gamma_1(c_0 t) = \Gamma_2(c_0' s), \\
a &\coloneqq \Gamma_1(c_1 t), \\
b &\coloneqq \Gamma_2(c_1' s),
\end{aligned}
$$

where $c_0, c_1, c_0', c_1' \in [0,1]$. Since Γ_1 and Γ_2 are geodesics, we have $c_0 t = c_0' s$.

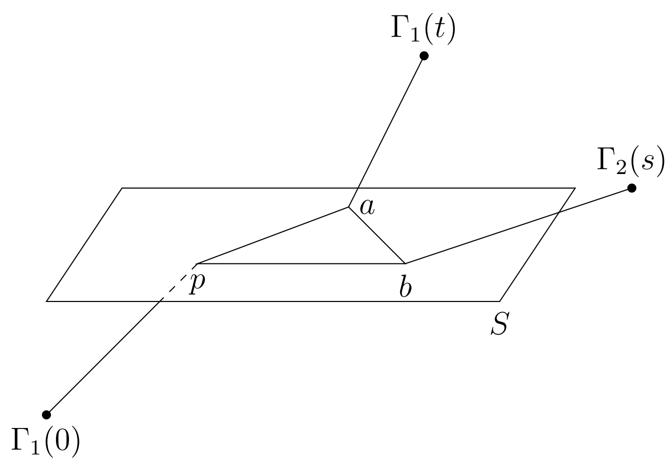

Figure 3.2: An example of geodesic triangles of $X * Y$.

Without loss of generality, we may assume that $c_1 \leq c_1'$. Since Γ_1, Γ_2 are geodesics with respect to the metric d_*, we have

$$\begin{aligned}
& c_1' \cdot d_*(\Gamma_1(t), \Gamma_2(s)) - d_*(a,b) \\
&= c_1'\{d_*(\Gamma_1(t), a) + d_*(a,b) + d_*(b, \Gamma_2(s))\} - d_*(a,b) \\
&= c_1' d_*(\Gamma_1(t), a) + c_1' d_*(b, \Gamma_2(s)) - (1-c_1') d_*(a,b) \\
&= c_1' d_*(\Gamma_1(t), \Gamma_1(c_1 t)) + c_1' d_*(\Gamma_2(c_1' s), \Gamma_2(s)) - (1-c_1') d_*(a,b) \\
&= c_1'(1-c_1) d_*(\Gamma_1(t), \Gamma_1(0)) + c_1'(1-c_1') d_*(\Gamma_2(0), \Gamma_2(s)) - (1-c_1') d_*(a,b).
\end{aligned}$$

Moreover, by $c_1 \leq c_1'$, the right-hand side of this equality can be estimated as follows:

$$\begin{aligned}
& c_1'(1-c_1) d_*(\Gamma_1(t), \Gamma_1(0)) + c_1'(1-c_1') d_*(\Gamma_2(0), \Gamma_2(s)) - (1-c_1') d_*(a,b) \\
&\geq c_1'(1-c_1') d_*(\Gamma_1(t), \Gamma_1(0)) + c_1'(1-c_1') d_*(\Gamma_2(0), \Gamma_2(s)) - (1-c_1') d_*(a,b) \\
&= (1-c_1')\{c_1' d_*(\Gamma_1(t), \Gamma_1(0)) + c_1' d_*(\Gamma_2(0), \Gamma_2(s)) - d_*(a,b)\} \\
&\geq (1-c_1')\{c_1 d_*(\Gamma_1(t), \Gamma_1(0)) + c_1' d_*(\Gamma_2(0), \Gamma_2(s)) - d_*(a,b)\} \\
&= (1-c_1')\{d_*(a, \Gamma_1(0)) + d_*(\Gamma_2(0), b) - d_*(a,b)\} \\
&\geq 0.
\end{aligned}$$

The last inequality follows from the triangle inequality. Then, we have

$$d_*(a,b) \leq c_1' \cdot d_*(\Gamma_1(t), \Gamma_2(s)). \tag{3.1}$$

We will show that there exist $E_* \geq 1$ and $C_* \geq 0$ depending only on E and C such that for all $c \in [0,1]$,

$$d_*(\Gamma_1(ct), \Gamma_2(cs)) \leq c E_* d_*(\Gamma_1(t), \Gamma_2(s)) + C_*.$$

We divide the proof into the following cases:

I). $\Gamma_1(ct) \in [\Gamma_2(0), \Gamma_2(cs)]$, or $\Gamma_2(cs) \in [\Gamma_1(0), \Gamma_1(ct)]$.

II). $\Gamma_1(ct) \in [p,a]$ and $\Gamma_2(cs) \in [p,b]$ (see Figure 3.3).

III). $\Gamma_1(ct) \in [p,a]$ and $\Gamma_2(cs) \in [b, \Gamma_2(s)]$ (see Figure 3.4).

IV). $\Gamma_1(ct) \in [a, \Gamma_1(t)]$ and $\Gamma_2(cs) \in [b, \Gamma_2(s)]$ (see Figure 3.5).

We consider case I). Note that $\Gamma_1(0) = \Gamma_2(0)$. In case I), we suppose that $\Gamma_1(0)$, $\Gamma_1(ct)$, and $\Gamma_2(cs)$ are on the same geodesic segment. Then we have that

$$\begin{aligned} d_*(\Gamma_1(ct), \Gamma_2(cs)) &= c|t - s| \\ &\leq c d_*(\Gamma_1(t), \Gamma_2(s)). \end{aligned}$$

In case II), we can put

$$\begin{aligned} \Gamma_1(ct) &= \gamma_S(p, a)(ct - c_0 t), \\ \Gamma_2(cs) &= \gamma_S(p, b)(cs - c_0' s). \end{aligned}$$

Set $T := (c_1 - c_0)t$ and $S := (c_1' - c_0')s$. Then we have

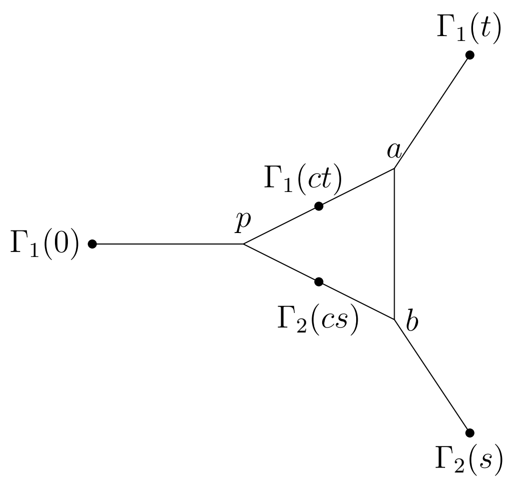

Figure 3.3: Case II)

$$\begin{aligned} p &= \gamma_S(p, a)\,(0) = \gamma_S(p, b)\,(0)\,, \\ \Gamma_1(ct) &= \gamma_S(p, a)\left(\frac{c - c_0}{c_1 - c_0} T\right), \qquad a = \gamma_S(p, a)\,(T)\,, \\ \Gamma_2(cs) &= \gamma_S(p, b)\left(\frac{c - c_0'}{c_1' - c_0'} S\right), \qquad b = \gamma_S(p, b)\,(S)\,. \end{aligned}$$

We define $a' \in [p, a]$ to be

$$a' \coloneqq \gamma_S(p, a)\left(\frac{c - c_0'}{c_1' - c_0'}T\right).$$

Firstly, by the coarsely convex inequality in S and inequality (3.1), we have

$$\begin{aligned}
d_*(a', \Gamma_2(cs)) &= d_*\left(\gamma_S(p, a)\left(\frac{c - c_0'}{c_1' - c_0'}T\right), \gamma_S(p, b)\left(\frac{c - c_0'}{c_1' - c_0'}S\right)\right) \\
&\leq \frac{c - c_0'}{c_1' - c_0'} \cdot E d_*(\gamma_S(p, a)(T), \gamma_S(p, b)(S)) + C \\
&= \frac{c - c_0'}{c_1' - c_0'} \cdot E d_*(a, b) + C \\
&\leq \frac{c c_1' - c_0' c_1'}{c_1' - c_0'} \cdot E d_*(\Gamma_1(t), \Gamma_2(s)) + C \\
&\leq c \cdot E d_*(\Gamma_1(t), \Gamma_2(s)) + C. \qquad (3.2)
\end{aligned}$$

Inequality (3.2) follows from $c_0' \leq c \leq c_1'$.

Next, we consider the distance between a' and $\Gamma_1(ct)$. Then

$$\begin{aligned}
d_*(a', \Gamma_1(ct)) &= d_*\left(\gamma_S(p, a)\left(\frac{c - c_0'}{c_1' - c_0'}T\right), \gamma_S(p, a)\left(\frac{c - c_0}{c_1 - c_0}T\right)\right) \\
&= \left|\frac{c - c_0'}{c_1' - c_0'}T - \frac{c - c_0}{c_1 - c_0}T\right|
\end{aligned}$$

By the triangle inequality $(c_1 - c_0)t \leq (c_1' - c_0')s + d_*(a, b)$ and inequality (3.1), we have

$$\begin{aligned}
\frac{c - c_0'}{c_1' - c_0'}T - \frac{c - c_0}{c_1 - c_0}T &= \frac{c - c_0'}{c_1' - c_0'}(c_1 - c_0)t - \frac{c - c_0}{c_1 - c_0}(c_1 - c_0)t \\
&\leq \frac{c - c_0'}{c_1' - c_0'}\{(c_1' - c_0')s + d_*(a, b)\} - (c - c_0)t \\
&\leq (c - c_0')s - (c - c_0)t + \frac{c - c_0'}{c_1' - c_0'} d_*(a, b) \\
&\leq c(s - t) + c d_*(\Gamma_1(t), \Gamma_2(s)) \\
&\leq c \cdot \{2 d_*(\Gamma_1(t), \Gamma_2(s))\}.
\end{aligned}$$

Similarly, by the triangle inequality $(c_1' - c_0')s - d_*(a,b) \leq (c_1 - c_0)t$ and inequality (3.1), we have

$$\begin{aligned}
\frac{c-c_0}{c_1-c_0}T - \frac{c-c_0'}{c_1'-c_0'}T &= \frac{c-c_0}{c_1-c_0}(c_1-c_0)t - \frac{c-c_0'}{c_1'-c_0'}(c_1-c_0)t \\
&\leq (c-c_0)t - \frac{c-c_0'}{c_1'-c_0'}\{(c_1'-c_0')s - d_*(a,b)\} \\
&= (c-c_0)t - (c-c_0')s + \frac{c-c_0'}{c_1'-c_0'}d_*(a,b) \\
&\leq c(t-s) + cd_*(\Gamma_1(t),\Gamma_2(s)) \\
&\leq c\cdot\{2d_*(\Gamma_1(t),\Gamma_2(s))\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
d_*(a',\Gamma_1(ct)) &= \left|\frac{c-c_0'}{c_1'-c_0'}T - \frac{c-c_0}{c_1-c_0}T\right| \\
&\leq c\cdot\{2d_*(\Gamma_1(t),\Gamma_2(s))\}. \qquad (3.3)
\end{aligned}$$

Combining (3.2) and (3.3) yields

$$\begin{aligned}
d_*(\Gamma_1(ct),\Gamma_2(cs)) &\leq d_*(\Gamma_1(ct),a') + d_*(a',\Gamma_2(cs)) \\
&\leq c\cdot\{2d_*(\Gamma_1(t),\Gamma_2(s))\} + c\cdot Ed_*(\Gamma_1(t),\Gamma_2(s)) + C \\
&= c(2+E)d_*(\Gamma_1(t),\Gamma_2(s)) + C.
\end{aligned}$$

We consider case III). In case III), we supposed that $\Gamma_1(ct) \in [p,a]$ and $\Gamma_2(cs) \in [b,\Gamma_2(s)]$, where $a = \Gamma_1(c_1t)$ and $b = \Gamma_2(c_1's)$. Let $t' := c_1t$ and $s' := c_1's$. We define

$$\begin{aligned}
a' &:= \Gamma_1(ct'), \\
b' &:= \Gamma_2(cs').
\end{aligned}$$

Since $a = \Gamma_1(t')$ and $b = \Gamma_2(s')$, we have $a' \in [\Gamma_1(0),p] \cup [p,a]$ and $b' \in [\Gamma_1(0),p] \cup [p,b]$. By the same argument as in case I) or case II), we have

$$\begin{aligned}
d_*(a',b') &= d_*(\Gamma_1(ct'),\Gamma_2(cs')) \\
&\leq c(2+E)d_*(\Gamma_1(t'),\Gamma_2(s')) + C \\
&= c(2+E)d_*(a,b) + C.
\end{aligned}$$

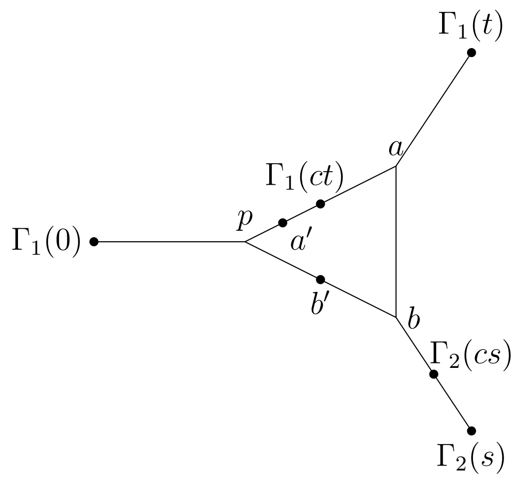

Figure 3.4: Case III)

We remark that $d_*(\Gamma_1(t), \Gamma_2(s)) = d_*(\Gamma_1(t), a) + d_*(a, b) + d_*(b, \Gamma_2(s))$. Then we have

$$\begin{aligned}
&d_*(\Gamma_1(ct), \Gamma_2(cs)) \\
&\leq d_*(\Gamma_1(ct), a') + d_*(a', b') + d_*(b', \Gamma_2(cs)) \\
&= d_*(\Gamma_1(ct), \Gamma_1(ct')) + d_*(\Gamma_1(ct'), \Gamma_2(cs')) + d_*(\Gamma_2(cs'), \Gamma_2(cs)) \\
&\leq c(t - t') + c(2 + E)d_*(\Gamma_1(t'), \Gamma_2(s')) + C + c(s - s') \\
&= cd_*(\Gamma_1(t), \Gamma_1(t')) + c(2 + E)d_*(\Gamma_1(t'), \Gamma_2(s')) + C + cd_*(\Gamma_2(s'), \Gamma_2(s)) \\
&\leq c(2 + E)\{d_*(\Gamma_1(t), a) + d_*(a, b) + d_*(b, \Gamma_2(s))\} + C \\
&= c(2 + E)d_*(\Gamma_1(t), \Gamma_2(s)) + C.
\end{aligned}$$

We consider case IV). In case IV), we supposed that $\Gamma_1(ct) \in [a, \Gamma_1(t)]$ and $\Gamma_2(cs) \in [b, \Gamma_2(s)]$, where $a = \Gamma_1(c_1 t)$, $b = \Gamma_2(c_1' s)$, and $c_1 \leq c_1'$. We remark that $c_1 \leq c_1' \leq c$. Then, by inequality (3.1), we have

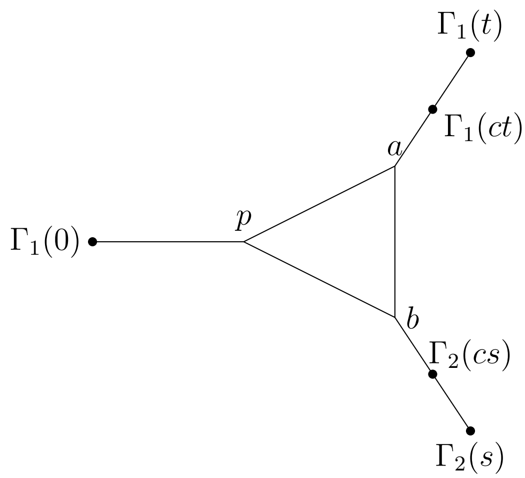

Figure 3.5: Case IV)

$$
\begin{aligned}
&d_*(\Gamma_1(ct), \Gamma_2(cs)) \\
&= d_*(\Gamma_1(ct), a) + d_*(a, b) + d_*(b, \Gamma_2(cs)) \\
&= d_*(\Gamma_1(ct), \Gamma_1(c_1 t)) + d_*(a, b) + d_*(\Gamma_2(c_1' s), \Gamma_2(cs)) \\
&\leq c\left(1 - \frac{c_1}{c}\right) t + c_1' d_*(\Gamma_1(t), \Gamma_2(s)) + c\left(1 - \frac{c_1'}{c}\right) s \\
&\leq c(1 - c_1)t + c d_*(\Gamma_1(t), \Gamma_2(s)) + c(1 - c_1')s \\
&= c\{d_*(\Gamma_1(t), a) + d_*(b, \Gamma_2(s))\} + c d_*(\Gamma_1(t), \Gamma_2(s)) \\
&\leq c\{2 d_*(\Gamma_1(t), \Gamma_2(s))\}.
\end{aligned}
$$

Therefore, for all $\Gamma_1, \Gamma_2 \in \mathcal{L}_*$ with $\Gamma_1 : [0, t] \to X * Y$ and $\Gamma_2 : [0, s] \to X * Y$,

$$d_*(\Gamma_1(ct), \Gamma_2(cs)) \leq c(2 + E) d_*(\Gamma_1(t), \Gamma_2(s)) + C \tag{3.4}$$

holds for all $c \in [0, 1]$.

Finally, we will show that there exist $E_{**} \geq 0$ and $C_{**} \geq 0$ such that for all $t' \in [0, t]$, $s' \in [0, s]$, and $c \in [0, 1]$,

$$d_*(\Gamma_1(ct'), \Gamma_2(cs')) \leq c E_{**} d_*(\Gamma_1(t'), \Gamma_2(s')) + C_{**}.$$

We recall that in general, $\Gamma_1|_{[0,t']}$ is not in $\mathcal{L}_*$. Let $\Gamma_1' \in \mathcal{L}_*$ be the geodesic segment from $\Gamma_1(0)$ to $\Gamma_1(t')$. By Lemma 3.1.3,

$$d_*(\Gamma_1(ct'), \Gamma_1'(ct')) \leq C. \tag{3.5}$$

holds for any $c \in [0,1]$. Let $\Gamma_2' \in \mathcal{L}_*$ be the geodesic segment from $\Gamma_2(0)$ to $\Gamma_2(s')$. By Lemma 3.1.3, we have that for all $c \in [0,1]$,

$$d_*(\Gamma_2(cs'), \Gamma_2'(cs')) \leq C. \tag{3.6}$$

Since Γ_1', $\Gamma_2' \in \mathcal{L}_*$ with $\Gamma_1' : [0,t'] \to X * Y$ and $\Gamma_2' : [0,s'] \to X * Y$, by (3.4), (3.5), and (3.6),

$$\begin{aligned}
&d_*(\Gamma_1(ct'), \Gamma_2(cs')) \\
&\leq d_*(\Gamma_1(ct'), \Gamma_1'(ct')) + d_*(\Gamma_1'(ct'), \Gamma_2'(cs')) + d_*(\Gamma_2'(cs'), \Gamma_2(cs')) \\
&\leq c(2+E)d_*(\Gamma_1'(t'), \Gamma_2'(s')) + 3C \\
&= c(2+E)d_*(\Gamma_1(t'), \Gamma_2(s')) + 3C.
\end{aligned}$$

Therefore, we obtain that (gCC2) holds.

Next, we will show that $\mathcal{L}_*$ satisfies (gCC3). Let $\Gamma_1 \in \mathcal{L}_*$ with $\Gamma_1 : [0,t] \to X * Y$ and let $\Gamma_2 \in \mathcal{L}_*$ with $\Gamma_2 : [0,s] \to X * Y$. We suppose that $\Gamma_1(t) = \Gamma_2(s)$. First, we will show that

$$d_*(\Gamma_1(ct), \Gamma_2(cs)) \leq (1-c)(2+E)d_*(\Gamma_1(0), \Gamma_2(0)) + C$$

holds for $c \in [0,1]$.

Let denote by $[\Gamma_1(0), \Gamma_1(t)]$, the image of Γ_1. We consider a geodesic triangle $[\Gamma_1(0), \Gamma_1(t)] \cup [\Gamma_1(0), \Gamma_2(0)] \cup [\Gamma_2(0), \Gamma_2(s)]$. There exist $S \in \mathcal{S}$ and $p_1, p_2, a \in S$ such that

$$\begin{gathered}
[\Gamma_1(0), \Gamma_1(t)] = [\Gamma_1(0), p_1] \cup [p_1, a] \cup [a, \Gamma_1(t)], \\
[\Gamma_2(0), \Gamma_2(s)] = [\Gamma_2(0), p_2] \cup [p_2, a] \cup [a, \Gamma_2(s)], \text{and}, \\
d_*(\Gamma_1(0), \Gamma_2(0)) = d_*(\Gamma_1(0), p_1) + d_*(p_1, p_2) + d_*(p_2, \Gamma_2(0))
\end{gathered}$$

holds. We set

$$\begin{aligned} p_1 &:= \Gamma_1(c_0 t), \\ p_2 &:= \Gamma_2(c_0' s), \\ a &:= \Gamma_1(c_1 t) = \Gamma_2(c_1' s), \end{aligned}$$

where $c_0, c_1, c_0', c_1' \in [0,1]$. Since Γ_1 and Γ_2 are geodesic segments, we have $(1-c_1)t = (1-c_1')s$.

Without loss of generality, we may assume that $1-c_0 \leq 1-c_0'$. By the same argument as in the proof of (3.1), we have

$$d_*(p_1, p_2) \leq (1-c_0') \cdot d_*(\Gamma_1(0), \Gamma_2(0)). \tag{3.7}$$

We divide the proof into the following cases:

2-I) $\Gamma_1(ct) \in \Gamma_2([cs, s])$, or $\Gamma_2(cs) \in \Gamma_1([ct, t])$.

2-II) $\Gamma_1(ct) \in [p_1, a]$ and $\Gamma_2(cs) \in [p_2, a]$ (see Figure 3.6).

2-III) $\Gamma_1(ct) \in [\Gamma_1(0), p_1]$ and $\Gamma_2(cs) \in [p_2, a]$ (see Figure 3.7).

2-IV) $\Gamma_1(ct) \in [\Gamma_1(0), p_1]$ and $\Gamma_2(cs) \in [\Gamma_2(0), p_2]$

We consider case 2-I). In case 2-I), we suppose that $\Gamma_1(t)$, $\Gamma_1(ct)$, and $\Gamma_2(cs)$ are on the same geodesic segment. Then we have

$$\begin{aligned} d_*(\Gamma_1(ct), \Gamma_2(cs)) &= c|t-s| \\ &\leq c d_*(\Gamma_1(0), \Gamma_2(0)). \end{aligned}$$

In case 2-II), we can put

$$\begin{aligned} \Gamma_1(ct) &= \gamma_S(p_1, a)\left(\frac{c-c_0}{c_1-c_0}T\right), \\ \Gamma_2(cs) &= \gamma_S(p_2, a)\left(\frac{c-c_0'}{c_1'-c_0'}S\right). \end{aligned}$$

Set $T := (c_1 - c_0)t$ and $S := (c_1' - c_0')s$. Note that

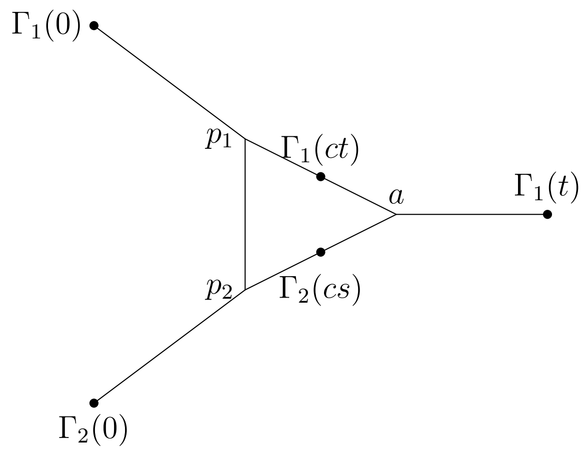

Figure 3.6: Case 2-II)

$$p_1 = \gamma_S(p_1, a)\,(0)\,, \qquad p_2 = \gamma_S(p_2, a)\,(0)\,,$$
$$\Gamma_1(ct) = \gamma_S(p_1, a)\left(\frac{c - c_0}{c_1 - c_0}T\right), \qquad \Gamma_2(cs) = \gamma_S(p_2, a)\left(\frac{c - c_0'}{c_1' - c_0'}S\right),$$
$$a = \gamma_S(p_1, a)\,(T) = \gamma_S(p_2, a)(S).$$

We define $a' \in [p, a]$ to be

$$a' \coloneqq \gamma_S(p_1, a)\left(\frac{c - c_0'}{c_1' - c_0'}T\right).$$

Since S is a (E, C)-geodesic coarsely convex space and inequality (3.7), we have

$$\begin{aligned}
d_*(a', \Gamma_2(cs)) &= d_*\left(\gamma_S(p_1, a)\left(\frac{c - c_0'}{c_1' - c_0'}T\right), \gamma_S(p_2, a)\left(\frac{c - c_0'}{c_1' - c_0'}S\right)\right) \\
&\leq \left(1 - \frac{c - c_0'}{c_1' - c_0'}\right) \cdot Ed_*(\gamma_S(p_1, a)(0), \gamma_S(p_2, a)(0)) + C \\
&= \frac{c_1' - c}{c_1' - c_0'} \cdot Ed_*(p_1, p_2) + C \\
&= \frac{(1 - c) - (1 - c_1')}{(1 - c_0') - (1 - c_1')} \cdot Ed_*(p_1, p_2) + C \\
&\leq \frac{(1 - c)(1 - c_0') - (1 - c_1')(1 - c_0')}{(1 - c_0') - (1 - c_1')} \cdot Ed_*(\Gamma_1(0), \Gamma_2(0)) + C \\
&\leq (1 - c) \cdot Ed_*(\Gamma_1(0), \Gamma_2(0)) + C. \qquad (3.8)
\end{aligned}$$

Inequality (3.8) follows from $c_0' \leq c \leq c_1'$.

Next, we consider the distance between a' and $\Gamma_1(ct)$. Then

$$\begin{aligned}
d_*(a', \Gamma_1(ct)) &= d_*\left(\gamma_S(p_1, a)\left(\frac{c - c_0'}{c_1' - c_0'}T\right), \gamma_S(p_1, a)\left(\frac{c - c_0}{c_1 - c_0}T\right)\right) \\
&= \left|\frac{c - c_0'}{c_1' - c_0'}T - \frac{c - c_0}{c_1 - c_0}T\right|.
\end{aligned}$$

Note that $T = (c_1 - c_0)t$ and $S = (c_1' - c_0')s$. By the triangle inequality $(c_1 - c_0)t \geq (c_1' - c_0')s - d_*(p_1, p_2)$, we have

$$\begin{aligned}
\left(\frac{c - c_0'}{c_1' - c_0'} - \frac{c - c_0}{c_1 - c_0}\right)(c_1 - c_0)t &= \left\{\left(1 - \frac{c - c_0}{c_1 - c_0}\right) - \left(1 - \frac{c - c_0'}{c_1' - c_0'}\right)\right\}(c_1 - c_0)t \\
&= \left(\frac{c_1 - c}{c_1 - c_0} - \frac{c_1' - c}{c_1' - c_0'}\right)(c_1 - c_0)t \\
&\leq (c_1 - c)t - \frac{c_1' - c}{c_1' - c_0'}(c_1 - c_0)t \\
&\leq (c_1 - c)t - \frac{c_1' - c}{c_1' - c_0'}\{(c_1' - c_0')s - d_*(p_1, p_2)\} \\
&\leq (c_1 - c)t - (c_1' - c)s + \frac{c_1' - c}{c_1' - c_0'}d_*(p_1, p_2). \qquad (3.9)
\end{aligned}$$

By the equation $(1-c_1)t = (1-c_1')s$, the sum of the first two terms of (3.9) can be estimated as follows:

$$\begin{aligned}(c_1 - c)t - (c_1' - c)s &= \{(1-c) - (1-c_1)\}t - \{(1-c) - (1-c_1')\}s \\ &= (1-c)(t-s) \\ &= (1-c)(d_*(\Gamma_1(0), \Gamma_1(t)) - d_*(\Gamma_2(0), \Gamma_2(s))) \\ &\leq (1-c)d_*(\Gamma_1(0), \Gamma_2(0)).\end{aligned}$$

By the inequality (3.7), we estimate the third term of (3.9)

$$\begin{aligned}\frac{c_1' - c}{c_1' - c_0'} d_*(p_1, p_2) &= \frac{(1-c) - (1-c_1')}{(1-c_0') - (1-c_1')} d_*(p_1, p_2) \\ &\leq \frac{(1-c)(1-c_0') - (1-c_1')(1-c_0')}{(1-c_0') - (1-c_1')} d_*(\Gamma_1(0), \Gamma_2(0)) \\ &\leq (1-c)d_*(\Gamma_1(0), \Gamma_2(0)).\end{aligned}$$

Thus, we obtain

$$\frac{c - c_0'}{c_1' - c_0'} T - \frac{c - c_0}{c_1 - c_0} T \leq (1-c)\{2d_*(\Gamma_1(0), \Gamma_2(0))\}.$$

By the similar argument, we have

$$\frac{c - c_0}{c_1 - c_0} T - \frac{c - c_0'}{c_1' - c_0'} T \leq (1-c)\{2d_*(\Gamma_1(0), \Gamma_2(0))\}.$$

Therefore,

$$\begin{aligned}d_*(a', \Gamma_1(c)) &= \left| \frac{c - c_0'}{c_1' - c_0'} T - \frac{c - c_0}{c_1 - c_0} T \right| \\ &\leq (1-c) \cdot \{2d_*(\Gamma_1(0), \Gamma_2(0))\}. \qquad (3.10)\end{aligned}$$

holds.

Combining (3.8) and (3.10) yields

$$\begin{aligned}d_*(\Gamma_1(c), \Gamma_2(c)) &\leq d_*(\Gamma_1(c), a') + d_*(a', \Gamma_2(c)) \\ &\leq (1-c)\{2d_*(\Gamma_1(0), \Gamma_2(0))\} + (1-c)Ed_*(\Gamma_1(0), \Gamma_2(0)) + C \\ &= (1-c)(2+E)d_*(\Gamma_1(0), \Gamma_2(0)) + C.\end{aligned}$$

We consider case 2-III). In case 2-III), we supposed that $\Gamma_1(ct) \in [\Gamma_1(0), p_1]$ and $\Gamma_2(cs) \in [p_2, a]$. Here $p_1 = \Gamma_1(c_0 t)$ and $p_2 = \Gamma_2(c_0' s)$. Let $\Gamma_1' = \Gamma(p_1, \Gamma_1(t))$ and $\Gamma_2' = \Gamma(p_2, \Gamma_2(s))$. Since p_1 and p_2 is identified points when concatenating geodesic segments, we have

$$\Gamma_1([c_0 t, t]) = \Gamma_1'([0, (1-c_0)t]).$$
$$\Gamma_2([c_0' s, s]) = \Gamma_2'([0, (1-c_0')s]).$$

Let $t' = (1-c_0)t$ and $s' = (1-c_0)s$ and let $a' = \Gamma_1'(ct')$ and $b' = \Gamma_2'(cs')$. Note that $a' \in [p_1, a] \cup [a, \Gamma_1(t)]$ and $b' \in [p_2, a] \cup [a, \Gamma_2(s)]$. By the same

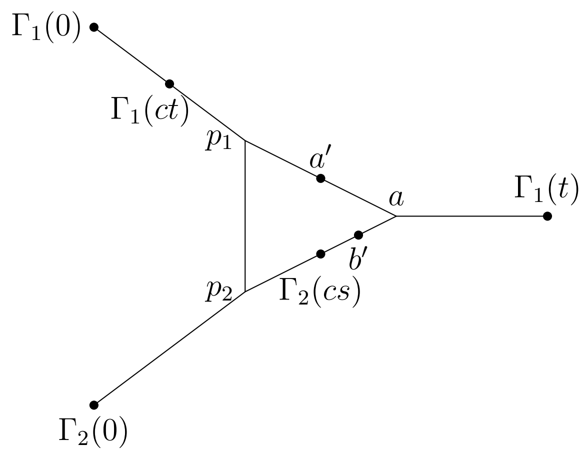

Figure 3.7: Case 2-III)

argument as in case 2-I) or 2-II), we have

$$\begin{aligned} d_*(a', b') &= d_*(\Gamma_1'(ct'), \Gamma_2'(cs')) \\ &\leq (1-c)(2+E) d_*(\Gamma_1'(0), \Gamma_2'(0)) + C \\ &= (1-c)(2+E) d_*(p_1, p_2) + C. \end{aligned}$$

Note that

$$\begin{aligned} d_*(\Gamma_1(0), \Gamma_2(0)) &= d_*(\Gamma_1(0), p_1) + d_*(p_1, p_2) + d_*(p_2, \Gamma_2(0)), \\ a' &= \Gamma_1'(ct') = \Gamma_1(c_0 t + (1-c_0)ct), \text{ and} \\ b' &= \Gamma_2'(cs') = \Gamma_2(c_0' s + (1-c_0')cs). \end{aligned}$$

Then we have

$$
\begin{aligned}
&d_*(\Gamma_1(ct), \Gamma_2(cs)) \\
&\le d_*(\Gamma_1(ct), a') + d_*(a', b') + d_*(b', \Gamma_2(cs)) \\
&= d_*(\Gamma_1(ct), \Gamma_1'(ct')) + (1-c)(2+E)d_*(p_1, p_2) + d_*(\Gamma_2'(cs'), \Gamma_2(cs)) + C \\
&= d_*(\Gamma_1(ct), \Gamma_1(c_0 t + (1-c_0)ct)) + (1-c)(2+E)d_*(p_1, p_2) \\
&\quad + d_*(\Gamma_2(cs), \Gamma_2(c_0' s + (1-c_0')cs)) + C \\
&= \{(c_0 + c - c_0 c) - c\}t + (1-c)(2+E)d_*(p_1, p_2) + \{(c_0' + c - c_0' c) - c\}s + C \\
&= (1-c)c_0 t + (1-c)c_0' s + (1-c)(2+E)d_*(p_1, p_2) + C \\
&= (1-c)\{d_*(\Gamma_1(0), p_1) + d_*(\Gamma_2(0), p_2)\} + (1-c)(2+E)d_*(p_1, p_2) + C \\
&= (1-c)(2+E)d_*(\Gamma_1(0), \Gamma_2(0)) + C.
\end{aligned}
$$

We consider case 2-IV). In case 2-IV), we supposed that $\Gamma_1(ct) \in [\Gamma_1(0), p_1]$ and $\Gamma_2(cs) \in [\Gamma_2(0), p_2]$. Here that $p_1 = \Gamma_1(c_0 t)$ and $p_2 = \Gamma_2(c_0' s)$. Note that $1 - c_0' \le 1 - c$. Then, by inequality (3.7), we have

$$
\begin{aligned}
&d_*(\Gamma_1(ct), \Gamma_2(cs)) \\
&= d_*(\Gamma_1(ct), p_1) + d_*(p_1, p_2) + d_*(p_2, \Gamma_2(cs)) \\
&= d_*(\Gamma_1(ct), \Gamma_1(c_0 t)) + d_*(p_1, p_2) + d_*(\Gamma_2(c_0' s), \Gamma_2(cs)) \\
&\le (c_0 - c)t + (1-c_0')d_*(\Gamma_1(0), \Gamma_2(0)) + (c_0' - c)s \\
&= \{(1-c) - (1-c_0)\}t + \{(1-c) - (1-c_0')\}s + (1-c_0')d_*(\Gamma_1(0), \Gamma_2(0)) \\
&= (1-c)\left(1 - \frac{1-c_0}{1-c}\right)t + (1-c)\left(1 - \frac{1-c_0'}{1-c}\right)s + (1-c_0')d_*(\Gamma_1(0), \Gamma_2(0)) \\
&\le (1-c)\{1 - (1-c_0)\}t + (1-c)\{1 - (1-c_0')\}s + (1-c_0')d_*(\Gamma_1(0), \Gamma_2(0)) \\
&= (1-c)c_0 t + (1-c)c_0' s + (1-c_0')d_*(\Gamma_1(0), \Gamma_2(0)) \\
&\le (1-c)d_*(\Gamma_1(0), p_1) + (1-c)d_*(\Gamma_2(0), p_2) + (1-c)d_*(\Gamma_1(0), \Gamma_2(0)) \\
&\le (1-c)\{2d_*(\Gamma_1(0), \Gamma_2(0))\}.
\end{aligned}
$$

Therefore,

$$
d_*(\Gamma_1(ct), \Gamma_2(cs)) \le (1-c)(2+E)d_*(\Gamma_1(0), \Gamma_2(0)) + C \tag{3.11}
$$

holds for all $c \in [0, 1]$.

For $t \in [0, 1]$, let $a = \Gamma_1(t)$. Let $\Gamma_1' = \Gamma(\Gamma_1(0), a)$ and $\Gamma_2' = \Gamma(\Gamma_2(0), a)$. Let $d = d_*(\Gamma_2(0), a)$. By Lemma 3.1.3,

$$d_*(\Gamma_1(ct), \Gamma_1'(ct)) \leq C.$$

holds for $c \in [0, 1]$. Then, we have

$$\begin{aligned} d_*(\Gamma_1(ct), \Gamma_2'(cd)) &\leq d_*(\Gamma_1(ct), \Gamma_1'(ct)) + d_*(\Gamma_1'(ct), \Gamma_2'(cd)) \\ &\leq (1-c)(2+E) d_*(\Gamma_1'(0), \Gamma_2'(0)) + 2C \qquad (3.12) \\ &= (1-c)(2+E) d_*(\Gamma_1(0), \Gamma_2(0)) + 2C \end{aligned}$$

Inequality (3.12) follows from (3.11). Therefore, $\mathcal{L}_*$ satisfies (gCC3). This completes the proof. □

Proof of Theorem 3.1.1. By Proposition 2.2.3 and 3.1.4, the free product $X * Y$ is a geodesic coarsely convex space, in particular, $X * Y$ is a $(1, 0, 2 + E, 6C, \mathcal{L}_*, \mathrm{id}_{\mathbb{R}_{\geq 0}})$-coarsely convex space. □

3.2 Group actions on free products of metric spaces

Let (X, d_X, e_X) and (Y, d_Y, e_Y) be metric spaces with base points e_X and e_Y, respectively. Let G and H be groups acting properly and cocompactly on X and Y, respectively. Bridson and Haefliger [2, Theorem II.11.18] construct a metric space $\overline{Z}$ on which the free product $G * H$ acts properly and cocompactly. Moreover, when X and Y are CAT(0) spaces, they showed that $\overline{Z}$ is a CAT(0) space. In this section, we will show that $\overline{Z}$ is isometric to the free product $X * Y$ with respect to the G-net $(G, o(e_X), e_X)$ and the H-net $(H, o(e_Y), e_Y)$, where $o(e_X)$ and $o(e_Y)$ are the orbit maps.

First, we briefly review their construction. Let $\Gamma = G * H$. Define Z by

$$Z := (\Gamma \times X) \bigsqcup (\Gamma \times [0, 1]) \bigsqcup (\Gamma \times Y).$$

Let $\overline{Z}$ be the quotient of Z by the equivalent relation generated by:

$$(\omega g, x) \sim (\omega, g(x)), (\omega h, y) \sim (\omega, h(y)),$$
$$(\omega, e_X) \sim (\omega, 0), (\omega, e_Y) \sim (\omega, 1)$$

for all $\omega \in G * H$, $g \in G$, $h \in H$, $x \in X$, and $y \in Y$. Let $\overline{X}$ be the quotient of $\Gamma \times X$ by the restriction of the above relation, and let $\overline{Y}$ be the quotient of $\Gamma \times Y$ by the restriction of the above relation. We remark that $\overline{X}$ is isometric to $(W_X \cup \{\epsilon\}) \times X$, where W_X is the set of words of $G * H$ such that the last letter of each word of W_X is in H, and $\overline{Y}$ is isometric to $(W_Y \cup \{\epsilon\}) \times Y$, where W_Y is the set of words of $G * H$ such that the last letter of each word of W_Y is in G.

Let $\omega = ugh$, where $u \in G * H$, $g \in G$, and $h \in H$. By the above equivalent relation, we have $(\omega, e_X) \sim (\omega, 0)$ and we have

$$\begin{aligned}(\omega, 1) &\sim (\omega, e_Y),\\ &= (ugh, e_Y),\\ &\sim (ug, h(e_Y)).\end{aligned}$$

Let $\tau = vghg'$, where $v \in G * H$, $g, g' \in G$, and $h \in H$. By the above equivalent relation, we have $(\tau, e_Y) \sim (\tau, 1)$ and we have

$$\begin{aligned}(\tau, 0) &\sim (\tau, e_X),\\ &= (ughg', e_X),\\ &\sim (ugh, g'(e_X)).\end{aligned}$$

Note that $(\epsilon, e_X) \sim (\epsilon, 0)$ and $(\epsilon, 1) \sim (\epsilon, e_Y)$. Therefore, $\overline{Z}$ consists of the following two types of components.

- The sheets consist of $\{\omega\} \times X$ and $\{\tau\} \times Y$, where $\omega \in W_X \cup \{\epsilon\}$ and $\tau \in W_Y \cup \{\epsilon\}$.
- The edges consist of the following three:

- There exists an edge $\{\epsilon\} \times [0,1]$ connecting $(\epsilon, e_X) \in \{\epsilon\} \times X$ and $(\epsilon, e_Y) \in \{\epsilon\} \times Y$
- Let $\omega \in W_X$ and $g \in G$. Then there exists an edge $\{\omega\} \times \{g\} \times [0,1]$ connecting $(\omega, g(e_X)) \in \{\omega\} \times X$ and $(\omega g, e_Y) \in \{\omega g\} \times Y$.
- Let $\tau \in W_Y$ and $h \in H$. Then there exists an edge $\{\tau\} \times \{h\} \times [0,1]$ connecting $(\tau, h(e_Y)) \in \{\tau\} \times Y$ and $(\tau h, e_X) \in \{\tau h\} \times X$.

Bridson and Haefliger [2, Theorem II.11.18] showed that $G * H$ acts properly and cocompactly on $\overline{Z}$.

We can easily show that $\overline{Z}$ is isometric to the free product $X * Y$ with respect to the G-net $(G, o(e_X), e_X)$ and the H-net $(H, o(e_Y), e_Y)$, where $o(e_X)$ and $o(e_Y)$ are the orbit maps. Therefore, we have

Proposition 3.2.1 ([2, Theorem II.11.18]). *Let X and Y be metric spaces. Let G and H be groups acting properly and cocompactly on X and Y, respectively. We associate X and Y with the G-net and the H-net, respectively. Then $G * H$ acts properly and cocompactly on $X * Y$.*

Corollary 3.2.2. *Let X and Y be geodesic coarsely convex spaces. Let G and H be groups acting properly and cocompactly on X and Y, respectively. Then $G * H$ is a coarsely convex group.*

Chapter 4

Application to the coarse Baum–Connes conjecture

4.1 Review of the coarse Baum–Connes conjecture

Let Y be a proper metric space. We say that Y satisfies the coarse Baum–Connes conjecture if the following coarse assembly map μ_Y of Y is an isomorphism:

$$\mu_Y \colon KX_*(Y) \to K_*(C^*(Y)).$$

Here, the left-hand side is the *coarse K-homology* of Y, and the right-hand side is the K-theory of the C^*-algebra $C^*(Y)$, called the *Roe algebra* of Y. Both are invariant under coarse equivalence, and the coarse assembly map behaves naturally for coarse maps. Therefore we have the following.

Proposition 4.1.1. *Let X and Y be proper metric spaces. We suppose that X and Y are coarsely equivalent. If X satisfies the coarse Baum–Connes conjecture, then so does Y.*

For details, see [11], [12], and [17]. Fukaya and Oguni show the following.

THEOREM 4.1.2 ([9, Theorem 1.3]). *Let X be a proper coarsely convex space. Then X satisfies the coarse Baum–Connes conjecture.*

We will apply Theorem 4.1.2 for free products of proper geodesic coarsely convex spaces.

LEMMA 4.1.3. *Let X and Y be proper metric spaces with nets. Then the free product $X * Y$ is a proper metric space.*

PROOF. Let (X, d_X) and (Y, d_Y) be proper metric spaces . Let (X_0, i_X, e_X) and (Y_0, i_Y, e_Y) be nets of X and Y, respectively. Set $a \in X * Y$ and $R \geq 0$. Let n be the height of the sheet that contains $a \in X * Y$. We define $B_*(a, R)$ to be

$$B_*(a, R) \coloneqq \{b \in X * Y \mid d_*(a, b) \leq R\}.$$

By the definition of nets, the height of the sheets that intersect $B_*(a, R)$ is in the interval $[n - [R] - 1, n + [R] + 1]$. Since X and Y are proper metric spaces, by the definition of nets, for any bounded closed subset K in each sheet, the preimage of K by the index maps is a finite set. Therefore, $B_*(a, R)$ is the finite union of bounded closed subsets in the sheets. Since X and Y are proper metric spaces, $B_*(a, R)$ is compact. □

Combining Theorem 3.1.1, Theorem 4.1.2, and Lemma 4.1.3, we obtain the following.

THEOREM 4.1.4. *Let X and Y be proper metric spaces with nets. If X and Y are geodesic coarsely convex spaces, then the free product $X * Y$ satisfies the coarse Baum–Connes conjecture.*

Let G and H be finitely generated groups acting properly and cocompactly on X and Y, respectively. As mentioned in Section 3.2, the free product $G * H$ acts properly and cocompactly on $X * Y$ with respect to the G-net and the H-net. Therefore, combining Theorem 4.1.4, the Švarc–Milnor Lemma, and Proposition 4.1.1, we obtain the following result.

THEOREM 4.1.5. *Let X and Y be proper metric spaces with nets. We suppose that X and Y are geodesic coarsely convex spaces. Let G and H be groups acting properly and cocompactly on X and Y, respectively. Then the free product $G * H$ satisfies the coarse Baum–Connes conjecture.*

Finally, we compare Theorem 4.1.4 and Theorem 4.1.5 with some known results for relatively hyperbolic groups, and spaces admitting a coarse embedding into a Hilbert space.

4.2 Relatively hyperbolic groups

Fukaya and Oguni [7] showed the following.

THEOREM 4.2.1 ([7, Theorem 1.1]). *Let G be a finitely generated group and $\mathbb{P} = \{P_1, \ldots, P_k\}$ be a finite family of infinite subgroups. Suppose that $(G, \mathbb{P})$ is a relatively hyperbolic group. If each subgroup P_i satisfies the coarse Baum–Connes conjecture, and admits a finite P_i-simplicial complex which is a universal space for proper actions, then G satisfies the coarse Baum–Connes conjecture.*

Let G and H be finitely generated groups. The free product $G * H$ is hyperbolic relative to $\{G, H\}$. If G and H act properly and cocompactly on any spaces listed in Table 1.1, then G and H admit finite G-simplicial (resp. H-simplicial) complexes which are universal spaces for proper actions. Therefore, $G * H$ satisfies the assumptions of Theorem 4.2.1. However, it is not known in general whether groups acting properly and cocompactly on geodesic coarsely convex spaces always admit finite G-simplicial complexes which are universal spaces for proper actions.

4.3 Spaces admitting a coarse embedding into a Hilbert space

The notion of coarse embedding into a Hilbert space is introduced by Gromov [10]. Yu [18] showed the following.

THEOREM 4.3.1 ([18, Theorem 1.1]). *Let Γ be a discrete metric space with bounded geometry. If Γ admits a uniform embedding into a Hilbert space, then the coarse Baum–Connes conjecture holds for Γ.*

Let X and Y be proper metric spaces. Let G and H be groups acting properly and cocompactly on X and Y, respectively. We suppose that X and Y admit coarse embeddings into Hilbert spaces. Then G and H also admit coarse embeddings into Hilbert spaces. By the work of Dadarlat–Guentner [4], the free product $G * H$ embeds coarsely into the Hilbert space. By Theorem 4.3.1, the free product $G * H$ satisfies the coarse Baum–Connes conjecture. We remark that $G * H$ acts properly and cocompactly on $X * Y$. Then, by the Švarc–Milnor Lemma and Proposition 4.1.1, it follows that $X * Y$ satisfies the coarse Baum–Connes conjecture.

We remark that in the above setting, all X, Y, and $X * Y$ are with bounded coarse geometry in the sense of [8, Definition A.2]. However, there exist geodesic coarsely convex spaces without bounded coarse geometry.

EXAMPLE 4.3.2. Let Γ be the Cayley graph of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ for some generating set. Since $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ is a hyperbolic group, Γ is a geodesic coarsely convex space.

For $p \in (0, \infty)$, let X_p be the proper Busemann space given in [8, Example 2.2]. As described in [8], X_p is constructed from the half-line $[0, \infty)$ by identifying each integer $n \in [0, \infty)$ with the origin of the n-dimensional l_p space. By Theorem 4.1.4, the free product $\Gamma * X_p$ satisfies the coarse Baum–Connes conjecture.

In [8, Appendix], it is shown that X_p is without bounded coarse geometry. Thus $\Gamma * X_p$ does not satisfy the assumptions of Theorem 4.3.1.

It also follows that X_p does not admit any proper cocompact actions by discrete groups. Therefore we cannot apply Theorem 4.2.1 to $\Gamma * X_p$.

Using expander graphs, Kondo [13] constructed a CAT(0) space which is not coarsely embeddable into a Hilbert space. The space given in [13] is not proper, however, with a slight modification, we obtain a proper CAT(0) space which is not coarsely embeddable into a Hilbert space. Following Kondo's argument [13], we give the construction of the space.

First, we briefly review expander graphs. Let $G = (V, E)$ be a graph. For $A \subset V$, we denote by $\partial^e A$, the set of edges connecting a vertex in A and a vertex in A^c, that is,

$$\partial^e A \coloneqq \{e \in E : \#(e \cap A) = 1\}.$$

The *Cheeger constant* $h(G)$ is defined by

$$h(G) \coloneqq \min\left\{\frac{\#\partial^e A}{\#A} : A \subset G, 0 < \#A \leq \frac{\#V}{2}\right\}.$$

The *girth* of G is the length of the shortest embedded cycle contained in G, denoted by $\mathrm{girth}(G)$.

A family of *expander graphs* is a sequence of finite connected graphs $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ satisfying the following conditions:

(1) $\#V_n \to \infty \quad (n \to \infty)$.

(2) There exists $k \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and $v \in V_n$, the degree of v is less than or equal to k.

(3) There exists $c > 0$ such that $h(G_n) > c$ for any $n \in \mathbb{N}$.

We remark that a family of expander graphs is not coarsely embeddable into Hilbert space.

Let a sequence of finite connected k-regular graphs $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ form a family of expander graphs satisfying $\mathrm{girth}(G_n) \to \infty (n \to \infty)$ while

$$\frac{\mathrm{diam}(G_n)}{\mathrm{girth}(G_n)}$$

remains bounded. Let ρ_n be the combinatorial distance on G_n and let

$$d_n \coloneqq \frac{2\pi}{\mathrm{girth}(G_n)}\rho_n.$$

Then each (G_n, d_n) is a CAT(1) space since we are considering the scaled distance d_n on G_n and the length of the shortest embedded circle in it is 2π.

For each G_n, let $\widetilde{G_n} \coloneqq G_n \times \mathbb{R}_{\geq 0}/G_n \times \{0\}$. We construct a metric space Y_0 from the half-line $[0, \infty)$ by identifying each integer $n \in [0, \infty)$ with $G_n \times \{0\}$ of $\widetilde{G_n}$ and the distance of Y_0 is defined by

$$d_{Y_0}((v_1, r_1), (v_2, r_2))^2 \coloneqq r_1^2 + r_2^2 - 2r_1 r_2 \cos(\min\{d_n(v_1, v_2), \pi\}),$$

when v_1 and v_2 are in the same G_n, and

$$d_{Y_0}((v_1, r_1), (v_2, r_2)) = r_1 + r_2 + |m - n|,$$

when $v_1 \in G_n, v_2 \in G_m$ for $n \neq m$. It is clear that (Y_0, d_{Y_0}) is proper. For each $n \in \mathbb{N}$, the metric space $(\widetilde{G_n}, d_{Y_0}|_{\widetilde{G_n}})$ is the Euclidean cone over G_n. Since G_n is a CAT(1) space, by Berestovskiĭ's theorem [1] (see also [2, Theorem II.3.14]), the Euclidean cone $(\widetilde{G_n}, d_{Y_0}|_{\widetilde{G_n}})$ is a CAT(0) space. Therefore, by using the gluing lemma [2, Theorem II.11.3] repeatedly, it follows that (Y_0, d_{Y_0}) is a proper CAT(0) space. Since (Y_0, d_{Y_0}) contains a bi-Lipschitz embedded family of expanders [13, Proposition 4.4 and Remark 4.6], this space is not coarsely embeddable into a Hilbert space.

Remark 4.3.3. Such a family of expanders is obtained from the Ramanujan graphs constructed by Lubotzky et al. [14].

Example 4.3.4. Let Γ be the Cayley graph of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ for some generating set. Since $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ is a hyperbolic group, Γ is a geodesic coarsely convex space. We remark that Γ is not a CAT(0) space. By Theorem 4.1.4, the free product $\Gamma * Y_0$ satisfies the coarse Baum–Connes conjecture.

The free product $\Gamma * Y_0$ is neither CAT(0) nor coarsely embeddable into any Hilbert space. Thus, $\Gamma * Y_0$ does not satisfy the assumptions of Theorem 4.3.1.

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