Several variational problems for nonlinear Schrödinger system with three wave interaction

3 波相互作用をもつ非線形シュレディンガー方程式系に対する種々の変分問題（英文）

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Preface

In 1982, Cazenave-Lions [10] studied the existence and stability of standing wave solutions for the following single nonlinear Schrödinger equation:

$$i\partial_t \Phi + \Delta\Phi + |\Phi|^{p-1}\Phi = 0 \quad \text{in } \mathbb{R}^N. \tag{0.1}$$

Here, the standing wave solutions is the solution of (0.1) of the form $\Phi(t,x) = e^{i\omega t}u(x)$. The solution has the spatially localized waveform which does not progress and oscillate. Starting with this study, a similar problems for the models with potentials and general nonlinearities have been actively studied. In recent years, the existence and stability of standing wave solutions for systems, such as two component interaction models describing the Bose-Einstein condensation phenomenon, have been actively investigated.

Under such circumstances, this thesis deals with the existence and asymptotic behavior of standing wave solutions for the following nonlinear Schrödinger system with three wave interaction:

$$\begin{cases} i\varepsilon\partial_t\Phi_1 + \varepsilon^2\Delta\Phi_1 - V_1(x)\Phi_1 + \beta|\Phi_1|^{p-1}\Phi_1 = -\alpha\Phi_3\bar{\Phi}_2 & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ i\varepsilon\partial_t\Phi_2 + \varepsilon^2\Delta\Phi_2 - V_2(x)\Phi_2 + \beta|\Phi_2|^{p-1}\Phi_2 = -\alpha\Phi_3\bar{\Phi}_1 & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ i\varepsilon\partial_t\Phi_3 + \varepsilon^2\Delta\Phi_3 - V_3(x)\Phi_3 + \beta|\Phi_3|^{p-1}\Phi_3 = -\alpha\Phi_1\Phi_2 & \text{in } \mathbb{R}\times\mathbb{R}^N. \end{cases} \tag{0.2}$$

This system was introduced by Colin-Colin-Ohta in [19] as a simplified model of a quasilinear Zakharov system studied in [15, 16]. This system describes the interaction between laser and plasma and is related to the Raman amplification in a plasma. The physical situation is as follows. When the incident laser field enters the plasma, it is backscattered by Raman-type processes. These two waves interact to create an electron plasma wave. The three waves combine to produce an ion density change that itself affects the three preceding waves. Here a solution of (0.2) of the form $(\Phi_1(t,x), \Phi_2(t,x), \Phi_3(t,x)) = (e^{i\lambda_1 t/\varepsilon}u_1(x), e^{i\lambda_2 t/\varepsilon}u_2(x), e^{i\lambda_3 t/\varepsilon}u_3(x))$ $(\lambda_3 = \lambda_1 + \lambda_2)$ is called a standing wave solution. Then $\mathbf{u} = (u_1, u_2, u_3)$ satisfies the following system:

$$\begin{cases} -\varepsilon^2 \Delta u_1 + (V_1(x) + \lambda_1)u_1 = \beta |u_1|^{p-1}u_1 + \alpha u_3 \bar{u}_2 & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta u_2 + (V_2(x) + \lambda_2)u_2 = \beta |u_2|^{p-1}u_2 + \alpha u_3 \bar{u}_1 & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta u_3 + (V_3(x) + \lambda_3)u_3 = \beta |u_3|^{p-1}u_3 + \alpha u_1 u_2 & \text{in } \mathbb{R}^N. \end{cases} \tag{$\mathcal{P}_\varepsilon^{\alpha,\beta}$}$$

In this thesis, we consider the fixed mass problem and the fixed frequency problem for $(\mathcal{P}_\varepsilon^{\alpha,\beta})$. The fixed mass problem is the problem of finding the solution $(\mathbf{u}, \lambda_1, \lambda_2, \lambda_3)$ of $(\mathcal{P}_\varepsilon^{\alpha,\beta})$ satisfying L^2–normalized condition $\int_{\mathbb{R}^N} |u_j|^2 = a_j$ for given $a_j > 0$ $(j = 1, 2, 3)$. The fixed frequency problem is the problem of finding the solution $\mathbf{u} = (u_1, u_2, u_3)$ of $(\mathcal{P}_\varepsilon^{\alpha,\beta})$ for fixed $\lambda_j \in \mathbb{R}$ $(j = 1, 2, 3)$. The fixed mass problem is dealt with in Part I and the fixed frequency problem is dealt with in Part II in this thesis.

0.1 Fixed mass problem

For the fixed mass problem, we define the following functional:

$$\begin{aligned} E_\varepsilon^{\alpha,\beta}(\mathbf{u}) := \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u_j|^2 + V_j(x)|u_j|^2 \\ - \frac{\beta}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |u_j|^{p+1} - \alpha \mathrm{Re} \int_{\mathbb{R}^N} u_1 u_2 \bar{u}_3, \end{aligned}$$

where $N \leq 3$, $1 < p < 1 + 4/N$, $\alpha, \beta, \varepsilon > 0$. We also impose the following conditions for the potential $V_j(x)$:

(V1) for all $j = 1, 2, 3$, $V_j \in L^\infty(\mathbb{R}^N; \mathbb{R})$.

(V2) for all $j = 1, 2, 3$, $V_j(x) \leq \lim_{|y| \to \infty} V_j(y) = 0$ (for almost every $x \in \mathbb{R}^N$).

We consider the following minimization problem of $E_\varepsilon^{\alpha,\beta}$ under L^2–normalized condition:

$$\xi_\varepsilon^{\alpha,\beta}(a) := \inf\{E_\varepsilon^{\alpha,\beta}(\mathbf{u}) \mid \mathbf{u} \in H, \quad \int_{\mathbb{R}^N} |u_j|^2 = a_j \ (j = 1, 2, 3)\},$$

where $H := H^1 \times H^1 \times H^1$, $H^1 := H^1(\mathbb{R}^N;\mathbb{C})$. Then a minimizer $\mathbf{u}$ for $\xi_\varepsilon^{\alpha,\beta}(a)$ satisfies $(\mathcal{P}_\varepsilon^{\alpha,\beta})$ and λ_j appears as a Lagrange multiplier.

We also consider the limit minimization problem:

$$\begin{aligned}
\xi_{\varepsilon,\infty}^{\alpha,\beta}(a) &:= \inf\{E_{\varepsilon,\infty}^{\alpha,\beta}(\mathbf{u}) \mid \mathbf{u} \in H, \quad \textstyle\int_{\mathbb{R}^N} |u_j|^2 = a_j \ (j=1,2,3)\},\\
E_{\varepsilon,\infty}^{\alpha,\beta}(\mathbf{u}) &:= \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} \varepsilon^2|\nabla u_j|^2 - \frac{\beta}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|u_j|^{p+1}\\
&\quad - \alpha \mathrm{Re}\int_{\mathbb{R}^N} u_1 u_2 \overline{u}_3.
\end{aligned}$$

To prove the existence of a minimizer for $\xi_\varepsilon^{\alpha,\beta}(a)$, it is extremely important to show the strict subadditivity condition

$$\xi_\varepsilon^{\alpha,\beta}(a) < \xi_\varepsilon^{\alpha,\beta}(b) + \xi_{\varepsilon,\infty}^{\alpha,\beta}(a-b) \tag{0.3}$$

for all $b=(b_1,b_2,b_3)$ with $b \neq a$ and $0 \leq b_j \leq a_j$ for all $j=1,2,3$ where $a=(a_1,a_2,a_3)$, $a_j > 0$ for all $j=1,2,3$.

For the single minimization problem with pure power nonlinearities, it is easy to show the strict subadditivity condition by using the scaling $u_\theta(x) = \theta u(x)$ for $\theta > 0$. Also, for the single minimization problem with general nonlinearities and without potentials, we can show that the strict subadditivity by using the scaling $u_\lambda(x) = u(\lambda x)$ for $\lambda > 0$ (see [46]). But for system minimization problems, it is more difficult to show the strict subadditivity condition. Ardila [4] showed the existence of minimizer for $\xi_\varepsilon^{\alpha,\beta}(a)$ under the condition $N=1$ and $V_j \equiv 0$ for $j=1,2,3$. Ardila [4] used the rearrangement techniques to obtain the strict subadditivity for $\xi_\varepsilon^{\alpha,\beta}(a)$. Kurata-Osada [31] showed the existence of minimizer for $\xi_\varepsilon^{\alpha,\beta}(a)$ under the condition $N \leq 3$ and (V1),(V2) and the following symmetric condition (V3):

(V3) for all $j=1,2,3$, $V_j(-x_1,x') = V_j(x_1,x')$ for almost every $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$,
$V_j(s,x') \leq V_j(t,x')$ for almost every $s,t \in \mathbb{R}$ with $0 \leq s < t$ and $x' \in \mathbb{R}^{N-1}$.

Kurata-Osada [31] used the coupled rearrangement techniques developed by Shibata [47] on another system. However, it is more difficult to show the strict subadditivity condition (0.3) without assuming symmetry for the potentials.

Recently, Ikoma-Miyamoto [27] established a method of showing the strict subadditivity condition for two component system arising Bose-Einstein condensates model without assuming symmetry for the potentials. In Chapter 1, we confirm that the method of [27] is also applicable to the three wave interaction model without assuming symmetry for the potentials.

Osada [42] obtained the following existence result of a minimizer (see Theorem 1.1 in Chapter 1):

Theorem 0.1. (Theorem 1.1 in [42], the existence of a minimizer for $\xi_1^{1,1}(a)$) Assume that $N \leq 3$, $1 < p < 1 + 4/N$ and (V1)–(V2) and $(V_1, V_2, V_3) \not\equiv (0,0,0)$ and $a_j > 0$ for all $j = 1, 2, 3$. Then for any minimizing sequence $\{\mathbf{u}_n\}_{n=1}^{\infty}$ for $\xi_1^{1,1}(a)$, up to a subsequence, there exists a minimizer $\mathbf{u} \in H$ for $\xi_1^{1,1}(a)$ such that

$$\|\mathbf{u}_n - \mathbf{u}\|_H \to 0.$$

0.2 Fixed frequency problem

In the fixed frequency problem, functions are considered as real-valued functions. Also, rewrite $V_j(x) + \lambda_j$ as $V_j(x)$. For fixed frequency problem, we define the following functional $I_\varepsilon^{\alpha,\beta}$ which characterizes the solution of $(\mathcal{P}_\varepsilon^{\alpha,\beta})$ as a critical point:

$$\begin{aligned} I_\varepsilon^{\alpha,\beta}(\mathbf{u}) := \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} \varepsilon^2|\nabla u_j|^2 + V_j(x)u_j^2 \\ - \frac{\beta}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|u_j|^{p+1} - \alpha\int_{\mathbb{R}^N} u_1u_2u_3. \end{aligned}$$

We consider $\mathbb{H} := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ as the space to consider the solution. Then we say that $\mathbf{u}$ is a ground state of $(\mathcal{P}_\varepsilon^{\alpha,\beta})$ if $\mathbf{u}$ is a nontrivial solution of $(\mathcal{P}_\varepsilon^{\alpha,\beta})$ and minimizes $I_\varepsilon^{\alpha,\beta}$ among all nontrivial solutions of $(\mathcal{P}_\varepsilon^{\alpha,\beta})$. To search a ground state solution, we consider the following constrained minimization problem:

$$\begin{aligned} c_\varepsilon^{\alpha,\beta} &:= \inf_{\mathbf{u}\in\mathcal{N}_\varepsilon^{\alpha,\beta}} I_\varepsilon^{\alpha,\beta}(\mathbf{u}), \\ \mathcal{N}_\varepsilon^{\alpha,\beta} &:= \{\mathbf{u} \in \mathbb{H} \setminus \{(0,0,0)\} \mid G_\varepsilon^{\alpha,\beta}(\mathbf{u}) = 0\}, \end{aligned}$$

$$G_\varepsilon^{\alpha,\beta}(\mathbf{u}) := \sum_{j=1}^{3} \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u_j|^2 + V_j(x) u_j^2 - \beta |u_j|^{p+1} - 3\alpha \int_{\mathbb{R}^N} u_1 u_2 u_3.$$

It is well-known that $\mathbf{u}$ is a ground state of $(\mathcal{P}_\varepsilon^{\alpha,\beta})$ if and only if $\mathbf{u}$ is a minimizer for $c_\varepsilon^{\alpha,\beta}$. Therefore, to show the existence of a ground state of $(\mathcal{P}_\varepsilon^{\alpha,\beta})$, it is sufficient to show the existence of a minimizer for $c_\varepsilon^{\alpha,\beta}$. In the following, we mention the historical background of the singular perturbation problem.

Rabinowitz [44] showed that there exists a ground state solution of

$$-\varepsilon^2 \Delta u + V(x) u = |u|^{p-1} u \quad \text{in } \mathbb{R}^N \tag{0.4}$$

for ε sufficiently small if $0 < \inf_{x \in \mathbb{R}^N} V(x) < \liminf_{|x| \to \infty} V(x)$. Here we say that u is a ground state of (0.4) if u is a nontrivial solution with least energy

$$\frac{1}{2} \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u|^2 + V(x) u^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}$$

among all nontrivial $H^1(\mathbb{R}^N)$ solutions of (0.4).

Wang [54] studied the concentration behavior of positive ground state solutions of (0.4). That solutions concentrate at a global minimum point of V as $\varepsilon \to +0$, have a unique local maximum (hence global maximum) point and exponential decay rapidly around the minimum point.

Lin-Wei [33] considered the following nonlinear Schrödinger system

$$\begin{cases} -\varepsilon^2 \Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \Omega, \\ -\varepsilon^2 \Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \Omega, \\ u_1, u_2 > 0 \quad \text{in } \Omega, \\ u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{0.5}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain. They showed that as $\varepsilon \to +0$, there are two spikes for both $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$, where $(u_{1,\varepsilon}, u_{2,\varepsilon})$ is a positive ground state of (0.5). If $\beta < 0$, the locations of two spikes reach a sphere-packing position (the positions that maximize the minimum distance from the boundary and the distance from each other) in the domain Ω. On the other hand, if $\beta > 0$, the locations of two spikes reach the position farthest from the boundary.

Lin-Wei [34] considered the following system with potentials:

$$\begin{cases} -\varepsilon^2\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbb{R}^N, \\ -\varepsilon^2\Delta u_2 + V_2(x)u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \mathbb{R}^N, \\ u_1, u_2 > 0 \quad \text{in } \mathbb{R}^N. \end{cases} \tag{0.6}$$

For this system, they showed the spikes are trapped at the minimum points of $V_j(x)$ if $\beta < 0$. On the other hand, if $\beta > 0$, they introduced a certain function $\rho(V_1(x), V_2(x); \beta)$ and the spikes are trapped at the minimum points of $\rho(V_1(x), V_2(x); \beta)$ or trapped at the minimum points of $V_j(x)$.

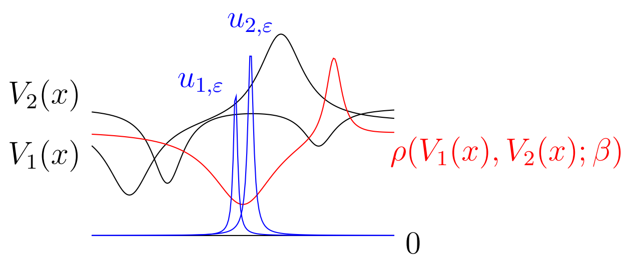

Figure 0.1: $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x); \beta) < d_1^{V_1,0} + d_1^{V_2,0}$

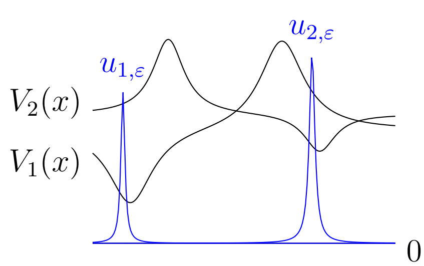

Figure 0.2: $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x); \beta) > d_1^{V_1,0} + d_1^{V_2,0}$

Here, $\rho(V_1(x_0), V_2(x_0); \beta)$ and $d_1^{V_j,0}$ are the least energies of the following equations respectively: Here, $\rho(V_1(x_0), V_2(x_0); \beta)$ and $d_1^{V_j,0}$ are the least energies of the following equations respectively:

$$\begin{cases} -\Delta u_1 + V_1(x_0)u_1 = u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + V_2(x_0)u_2 = u_2^3 + \beta u_1^2 u_2 & \text{in } \mathbb{R}^N, \\ u_1 > 0, \quad u_2 > 0 & \text{in } \mathbb{R}^N \end{cases}$$

and

$$\begin{cases} -\Delta u + V_{j,0} u = u^3 & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N. \end{cases}$$

The least energy means the energy which ground state has.

Montefusco-Pellacci-Squassina [39] considered (0.6) for the case $N = 3$. They showed that the least energy solution of (0.6) converges (up to scalings) to a least energy solution of corresponding limit problem as $\varepsilon \to +0$. They adopt a definition of Nehari manifolds similar to Pomponio [43] and ours. They also proved that if β is sufficiently large, then the limit state is vector, on the other hand, if β is sufficiently small, then the limit state is scalar.

We now introduce the main result in the setting of the singular perturbation problem. In the following, we state the main results in Chapter 4. We assume $\beta = 1$. To state main results in Chapter 4, we also consider the following system and define the following corresponding functional:

$$\begin{cases} -\Delta v_1 + \lambda_1 v_1 = |v_1|^{p-1} v_1 + \alpha v_2 v_3, \\ -\Delta v_2 + \lambda_2 v_2 = |v_2|^{p-1} v_2 + \alpha v_1 v_3, \\ -\Delta v_3 + \lambda_3 v_3 = |v_3|^{p-1} v_3 + \alpha v_1 v_2, \end{cases} \tag{$\tilde{\mathcal{P}}^{\lambda,\alpha}$}$$

$$\begin{aligned} \tilde{I}^{\lambda,\alpha}(\mathbf{v}) := &\frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla v_j|^2 + \lambda_j v_j^2 \\ &- \frac{1}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |v_j|^{p+1} - \alpha \int_{\mathbb{R}^N} v_1 v_2 v_3, \end{aligned}$$

where $\lambda := (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_j > 0$ $(j = 1, 2, 3)$. Define the least energy as follows:

$$\begin{aligned} \rho(\lambda_1, \lambda_2, \lambda_3; \alpha) &:= \inf_{\mathbf{v} \in \tilde{\mathcal{N}}^{\lambda,\alpha}} \tilde{I}^{\lambda,\alpha}(\mathbf{v}), \\ \tilde{\mathcal{N}}^{\lambda,\alpha} &:= \{\mathbf{v} \in \mathbb{H} \setminus \{(0,0,0)\} \mid \tilde{G}^{\lambda,\alpha}(\mathbf{v}) = 0\}, \\ \tilde{G}^{\lambda,\alpha}(\mathbf{v}) &:= \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla v_j|^2 + \lambda_j v_j^2 - |v_j|^{p+1} - 3\alpha \int_{\mathbb{R}^N} v_1 v_2 v_3. \end{aligned}$$

We assume the following condition for the potentials:

(V1) for all $j = 1, 2, 3$, $V_j \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$, (V2) for all $j = 1, 2, 3$, $0 < V_{j,0} := \inf_{x\in\mathbb{R}^N} V_j(x) < \lim_{|x|\to\infty} V_j(x) =: V_{j,\infty}$.

$(\mathrm{C1})_\alpha$ $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) < \rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha)$.

We now state main results for the singular perturbation problem. First, we state the existence of a ground state of $(\mathcal{P}_\varepsilon^{\alpha,1})$ for ε sufficiently small (see Theorem 4.3 in Chapter 4).

Theorem 0.2. We assume that (V1),(V2) and fix α so that $(\mathrm{C1})_\alpha$ holds. Then it follows that

$$c_\varepsilon^{\alpha,1} \leq \varepsilon^N \left(\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) + o(1) \right), \quad \text{as } \varepsilon \to +0.$$

Moreover, there exists a non-negative ground state $\mathbf{u}$ of $(\mathcal{P}_\varepsilon^{\alpha,1})$ for ε sufficiently small.

Next, we state the precise asymptotic behavior of a ground state of $(\mathcal{P}_\varepsilon^{\alpha,1})$ as $\varepsilon \to +0$. To obtain the asymptotic behavior precisely, we introduce the following condition:

$(\mathrm{C2})_\alpha$ $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) < \min_{j=1,2,3} c_1^{V_{j,0}}$,

where

$$\begin{aligned}
\lambda &> 0, \\
I_1^\lambda(u) &:= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}, \\
c_1^\lambda &:= \inf_{u\in\mathcal{N}_1^\lambda} I_1^\lambda(u), \\
\mathcal{N}_1^\lambda &:= \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid G_1^\lambda(u) = 0\}, \\
G_1^\lambda(u) &:= \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 - |u|^{p+1}.
\end{aligned}$$

We also consider the following equation associated the above minimization problem:

$$-\Delta u + \lambda u = |u|^{p-1} u \qquad (\mathcal{P}_1^\lambda)$$

Now, we state the precise asymptotic behavior for a non-negative ground state of $(\mathcal{P}_\varepsilon^{\alpha,1})$ as $\varepsilon \to +0$ (see Theorem 4.5 in Chapter 4).

Theorem 0.3. We assume that (V1),(V2) and fix α so that $(\mathrm{C1})_\alpha$ and $(\mathrm{C2})_\alpha$ hold. Let $\{\varepsilon_n\}_{n=1}^\infty \subset (0,\infty)$ with $\varepsilon_n \to 0$ as $n \to \infty$ and let $\mathbf{u}_n$ be a non-negative ground state of $(\mathcal{P}_{\varepsilon_n}^{\alpha,1})$. Let $x_{j,n}$ be a maximum point of $u_{j,n}$.

(1) Then, it follows that $\{x_{j,n}\}_{n=1}^\infty$ is bounded for all $j = 1,2,3$.

(2) It holds that

$$c_\varepsilon^{\alpha,1} = \varepsilon^N \left(\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) + o(1) \right), \quad \text{as } \varepsilon \to +0.$$

(3) Furthermore, up to a subsequence, there exist $\mathbf{W}_0 \in \mathbb{H}$ and $x_0 \in \mathbb{R}^N$ such that

$$\begin{aligned}
&x_{j,n} \to x_0,\\
&\frac{|x_{j,n} - x_{k,n}|}{\varepsilon_n} \to 0, \quad \text{as } n \to \infty,\ j \neq k,\\
&\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) = \rho(V_1(x_0), V_2(x_0), V_3(x_0); \alpha),\\
&u_{j,n}(x_{j,n} + \varepsilon_n y) \to W_{j,0} \quad \text{in } H^1(\mathbb{R}^N),\\
&\mathbf{W}_0 \text{ is a ground state of } (\tilde{\mathcal{P}}^{\mathbf{V}(x_0),\alpha})\\
&W_{j,0} \text{ is positive, radially symmetric and strictly decreasing}\\
&\text{for all } j = 1,2,3,
\end{aligned}$$

where $\mathbf{V}(x_0) = (V_1(x_0), V_2(x_0), V_3(x_0))$.

(4) Moreover, for any $0 < \eta < V_0$, there exists $C_\eta > 0$ such that

$$u_{j,n}(x) \le C_\eta e^{-\sqrt{\eta}|x - x_{j,n}|/\varepsilon_n} \quad \text{for all } x \in \mathbb{R}^N,\ n \in \mathbb{N},\ j = 1,2,3,$$

where $V_0 := \min\{V_{1,0}, V_{2,0}, V_{3,0}\}$.

Finally, we state the main result in the asymptotic behavior of a non-negative ground state of $(\mathcal{P}_\varepsilon^{\alpha,1})$ as $\varepsilon \to +0$ for the case where $(\mathrm{C2})_\alpha$ does not hold (see Theorem 4.6 in Chapter 4). When $(\mathrm{C2})_\alpha$ does not hold, the following condition holds (see Lemma 4.15 and Proposition 4.18):

$(\mathrm{C3})_\alpha$ $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) = \min_{j=1,2,3} c_1^{V_{j,0}}$.

Theorem 0.4. We assume that (V1),(V2) and fix α so that $(C1)_\alpha$ and $(C3)_\alpha$ hold. In addition, we assume that there exists $\alpha' > \alpha$ such that $(C3)_{\alpha'}$ holds. Let $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,\infty)$ such that $\varepsilon_n \to +0$ and let $\mathbf{u}_n$ be a non-negative ground state for $(\mathcal{P}_{\varepsilon_n}^{\alpha,1})$. Let $x_{j,n}$ be a maximum point of $u_{j,n}$. Then, up to a subsequence, there exist $l_0 \in \{1,2,3\}$ and $x_{l_0,0} \in \mathbb{R}^N$ such that

$$\begin{aligned}
&x_{l_0,n} \to x_{l_0,0}, \quad V_{l_0}(x_{l_0,0}) = V_{l_0,0} = V_0,\\
&c_\varepsilon^{\alpha,1} = \varepsilon^N \left(\min_{j=1,2,3} c_1^{V_{j,0}} + o(1) \right) = \varepsilon^N \left(c_1^{V_{l_0,0}} + o(1) \right), \quad \text{as } \varepsilon \to +0,\\
&u_{l_0,n}(x_{l_0,n} + \varepsilon_n y) \to W \quad \text{in } H^1(\mathbb{R}^N),\\
&u_{j,n}(x_{j,n} + \varepsilon_n y) \to 0 \quad \text{in } H^1(\mathbb{R}^N) \quad j \neq l_0,
\end{aligned}$$

where W is the unique solution of the following equation:

$$\begin{cases}
-\Delta W + V_0 W = W^p \quad \text{in } \mathbb{R}^N,\\
W > 0 \quad \text{in } \mathbb{R}^N,\\
W(0) = \max_{x\in\mathbb{R}^N} W(x),\\
W(x) \to 0, \quad \text{as } |x| \to \infty.
\end{cases}$$

In the problem considered Lin-Wei [34], they consider the least energy solution among all vector solutions (the solution which has all components are non-zero) of

$$\begin{cases}
-\varepsilon^2 \Delta u_1 + V_1(x) u_1 = u_1^3 + \beta u_1 u_2^2,\\
-\varepsilon^2 \Delta u_2 + V_2(x) u_2 = u_2^3 + \beta u_1^2 u_2.
\end{cases}$$

On the other hand, in our setting, we consider the least energy solution among all nontrivial solutions (includes scalar solution (the solution which only one component survive)) of

$$\begin{cases}
-\varepsilon^2 \Delta u_1 + V_1(x) u_1 = |u_1|^{p-1} u_1 + \alpha u_3 u_2 \quad \text{in } \mathbb{R}^N,\\
-\varepsilon^2 \Delta u_2 + V_2(x) u_2 = |u_2|^{p-1} u_2 + \alpha u_3 u_1 \quad \text{in } \mathbb{R}^N,\\
-\varepsilon^2 \Delta u_3 + V_3(x) u_3 = |u_3|^{p-1} u_3 + \alpha u_1 u_2 \quad \text{in } \mathbb{R}^N.
\end{cases} \qquad (\mathcal{P}_\varepsilon^{\alpha,1})$$

Therefore, in the result of Theorem 0.4, the case which each component of ground states survives and converges to a minimum point of corresponding potential respectively as in the result in Lin-Wei [34] does not occur.

The difference in the range of N and p arising the fixed mass problem and the fixed frequency problem is due to the difference in the conditions for ensuring that the functional is bounded below due to the difference in the constraints. The purpose of this thesis is to analyze the existence and asymptotic behavior of the solutions of minimization problems for the fixed mass problem and the fixed frequency problem using variational methods.

The rest of the thesis is organized as follows.

In Part I, we consider the fixed mass problem. Part I consists of Chapter 1 and 2. In Part II, we consider the fixed frequency problem. Part II consists of Chapter 3 and 4.

In Chapter 1, we show that the existence of a minimizer for $\xi_1^{1,1}(a)$ under without assuming symmetry for $V_j(x)$. This result is an extension of the result of Kurata-Osada [31]. In doing so, a technique interaction estimate developed by Ikoma and Miyamoto in [26] plays an important role. Chapter 1 is based on the result in Osada [42] and Kurata-Osada [31].

In Chapter 2, we consider the asymptotic behavior of a minimizer for $\xi_1^{\alpha,\beta}(a)$ as $\beta \to \infty$ under supposing $\alpha = \beta^\kappa$ for given $\kappa \in \mathbb{R}$. We show that the asymptotic behavior of a minimizer can be classified into five types depending on the size of $\kappa \in \mathbb{R}$. Moreover, we investigate the asymptotic behavior of a minimizer for $\xi_1^{\alpha,\beta}(a)$ as $\alpha \to \infty$ under supposing $\beta = \alpha^\tau$ for given $\tau \in \mathbb{R}$. This is an extension of the result of [31]. Chapter 2 is based on the result in Osada [41].

In Chapter 3, we investigate the asymptotic behavior of a ground state of $(\mathcal{P}_1^{\alpha,1})$ as $\alpha \to \infty$. Moreover, we obtain the result that there exists a positive constant α^* such that all ground states of $(\mathcal{P}_1^{\alpha,1})$ is scalar (a state in which only one component survives) if $0 \le \alpha < \alpha^*$, and all ground states of $(\mathcal{P}_1^{\alpha,1})$ is vector (a state in which all components survive) if $\alpha > \alpha^*$. Chapter 3 is based on the result in Kurata-Osada [30].

In Chapter 4, we consider the existence of a non-negative ground state of $(\mathcal{P}_\varepsilon^{\alpha,1})$ for ε sufficiently small and the asymptotic behavior of the non-negative ground state of $(\mathcal{P}_\varepsilon^{\alpha,1})$ as $\varepsilon \to +0$. In particular, it is clarified that the asymptotic form becomes spike-like, and the position of the spike is determined by the shape of the potential $V_j(x)$ and the attractive force of the three wave interaction α. Chapter 4 is based on the result in Osada [40].

Contents

Part I

Fixed mass problem

Chapter 1

Existence of a minimizer for a nonlinear Schrödinger system with three wave interaction under non-symmetric potentials

1.1 Introduction

We consider the following L^2-constrained minimization problem associated with a nonlinear Schrödinger system with three wave interaction: for $a = (a_1, a_2, a_3),\ a_1, a_2, a_3 \geq 0$,

$$\begin{aligned} \xi(a) &:= \inf\{E(\mathbf{u}) \mid \mathbf{u} \in M(a)\}, \qquad (1.1)\\ E(\mathbf{u}) &:= \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla u_j|^2 + V_j(x)|u_j|^2\,dx \\ &\quad - \frac{\beta}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |u_j|^{p+1}\,dx - \alpha\mathrm{Re}\int_{\mathbb{R}^N} u_1u_2\overline{u}_3\,dx, \\ M(a) &:= \{\mathbf{u} \in H \mid \|u_j\|_2^2 = a_j \quad (j = 1, 2, 3)\}, \end{aligned}$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $H := H^1 \times H^1 \times H^1$, $H^1 := H^1(\mathbb{R}^N; \mathbb{C})$, $1 \le N \le 3$, $1 < p < 1 + 4/N$, $\alpha, \beta > 0$ and each potential V_j satisfies the following conditions:

(V1) for all $j = 1, 2, 3$, $V_j \in L^\infty(\mathbb{R}^N; \mathbb{R})$.

(V2) for all $j = 1, 2, 3$, $V_j(x) \le \lim_{|y|\to\infty} V_j(y) = 0$ (for almost every $x \in \mathbb{R}^N$).

The minimization problem (1.1) is related to the existence of a standing wave solution of the nonlinear Schrödinger system with three wave interaction:

$$\begin{cases} i\partial_t v_1 - V_1(x)v_1 + \Delta v_1 + \beta|v_1|^{p-1}v_1 = -\alpha\bar{v}_2 v_3, & (1.2)\\ i\partial_t v_2 - V_2(x)v_2 + \Delta v_2 + \beta|v_2|^{p-1}v_2 = -\alpha\bar{v}_1 v_3, & (1.3)\\ i\partial_t v_3 - V_3(x)v_3 + \Delta v_3 + \beta|v_3|^{p-1}v_3 = -\alpha v_1 v_2. & (1.4) \end{cases}$$

As explained in [4], once we show the existence of a minimizer of (1.1), we can also show the existence of a minimizer of the energy E under the constraints

$$\|u_1\|_2^2 + \|u_3\|_2^2 = a_1, \quad \|u_2\|_2^2 + \|u_3\|_2^2 = a_2,$$

for given $a_1 > 0$ and $a_2 > 0$. Then if $\mathbf{u}$ is a minimizer of the energy E under the constraints $\|u_1\|_2^2 + \|u_3\|_2^2 = a_1$ and $\|u_2\|_2^2 + \|u_3\|_2^2 = a_2$, there exist ω_1 and ω_2 such that

$$\begin{cases} -\Delta u_1 + (\omega_1 + V_1(x))u_1 - \beta|u_1|^{p-1}u_1 = \alpha\bar{u}_2 u_3,\\ -\Delta u_2 + (\omega_2 + V_2(x))u_2 - \beta|u_2|^{p-1}u_2 = \alpha\bar{u}_1 u_3,\\ -\Delta u_3 + (\omega_3 + V_3(x))u_3 - \beta|u_3|^{p-1}u_3 = \alpha u_1 u_2, \end{cases}$$

where $\omega_3 = \omega_1 + \omega_2$. That is,

$$(e^{i\omega_1 t}u_1(x), e^{i\omega_2 t}u_2(x), e^{i(\omega_1+\omega_2)t}u_3(x))$$

is a standing wave solution of (1.2)–(1.4). In that sense, it is important to show the existence of a minimizer of the minimization problem (1.1).

The system (1.2)–(1.4) was introduced by Colin-Colin-Ohta [19] with $V_j(x) \equiv 0$ and $\beta = 1$ (see also [15, 16]). Colin-Colin-Ohta [19] showed that the standing wave solutions $(e^{i\omega t}\varphi, 0, 0)$ and $(0, e^{i\omega t}\varphi, 0)$ is orbitally stable for all $\alpha > 0$, where $\omega > 0$ and φ is the unique positive radial solution of

$$-\Delta v + \omega v - |v|^{p-1}v = 0 \quad \text{in } \mathbb{R}^N.$$

On the other hand, $(0, 0, e^{i\omega t}\varphi)$ is orbitally stable if $0 < \alpha < \alpha^*$ and is orbitally unstable if $\alpha > \alpha^*$ where α^* is suitable positive constant (see [19] for more detail).

We say that $\{\mathbf{u}_n\}_{n=1}^{\infty} \subset M(a)$ is a minimizing sequence for $\xi(a)$ if $E(\mathbf{u}_n) \to \xi(a)$ as $n \to \infty$. We state the main result in this chapter.

Theorem 1.1. Assume that $N \leq 3$, $1 < p < 1 + 4/N$, $\alpha, \beta > 0$ and (V1)–(V2) and $(V_1, V_2, V_3) \not\equiv (0, 0, 0)$ and $a_j > 0$ for all $j = 1, 2, 3$. Then for any minimizing sequence $\{\mathbf{u}_n\}_{n=1}^{\infty} \subset M(a)$ for $\xi(a)$, up to a subsequence, there exists a minimizer $\mathbf{u} \in H$ for $\xi(a)$ such that

$$\|\mathbf{u}_n - \mathbf{u}\|_H \to 0.$$

We also consider the limit minimization problem:

$$\begin{aligned}
\xi_\infty(a) &:= \inf\{E_\infty(\mathbf{u}) \mid \mathbf{u} \in M(a)\}, \\
E_\infty(\mathbf{u}) &:= \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla u_j|^2\,dx - \frac{\beta}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |u_j|^{p+1}\,dx \\
&\quad - \alpha \mathrm{Re}\int_{\mathbb{R}^N} u_1 u_2 \overline{u}_3\,dx.
\end{aligned}$$

To prove Theorem 1.1, it is extremely important to show the strict subadditivity condition

$$\xi(a) < \xi(b) + \xi_\infty(a-b) \tag{1.5}$$

for all $b = (b_1, b_2, b_3)$ with $b \neq a$ and $0 \leq b_j \leq a_j$ for all $j = 1, 2, 3$ where $a = (a_1, a_2, a_3)$, $a_j > 0$ for all $j = 1, 2, 3$. In previous paper [31], we showed the existence of a minimizer under the conditions (V1),(V2) and the additional assumption:

(V3) for all $j = 1, 2, 3$, $V_j(-x_1, x') = V_j(x_1, x')$ for almost every $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$,
$V_j(s, x') \leq V_j(t, x')$ for almost every $s, t \in \mathbb{R}$ with $0 \leq s < t$ and $x' \in \mathbb{R}^{N-1}$.

Thus Theorem 1.1 improves the result in [31]. The condition $V(x) \leq \lim_{|y|\to\infty} V(y) = 0$ (for almost every $x \in \mathbb{R}^N$) is almost necessary even for the scalar case (see e.g. [35, 36]). However it is known that the condition (V3) is not necessary for the scalar case or some systems (see e.g. [26, 27]). The key point of the proof of Theorem 1.1 is to show a quantitative estimate (1.41) (see the proof of Proposition 1.9) for our system, which implies the strict subadditivity (1.5) without symmetric condition (V3) by using the idea in [26, 27].

For the single minimization problem with pure power nonlinearities, it is easy to show the strict subadditivity condition by using the scaling $u_\theta(x) = \theta u(x)$ for $\theta > 0$. Also, for the single minimization problem with general nonlinearities and without potentials, we can show that the strict subadditivity by using the scaling $u_\lambda(x) = u(\lambda x)$ for $\lambda > 0$ (see [46]). But for system minimization problems, it is more difficult to show the strict subadditivity condition. Ardila [4] showed the existence of a minimizer for $\xi(a)$ under the condition $N = 1$ and $V_j \equiv 0$ for $j = 1, 2, 3$. Ardila [4] used the rearrangement techniques to obtain the strict subadditivity for $\xi(a)$. Kurata-Osada [31] showed the existence of a minimizer for $\xi(a)$ under the condition $N \leq 3$ and (V1),(V2) and the symmetric condition (V3). Kurata-Osada [31] used the coupled rearrangement techniques developed by Shibata [47] (see Gou [23] and Gou-Jeanjean [24] for other studies using coupled rearrangement). However, it is more difficult to show the strict subadditivity condition (1.5) without assuming symmetry for the potentials. Recently, Ikoma-Miyamoto [27] established a method of showing the strict subadditivity condition for two component system arising Bose-Einstein condensates model without assuming symmetry for the potentials. So in this chapter, we prove Theorem 1.1 based on the technique due to [27].

We also mention other studies on nonlinear Schrödinger system with three wave interaction. Pomponio [43] studied the existence of vector ground state of the system

$$\begin{cases} -\Delta u_1 + V_1(x)u_1 - |u_1|^{p-1}u_1 = \alpha u_2\overline{u}_3, & (1.6) \\ -\Delta u_2 + V_2(x)u_2 - |u_2|^{p-1}u_2 = \alpha u_1\overline{u}_3, & (1.7) \\ -\Delta u_3 + V_3(x)u_3 - |u_3|^{p-1}u_3 = \alpha u_1 u_2, & (1.8) \end{cases}$$

for $\alpha > 0$ sufficiently large. Here, $N \leq 5$, $2 < p < 2^* - 1$, $2^* := \infty$ $(N = 1, 2)$, $2^* := 2N/(N-2)$ $(N \geq 3)$ and the potential V_j satisfies a suitable conditions (see [43] for more details). After that, Kurata-Osada [30] showed that the asymptotic expansion of ground state energy as $\alpha \to \infty$ and there exists a positive threshold α^* such that all ground state for (1.6)–(1.8) is scalar if $0 \le \alpha < \alpha^*$ and is vector if $\alpha > \alpha^*$. Moreover, Osada [41] showed the asymptotic expansion of $\xi(a)$ as $\beta \to \infty$ with $\alpha = \beta^\theta$ ($\theta \in \mathbb{R}$).

The rest of this chapter is organized as follows: In Section 1.2, we note that a property of a minimizing sequence for $\xi(a)$ and an exponential decay estimate for a non-negative solution of a corresponding nonlinear elliptic system (see Lemma 1.8). In Section 1.3, we prove the strict subadditivity for $\xi(a)$ by using the idea in [27]. In Section 1.4 we prove Theorem 1.1. In Appendix, we prove the existence of a minimizer for $\xi(a)$ under symmetric conditions for potentials $V_j(x)$. Although this result have been proved in [31], we give the proof for the reader's convenience.

Notation

$$
\begin{aligned}
&|\mathbf{u}| = (|u_1|, |u_2|, |u_3|),\\
&(u, v)_2 := \int_{\mathbb{R}^N} u\overline{v}\, dx,\\
&\|u\|_2^2 = (u, u)_2,\\
&(u, v)_{H^1} := \int_{\mathbb{R}^N} \nabla u \cdot \overline{\nabla v} + u\overline{v}\, dx,\\
&\|u\|_{H^1}^2 := (u, u)_{H^1},\\
&(\mathbf{u}, \mathbf{v})_H := \sum_{j=1}^{3} (u_j, v_j)_{H^1},\\
&\|\mathbf{u}\|_H^2 := (\mathbf{u}, \mathbf{u})_H,\\
&\langle u, v\rangle_{V_j} := \mathrm{Re} \int_{\mathbb{R}^N} \nabla u \cdot \overline{\nabla v} + V_j(x) u\overline{v}\, dx,\\
&F_{V_j}(u) := \langle u, u\rangle_{V_j}.
\end{aligned}
$$

1.2 Preliminaries

From now on, in this chapter, we assume that $\alpha = \beta = 1$ for simplicity.

1.2.1 Existence of the nice minimizing sequence

The following lemma can be proved as in Lemma 2.2 in [27].

Lemma 1.2. Assume that $N \leq 3$, $1 < p < 1 + 4/N$ and (V1),(V2) and let $\{\mathbf{u}_n\}_{n=1}^{\infty} \subset M(a)$ be a minimizing sequence for $\xi(a)$. Then $\{|u_n|\}_{n=1}^{\infty}$ is also a minimizing sequence for $\xi(a)$. Moreover, if $\{|\mathbf{u}_n|\}_{n=1}^{\infty}$ has a strongly convergent subsequence in H, then $\{\mathbf{u}_n\}_{n=1}^{\infty}$ has also a strongly convergent subsequence in H.

Although the following lemma can be proved as in Lemma 2.3 in [27], we give a proof for the reader's convenience according to the setting of this chapter.

Lemma 1.3. Suppose that (V1) and let $\{\mathbf{u}_n\}_{n=1}^{\infty} \subset M(a)$ be a minimizing sequence for $\xi(a)$. Then there exist $\{\mathbf{v}_n\}_{n=1}^{\infty} \subset M(a)$ and $\{\lambda_{j,n}\} \subset \mathbb{R}$ such that $\{\lambda_{j,n}\}_{n=1}^{\infty}$ are bounded and

$$\|\mathbf{u}_n - \mathbf{v}_n\|_H \to 0,$$
$$E'(\mathbf{v}_n) + \lambda_{1,n} Q_1'(\mathbf{v}_n) + \lambda_{2,n} Q_2'(\mathbf{v}_n) + \lambda_{3,n} Q_3'(\mathbf{v}_n) \to 0 \quad \text{strongly in } H^*,$$

where

$$E'(\mathbf{u})[\mathbf{v}] = \mathrm{Re} \sum_{j=1}^{3} \int_{\mathbb{R}^N} \nabla u_j \cdot \overline{\nabla v_j} + V_j(x) u_j \overline{v}_j \, dx - \mathrm{Re} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |u_j|^{p-1} u_j \overline{v_j} \, dx$$
$$- \mathrm{Re} \int_{\mathbb{R}^N} v_1 u_2 \overline{u}_3 + u_1 v_2 \overline{u}_3 + u_1 u_2 \overline{v}_3 \, dx,$$
$$Q_j(\mathbf{u}) := \frac{1}{2} \|u_j\|_2^2, \quad Q_j'(\mathbf{u})[\mathbf{v}] = \mathrm{Re}(u_j, v_j)_2.$$

Proof. Applying Ekeland's variational principle for E and $\{\mathbf{u}_n\}$ on $M(a)$ (see [38, Theorem 4.1 and Remark 4.1]), there exists $\{\mathbf{v}_n\}_{n=1}^{\infty} \subset M(a)$ such that for $\varepsilon_n := E(\mathbf{u}_n) - \xi(a) \geq 0$,

$$\begin{aligned} &E(\mathbf{v}_n) \leq E(\mathbf{u}_n), \quad \|\mathbf{u}_n - \mathbf{v}_n\|_H \leq \sqrt{\varepsilon_n}, \\ &E(\mathbf{v}_n) \leq E(\mathbf{w}) + \sqrt{\varepsilon_n}\|\mathbf{v}_n - \mathbf{w}\|_H \quad \text{for all } \mathbf{w} \in M(a). \end{aligned} \tag{1.9}$$

Thus, $\{\mathbf{v}_n\}_{n=1}^{\infty}$ also a minimizing sequence for $\xi(a)$.

For $\mathbf{u} \in M(a)$, set

$$T_{\mathbf{u}}M(a) := \{\mathbf{v} \in H \mid \mathrm{Re}(u_j, v_j)_2 = 0 \quad (j = 1, 2, 3)\}.$$

By the Riesz representation theorem, there exists unique $\tilde{u}_j \in H^1$ such that

$$\begin{aligned} &(u_j, v_j)_2 = (\tilde{u}_j, v_j)_{H^1} \quad \text{for all } v_j \in H^1, \\ &\|u_j\|_2 = \|\tilde{u}_j\|_{H^1}. \end{aligned} \tag{1.10}$$

Set $\nabla Q_1(\mathbf{u}) := (\tilde{u}_1, 0, 0)$, $\nabla Q_2(\mathbf{u}) := (0, \tilde{u}_2, 0)$, $\nabla Q_3(\mathbf{u}) := (0, 0, \tilde{u}_3)$ and

$$\begin{aligned} &\mathrm{span}\ \{\nabla Q_1(\mathbf{u}), \nabla Q_2(\mathbf{u}), \nabla Q_3(\mathbf{u})\} \\ &:= \{c_1\nabla Q_1(\mathbf{u}) + c_2\nabla Q_2(\mathbf{u}) + c_3\nabla Q_3(\mathbf{u}) \mid c_1, c_2, c_3 \in \mathbb{R}\}. \end{aligned}$$

Then we have

$$\begin{aligned} T_{\mathbf{u}}M(a) &= \{\mathbf{v} \in H \mid \mathrm{Re}(\nabla Q_j(\mathbf{u}), \mathbf{v})_H = 0 \quad (j = 1, 2, 3)\} \\ &= \mathrm{span}\ \{\nabla Q_1(\mathbf{u}), \nabla Q_2(\mathbf{u}), \nabla Q_3(\mathbf{u})\}^{\perp}, \\ H &= T_{\mathbf{u}}M(a) \oplus \mathrm{span}\ \{\nabla Q_1(\mathbf{u}), \nabla Q_2(\mathbf{u}), \nabla Q_3(\mathbf{u})\} \end{aligned} \tag{1.11}$$

Noting that (1.11), for all $\mathbf{u} \in H$, there exist $\mathbf{h} \in T_{\mathbf{v}_n}M(a)$ and $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\mathbf{u} = \mathbf{h} + c_1\nabla Q_1(\mathbf{v}_n) + c_2\nabla Q_2(\mathbf{v}_n) + c_3\nabla Q_3(\mathbf{v}_n).$$

Setting $\lambda_{j,n} := -E'(\mathbf{v}_n)[\nabla Q_j(\mathbf{v}_n)]/\|\nabla Q_j(\mathbf{v}_n)\|_H^2$, we have

$$(E'(\mathbf{v}_n) + \lambda_{1,n}Q_1'(\mathbf{v}_n) + \lambda_{2,n}Q_2'(\mathbf{v}_n) + \lambda_{3,n}Q_3'(\mathbf{v}_n))[\mathbf{u}] = E'(\mathbf{v}_n)[\mathbf{h}]. \tag{1.12}$$

Here, we define $c(t) : (-\varepsilon, \varepsilon) \to M(a)$ as follows:

$$c(t) = \left(\sqrt{a_1}\frac{v_{1,n} + th_1}{\|v_{1,n} + th_1\|_2}, \sqrt{a_2}\frac{v_{2,n} + th_2}{\|v_{2,n} + th_2\|_2}, \sqrt{a_3}\frac{v_{3,n} + th_3}{\|v_{3,n} + th_3\|_2}\right).$$

Then $c(t)$ is a C^1-curve satisfying $c(0) = \mathbf{v}_n$ and $c'(0) = \mathbf{h}$. From (1.9),

$$\frac{E(c(t)) - E(\mathbf{v}_n)}{t} \geq -\sqrt{\varepsilon_n} \left\| \frac{c(t) - \mathbf{v}_n}{t} \right\|_H \quad \text{if } t > 0, \tag{1.13}$$

$$\frac{E(c(t)) - E(\mathbf{v}_n)}{t} \leq \sqrt{\varepsilon_n} \left\| \frac{c(t) - \mathbf{v}_n}{t} \right\|_H \quad \text{if } t < 0. \tag{1.14}$$

Since E is Fréchet differentiable,

$$\frac{E(c(t)) - E(\mathbf{v}_n)}{t} \to E'(\mathbf{v}_n)[\mathbf{h}], \quad \text{as } t \to 0. \tag{1.15}$$

From (1.13)–(1.15), we have

$$|E'(\mathbf{v}_n)[\mathbf{h}]| \leq \sqrt{\varepsilon_n} \|\mathbf{h}\|_H. \tag{1.16}$$

Thus from (1.12) and (1.16),

$$\begin{aligned} &\|E'(\mathbf{v}_n) + \lambda_{1,n} Q_1'(\mathbf{v}_n) + \lambda_{2,n} Q_2'(\mathbf{v}_n) + \lambda_{3,n} Q_3'(\mathbf{v}_n)\|_{H^*} \\ = &\sup_{\mathbf{u} \in H,\ \|\mathbf{u}\|_H \leq 1} |(E'(\mathbf{v}_n) + \lambda_{1,n} Q_1'(\mathbf{v}_n) + \lambda_{2,n} Q_2'(\mathbf{v}_n) + \lambda_{3,n} Q_3'(\mathbf{v}_n))[\mathbf{u}]| \\ \leq &\sqrt{\varepsilon_n} \to 0. \end{aligned}$$

Since $\{\mathbf{v}_n\}$ is bounded in H and (1.10) and $\mathbf{v}_n \in M(a)$, there exista $M > 0$ such that

$$\begin{aligned} &\|\nabla Q_j(\mathbf{v}_n)\|_H = \sqrt{a_j}, \\ &|E'(\mathbf{v}_n)[\nabla Q_j(\mathbf{v}_n)]| \leq M \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Therefore $\{\lambda_{j,n}\}$ is bounded. □

1.2.2 Exponential decay estimate

We introduce the following terminology for convenience.

Definition 1.4. Let f be a non-negative function defined on $\mathbb{R}^N$ and $\lambda > 0$. We say that f has an essentially exponential decay order $\sqrt{\lambda}$ if for all $0 < \sqrt{\eta_1} < \sqrt{\lambda} < \sqrt{\eta_2}$, there exist $C_{\eta_1}, C_{\eta_2} > 0$ such that

$$C_{\eta_2} e^{-\sqrt{\eta_2}|x|} \leq f(x) \leq C_{\eta_1} e^{-\sqrt{\eta_1}|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

Definition 1.5. Let $\{a_n\}_{n=1}^\infty$ be a non-negative sequence and $\lambda > 0$. We say that $\{a_n\}_{n=1}^\infty$ has an essentially exponential decay order $\sqrt{\lambda}$ if for all $0 < \sqrt{\eta_1} < \sqrt{\lambda} < \sqrt{\eta_2}$, there exist $C_{\eta_1}, C_{\eta_2} > 0$ such that

$$C_{\eta_2} e^{-\sqrt{\eta_2} n} \le a_n \le C_{\eta_1} e^{-\sqrt{\eta_1} n} \quad \text{for all } n \in \mathbb{N}.$$

We give a simple proof of the following weak version of the interaction estimate due to Bahri-Li [5].

Lemma 1.6. Let f and g be non-negative functions and $\lambda_1, \lambda_2 > 0$. We assume that f and g have an essentially exponential decay order $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ respectively. Set

$$a_n := \int_{\mathbb{R}^N} f(x) g(x - ne_1)\, dx.$$

Then $\{a_n\}_{n=1}^\infty$ has an essentially exponential decay order $\min\{\sqrt{\lambda_1}, \sqrt{\lambda_2}\}$.

Proof. Without loss of generality, we may assume that $\lambda_1 \le \lambda_2$. We first prove that for any $0 < \sqrt{\eta_1} < \sqrt{\lambda_1}$, there exists $C_{\eta_1} > 0$ such that

$$a_n \le C_{\eta_1} e^{-\sqrt{\eta_1} n} \quad \text{for all } n \in \mathbb{N}.$$

Indeed, since f and g have an essentially exponential decay order $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$, it follows that for any $0 < \sqrt{\eta_3} < \sqrt{\lambda_1}$, there exist $\sqrt{\eta_3} < \sqrt{\eta_4} < \sqrt{\lambda_2}$, $C_{\eta_3}, C_{\eta_4} > 0$ such that

$$f(x) \le C_{\eta_3} e^{-\sqrt{\eta_3}|x|} \quad \text{for all } x \in \mathbb{R}^N,$$
$$g(x) \le C_{\eta_4} e^{-\sqrt{\eta_4}|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

Thus, we have

$$\begin{aligned} a_n &\le C_{\eta_3} C_{\eta_4} \int_{\mathbb{R}^N} e^{-\sqrt{\eta_3}|x+ne_1|} e^{-\sqrt{\eta_4}|x|}\, dx \\ &\le C_{\eta_3} C_{\eta_4} \int_{\mathbb{R}^N} e^{-(\sqrt{\eta_4}-\sqrt{\eta_3})|x|}\, dx\, e^{-\sqrt{\eta_3} n} \quad \text{for all } 0 < \sqrt{\eta_3} < \sqrt{\lambda_1}. \end{aligned}$$

Secondly, we prove that for any $\sqrt{\eta_2} > \sqrt{\lambda_1}$, there exists $C_{\eta_2} > 0$ such that

$$a_n \geq C_{\eta_2} e^{-\sqrt{\eta_2} n} \quad \text{for all } n \in \mathbb{N}.$$

Since f and g have an essentially exponential decay order $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$, for any $\sqrt{\eta_5} > \sqrt{\lambda_1}$, $\sqrt{\eta_6} > \sqrt{\lambda_2}$, there exist $C_{\eta_5} > 0$ and $C_{\eta_6} > 0$ such that

$$f(x) \geq C_{\eta_5} e^{-\sqrt{\eta_5}|x|} \quad \text{for all } x \in \mathbb{R}^N,$$
$$g(x) \geq C_{\eta_6} e^{-\sqrt{\eta_6}|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

Thus, we have

$$\begin{aligned} a_n &\geq C_{\eta_5} C_{\eta_6} \int_{\mathbb{R}^N} e^{-\sqrt{\eta_5}|x+ne_1|} e^{-\sqrt{\eta_6}|x|}\, dx \\ &\geq C_{\eta_5} C_{\eta_6} \int_{\mathbb{R}^N} e^{-(\sqrt{\eta_5}+\sqrt{\eta_6})|x|}\, dx\, e^{-\sqrt{\eta_5} n}. \end{aligned}$$

□

Lemma 1.7. Let f and g be a non-negative functions and $p, q \geq 0$ with $(p, q) \neq (0, 0)$ and $\lambda_1, \lambda_2 > 0$. We assume that f and g have an essentially exponential decay order $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ respectively. Then it follows that $f^p g^q$ has an essentially exponential decay order $p\sqrt{\lambda_1} + q\sqrt{\lambda_2}$.

To show the strict subadditivity for $\xi(a)$, we prove the exponential decay estimate for the non-negative solution of the following system (1.17)–(1.19).

Lemma 1.8. (cf Lemma 3.1 in [27]) Let $u_1, u_2, u_3 \in H^1(\mathbb{R}^N)$ be a non-negative weak solution of the following elliptic system:

$$\begin{cases} -\Delta u_1 + (\lambda_1 + V_1(x)) u_1 = u_1^p + u_2 u_3, & (1.17) \\ -\Delta u_2 + (\lambda_2 + V_2(x)) u_2 = u_2^p + u_1 u_3, & (1.18) \\ -\Delta u_3 + (\lambda_3 + V_3(x)) u_3 = u_3^p + u_1 u_2, & (1.19) \end{cases}$$

where $N \leq 3$, $1 < p < 1 + 4/N$, $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$, V_j satisfies (V1) and (V2). Then it follows that

(i) if $u_1 > 0$, then u_1 has an essentially exponential decay $\sqrt{\lambda_1}$.

(ii) if $u_2 > 0$, then u_2 has an essentially exponential decay $\sqrt{\lambda_2}$.

(iii) if $u_1 = u_2 = 0$ and $u_3 > 0$, then u_3 has an essentially exponential decay $\sqrt{\lambda_3}$.

(iv) if $u_1, u_2, u_3 > 0$, then u_3 has an essentially exponential decay $\sqrt{\lambda_4} := \min\{\sqrt{\lambda_1} + \sqrt{\lambda_2}, \sqrt{\lambda_3}\}$.

Proof. (i): For the case of $u_1 > 0, u_2 = u_3 = 0$, we can prove easily (i). So we assume that $u_1, u_2, u_3 \not\equiv 0$. By the strong maximum principle, it follows that $u_1, u_2, u_3 > 0$. First, we prove upper estimate for u_1. For all $0 < \eta_1 < \lambda_1$, there exists $\varepsilon > 0$ such that $0 < \eta_1 < \lambda_1 - \varepsilon$. Set $u := u_1 + u_2$, $V := V_1 + V_2$. Note that $u_1, u_2 \geq 0$, $V_1, V_2 \leq 0$ and $0 < \lambda_1 \leq \lambda_2$. From (1.17) and (1.18),

$$\begin{aligned}
&-\Delta u + (\lambda_1 + V(x))u \\
&\leq -\Delta u_1 - \Delta u_2 + \lambda_1 u_1 + \lambda_2 u_2 + V_1(x)u_1 + V_2(x)u_2 \\
&= u_1^p + u_2^p + (u_1 + u_2)u_3 \\
&\leq 2u^p + uu_3.
\end{aligned} \tag{1.20}$$

From $N \leq 3$ and the elliptic regularity, $u_j \in H^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and

$$\lim_{|x|\to\infty} u_j(x) = 0.$$

Note that $V_j(x) \to 0$ as $|x| \to \infty$. There exists $R > 0$ such that if $|x| \geq R$ then

$$-V(x) + 2u^{p-1} + u_3 \leq \varepsilon.$$

From (1.20), if $|x| \geq R$, then

$$-\Delta u + (\lambda_1 - \varepsilon)u \leq 0.$$

On the other hand, we define the comparison function ψ as

$$\psi(x) := Ce^{-\sqrt{\eta_1}|x|}.$$

Noting $\eta_1 < \lambda_1 - \varepsilon$, we have

$$-\Delta\psi + (\lambda_1 - \varepsilon)\psi = (\sqrt{\eta_1}(N-1)/|x| + \lambda_1 - \varepsilon - \eta_1)Ce^{-\sqrt{\eta_1}|x|} \geq 0.$$

Thus

$$-\Delta u + (\lambda_1 - \varepsilon)u \le -\Delta\psi + (\lambda_1 - \varepsilon)\psi \quad \text{for all } |x| \ge R.$$

From the comparison principle, there exists $C_{\eta_1} > 0$ such that

$$u_1(x) \le u(x) \le C_{\eta_1} e^{-\sqrt{\eta_1}|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

Secondly, we prove lower estimate for u_1. Since $u_1 \ge 0$ and $V_1 \le 0$, we have

$$-\Delta u_1 + \lambda_1 u_1 \ge 0.$$

By the same argument as above, we can prove that for all $\eta_2 > \lambda_1$, there exists $C_{\eta_2} > 0$ such that

$$u_1(x) \ge C_{\eta_2} e^{-\sqrt{\eta_2}|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

(ii),(iii): They can be proved by the same argument as in (i).

(iv): We first prove upper estimate for u_3. For all $0 < \sqrt{\eta_5} < \sqrt{\lambda_4} := \min\{\sqrt{\lambda_1} + \sqrt{\lambda_2}, \sqrt{\lambda_3}\}$, there exist $0 < \sqrt{\eta_1} < \sqrt{\lambda_1}$ and $0 < \sqrt{\eta_3} < \sqrt{\lambda_2}$ such that

$$\sqrt{\eta_5} < \sqrt{\eta_1} + \sqrt{\eta_3}.$$

From the upper bound for u_1, u_2, we have

$$u_1(x) \le C_{\eta_1} e^{-\sqrt{\eta_1}|x|} \quad \text{for all } x \in \mathbb{R}^N,$$
$$u_2(x) \le C_{\eta_3} e^{-\sqrt{\eta_3}|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

From (1.19),

$$-\Delta u_3 + (\lambda_3 + V_3(x))u_3 - u_3^p \le C_{\eta_1} C_{\eta_2} e^{-(\sqrt{\eta_1}+\sqrt{\eta_3})|x|}.$$

Since $\eta_5 < \lambda_3$, thre exists $\varepsilon > 0$ such that $\eta_5 < \lambda_3 - \varepsilon$. Note that

$$\lim_{|x|\to\infty} u_3(x) = \lim_{|x|\to\infty} V_3(x) = 0.$$

Thus, for $|x|$ sufficiently large,

$$V_3(x) - u_3^{p-1} \geq -\varepsilon.$$

Hence, for $|x|$ sufficiently large, we have

$$-\Delta u_3 + (\lambda_3 - \varepsilon)u_3 \leq C_{\eta_1} C_{\eta_3} e^{-(\sqrt{\eta_1}+\sqrt{\eta_3})|x|}. \tag{1.21}$$

On the other hand, we set $\psi(x) := Ce^{-\sqrt{\eta_5}|x|}$. Then

$$-\Delta\psi + (\lambda_3 - \varepsilon)\psi = (\sqrt{\eta_5}(N-1)/|x| + \lambda_3 - \varepsilon - \eta_5)Ce^{-\sqrt{\eta_5}|x|}. \tag{1.22}$$

Since $\sqrt{\eta_5} < \sqrt{\eta_1} + \sqrt{\eta_3}$, for $|x|$ sufficiently large, it follows from (1.21) and (1.22) that

$$-\Delta u_3 + (\lambda_3 - \varepsilon)u_3 \leq -\Delta\psi + (\lambda_3 - \varepsilon)\psi.$$

By the comparison principle, there exists $C_{\eta_5} > 0$ such that

$$u_3(x) \leq C_{\eta_5} e^{-\sqrt{\eta_5}|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

Secondly, we show the lower estimate for u_3. Let $\sqrt{\eta_6} > \sqrt{\lambda_4} := \min\{\sqrt{\lambda_1} + \sqrt{\lambda_2}, \sqrt{\lambda_3}\}$. If $\lambda_4 = \lambda_3$, then we can prove the lower estimate for u_3 by the same argument as in the lower estimate for u_1. So we consider the case of $\sqrt{\lambda_4} = \sqrt{\lambda_1} + \sqrt{\lambda_2}$. Then there exist $\sqrt{\eta_2} > \sqrt{\lambda_1}$ and $\sqrt{\eta_4} > \sqrt{\lambda_2}$ such that

$$\sqrt{\eta_6} > \sqrt{\eta_2} + \sqrt{\eta_4}.$$

From (1.19), we have

$$\begin{aligned} -\Delta u_3 + \lambda_3 u_3 &= -V_3(x)u_3 + u_3^p + u_1 u_2 \\ &\geq u_1 u_2. \end{aligned}$$

From the lower estimate for u_1, u_2, there exist $C_{\eta_2}, C_{\eta_4} > 0$ such that

$$\begin{aligned} u_1(x) &\geq C_{\eta_2} e^{-\sqrt{\eta_2}|x|} \quad \text{for all } x \in \mathbb{R}^N, \\ u_2(x) &\geq C_{\eta_4} e^{-\sqrt{\eta_4}|x|} \quad \text{for all } x \in \mathbb{R}^N. \end{aligned}$$

Thus

$$-\Delta u_3 + \lambda_3 u_3 \geq C_{\eta_2} C_{\eta_4} e^{-(\sqrt{\eta_2}+\sqrt{\eta_4})|x|}.$$

On the other hand, we set $\psi(x) := Ce^{-\sqrt{\eta_6}|x|}$. Then

$$-\Delta\psi + \lambda_3\psi = (\sqrt{\eta_6}(N-1)/|x| + \lambda_3 - \eta_6)Ce^{-\sqrt{\eta_6}|x|}.$$

Since $\sqrt{\eta_6} > \sqrt{\eta_2} + \sqrt{\eta_4}$, for $|x|$ sufficiently large,

$$-\Delta u_3 + \lambda_3 u_3 \geq -\Delta\psi + \lambda_3\psi.$$

By the comparison principle, there exists $C_{\eta_6} > 0$ such that

$$u_3(x) \geq C_{\eta_6} e^{-\sqrt{\eta_6}|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

□

1.3 Strict subadditivity for $\xi(a)$

The next proposition plays a crucial role to prove Theorem 1.1.

Proposition 1.9. Let $a, b, c \in \mathbb{R}^3_{\geq 0}$ satisfying $b, c \neq (0,0,0)$ and $a_j = b_j + c_j > 0$ and let $\mathbf{u}_0$ and $\mathbf{w}_0$ be a minimizer for $\xi(b)$ and $\xi_\infty(c)$ respectively satisfying

$$u_{j,0} \geq 0, \quad w_{j,0} \geq 0 \text{ a.e. in } \mathbb{R}^N,$$

$$\begin{cases} -\Delta u_{1,0} + (\lambda_{1,0} + V_1(x))u_{1,0} - u_{1,0}^p = u_{2,0}u_{3,0} & \text{in } \mathbb{R}^N, & (1.23)\\ -\Delta u_{2,0} + (\lambda_{2,0} + V_2(x))u_{2,0} - u_{2,0}^p = u_{1,0}u_{3,0} & \text{in } \mathbb{R}^N, & (1.24)\\ -\Delta u_{3,0} + (\lambda_{3,0} + V_3(x))u_{3,0} - u_{3,0}^p = u_{1,0}u_{2,0} & \text{in } \mathbb{R}^N, & (1.25)\end{cases}$$

$$\begin{cases} -\Delta w_{1,0} + \lambda_{1,0}w_{1,0} - w_{1,0}^p = w_{2,0}w_{3,0} & \text{in } \mathbb{R}^N, & (1.26)\\ -\Delta w_{2,0} + \lambda_{2,0}w_{2,0} - w_{2,0}^p = w_{1,0}w_{3,0} & \text{in } \mathbb{R}^N, & (1.27)\\ -\Delta w_{3,0} + \lambda_{3,0}w_{3,0} - w_{3,0}^p = w_{1,0}w_{2,0} & \text{in } \mathbb{R}^N, & (1.28)\end{cases}$$

where $0 < \lambda_{1,0} \leq \lambda_{2,0} \leq \lambda_{3,0}$ for all $j = 1, 2, 3$. Then we have

$$\xi(a) < \xi(b) + \xi_\infty(c).$$

Proof. We borrowed the ideas from Theorem 1.1 in Ikoma-Miyamoto [27]. Set

$$w_{j,n}(x) := w_{j,0}(x - ne_1), \quad \tau_{j,n} := \frac{\sqrt{a_j}}{\|u_{j,0} + w_{j,n}\|_2}, \quad \kappa_{j,n} := (u_{j,0}, w_{j,n})_2.$$

Remark that

$$\begin{aligned}&(\tau_{1,n}(u_{1,0} + w_{1,n}), \tau_{2,n}(u_{2,0} + w_{2,n}), \tau_{3,n}(u_{3,0} + w_{3,n})) \in M(a),\\ &0 \le \kappa_{j,n} \to 0.\end{aligned}$$

By the strong maximum principle, it is sufficient to consider the cases

(i) $\mathbf{u}_0 = (u_{1,0}, 0, 0), \quad \mathbf{w}_0 = (w_{1,0}, w_{2,0}, w_{3,0}),$

(ii) $\mathbf{u}_0 = (0, u_{2,0}, 0), \quad \mathbf{w}_0 = (w_{1,0}, w_{2,0}, w_{3,0}),$

(iii) $\mathbf{u}_0 = (0, 0, u_{3,0}), \quad \mathbf{w}_0 = (w_{1,0}, w_{2,0}, w_{3,0}),$

(iv) $\mathbf{u}_0 = (u_{1,0}, u_{2,0}, u_{3,0}), \quad \mathbf{w}_0 = (w_{1,0}, 0, 0),$

(v) $\mathbf{u}_0 = (u_{1,0}, u_{2,0}, u_{3,0}), \quad \mathbf{w}_0 = (0, w_{2,0}, 0),$

(vi) $\mathbf{u}_0 = (u_{1,0}, u_{2,0}, u_{3,0}), \quad \mathbf{w}_0 = (0, 0, w_{3,0}),$

(vii) $\mathbf{u}_0 = (u_{1,0}, u_{2,0}, u_{3,0}), \quad \mathbf{w}_0 = (w_{1,0}, w_{2,0}, w_{3,0}).$

So it is sufficient to consider the cases

(A) $\kappa_{1,n} > 0,$

(B) $\kappa_{1,n} = 0 < \kappa_{2,n},$

(C) $\kappa_{1,n} = \kappa_{2,n} = 0 < \kappa_{3,n}.$

Noting that

$$E(\tau_{1,n}(u_{1,0} + w_{1,n}), \tau_{2,n}(u_{2,0} + w_{2,n}), \tau_{3,n}(u_{3,0} + w_{3,n}))$$

$$
\begin{aligned}
&= \sum_{j=1}^{3} \frac{\tau_{j,n}^2}{2} F_{V_j}(u_{j,0} + w_{j,n}) - \sum_{j=1}^{3} \frac{\tau_{j,n}^{p+1}}{p+1} \int_{\mathbb{R}^N} (u_{j,0} + w_{j,n})^{p+1}\, dx \\
&\quad - \tau_{1,n}\tau_{2,n}\tau_{3,n} \int_{\mathbb{R}^N} (u_{1,0} + w_{1,n})(u_{2,0} + w_{2,n})(u_{3,0} + w_{3,n})\, dx,
\end{aligned}
$$

we compute each term.

(Step 1) Since $a_j = b_j + c_j$, we have

$$\|u_{j,0} + w_{j,n}\|_2^2 = a_j + 2\kappa_{j,n}.$$

Hence,

$$\tau_{j,n}^q = \left(\frac{a_j}{a_j + 2\kappa_{j,n}} \right)^{q/2} = 1 - \frac{q}{a_j}\kappa_{j,n} + O(\kappa_{j,n}^2).$$

First, we estimate each term as Claim A, B and C as follows.

Claim A. There exists $\theta_1 \in (0,1)$ such that

$$
\begin{aligned}
&\frac{\tau_{j,n}^2}{2} F_{V_j}(u_{j,0} + w_{j,n}) \\
&\le \left(\frac{1}{2} - \frac{\kappa_{j,n}}{a_j} \right) (F_{V_j}(u_{j,0}) + \|\nabla w_{j,0}\|_2^2) - \lambda_{j,0}\kappa_{j,n} + \int_{\mathbb{R}^N} u_{j,0}^p w_{j,n}\, dx \\
&\quad + \begin{cases} \int_{\mathbb{R}^N} w_{1,n} u_{2,0} u_{3,0}\, dx & (j=1) \\ \int_{\mathbb{R}^N} u_{1,0} w_{2,n} u_{3,0}\, dx & (j=2) \\ \int_{\mathbb{R}^N} u_{1,0} u_{2,0} w_{3,n}\, dx & (j=3) \end{cases} + o(\kappa_{j,n}^{1+\theta_1}), \quad \text{if } \kappa_{j,n} > 0. \qquad (1.29)
\end{aligned}
$$

Claim B. There exists $\theta_2 \in (0,1)$ such that

$$
\begin{aligned}
&\frac{\tau_{j,n}^{p+1}}{p+1} \int_{\mathbb{R}^N} (u_{j,0} + w_{j,n})^{p+1}\, dx \\
&\ge \left(\frac{1}{p+1} - \frac{\kappa_{j,n}}{a_j} \right) \int_{\mathbb{R}^N} u_{j,0}^{p+1} + w_{j,0}^{p+1}\, dx \\
&\quad + \int_{\mathbb{R}^N} u_{j,0}^p w_{j,n} + u_{j,0} w_{j,n}^p\, dx + o(\kappa_{j,n}^{1+\theta_2}), \quad \text{if } \kappa_{j,n} > 0. \qquad (1.30)
\end{aligned}
$$

Claim C. There exists $\theta_3 \in (0,1)$ such that

$$\begin{aligned}
&\tau_{1,n}\tau_{2,n}\tau_{3,n} \int_{\mathbb{R}^N} (u_{1,0} + w_{1,n})(u_{2,0} + w_{2,n})(u_{3,0} + w_{3,n})\, dx \\
\geq &\left(1 - \left(\sum_{j=1}^{3} \frac{\kappa_{j,n}}{a_j}\right)\right) \int_{\mathbb{R}^N} u_{1,0}u_{2,0}u_{3,0} + w_{1,0}w_{2,0}w_{3,0}\, dx \\
&+ \int_{\mathbb{R}^N} w_{1,n}u_{2,0}u_{3,0} + u_{1,0}w_{2,n}u_{3,0} + u_{1,0}u_{2,0}w_{3,n} + w_{1,n}w_{2,n}u_{3,0}\, dx \\
&+ \begin{cases} o(\kappa_{1,n}^{1+\theta_3}) & \text{if } \kappa_{1,n} > 0, \\ o(\kappa_{2,n}^{1+\theta_3}) & \text{if } \kappa_{1,n} = 0 < \kappa_{2,n}, \\ o(\kappa_{3,n}^{1+\theta_3}) & \text{if } \kappa_{1,n} = \kappa_{2,n} = 0 < \kappa_{3,n}. \end{cases}
\end{aligned} \tag{1.31}$$

Proof of Claim A. It follows from (3.44) in [27] that

$$\begin{aligned}
&\frac{\tau_{j,n}^2}{2} F_{V_j}(u_{j,0} + w_{j,n}) \\
\leq &\left(\frac{1}{2} - \frac{\kappa_{j,n}}{a_j}\right) (F_{V_j}(u_{j,0}) + \|\nabla w_{j,0}\|_2^2) + \langle u_{j,0}, w_{j,n} \rangle_{V_j} \\
&+ O(\kappa_{j,n}^2) + O(\kappa_{j,n} \langle u_{j,0}, w_{j,n} \rangle_{V_j}).
\end{aligned}$$

Since $\mathbf{u}_0$ is a solution of (1.23)–(1.25), one sees

$$\begin{aligned}
\langle u_{j,0}, w_{j,n} \rangle_{V_j} = &-\lambda_{j,0}\kappa_{j,n} + \int_{\mathbb{R}^N} u_{j,0}^p w_{j,n}\, dx \\
&+ \begin{cases} \int_{\mathbb{R}^N} w_{1,n}u_{2,0}u_{3,0}\, dx & (j=1) \\ \int_{\mathbb{R}^N} u_{1,0}w_{2,n}u_{3,0}\, dx & (j=2) \\ \int_{\mathbb{R}^N} u_{1,0}u_{2,0}w_{3,n}\, dx & (j=3) \end{cases}.
\end{aligned} \tag{1.32}$$

Now, we prove that if $\kappa_{j,n} > 0$, then there exists $\theta_{1,j} \in (0,1)$ such that $\langle u_{j,0}, w_{j,n} \rangle_{V_j} = o(\kappa_{j,n}^{\theta_{1,j}})$.

Indeed, if $\kappa_{1,n} > 0$, then $u_{1,0}, w_{1,0} > 0$. From Lemma 1.8, $u_{1,0}$ and $w_{1,0}$ have an essentially exponential decay order $\sqrt{\lambda_{1,0}}$. Thus from Lemma 1.6 and 1.7, there exist $\eta_1, \eta_2 > 0$ and $C_1, C_2 > 0$ such that

$$C_1 e^{-\sqrt{\eta_1} n} \leq \kappa_{1,n} \quad \text{for all } n \in \mathbb{N}, \tag{1.33}$$

$$\int_{\mathbb{R}^N} u_{1,0}^p w_{1,n}\,dx \le C_2 e^{-\sqrt{\eta_2}n} \quad \text{for all } n \in \mathbb{N}. \tag{1.34}$$

If $u_{2,0}, u_{3,0} > 0$, then $u_{2,0}$ and $u_{3,0}$ have an essentially exponential decay order $\sqrt{\lambda_2}$ and $\sqrt{\lambda_4}$ respectively. Hence from Lemma 1.6 and 1.7, there exist $\eta_3 > 0$ and $C_3 > 0$ such that

$$\int_{\mathbb{R}^N} w_{1,n}u_{2,0}u_{3,0}\,dx \le C_3 e^{-\sqrt{\eta_3}n} \quad \text{for all } n \in \mathbb{N}. \tag{1.35}$$

From (1.32)–(1.35), there exists $0 < \theta_{1,1} < 1$ such that

$$\langle u_{1,0}, w_{1,n}\rangle_{V_1} = o(\kappa_{1,n}^{\theta_{1,1}}).$$

We can prove the cases $\kappa_{2,n} > 0$ or $\kappa_{3,n} > 0$ by the same argument.

Set $\theta_1 := \min\{\theta_{1,1}, \theta_{1,2}, \theta_{1,3}\} \in (0,1)$. We have $\langle u_{j,0}, w_{j,n}\rangle_{V_j} = o(\kappa_{j,n}^{\theta_1})$. Hence, if $\kappa_{j,n} > 0$,

$$\kappa_{j,n}\langle u_{j,0}, w_{j,n}\rangle_{V_j} = o(\kappa_{j,n}^{1+\theta_1}).$$

Thus we have (1.29).

Proof of Claim B. We can prove (1.30) by the same argument as in page 21 in [27]. So we omit the details.

Proof of Claim C. It follows that

$$\begin{aligned}
&\tau_{1,n}\tau_{2,n}\tau_{3,n}\int_{\mathbb{R}^N}(u_{1,0}+w_{1,n})(u_{2,0}+w_{2,n})(u_{3,0}+w_{3,n})\,dx\\
&\ge \left(1-\frac{\kappa_{1,n}}{a_1}+O(\kappa_{1,n}^2)\right)\left(1-\frac{\kappa_{2,n}}{a_2}+O(\kappa_{2,n}^2)\right)\left(1-\frac{\kappa_{3,n}}{a_3}+O(\kappa_{3,n}^2)\right)\\
&\quad\times\int_{\mathbb{R}^N}(u_{1,0}u_{2,0}u_{3,0}+w_{1,0}w_{2,0}w_{3,0}+w_{1,n}u_{2,0}u_{3,0}+u_{1,0}w_{2,n}u_{3,0}\\
&\qquad+u_{1,0}u_{2,0}w_{3,n}+w_{1,n}w_{2,n}u_{3,0})\,dx\\
&=\int_{\mathbb{R}^N}(u_{1,0}u_{2,0}u_{3,0}+w_{1,0}w_{2,0}w_{3,0}+w_{1,n}u_{2,0}u_{3,0}+u_{1,0}w_{2,n}u_{3,0}\\
&\quad+u_{1,0}u_{2,0}w_{3,n}+w_{1,n}w_{2,n}u_{3,0})\,dx
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{3}\frac{\kappa_{j,n}}{a_j}\int_{\mathbb{R}^N}(u_{1,0}u_{2,0}u_{3,0}+w_{1,0}w_{2,0}w_{3,0})\,dx\\
&-\sum_{j=1}^{3}\frac{\kappa_{j,n}}{a_j}\int_{\mathbb{R}^N}(w_{1,n}u_{2,0}u_{3,0}+u_{1,0}w_{2,n}u_{3,0}+u_{1,0}u_{2,0}w_{3,n}\\
&\quad+w_{1,n}w_{2,n}u_{3,0})\,dx+O(\kappa_{1,n}^2+\kappa_{2,n}^2+\kappa_{3,n}^2).
\end{aligned}
$$

Here we show that there exists $\theta_{3,j}\in(0,1)$ such that

$$
\begin{aligned}
&\sum_{j=1}^{3}\frac{\kappa_{j,n}}{a_j}\int_{\mathbb{R}^N} w_{1,n}u_{2,0}u_{3,0}+u_{1,0}w_{2,n}u_{3,0}+u_{1,0}u_{2,0}w_{3,n}+w_{1,n}w_{2,n}u_{3,0}\,dx\\
&=\begin{cases} o(\kappa_{1,n}^{1+\theta_{3,1}}) \text{ if } \kappa_{1,n}>0,\\ o(\kappa_{2,n}^{1+\theta_{3,2}}) \text{ if } \kappa_{1,n}=0<\kappa_{2,n},\\ o(\kappa_{3,n}^{1+\theta_{3,3}}) \text{ if } \kappa_{1,n}=\kappa_{2,n}=0<\kappa_{3,n}.\end{cases} \qquad (1.36)
\end{aligned}
$$

If $\kappa_{1,n}>0$, from Lemma 1.6–Lemma 1.8, there exist $\eta_1>0$ and $C_1>0$ such that

$$\kappa_{1,n}\geq C_1e^{-\sqrt{\eta_1}n}\quad\text{for all } n\in\mathbb{N}. \qquad (1.37)$$

If $\int_{\mathbb{R}^N} w_{1,n}u_{2,0}u_{3,0}\,dx>0$, then from Lemma 1.6–Lemma 1.8, there exist $\eta_2>0$ and $C_2>0$ such that

$$\int_{\mathbb{R}^N} w_{1,n}u_{2,0}u_{3,0}\,dx\leq C_2e^{-\sqrt{\eta_2}n}\quad\text{for all } n\in\mathbb{N}.$$

Since other terms can be estimated in the same way, there exist $\eta_3>0$ and $C_3>0$ such that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(w_{1,n}u_{2,0}u_{3,0}+u_{1,0}w_{2,n}u_{3,0}+u_{1,0}u_{2,0}w_{3,n}\\
&\quad+w_{1,n}w_{2,n}u_{3,0})\,dx\leq C_3e^{-\sqrt{\eta_3}n}\quad\text{for all } n\in\mathbb{N}. \qquad (1.38)
\end{aligned}
$$

If $\kappa_{2,n},\kappa_{3,n}>0$, it follows from $\sqrt{\lambda_{1,0}}\leq\sqrt{\lambda_{2,0}}\leq\sqrt{\lambda_{4,0}}:=\min\{\sqrt{\lambda_{1,0}}+\sqrt{\lambda_{2,0}},\sqrt{\lambda_{3,0}}\}$ that for all $\theta\in(0,1)$,

$$\kappa_{2,n}=o(\kappa_{1,n}^{\theta}),\quad \kappa_{3,n}=o(\kappa_{1,n}^{\theta}).$$

In any case, it follows from (1.37),(1.38) that there exists $\theta_{3,1} \in (0,1)$ such that

$$\sum_{j=1}^{3} \frac{\kappa_{j,n}}{a_j} \int_{\mathbb{R}^N} (w_{1,n}u_{2,0}u_{3,0} + u_{1,0}w_{2,n}u_{3,0} + u_{1,0}u_{2,0}w_{3,n} + w_{1,n}w_{2,n}u_{3,0})\,dx = o(\kappa_{1,n}^{1+\theta_{3,1}}).$$

For the case $\kappa_{1,n} = 0 < \kappa_{2,n}$ or $\kappa_{1,n} = \kappa_{2,n} = 0 < \kappa_{3,n}$, (1.36) can be proved by the same argument.

In the same way, it follows that there exists $\theta_3 \in (0, \theta_{3,j})$ such that

$$\kappa_{1,n}^2 + \kappa_{2,n}^2 + \kappa_{3,n}^2 = \begin{cases} o(\kappa_{1,n}^{1+\theta_3}) \text{ if } \kappa_{1,n} > 0, \\ o(\kappa_{2,n}^{1+\theta_3}) \text{ if } \kappa_{1,n} = 0 < \kappa_{2,n}, \\ o(\kappa_{3,n}^{1+\theta_3}) \text{ if } \kappa_{1,n} = \kappa_{2,n} = 0 < \kappa_{3,n}. \end{cases}$$

Therefore we have (1.31).

(Step 2) From (1.29)–(1.31), setting $\theta_4 := \min\{\theta_1, \theta_2, \theta_3\}/2 \in (0,1)$, we have

$$\begin{aligned} &E(\tau_{1,n}(u_{1,0} + w_{1,n}), \tau_{2,n}(u_{2,0} + w_{2,n}), \tau_{3,n}(u_{3,0} + w_{3,n})) \\ \le\ & \frac{1}{2}\sum_{j=1}^{3}(F_{V_j}(u_{j,0}) + \|\nabla w_{j,0}\|_2^2) - \sum_{j=1}^{3} \frac{\kappa_{j,n}}{a_j}(\lambda_{j,0}a_j + F_{V_j}(u_{j,0}) + \|\nabla w_{j,0}\|_2^2) \\ &+ \sum_{j=1}^{3} \int_{\mathbb{R}^N} u_{j,0}^p w_{j,n}\,dx + \int_{\mathbb{R}^N} w_{1,n}u_{2,0}u_{3,0} + u_{1,0}w_{2,n}u_{3,0} + u_{1,0}u_{2,0}w_{3,n}\,dx \\ &- \frac{1}{p+1}\sum_{j=1}^{3} \int_{\mathbb{R}^N} u_{j,0}^{p+1} + w_{j,0}^{p+1}\,dx - \sum_{j=1}^{3} \int_{\mathbb{R}^N} u_{j,0}^p w_{j,n} + u_{j,0}w_{j,n}^p\,dx \\ &+ \sum_{j=1}^{3} \frac{\kappa_{j,n}}{a_j} \int_{\mathbb{R}^N} u_{j,0}^{p+1} + w_{j,0}^{p+1}\,dx - \int_{\mathbb{R}^N} u_{1,0}u_{2,0}u_{3,0} + w_{1,0}w_{2,0}w_{3,0}\,dx \\ &- \int_{\mathbb{R}^N} w_{1,n}u_{2,0}u_{3,0} + u_{1,0}w_{2,n}u_{3,0} + u_{1,0}u_{2,0}w_{3,n} + w_{1,n}w_{2,n}u_{3,0}\,dx \\ &+ \sum_{j=1}^{3} \frac{\kappa_{j,n}}{a_j} \int_{\mathbb{R}^N} u_{1,0}u_{2,0}u_{3,0} + w_{1,0}w_{2,0}w_{3,0}\,dx \end{aligned}$$

$$
\begin{aligned}
&+\begin{cases} o(\kappa_{1,n}^{1+\theta_4}) \text{ if } \kappa_{1,n}>0,\\ o(\kappa_{2,n}^{1+\theta_4}) \text{ if } \kappa_{1,n}=0<\kappa_{2,n},\\ o(\kappa_{3,n}^{1+\theta_4}) \text{ if } \kappa_{1,n}=\kappa_{2,n}=0<\kappa_{3,n}\end{cases}\\
=\,& E(\mathbf{u}_0)+E_\infty(\mathbf{w}_0)-\sum_{j=1}^{3}\frac{\kappa_{j,n}}{a_j}(\lambda_{j,0}a_j+F_{V_j}(u_{j,0})+\|\nabla w_{j,0}\|_2^2\\
&-\int_{\mathbb{R}^N} u_{j,0}^{p+1}+w_{j,0}^{p+1}\,dx-\int_{\mathbb{R}^N} u_{1,0}u_{2,0}u_{3,0}+w_{1,0}w_{2,0}w_{3,0}\,dx)\\
&-\sum_{j=1}^{3}\int_{\mathbb{R}^N} u_{j,0}w_{j,n}^p\,dx-\int_{\mathbb{R}^N} w_{1,n}w_{2,n}u_{3,0}\,dx\\
&+\begin{cases} o(\kappa_{1,n}^{1+\theta_4}) \text{ if } \kappa_{1,n}>0,\\ o(\kappa_{2,n}^{1+\theta_4}) \text{ if } \kappa_{1,n}=0<\kappa_{2,n},\\ o(\kappa_{3,n}^{1+\theta_4}) \text{ if } \kappa_{1,n}=\kappa_{2,n}=0<\kappa_{3,n}.\end{cases}
\end{aligned}
\tag{1.39}
$$

(Step 3) By (1.23)–(1.25) and (1.26)–(1.28), we have

$$
\begin{aligned}
F_{V_j}(u_{j,0})&=-\lambda_{j,0}\|u_{j,0}\|_2^2+\int_{\mathbb{R}^N} u_{j,0}^{p+1}+u_{1,0}u_{2,0}u_{3,0}\,dx,\\
\|\nabla w_{j,0}\|_2^2&=-\lambda_{j,0}\|w_{j,0}\|_2^2+\int_{\mathbb{R}^N} w_{j,0}^{p+1}+w_{1,0}w_{2,0}w_{3,0}\,dx.
\end{aligned}
$$

Recalling that $a_j=\|u_{j,0}\|_2^2+\|w_{j,0}\|_2^2$, we have

$$
\begin{aligned}
F_{V_j}(u_{j,0})+\|\nabla w_{j,0}\|_2^2=-\lambda_{j,0}a_j&+\int_{\mathbb{R}^N} u_{j,0}^{p+1}+w_{j,0}^{p+1}\,dx\\
&+\int_{\mathbb{R}^N} u_{1,0}u_{2,0}u_{3,0}+w_{1,0}w_{2,0}w_{3,0}\,dx.
\end{aligned}
\tag{1.40}
$$

From (1.39) and (1.40), we have

$$
\xi(a)\le\xi(b)+\xi_\infty(c)-R_n+\begin{cases} o(\kappa_{1,n}^{1+\theta_4}) \text{ if } \kappa_{1,n}>0,\\ o(\kappa_{2,n}^{1+\theta_4}) \text{ if } \kappa_{1,n}=0<\kappa_{2,n},\\ o(\kappa_{3,n}^{1+\theta_4}) \text{ if } \kappa_{1,n}=\kappa_{2,n}=0<\kappa_{3,n},\end{cases}
\tag{1.41}
$$

where

$$
R_n:=\sum_{j=1}^{3}\int_{\mathbb{R}^N} u_{j,0}w_{j,n}^p\,dx+\int_{\mathbb{R}^N} w_{1,n}w_{2,n}u_{3,0}\,dx.
$$

(Step 4) Now, we prove that

$$\kappa_{1,n}^{1+\theta_4} = o(R_n) \text{ if } \kappa_{1,n} > 0, \tag{1.42}$$
$$\kappa_{2,n}^{1+\theta_4} = o(R_n) \text{ if } \kappa_{1,n} = 0 < \kappa_{2,n}, \tag{1.43}$$
$$\kappa_{3,n}^{1+\theta_4} = o(R_n) \text{ if } \kappa_{1,n} = \kappa_{2,n} = 0 < \kappa_{3,n}. \tag{1.44}$$

Proof of (1.42) Since $\kappa_{1,n} > 0$, from Lemma 1.8, $u_{1,0}$ and $w_{1,0}$ have an essentially exponential decay order $\sqrt{\lambda_{1,0}}$. From Lemma 1.6 and Lemma 1.7, $\int_{\mathbb{R}^N} u_{1,0} w_{1,n}^p \, dx$ has an essentially exponential decay order $\min\{\sqrt{\lambda_{1,0}}, p\sqrt{\lambda_{1,0}}\} = \sqrt{\lambda_{1,0}}$. Since $\kappa_{1,n}$ has also an essentially exponential decay order $\sqrt{\lambda_{1,0}}$, for all $\theta > 1$, there exists $C_\theta > 0$ such that

$$\int_{\mathbb{R}^N} u_{1,0} w_{1,n}^p \, dx \geq C_\theta \kappa_{1,n}^\theta.$$

Assume $\theta < 1 + \theta_4$. Then we have

$$\kappa_{1,n}^{1+\theta_4}/R_n \leq C_\theta^{-1} \kappa_{1,n}^{1+\theta_4-\theta} \to 0.$$

Proof of (1.43) (1.43) can be proved as in (1.42).

Proof of (1.44) Since $\kappa_{1,n} = \kappa_{2,n} = 0 < \kappa_{3,n}$, there are two possibilities,

(a) $u_{1,0}, u_{2,0}, u_{3,0}, w_{3,0} > 0$ and $w_{1,0} = w_{2,0} = 0$.

(b) $u_{3,0}, w_{1,0}, w_{2,0}, w_{3,0} > 0$ and $u_{1,0} = u_{2,0} = 0$.

Case (a) By Lemma 1.8, $u_{3,0}$ and $w_{3,0}$ have an essentially exponential decay order $\sqrt{\lambda_{4,0}} := \min\{\sqrt{\lambda_{1,0}} + \sqrt{\lambda_{2,0}}, \sqrt{\lambda_{3,0}}\}$ and $\sqrt{\lambda_{3,0}}$ respectively. From $\sqrt{\lambda_{4,0}} \leq \sqrt{\lambda_{3,0}}$ and Lemma 1.6 and Lemma 1.7, $\int_{\mathbb{R}^N} u_{3,0} w_{3,n}^p \, dx$ has an essentially exponential decay order $\min\{\sqrt{\lambda_{4,0}}, p\sqrt{\lambda_{3,0}}\} = \sqrt{\lambda_{4,0}}$. Since $\kappa_{3,n}$ has also an essentially exponential decay order $\sqrt{\lambda_{4,0}}$, we can prove (1.44) in the case (a) by the same argument as in (1.42).

Case (b) By (1.25) and (1.28) with $u_{1,0} = u_{2,0} = 0$, one sees

$$\int_{\mathbb{R}^N} u_{3,0}w_{3,n}^p + w_{1,n}w_{2,n}u_{3,0}\,dx \geq \int_{\mathbb{R}^N} u_{3,0}^p w_{3,n}\,dx.$$

Since $\int_{\mathbb{R}^N} u_{3,0}^p w_{3,n}\,dx$ and $\kappa_{3,n}$ have an essentially exponential decay order $\min\{p\sqrt{\lambda_{3,0}}, \sqrt{\lambda_{4,0}}\} = \sqrt{\lambda_{4,0}}$, for all $\theta > 1$, there exists $C_\theta > 0$ such that

$$\int_{\mathbb{R}^N} u_{3,0}^p w_{3,n}\,dx \geq C_\theta \kappa_{3,n}^\theta.$$

Thus we can prove (1.44) in the case (b) by the same argument as in (1.42).

From (1.41) and (1.42)–(1.44), we have

$$\xi(a) < \xi(b) + \xi_\infty(c).$$

□

1.4 Proof of Theorem 1.1

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{\mathbf{u}_n\}_{n=1}^\infty \subset M(a)$ be a minimizing sequence for $\xi(a)$. By Lemma 1.2 and 1.3, we may assume that

$$\|E'(\mathbf{u}_n) + \lambda_{1,n}Q_1'(\mathbf{u}_n) + \lambda_{2,n}Q_2'(\mathbf{u}_n) + \lambda_{3,n}Q_3'(\mathbf{u}_n)\|_{H^*} \to 0,\ \|(u_{j,n})_-\|_2 \to 0. \tag{1.45}$$

In addition, since $\{\lambda_{j,n}\}_{n=1}^\infty$ and $\{\mathbf{u}_n\}_{n=1}^\infty$ is bounded, we may assume that

$$\lambda_{j,n} \to \lambda_{j,0}, \quad \mathbf{u}_n \rightharpoonup \mathbf{u}_0 \text{ weakly in } H. \tag{1.46}$$

Remark that

$$b_j := \|u_{j,0}\|_2^2 \leq \liminf_{n\to\infty} \|u_{j,n}\|_2^2 = a_j.$$

Set $b := (b_1, b_2, b_3)$. If $b = a$, we have $\|\mathbf{u}_n - \mathbf{u}_0\|_H \to 0$ as in [31].

Now, we aim to prove that $b = a$. We prove by contradiction. Assume that $b \neq a$. As in Theorem 1.10 in Appendix in this chapter, we have

$$\begin{aligned}
&\|u_{j,n}\|_2^2 = \|u_{j,0}\|_2^2 + \|u_{j,n} - u_{j,0}\|_2^2 + o(1),\\
&E(\mathbf{u}_n) = E(\mathbf{u}_0) + E_\infty(\mathbf{u}_n - \mathbf{u}_0) + o(1),\\
&\xi(a) = \xi(b) + \xi_\infty(a-b),\\
&E(\mathbf{u}_0) = \xi(b),\\
&\lim_{n\to\infty} E_\infty(\mathbf{u}_n - \mathbf{u}_0) = \xi_\infty(a-b).
\end{aligned} \tag{1.47}$$

In addition, since Lemma 1.13 in Appendix in this chapter and (1.47), we have $b \neq (0,0,0)$. Since $\{\mathbf{u}_n - \mathbf{u}_0\}_{n=1}^\infty$ is a minimizing sequence for $\xi_\infty(a-b)$, up to a subsequence, there exist $R > 0$ and $\varepsilon > 0$ and $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ such that

$$\int_{|x-y_n|<R} \sum_{j=1}^{3} |u_{j,n}(x) - u_{j,0}(x)|^2\, dx \geq \varepsilon \quad \text{for all } n \in \mathbb{N}$$

by Lemma 4.2 in [31]. Since $u_{j,n} \to u_{j,0}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, up to a subsequence, we have $|y_n| \to \infty$. Since $\{(\mathbf{u}_n - \mathbf{u}_0)(\cdot + y_n)\}$ is bounded in H, there exists $\mathbf{w}_0 \in H$ such that

$$(\mathbf{u}_n - \mathbf{u}_0)(\cdot + y_n) \rightharpoonup \mathbf{w}_0 \text{ weakly in } H.$$

Then we have

$$\mathbf{u}_n(\cdot + y_n) \rightharpoonup \mathbf{w}_0 \text{ weakly in } H. \tag{1.48}$$

Moreover, as in Theorem 1.10 for the case without potentials in Appendix in this chapter, up to a subsequence, we have

$$\begin{aligned}
&c = a - b,\\
&E_\infty(\mathbf{w}_0) = \xi_\infty(c),\\
&\|(\mathbf{u}_n - \mathbf{u}_0)(\cdot + y_n) - \mathbf{w}_0\|_H \to 0
\end{aligned}$$

where $c_j := \|w_{j,0}\|_2^2$, $c := (c_1, c_2, c_3)$.

From (1.45)–(1.48), we derive that

$$u_{j,0} \geq 0, \quad w_{j,0} \geq 0 \text{ a.e. in } \mathbb{R}^N,$$

$$\begin{cases} -\Delta u_{1,0} + (\lambda_{1,0} + V_1(x))u_{1,0} - u_{1,0}^p = u_{2,0}u_{3,0}, \\ -\Delta u_{2,0} + (\lambda_{2,0} + V_2(x))u_{2,0} - u_{2,0}^p = u_{1,0}u_{3,0}, \\ -\Delta u_{3,0} + (\lambda_{3,0} + V_3(x))u_{3,0} - u_{3,0}^p = u_{1,0}u_{2,0}, \end{cases}$$

$$\begin{cases} -\Delta w_{1,0} + \lambda_{1,0}w_{1,0} - w_{1,0}^p = w_{2,0}w_{3,0}, \\ -\Delta w_{2,0} + \lambda_{2,0}w_{2,0} - w_{2,0}^p = w_{1,0}w_{3,0}, \\ -\Delta w_{3,0} + \lambda_{3,0}w_{3,0} - w_{3,0}^p = w_{1,0}w_{2,0}. \end{cases}$$

Since $\|u_{j,0}\|_2^2 + \|w_{j,0}\|_2^2 = a_j > 0$, we have $u_{j,0} \not\equiv 0$ or $w_{j,0} \not\equiv 0$. By Lemma 3.7 in [27], we have $\lambda_{j,0} > 0$ for all $j = 1, 2, 3$. Without loss of generality, we may assume that $0 < \lambda_{1,0} \leq \lambda_{2,0} \leq \lambda_{3,0}$. From Proposition 1.9, we have

$$\xi(a) < \xi(b) + \xi_\infty(a - b).$$

This contradicts (1.47). So we have $b = a$. □

1.5 Appendix

In this Appendix, we prove that the strict subadditivity of ξ and the existence of a minimizer for $\xi(a)$ under symmetric conditions for the potentials (see Proposition 1.12 and Theorem 1.10). This result is an extension of Ardila's result [4] to a model with higher spatial dimensions and potentials. Although this result was obtained in Kurata-Osada [31], for the reader's convenience, we mention this symmetric case result and the method used in the proofs.

We assume the following condition (V3) in addition to the assumptions in Chapter 1.

(V3) for all $j = 1, 2, 3$, $V_j(-x_1, x') = V_j(x_1, x')$ for almost every $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$,
$V_j(s, x') \leq V_j(t, x')$ for almost every $s, t \in \mathbb{R}$ with $0 \leq s < t$ and $x' \in \mathbb{R}^{N-1}$.

The following Theorem is mentioned as Theorem 1.1 in [31].

Theorem 1.10. (the existence of a minimizer for $\xi(a)$) Suppose a_1, a_2, $a_3 > 0$, $N = 1, 2, 3$, $1 < p < 1 + 4/N$ and $V_j(x)$ $(j = 1, 2, 3)$ satisfies the conditions (V1)–(V3), respectively. Then, any minimizing sequence $\{\mathbf{u}_n\}_{n=1}^{\infty}$ for $\xi(a_1, a_2, a_3)$ is relatively compact in H up to translations. That is, there exist $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ and $\mathbf{u} \in H$ such that $\{\mathbf{u}_n(\cdot + y_n)\}_{n=1}^{\infty}$ has a subsequence converging strongly in H to $\mathbf{u}$. Moreover, $\mathbf{u}$ is a minimizer for $\xi(a_1, a_2, a_3)$.

Remark 1.1. In Theorem 1.10, if there exists $j \in \{1, 2, 3\}$ such that $V_j \not\equiv 0$, then we can take $y_n = 0$ for all $n \in \mathbb{N}$.

For $x \in \mathbb{R}^N$, we denote by $x = (x_1, x')$ ($x_1 \in \mathbb{R},\ x' \in \mathbb{R}^{N-1}$) and also we denote by $\mathcal{L}^1$ the 1-dimensional Lebesgue measure.

First, we state the definition of the coupled rearrangement developed by Shibata [47].

Definition 1.11. (coupled rearrangement, cf. Shibata [47]) Let u, v be measurable functions defined on $\mathbb{R}^N$ such that

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0.$$

Then the coupled rearrangement $u \star v$ is defined by

$$(u \star v)(x_1, x') := \int_0^\infty \chi_{\{|u(\cdot,x')|>t\}\star\{|v(\cdot,x')|>t\}}(x_1)\, dt, \quad x_1 \in \mathbb{R},\ x' \in \mathbb{R}^{N-1}$$

for any measurable subsets $A, B \subset \mathbb{R}$, where $A \star B$ is defined as follows:

$$A \star B := \left(-(\mathcal{L}^1(A) + \mathcal{L}^1(B))/2, (\mathcal{L}^1(A) + \mathcal{L}^1(B))/2\right).$$

The following proposition is mentioned as Proposition 3 in [31].

Proposition 1.12. (Strict subadditivity for ξ_∞) Assume the conditions (V1)–(V3) for $V_j(x),\ j = 1, 2, 3$. Let $b_1, b_2, b_3, c_1, c_2, c_3 \geq 0$ and we assume $\xi_\infty(b_1, b_2, b_3)$ and $\xi_\infty(c_1, c_2, c_3)$ have a minimizer respectively. If $b_1, c_1 > 0$ or $b_2, c_2 > 0$ or $b_3, c_3 > 0$, then we have

$$\xi(b_1 + c_1, b_2 + c_2, b_3 + c_3) < \xi(b_1, b_2, b_3) + \xi_\infty(c_1, c_2, c_3),$$

where we denote by ξ_∞ instead of ξ if $V_j \equiv 0$ for all $j = 1, 2, 3$.

Proof. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are minimizers for $\xi(b_1, b_2, b_3)$ and $\xi_\infty(c_1, c_2, c_3)$ respectively. We consider only the case $b_1 > 0$ and $c_1 > 0$. Since $b_1 > 0$ and $c_1 > 0$, from Lemma 3.3 in [31], $\mathbf{u}^\star$ and $\mathbf{v}^\star$ are minimizers for $\xi(b_1, b_2, b_3)$ and $\xi_\infty(c_1, c_2, c_3)$ respectively and for all $j \in \{1, 2, 3\}$,

$$u_1^\star > 0 \quad \text{almost everywhere in } \mathbb{R}^N, \quad u_2^\star, u_3^\star \geq 0 \quad \text{for almost every } x \in \mathbb{R}^N, \tag{1.49}$$

$$u_j^\star \in H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N), \qquad \lim_{|x| \to \infty} u_j^\star(x) = 0, \tag{1.50}$$

$$u_j^\star(-x_1, x') = u_j^\star(x_1, x') \quad \text{for almost every } x_1 \in \mathbb{R} \text{ and} \quad \text{for almost every } x' \in \mathbb{R}^{N-1}, \tag{1.51}$$

$$u_j^\star(s, x') \geq u_j^\star(t, x') \quad \text{for almost every } s, t \in \mathbb{R} \text{ with } 0 \leq s < t \text{ and} \quad \text{for almost every } x' \in \mathbb{R}^{N-1}, \tag{1.52}$$

and

$$v_1^\star > 0 \quad \text{almost everywhere in } \mathbb{R}^N, \quad v_2^\star, v_3^\star \geq 0 \quad \text{almost everywhere in } \mathbb{R}^N, \tag{1.53}$$

$$v_j^\star \in H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N), \qquad \lim_{|x| \to \infty} v_j^\star(x) = 0, \tag{1.54}$$

$$v_j^\star(-x_1, x') = v_j^\star(x_1, x') \quad \text{for almost every } x_1 \in \mathbb{R} \text{ and} \quad \text{for almost every } x' \in \mathbb{R}^{N-1}, \tag{1.55}$$

$$v_j^\star(s, x') \geq v_j^\star(t, x') \quad \text{almost every } s, t \in \mathbb{R} \text{ with } 0 \leq s < t \text{ and} \quad \text{for almost every } x' \in \mathbb{R}^{N-1}, \tag{1.56}$$

where $u^\star$ is the Steiner rearrangement of u with respect to the hyperplane $x_1 = 0$. We refer [28] for the definition of the Steiner rearrangement. From (1.49)–(1.56), Lemma 2.12, Corollary 1 in [31] and the results in [47], we have

$$\int_{\mathbb{R}^N} |\nabla(u_1^\star \star v_1^\star)|^2 \, dx < \int_{\mathbb{R}^N} |\nabla u_1^\star|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v_1^\star|^2 \, dx, \tag{1.57}$$

$$\int_{\mathbb{R}^N} |\nabla(u_j^\star \star v_j^\star)|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla u_j^\star|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v_j^\star|^2 \, dx \quad (j = 2, 3), \tag{1.58}$$

$$\int_{\mathbb{R}^N} (-V_j(x))(u_j^\star)^2 \, dx \leq \int_{\mathbb{R}^N} (-V_j(x))(u_j^\star \star v_j^\star)^2 \, dx, \tag{1.59}$$

$$\int_{\mathbb{R}^N} u_1^\star u_2^\star u_3^\star \, dx + \int_{\mathbb{R}^N} v_1^\star v_2^\star v_3^\star \, dx \leq \int_{\mathbb{R}^N} (u_1^\star \star v_1^\star)(u_2^\star \star v_2^\star)(u_3^\star \star v_3^\star) \, dx, \tag{1.60}$$

$$\int_{\mathbb{R}^N} |u_j^\star \star v_j^\star|^q \, dx = \int_{\mathbb{R}^N} |u_j^\star|^q \, dx + \int_{\mathbb{R}^N} |v_j^\star|^q \, dx \quad (q = 2, p+1). \tag{1.61}$$

From (1.57)–(1.61), it holds that

$$\begin{aligned} \xi(b_1 + c_1, b_2 + c_2, b_3 + c_3) &\leq E(u_1^\star \star v_1^\star, u_2^\star \star v_2^\star, u_3^\star \star v_3^\star) \\ &< E(u_1^\star, u_2^\star, u_3^\star) + E_\infty(v_1^\star, v_2^\star, v_3^\star) \\ &= \xi(b_1, b_2, b_3) + \xi_\infty(c_1, c_2, c_3). \end{aligned}$$

□

Now we prove Theorem 1.10 for the case without potentials.

Proof of Theorem 1.10 for the case without potentials. Let $\{\mathbf{u}_n\}_{n=1}^\infty \subset H$ be a minimizing sequence for $\xi_\infty(a_1, a_2, a_3)$. The proof proceeds in five steps:

(Step 1) First, we prove that taking a subsequence, there exist $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\mathbf{u} \in H$ such that

$$\begin{aligned} &u_{j,n}(\cdot + x_n) \rightharpoonup u_j \text{ weakly in } H^1(\mathbb{R}^N) \quad (j = 1, 2, 3,\ n \to \infty), \\ &u_1 \not\equiv 0. \end{aligned}$$

Since $a_1 > 0$, by Lemma 4.2 in [31], taking a subsequence, there exist $\varepsilon_1 > 0$ and $R_1 > 0$ such that

$$\sup_{y \in \mathbb{R}^N} \int_{|x-y|<R_1} |u_{1,n}|^2 \, dx \geq \varepsilon_1 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, there exists $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$ such that

$$\int_{|x-x_n|<R_1} |u_{1,n}|^2 \, dx > \frac{\varepsilon_1}{2} \quad \text{for all } n \in \mathbb{N}.$$

On the other hand, since $\{\mathbf{u}_n\}_{n=1}^\infty$ is bounded in H from Lemma 2.4 in [31], taking a subsequence, there exists $\mathbf{u} \in H$ such that

$$u_{j,n}(\cdot + x_n) \rightharpoonup u_j \text{ weakly in } H^1(\mathbb{R}^N) \quad (j = 1, 2, 3).$$

Now, we remark that

$$\int_{|x|<R_1} |u_{1,n}(x+x_n)|^2\,dx = \int_{|x-x_n|<R_1} |u_{1,n}|^2\,dx > \frac{\varepsilon_1}{2} \quad \text{for all } n \in \mathbb{N}.$$

Since the embedding $H^1(\{|x|<R_1\}) \subset L^2(\{|x|<R_1\})$ is compact, we have

$$\int_{|x|<R_1} |u_1|^2\,dx \geq \frac{\varepsilon_1}{2} > 0,$$

that is, we have $u_1 \not\equiv 0$.

(Step 2) Let $b_1 := \|u_1\|_2^2\ (>0)$, $b_2 := \|u_2\|_2^2$, $b_3 := \|u_3\|_2^2$. Moreover, we set $v_{j,n}(x) := u_{j,n}(x+x_n) - u_j(x)$ $(j=1,2,3)$. Then, we prove that $\mathbf{u}$ is a minimizer for $\xi_\infty(b_1,b_2,b_3)$ and taking a subsequence, the followings hold:

$$\begin{aligned} &\|u_{j,n}\|_2^2 = \|u_j\|_2^2 + \|v_{j,n}\|_2^2 + o(1) \quad (j=1,2,3,\ \text{as } n\to\infty), \\ &E_\infty(\mathbf{u}_n) = E_\infty(\mathbf{u}) + E_\infty(\mathbf{v}_n) + o(1) \quad (\text{as } n\to\infty), \\ &\xi_\infty(a_1,a_2,a_3) = \xi_\infty(b_1,b_2,b_3) + \xi_\infty(a_1-b_1, a_2-b_2, a_3-b_3). \end{aligned}$$

Claim 1. Taking a subsequence, for $q=2$ and $q=p+1$, we have

$$\int_{\mathbb{R}^N} |u_{j,n}|^q\,dx = \int_{\mathbb{R}^N} |u_j|^q\,dx + \int_{\mathbb{R}^N} |v_{j,n}|^q\,dx + o(1) \quad (\text{as } n\to\infty).$$

This statement follows immediately from the Brezis-Lieb Lemma (see e.g. [7]).

Claim 2.

$$\int_{\mathbb{R}^N} (|\nabla u_{j,n}|^2 - |\nabla u_j|^2 - |\nabla v_{j,n}|^2)\,dx = o(1) \quad (\text{as } n\to\infty).$$

Indeed, note that

$$\int_{\mathbb{R}^N} (|\nabla u_{j,n}|^2 - |\nabla u_j|^2 - |\nabla v_{j,n}|^2)\,dx = 2\mathrm{Re}\int_{\mathbb{R}^N} \nabla u_j \cdot \overline{\nabla v_{j,n}}\,dx,$$

where $\nabla u \cdot \overline{\nabla v} := \sum_{k=1}^N \frac{\partial u}{\partial x_k}\overline{\frac{\partial v}{\partial x_k}}$. Since $v_{j,n} \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$, it follows

$$\int_{\mathbb{R}^N} \nabla u_j \cdot \overline{\nabla v_{j,n}}\,dx = o(1) \quad (\text{as } n\to\infty).$$

Hence we get Claim 2.

Claim 3.

$$\int_{\mathbb{R}^N} u_{1,n}u_{2,n}\overline{u}_{3,n}\,dx = \int_{\mathbb{R}^N} u_1u_2\overline{u}_3\,dx + \int_{\mathbb{R}^N} v_{1,n}v_{2,n}\overline{v}_{3,n}\,dx + o(1) \quad (\text{as } n \to \infty).$$

Actually, since

$$\begin{aligned}
&\int_{\mathbb{R}^N} u_{1,n}u_{2,n}\overline{u}_{3,n}\,dx = \int_{\mathbb{R}^N} (u_1+v_{1,n})(u_2+v_{2,n})(\overline{u}_3+\overline{v}_{3,n})\,dx \\
&= \int_{\mathbb{R}^N} u_1u_2\overline{u}_3\,dx + \int_{\mathbb{R}^N} u_1u_2\overline{v}_{3,n}\,dx + \int_{\mathbb{R}^N} u_1v_{2,n}\overline{u}_3\,dx + \int_{\mathbb{R}^N} u_1v_{2,n}\overline{v}_{3,n}\,dx \\
&+ \int_{\mathbb{R}^N} v_{1,n}u_2\overline{u}_3\,dx + \int_{\mathbb{R}^N} v_{1,n}u_2\overline{v}_{3,n}\,dx + \int_{\mathbb{R}^N} v_{1,n}v_{2,n}\overline{u}_3\,dx + \int_{\mathbb{R}^N} v_{1,n}v_{2,n}\overline{v}_{3,n}\,dx,
\end{aligned}$$

it suffices to prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} v_{1,n}u_2\overline{u}_3\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} u_1v_{2,n}\overline{u}_3\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} u_1u_2\overline{v}_{3,n}\,dx = 0, \tag{1.62}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} u_1v_{2,n}\overline{v}_{3,n}\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} v_{1,n}u_2\overline{v}_{3,n}\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} v_{1,n}v_{2,n}\overline{u}_3\,dx = 0. \tag{1.63}$$

Since we can prove these easily by using $v_{j,n} \to 0$ $(j = 1, 2, 3)$ in $L^3_{\text{loc}}(\mathbb{R}^N)$, we omit the details.

From Claim 1–Claim3, it follows that

$$E_\infty(\mathbf{u}_n) = E_\infty(\mathbf{u}) + E_\infty(\mathbf{v}_n) + o(1) \quad (\text{as } n \to \infty). \tag{1.64}$$

From Claim 1 we have also

$$\|u_{j,n}\|_2^2 = \|u_j\|_2^2 + \|v_{j,n}\|_2^2 + o(1) \quad (\text{as } n \to \infty).$$

Letting $n \to \infty$ in (1.64), from Lemma 2.6 and Lemma 3.2 in [31], we have

$$\begin{aligned}
\xi_\infty(a_1,a_2,a_3) &= E_\infty(\mathbf{u}) + \lim_{n\to\infty} E_\infty(\mathbf{v}_n) \\
&\geq \xi_\infty(b_1,b_2,b_3) + \lim_{n\to\infty} \xi_\infty(\|v_{1,n}\|_2^2, \|v_{2,n}\|_2^2, \|v_{3,n}\|_2^2)
\end{aligned}$$

$$= \xi_\infty(b_1, b_2, b_3) + \xi_\infty(a_1 - b_1, a_2 - b_2, a_3 - b_3)$$
$$\geq \xi_\infty(a_1, a_2, a_3).$$

Therefore,

$$E_\infty(\mathbf{u}) = \xi_\infty(b_1, b_2, b_3), \tag{1.65}$$
$$\lim_{n\to\infty} E_\infty(\mathbf{v}_n) = \xi_\infty(a_1 - b_1, a_2 - b_2, a_3 - b_3), \tag{1.66}$$
$$\xi_\infty(a_1, a_2, a_3) = \xi_\infty(b_1, b_2, b_3) + \xi_\infty(a_1 - b_1, a_2 - b_2, a_3 - b_3). \tag{1.67}$$

That is, $\mathbf{u}$ is a minimizer for $\xi_\infty(b_1, b_2, b_3)$.

(Step 3) We prove $b_1 = a_1$. Suppose by contradiction that $b_1 < a_1$. By the same argument as in (Step 1), then there exist a minimizing sequence $\{\mathbf{v}_n\}_{n=1}^\infty$ for $\xi_\infty(a_1 - b_1, a_2 - b_2, a_3 - b_3)$ and a function $\mathbf{v} \in H$ with $v_1 \not\equiv 0$ such that

$$v_{j,n} \rightharpoonup v_j \text{ weakly in } H^1(\mathbb{R}^N) \quad (j = 1, 2, 3).$$

Set $c_1 := \|v_1\|_2^2 (> 0)$, $c_2 := \|v_2\|_2^2$, $c_3 := \|v_3\|_2^2$. Using the same argument as in (Step 2), we have

$$\begin{aligned} \xi_\infty(a_1 - b_1, a_2 - b_2, a_3 - b_3) &= \xi_\infty(c_1, c_2, c_3) + \\ &+ \xi_\infty(a_1 - b_1 - c_1, a_2 - b_2 - c_2, a_3 - b_3 - c_3), \\ E_\infty(\mathbf{v}) &= \xi_\infty(c_1, c_2, c_3). \end{aligned} \tag{1.68}$$

Now, $\mathbf{u}$ and $\mathbf{v}$ are minimizers for $\xi_\infty(b_1, b_2, b_3)$ and $\xi_\infty(c_1, c_2, c_3)$ respectively. Furthermore since $b_1 > 0$ and $c_1 > 0$, from Proposition 1.12, it follows that

$$\xi_\infty(b_1 + c_1, b_2 + c_2, b_3 + c_3) < \xi_\infty(b_1, b_2, b_3) + \xi_\infty(c_1, c_2, c_3). \tag{1.69}$$

Therefore, combining (1.67),(1.68),(1.69) and using Lemma 3.2 in [31], we arrive at

$$\begin{aligned} &\xi_\infty(a_1, a_2, a_3) = \xi_\infty(b_1, b_2, b_3) + \xi_\infty(a_1 - b_1, a_2 - b_2, a_3 - b_3) \\ &= \xi_\infty(b_1, b_2, b_3) + \xi_\infty(c_1, c_2, c_3) + \xi_\infty(a_1 - b_1 - c_1, a_2 - b_2 - c_2, a_3 - b_3 - c_3) \\ &> \xi_\infty(b_1 + c_1, b_2 + c_2, b_3 + c_3) + \xi_\infty(a_1 - b_1 - c_1, a_2 - b_2 - c_2, a_3 - b_3 - c_3) \\ &\geq \xi_\infty(a_1, a_2, a_3). \end{aligned}$$

This is a contradiction. Hence, it follows that $b_1 = a_1$.

(Step 4) We prove $b_2 = a_2$ and $b_3 = a_3$. First, we show $b_2 > 0$. We assume that $b_2 = 0$, then by $b_1 = a_1$ and the subadditivity for S_∞ (see Lions [35, 36] for more details), it follows that

$$\begin{aligned}\xi_\infty(a_1, a_2, a_3) &= \xi_\infty(a_1, 0, b_3) + \xi_\infty(0, a_2, a_3 - b_3)\\ &= S_\infty(a_1) + S_\infty(b_3) + S_\infty(a_2) + S_\infty(a_3 - b_3)\\ &\geq S_\infty(a_1) + S_\infty(a_2) + S_\infty(a_3),\end{aligned}$$

where

$$\begin{aligned}S_\infty(a) &:= \inf\{J_\infty(u) \mid u \in H^1(\mathbb{R}^N),\ \|u\|_2^2 = a\},\\ J_\infty(u) &:= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\, dx.\end{aligned}$$

This contradicts Lemma 4.1 in [31] (i). Thus, we have $b_2 > 0$. We can show $b_2 = a_2$ as in (Step 3). In a similar way, we can also prove $b_3 = a_3$.

(Step 5) We prove that $\lim_{n\to\infty} \|\mathbf{u}_n(\cdot + x_n) - \mathbf{u}\|_H = 0$ and $\mathbf{u}$ is minimizer for $\xi_\infty(a_1, a_2, a_3)$. From (Step 1)–(Step 4), we have proved

$$\begin{aligned}&u_{j,n}(\cdot + x_n) \rightharpoonup u_j \text{ weakly in } H^1(\mathbb{R}^N) \quad (j = 1, 2, 3),\\ &\|u_1\|_2^2 = a_1, \quad \|u_2\|_2^2 = a_2, \quad \|u_3\|_2^2 = a_3.\end{aligned}$$

From (1.65) and $b_1 = a_1, b_2 = a_2, b_3 = a_3$, $\mathbf{u}$ is a minimizer for $\xi_\infty(a_1, a_2, a_3)$. Moreover, noting that $\|u_{1,n}\|_2^2 \to a_1$, $\|u_{2,n}\|_2^2 \to a_2$ and $\|u_{3,n}\|_2^2 \to a_3$ $(n \to \infty)$, we have

$$\|u_{j,n}(\cdot + x_n) - u_j\|_2 \to 0 \quad (n \to \infty), \tag{1.70}$$

that is,

$$\|v_{j,n}\|_2 \to 0 \quad (n \to \infty). \tag{1.71}$$

Since $\{v_{j,n}\}_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$, by Gagliardo-Nirenberg's inequality, it holds that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |v_{j,n}|^{p+1}\, dx = 0, \tag{1.72}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} v_{1,n} v_{2,n} \overline{v}_{3,n}\, dx = 0. \tag{1.73}$$

Indeed, by Gagliardo-Nirenberg's inequality, for each $q = p + 1$ and $q = 3$, there exists a constant $C(q, N) > 0$ respectively such that

$$\|v_{j,n}\|_q^q \leq C(q, N)\|\nabla v_{j,n}\|_2^{N(q-2)/2}\|v_{j,n}\|_2^{q-N(q-2)/2}.$$

Note that $\{v_{j,n}\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^N)$ and (1.71), it follows that

$$\|v_{j,n}\|_{p+1}^{p+1} \to 0, \quad \|v_{j,n}\|_3^3 \to 0. \tag{1.74}$$

From this, we have (1.72) and (1.73). From $b_1 = a_1, b_2 = a_2, b_3 = a_3$, (1.65),(1.66),(1.72),(1.73), we have

$$E_\infty(\mathbf{u}) = \xi_\infty(a_1, a_2, a_3),$$
$$\lim_{n\to\infty} \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 \, dx = 0.$$

From this and (1.70), the conclusion follows. □

Next, we prove the existence of a minimize for $\xi(a)$ with potential.

We note that the relationship between ξ and ξ_∞. This lemma guarantees that minimizing sequence for $\xi(a_1, a_2, a_3)$ with potentials does not vanish. This lemma is mentioned as Lemma 4.5 in [31].

Lemma 1.13. Let $a_1, a_2, a_3 > 0$. Then it follows that $\xi(a_1, a_2, a_3) < \xi_\infty(a_1, a_2, a_3)$.

***Proof*.** From Lemma 4.4 in [31], there exists a minimizer $\mathbf{u} = (u_1, u_2, u_3)$ for $\xi_\infty(a_1, a_2, a_3)$ such that $u_j > 0$ almost everywhere in $\mathbb{R}^N$. Since $V_1 \not\equiv 0$ or $V_2 \not\equiv 0$ or $V_3 \not\equiv 0$, it follows that

$$\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j(x)u_j^2 \, dx < 0.$$

Thus we have

$$\xi(a_1, a_2, a_3) \leq E(\mathbf{u}) < E_\infty(\mathbf{u}) = \xi_\infty(a_1, a_2, a_3).$$

□

Now we prove Theorem 1.10 for the case with potentials.

Proof of Theorem 1.10 for the case with potentials. Let $\{\mathbf{u}_n\}_{n=1}^{\infty} \subset H$ be a minimizing sequence for $\xi(a_1, a_2, a_3)$. Since $\{\mathbf{u}_n\}_{n=1}^{\infty}$ is bounded in H, there exists $\mathbf{u} \in H$ such that

$$u_{j,n} \rightharpoonup u_j \text{ weakly in } H^1(\mathbb{R}^N) \quad (j = 1, 2, 3,\ n \to \infty).$$

(Step 1) Set $b_1 := \|u_1\|_2^2$, $b_2 := \|u_2\|_2^2$, $b_3 := \|u_3\|_2^2$. We prove that $\mathbf{u}$ is a minimizer for $\xi(b_1, b_2, b_3)$ and taking a subsequence, the followings hold:

$$\begin{aligned}
&\|u_{j,n}\|_2^2 = \|u_j\|_2^2 + \|u_{j,n} - u_j\|_2^2 + o(1) \quad (j = 1, 2, 3,\ \text{as } n \to \infty),\\
&E(\mathbf{u}_n) = E(\mathbf{u}) + E_\infty(\mathbf{u}_n - \mathbf{u}) + o(1) \quad (\text{as } n \to \infty),\\
&\xi(a_1, a_2, a_3) = \xi(b_1, b_2, b_3) + \xi_\infty(a_1 - b_1, a_2 - b_2, a_3 - b_3),\\
&b_1 > 0 \text{ or } b_2 > 0 \text{ or } b_3 > 0.
\end{aligned}$$

From the same argument as in the Step 2 of the proof of Theorem 1.10 for the case without potentials, taking a subsequence, we have

$$\int_{\mathbb{R}^N} |u_{j,n}|^2\,dx = \int_{\mathbb{R}^N} |u_j|^2\,dx + \int_{\mathbb{R}^N} |u_{j,n} - u_j|^2\,dx + o(1) \quad (\text{as } n \to \infty), \tag{1.75}$$

$$E_\infty(\mathbf{u}_n) = E_\infty(\mathbf{u}) + E_\infty(\mathbf{u}_n - \mathbf{u}) + o(1) \quad (\text{as } n \to \infty). \tag{1.76}$$

Moreover, from Lemma 2.7 in [31], it follows that

$$\int_{\mathbb{R}^N} V_j(x)|u_{j,n}|^2\,dx = \int_{\mathbb{R}^N} V_j(x)|u_j|^2\,dx + o(1) \quad (\text{as } n \to \infty,\ j = 1, 2, 3). \tag{1.77}$$

From (1.76),(1.77), we have

$$E(\mathbf{u}_n) = E(\mathbf{u}) + E_\infty(\mathbf{u}_n - \mathbf{u}) + o(1) \quad (\text{as } n \to \infty). \tag{1.78}$$

By the same argument as in the proof of Theorem 1.10 for the case without potentials, we can prove that

$$E(\mathbf{u}) = \xi(b_1, b_2, b_3),$$

$$\xi(a_1, a_2, a_3) = \xi(b_1, b_2, b_3) + \xi_\infty(a_1 - b_1, a_2 - b_2, a_3 - b_3). \tag{1.79}$$

That is, $\mathbf{u}$ is a minimizer for $\xi(b_1, b_2, b_3)$. Suppose $b_1 = b_2 = b_3 = 0$. From $\xi(0,0,0) = 0$ and (1.79),

$$\xi(a_1, a_2, a_3) = \xi_\infty(a_1, a_2, a_3).$$

This contradicts Lemma 1.13. Therefore we have $b_1 > 0$ or $b_2 > 0$ or $b_3 > 0$.

(Step 2) From (Step 1), $b_1 > 0$ or $b_2 > 0$ or $b_3 > 0$. We consider only the case $b_1 > 0$. We prove $b_1 = a_1$. Suppose $b_1 < a_1$. From (Step 1), $\mathbf{u}$ is a minimizer for $\xi(b_1, b_2, b_3)$ and from Lemma 4.3 in [31], $\xi_\infty(a_1 - b_1, a_2 - b_2, a_3 - b_3)$ has a minimizer. Since $b_1 > 0$, $a_1 - b_1 > 0$, from Proposition 1.12, we have

$$\xi(a_1, a_2, a_3) < \xi(b_1, b_2, b_3) + \xi_\infty(a_1 - b_1, a_2 - b_2, a_3 - b_3).$$

This contradicts (1.79).

(Conclusion) We can prove $b_2 = a_2$ and $b_3 = a_3$ by the same argument as in the Step 4 of the proof of Theorem 1.10 for the case without potentials. Then we can prove that $\lim_{n\to\infty} \|\mathbf{u}_n - \mathbf{u}\|_{H^1} = 0$ and $\mathbf{u}$ is a minimizer for $\xi(a_1, a_2, a_3)$ by the same argument as in the Step 5 of the proof of Theorem 1.10 for the case without potentials. □

Chapter 2

Energy asymptotic expansion for a system of nonlinear Schrödinger equations with three wave interaction

2.1 Introduction

Recently, there are many studies on the existence of standing waves and their stability for the nonlinear Schrödinger system with three wave interaction (see Colin-Colin-Ohta [18, 19], Pomponio [43], Ardila [4], Kurata-Osada [31] and the references therein) and related systems (see e.g. Gou-Jeanjean [24], Bhattarai [6], Zhao-Zhao-Shi [60] and the references therein).

In particular, the L^2-constrained variational problems associated with the systems and the orbital stability of ground states have been studied by many works (e.g. Bhattarai [6], Gou-Jeanjean [24], Ardila [4], Kurata-Osada [31]). In this chapter, we focus on the following L^2-constrained variational problem:

$$\xi_\alpha^\beta(a_1, a_2, a_3) := \inf\{E_\alpha^\beta(\mathbf{u}) \mid \mathbf{u} \in H, \\ \|u_1\|_2^2 = a_1,\ \|u_2\|_2^2 = a_2,\ \|u_3\|_2^2 = a_3\},$$

$$E_\alpha^\beta(\mathbf{u}) := \frac{1}{2}\sum_{j=1}^3 \int_{\mathbb{R}^N} |\nabla u_j|^2\,dx + \frac{1}{2}\sum_{j=1}^3 \int_{\mathbb{R}^N} V_j(x)|u_j|^2\,dx$$
$$- \frac{\beta}{p+1}\sum_{j=1}^3 \int_{\mathbb{R}^N} |u_j|^{p+1}\,dx - \alpha\mathrm{Re}\int_{\mathbb{R}^N} u_1u_2\overline{u}_3\,dx,$$

where $\mathbf{u} := (u_1, u_2, u_3)$, $\overline{u}_3$ is the complex conjugate of u_3, $H := H^1 \times H^1 \times H^1$, $H^1 := H^1(\mathbb{R}^N;\mathbb{C})$, $\alpha, \beta > 0$, $N = 1, 2, 3$, $1 < p < 1 + 4/N$, $a_1, a_2, a_3 > 0$ and each potential V_j $(j = 1, 2, 3)$ satisfies some suitable conditions. In this chapter, we assume only one of the following conditions for the potentials V_j $(j = 1, 2, 3)$.

(V1) $V \in L^\infty(\mathbb{R}^N;\mathbb{R})$.

(V2) $V \in C(\mathbb{R}^N;\mathbb{R})$ and $V(x) \le \lim_{|y|\to\infty} V(y) = 0$, for all $x \in \mathbb{R}^N$.

In the previous paper [31], for the case $\beta = 1$, we studied the energy asymptotic expansion of $\xi_\alpha^1(a_1, a_2, a_3)$ as $\alpha \to \infty$. In this chapter, we consider the asymptotic expansion of the energy $\xi_\alpha^\beta(a_1, a_2, a_3)$ as $\beta \to \infty$ with $\alpha = \beta^\kappa$ for a given $\kappa \in \mathbb{R}$.

To state the main result in this chapter in details, we define the following variational problems:

$$\begin{aligned}
\Sigma_0(a_1, a_2, a_3) &:= \inf\{E^0(\mathbf{u}) \mid \mathbf{u} \in H,\ \|u_1\|_2^2 = a_1,\ \|u_2\|_2^2 = a_2,\ \|u_3\|_2^2 = a_3\},\\
\Sigma_1(a_1, a_2, a_3) &:= \sup\{E^1(\mathbf{u}) \mid \mathbf{u} \text{ is a minimizer for } \Sigma_0(a_1, a_2, a_3)\},\\
\xi_\infty(a_1, a_2, a_3) &:= \inf\{E_\infty(\mathbf{u}) \mid \mathbf{u} \in H,\ \|u_1\|_2^2 = a_1,\ \|u_2\|_2^2 = a_2,\ \|u_3\|_2^2 = a_3\},\\
S_\infty(a_j) &:= \inf\{J_\infty(u) \mid u \in H^1(\mathbb{R}^N),\ \|u\|_2^2 = a_j\} \quad (j = 1, 2, 3),\\
S^1(a_1, a_2, a_3) &:= \sup\{J^1(\mathbf{u}) \mid u_1, u_2, u_3 \text{ are minimizers for}\\
&\qquad S_\infty(a_1), S_\infty(a_2), S_\infty(a_3) \text{ respectively}\},
\end{aligned}$$

where

$$E^0(\mathbf{u}) := \frac{1}{2}\sum_{j=1}^3 \int_{\mathbb{R}^N} |\nabla u_j|^2\,dx - \mathrm{Re}\int_{\mathbb{R}^N} u_1u_2\overline{u}_3\,dx,$$

$$E^1(\mathbf{u}) := \frac{1}{p+1}\sum_{j=1}^3 \int_{\mathbb{R}^N} |u_j|^{p+1}\,dx,$$

$$E_\infty(\mathbf{u}) := \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla u_j|^2\,dx - \frac{1}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |u_j|^{p+1}\,dx - \mathrm{Re}\int_{\mathbb{R}^N} u_1 u_2 \overline{u}_3\,dx,$$
$$J_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx,$$
$$J^1(\mathbf{u}) := \mathrm{Re}\int_{\mathbb{R}^N} u_1 u_2 \overline{u}_3\,dx.$$

Remark 2.1. Let $N \le 3$, $1 < p < 1 + 4/N$, $\alpha, \beta > 0$. Under the following three assumptions on V_j $(j = 1, 2, 3)$:

- $V \in L^\infty(\mathbb{R}^N; \mathbb{R})$,
- $V(x) \le \lim_{|y|\to\infty} V(y) = 0$ (a.e. $x \in \mathbb{R}^N$),
- $V(-x_1, x') = V(x_1, x')$ (a.e. $x_1 \in \mathbb{R},\ x' \in \mathbb{R}^{N-1}$),
  $V(s, x') \le V(t, x')$ (a.e. $s, t \in \mathbb{R}$ with $0 \le s < t$, a.e. $x' \in \mathbb{R}^{N-1}$),

the existence of a minimizer for $\xi_\alpha^\beta(a_1, a_2, a_3)$ is known (see [31]).

See also [31] about the existence of minimizer for $\Sigma_0(a_1, a_2, a_3)$ under the additional condition $N \le 2$. Moreover, since it is easy to check that the set of minimizers for $\Sigma_0(a_1, a_2, a_3)$ is uniformly bounded in H, it follows that $\Sigma_1(a_1, a_2, a_3) < \infty$.

Remark 2.2. When $N \in \mathbb{N}$, $1 < p < 1 + 4/N$, for all $a_j > 0$, it is well-known that there exists a unique positive, radial symmetric and strictly decreasing minimizer $\Psi_{a_j} \in H^1(\mathbb{R}^N)$ for $S_\infty(a_j)$ such that for all minimizer u for $S_\infty(a_j)$, there exist $y \in \mathbb{R}^N$ and $\theta \in \mathbb{R}$ such that

$$u(x) = e^{i\theta}\Psi_{a_j}(x + y)$$

(see [14, 21, 32]).

Unless otherwise noted, Ψ_{a_j} means the one in Remark 2.2. Also, we set $\mathbf{\Psi} := (\Psi_{a_1}, \Psi_{a_2}, \Psi_{a_3})$. Note that $\mathbf{\Psi}$ is a maximizer for $S^1(a_1, a_2, a_3)$. See Lemma 2.4 for the proof.

For a given $\kappa \in \mathbb{R}$, as $\alpha = \beta^\kappa$ we define for simplicity

$$E^\beta(\mathbf{u}) := E^\beta_{\beta^\kappa}(\mathbf{u}),$$

$$\xi^\beta(a_1,a_2,a_3) := \xi^\beta_{\beta^\kappa}(a_1,a_2,a_3).$$

We show that there exist two critical numbers

$$\kappa_1 := (4-N)/(4-N(p-1)), \quad \kappa_2 := -N/(4-N(p-1))$$

such that the asymptotic expansion of $\xi^\beta(a_1,a_2,a_3)$ as $\beta\to\infty$ are different in the following five cases:
(i) $\kappa>\kappa_1$, (ii) $\kappa=\kappa_1$, (iii) $\kappa_2<\kappa<\kappa_1$, (iv) $\kappa=\kappa_2$, (v) $\kappa<\kappa_2$.
We say $\{\mathbf{u}_n\}_{n=1}^\infty$ is a minimizing sequence for $\xi^{\beta_n}(a_1,a_2,a_3)$ with $\beta_n\to\infty$ if

$$\begin{aligned}&\|u_{1,n}\|_2^2=a_1, \quad \|u_{2,n}\|_2^2=a_2, \quad \|u_{3,n}\|_2^2=a_3,\\ &E^{\beta_n}(\mathbf{u}_n)=\xi^{\beta_n}(a_1,a_2,a_3)+o(1), \quad \text{as } n\to\infty.\end{aligned}$$

We also study the asymptotic behavior of minimizing sequences $\{\mathbf{u}_n\}$ by using the rescaled functions of two types:

$$\mathbf{w}_n(x) := \beta_n^{-\kappa N/(4-N)}\mathbf{u}_n(\beta_n^{-2\kappa/(4-N)}x) \tag{2.1}$$

for the case (i) and

$$\mathbf{v}_n(x) := \beta_n^{-N/(4-N(p-1))}\mathbf{u}_n(\beta_n^{-2/(4-N(p-1))}x) \tag{2.2}$$

for the cases (ii)–(v), respectively.

Now we state the main result in this chapter.

Theorem 2.1. Let $N=1,2,3$, $1<p<1+4/N$ and let $\{\mathbf{u}_n\}_{n=1}^\infty$ be a minimizing sequence for $\xi^{\beta_n}(a_1,a_2,a_3)$ with $\beta_n\to\infty$. Then we have the asymptotic expansion of $\xi^\beta(a_1,a_2,a_3)=\xi^\beta_{\beta^\kappa}(a_1,a_2,a_3)$ as $\beta\to\infty$ in the five cases as follows:

(i) For the case $\kappa>\kappa_1$, assume $N\le 2$ and the condition (V1) for each potential V_j $(j=1,2,3)$. Then

$$\begin{aligned}\xi^\beta(a_1,a_2,a_3) &= \beta^{4\kappa/(4-N)}\Sigma_0(a_1,a_2,a_3) - \beta^{\kappa N(p-1)/(4-N)+1}\Sigma_1(a_1,a_2,a_3)\\ &\quad + o(\beta^{\kappa N(p-1)/(4-N)+1}), \quad \text{as } \beta\to\infty.\end{aligned}$$

Moreover, for the rescaled function $\mathbf{w}_n$ defined by (2.1), up to a subsequence, there exist $\{y_n\}_{n=1}^\infty\subset\mathbb{R}^N$ and a maximizer $\mathbf{w}$ for $\Sigma_1(a_1,a_2,a_3)$ such that

$$\|\mathbf{w}_n(\cdot+y_n)-\mathbf{w}\|_{H^1}\to 0, \quad \text{as } n\to\infty.$$

(ii) For the case $\kappa = \kappa_1$, assume the condition (V2) for each potential V_j $(j = 1, 2, 3)$ and $(V_1, V_2, V_3) \not\equiv (0, 0, 0)$. Then it holds that

$$\begin{aligned}\xi^{\beta}(a_1, a_2, a_3) &= \beta^{4/(4-N(p-1))}\xi_\infty(a_1, a_2, a_3)\\ &\quad + \frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)a_1 + V_2(x)a_2 + V_3(x)a_3\} + o(1), \quad \text{as } \beta \to \infty.\end{aligned}$$

Moreover, for the rescaled function $\mathbf{v}_n$ defined by (2.2), up to a subsequence, there exist $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$, a minimizer $\mathbf{v}$ for $\xi_\infty(a_1, a_2, a_3)$ and $z_0 \in \mathbb{R}^N$ such that

$$\begin{aligned}&\|\mathbf{v}_n(\cdot + y_n) - \mathbf{v}\|_{H^1} \to 0, \quad y_n/\beta_n^{2/(4-N(p-1))} \to z_0 \text{ in } \mathbb{R}^N, \quad \text{as } n \to \infty,\\ &\min_{x\in\mathbb{R}^N}\{V_1(x)a_1 + V_2(x)a_2 + V_3(x)a_3\} = V_1(z_0)a_1 + V_2(z_0)a_2 + V_3(z_0)a_3.\end{aligned}$$

(iii) For the case $\kappa_2 < \kappa < \kappa_1$, assume the condition (V1) for each potential V_j $(j = 1, 2, 3)$. Then

$$\begin{aligned}\xi^{\beta}(a_1, a_2, a_3) &= \beta^{4/(4-N(p-1))}(S_\infty(a_1) + S_\infty(a_2) + S_\infty(a_3))\\ &\quad - \beta^{N/(4-N(p-1))+\kappa}S^1(a_1, a_2, a_3) + o(\beta^{N/(4-N(p-1))+\kappa}), \quad \text{as } \beta \to \infty.\end{aligned}$$

Moreover, for the rescaled function $\mathbf{v}_n$ defined by (2.2), up to a subsequence, there exist $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$, and $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ such that

$$\begin{aligned}&\|v_{j,n}(\cdot + y_n) - e^{i\theta_j}\Psi_j\|_{H^1} \to 0, \quad \text{as } n \to \infty,\\ &\theta_1 + \theta_2 = \theta_3,\end{aligned}$$

where $\Psi_1 = \Psi_{a_1}$, $\Psi_2 = \Psi_{a_2}$, $\Psi_3 = \Psi_{a_3}$.

(iv) For the case $\kappa = \kappa_2$, assume that the condition (V2) for each potential V_j $(j = 1, 2, 3)$, $(V_1, V_2, V_3) \not\equiv (0, 0, 0)$. We also assume that V_j has a unique minimum point $z_{j,0}$ and $z_{1,0} = z_{2,0} = z_{3,0} =: z_0$. Then

$$\begin{aligned}\xi^{\beta}(a_1, a_2, a_3) &= \beta^{4/(4-N(p-1))}(S_\infty(a_1) + S_\infty(a_2) + S_\infty(a_3))\\ &\quad - S^1(a_1, a_2, a_3) + \frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)a_1 + V_2(x)a_2 + V_3(x)a_3\} + o(1),\\ &\quad \text{as } \beta \to \infty.\end{aligned}$$

Moreover, for the rescaled function $\mathbf{v}_n$ defined by (2.2), up to a subsequence, there exist $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$, and $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ such that

$$\begin{aligned}
&\|v_{j,n}(\cdot + y_n) - e^{i\theta_j}\Psi_j\|_{H^1} \to 0, \quad \text{as } n \to \infty, \\
&\theta_1 + \theta_2 = \theta_3, \\
&y_n/\beta_n^{2/(4-N(p-1))} \to z_0 \text{ in } \mathbb{R}^N, \quad \text{as } n \to \infty,
\end{aligned}$$

where $\Psi_1 = \Psi_{a_1}$, $\Psi_2 = \Psi_{a_2}$, $\Psi_3 = \Psi_{a_3}$.

(v) For the case $\kappa < \kappa_2$, assume that the condition (V2) for each potential V_j $(j = 1, 2, 3)$ and $(V_1, V_2, V_3) \not\equiv (0, 0, 0)$. Then

$$\begin{aligned}
\xi^\beta(a_1, a_2, a_3) = {} & \beta^{4/(4-N(p-1))}(S_\infty(a_1) + S_\infty(a_2) + S_\infty(a_3)) \\
& + \frac{1}{2}\left(\min_{x\in\mathbb{R}^N} V_1(x)a_1 + \min_{x\in\mathbb{R}^N} V_2(x)a_2 + \min_{x\in\mathbb{R}^N} V_3(x)a_3\right) + o(1), \\
& \text{as } \beta \to \infty.
\end{aligned}$$

Moreover, for the rescaled function $\mathbf{v}_n$ defined by (2.2), up to a subsequence, there exist $\{y_n^{(j)}\}_{n=1}^{\infty} \subset \mathbb{R}^N$ $(j = 1, 2, 3)$, and $\theta_j \in \mathbb{R}$ $(j = 1, 2, 3)$ and $z_{j,0} \in \mathbb{R}^N$ $(j = 1, 2, 3)$ such that

$$\begin{aligned}
&\|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j}\Psi_j\|_{H^1} \to 0, \quad \text{as } n \to \infty, \\
&y_n^{(j)}/\beta_n^{2/(4-N(p-1))} \to z_{j,0} \text{ in } \mathbb{R}^N, \quad \text{as } n \to \infty, \\
&\min_{x\in\mathbb{R}^N} V_j(x) = V_j(z_{j,0}),
\end{aligned}$$

where $\Psi_1 = \Psi_{a_1}$, $\Psi_2 = \Psi_{a_2}$, $\Psi_3 = \Psi_{a_3}$.

Remark 2.3. By Theorem 2.1, we can say that the effect of the three wave interaction appears in the first order term in the case $\kappa \geq \kappa_1$ and in the second order term in the case $\kappa_2 \leq \kappa < \kappa_1$, but disappears in the case $\kappa < \kappa_2$. We also emphasize that we use the different rescaled functions in the case (ii)–(v) and in the case (i), respectively, to obtain the asymptotic behavior of minimizing sequences precisely.

The rest of this chapter is organized as follows: In Section 2.2, we prepare the characterization of $S^1(a_1, a_2, a_3)$ to prove Theorem 2.1 in the cases (iii) and (iv). In Section 2.3, we prove Theorem 2.1 concerning the asymptotic expansion of $\xi^\beta(a_1, a_2, a_3)$ and the asymptotic behavior of a minimizing sequence for the cases (i)–(v). In Appendix, we note that the asymptotic expansion of $\xi_\alpha^{\alpha^\tau}$ as $\alpha \to \infty$ for a given $\tau \leq 0$ and the asymptotic behavior of a minimizing sequence for $\xi_{\alpha_n}^{\alpha_n^\tau}$ where $\alpha_n \to \infty$.

2.2 Preliminaries

For simplicity, we prove Theorem 2.1 as $a_1 = a_2 = a_3 = 1$. So for simplicity, we write $\xi^\beta(a_1, a_2, a_3)$, $S_\infty(a_1)$, $S^1(a_1, a_2, a_3)$, $\xi_\infty(a_1, a_2, a_3)$, $\Sigma_0(a_1, a_2, a_3)$ and $\Sigma_1(a_1, a_2, a_3)$ as ξ^β, S_∞, S^1, ξ_∞, Σ_0 and Σ_1. Moreover, when $a_1 = 1$, Ψ_{a_1} in Remark 2.2 is abbreviated as Ψ.

As stated in Remark 2.2, the following compactness of the minimizing sequence for S_∞ is known (see Lions [35, 36]).

Lemma 2.2. Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence for S_∞. Then up to a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\theta \in \mathbb{R}$ such that

$$\|u_n(\cdot + y_n) - e^{i\theta}\Psi\|_{H^1} \to 0, \quad \text{as } n \to \infty.$$

Here, we note that the fact on rearrangements (see [8]).

Lemma 2.3. We assume that $N \in \mathbb{N}$ and let $f, g, h \in C(\mathbb{R}^N)$ be functions such that positive, radial symmetric and strictly decreasing and

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} g(x) = \lim_{|x|\to\infty} h(x) = 0,$$
$$\int_{\mathbb{R}^N} f(x)g(x)h(x)\,dx < \infty.$$

For $y_0, y_1 \in \mathbb{R}^N$, if $y_0 \neq 0$ or $y_1 \neq 0$, then

$$\int_{\mathbb{R}^N} f(x)g(x-y_0)h(x-y_1)\,dx < \int_{\mathbb{R}^N} f(x)g(x)h(x)\,dx$$

holds.

Lemma 2.4. (characterization of maximizer for S^1) Let $\mathbf{u}$ be a maximizer for S^1. Then there exist $y \in \mathbb{R}^N$ and $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ with $\theta_1 + \theta_2 = \theta_3$ such that

$$\mathbf{u} = (e^{i\theta_1}\Psi(\cdot + y), e^{i\theta_2}\Psi(\cdot + y), e^{i\theta_3}\Psi(\cdot + y)),$$

$$S^1 = \int_{\mathbb{R}^N} \Psi^3 \, dx \ (> 0).$$

Proof. By the definition of S^1,

$$\begin{aligned} S^1 &= \sup_{\theta_1,\theta_2,\theta_3 \in \mathbb{R}} \mathrm{Re}(e^{i(\theta_1+\theta_2-\theta_3)}) \sup_{z_1,z_2 \in \mathbb{R}^N} \int_{\mathbb{R}^N} \Psi(x)\Psi(x+z_1)\Psi(x+z_2) \, dx \\ &= \sup_{z_1,z_2 \in \mathbb{R}^N} \int_{\mathbb{R}^N} \Psi(x)\Psi(x+z_1)\Psi(x+z_2) \, dx \end{aligned}$$

with $\theta_1 + \theta_2 = \theta_3$. From Lemma 2.3, we have

$$\sup_{z_1,z_2 \in \mathbb{R}^N} \int_{\mathbb{R}^N} \Psi(x)\Psi(x+z_1)\Psi(x+z_2) \, dx = \int_{\mathbb{R}^N} \Psi(x)\Psi(x)\Psi(x) \, dx$$

and the supremum is attained only for the case $z_1 = z_2 = 0$. Thus

$$S^1 = \int_{\mathbb{R}^N} \Psi(x)^3 \, dx \ (> 0).$$

□

We note the following compactness of minimizing sequence for ξ_∞.

Lemma 2.5. **([31])** Let $N \le 3$, $1 < p < 1 + 4/N$. Let $\{\mathbf{u}_n\}_{n=1}^\infty$ be a minimizing sequence for ξ_∞. Then up to a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and a minimizer $\mathbf{u}$ for ξ_∞ such that

$$\|u_{j,n}(\cdot + y_n) - u_j\|_{H^1} \to 0, \quad \text{as } n \to \infty.$$

2.3 Proof of Theorem 2.1

Throughout this section, we assume that $N \le 3$, $1 < p < 1 + 4/N$, $\beta > 0$, $\alpha = \beta^\kappa$ with $\kappa \in \mathbb{R}$ and $a_1 = a_2 = a_3 = 1$. First, we give the proof of the cases (ii)–(v) of Theorem 2.1. Finally, we give the proof of the case (i) of Theorem 2.1.

To show the results in the cases (ii)–(v), we rescale the function $\mathbf{u}$ as (2.2), the functional E^β and its energy ξ^β as follows:

Let $\mathbf{u}$ be a function such that

$$\|u_1\|_2^2 = \|u_2\|_2^2 = \|u_3\|_2^2 = 1.$$

We rescale the function $\mathbf{u}$ as follows:

$$\mathbf{v}(x) := \beta^{-N/(4-N(p-1))}\mathbf{u}(\beta^{-2/(4-N(p-1))}x).$$

Then it follows that

$$\|v_1\|_2^2 = \|v_2\|_2^2 = \|v_3\|_2^2 = 1$$

and

$$E^\beta(\mathbf{u}) = \beta^{4/(4-N(p-1))}\tilde{E}^\beta(\mathbf{v}), \quad \xi^\beta = \beta^{4/(4-N(p-1))}\tilde{\xi}^\beta,$$

where

$$\begin{aligned}
\tilde{E}^\beta(\mathbf{v}) :=& \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_j|^2\,dx - \frac{1}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |v_j|^{p+1}\,dx \\
&- \beta^{(N-4)/(4-N(p-1))+\kappa}\mathrm{Re}\int_{\mathbb{R}^N} v_1 v_2 \overline{v}_3\,dx \\
&+ \frac{1}{\beta^{4/(4-N(p-1))}}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|v_j|^2\,dx, \\
\tilde{\xi}^\beta :=& \inf\{\tilde{E}^\beta(\mathbf{v}) \mid \mathbf{v} \in H,\ \|v_j\|_2^2 = 1\ (j=1,2,3)\}.
\end{aligned}$$

So it is sufficient to prove the energy expansion of $\tilde{\xi}^\beta$ and the asymptotic behavior of $\mathbf{v}_n$ to prove the cases (ii)–(v) in Theorem 2.1.

2.3.1 Proof of Theorem 2.1 (ii)

For the case $\kappa = \kappa_1$, we have

$$\tilde{E}^\beta(\mathbf{v}) = E_\infty(\mathbf{v}) + \beta^{-4/(4-N(p-1))}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|v_j|^2\,dx.$$

Upper bound

Lemma 2.6. (upper bound for $\tilde{\xi}^\beta$) Under the assumptions in the case (ii), it follows that

$$\tilde{\xi}^\beta \leq \xi_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} \\ +o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta\to\infty.$$

Proof. From Lemma 2.5, there exists a minimizer $\mathbf{v}$ for ξ_∞. Let $x_0\in\mathbb{R}^N$ be a point which attains

$$\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}.$$

For $\beta>0$, we set

$$\varphi_\beta(x) := \mathbf{v}(x-\beta^{2/(4-N(p-1))}x_0).$$

Then it holds that

$$\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|\varphi_{j,\beta}(x)|^2\,dx \\ = \int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}+x_0\right)|v_j(x)|^2\,dx.$$

From (V2), it follows that

$$\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}+x_0\right)|v_j(x)|^2\,dx \\ \to \int_{\mathbb{R}^N} V_j(x_0)|v_j(x)|^2\,dx, \quad \text{as } \beta\to\infty.$$

Then we have

$$\tilde{\xi}^\beta \leq \tilde{E}^\beta(\varphi_\beta) \\ = \xi_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|\varphi_{j,\beta}(x)|^2\,dx \\ = \xi_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} \\ +o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta\to\infty.$$

□

Lower bound and the completion of the proof of Theorem 2.1 (ii)

Theorem 2.1 (ii) with $a_1 = a_2 = a_3 = 1$ is reduced to the following lemma.

Lemma 2.7. Under the assumptions in the case (ii), it follows that

$$\tilde{\xi}^{\beta} = \xi_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} + o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta\to\infty.$$

Moreover, for the rescaled function $\mathbf{v}_n$ defined by (2.2), up to a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$, a minimizer $\mathbf{v}$ for ξ_∞ and $z_0 \in \mathbb{R}^N$ such that

$$\|\mathbf{v}_n(\cdot+y_n) - \mathbf{v}\|_{H^1} \to 0, \quad y_n/\beta_n^{2/(4-N(p-1))} \to z_0 \text{ in } \mathbb{R}^N, \quad \text{as } n\to\infty,$$
$$\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} = V_1(z_0)+V_2(z_0)+V_3(z_0).$$

Proof. Note that $\mathbf{v}_n$ satisfies

$$\|v_{1,n}\|_2^2 = \|v_{2,n}\|_2^2 = \|v_{3,n}\|_2^2 = 1,$$
$$\tilde{E}^{\beta_n}(\mathbf{v}_n) = \tilde{\xi}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}),$$

where $\beta_n\to\infty$. From Lemma 2.6, it follows that

$$\begin{aligned}
&\xi_\infty + o(1)\\
&\geq \tilde{\xi}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}) = \tilde{E}^{\beta_n}(\mathbf{v}_n)\\
&= E_\infty(\mathbf{v}_n) + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\sum_{j=1}^3\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta_n^{2/(4-N(p-1))}}\right)|v_{j,n}|^2\,dx\\
&\geq \xi_\infty + o(1).
\end{aligned} \tag{2.3}$$

Therefore $\{\mathbf{v}_n\}_{n=1}^\infty$ is a minimizing sequence for ξ_∞. From Lemma 2.5, up to a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\mathbf{v}\in H$ such that

$$\|\mathbf{v}_n(\cdot+y_n) - \mathbf{v}\|_{H^1} \to 0, \quad \text{as } n\to\infty$$
$$\mathbf{v} \text{ is a minimizer for } \xi_\infty.$$

Since $\|v_{j,n}(\cdot+y_n) - v_j\|_2 \to 0$ (as $n\to\infty$), up to a subsequence, there exists $g_j \in L^2(\mathbb{R}^N)$ such that

$$v_{j,n}(x+y_n) \to v_j(x), \quad \text{as } n\to\infty, \text{ a.e. } x\in\mathbb{R}^N,$$

$$|v_{j,n}(x+y_n)| \le g_j(x), \quad \text{for all } n \in \mathbb{N}, \text{ a.e. } x \in \mathbb{R}^N.$$

Claim. $\{y_n/\beta_n^{2/(4-N(p-1))}\}_{n=1}^{\infty}$ is bounded.

If not, up to a subsequence, $|y_n|/\beta_n^{2/(4-N(p-1))} \to \infty$ (as $n \to \infty$). From (V2), it holds that

$$\int_{\mathbb{R}^N} V_j\left(\frac{x+y_n}{\beta_n^{2/(4-N(p-1))}}\right)|v_{j,n}(x+y_n)|^2\,dx \to 0, \quad \text{as } n \to \infty.$$

From Lemma 2.6, we have

$$\begin{aligned}
&\xi_\infty + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} + o(\beta_n^{-4/(4-N(p-1))})\\
&\ge \tilde{\xi}^{\beta_n} = \tilde{E}^{\beta_n}(\mathbf{v}_n) + o(\beta_n^{-4/(4-N(p-1))})\\
&\ge \xi_\infty + o(\beta_n^{-4/(4-N(p-1))}), \quad \text{as } n \to \infty.
\end{aligned}$$

Then we have

$$\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} \ge 0.$$

On the other hand, since $V_j(x) \le 0$ (for all $x \in \mathbb{R}^N$) and $V_1 \not\equiv 0$ or $V_2 \not\equiv 0$ or $V_3 \not\equiv 0$, it follows that

$$\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} < 0.$$

This is a contradiction. Thus the claim holds. Therefore, up to a subsequence, there exists $z_0 \in \mathbb{R}^N$ such that

$$y_n/\beta_n^{2/(4-N(p-1))} \to z_0, \quad \text{as } n \to \infty.$$

From (V2), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} V_j\left(\frac{x+y_n}{\beta_n^{2/(4-N(p-1))}}\right)|v_{j,n}(x+y_n)|^2\,dx\\
&\to \int_{\mathbb{R}^N} V_j(z_0)|v_j(x)|^2\,dx, \quad \text{as } n \to \infty.
\end{aligned} \tag{2.4}$$

From (2.3)–(2.4), we have

$$
\begin{aligned}
&\xi_\infty + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq \tilde{\xi}^{\beta_n} = \tilde{E}^{\beta_n}(\mathbf{v}_n) + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq \xi_\infty + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}(V_1(z_0)+V_2(z_0)+V_3(z_0)) + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq \xi_\infty + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} + o(\beta_n^{-4/(4-N(p-1))}), \\
&\hspace{30em}\text{as } n\to\infty.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} = V_1(z_0)+V_2(z_0)+V_3(z_0), \\
&\lim_{n\to\infty}\beta_n^{4/(4-N(p-1))}(\tilde{\xi}^{\beta_n}-\xi_\infty) = \frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}.
\end{aligned}
$$

Since $\{\beta_n\}_{n=1}^\infty$ is arbitrary sequence satisfying $\beta_n\to\infty$ (as $n\to\infty$), we have

$$
\begin{aligned}
\tilde{\xi}^{\beta} = \xi_\infty &+ \beta^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} \\
&+ o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta\to\infty.
\end{aligned}
$$

□

Remark 2.4. The result of Theorem 2.1 (ii) indicates that $\mathbf{u}_n$ concentrates at z_0. Indeed, $\mathbf{u}_n$ behaves like

$$
\begin{aligned}
\mathbf{u}_n(x) &= \beta_n^{N/(4-N(p-1))}\mathbf{v}_n(\beta_n^{2/(4-N(p-1))}x) \\
&\sim \beta_n^{N/(4-N(p-1))}\mathbf{v}(\beta_n^{2/(4-N(p-1))}x - y_n) \\
&\sim \beta_n^{N/(4-N(p-1))}\mathbf{v}(\beta_n^{2/(4-N(p-1))}(x - z_0)), \quad \text{as } \beta_n\to\infty.
\end{aligned}
$$

2.3.2 Proof of Theorem 2.1 (iii)

Note that for the case (iii)

$$
-4/(4-N(p-1)) < (N-4)/(4-N(p-1)) + \kappa < 0
$$

and

$$\tilde{E}^{\beta}(\mathbf{v}) = \sum_{j=1}^{3} J_{\infty}(v_j) - \beta^{(N-4)/(4-N(p-1))+\kappa} J^1(\mathbf{v})$$
$$+ \beta^{-4/(4-N(p-1))} \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} V_j \left(\frac{x}{\beta^{2/(4-N(p-1))}} \right) |v_j|^2 \, dx.$$

First, we prove the upper bound for $\tilde{\xi}^{\beta}$. Taking $\mathbf{\Psi} = (\Psi, \Psi, \Psi)$, where Ψ is the function Ψ_{a_j} defined in Remark 2.2 with $a_j = 1$, under the assumption in the case (iii), from Lemma 2.4, it is easy to obtain

$$\tilde{\xi}^{\beta} \leq \tilde{E}^{\beta}(\mathbf{\Psi}) \leq 3S_{\infty} - \beta^{(N-4)/(4-N(p-1))+\kappa} S^1, \quad \text{as } \beta \to \infty.$$

Theorem 2.1 (iii) with $a_1 = a_2 = a_3 = 1$ is reduced to the following lemma.

Lemma 2.8. Under the assumption in the case (iii), it holds that

$$\tilde{\xi}^{\beta} = 3S_{\infty} - \beta^{(N-4)/(4-N(p-1))+\kappa} S^1 + o(\beta^{(N-4)/(4-N(p-1))+\kappa}), \quad \text{as } \beta \to \infty.$$

Moreover, for the rescaled function $\mathbf{v}_n$ defined by (2.2), up to a subsequence, there exist $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$, and $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ such that

$$\|v_{j,n}(\cdot + y_n) - e^{i\theta_j}\Psi\|_{H^1} \to 0, \quad \text{as } n \to \infty, \ j = 1, 2, 3,$$
$$\theta_1 + \theta_2 = \theta_3.$$

Proof. **(Step 1)** Note that $\mathbf{v}_n$ satisfies

$$\|v_{1,n}\|_2^2 = \|v_{2,n}\|_2^2 = \|v_{3,n}\|_2^2 = 1, \tag{2.5}$$
$$\tilde{E}^{\beta_n}(\mathbf{v}_n) = \tilde{\xi}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}). \tag{2.6}$$

From the upper bound for $\tilde{\xi}^{\beta}$, it holds that

$$\begin{aligned}
&3S_\infty + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq \tilde{\xi}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}) = \tilde{E}^{\beta_n}(\mathbf{v}_n) \\
&\geq J_\infty(v_{1,n}) + J_\infty(v_{2,n}) + J_\infty(v_{3,n}) \\
&\quad + O(1/\beta_n^{4/(4-N(p-1))}) - \beta_n^{(N-4)/(4-N(p-1))+\kappa}\mathrm{Re}\int_{\mathbb{R}^N} v_{1,n}v_{2,n}\overline{v}_{3,n}\,dx \\
&\geq \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla v_{j,n}|^2\,dx - \frac{1}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|v_{j,n}|^{p+1}\,dx \\
&\quad + O(1/\beta_n^{4/(4-N(p-1))}) - \frac{\beta_n^{(N-4)/(4-N(p-1))+\kappa}}{3}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|v_{j,n}|^3\,dx.
\end{aligned} \tag{2.7}$$

Here we note that $N \leq 3$, $1 < p < 1+4/N$ and $(N-4)/(4-N(p-1))+\kappa < 0$. Then for n sufficiently large, it follows that $\beta_n^{(N-4)/(4-N(p-1))+\kappa} \leq 1$. From Gagliardo-Nirenberg's inequality (see Adams [1]) and (2.5), for $q = p+1$ and $q = 3$, we have

$$\begin{aligned}
\|v_{j,n}\|_q^q &\leq C(N,q)\|\nabla v_{j,n}\|_2^{N(q-2)/2}\|v_{j,n}\|_2^{q-N(q-2)/2} \\
&\leq \varepsilon\|\nabla v_{j,n}\|_2^2 + C(\varepsilon,N,q), \quad \text{for all } \varepsilon > 0.
\end{aligned} \tag{2.8}$$

Here $C(N,q), C(\varepsilon,N,q) > 0$ is a constant. From (2.7),(2.8), we have

$$3S_\infty + O(1) \geq \left(\frac{1}{2} - \frac{1}{p+1}\varepsilon - \frac{1}{3}\varepsilon\right)\sum_{j=1}^{3}\|\nabla v_{j,n}\|_2^2.$$

Fix $\varepsilon > 0$ such that $1/2 - \varepsilon/(p+1) - \varepsilon/3 > 0$. Combining with (2.5), we find that there exists a positive constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\sum_{j=1}^{3}\|v_{j,n}\|_{H^1}^2 \leq C. \tag{2.9}$$

(Step 2) From the upper bound for $\tilde{\xi}^{\beta}$, we have

$$\begin{aligned}
3S_\infty \geq \tilde{\xi}^{\beta_n} &= \tilde{E}^{\beta_n}(\mathbf{v}_n) + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq J_\infty(v_{1,n}) + J_\infty(v_{2,n}) + J_\infty(v_{3,n}) \\
&\quad + O(1/\beta_n^{4/(4-N(p-1))}) \\
&\quad - \beta_n^{(N-4)/(4-N(p-1))+\kappa}\mathrm{Re}\int_{\mathbb{R}^N} v_{1,n}v_{2,n}\overline{v}_{3,n}\,dx.
\end{aligned} \tag{2.10}$$

From (2.9) and $N \leq 3$, $1 < p < 1+4/N$ and $(N-4)/(4-N(p-1))+\kappa < 0$, we deduce that

$$\beta_n^{(N-4)/(4-N(p-1))+\kappa}\mathrm{Re}\int_{\mathbb{R}^N} v_{1,n}v_{2,n}\overline{v}_{3,n}\,dx = o(1), \quad \text{as } n \to \infty.$$

From (2.5),(2.10) and the definition of S_∞, we have

$$\begin{aligned} 3S_\infty &\geq \tilde{\xi}^{\beta_n} \geq J_\infty(v_{1,n}) + J_\infty(v_{2,n}) + J_\infty(v_{3,n}) + o(1) \\ &\geq 3S_\infty + o(1), \quad \text{as } n \to \infty. \end{aligned}$$

Thus we have

$$\lim_{n\to\infty} J_\infty(v_{j,n}) = S_\infty, \quad j = 1,2,3.$$

Thus $\{v_{1,n}\}_{n=1}^\infty$,$\{v_{2,n}\}_{n=1}^\infty$,$\{v_{3,n}\}_{n=1}^\infty$ are minimizing sequences for S_∞. From Lemma 2.2, up to a subsequence, there exist $\{y_n^{(j)}\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\theta_j \in \mathbb{R}$ such that

$$\|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j}\Psi\|_{H^1} \to 0, \quad \text{as } n \to \infty,\ j = 1,2,3. \tag{2.11}$$

(Step 3) Set

$$\begin{aligned} &\Psi_{j,n} := e^{i\theta_j}\Psi(\cdot - y_n^{(j)}),\ j = 1,2,3 \\ &\mathbf{\Psi}_n := (\Psi_{1,n}, \Psi_{2,n}, \Psi_{3,n}). \end{aligned}$$

From (2.11) and $\{\mathbf{v}_n\}_{n=1}^\infty$ and $\{\mathbf{\Psi}_n\}_{n=1}^\infty$ are bounded in H, we have

$$\begin{aligned} &|J^1(\mathbf{v}_n) - J^1(\mathbf{\Psi}_n)| \\ &\leq \int_{\mathbb{R}^N} |v_{1,n}||v_{2,n}||v_{3,n} - \Psi_{3,n}|\,dx + \int_{\mathbb{R}^N} |v_{1,n}||v_{2,n} - \Psi_{2,n}||\Psi_{3,n}|\,dx \\ &\quad + \int_{\mathbb{R}^N} |v_{1,n} - \Psi_{1,n}||\Psi_{2,n}||\Psi_{3,n}|\,dx \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{2.12}$$

Moreover, since $\Psi_{j,n}$ is a minimizer for S_∞, it follows that

$$J^1(\mathbf{\Psi}_n) \leq S^1.$$

From the upper bound for $\tilde{\xi}^\beta$, it follows that

$$3S_\infty - \beta_n^{(N-4)/(4-N(p-1))+\kappa} S^1$$

$$
\begin{aligned}
&\geq \tilde{\xi}^{\beta_n} = \tilde{E}^{\beta_n}(\mathbf{v}_n) + o(\beta_n^{-4/(4-N(p-1))}) \\
&= J_\infty(v_{1,n}) + J_\infty(v_{2,n}) + J_\infty(v_{3,n}) - \beta_n^{(N-4)/(4-N(p-1))+\kappa} J^1(\mathbf{v}_n) \\
&\quad + o(\beta_n^{(N-4)/(4-N(p-1))+\kappa}) \\
&\geq 3S_\infty - \beta_n^{(N-4)/(4-N(p-1))+\kappa} J^1(\mathbf{\Psi}_n) + o(\beta_n^{(N-4)/(4-N(p-1))+\kappa}) \\
&\geq 3S_\infty - \beta_n^{(N-4)/(4-N(p-1))+\kappa} S^1 + o(\beta_n^{(N-4)/(4-N(p-1))+\kappa}), \quad \text{as } n \to \infty.
\end{aligned}
$$

Thus we have

$$
\lim_{n\to\infty} \frac{\tilde{\xi}^{\beta_n} - 3S_\infty}{\beta_n^{(N-4)/(4-N(p-1))+\kappa}} = -S^1, \quad \lim_{n\to\infty} J^1(\mathbf{\Psi}_n) = S^1. \tag{2.13}
$$

Since $\{\beta_n\}_{n=1}^\infty$ is arbitrary sequence satisfying $\beta_n \to \infty$, we have

$$
\tilde{\xi}^\beta = 3S_\infty - \beta^{(N-4)/(4-N(p-1))+\kappa} S^1 + o(\beta^{(N-4)/(4-N(p-1))+\kappa}), \quad \text{as } \beta \to \infty.
$$

(Step 4) From (2.13), it follows that

$$
\mathrm{Re}(e^{i(\theta_1+\theta_2-\theta_3)}) \lim_{n\to\infty} \int_{\mathbb{R}^N} \Psi(x)\Psi(x + y_n^{(1)} - y_n^{(2)})\Psi(x + y_n^{(1)} - y_n^{(3)})\, dx = S^1. \tag{2.14}
$$

We prove $\{y_n^{(1)} - y_n^{(2)}\}_{n=1}^\infty$ and $\{y_n^{(1)} - y_n^{(3)}\}_{n=1}^\infty$ are bounded in $\mathbb{R}^N$. If not, for example, if $\{y_n^{(1)} - y_n^{(2)}\}_{n=1}^\infty$ is not bounded, up to a subsequence, then it holds that

$$
|y_n^{(1)} - y_n^{(2)}| \to \infty, \quad \text{as } n \to \infty.
$$

From Remark 2.2, $\Psi \in L^2(\mathbb{R}^N)$ is radial symmetric and decreasing, it holds that

$$
\lim_{|x|\to\infty} \Psi(x) = 0.
$$

Thus for all $\varepsilon > 0$, there exists $R > 0$ such that

$$
|x| \geq R \Longrightarrow \Psi(x) < \varepsilon.
$$

In addition, since $|y_n^{(1)} - y_n^{(2)}| \to \infty$ (as $n \to \infty$), for n sufficiently large, we have

$$
\Psi(x + y_n^{(1)} - y_n^{(2)}) < \varepsilon, \quad \text{for all } |x| < R.
$$

Thus for n sufficiently large, it follows that

$$
\begin{aligned}
|J^1(\mathbf{\Psi}_n)| &= |J^1(\Psi(\cdot - y_n^{(1)}), \Psi(\cdot - y_n^{(2)}), \Psi(\cdot - y_n^{(3)}))| \\
&\leq \int_{\mathbb{R}^N} \Psi(x)\Psi(x + y_n^{(1)} - y_n^{(2)})\Psi(x + y_n^{(1)} - y_n^{(3)})\, dx \\
&\leq \varepsilon \int_{|x|<R} \Psi(x)\Psi(x + y_n^{(1)} - y_n^{(3)})\, dx \\
&\quad + \varepsilon \int_{|x|\geq R} \Psi(x + y_n^{(1)} - y_n^{(2)})\Psi(x + y_n^{(1)} - y_n^{(3)})\, dx \\
&\leq \varepsilon \|\Psi\|_2^2 + \varepsilon \|\Psi\|_2^2 = 2\varepsilon.
\end{aligned}
$$

Thus we have

$$\lim_{n\to\infty} J^1(\mathbf{\Psi}_n) = 0.$$

Although

$$\lim_{n\to\infty} J^1(\mathbf{\Psi}_n) = S^1,$$

this is a contradiction to $S^1 > 0$ from Lemma 2.4. Therefore $\{y_n^{(1)} - y_n^{(2)}\}_{n=1}^{\infty}$ is bounded. We can prove that $\{y_n^{(1)} - y_n^{(3)}\}_{n=1}^{\infty}$ is bounded in the same way. Hence up to a subsequence, there exist $y^{(2)}, y^{(3)} \in \mathbb{R}^N$ such that

$$
\begin{aligned}
y_n^{(1)} - y_n^{(2)} \to y^{(2)}, &\quad \text{as } n \to \infty, \\
y_n^{(1)} - y_n^{(3)} \to y^{(3)}, &\quad \text{as } n \to \infty.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \Psi(x)\Psi(x + y_n^{(1)} - y_n^{(2)})\Psi(x + y_n^{(1)} - y_n^{(3)})\, dx \\
&\to \int_{\mathbb{R}^N} \Psi(x)\Psi(x + y^{(2)})\Psi(x + y^{(3)})\, dx, \quad \text{as } n \to \infty.
\end{aligned}
$$

From (2.14), it holds that

$$\mathrm{Re}(e^{i(\theta_1+\theta_2-\theta_3)}) \int_{\mathbb{R}^N} \Psi(x)\Psi(x + y^{(2)})\Psi(x + y^{(3)})\, dx = S^1.$$

Therefore $(e^{i\theta_1}\Psi, e^{i\theta_2}\Psi(\cdot + y^{(2)}), e^{i\theta_3}\Psi(\cdot + y^{(3)}))$ is a maximizer for S^1. From Lemma 2.4, $y^{(2)} = y^{(3)} = 0$ and we may assume that $\theta_1 + \theta_2 = \theta_3$.

Moreover we have

$$\|v_{j,n}(\cdot + y_n^{(1)}) - e^{i\theta_j}\Psi\|_{H^1} \to 0, \quad \text{as } n \to \infty,\ j = 2, 3.$$

Indeed, setting $z_n^{(j)} := y_n^{(1)} - y_n^{(j)}$ $(j = 2, 3)$, we have

$$\begin{aligned}
&\|v_{j,n}(\cdot + y_n^{(1)}) - e^{i\theta_j}\Psi\|_{H^1} = \|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j}\Psi(\cdot - z_n^{(j)})\|_{H^1} \\
&\le \|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j}\Psi\|_{H^1} + \|e^{i\theta_j}\Psi - e^{i\theta_j}\Psi(\cdot - z_n^{(j)})\|_{H^1} \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

□

2.3.3 Proof of Theorem 2.1 (iv)

For the case $\kappa = \kappa_2$, we have

$$\begin{aligned}
\tilde{E}^\beta(\mathbf{v}) = \sum_{j=1}^{3} J_\infty(v_j) - \beta^{-4/(4-N(p-1))} &\times \\
\times \left(J^1(\mathbf{v}) - \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|v_j|^2\,dx \right)&.
\end{aligned}$$

For the proof of the upper bound, we use the following test function:

$$\varphi_\beta(x) := \mathbf{\Psi}(x - \beta^{2/(4-N(p-1))}z_0),$$

where z_0 is unique minimum point of V_j. By using the arguments used in Theorem 2.1 (ii), we can prove the upper bound:

$$\begin{aligned}
\tilde{\xi}^\beta \le \tilde{E}^\beta(\varphi_\beta) &= 3S_\infty - \beta^{-4/(4-N(p-1))} \times \\
&\times \left(S^1 - \frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x) + V_2(x) + V_3(x)\} \right) \\
&+ o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta \to \infty.
\end{aligned}$$

For the proof of the lower bound, note that the rescaled function $\mathbf{v}_n$ defined by (2.2) satisfies

$$\|v_{1,n}\|_2^2 = \|v_{2,n}\|_2^2 = \|v_{3,n}\|_2^2 = 1,$$

$$\tilde{E}^{\beta_n}(\mathbf{v}_n) = \tilde{\xi}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}),$$

where $\beta_n \to \infty$. By the similar argument as in the proof of Theorem 2.1 (iii), it holds that $\{\mathbf{v}_n\}_{n=1}^\infty$ is bounded in H and each $\{v_{j,n}\}_{n=1}^\infty$ is a minimizing sequence for S_∞. Therefore up to a subsequence, there exist $\{y_n^{(j)}\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\theta_j \in \mathbb{R}$ such that

$$\|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j}\Psi\|_{H^1} \to 0.$$

From the upper bound for $\tilde{\xi}^{\beta_n}$, we have

$$\begin{aligned}
&3S_\infty - \beta_n^{-4/(4-N(p-1))}\times \\
&\quad\times (S^1 - \frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}) + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq \tilde{\xi}^{\beta_n} \\
&\geq 3S_\infty - \beta_n^{-4/(4-N(p-1))}\times \\
&\quad\times \left(J^1(\mathbf{v}_n) - \frac{1}{2}\sum_{j=1}^3 \int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta_n^{2/(4-N(p-1))}}\right)|v_{j,n}|^2\,dx\right) \\
&\quad + o(\beta_n^{-4/(4-N(p-1))}).
\end{aligned}$$

Since

$$\begin{aligned}
J^1(\mathbf{v}_n) &= J^1(e^{i\theta_1}\Psi(\cdot - y_n^{(1)}), e^{i\theta_2}\Psi(\cdot - y_n^{(2)}), e^{i\theta_3}\Psi(\cdot - y_n^{(3)})) + o(1) \\
&\leq S^1 + o(1),
\end{aligned}$$

by the same argument as in Theorem 2.1 (ii) and (iii), we have

$$\begin{aligned}
&y_n^{(j)}/\beta_n^{2/(4-N(p-1))} \to z_{j,0} = z_0, \\
&\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta_n^{2/(4-N(p-1))}}\right)|v_{j,n}|^2\,dx \to V_j(z_{j,0}) = V_j(z_0).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&3S_\infty - \beta_n^{-4/(4-N(p-1))}(S^1 - \frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}) \\
&\quad + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq \tilde{\xi}^{\beta_n}
\end{aligned}$$

$$
\begin{aligned}
&\geq 3S_\infty - \beta_n^{-4/(4-N(p-1))}\times \\
&\times \Big(J^1(e^{i\theta_1}\Psi(\cdot - y_n^{(1)}), e^{i\theta_2}\Psi(\cdot - y_n^{(2)}), e^{i\theta_3}\Psi(\cdot - y_n^{(3)})) \\
&- \frac{1}{2}\{V_1(z_0)+V_2(z_0)+V_3(z_0)\}\Big) + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq 3S_\infty - \beta_n^{-4/(4-N(p-1))}(S^1 - \frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}) \\
&\quad + o(\beta_n^{-4/(4-N(p-1))}).
\end{aligned}
$$

By the same argument as in Theorem 2.1 (iii), we have

$$
\begin{aligned}
&\tilde{\xi}^\beta = 3S_\infty - \beta^{-4/(4-N(p-1))}(S^1 - \frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}) \\
&\quad + o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta\to\infty, \\
&\theta_1+\theta_2 = \theta_3 + 2k\pi, \quad k\in\mathbb{Z}, \\
&y_n^{(1)} - y_n^{(2)} \to 0, \quad y_n^{(1)} - y_n^{(3)} \to 0, \\
&\|v_{j,n}(\cdot + y_n) - e^{i\theta_j}\Psi\|_{H^1} \to 0, \\
&y_n/\beta_n^{2/(4-N(p-1))} \to z_0,
\end{aligned}
$$

where $y_n = y_n^{(1)}$.

2.3.4 Proof of Theorem 2.1 (v)

For the case (v) $\kappa < \kappa_2$, note that

$$
(N-4)/(4-N(p-1)) + \kappa < -4/(4-N(p-1))
$$

and

$$
\begin{aligned}
\tilde{E}^\beta(\mathbf{v}) = \sum_{j=1}^3 J_\infty(v_j) + \beta^{-4/(4-N(p-1))}\frac{1}{2}\sum_{j=1}^3\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|v_j|^2\,dx \\
- \beta^{(N-4)/(4-N(p-1))+\kappa}J^1(\mathbf{v}).
\end{aligned}
$$

First we prove the upper bound for $\tilde{\xi}^\beta$. Let $x_{j,0}\in\mathbb{R}^N$ such that $\min_{x\in\mathbb{R}^N} V_j(x) = V_j(x_{j,0})$ for all $j=1,2,3$.

Set $v_j(x) = \Psi(x - \beta^{2/(4-N(p-1))}x_{j,0})$, $\mathbf{v} = (v_1, v_2, v_3)$. Then we have

$$\begin{aligned}
\tilde{\xi}^\beta &\le \tilde{E}^\beta(\mathbf{v}) \\
&= 3S_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j(\beta^{-2/(4-N(p-1))}x + x_{j,0})|\Psi|^2\,dx \\
&+ o(\beta^{-4/(4-N(p-1))}) \\
&= 3S_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\{\min_{x\in\mathbb{R}^N} V_1(x) + \min_{x\in\mathbb{R}^N} V_2(x) + \min_{x\in\mathbb{R}^N} V_3(x)\} \\
&+ o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta \to \infty.
\end{aligned}$$

Next, we prove the lower bound for $\tilde{\xi}^\beta$. Recall that the rescaled function $\mathbf{v}_n$ defined by (2.2) satisfies

$$\begin{aligned}
&\|v_{1,n}\|_2^2 = \|v_{2,n}\|_2^2 = \|v_{3,n}\|_2^2 = 1, \\
&\tilde{E}^{\beta_n}(\mathbf{v}_n) = \tilde{\xi}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}),
\end{aligned}$$

where $\beta_n \to \infty$. Since $\{v_{j,n}\}_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$, by the same argument as in the proof of Theorem 2.1 (iii) and (iv), $\{v_{j,n}\}_{n=1}^\infty$ is a minimizing sequence for S_∞. Thus up to a subsequence, there exist $\{y_n^{(j)}\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\theta_j \in \mathbb{R}$ such that

$$\|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j}\Psi\|_{H^1} \to 0.$$

By the same argument as in the proof of Theorem 2.1 (ii), since $\{y_n^{(j)}/\beta_n^{2/(4-N(p-1))}\}_{n=1}^\infty$ is bounded, up to a subsequence, there exists $z_{j,0} \in \mathbb{R}^N$ such that

$$y_n^{(j)}/\beta_n^{2/(4-N(p-1))} \to z_{j,0}.$$

Moreover we have

$$\int_{\mathbb{R}^N} V_j(\beta_n^{-2/(4-N(p-1))}x)|v_{j,n}|^2\,dx \to V_j(z_{j,0}).$$

From the upper bound for $\tilde{\xi}^\beta$, it follows that

$$3S_\infty + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\{\min_{x\in\mathbb{R}^N} V_1(x) + \min_{x\in\mathbb{R}^N} V_2(x) + \min_{x\in\mathbb{R}^N} V_3(x)\}$$

$$
\begin{aligned}
&\quad + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq \tilde{\xi}^{\beta_n} \\
&\geq 3S_\infty + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\{V_1(z_{1,0}) + V_2(z_{2,0}) + V_3(z_{3,0})\} + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq 3S_\infty + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\{\min_{x\in\mathbb{R}^N} V_1(x) + \min_{x\in\mathbb{R}^N} V_2(x) + \min_{x\in\mathbb{R}^N} V_3(x)\} \\
&\quad + o(\beta_n^{-4/(4-N(p-1))}).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\tilde{\xi}^\beta = 3S_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\{\min_{x\in\mathbb{R}^N} V_1(x) + \min_{x\in\mathbb{R}^N} V_2(x) + \min_{x\in\mathbb{R}^N} V_3(x)\} \\
+ o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta \to \infty,
\end{aligned}
$$

$$
\min_{x\in\mathbb{R}^N} V_j(x) = V_j(z_{j,0}).
$$

2.3.5 Proof of Theorem 2.1 (i)

Let $\mathbf{u}$ be a function such that

$$
\|u_1\|_2^2 = \|u_2\|_2^2 = \|u_3\|_2^2 = 1.
$$

We consider the rescaled function $\mathbf{w}$ as (2.1) such that

$$
\mathbf{w}(x) := \beta^{-\kappa N/(4-N)}\mathbf{u}(\beta^{-2\kappa/(4-N)}x).
$$

Then it follows that

$$
\|w_1\|_2^2 = \|w_2\|_2^2 = \|w_3\|_2^2 = 1
$$

and

$$
E^\beta(\mathbf{u}) = \beta^{4\kappa/(4-N)}\tilde{F}^\beta(\mathbf{w}), \quad \xi^\beta = \beta^{4\kappa/(4-N)}\tilde{K}^\beta
$$

where

$$
\begin{aligned}
\tilde{F}^\beta(\mathbf{w}) := E^0(\mathbf{w}) - \beta^{\kappa(N(p-1)-4)/(4-N)+1}E^1(\mathbf{w}) \\
+ \beta^{-4\kappa/(4-N)}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j(\beta^{-2\kappa/(4-N)}x)|w_j|^2\,dx,
\end{aligned}
$$

$$\tilde{K}^\beta := \inf\{\tilde{F}^\beta(\mathbf{w}) \mid \mathbf{w} \in H, \quad \|w_j\|_2^2 = 1 \ (j = 1, 2, 3)\}.$$

For the case (i) $\kappa > \kappa_1$, note that

$$-4\kappa/(4-N) < \kappa(N(p-1)-4)/(4-N) + 1 < 0.$$

We first prove the upper bound for $\tilde{K}^\beta$. Let $\mathbf{W}_n$ be a maximizing sequence for Σ_1, that is, $\mathbf{W}_n$ satisfies

$$\begin{aligned} &\mathbf{W}_n \text{ is a minimizer for } \Sigma_0, \\ &E^1(\mathbf{W}_n) \to \Sigma_1, \quad \text{as } n \to \infty. \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{K}^\beta \le \tilde{F}^\beta(\mathbf{W}_n) &= E^0(\mathbf{W}_n) - \beta^{\kappa(N(p-1)-4)/(4-N)+1} E^1(\mathbf{W}_n) \\ &\quad + \beta^{-4\kappa/(4-N)} \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} V_j(\beta^{-2\kappa/(4-N)} x) |W_{j,n}|^2 \, dx \\ &\le \Sigma_0 - \beta^{\kappa(N(p-1)-4)/(4-N)+1} E^1(\mathbf{W}_n). \end{aligned}$$

Then letting $n \to \infty$, we have

$$\tilde{K}^\beta \le \Sigma_0 - \beta^{\kappa(N(p-1)-4)/(4-N)+1} \Sigma_1.$$

Next we prove the lower bound for $\tilde{K}^\beta$. Note that the rescaled function $\mathbf{w}_n$ defined by (2.1) satisfies

$$\begin{aligned} &\|w_{1,n}\|_2^2 = \|w_{2,n}\|_2^2 = \|w_{3,n}\|_2^2 = 1, \\ &\tilde{F}^{\beta_n}(\mathbf{w}_n) = \tilde{K}^{\beta_n} + o(\beta_n^{-4\kappa/(4-N)}), \end{aligned}$$

where $\beta_n \to \infty$ as $n \to \infty$. Since $\{w_{j,n}\}_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$, by the same argument as in Theorem 2.1 (iii), $\{\mathbf{w}_n\}_{n=1}^\infty$ is a minimizing sequence for Σ_0. From the compactness of minimizing sequence for Σ_0 (see Kurata-Osada [31]), up to a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and a minimizer $\mathbf{w}$ for Σ_0 such that

$$\|\mathbf{w}_n(\cdot + y_n) - \mathbf{w}\|_{H^1} \to 0, \quad \text{as } n \to \infty.$$

From the upper bound for $\tilde{K}^\beta$, we have

$$\begin{aligned}
&\Sigma_0 - \beta_n^{\kappa(N(p-1)-4)/(4-N)+1}\Sigma_1 \\
&\geq \tilde{K}^{\beta_n} = \tilde{F}^{\beta_n}(\mathbf{w}_n) + o(\beta_n^{-4\kappa/(4-N)}) \\
&\geq \Sigma_0 - \beta_n^{\kappa(N(p-1)-4)/(4-N)+1}E^1(\mathbf{w}_n) \\
&\qquad + \beta_n^{-4\kappa/(4-N)}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j(\beta_n^{-2\kappa/(4-N)}x)|w_{j,n}|^2\,dx \\
&= \Sigma_0 - \beta_n^{\kappa(N(p-1)-4)/(4-N)+1}E^1(\mathbf{w}) + o(\beta_n^{\kappa(N(p-1)-4)/(4-N)+1}) \\
&\geq \Sigma_0 - \beta_n^{\kappa(N(p-1)-4)/(4-N)+1}\Sigma_1 + o(\beta_n^{\kappa(N(p-1)-4)/(4-N)+1}).
\end{aligned}$$

Thus we have

$$\tilde{K}^\beta = \Sigma_0 - \beta^{\kappa(N(p-1)-4)/(4-N)+1}\Sigma_1 + o(\beta^{\kappa(N(p-1)-4)/(4-N)+1}), \quad \text{as } \beta \to \infty$$

and $\mathbf{w}$ is a maximizer for Σ_1.

2.4 Appendix

We remark the another asymptotic expansion of the energy $\xi_\alpha^\beta(a_1, a_2, a_3)$ as $\alpha \to \infty$ with $\beta = \alpha^\tau$ for a given $\tau \in \mathbb{R}$. For $\tau > 0$, the result of asymptotic expansion of $\xi_\alpha^\beta(a_1, a_2, a_3)$ as $\alpha \to \infty$ with $\beta = \alpha^\tau$ is included in Theorem 2.1. So we consider the case $\tau \leq 0$. For a given $\tau \leq 0$, as $\beta = \alpha^\tau$ define

$$\begin{aligned}
E_\alpha(\mathbf{u}) &:= E_\alpha^{\alpha^\tau}(\mathbf{u}), \\
\xi_\alpha(a_1, a_2, a_3) &:= \xi_\alpha^{\alpha^\tau}(a_1, a_2, a_3).
\end{aligned}$$

Let $\{\alpha_n\}_{n=1}^\infty$ be a positive number sequence such that $\alpha_n \to \infty$ as $n \to \infty$. We say that $\{\mathbf{u}_n\}_{n=1}^\infty$ is a minimizing sequence for $\xi_{\alpha_n}(a_1, a_2, a_3)$ if

$$\begin{aligned}
&\|u_{1,n}\|_2^2 = a_1, \quad \|u_{2,n}\|_2^2 = a_2, \quad \|u_{3,n}\|_2^2 = a_3, \\
&E_{\alpha_n}(\mathbf{u}_n) = \xi_{\alpha_n}(a_1, a_2, a_3) + o(1), \quad \text{as } n \to \infty.
\end{aligned}$$

We use the rescaled function $\mathbf{w}_n$ defined by (2.1) to analyze the asymptotic expansion for $\xi_\alpha(a_1, a_2, a_3)$ as $\alpha \to \infty$. The asymptotic expansion up to the first term for $\xi_\alpha(a_1, a_2, a_3)$ for the case $\tau = 0$ is treated in Kurata-Osada [31].

Proposition 1. *(I)* $-N(p-1)/(4-N) < \tau \le 0$
Assume that $N \le 2$. *Then it holds that*

$$\xi_\alpha(a_1,a_2,a_3) = \alpha^{4/(4-N)}\Sigma_0(a_1,a_2,a_3) - \alpha^{N(p-1)/(4-N)+\tau}\Sigma_1(a_1,a_2,a_3) + o(\alpha^{N(p-1)/(4-N)+\tau}), \quad \text{as } \alpha \to \infty.$$

Moreover, let $\mathbf{u}_n$ *be a minimizing sequence for* $\xi_{\alpha_n}(a_1,a_2,a_3)$ *where* $\alpha_n \to \infty$. *For the rescaled function* $\mathbf{w}_n$ *defined by* (2.1), *up to a subsequence, there exist a maximizer* $\mathbf{w}$ *for* $\Sigma_1(a_1,a_2,a_3)$ *and* $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ *such that*

$$\|\mathbf{w}_n(\cdot + y_n) - \mathbf{w}\|_{H^1} \to 0, \quad \text{as } n \to \infty.$$

(II) $\tau = -N(p-1)/(4-N)$
Assume that $N \le 2$, *(V2) and* $(V_1,V_2,V_3) \not\equiv (0,0,0)$. *Then it holds that*

$$\xi_\alpha(a_1,a_2,a_3) = \alpha^{4/(4-N)}\Sigma_0(a_1,a_2,a_3) - \Sigma_1(a_1,a_2,a_3) + \frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)a_1 + V_2(x)a_2 + V_3(x)a_3\} + o(1), \quad \text{as } \alpha \to \infty.$$

Moreover let $\mathbf{u}_n$ *be a minimizing sequence for* $\xi_{\alpha_n}(a_1,a_2,a_3)$ *where* $\alpha_n \to \infty$. *For the rescaled function* $\mathbf{w}_n$ *defined by* (2.1), *up to a subsequence, there exist a maximizer* $\mathbf{w}$ *for* $\Sigma_1(a_1,a_2,a_3)$, $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ *and* $z_0 \in \mathbb{R}^N$ *such that*

$$\|w_{j,n}(\cdot + y_n) - w_j\|_{H^1} \to 0, \quad \text{as } n \to \infty,$$
$$y_n/\alpha_n^{2/(4-N)} \to z_0, \quad \text{as } n \to \infty,$$
$$\min_{x\in\mathbb{R}^N}\{V_1(x)a_1 + V_2(x)a_2 + V_3(x)a_3\} = V_1(z_0)a_1 + V_2(z_0)a_2 + V_3(z_0)a_3.$$

(III) $\tau < -N(p-1)/(4-N)$
Assume that (V2) and $(V_1,V_2,V_3) \not\equiv (0,0,0)$. *Then it holds that*

$$\xi_\alpha(a_1,a_2,a_3) = \alpha^{4/(4-N)}\Sigma_0(a_1,a_2,a_3) + \frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)a_1 + V_2(x)a_2 + V_3(x)a_3\} + o(1), \quad \text{as } \alpha \to \infty.$$

Moreover let $\mathbf{u}_n$ *be a minimizing sequence for* $\xi_{\alpha_n}(a_1, a_2, a_3)$ *where* $\alpha_n \to \infty$. *For the rescaled function* $\mathbf{w}_n$ *defined by* (2.1)*, up to a subsequence, there exist a minimizer* $\mathbf{w}$ *for* $\Sigma_0(a_1, a_2, a_3)$, $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ *and* $z_0 \in \mathbb{R}^N$ *such that*

$$\|w_{j,n}(\cdot + y_n) - w_j\|_{H^1} \to 0, \quad \text{as } n \to \infty,$$
$$y_n/\alpha_n^{2/(4-N)} \to z_0, \quad \text{as } n \to \infty,$$
$$\min_{x \in \mathbb{R}^N} \{V_1(x)a_1 + V_2(x)a_2 + V_3(x)a_3\} = V_1(z_0)a_1 + V_2(z_0)a_2 + V_3(z_0)a_3.$$

Since we can prove Proposition 1 in a similar way as in the proof of Theorem 2.1, we omit the details. We note that we assume an additional condition for the bottom of the potentials in the case (iv) in Theorem 2.1. But we do not need the additional condition in Proposition 1 since the compactness of the minimizing sequence of a minimization problem for appearing in the first term of the asymptotic expansion of ξ_α aligns the translations for each component.

Part II

Fixed frequency problem

Chapter 3

Asymptotic expansion of the ground state energy for nonlinear Schrödinger system with three wave interaction

3.1 Introduction

In [43], Pomponio studied the existence of a vector ground state of the nonlinear Schrödinger system with three wave interaction:

$$\begin{cases} -\Delta u_1 + V_1(x)u_1 - |u_1|^{p-1}u_1 = \alpha u_2 u_3 & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + V_2(x)u_2 - |u_2|^{p-1}u_2 = \alpha u_1 u_3 & \text{in } \mathbb{R}^N, \\ -\Delta u_3 + V_3(x)u_3 - |u_3|^{p-1}u_3 = \alpha u_1 u_2 & \text{in } \mathbb{R}^N, \end{cases} \qquad (\mathcal{P}_{\mathbf{V}})$$

where $\mathbf{u} := (u_1, u_2, u_3)$, u_1, u_2, u_3 are real-valued functions, $\alpha \in \mathbb{R}$, $N \in \mathbb{N}, N \leq 5$ and $2 \leq p < 2^* - 1$, where 2^* is defined as follows:

$$2^* := \begin{cases} \infty & (N = 1, 2), \\ 2N/(N-2) & (N \geq 3). \end{cases}$$

Here the potential $\mathbf{V} = (V_1, V_2, V_3)$ satisfy the following conditions:

(V1) for all $i = 1, 2, 3$, $V_i \in L^\infty(\mathbb{R}^N; \mathbb{R})$.

(V2) for all $i = 1, 2, 3$, $V_i(x) \leq \lim_{|y|\to\infty} V_i(y) =: V_{i,\infty} \in \mathbb{R}$, for almost every $x \in \mathbb{R}^N$.

(V3) for all $i = 1, 2, 3$, $0 < C_i \leq V_i(x)$, for almost every $x \in \mathbb{R}^N$.

We note that the results in [43] holds even if $p = 2$, although the condition $p > 2$ was assumed in [43]. In particular, in [43], the existence of a vector ground state for α sufficiently large was shown. Also, in [43], the ground state converges to the scalar ground state as $\alpha \to 0$ was shown. However, in [43], it was not clear whether the ground state is scalar or not for small α. In this chapter, we give a positive answer to this question and moreover establish a precise asymptotic expansion of the ground state energy for $\alpha \to \infty$.

We set $\mathbb{H} = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. The solution of $(\mathcal{P}_{\mathbf{V}})$ is characterized as a critical point of the functional $I_{\mathbf{V}} : \mathbb{H} \to \mathbb{R}$ defined as follows:

$$
\begin{aligned}
I_{\mathbf{V}}(\mathbf{u}) &:= \sum_{i=1}^{3} I_{V_i}(u_i) - \alpha \int_{\mathbb{R}^N} u_1 u_2 u_3, \\
I_{V_i}(u_i) &:= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_i|^2 + V_i(x) u_i^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |u_i|^{p+1}.
\end{aligned}
$$

Now we set

$$
c_{\mathbf{V}} := \inf_{\mathbf{u} \in \mathcal{N}_{\mathbf{V}}} I_{\mathbf{V}}(\mathbf{u}),
$$

where

$$
\begin{aligned}
\mathcal{N}_{\mathbf{V}} &:= \{\mathbf{u} \in \mathbb{H} \setminus \{(0,0,0)\} \mid G_{\mathbf{V}}(\mathbf{u}) = 0\}, \\
G_{\mathbf{V}}(\mathbf{u}) &:= \sum_{i=1}^{3} \int_{\mathbb{R}^N} |\nabla u_i|^2 + V_i(x) u_i^2 - |u_i|^{p+1} - 3\alpha \int_{\mathbb{R}^N} u_1 u_2 u_3.
\end{aligned}
$$

Definition 3.1. A solution $\mathbf{u} = (u_1, u_2, u_3) \not\equiv (0,0,0)$ of $(\mathcal{P}_{\mathbf{V}})$ is called a scalar solution if there exist $i, j \in \{1, 2, 3\}$ with $i \neq j$ such that $u_i \equiv u_j \equiv 0$; while a solution $\mathbf{u}$ of $(\mathcal{P}_{\mathbf{V}})$ is called a vector solution if $u_1 \neq 0$, $u_2 \neq 0$ and $u_3 \neq 0$.

Definition 3.2. We will say that $\mathbf{u}$ is a ground state of $(\mathcal{P}_{\mathbf{V}})$ if $\mathbf{u}$ is a non-trivial solution of $(\mathcal{P}_{\mathbf{V}})$ and $I_{\mathbf{V}}(\mathbf{u}) \leq I_{\mathbf{V}}(\mathbf{w})$ holds for all non-trivial solution $\mathbf{w}$ of $(\mathcal{P}_{\mathbf{V}})$.

To state the asymptotic expansion precisely, we consider the following limit problem:

$$\begin{cases} -\Delta u_1 + V_1(x)u_1 = u_2u_3 & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + V_2(x)u_2 = u_1u_3 & \text{in } \mathbb{R}^N, \\ -\Delta u_3 + V_3(x)u_3 = u_1u_2 & \text{in } \mathbb{R}^N. \end{cases} \qquad (\tilde{\mathcal{P}}_\infty)$$

The solution of $(\tilde{\mathcal{P}}_\infty)$ is characterized as a critical point of the functional $\tilde{I}_{\mathbf{V},\infty} : \mathbb{H} \to \mathbb{R}$ defined as follows:

$$\tilde{I}_{\mathbf{V},\infty}(\mathbf{u}) := \sum_{i=1}^{3} \tilde{I}_{V_i,\infty}(u_i) - \int_{\mathbb{R}^N} u_1u_2u_3,$$
$$\tilde{I}_{V_i,\infty}(u_i) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_i|^2 + V_i(x)u_i^2.$$

Moreover we set

$$\tilde{c}_{\mathbf{V},\infty} := \inf_{\mathbf{u}\in\tilde{\mathcal{N}}_{\mathbf{V},\infty}} \tilde{I}_{\mathbf{V},\infty}(\mathbf{u}),$$

where

$$\tilde{\mathcal{N}}_{\mathbf{V},\infty} := \{\mathbf{u} \in \mathbb{H} \setminus \{(0,0,0)\} \mid \tilde{G}_{\mathbf{V},\infty}(\mathbf{u}) = 0\},$$
$$\tilde{G}_{\mathbf{V},\infty}(\mathbf{u}) := \sum_{i=1}^{3} \int_{\mathbb{R}^N} |\nabla u_i|^2 + V_i(x)u_i^2 - 3\int_{\mathbb{R}^N} u_1u_2u_3.$$

We define also a vector solution and a ground state for the problem $(\tilde{\mathcal{P}}_\infty)$ similarly.

First, we note the compactness of the minimizing sequence for $\tilde{c}_{\mathbf{V},\infty}$.

Proposition 3.3. Let $\{\mathbf{u}_n\}_{n=1}^\infty \subset \tilde{\mathcal{N}}_{\mathbf{V},\infty}$ be a minimizing sequence for $\tilde{c}_{\mathbf{V},\infty}$. Then up to a subsequence, there exist $\{\xi_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\mathbf{u} \in \mathbb{H}$ such that

$$\|u_{i,n}(\cdot + \xi_n) - u_i\|_{H^1} \to 0, \quad \text{as } n \to \infty.$$

We can take $\xi_n = 0$ for all $n \in \mathbb{N}$ if $(V_1, V_2, V_3) \not\equiv (V_{1,\infty}, V_{2,\infty}, V_{3,\infty})$. Moreover $\mathbf{u}$ is a minimizer for $\tilde{c}_{\mathbf{V},\infty}$, that is, $\mathbf{u}$ is a ground state of $(\tilde{\mathcal{P}}_\infty)$.

Note that any ground state $\mathbf{u}$ to $(\tilde{\mathcal{P}}_\infty)$ should be a vector solution of $(\tilde{\mathcal{P}}_\infty)$, that is $\mathbf{u}$ satisfies $(\tilde{\mathcal{P}}_\infty)$ and $u_i \not\equiv 0$ for all $i = 1, 2, 3$.

Now we write $c_{\mathbf{V}}$ and $\tilde{c}_{\mathbf{V},\infty}$ as c_α and $\tilde{c}_\infty$ for simplicity. Using Proposition 3.3, we established the following asymptotic expansion to c_α as $\alpha \to \infty$.

Theorem 3.4. Let $\alpha > 0$. Then

$$c_\alpha = \tilde{c}_\infty/\alpha^2 + o(1/\alpha^2), \quad \text{as } \alpha \to \infty.$$

Moreover, let $\{\alpha_n\}_{n=1}^\infty \subset (0, \infty)$ be a sequence such that $\alpha_n \to \infty$ as $n \to \infty$ and $\mathbf{u}_n$ a minimizer for c_{α_n}. Then up to a subsequence, there exist a minimizer $\mathbf{u}$ for $\tilde{c}_\infty$ and a sequence $\{\xi_n\}_{n=1}^\infty \subset \mathbb{R}^N$ such that

$$\|\alpha_n u_{i,n}(\cdot + \xi_n) - u_i\|_{H^1} \to 0, \quad \text{as } n \to \infty.$$

We can take $\xi_n = 0$ for all $n \in \mathbb{N}$ if $(V_1, V_2, V_3) \not\equiv (V_{1,\infty}, V_{2,\infty}, V_{3,\infty})$.

From Remark 1.1 in [43], it follows that c_α is an even function on $\mathbb{R}$. So we consider only for the case $\alpha \in [0, \infty)$.

We show the existence of the positive threshold α^* as follows.

Theorem 3.5. c_α is non-increasing and continuous on $[0, \infty)$. In addition, there exists $\alpha^* > 0$ such that $c_\alpha = c_0$ if $0 \le \alpha \le \alpha^*$ and $c_\alpha < c_0$ if $\alpha > \alpha^*$. Moreover, for $\alpha > \alpha^*$, all minimizer for c_α is a vector solution of $(\mathcal{P}_{\mathbf{V}})$ and c_α is strictly decreasing on (α^*, ∞). For $\alpha \in [0, \alpha^*)$, all minimizer for c_α is a scalar solution of $(\mathcal{P}_{\mathbf{V}})$.

Remark 3.1. For the case $V_1 \equiv V_2 \equiv V_3 \equiv V$, we give an upper bound of the threshold α^*.

(i) For the case $p = 2$.
It holds that $\alpha^* \le \sqrt{3} - 1$.

(ii) For the case $p \neq 2$.
Let α_0 be a unique positive solution of

$$\frac{A(\overline{u})^2}{\alpha^2 C(\overline{u})^2} \left\{ \frac{1}{2} + \frac{p-2}{p+1} \frac{A(\overline{u})^{p-1}}{\alpha^{p-1} C(\overline{u})^{p-1}} \right\} \frac{2(p+1)}{p-1} = 1,$$

where $\overline{u}$ is a positive ground state of

$$-\Delta u + V(x)u - |u|^{p-1}u = 0$$

and

$$A(\overline{u}) := \int_{\mathbb{R}^N} |\nabla \overline{u}|^2 + V(x)\overline{u}^2, \quad C(\overline{u}) := \int_{\mathbb{R}^N} |\overline{u}|^3.$$

Then we have $\alpha^* \leq \alpha_0$.

See the Appendix in [30] for the proof of Remark 3.1.

For the physical background of the nonlinear Schrödinger system with three wave interaction, see Colin-Colin [15, 16] and Colin-Colin-Ohta [18, 19]. In particular, in [18, 19], they studied the orbital stability of a standing wave for a nonlinear Schrödinger system with three wave interaction. More precisely, they revealed that the stability and instability of the standing wave solution depended on the size of the coupling parameter α (see also Colin-Ohta [17]).

Recently, there are several works on the nonlinear Schrödinger system with three wave interaction and related models. For the L^2-constrained variational problem associated with this system, see Ardila [4], Kurata-Osada [31], Osada [41] and for other related models, see e.g. Tian-Wang-Zhao [50], Wang [52], Zhao-Zhao-Shi [60] and the references therein.

The rest of this chapter is organized as follows. In Section 3.2, we show the compactness of the minimizing sequence for $\tilde{c}_{\mathbf{V},\infty}$. In Section 3.3, we prove the asymptotic behavior of a ground state of $(\mathcal{P}_{\mathbf{V}})$ and its energy c_α as $\alpha \to \infty$. In Section 3.4, we prove the existence of the positive threshold α^* of α such that the ground state of $(\mathcal{P}_{\mathbf{V}})$ is a scalar solution if $0 \leq \alpha < \alpha^*$, whereas the ground state is a vector solution if $\alpha > \alpha^*$. In Appendix, we give the proof of the continuity of c_α on the parameter $\alpha \in [0, \infty)$.

Notation

- For $r > 0$ and $x_0 \in \mathbb{R}^N$, we define $B_r(x_0) := \{x \in \mathbb{R}^N \mid |x - x_0| < r\}$.
- We denote by $H^1(\mathbb{R}^N)$ the set of real valued H^1 function.

- We denote by $\|\cdot\|_{H^1}$ the norm of $H^1(\mathbb{R}^N)$.
- We set $\mathbb{H} := H^1(\mathbb{R}^N)^3$, and set $\|\mathbf{u}\|_{\mathbb{H}}^2 := \sum_{i=1}^3 \|u_i\|_{H^1}^2$ for $\mathbf{u} \in \mathbb{H}$.

3.2 Proof of Proposition 3.3

To show Proposition 3.3, we need the following lemma which can be proved as in [43].

Lemma 3.6. Let $\{\mathbf{u}_n\}_{n=1}^\infty \subset \tilde{\mathcal{N}}_{\mathbf{V},\infty}$ be a minimizing sequence for $\tilde{c}_{\mathbf{V},\infty}$. Then $\{\mathbf{u}_n\}_{n=1}^\infty$ is bounded in $\mathbb{H}$ and it does not vanish, that is, there exists $r > 0$ such that

$$\lim_{n\to\infty} \sup_{\xi\in\mathbb{R}^N} \int_{B_r(\xi)} u_{1,n}^2 + u_{2,n}^2 + u_{3,n}^2 \neq 0.$$

We also note that, since

$$\begin{aligned} \tilde{I}_{\mathbf{V},\infty}(\mathbf{u}) &= \frac{1}{6}\sum_{i=1}^3 \int_{\mathbb{R}^N} |\nabla u_i|^2 + V_i(x)u_i^2 \\ &\geq C\|\mathbf{u}\|_{\mathbb{H}}^2, \quad \text{for all } \mathbf{u} \in \tilde{\mathcal{N}}_{\mathbf{V},\infty}, \end{aligned} \tag{3.1}$$

and there exists $C > 0$ such that $\|\mathbf{u}\|_{\mathbb{H}} \geq C$ for all $\mathbf{u} \in \tilde{\mathcal{N}}_{\mathbf{V},\infty}$ (see [43, Lemma 2.1]), $\tilde{c}_{\mathbf{V},\infty}$ is a positive constant.

We define the functional $\tilde{J}_{\mathbf{V},\infty} : \mathbb{H} \to \mathbb{R}$ as follows:

$$\tilde{J}_{\mathbf{V},\infty}(\mathbf{u}) := \frac{1}{6}\sum_{i=1}^3 \int_{\mathbb{R}^N} |\nabla u_i|^2 + V_i(x)u_i^2.$$

We need also the following lemma.

Lemma 3.7. (cf. [29, Lemma 1]) If $\tilde{G}_{\mathbf{V},\infty}(\mathbf{u}) < 0$, then it follows that $\tilde{J}_{\mathbf{V},\infty}(\mathbf{u}) > \tilde{c}_{\mathbf{V},\infty}$.

Proof. Since $\tilde{G}_{\mathbf{V},\infty}(\mathbf{u}) < 0$, it follows that $\mathbf{u} \neq (0,0,0)$. Therefore, it holds that $\int_{\mathbb{R}^N} u_1 u_2 u_3 > 0$. Thus there exists unique $\bar{t} > 0$ such that $\tilde{G}_{\mathbf{V},\infty}(\bar{t}\mathbf{u}) = 0$ by the same argument as in Lemma 2.3 in [43]. We set

$$\begin{aligned} &\tilde{G}_{\mathbf{V},\infty}(t\mathbf{u}) =: At^2 - Ct^3, \\ &A := \sum_{i=1}^{3} \int_{\mathbb{R}^N} |\nabla u_i|^2 + V_i(x) u_i^2 > 0, \\ &C := 3\alpha \int_{\mathbb{R}^N} u_1 u_2 u_3 > 0. \end{aligned}$$

Then $\tilde{G}_{\mathbf{V},\infty}(t\mathbf{u}) > 0$ if $0 < t < \bar{t}$ and $\tilde{G}_{\mathbf{V},\infty}(t\mathbf{u}) < 0$ if $t > \bar{t}$. Since $\tilde{G}_{\mathbf{V},\infty}(\mathbf{u}) < 0$, it follows that $\bar{t} < 1$. Hence we obtain

$$\tilde{c}_{\mathbf{V},\infty} \leq \tilde{J}_{\mathbf{V},\infty}(\bar{t}\mathbf{u}) = \bar{t}^2 \tilde{J}_{\mathbf{V},\infty}(\mathbf{u}) < \tilde{J}_{\mathbf{V},\infty}(\mathbf{u}).$$

□

Now we prove the compactness of the minimizing sequence for $\tilde{c}_{\mathbf{V},\infty}$ (cf. [29, Lemma 3]).

Proof of Proposition 3.3. (Step 1) First we show for the constant potential case: $\mathbf{V} = \mathbf{V}_\infty$.

Let $\{\mathbf{u}_n\}_{n=1}^\infty \subset \tilde{\mathcal{N}}_{\mathbf{V}_\infty,\infty}$ be a minimizing sequence for $\tilde{c}_{\mathbf{V}_\infty,\infty}$. From (3.1) and Lemma 3.6, $\{\mathbf{u}_n\}_{n=1}^\infty$ is bounded in $\mathbb{H}$ and it does not vanish, that is, up to a subsequence, there exist $C > 0$, $r > 0$ and $\{\xi_n\}_{n=1}^\infty \subset \mathbb{R}^N$ such that

$$\int_{B_r(\xi_n)} u_{1,n}^2 + u_{2,n}^2 + u_{3,n}^2 \geq C, \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Since $\{\mathbf{u}_n(\cdot + \xi_n)\}_{n=1}^\infty$ is bounded in $\mathbb{H}$, up to a subsequence, there exists $\mathbf{u} \in \mathbb{H}$ such that for $i = 1, 2, 3$,

$$\begin{aligned} &u_{i,n}(\cdot + \xi_n) \rightharpoonup u_i \text{ weakly in } H^1(\mathbb{R}^N), \\ &u_{i,n}(\cdot + \xi_n) \to u_i \text{ a.e. in } \mathbb{R}^N, \\ &u_{i,n}(\cdot + \xi_n) \to u_i \text{ in } L^q_{\text{loc}}(\mathbb{R}^N), \quad 1 \leq q < 2^*. \end{aligned}$$

By the lower semicontinuity, we get $\tilde{J}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) \leq \tilde{c}_{\mathbf{V}_\infty,\infty}$. From (3.2), we have $\mathbf{u} \neq (0,0,0)$. By the Brezis-Lieb Lemma, it follows that

$$0 = \tilde{G}_{\mathbf{V}_\infty,\infty}(\mathbf{u}_n) = \tilde{G}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) + \tilde{G}_{\mathbf{V}_\infty,\infty}(\mathbf{u}_n(\cdot+\xi_n) - \mathbf{u}) + o_n(1), \tag{3.3}$$

$$\tilde{J}_{\mathbf{V}_\infty,\infty}(\mathbf{u}_n) = \tilde{J}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) + \tilde{J}_{\mathbf{V}_\infty,\infty}(\mathbf{u}_n(\cdot+\xi_n) - \mathbf{u}) + o_n(1). \tag{3.4}$$

We prove $\tilde{G}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) = 0$ by contradiction.

Case 1. The case of $\tilde{G}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) > 0$. From (3.3), for n sufficiently large, it follows that $\tilde{G}_{\mathbf{V}_\infty,\infty}(\mathbf{u}_n(\cdot+\xi_n) - \mathbf{u}) < 0$. From Lemma 3.7, for n sufficiently large, we have $\tilde{J}_{\mathbf{V}_\infty,\infty}(\mathbf{u}_n(\cdot+\xi_n) - \mathbf{u}) > \tilde{c}_{\mathbf{V}_\infty,\infty}$. From (3.4), it holds that $\tilde{J}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) = 0$. Thus we have $\mathbf{u} = (0,0,0)$. This contradicts $\mathbf{u} \neq (0,0,0)$.

Case 2. The case of $\tilde{G}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) < 0$. From Lemma 3.7, we have $\tilde{J}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) > \tilde{c}_{\mathbf{V}_\infty,\infty}$. This contradicts $\tilde{J}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) \leq \tilde{c}_{\mathbf{V}_\infty,\infty}$.

From the above, we have $\tilde{G}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) = 0$. Since $\mathbf{u} \neq (0,0,0)$, it follows that $\mathbf{u} \in \tilde{\mathcal{N}}_{\mathbf{V}_\infty,\infty}$. Thus we have

$$\tilde{c}_{\mathbf{V}_\infty,\infty} \leq \tilde{I}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) \leq \liminf_{n\to\infty} \tilde{I}_{\mathbf{V}_\infty,\infty}(\mathbf{u}_n) = \tilde{c}_{\mathbf{V}_\infty,\infty}.$$

Since

$$u_{i,n}(\cdot+\xi_n) \rightharpoonup u_i \quad \text{weakly in } H^1(\mathbb{R}^N),$$
$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_{i,n}|^2 + V_{i,\infty} u_{i,n}^2 = \int_{\mathbb{R}^N} |\nabla u_i|^2 + V_{i,\infty} u_i^2,$$

we have $\|u_{i,n}(\cdot+\xi_n) - u_i\|_{H^1} \to 0$ (as $n \to \infty$). Therefore, we have $\mathbf{u} \in \tilde{\mathcal{N}}_{\mathbf{V}_\infty,\infty}$ and $\tilde{I}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) = \tilde{c}_{\mathbf{V}_\infty,\infty}$. The remaining part of the statement can be proved in the standard argument (see e.g. [43]).

Suppose that $\mathbf{v} = (v_1, v_2, v_3)$ is a ground state of $(\tilde{\mathcal{P}}_\infty)$. Since $\int_{\mathbb{R}^N} v_1 v_2 v_3 > 0$, it holds that $v_i \neq 0$ for all $i = 1,2,3$.

(Step 2) Next, we show for the case $\mathbf{V} \not\equiv \mathbf{V}_\infty$. We note that it follows that $\tilde{c}_{\mathbf{V},\infty} < \tilde{c}_{\mathbf{V}_\infty,\infty}$ if $\mathbf{V} \not\equiv \mathbf{V}_\infty$. Indeed, from Proposition 3.3 for the constant potential case, there exists a vector ground state $\mathbf{u}$ to $(\tilde{\mathcal{P}}_\infty)$ for $\mathbf{V} = \mathbf{V}_\infty$. Let $\bar{t} > 0$ be a positive constant such that $\bar{t}\mathbf{u} \in \tilde{\mathcal{N}}_{\mathbf{V},\infty}$. Then we have

$$\tilde{c}_{\mathbf{V},\infty} \leq \tilde{I}_{\mathbf{V},\infty}(\bar{t}\mathbf{u}) < \tilde{I}_{\mathbf{V}_\infty,\infty}(\bar{t}\mathbf{u}) \leq \tilde{I}_{\mathbf{V}_\infty,\infty}(\mathbf{u}) = \tilde{c}_{\mathbf{V}_\infty,\infty}.$$

Let $\{\mathbf{u}_n\}_{n=1}^{\infty} \subset \tilde{\mathcal{N}}_{\mathbf{V},\infty}$ be a minimizing sequence for $\tilde{c}_{\mathbf{V},\infty}$. We can show that $\{\mathbf{u}_n\}_{n=1}^{\infty}$ is bounded in $\mathbb{H}$ as in Step 1. Up to a subsequence, we have

$$\begin{aligned} &u_{i,n} \rightharpoonup u_i \text{ weakly in } H^1(\mathbb{R}^N), \\ &u_{i,n} \to u_i \text{ a.e. in } \mathbb{R}^N, \\ &u_{i,n} \to u_i \text{ in } L^q_{\mathrm{loc}}(\mathbb{R}^N), \quad 1 \le q < 2^*. \end{aligned}$$

Let $\{t_n\}_{n=1}^{\infty} \subset (0,\infty)$ be a sequence such that $t_n\mathbf{u}_n \in \tilde{\mathcal{N}}_{\mathbf{V}_\infty,\infty}$. Thus

$$\begin{aligned} \tilde{c}_{\mathbf{V},\infty} + o_n(1) &= \tilde{I}_{\mathbf{V},\infty}(\mathbf{u}_n) \ge \tilde{I}_{\mathbf{V},\infty}(t_n\mathbf{u}_n) \\ &= \tilde{I}_{\mathbf{V}_\infty,\infty}(t_n\mathbf{u}_n) + \frac{t_n^2}{2}\sum_{i=1}^{3}\int_{\mathbb{R}^N}(V_i(x) - V_{i,\infty})u_{i,n}^2 \\ &\ge \tilde{c}_{\mathbf{V}_\infty,\infty} + \frac{t_n^2}{2}\sum_{i=1}^{3}\int_{\mathbb{R}^N}(V_i(x) - V_{i,\infty})u_{i,n}^2. \end{aligned}$$

Suppose that $\mathbf{u} \equiv (0,0,0)$. Since $\{t_n\}_{n=1}^{\infty}$ is bounded (the proof is the same as in Lemma 3.6 in [43]) and $u_{i,n} \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and (V2),

$$\lim_{n\to\infty} \frac{t_n^2}{2}\sum_{i=1}^{3}\int_{\mathbb{R}^N}(V_i(x) - V_{i,\infty})u_{i,n}^2 = 0.$$

Hence we have $\tilde{c}_{\mathbf{V},\infty} \ge \tilde{c}_{\mathbf{V}_\infty,\infty}$. This is a contradiction to $\tilde{c}_{\mathbf{V},\infty} < \tilde{c}_{\mathbf{V}_\infty,\infty}$. Thus $\mathbf{u} \not\equiv (0,0,0)$. The rest of this proof is proved by the same argument as in Step 1. □

3.3 Proof of Theorem 3.4

Hereafter, we write $I_{\mathbf{V}}$, $\mathcal{N}_{\mathbf{V}}$, $G_{\mathbf{V}}$, $c_{\mathbf{V}}$, $\tilde{I}_{\mathbf{V},\infty}$, $\tilde{\mathcal{N}}_{\mathbf{V},\infty}$ and $\tilde{c}_{\mathbf{V},\infty}$ as I_α, $\mathcal{N}_\alpha$, G_α, c_α, $\tilde{I}_\infty$, $\tilde{\mathcal{N}}_\infty$ and $\tilde{c}_\infty$. Here we rescale the energy functional I_α and G_α and the infimum c_α as follows: Let $\mathbf{u} \in \mathbb{H}$. Set $\mathbf{w} = \alpha\mathbf{u}$. Then it follows that

$$\begin{aligned} I_\alpha(\mathbf{u}) &= \frac{1}{\alpha^2}\tilde{I}_\alpha(\mathbf{w}), \\ G_\alpha(\mathbf{u}) &= \frac{1}{\alpha^2}\tilde{G}_\alpha(\mathbf{w}), \end{aligned}$$

$$c_\alpha = \frac{1}{\alpha^2}\tilde{c}_\alpha,$$

where

$$\begin{aligned}\tilde{I}_\alpha(\mathbf{w}) &:= \frac{1}{2}\sum_{i=1}^{3}\int_{\mathbb{R}^N} |\nabla w_i|^2 + V_i(x)w_i^2 \\ &\qquad - \frac{1}{p+1}\frac{1}{\alpha^{p-1}}\sum_{i=1}^{3}\int_{\mathbb{R}^N} |w_i|^{p+1} - \int_{\mathbb{R}^N} w_1w_2w_3, \\ \tilde{c}_\alpha &:= \inf_{\mathbf{w}\in\tilde{\mathcal{N}}_\alpha} \tilde{I}_\alpha(\mathbf{w}), \\ \tilde{\mathcal{N}}_\alpha &:= \{\mathbf{w}\in\mathbb{H}\setminus\{(0,0,0)\} \mid \tilde{G}_\alpha(\mathbf{w}) = 0\}, \\ \tilde{G}_\alpha(\mathbf{w}) &:= \sum_{i=1}^{3}\int_{\mathbb{R}^N} |\nabla w_i|^2 + V_i(x)w_i^2 - \frac{1}{\alpha^{p-1}}|w_i|^{p+1} - 3\int_{\mathbb{R}^N} w_1w_2w_3.\end{aligned}$$

Therefore, the proof of Theorem 3.4 is reduced to the proof of the following proposition.

Proposition 3.8. It follows that $\tilde{c}_\alpha \to \tilde{c}_\infty$ as $\alpha \to \infty$. Moreover let $\{\alpha_n\}_{n=1}^\infty \subset (0,\infty)$ be a sequence such that $\alpha_n \to \infty$ as $n \to \infty$ and $\mathbf{w}_n$ a minimizer for $\tilde{c}_{\alpha_n}$. Then up to a subsequence, there exist a minimizer $\mathbf{w}$ for $\tilde{c}_\infty$ and $\{\xi_n\}_{n=1}^\infty \subset \mathbb{R}^N$ such that

$$\|w_{i,n}(\cdot + \xi_n) - w_i\|_{H^1} \to 0, \quad \text{as } n \to \infty.$$

We can take $\xi_n = 0$ for all $n \in \mathbb{N}$ if $\mathbf{V} \not\equiv \mathbf{V}_\infty$.

***Proof*.** **(Step 1)** First we note the upper bound $\tilde{c}_\alpha \le \tilde{c}_\infty$ for all $\alpha > 0$. Let $\mathbf{w}$ be a minimizer for $\tilde{c}_\infty$ and s_α a positive constant such that $s_\alpha\mathbf{w} \in \tilde{\mathcal{N}}_\alpha$. Then we have

$$\tilde{c}_\infty = \tilde{I}_\infty(\mathbf{w}) \ge \tilde{I}_\infty(s_\alpha\mathbf{w}) \ge \tilde{I}_\alpha(s_\alpha\mathbf{w}) \ge \tilde{c}_\alpha.$$

(Step 2) Next, we show the lower bound $\tilde{c}_\alpha \ge \tilde{c}_\infty + o(1)$ as $\alpha \to \infty$. To show this, first we show that there exists $C > 0$ such that for all $\alpha \ge 1$ and $\mathbf{w} \in \tilde{\mathcal{N}}_\alpha$, $\|\mathbf{w}\|_{\mathbb{H}} \ge C$. Let $\alpha \ge 1$ and $\mathbf{w} \in \tilde{\mathcal{N}}_\alpha$. Then we have

$$C\|\mathbf{w}\|_{\mathbb{H}}^2 \le \sum_{i=1}^{3}\int_{\mathbb{R}^N} |\nabla w_i|^2 + V_i(x)w_i^2 = \frac{1}{\alpha^{p-1}}\sum_{i=1}^{3}\int_{\mathbb{R}^N} |w_i|^{p+1} + 3\int_{\mathbb{R}^N} w_1w_2w_3$$

$$\leq C(\|\mathbf{w}\|_{\mathbb{H}}^{p+1} + \|\mathbf{w}\|_{\mathbb{H}}^{3}).$$

Thus, there exists $C > 0$ such that for all $\alpha \geq 1$, $\|\mathbf{w}\|_{\mathbb{H}} \geq C$ by the same argument as in Lemma 2.1 in [43].

Let $\mathbf{w}_\alpha$ be a minimizer for $\tilde{c}_\alpha$. Next we show that there exist $\alpha_0 \geq 1$ and $C > 0$ such that for all $\alpha \geq \alpha_0$,

$$\int_{\mathbb{R}^N} w_{1,\alpha} w_{2,\alpha} w_{3,\alpha} \geq C.$$

Indeed, from the upper bound for $\tilde{c}_\alpha$,

$$\begin{aligned}\tilde{c}_\infty \geq \tilde{c}_\alpha &= \frac{1}{6}\sum_{i=1}^{3}\int_{\mathbb{R}^N} |\nabla w_{i,\alpha}|^2 + V_i(x) w_{i,\alpha}^2 + \frac{1}{\alpha^{p-1}}\frac{p-2}{3(p+1)}\int_{\mathbb{R}^N} |w_{i,\alpha}|^{p+1} \\ &\geq C\|\mathbf{w}_\alpha\|_{\mathbb{H}}^2, \quad \text{for all } \alpha > 0.\end{aligned}$$

Thus $\{\mathbf{w}_\alpha\}_{\alpha>0}$ is bounded in $\mathbb{H}$. Since $\mathbf{w}_\alpha \in \tilde{\mathcal{N}}_\alpha$,

$$\begin{aligned}C \leq C\|\mathbf{w}_\alpha\|_{\mathbb{H}}^2 &\leq \sum_{i=1}^{3}\int_{\mathbb{R}^N} |\nabla w_{i,\alpha}|^2 + V_i(x) w_{i,\alpha}^2 \\ &= \frac{1}{\alpha^{p-1}}\sum_{i=1}^{3}\int_{\mathbb{R}^N} |w_{i,\alpha}|^{p+1} + 3\int_{\mathbb{R}^N} w_{1,\alpha} w_{2,\alpha} w_{3,\alpha} \\ &\leq \frac{1}{\alpha^{p-1}} C + 3\int_{\mathbb{R}^N} w_{1,\alpha} w_{2,\alpha} w_{3,\alpha}.\end{aligned}$$

Hence for α sufficiently large, we have

$$\int_{\mathbb{R}^N} w_{1,\alpha} w_{2,\alpha} w_{3,\alpha} \geq C. \tag{3.5}$$

For α sufficiently large, let $t_\alpha > 0$ be a positive constant $t_\alpha \mathbf{w}_\alpha \in \tilde{\mathcal{N}}_\infty$ (this fact is proved by the same argument as in Lemma 2.3 in [43]). It follows that

$$\sum_{i=1}^{3}\int_{\mathbb{R}^N} |\nabla w_{i,\alpha}|^2 + V_i(x) w_{i,\alpha}^2 = 3t_\alpha \int_{\mathbb{R}^N} w_{1,\alpha} w_{2,\alpha} w_{3,\alpha}. \tag{3.6}$$

Since $\mathbf{w}_\alpha \in \tilde{\mathcal{N}}_\alpha$, we have

$$\sum_{i=1}^{3}\int_{\mathbb{R}^N}|\nabla w_{i,\alpha}|^2+V_i(x)w_{i,\alpha}^2=\frac{1}{\alpha^{p-1}}\sum_{i=1}^{3}\int_{\mathbb{R}^N}|w_{i,\alpha}|^{p+1}+3\int_{\mathbb{R}^N}w_{1,\alpha}w_{2,\alpha}w_{3,\alpha}. \tag{3.7}$$

From (3.5),(3.6),(3.7),

$$3(t_\alpha-1)\int_{\mathbb{R}^N}w_{1,\alpha}w_{2,\alpha}w_{3,\alpha}=\frac{1}{\alpha^{p-1}}\sum_{i=1}^{3}\int_{\mathbb{R}^N}|w_{i,\alpha}|^{p+1},$$

$$0\le t_\alpha-1\le\frac{C}{\alpha^{p-1}},$$

that is, $t_\alpha \to 1$ as $\alpha\to\infty$. Thus we have

$$\tilde{c}_\alpha=\tilde{I}_\alpha(\mathbf{w}_\alpha)\ge\tilde{I}_\alpha(t_\alpha\mathbf{w}_\alpha)=\tilde{I}_\infty(t_\alpha\mathbf{w}_\alpha)+o(1)\ge\tilde{c}_\infty+o(1),\quad \text{as } \alpha\to\infty.$$

Hence we have

$$\tilde{c}_\alpha\to\tilde{c}_\infty,\quad \text{as } \alpha\to\infty,$$
$$\tilde{I}_\infty(t_\alpha\mathbf{w}_\alpha)\to\tilde{c}_\infty,\quad \text{as } \alpha\to\infty.$$

(Step 3) Let $\{\alpha_n\}_{n=1}^\infty\subset(0,\infty)$ be a sequence such that $\alpha_n\to\infty$ as $n\to\infty$. Let $\mathbf{w}_n$ be a minimizer for $\tilde{c}_{\alpha_n}$. By the same argument as in Step 2, for α_n sufficiently large, there exists $t_n>0$ such that

$$\|\mathbf{w}_n\|_{\mathbb{H}}^2\le C,\quad \text{for all } n\in\mathbb{N}, \tag{3.8}$$
$$t_n\to 1,\ \text{as } n\to\infty, \tag{3.9}$$
$$t_n\mathbf{w}_n\in\tilde{\mathcal{N}}_\infty,\quad \text{for all } n\in\mathbb{N},$$
$$\tilde{I}_\infty(t_n\mathbf{w}_n)\to\tilde{c}_\infty,\quad \text{as } n\to\infty.$$

Thus from Proposition 3.3, up to a subsequence, there exist a minimizer $\mathbf{w}$ for $\tilde{c}_\infty$ and $\{\xi_n\}_{n=1}^\infty\subset\mathbb{R}^N$ (if $\mathbf{V}\not\equiv\mathbf{V}_\infty$, then we can take $\xi_n=0$ for all $n\in\mathbb{N}$) such that

$$\|t_nw_{i,n}(\cdot+\xi_n)-w_i\|_{H^1}\to 0,\quad \text{as } n\to\infty.$$

Moreover noting that (3.8) and (3.9),

$$\|t_nw_{i,n}(\cdot+\xi_n)-w_{i,n}(\cdot+\xi_n)\|_{H^1}\to 0,\quad \text{as } n\to\infty.$$

Thus we have

$$\|w_{i,n}(\cdot+\xi_n)-w_i\|_{H^1}\to 0,\quad \text{as } n\to\infty.$$

□

3.4 Proof of Theorem 3.5

Before going to give the proof of Theorem 3.5, we give several remarks in the following lemmas.

Lemma 3.9. Let $\alpha \in \mathbb{R}$ and $\mathbf{u}$ a minimizer for c_α such that $\alpha \int_{\mathbb{R}^N} u_1 u_2 u_3 = 0$ (if $\alpha = 0$, it satisfies automatically). Then it follows that $c_\alpha = \min_{i=1,2,3} c(i)$ and $\mathbf{u} = (u_1, 0, 0)$ or $(0, u_2, 0)$ or $(0, 0, u_3)$. Here

$$\mathcal{N}(i) := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid I'_{V_i}(u)[u] = 0\},$$
$$c(i) := \inf_{u \in \mathcal{N}(i)} I_{V_i}(u) > 0.$$

Proof. Since $\mathbf{u}$ is a ground state of $(\mathcal{P}_{\mathbf{V}})$,

$$\begin{cases} -\Delta u_1 + V_1(x) u_1 - |u_1|^{p-1} u_1 = \alpha u_2 u_3, \\ -\Delta u_2 + V_2(x) u_2 - |u_2|^{p-1} u_2 = \alpha u_1 u_3, \\ -\Delta u_3 + V_3(x) u_3 - |u_3|^{p-1} u_3 = \alpha u_1 u_2 \end{cases}$$

holds. Since $\alpha \int_{\mathbb{R}^N} u_1 u_2 u_3 = 0$, we have

$$\int_{\mathbb{R}^N} |\nabla u_i|^2 + V_i(x) u_i^2 - \int_{\mathbb{R}^N} |u_i|^{p+1} = 0.$$

Then it follows that

$$I_{V_i}(u_i) = \frac{p-1}{2(p+1)} \int_{\mathbb{R}^N} |\nabla u_i|^2 + V_i(x) u_i^2 \geq 0.$$

Since $\mathbf{u} \neq (0,0,0)$, there exists $i \in \{1,2,3\}$ such that $I_{V_i}(u_i) \geq c(i)$ (> 0). Therefore we have

$$c_\alpha = I_\alpha(\mathbf{u}) = \sum_{i=1}^{3} I_{V_i}(u_i) \geq \min_{i=1,2,3} c(i).$$

Let $i_0 \in \{1,2,3\}$ be an index such that $c(i_0) = \min_{i=1,2,3} c(i)$. Let $\overline{u} \in \mathcal{N}(i_0)$ be a function such that $I_{V_{i_0}}(\overline{u}) = \inf_{u \in \mathcal{N}(i_0)} I_{V_{i_0}}(u) (= c(i_0))$. For simplicity, we assume that $i_0 = 1$. Since $(\overline{u}, 0, 0) \in \mathcal{N}_\alpha$, we have

$$c_\alpha \leq I_\alpha(\overline{u}, 0, 0) = I_{V_1}(\overline{u}) = \min_{i=1,2,3} c(i).$$

Hence it holds $c_\alpha = \min_{i=1,2,3} c(i)$.

If $u_1 \neq 0$, $u_2 \neq 0$ and $u_3 \neq 0$, $c_\alpha = I_\alpha(\mathbf{u}) \geq \sum_{i=1}^{3} c(i) > \min_{i=1,2,3} c(i)$. This is a contradiction to $c_\alpha = \min_{i=1,2,3} c(i)$. We can also rule out the case that two components are non-zero. Therefore it follows that $\mathbf{u} = (u_1, 0, 0)$ or $(0, u_2, 0)$ or $(0, 0, u_3)$. □

Lemma 3.10. $(\mathcal{P}_{\mathbf{V}})$ has a ground state.

Proof. For the case $\mathbf{V} \equiv \mathbf{V}_\infty$ or the case $c_{\mathbf{V}} < c_{\mathbf{V}_\infty}$, the result of [43] implies the existence of a ground state of $(\mathcal{P}_{\mathbf{V}})$. We also note that the statement holds even for the general cases for the sake of completeness. So let $\mathbf{V} \not\equiv \mathbf{V}_\infty$. If $c_{\mathbf{V}_\infty}$ has a vector ground state, then $c_{\mathbf{V}} < c_{\mathbf{V}_\infty}$ holds and hence $(\mathcal{P}_{\mathbf{V}})$ has a ground state. Finally, assume that all ground state of $(\mathcal{P}_{\mathbf{V}_\infty})$ is a scalar solution. Then for $V_{i_0,\infty} = \min_{i=1,2,3} V_{i,\infty}$ we have a scalar ground state $\mathbf{u}$ to $(\mathcal{P}_{\mathbf{V}_\infty})$ with $u_{i_0} \neq 0$. Moreover, we may assume $V_{i_0} \equiv V_{i_0,\infty}$ and $c_{\mathbf{V}} = c_{\mathbf{V}_\infty}$. So $\mathbf{u}$ itself is a ground state of $(\mathcal{P}_{\mathbf{V}})$. □

Although Pomponio [43] proved the following lemma only for the cases $\mathbf{V} \equiv \mathbf{V}_\infty$ or $V_i \not\equiv V_{i,\infty}$ for all $i = 1, 2, 3$, the same argument yields the following statement even for the general cases.

Lemma 3.11. Let $\{\alpha_n\}_{n=1}^\infty \subset (0, \infty)$ with $\alpha_n \to 0$ as $n \to \infty$ and $\mathbf{u}_n$ a ground state of $(\mathcal{P}_{\mathbf{V}})$ for $\alpha = \alpha_n$. Then there exist $i \neq j$ such that $u_{i,n}, u_{j,n} \to 0$ in $H^1(\mathbb{R}^N)$.

Proof of Theorem 3.5. Claim 1. c_α is non-increasing on $[0, \infty)$. We can prove this claim by the same argument as in the proof of Lemma 2.5 in [43].

Claim 2. c_α is continuous on $[0, \infty)$. We omit the proof. See the proof of the continuity of c_α in Appendix in this chapter.

Claim 3. There exists $\alpha^* \geq 0$ such that $c_\alpha = c_0$ $(0 \leq \alpha \leq \alpha^*)$ and $c_\alpha < c_0$ $(\alpha > \alpha^*)$. $\{\alpha \geq 0 \mid c_\alpha = c_0\}$ is bounded closed interval. Indeed since $c_\alpha = \tilde{c}_\infty/\alpha^2 + o(1/\alpha^2)$ as $\alpha \to \infty$, it is bounded set. Being closed set follows from the continuity of c_α. Being interval follows from monotonicity of c_α. Hence there exists a maximum of $\{\alpha \geq 0 \mid c_\alpha = c_0\}$. We define

$\alpha^* := \max\{\alpha \geq 0 \mid c_\alpha = c_0\}$. Then $c_\alpha = c_0$ if $0 \leq \alpha \leq \alpha^*$, and $c_\alpha < c_0$ if $\alpha > \alpha^*$.

Claim 4. $\alpha^* > 0$. If not, there exists $\alpha_n > 0$ with $\alpha_n \to 0$ as $n \to \infty$ such that $(u_{1,n}, u_{2,n}, u_{3,n})$ is a ground state of $(\mathcal{P}_{\mathbf{V}})$ with $u_{i,n} \neq 0$ for any $i = 1, 2, 3$. By Lemma 3.11, we may assume $u_{2,n}, u_{3,n} \to 0$ in $H^1(\mathbb{R}^N)$. We also have $\{u_{1,n}\}_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$. Now, we have

$$\int_{\mathbb{R}^N} |\nabla u_{2,n}|^2 + V_2(x)u_{2,n}^2 = \int_{\mathbb{R}^N} |u_{2,n}|^{p+1} + \alpha_n \int_{\mathbb{R}^N} u_{1,n}u_{2,n}u_{3,n},$$
$$\int_{\mathbb{R}^N} |\nabla u_{3,n}|^2 + V_3(x)u_{3,n}^2 = \int_{\mathbb{R}^N} |u_{3,n}|^{p+1} + \alpha_n \int_{\mathbb{R}^N} u_{1,n}u_{2,n}u_{3,n}.$$

It follows

$$C_1(\|u_{2,n}\|_{H^1}^2 + \|u_{3,n}\|_{H^1}^2) \leq \int_{\mathbb{R}^N} |u_{2,n}|^{p+1} + |u_{3,n}|^{p+1} + 2\alpha_n \int_{\mathbb{R}^N} u_{1,n}u_{2,n}u_{3,n}.$$

Here we note by the Sobolev embedding theorem

$$\begin{aligned} 2\int_{\mathbb{R}^N} |u_{1,n}||u_{2,n}||u_{3,n}| &\leq \int_{\mathbb{R}^N} |u_{1,n}|(|u_{2,n}|^2 + |u_{3,n}|^2) \\ &\leq C_2\|u_{1,n}\|_{L^3}(\|u_{2,n}\|_{H^1}^2 + \|u_{3,n}\|_{H^1}^2) \\ &\leq C_3(\|u_{2,n}\|_{H^1}^2 + \|u_{3,n}\|_{H^1}^2). \end{aligned}$$

Take $\alpha_0 > 0$ so that $\alpha_0 C_3 \leq C_1/2$. Then, for $0 < \alpha_n < \alpha_0$, we have

$$\begin{aligned} \frac{C_1}{2}(\|u_{2,n}\|_{H^1}^2 + \|u_{3,n}\|_{H^1}^2) &\leq \int_{\mathbb{R}^N} |u_{2,n}|^{p+1} + |u_{3,n}|^{p+1} \\ &\leq C_4\left(\|u_{2,n}\|_{H^1}^2 + \|u_{3,n}\|_{H^1}^2\right)^{(p+1)/2}. \end{aligned}$$

Since $u_{2,n} \neq 0$ and $u_{3,n} \neq 0$, we obtain

$$\frac{1}{2} \leq C\left(\|u_{2,n}\|_{H^1}^2 + \|u_{3,n}\|_{H^1}^2\right)^{(p-1)/2}$$

which contradicts $u_{2,n}, u_{3,n} \to 0$ in $H^1(\mathbb{R}^N)$.

Claim 5. For $\alpha > \alpha^*$, for any minimizer for c_α is a vector solution and c_α is strictly decreasing. Moreover, for any minimizer for c_α on $(0, \alpha^*)$ is a scalar

solution.
Let $\alpha > \alpha^*$. It follows that $c_\alpha < c_0$. Let $\mathbf{u}$ be a minimizer for c_α. From $c_\alpha < c_0$, $\mathbf{u}$ is a vector solution. Let $\alpha_1, \alpha_2 > \alpha^*$ with $\alpha_1 < \alpha_2$. Let $\mathbf{u}$ be a minimizer for c_{α_1}. Since $c_{\alpha_1} < c_0$, $\mathbf{u}$ is a vector solution and it satisfies

$$\max_{t>0} I_{\alpha_1}(t\mathbf{u}) = I_{\alpha_1}(\mathbf{u}),$$
$$\int_{\mathbb{R}^N} u_1 u_2 u_3 > 0.$$

Let $\bar{t} > 0$ be a positive constant such that $\bar{t}\mathbf{u} \in \mathcal{N}_{\alpha_2}$. Then we have

$$c_{\alpha_1} = I_{\alpha_1}(\mathbf{u}) \geq I_{\alpha_1}(\bar{t}\mathbf{u}) > I_{\alpha_2}(\bar{t}\mathbf{u}) \geq c_{\alpha_2}.$$

Now we show that for any minimizer for c_α on $(0, \alpha^*)$ is scalar solution. If not, there exists $\alpha_1 \in (0, \alpha^*)$ such that there exists a minimizer for c_{α_1} such that it is vector solution. Then as before, $c_{\alpha_1} > c_{\alpha_2}$ for $\alpha_1 < \alpha_2 < \alpha^*$. This is a contradiction to $c_\alpha = c_0$ for all $\alpha \in [0, \alpha^*]$. □

3.5 Appendix

In this Appendix, we prove the continuity of c_α on the parameter $\alpha \in [0, \infty)$. For a similar argument, see Zhao-Zhao-Shi [60, Lemma 4.1].

Proof of the continuity of c_α on $[0, \infty)$. Let $\alpha_0 \geq 0$. We now prove that the continuity of c_α at $\alpha = \alpha_0$. Assume that $\alpha \geq 0$ and α is sufficiently close to α_0. From [43], there exists a non-negative ground state $\mathbf{u}$ to $(\mathcal{P}_{\mathbf{V}})$ with $\alpha = \alpha_0$. In addition, let $t_\alpha > 0$ be a positive constant such that $t_\alpha \mathbf{u} \in \mathcal{N}_\alpha$. Then we have

$$\sum_{i=1}^{3} \int_{\mathbb{R}^N} |\nabla u_i|^2 + V_i(x) u_i^2 - t_\alpha^{p-1} \sum_{i=1}^{3} \int_{\mathbb{R}^N} |u_i|^{p+1} - 3 t_\alpha \alpha \int_{\mathbb{R}^N} u_1 u_2 u_3 = 0.$$

Therefore $\{t_\alpha\}_\alpha$ is bounded. So we have

$$(\alpha - \alpha_0) \int_{\mathbb{R}^N} t_\alpha u_1 t_\alpha u_2 t_\alpha u_3 = o(1), \quad \text{as } \alpha \to \alpha_0.$$

Thus we have

$$c_\alpha \le I_\alpha(t_\alpha \mathbf{u}) = I_{\alpha_0}(t_\alpha \mathbf{u}) + o(1) \le I_{\alpha_0}(\mathbf{u}) + o(1) = c_{\alpha_0} + o(1), \quad \text{as } \alpha \to \alpha_0. \tag{3.10}$$

On the other hand, let $\{\alpha_n\}_{n=1}^\infty$ be any non-negative sequence such that $\alpha_n \to \alpha_0$ as $n \to \infty$ and $\mathbf{u}_{\alpha_n}$ a non-negative minimizer for c_{α_n}. Then

$$\begin{aligned} &\frac{1}{6}\sum_{i=1}^{3}\int_{\mathbb{R}^N} |\nabla u_{i,\alpha_n}|^2 + V_i(x)u_{i,\alpha_n}^2 + \frac{p-2}{3(p+1)}\sum_{i=1}^{3}\int_{\mathbb{R}^N} |u_{i,\alpha_n}|^{p+1} \\ &= I_{\alpha_n}(\mathbf{u}_{\alpha_n}) = c_{\alpha_n} \le c_{\alpha_0} + o(1), \quad \text{as } n \to \infty. \end{aligned}$$

Hence $\{\mathbf{u}_{\alpha_n}\}_{n=1}^\infty$ is bounded in $\mathbb{H}$. Let $t_{\alpha_n} > 0$ be a positive constant such that $t_{\alpha_n}\mathbf{u}_{\alpha_n} \in \mathcal{N}_{\alpha_0}$. Then

$$\begin{aligned} &\sum_{i=1}^{3}\int_{\mathbb{R}^N} |\nabla u_{i,\alpha_n}|^2 + V_i(x)u_{i,\alpha_n}^2 - t_{\alpha_n}^{p-1}\sum_{i=1}^{3}\int_{\mathbb{R}^N} |u_{i,\alpha_n}|^{p+1} \\ &\quad - 3t_{\alpha_n}\alpha_0\int_{\mathbb{R}^N} u_{1,\alpha_n}u_{2,\alpha_n}u_{3,\alpha_n} = 0. \end{aligned} \tag{3.11}$$

Since c_α is non-increasing on $[0,\infty)$, there exists $\sigma > 0$ such that $c_{\alpha_0+\sigma} \le c_{\alpha_n}$ for all $n \in \mathbb{N}$. Therefore up to a subsequence,

$$\sum_{i=1}^{3}\int_{\mathbb{R}^N} |u_{i,\alpha_n}|^{p+1} \ge C, \quad \text{for all } n \in \mathbb{N}. \tag{3.12}$$

Indeed, if

$$\lim_{n\to\infty}\sum_{i=1}^{3}\int_{\mathbb{R}^N} |u_{i,\alpha_n}|^{p+1} = 0,$$

by Hölder's inequality, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} u_{1,\alpha_n}u_{2,\alpha_n}u_{3,\alpha_n} = 0.$$

Therefore

$$0 < c_{\alpha_0+\sigma} \le c_{\alpha_n} = I_{\alpha_n}(\mathbf{u}_{\alpha_n}) - \frac{1}{2}G_{\alpha_n}(\mathbf{u}_{\alpha_n})$$

$$= \frac{p-1}{2(p+1)} \sum_{i=1}^{3} \int_{\mathbb{R}^N} |u_{i,\alpha_n}|^{p+1} + \frac{\alpha_n}{2} \int_{\mathbb{R}^N} u_{1,\alpha_n} u_{2,\alpha_n} u_{3,\alpha_n} \to 0, \quad \text{as } n \to \infty.$$

This is a contradiction. Thus (3.12) holds. From (3.11), $\{t_{\alpha_n}\}_{n=1}^{\infty}$ is bounded. Thus

$$\begin{aligned} c_{\alpha_0} &\le I_{\alpha_0}(t_{\alpha_n}\mathbf{u}_{\alpha_n}) = I_{\alpha_n}(t_{\alpha_n}\mathbf{u}_{\alpha_n}) + o(1) \\ &\le I_{\alpha_n}(\mathbf{u}_{\alpha_n}) + o(1) = c_{\alpha_n} + o(1), \quad \text{as } n \to \infty. \end{aligned} \tag{3.13}$$

From (3.10) and (3.13), it implies that $c_\alpha \to c_{\alpha_0}$ as $\alpha \to \alpha_0$. □

Chapter 4

A singular perturbation problem for a nonlinear Schrödinger system with three wave interaction

4.1 Introduction and main results

In this chapter, we consider the following nonlinear Schrödinger system with three wave interaction:

$$\begin{cases} -\varepsilon^2 \Delta u_1 + V_1(x) u_1 = |u_1|^{p-1} u_1 + \alpha u_2 u_3 & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta u_2 + V_2(x) u_2 = |u_2|^{p-1} u_2 + \alpha u_1 u_3 & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta u_3 + V_3(x) u_3 = |u_3|^{p-1} u_3 + \alpha u_1 u_2 & \text{in } \mathbb{R}^N, \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

where $N \leq 5$, $2 \leq p < 2^* - 1$, $2^* = \infty$ $(N \leq 2)$, $2^* = 2N/(N-2)$ $(N \geq 3)$, $\varepsilon > 0$, $\alpha \geq 0$. We also assume the following basic conditions for the potentials V_j $(j = 1, 2, 3)$:

(V1) for all $j = 1, 2, 3$, $V_j \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$,

(V2) for all $j = 1, 2, 3$, $0 < V_{j,0} := \inf_{x \in \mathbb{R}^N} V_j(x) < \lim_{|x| \to \infty} V_j(x) =: V_{j,\infty}$.

We define the following functional and least energy for $(\mathcal{P}_\varepsilon)$:

$$\begin{aligned}
\mathbf{u} &:= (u_1, u_2, u_3), \quad \mathbb{H} := H^1(\mathbb{R}^N)^3, \\
I_\varepsilon(\mathbf{u}) &:= \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} \varepsilon^2|\nabla u_j|^2 + V_j(x)u_j^2 \\
&\qquad - \frac{1}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|u_j|^{p+1} - \alpha\int_{\mathbb{R}^N} u_1u_2u_3, \\
c_\varepsilon &:= \inf_{\mathbf{u}\in\mathcal{N}_\varepsilon} I_\varepsilon(\mathbf{u}), \\
\mathcal{N}_\varepsilon &:= \{\mathbf{u}\in\mathbb{H}\setminus\{(0,0,0)\} \mid G_\varepsilon(\mathbf{u}) = 0\}, \\
G_\varepsilon(\mathbf{u}) &:= \sum_{j=1}^{3}\int_{\mathbb{R}^N} \varepsilon^2|\nabla u_j|^2 + V_j(x)u_j^2 - |u_j|^{p+1} - 3\alpha\int_{\mathbb{R}^N} u_1u_2u_3.
\end{aligned}$$

$(\mathcal{P}_\varepsilon)$ is related to standing wave solutions of the following time dependent nonlinear Schrödinger system with three wave interaction:

$$\begin{cases}
i\varepsilon\partial_t v_1 + \varepsilon^2\Delta v_1 - \tilde{V}_1(x)v_1 + |v_1|^{p-1}v_1 = -\alpha\bar{v}_2v_3 & \text{in } \mathbb{R}\times\mathbb{R}^N, \\
i\varepsilon\partial_t v_2 + \varepsilon^2\Delta v_2 - \tilde{V}_2(x)v_2 + |v_2|^{p-1}v_2 = -\alpha\bar{v}_1v_3 & \text{in } \mathbb{R}\times\mathbb{R}^N, \\
i\varepsilon\partial_t v_3 + \varepsilon^2\Delta v_3 - \tilde{V}_3(x)v_3 + |v_3|^{p-1}v_3 = -\alpha v_1v_2 & \text{in } \mathbb{R}\times\mathbb{R}^N.
\end{cases} \tag{4.1}$$

Indeed, if $(v_1(t,x), v_2(t,x), v_3(t,x)) = (e^{i\omega_1 t/\varepsilon}u_1(x), e^{i\omega_2 t/\varepsilon}u_2(x), e^{i\omega_3 t/\varepsilon}u_3(x))$ with $\omega_3 = \omega_1 + \omega_2$ is a solution of (4.1), then (u_1, u_2, u_3) satisfies $(\mathcal{P}_\varepsilon)$ with $V_j(x) = \omega_j + \tilde{V}_j(x)$, where u_1, u_2, u_3 are real-valued functions. The system (4.1) was introduced by Colin-Colin-Ohta [19] with $V_j(x) \equiv 0$ and $\varepsilon = 1$ (see also [15, 16]). Colin-Colin-Ohta [18, 19] showed that the standing wave solutions $(e^{i\omega t}\varphi, 0, 0)$ and $(0, e^{i\omega t}\varphi, 0)$ are orbitally stable for all $\alpha > 0$, where $\omega > 0$ and φ is the unique positive radial solution of

$$-\Delta v + \omega v - |v|^{p-1}v = 0 \quad \text{in } \mathbb{R}^N.$$

On the other hand, $(0, 0, e^{i\omega t}\varphi)$ is orbitally stable if $0 < \alpha < \alpha^*$ and is orbitally unstable if $\alpha > \alpha^*$ where α^* is a suitable positive constant (see [18, 19] for more detail). For other studies on nonlinear Schrödinger system with three wave interaction, see [4, 30, 31, 37, 41, 42, 43, 51] and the references therein.

Rabinowitz [44] showed that there exists a ground state solution of

$$-\varepsilon^2 \Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N \tag{4.2}$$

for ε sufficiently small if $0 < \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to\infty} V(x)$, where f satisfies suitable conditions (see [44] for more detail). Here we say that u is a ground state of (4.2) if u is a nontrivial solution with least energy

$$\frac{1}{2}\int_{\mathbb{R}^N} \varepsilon^2|\nabla u|^2 + V(x)u^2 - \int_{\mathbb{R}^N} F(x,u)$$

among all nontrivial $H^1(\mathbb{R}^N)$ solutions of (4.2), where

$$F(x,u) := \int_0^u f(x,t)\,dt.$$

Wang [54] studied the concentration behavior of positive ground state solutions of (4.2) for the case $f(x,u) = |u|^{p-1}u$. That solutions concentrate at a global minimum point of V as $\varepsilon \to +0$, have a unique local maximum (hence global maximum) point and exponential decay rapidly around the minimum point.

Lin-Wei [33] considered the following nonlinear Schrödinger system

$$\begin{cases} -\varepsilon^2\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \Omega, \\ -\varepsilon^2\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \Omega, \\ u_1, u_2 > 0 \quad \text{in } \Omega, \\ u_1 = u_2 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain. They showed that as $\varepsilon \to +0$, there are two spikes for both $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$, where $(u_{1,\varepsilon}, u_{2,\varepsilon})$ is a ground state of (4.3). If $\beta < 0$, the locations of two spikes reach a sphere-packing position (the positions that maximize the minimum distance from the boundary and the distance from each other) in the domain Ω. On the other hand, if $\beta > 0$, the locations of two spikes reach the innermost part (the farthest part from the boundary) of the domain.

Lin-Wei [34] considered the following system with potentials:

$$\begin{cases} -\varepsilon^2\Delta u_1 + V_1(x) u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbb{R}^N, \\ -\varepsilon^2\Delta u_2 + V_2(x) u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \mathbb{R}^N, \\ u_1, u_2 > 0 \quad \text{in } \mathbb{R}^N. \end{cases} \tag{4.4}$$

For this system, they showed the spikes are trapped at the minimum points of $V_j(x)$ if $\beta < 0$. On the other hand, if $\beta > 0$, they introduced a certain function $\rho(V_1(x), V_2(x); \beta)$ and the spikes are trapped at the minimum points of $\rho(V_1(x), V_2(x); \beta)$ or trapped at the minimum points of $V_j(x)$.

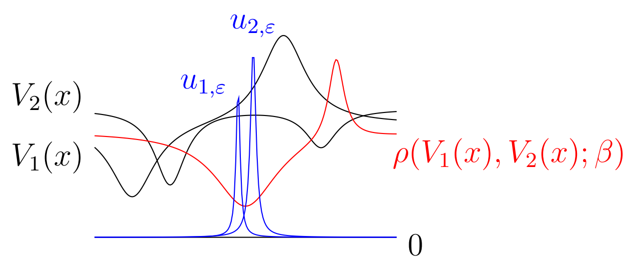

Figure 4.1: $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x); \beta) < d_1^{V_{1,0}} + d_1^{V_{2,0}}$

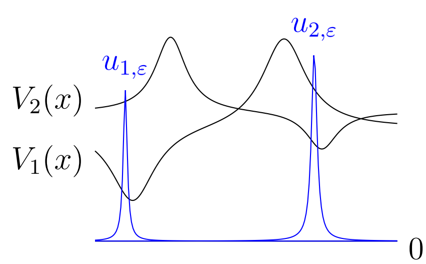

Figure 4.2: $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x); \beta) > d_1^{V_{1,0}} + d_1^{V_{2,0}}$

Here, $\rho(V_1(x_0), V_2(x_0); \beta)$ and $d_1^{V_{j,0}}$ are the least energies of the following equations respectively:

$$\begin{cases} -\Delta u_1 + V_1(x_0)u_1 = u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + V_2(x_0)u_2 = u_2^3 + \beta u_1^2 u_2 & \text{in } \mathbb{R}^N, \\ u_1 > 0, \quad u_2 > 0 & \text{in } \mathbb{R}^N \end{cases}$$

and

$$\begin{cases} -\Delta u + V_{j,0}u = u^3 & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N. \end{cases}$$

The least energy means the energy which ground state has.

Montefusco-Pellacci-Squassina [39] considered (4.4) for the case $N = 3$. They showed that the least energy solution of (4.4) converges (up to scalings) to a least energy solution of corresponding limit problem as $\varepsilon \to +0$. They adopt a definition of Nehari manifolds similar to Pomponio [43] and ours. They also proved that if β is sufficiently large, then the limit state is vector, on the other hand, if β is sufficiently small, then the limit state is scalar.

For other studies on concentration behavior and related studies, see [2, 11, 12, 13, 48, 49, 53, 55, 56, 58, 59] and the references therein.

To state main results in this chapter, we also consider the following system and define the following functional:

$$\begin{cases} -\Delta v_1 + \lambda_1 v_1 = |v_1|^{p-1} v_1 + \alpha v_2 v_3, \\ -\Delta v_2 + \lambda_2 v_2 = |v_2|^{p-1} v_2 + \alpha v_1 v_3, \\ -\Delta v_3 + \lambda_3 v_3 = |v_3|^{p-1} v_3 + \alpha v_1 v_2, \end{cases} \tag{$\tilde{\mathcal{P}}^{\lambda,\alpha}$}$$

$$\tilde{I}^{\lambda,\alpha}(\mathbf{v}) := \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla v_j|^2 + \lambda_j v_j^2 \tag{4.5}$$

$$- \frac{1}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |v_j|^{p+1} - \alpha \int_{\mathbb{R}^N} v_1 v_2 v_3, \tag{4.6}$$

where $\lambda := (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_j > 0$ $(j = 1, 2, 3)$. Define the least energy as follows:

$$\rho(\lambda_1, \lambda_2, \lambda_3; \alpha) := \inf_{\mathbf{v} \in \tilde{\mathcal{N}}^{\lambda,\alpha}} \tilde{I}^{\lambda,\alpha}(\mathbf{v}),$$

$$\tilde{\mathcal{N}}^{\lambda,\alpha} := \{\mathbf{v} \in \mathbb{H} \setminus \{(0,0,0)\} \mid \tilde{G}^{\lambda,\alpha}(\mathbf{v}) = 0\},$$

$$\tilde{G}^{\lambda,\alpha}(\mathbf{v}) := \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla v_j|^2 + \lambda_j v_j^2 - |v_j|^{p+1} - 3\alpha \int_{\mathbb{R}^N} v_1 v_2 v_3.$$

Definition 4.1. A solution $\mathbf{u} = (u_1, u_2, u_3)$ of $(\mathcal{P}_\varepsilon)$ is called a scalar solution if there exists $j_0 \in \{1, 2, 3\}$ such that $u_{j_0} \neq 0$ and $u_j = 0$ for all $j \neq j_0$; while a solution $\mathbf{u}$ of $(\mathcal{P}_\varepsilon)$ is called a vector solution if $u_j \neq 0$ for all $j = 1, 2, 3$.

Definition 4.2. We say that $\mathbf{u}$ is a nontrivial solution of $(\mathcal{P}_\varepsilon)$ if $\mathbf{u}$ satisfies $(\mathcal{P}_\varepsilon)$ and $\mathbf{u} \neq (0, 0, 0)$. We say that $\mathbf{u}$ is a ground state of $(\mathcal{P}_\varepsilon)$ if $\mathbf{u}$ is a nontrivial solution of $(\mathcal{P}_\varepsilon)$ with least energy $I_\varepsilon(\mathbf{u})$ among all nontrivial $\mathbb{H}$ solutions of $(\mathcal{P}_\varepsilon)$. We say that $\mathbf{u}$ is a minimizer for c_ε if $\mathbf{u} \in \mathcal{N}_\varepsilon$ and $I_\varepsilon(\mathbf{u}) = c_\varepsilon$. We say that $\mathbf{u}$ is a non-negative ground state of $(\mathcal{P}_\varepsilon)$ if $\mathbf{u}$ is a ground state of $(\mathcal{P}_\varepsilon)$ and $u_j \geq 0$ in $\mathbb{R}^N$ for all $j = 1, 2, 3$. Similarly, we say that $\mathbf{u}$ is a non-negative minimizer for c_ε if $\mathbf{u}$ is a minimizer for c_ε and $u_j \geq 0$ in $\mathbb{R}^N$ for all $j = 1, 2, 3$.

Remark 4.1. $\mathbf{u}$ is a minimizer for c_ε if and only if $\mathbf{u}$ is a ground state of $(\mathcal{P}_\varepsilon)$ (see for example, [3, 57]). Similar results hold for the minimization problem $\tilde{c}_\varepsilon$ and $\rho(\lambda_1, \lambda_2, \lambda_3; \alpha)$.

Remark 4.2. Suppose $\lambda_j > 0$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. From [43], $(\tilde{\mathcal{P}}^{\lambda,\alpha})$ has a non-negative ground state.

We assume the following additional condition for the potentials:

$(\mathrm{C1})_\alpha$ $\inf_{x \in \mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) < \rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha)$.

We now state main results in this chapter. First, we state the existence of a ground state of $(\mathcal{P}_\varepsilon)$ for ε sufficiently small.

Theorem 4.3. We assume that (V1),(V2) and fix α so that $(\mathrm{C1})_\alpha$ holds. Then it follows that

$$c_\varepsilon \leq \varepsilon^N \left(\inf_{x \in \mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) + o(1) \right), \quad \text{as } \varepsilon \to +0.$$

Moreover, there exists a non-negative ground state $\mathbf{u}$ of $(\mathcal{P}_\varepsilon)$ for ε sufficiently small.

Remark 4.3. (1) We can show the potentials $V_j(x)$ $(j = 1, 2, 3)$ satisfies $(\mathrm{C1})_\alpha$ for all $\alpha \geq 0$ if we assume the following condition (V3):

(V3) there exists $y_0 \in \mathbb{R}^N$ such that $0 < V_j(y_0) < V_{j,\infty}$ for all $j = 1, 2, 3$.

Indeed, from (V3) and Lemma 4.9, which will be described later, it follows that

$$\rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha) > \rho(V_1(y_0), V_2(y_0), V_3(y_0); \alpha).$$

Hence, it holds that

$$\rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha) > \inf_{x \in \mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha).$$

(2) We consider the following condition:

(V4) there exists $z_0 \in \mathbb{R}^N$ such that

$$V_j(z_0) = \min_{x \in \mathbb{R}^N} V_j(x) = V_{j,0} \quad \text{for all } j = 1, 2, 3.$$

It is easy to see that (V2) and (V4) imply (V3).

(3) We consider the following condition:

(V3)′ there exists $y_0 \in \mathbb{R}^N$ such that for all $j \in \{1, 2, 3\}$, $0 < V_j(y_0) \leq V_{j,\infty}$ and $(V_1(y_0), V_2(y_0), V_3(y_0)) \neq (V_{1,\infty}, V_{2,\infty}, V_{3,\infty})$.

Then from Lemma 4.9, if $\alpha > \alpha^*_{\mathbf{V}_\infty}$ and (V3)′, then $(C1)_\alpha$ holds, where

$$\alpha^*_{\mathbf{V}_\infty} := \max\{\alpha \geq 0 \mid \rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha) = \rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; 0)\}.$$

(4) To clear the dependence on α, we write c_ε as $c_{\varepsilon,\alpha}$, if necessary. We note that $c_{\varepsilon,\alpha}$ is an even function with respect to α. So we only consider the case of $\alpha \geq 0$.

Next, we state the asymptotic behavior of a ground state of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \to +0$.

Theorem 4.4. We assume that (V1),(V2) and fix α so that $(C1)_\alpha$ holds. Let $\{\varepsilon_n\}_{n=1}^\infty \subset (0, \infty)$ with $\varepsilon_n \to 0$ as $n \to \infty$ and let $\mathbf{u}_n$ be a non-negative ground state of $(\mathcal{P}_{\varepsilon_n})$. Let $x_{j,n}$ be a maximum point of $u_{j,n}$. Then, up to a subsequence, there exist $l_0 \in \{1, 2, 3\}$, $x_{l_0,0} \in \mathbb{R}^N$ and $\mathbf{U}^{(l_0)} = (U_1^{(l_0)}, U_2^{(l_0)}, U_3^{(l_0)}) \in \mathbb{H}$ such that

(0) $u_{j,n}(x_{l_0,n} + \varepsilon_n y) \rightharpoonup U_j^{(l_0)}(y)$ weakly in $H^1(\mathbb{R}^N)$ $(j = 1, 2, 3)$ and $U_{l_0}^{(l_0)} \neq 0$.

(1) $\{x_{l_0,n}\}_{n=1}^{\infty}$ is bounded.

(2) It follows that

$$c_\varepsilon = \varepsilon^N \left(\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) + o(1) \right), \quad \text{as } \varepsilon \to +0.$$

(3) It follows that

$$\begin{aligned}
&x_{l_0,n} \to x_{l_0,0}, \\
&\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) = \rho(V_1(x_{l_0,0}), V_2(x_{l_0,0}), V_3(x_{l_0,0}); \alpha), \\
&u_{j,n}(x_{l_0,n} + \varepsilon_n y) \to U_j^{(l_0)}(y) \quad \text{in } H^1(\mathbb{R}^N), \\
&\mathbf{U}^{(l_0)} \text{ is a ground state of } (\tilde{\mathcal{P}}^{\mathbf{V}(x_{l_0,0}),\alpha}).
\end{aligned}$$

Next, we state the precise asymptotic behavior of a ground state of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \to +0$. To obtain the asymptotic behavior precisely, we introduce the following condition:

$(\mathrm{C2})_\alpha$ $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) < \min_{j=1,2,3} c_1^{V_{j,0}}$,

where

$$\begin{aligned}
\lambda &> 0, \\
I_1^\lambda(u) &:= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}, \\
c_1^\lambda &:= \inf_{u\in\mathcal{N}_1^\lambda} I_1^\lambda(u), \\
\mathcal{N}_1^\lambda &:= \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid G_1^\lambda(u) = 0\}, \\
G_1^\lambda(u) &:= \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 - |u|^{p+1}.
\end{aligned}$$

We also consider the following equation associated the above minimization problem:

$$-\Delta u + \lambda u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N. \qquad (\mathcal{P}_1^\lambda)$$

Now, we state the precise asymptotic behavior for a non-negative ground state of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \to +0$.

Theorem 4.5. We assume that (V1),(V2) and fix α so that $(C1)_\alpha$ and $(C2)_\alpha$ hold. Let $\{\varepsilon_n\}_{n=1}^\infty \subset (0,\infty)$ with $\varepsilon_n \to 0$ as $n \to \infty$ and let $\mathbf{u}_n$ be a non-negative ground state of $(\mathcal{P}_{\varepsilon_n})$. Let $x_{j,n}$ be a maximum point of $u_{j,n}$.

(1) Then, it follows that $\{x_{j,n}\}_{n=1}^\infty$ is bounded for all $j = 1, 2, 3$.

(2) It holds that

$$c_\varepsilon = \varepsilon^N \left(\inf_{x \in \mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) + o(1) \right), \quad \text{as } \varepsilon \to +0.$$

(3) Furthermore, up to a subsequence, there exist $\mathbf{W}_0 \in \mathbb{H}$ and $x_0 \in \mathbb{R}^N$ such that

$$\begin{aligned}
&x_{j,n} \to x_0, \\
&\frac{|x_{j,n} - x_{k,n}|}{\varepsilon_n} \to 0, \quad \text{as } n \to \infty,\ j \neq k, \\
&\inf_{x \in \mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) = \rho(V_1(x_0), V_2(x_0), V_3(x_0); \alpha), \\
&u_{j,n}(x_{j,n} + \varepsilon_n y) \to W_{j,0}(y) \quad \text{in } H^1(\mathbb{R}^N), \\
&\mathbf{W}_0 \text{ is a ground state of } (\tilde{\mathcal{P}}^{\mathbf{V}(x_0),\alpha}) \\
&W_{j,0} \text{ is positive, radially symmetric and strictly decreasing} \\
&\text{for all } j = 1, 2, 3,
\end{aligned}$$

where $\mathbf{V}(x_0) = (V_1(x_0), V_2(x_0), V_3(x_0))$.

(4) Moreover, for any $0 < \eta < V_0$, there exists $C_\eta > 0$ such that

$$u_{j,n}(x) \leq C_\eta e^{-\sqrt{\eta}|x - x_{j,n}|/\varepsilon_n} \quad \text{for all } x \in \mathbb{R}^N,\ n \in \mathbb{N},\ j = 1, 2, 3,$$

where $V_0 := \min\{V_{1,0}, V_{2,0}, V_{3,0}\}$.

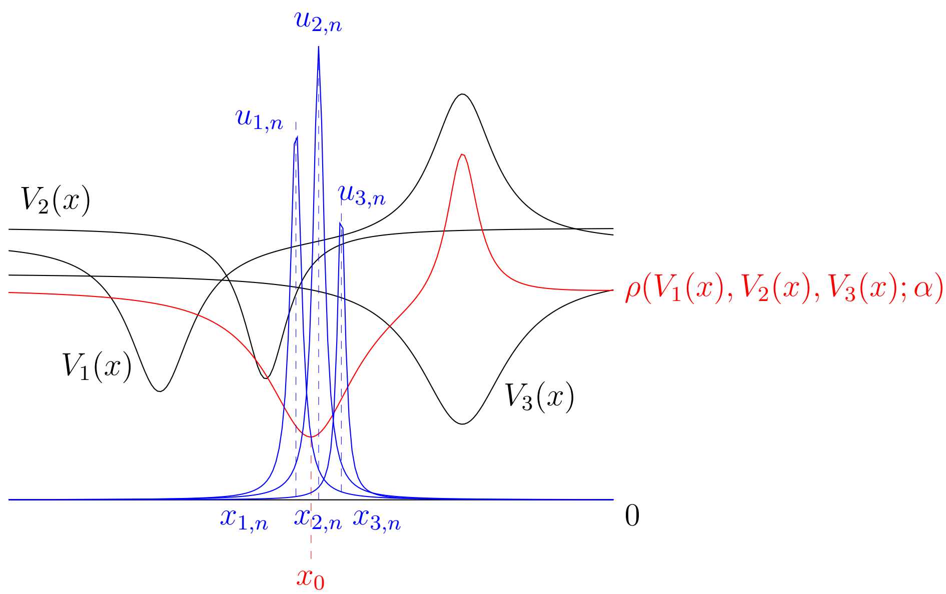

Figure 4.3: Illustration of the result of Theorem 4.5

Remark 4.4. We can define a value $\alpha^* \geq 0$ as in Section 4.7. The definition of α^*, $(\mathrm{C2})_\alpha$ holds if $\alpha > \alpha^*$ and $(\mathrm{C2})_\alpha$ does not hold if $0 \leq \alpha \leq \alpha^*$. From Remark 4.3, (V3) implies that $(\mathrm{C1})_\alpha$ holds for all $\alpha > 0$. So Theorem 4.5 holds if we assume that (V1),(V2),(V3) and $\alpha > \alpha^*$. See Theorem 4.6 for the precise asymptotic behavior of a ground state, when $(\mathrm{C2})_\alpha$ does not hold.

In the following, we consider the case where $(\mathrm{C2})_\alpha$ does not hold. When $(\mathrm{C2})_\alpha$ does not hold, the following condition holds (see Lemma 4.15 and Proposition 4.18):

$(\mathrm{C3})_\alpha$ $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) = \min_{j=1,2,3} c_1^{V_{j,0}}$.

Theorem 4.6. We assume that (V1),(V2) and fix α so that $(\mathrm{C1})_\alpha$ and $(\mathrm{C3})_\alpha$ hold. In addition, we assume that there exists $\alpha' > \alpha$ such that $(\mathrm{C3})_{\alpha'}$ holds. Let $\{\varepsilon_n\}_{n=1}^\infty \subset (0,\infty)$ such that $\varepsilon_n \to +0$ and let $\mathbf{u}_n$ be a non-negative ground state for $(\mathcal{P}_{\varepsilon_n})$. Let $x_{j,n}$ be a maximum point of $u_{j,n}$. Then, up to a subsequence, there exist $l_0 \in \{1,2,3\}$ and $x_{l_0,0} \in \mathbb{R}^N$ such that

$$x_{l_0,n} \to x_{l_0,0}, \quad V_{l_0}(x_{l_0,0}) = V_{l_0,0} = V_0,$$

$$
\begin{aligned}
&c_\varepsilon = \varepsilon^N \left(\min_{j=1,2,3} c_1^{V_{j,0}} + o(1) \right) = \varepsilon^N \left(c_1^{V_{l_0,0}} + o(1) \right), \quad \text{as } \varepsilon \to +0,\\
&u_{l_0,n}(x_{l_0,n} + \varepsilon_n y) \to W \quad \text{in } H^1(\mathbb{R}^N),\\
&u_{j,n}(x_{j,n} + \varepsilon_n y) \to 0 \quad \text{in } H^1(\mathbb{R}^N) \quad j \neq l_0,
\end{aligned}
$$

where W is the unique solution of the following equation:

$$
\begin{cases}
-\Delta W + V_0 W = W^p \quad \text{in } \mathbb{R}^N,\\
W > 0 \quad \text{in } \mathbb{R}^N,\\
W(0) = \max_{x\in\mathbb{R}^N} W(x),\\
W(x) \to 0, \quad \text{as } |x| \to \infty.
\end{cases}
$$

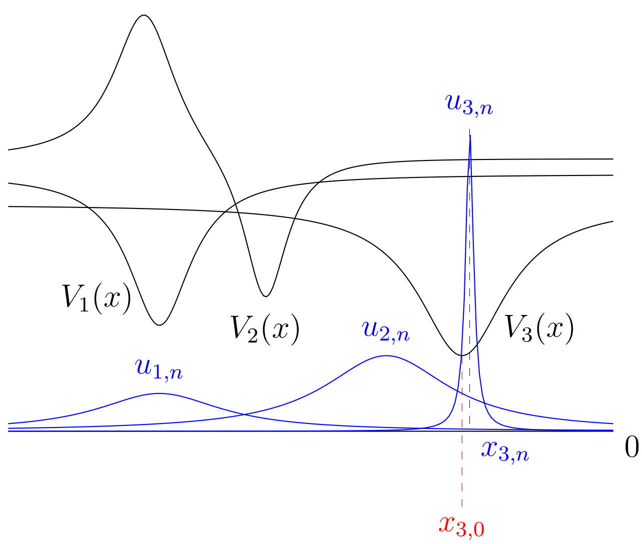

Figure 4.4: Illustration of The result of Theorem 4.6

In the problem considered Lin-Wei [34], they consider the least energy solution among all vector solutions (the solution which has all components are non-zero) of

$$
\begin{cases}
-\varepsilon^2 \Delta u_1 + V_1(x) u_1 = u_1^3 + \beta u_1 u_2^2 \quad \text{in } \mathbb{R}^N,\\
-\varepsilon^2 \Delta u_2 + V_2(x) u_2 = u_2^3 + \beta u_1^2 u_2 \quad \text{in } \mathbb{R}^N.
\end{cases}
$$

On the other hand, in our setting, we consider the least energy solution among all nontrivial solutions (includes scalar solution (the solution which only one component survives)) of

$$\begin{cases} -\varepsilon^2\Delta u_1 + V_1(x)u_1 = |u_1|^{p-1}u_1 + \alpha u_3 u_2 & \text{in } \mathbb{R}^N, \\ -\varepsilon^2\Delta u_2 + V_2(x)u_2 = |u_2|^{p-1}u_2 + \alpha u_3 u_1 & \text{in } \mathbb{R}^N, \\ -\varepsilon^2\Delta u_3 + V_3(x)u_3 = |u_3|^{p-1}u_3 + \alpha u_1 u_2 & \text{in } \mathbb{R}^N. \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

Therefore, in the result of Theorem 4.6, the case which each component of ground states survives and converges to a minimum point of corresponding potential respectively as in the result in Lin-Wei [34] does not occur.

Remark 4.5. (1) The unique solution W can be represented using w in (4.28) as follows:

$$W(x) = V_0^{1/(p-1)} w(V_0^{1/2} x).$$

(2) If (V1),(V2),(V3) hold, then $\alpha^* > 0$ holds (see Proposition 4.18).

(3) Theorem 4.6 holds if we assume that (V1),(V2),(V3) and $0 \le \alpha < \alpha^*$.

In particular, we have the following corollary: To clear the dependence on α, we write $(\mathcal{P}_\varepsilon)$ as $(\mathcal{P}_{\varepsilon,\alpha})$, if necessary.

Corollary 4.7. Suppose (V1),(V2),(V3). Then $\alpha^* > 0$ (see Proposition 4.18) and the following cases hold:

(i) If $\alpha > \alpha^*$, then Theorem 4.5 holds and the asymptotic limit of a ground state of $(\mathcal{P}_{\varepsilon,\alpha})$ is vector.

(ii) If $0 \le \alpha < \alpha^*$, then Theorem 4.6 holds and the asymptotic limit of a ground state of $(\mathcal{P}_{\varepsilon,\alpha})$ is scalar.

Remark 4.6. We can show that all the ground states of $(\mathcal{P}_{\varepsilon,\alpha})$ are scalar for ε sufficiently small and for α sufficiently small. On the other hand, all the ground states of $(\mathcal{P}_{\varepsilon,\alpha})$ are vector for ε sufficiently small and for α sufficiently large (see Proposition 4.19).

The rest of this chapter is organized as follows. In Section 4.2, we prove the existence of a non-negative ground state of $(\mathcal{P}_\varepsilon)$ for ε sufficiently small. In Section 4.3, we prove the asymptotic behavior of a non-negative ground state of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \to +0$ without $(\mathrm{C2})_\alpha$ and $(\mathrm{C3})_\alpha$. In Section 4.4, we show the asymptotic behavior of a non-negative ground state of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \to +0$ under $(\mathrm{C2})_\alpha$. In Section 4.5, we prove the asymptotic behavior of a non-negative ground state of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \to +0$ under $(\mathrm{C3})_\alpha$. In Section 4.6, we study the asymptotic behavior of $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha)$ as $\alpha \to \infty$. In Section 4.7, we show the existence of the positive threshold α^* for $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha)$, which divides the asymptotic behavior of a ground state of $(\mathcal{P}_{\varepsilon,\alpha})$ for $\alpha > \alpha^*$ and $0 \le \alpha < \alpha^*$. In Section 4.8, we give two thresholds for $c_{\varepsilon,\alpha}$ and consider when all the ground states of $(\mathcal{P}_{\varepsilon,\alpha})$ are scalar or vector. In Appendix, we give the outline of the proof of the radial symmetry and monotonicity of classical solutions of elliptic systems in the case $N = 1$.

Notation

$$\begin{aligned}
\mathbb{H} &:= H^1(\mathbb{R}^N)^3,\\
\mathbf{u} &:= (u_1, u_2, u_3),\\
\|\mathbf{u}\|_{\mathbb{H}}^2 &:= \sum_{j=1}^{3} \|u_j\|_{H^1}^2,\\
(\mathbf{u},\mathbf{v})_{\mathbb{H}} &:= \sum_{j=1}^{3} (u_j, v_j)_{H^1},\\
\mathbf{u}_n &:= (u_{1,n}, u_{2,n}, u_{3,n}).
\end{aligned}$$

- We say that $\mathbf{u}_n \to \mathbf{u}$ in $\mathbb{H}$ if

$$\|\mathbf{u}_n - \mathbf{u}\|_{\mathbb{H}} \to 0.$$

- We say that $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $\mathbb{H}$ if

$$(\mathbf{u}_n, \mathbf{v})_{\mathbb{H}} \to (\mathbf{u}, \mathbf{v})_{\mathbb{H}} \quad \text{for all } \mathbf{v} \in \mathbb{H}.$$

- We also set

$$\mathbf{V}_0 := (V_{1,0}, V_{2,0}, V_{3,0}),$$

$$V_0 := \min\{V_{1,0}, V_{2,0}, V_{3,0}\}.$$

- Let $x_{j,n}$ be a maximum point of $u_{j,n}$, that is,

$$u_{j,n}(x_{j,n}) = \max_{x\in\mathbb{R}^N} u_{j,n}(x).$$

- We rescale $\mathbf{u}_n$ as follows:

$$\mathbf{U}_n^{(l)}(y) := \mathbf{u}_n(x_{l,n} + \varepsilon_n y) \quad (l = 1, 2, 3),$$
$$U_{j,n}^{(l)}(y) := u_{j,n}(x_{l,n} + \varepsilon_n y) \quad (j, l = 1, 2, 3).$$

- In particular, if $j = l$, we define

$$W_{j,n} := U_{j,n}^{(j)},$$
$$\mathbf{W}_n := (W_{1,n}, W_{2,n}, W_{3,n}).$$

4.2 Proof of Theorem 4.3

To prove Theorem 4.3, we prove the following three lemmas needed later.

Lemma 4.8. $\rho : (0,\infty)^3 \to \mathbb{R}$ is continuous.

Proof. From Lemma 3.7 in Pomponio [43], ρ is continuous on $(0,\infty)^3$. □

From (V1),(V2),(C1)$_\alpha$ and continuity of ρ, there exists a point $z_0 \in \mathbb{R}^N$ such that

$$\rho(V_1(z_0), V_2(z_0), V_3(z_0); \alpha) = \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha).$$

Lemma 4.9. Let $\alpha \geq 0$.
(1) If $0 < \lambda_j < \lambda'_j$ for all $j = 1, 2, 3$, then

$$\rho(\lambda_1, \lambda_2, \lambda_3; \alpha) < \rho(\lambda'_1, \lambda'_2, \lambda'_3; \alpha).$$

(2) If $0 < \lambda_j \leq \lambda'_j$ for all $j = 1, 2, 3$, then

$$\rho(\lambda_1, \lambda_2, \lambda_3; \alpha) \leq \rho(\lambda'_1, \lambda'_2, \lambda'_3; \alpha).$$

(3) If $\alpha > \alpha^*_{\lambda'}$ and $0 < \lambda_j \le \lambda'_j$ and $\lambda \neq \lambda'$, then

$$\rho(\lambda_1, \lambda_2, \lambda_3; \alpha) < \rho(\lambda'_1, \lambda'_2, \lambda'_3; \alpha),$$

where

$$\begin{aligned}
&\lambda = (\lambda_1, \lambda_2, \lambda_3), \quad \lambda' = (\lambda'_1, \lambda'_2, \lambda'_3),\\
&\alpha^*_{\lambda'} := \max\{\alpha \ge 0 \mid \rho(\lambda'_1, \lambda'_2, \lambda'_3; \alpha) = \rho(\lambda'_1, \lambda'_2, \lambda'_3; 0)\}.
\end{aligned}$$

Proof. (1) From Remark 4.1 and 4.2, there exists a non-negative minimizer $\mathbf{v}_0$ for $\rho(\lambda'_1, \lambda'_2, \lambda'_3; \alpha)$. Let $t_0 > 0$ be a number such that $t_0\mathbf{v}_0 \in \tilde{\mathcal{N}}^{\lambda,\alpha}$, where $\lambda := (\lambda_1, \lambda_2, \lambda_3)$ (see Pomponio [43]). Since $\mathbf{v}_0 \neq (0, 0, 0)$, then we have

$$\sum_{j=1}^{3} \int_{\mathbb{R}^N} \lambda_j v_{j,0}^2 < \sum_{j=1}^{3} \int_{\mathbb{R}^N} \lambda'_j v_{j,0}^2.$$

Since $\tilde{I}^{\lambda',\alpha}(t_0\mathbf{v}_0) \le \tilde{I}^{\lambda',\alpha}(\mathbf{v}_0)$, it holds that

$$\rho(\lambda'_1, \lambda'_2, \lambda'_3; \alpha) = \tilde{I}^{\lambda',\alpha}(\mathbf{v}_0) \ge \tilde{I}^{\lambda',\alpha}(t_0\mathbf{v}_0) > \tilde{I}^{\lambda,\alpha}(t_0\mathbf{v}_0) \ge \rho(\lambda_1, \lambda_2, \lambda_3; \alpha).$$

(2) We can show (2) by the same argument as in (1).
(3) Suppose $\alpha > \alpha^*_{\lambda'}$. Let $\mathbf{v}_0$ be a non-negative minimizer for $\rho(\lambda'_1, \lambda'_2, \lambda'_3; \alpha)$. From Theorem 1.4 in [30], all the minimizers of $\rho(\lambda'_1, \lambda'_2, \lambda'_3; \alpha)$ are vector if $\alpha > \alpha^*_{\lambda'}$. Hence $v_{j,0} \neq 0$ for all $j = 1, 2, 3$. Then

$$\sum_{j=1}^{3} \int_{\mathbb{R}^N} \lambda_j v_{j,0}^2 < \sum_{j=1}^{3} \int_{\mathbb{R}^N} \lambda'_j v_{j,0}^2.$$

Hence we can prove (3) by the same argument as in (1). □

For $\mathbf{u} = (u_1, u_2, u_3)$, we set

$$\mathbf{v}(y) = \mathbf{u}(\varepsilon y). \tag{4.7}$$

We consider the following system and define the following functional and least energy:

$$\begin{cases}
-\Delta v_1 + V_1(\varepsilon y)v_1 = |v_1|^{p-1}v_1 + \alpha v_2 v_3,\\
-\Delta v_2 + V_2(\varepsilon y)v_2 = |v_2|^{p-1}v_2 + \alpha v_1 v_3,\\
-\Delta v_3 + V_3(\varepsilon y)v_3 = |v_3|^{p-1}v_3 + \alpha v_1 v_2,
\end{cases} \tag{$\tilde{\mathcal{P}}_\varepsilon$}$$

$$\begin{aligned}
\mathbf{v} &:= (v_1, v_2, v_3),\\
\tilde{I}_\varepsilon(\mathbf{v}) &:= \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla v_j|^2 + V_j(\varepsilon y)v_j^2 - \frac{1}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|v_j|^{p+1} - \alpha\int_{\mathbb{R}^N} v_1v_2v_3,\\
\tilde{c}_\varepsilon &:= \inf_{\mathbf{v}\in\tilde{\mathcal{N}}_\varepsilon}\tilde{I}_\varepsilon(\mathbf{v}),\\
\tilde{\mathcal{N}}_\varepsilon &:= \{\mathbf{v}\in\mathbb{H}\setminus\{(0,0,0)\} \mid \tilde{G}_\varepsilon(\mathbf{v}) = 0\},\\
\tilde{G}_\varepsilon(\mathbf{v}) &:= \sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla v_j|^2 + V_j(\varepsilon y)v_j^2 - |v_j|^{p+1} - 3\alpha\int_{\mathbb{R}^N} v_1v_2v_3.
\end{aligned}$$

We note that under (4.7),

$$\begin{aligned}
I_\varepsilon(\mathbf{u}) &= \varepsilon^N\tilde{I}_\varepsilon(\mathbf{v}),\\
G_\varepsilon(\mathbf{u}) &= \varepsilon^N\tilde{G}_\varepsilon(\mathbf{v}),\\
c_\varepsilon &= \varepsilon^N\tilde{c}_\varepsilon.
\end{aligned}$$

We now prove the upper bound for $\tilde{c}_\varepsilon$.

Lemma 4.10. We assume that (V1),(V2) and $(\mathrm{C1})_\alpha$. Then the followings hold:

(i)

$$\tilde{c}_\varepsilon \le \inf_{x\in\mathbb{R}^N}\rho(V_1(x), V_2(x), V_3(x);\alpha) + o(1), \quad \text{as } \varepsilon\to+0.$$

(ii) For all $\varepsilon > 0$ and $\alpha \ge 0$, it follows that

$$\tilde{c}_{\varepsilon,\alpha} \le \rho(V_{1,\max}, V_{2,\max}, V_{3,\max}; 0),$$

where $V_{j,\max} = \sup_{x\in\mathbb{R}^N} V_j(x)$.

(iii) Let $\mathbf{u}_{\varepsilon,\alpha} = (u_{1,\varepsilon,\alpha}, u_{2,\varepsilon,\alpha}, u_{3,\varepsilon,\alpha})$ be a non-negative ground state of $(\mathcal{P}_{\varepsilon,\alpha})$ and let $x_{l,\varepsilon,\alpha}$ be a maximum point of $u_{l,\varepsilon,\alpha}$. Set $\mathbf{U}^{(l)}_{\varepsilon,\alpha}(y) = \mathbf{u}_{\varepsilon,\alpha}(x_{l,\varepsilon,\alpha} + \varepsilon y)$. Then,

$$\sup_{\varepsilon>0,\ \alpha\ge0}\|\mathbf{U}^{(l)}_{\varepsilon,\alpha}\|_{\mathbb{H}} < \infty.$$

(iv) Let $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,\infty)$ such that $\varepsilon_n \to +0$ and let $\{\alpha_n\}_{n=1}^{\infty} \subset [0,\infty)$ be a bounded sequence. Let $\mathbf{u}_n$ be a non-negative ground state of $(\mathcal{P}_{\varepsilon_n,\alpha_n})$ and let $x_{l,n}$ be a maximum point of $u_{l,n}$. Set $\mathbf{U}_n^{(l)}(y) = \mathbf{u}_n(x_{l,n} + \varepsilon_n y)$. From (iii), up to a subsequence, there exists $\mathbf{U}^{(l)}$ such that

$$\mathbf{U}_n^{(l)} \rightharpoonup \mathbf{U}^{(l)} \quad \text{weakly in } \mathbb{H}.$$

Then, it follows that

$$\begin{aligned} &\sup_{n\in\mathbb{N}} \|U_{j,n}^{(l)}\|_{L^\infty(\mathbb{R}^N)} < \infty, \\ &U_{j,n}^{(l)} \in C^2(\mathbb{R}^N), \\ &U_{j,n}^{(l)} \to U_j^{(l)} \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^N). \end{aligned}$$

(v) In addition to the condition (iv), we suppose that $\|U_{j,n}^{(l)} - U_j^{(l)}\|_{H^1} \to 0$. Then it holds that

$$\sup_{n\in\mathbb{N}} U_{j,n}^{(l)}(y_0) \to 0, \quad \text{as } |y_0| \to \infty, \text{ for all } j = 1,2,3.$$

Proof. (i) Let $z_0 \in \mathbb{R}^N$ be a point which attains the

$$\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha),$$

and set $\lambda_{j,0} := V_j(z_0)$ (> 0). Let $\mathbf{w}_0$ be a non-negative minimizer for $\rho(\lambda_{1,0}, \lambda_{2,0}, \lambda_{3,0}; \alpha)$. Set $\mathbf{v}_{0,\varepsilon}(y) := \mathbf{w}_0(y - z_0/\varepsilon)$. Let $t_{0,\varepsilon} > 0$ be a number such that $t_{0,\varepsilon}\mathbf{v}_{0,\varepsilon} \in \tilde{\mathcal{N}}_\varepsilon$. Then we have

$$\begin{aligned} &t_{0,\varepsilon}^2 \sum_{j=1}^3 \int_{\mathbb{R}^N} |\nabla v_{j,0,\varepsilon}|^2 + V_j(\varepsilon y) v_{j,0,\varepsilon}^2 \\ &= t_{0,\varepsilon}^{p+1} \sum_{j=1}^3 \int_{\mathbb{R}^N} |v_{j,0,\varepsilon}|^{p+1} + 3t_{0,\varepsilon}^3 \alpha \int_{\mathbb{R}^N} v_{1,0,\varepsilon} v_{2,0,\varepsilon} v_{3,0,\varepsilon}, \end{aligned}$$

that is,

$$\begin{aligned} &\sum_{j=1}^3 \int_{\mathbb{R}^N} |\nabla w_{j,0}|^2 + V_j(z_0 + \varepsilon y) w_{j,0}^2 \\ &= t_{0,\varepsilon}^{p-1} \sum_{j=1}^3 |w_{j,0}|^{p+1} + 3t_{0,\varepsilon} \alpha \int_{\mathbb{R}^N} w_{1,0} w_{2,0} w_{3,0}. \end{aligned} \tag{4.8}$$

From (4.8), $\{t_{0,\varepsilon}\}_\varepsilon$ is bounded. Moreover, we have

$$\begin{aligned}
&\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) = \rho(V_1(z_0), V_2(z_0), V_3(z_0); \alpha) \\
&= \rho(\lambda_{1,0}, \lambda_{2,0}, \lambda_{3,0}; \alpha) = \tilde{I}^{\lambda_0,\alpha}(\mathbf{w}_0) \geq \tilde{I}^{\lambda_0,\alpha}(t_{0,\varepsilon}\mathbf{w}_0) \\
&= \frac{t_{0,\varepsilon}^2}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla w_{j,0}|^2 + \lambda_{j,0} w_{j,0}^2 \\
&\quad - \frac{t_{0,\varepsilon}^{p+1}}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |w_{j,0}|^{p+1} - t_{0,\varepsilon}^3 \alpha \int_{\mathbb{R}^N} w_{1,0} w_{2,0} w_{3,0}
\end{aligned}$$

and

$$\left| t_{0,\varepsilon}^2 \int_{\mathbb{R}^N} V_j(z_0 + \varepsilon y) w_{j,0}^2 - t_{0,\varepsilon}^2 \int_{\mathbb{R}^N} \lambda_{j,0} w_{j,0}^2 \right| \to 0 \quad (\varepsilon \to +0),$$

where $\lambda_0 := (\lambda_{1,0}, \lambda_{2,0}, \lambda_{3,0})$. Thus, we have

$$\begin{aligned}
\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) &\geq \tilde{I}_\varepsilon(t_{0,\varepsilon}\mathbf{v}_{0,\varepsilon}) + o(1) \\
&\geq \tilde{c}_\varepsilon + o(1), \quad \text{as } \varepsilon \to +0.
\end{aligned}$$

(ii) From Lemma 4.9, we have

$$\tilde{c}_{\varepsilon,\alpha} \leq \rho(V_{1,\max}, V_{2,\max}, V_{3,\max}; \alpha).$$

From Lemma 2.5 in [43], we have

$$\rho(V_{1,\max}, V_{2,\max}, V_{3,\max}; \alpha) \leq \rho(V_{1,\max}, V_{2,\max}, V_{3,\max}; 0).$$

Hence we obtain the conclusion.

(iii) Let $\mathbf{v}_{\varepsilon,\alpha}(y) := \mathbf{u}_{\varepsilon,\alpha}(\varepsilon y)$. Then $\mathbf{v}_{\varepsilon,\alpha}$ is a ground state of $(\tilde{\mathcal{P}}_{\varepsilon,\alpha})$ and

$$\mathbf{U}_{\varepsilon,\alpha}^{(l)}(y) = \mathbf{v}_{\varepsilon,\alpha}(y + x_{l,\varepsilon,\alpha}/\varepsilon), \quad (l = 1, 2, 3).$$

Thus, from (ii), we have

$$\begin{aligned}
&\rho(V_{1,\max}, V_{2,\max}, V_{3,\max}; 0) \\
&\geq \tilde{c}_{\varepsilon,\alpha} = \tilde{I}_{\varepsilon,\alpha}(\mathbf{v}_{\varepsilon,\alpha})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla v_{j,\varepsilon,\alpha}|^2 + V_j(\varepsilon y)v_{j,\varepsilon,\alpha}^2 \\
&\quad - \frac{1}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N} v_{j,\varepsilon,\alpha}^{p+1} - \alpha\int_{\mathbb{R}^N} v_{1,\varepsilon,\alpha}v_{2,\varepsilon,\alpha}v_{3,\varepsilon,\alpha}, \quad \text{as } \varepsilon_n \to +0.
\end{aligned}
$$

Since $\mathbf{v}_{\varepsilon,\alpha}$ is a ground state of $(\tilde{\mathcal{P}}_{\varepsilon,\alpha})$,

$$
\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla v_{j,\varepsilon,\alpha}|^2 + V_j(\varepsilon y)v_{j,\varepsilon,\alpha}^2 - v_{j,\varepsilon,\alpha}^{p+1} = 3\alpha\int_{\mathbb{R}^N} v_{1,\varepsilon,\alpha}v_{2,\varepsilon,\alpha}v_{3,\varepsilon,\alpha}
$$

and (V2), we have

$$
\begin{aligned}
&\rho(V_{1,\max}, V_{2,\max}, V_{3,\max}; 0) \\
&\geq \tilde{c}_{\varepsilon,\alpha} = \tilde{I}_{\varepsilon,\alpha}(\mathbf{v}_{\varepsilon,\alpha}) \\
&= \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla v_{j,\varepsilon,\alpha}|^2 + V_j(\varepsilon y)v_{j,\varepsilon,\alpha}^2 + \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N} v_{j,\varepsilon,\alpha}^{p+1} \\
&= \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla U_{j,\varepsilon,\alpha}^{(l)}|^2 + V_j(x_{l,\varepsilon,\alpha}+\varepsilon y)(U_{j,\varepsilon,\alpha}^{(l)})^2 + \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N} (U_{j,\varepsilon,\alpha}^{(l)})^{p+1} \\
&\geq \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla U_{j,\varepsilon,\alpha}^{(l)}|^2 + V_{j,0}(U_{j,\varepsilon,\alpha}^{(l)})^2.
\end{aligned}
$$

Hence it follows that

$$
\sup_{\varepsilon>0,\alpha\geq 0}\|\mathbf{U}_{\varepsilon,\alpha}^{(l)}\|_{\mathbb{H}} < \infty.
$$

(iv) Since $\mathbf{u}_n$ is a non-negative ground state of $(\mathcal{P}_{\varepsilon_n,\alpha_n})$ and $\mathbf{U}_n^{(l)}(y) = \mathbf{u}(x_{l,n}+\varepsilon_n y)$, we have

$$
\begin{cases}
-\Delta U_{1,n}^{(l)} + V_1(x_{l,n}+\varepsilon_n y)U_{1,n}^{(l)} = (U_{1,n}^{(l)})^p + \alpha_n U_{2,n}^{(l)}U_{3,n}^{(l)}, \\
-\Delta U_{2,n}^{(l)} + V_2(x_{l,n}+\varepsilon_n y)U_{2,n}^{(l)} = (U_{2,n}^{(l)})^p + \alpha_n U_{1,n}^{(l)}U_{3,n}^{(l)}, \\
-\Delta U_{3,n}^{(l)} + V_3(x_{l,n}+\varepsilon_n y)U_{3,n}^{(l)} = (U_{3,n}^{(l)})^p + \alpha_n U_{1,n}^{(l)}U_{2,n}^{(l)}, \\
U_{j,n}^{(l)} \geq 0, \quad (l = 1,2,3).
\end{cases}
\tag{4.9}
$$

Set $U_n^{(l)} := U_{1,n}^{(l)} + U_{2,n}^{(l)}$, then we have

$$\begin{aligned} -\Delta U_n^{(l)} &\leq -\Delta U_{1,n}^{(l)} - \Delta U_{2,n}^{(l)} + V_1(x_{l,n} + \varepsilon_n y) U_{1,n}^{(l)} + V_2(x_{l,n} + \varepsilon_n y) U_{2,n}^{(l)} \\ &\leq 2(U_n^{(l)})^p + \alpha_n U_{3,n}^{(l)} U_n^{(l)} \leq (2(U_n^{(l)})^{p-1} + C_1 U_{3,n}^{(l)}) U_n^{(l)}, \end{aligned}$$

where $C_1 = \sup_{n\in\mathbb{N}} \alpha_n < \infty$. Note $U_n^{(l)}$ is a subsolution of $\Delta u + c(x)u = 0$ with $c(x) = 2(U_n^{(l)})^{p-1} + C_1 U_{3,n}^{(l)}$. Note that $U_n^{(l)} \in L^q_{\text{loc}}(\mathbb{R}^N)$ for some $N/2 < q < 2^*$ and $U_{3,n}^{(l)} \in L^3(\mathbb{R}^N)$ and $3 > N/2$ since $N \leq 5$. By the one-sided Harnack inequality, we have

$$\max_{B_1(y_0)} U_n^{(l)} \leq C \left(\int_{B_2(y_0)} (U_n^{(l)})^2 \right)^{1/2},$$

where y_0 is an arbitrary point in $\mathbb{R}^N$, C is a constant depending only on N, p and M where M is a bound of $\|U_n^{(l)}\|_{H^1}$ and $\|U_{3,n}^{(l)}\|_{H^1}$ and independent of n (see [22, 45]). Then

$$\max_{B_1(y_0)} U_n^{(l)} \leq CM.$$

Hence $\{U_{j,n}^{(l)}\}_{n=1}^{\infty}$ $(j = 1, 2)$ is bounded in $L^\infty(\mathbb{R}^N)$. Similarly, it follows that $\{U_{3,n}^{(l)}\}_{n=1}^{\infty}$ is bounded in $L^\infty(\mathbb{R}^N)$.
From (4.9) and $V_j \in C^1(\mathbb{R}^N)$ and by the elliptic regularity, it follows that

$$U_{j,n}^{(l)} \in C^2(\mathbb{R}^N), \quad U_{j,n}^{(l)} \to U_j^{(l)} \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^N).$$

(v) Moreover, since $U_{j,n}^{(l)} \to U_j^{(l)}$ in $H^1(\mathbb{R}^N)$, up to a subsequence, there exists $g_j \in L^2(\mathbb{R}^N)$ such that

$$U_{j,n}^{(l)} \leq g_j \quad \text{a.e. } x \in \mathbb{R}^N, \text{ for all } n \in \mathbb{N}.$$

Thus, we have

$$\sup_{n\in\mathbb{N}} \int_{B_2(y_0)} (U_n^{(l)})^2 \leq \int_{B_2(y_0)} (g_1 + g_2)^2 \to 0, \quad \text{as } |y_0| \to \infty.$$

Hence, we obtain

$$\sup_{n\in\mathbb{N}} \max_{B_1(y_0)} U_{j,n}^{(l)} \to 0, \quad \text{as } |y_0| \to \infty, \ j = 1, 2,$$

that is,

$$\sup_{n\in\mathbb{N}} U_{j,n}^{(l)}(y_0) \to 0, \quad \text{as } |y_0| \to \infty,\ j = 1, 2.$$

Similarly, it follows that

$$\sup_{n\in\mathbb{N}} U_{3,n}^{(l)}(y_0) \to 0, \quad \text{as } |y_0| \to \infty.$$

□

Now, we prove the existence of a ground state of $(\mathcal{P}_\varepsilon)$.

Proof of Theorem 4.3. From $(\mathrm{C1})_\alpha$ and Lemma 4.10 (i), we have

$$\tilde{c}_\varepsilon < \rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha) \tag{4.10}$$

for ε sufficiently small. From (4.10), there exists a ground state $\mathbf{u}$ of $(\tilde{\mathcal{P}}_\varepsilon)$ (see the argument as in Pomponio [43]). $|\mathbf{u}| = (|u_1|, |u_2|, |u_3|)$ is also a ground state of $(\mathcal{P}_\varepsilon)$. Hence, there exists a non-negative ground state of $(\mathcal{P}_\varepsilon)$. □

4.3 Proof of Theorem 4.4

Let $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0, 1)$ such that $\varepsilon_n \to +0$ and let $\mathbf{u}_n$ be a non-negative ground state of $(\mathcal{P}_{\varepsilon_n})$. Let $x_{j,n}$ be a point such that

$$u_{j,n}(x_{j,n}) = \max_{x\in\mathbb{R}^N} u_{j,n}(x).$$

Lemma 4.11. We assume that (V1),(V2). Then it follows that

$$\begin{aligned} V_0 \le\ & 2(u_{1,n}(x_{1,n}) + u_{2,n}(x_{2,n}) + u_{3,n}(x_{3,n}))^{p-1} \\ & + \alpha(u_{1,n}(x_{1,n}) + u_{2,n}(x_{2,n}) + u_{3,n}(x_{3,n})). \end{aligned} \tag{4.11}$$

Proof. Since $\mathbf{u}_n \ne (0, 0, 0)$, we may assume that $u_{1,n}(x_{1,n}) \ne 0$. From $V_j \in C^1(\mathbb{R}^N)$ and by the elliptic regularity, $u_{j,n} \in C^2(\mathbb{R}^N)$. Since $\Delta u_{j,n}(x_{j,n}) \le 0$, we have

$$V_1(x_{1,n})u_{1,n}(x_{1,n}) \le u_{1,n}(x_{1,n})^p + \alpha u_{2,n}(x_{1,n})u_{3,n}(x_{1,n})$$

$$
\begin{aligned}
&\leq u_{1,n}(x_{1,n})^p + \alpha u_{2,n}(x_{2,n})u_{3,n}(x_{3,n}),\\
V_2(x_{2,n})u_{2,n}(x_{2,n}) &\leq u_{2,n}(x_{2,n})^p + \alpha u_{1,n}(x_{2,n})u_{3,n}(x_{2,n})\\
&\leq u_{2,n}(x_{2,n})^p + \alpha u_{1,n}(x_{1,n})u_{3,n}(x_{3,n}).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
V_0(u_{1,n}(x_{1,n}) + u_{2,n}(x_{2,n})) \leq\ & 2(u_{1,n}(x_{1,n}) + u_{2,n}(x_{2,n}))^p\\
&+ \alpha(u_{1,n}(x_{1,n}) + u_{2,n}(x_{2,n}))u_{3,n}(x_{3,n}).
\end{aligned}
$$

Since $u_{1,n}(x_{1,n}) + u_{2,n}(x_{2,n}) \neq 0$, we have

$$V_0 \leq 2(u_{1,n}(x_{1,n}) + u_{2,n}(x_{2,n}))^{p-1} + \alpha u_{3,n}(x_{3,n}).$$

Then it follows that

$$
\begin{aligned}
V_0 \leq\ & 2(u_{1,n}(x_{1,n}) + u_{2,n}(x_{2,n}) + u_{3,n}(x_{3,n}))^{p-1}\\
&+ \alpha(u_{1,n}(x_{1,n}) + u_{2,n}(x_{2,n}) + u_{3,n}(x_{3,n})).
\end{aligned}
$$

□

Set

$$\mathbf{U}_n^{(l)}(y) := \mathbf{u}_n(x_{l,n} + \varepsilon_n y), \quad (l = 1, 2, 3).$$

Proof of Theorem 4.4. (0) From Lemma 4.10 (iii),(iv), $\{\mathbf{U}_n^{(l)}\}_n$ is bounded in $\mathbb{H}$ and up to a subsequence, there exists $\mathbf{U}^{(l)} \in \mathbb{H}$ such that

$$
\begin{aligned}
&\mathbf{U}_n^{(l)} \rightharpoonup \mathbf{U}^{(l)} \quad \text{weakly in } \mathbb{H},\\
&U_{j,n}^{(l)} \to U_j^{(l)} \quad \text{in } C_{\mathrm{loc}}(\mathbb{R}^N).
\end{aligned} \tag{4.12}
$$

From (4.11) and (4.12), we have

$$V_0 \leq 2(U_1^{(1)}(0) + U_2^{(2)}(0) + U_3^{(3)}(0))^{p-1} + \alpha(U_1^{(1)}(0) + U_2^{(2)}(0) + U_3^{(3)}(0)).$$

Thus we have

$$(U_1^{(1)}(0), U_2^{(2)}(0), U_3^{(3)}(0)) \neq (0, 0, 0).$$

Therefore, there exists $l_0 \in \{1, 2, 3\}$ such that $\mathbf{U}^{(l_0)} \neq (0, 0, 0)$.

(1) Now we show that $\sup_{n\in\mathbb{N}}|x_{l_0,n}| < \infty$. Suppose that $\sup_{n\in\mathbb{N}}|x_{l_0,n}| = \infty$. Then up to a subsequence,

$$|x_{l_0,n}| \to \infty.$$

Since $\lim_{|x|\to\infty} V_j(x) = V_{j,\infty}$ and $U_{j,n}^{(l_0)} \to U_j^{(l_0)}$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, it follows that for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V_j(x_{l_0,n} + \varepsilon_n y) U_{j,n}^{(l_0)} \varphi \to \int_{\mathbb{R}^N} V_{j,\infty} U_j^{(l_0)} \varphi.$$

Since $\mathbf{u}_n$ is a ground state of $(\mathcal{P}_{\varepsilon_n})$, we have

$$\begin{cases} -\varepsilon_n^2 \Delta u_{1,n} + V_1(x) u_{1,n} = u_{1,n}^p + \alpha u_{2,n} u_{3,n}, \\ -\varepsilon_n^2 \Delta u_{2,n} + V_2(x) u_{2,n} = u_{2,n}^p + \alpha u_{1,n} u_{3,n}, \\ -\varepsilon_n^2 \Delta u_{3,n} + V_3(x) u_{3,n} = u_{3,n}^p + \alpha u_{1,n} u_{2,n}. \end{cases}$$

Hence, we have

$$\begin{cases} -\Delta U_1^{(l_0)} + V_{1,\infty} U_1^{(l_0)} = (U_1^{(l_0)})^p + \alpha U_2^{(l_0)} U_3^{(l_0)}, \\ -\Delta U_2^{(l_0)} + V_{2,\infty} U_2^{(l_0)} = (U_2^{(l_0)})^p + \alpha U_1^{(l_0)} U_3^{(l_0)}, \\ -\Delta U_3^{(l_0)} + V_{3,\infty} U_3^{(l_0)} = (U_3^{(l_0)})^p + \alpha U_1^{(l_0)} U_2^{(l_0)}, \\ U_j^{(l_0)} \geq 0. \end{cases}$$

Since $U_j^{(l_0)} \in H^1(\mathbb{R}^N)$, for all $\delta > 0$, there exists $R > 0$ such that

$$\begin{aligned} &\frac{1}{6} \sum_{j=1}^{3} \left| \int_{B_R} |\nabla U_j^{(l_0)}|^2 + V_{j,\infty} (U_j^{(l_0)})^2 - \int_{\mathbb{R}^N} |\nabla U_j^{(l_0)}|^2 + V_{j,\infty} (U_j^{(l_0)})^2 \right| \\ &\quad + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \left| \int_{B_R} (U_j^{(l_0)})^{p+1} - \int_{\mathbb{R}^N} (U_j^{(l_0)})^{p+1} \right| < \delta. \end{aligned}$$

We have

$$\begin{aligned} c_{\varepsilon_n} &= I_{\varepsilon_n}(\mathbf{u}_n) \\ &\geq \frac{1}{6} \sum_{j=1}^{3} \int_{B_{\varepsilon_n R}(x_{l_0,n})} \varepsilon_n^2 |\nabla u_{j,n}|^2 + V_j(x) u_{j,n}^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{B_{\varepsilon_n R}(x_{l_0,n})} u_{j,n}^{p+1} \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_n^N \left\{ \frac{1}{6} \sum_{j=1}^{3} \int_{B_R} |\nabla U_{j,n}^{(l_0)}|^2 + V_j(x_{l_0,n} + \varepsilon_n y)(U_{j,n}^{(l_0)})^2 \right. \\
&\qquad \left. + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{B_R} (U_{j,n}^{(l_0)})^{p+1} \right\} \\
&= \varepsilon_n^N \left\{ \frac{1}{6} \sum_{j=1}^{3} \int_{B_R} |\nabla U_{j,n}^{(l_0)}|^2 + V_{j,\infty}(U_{j,n}^{(l_0)})^2 \right. \\
&\qquad \left. + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{B_R} (U_{j,n}^{(l_0)})^{p+1} + o(1) \right\}, \quad \text{as } n \to \infty.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\liminf_{n\to\infty} \frac{c_{\varepsilon_n}}{\varepsilon_n^N} \\
&\geq \liminf_{n\to\infty} \left\{ \frac{1}{6} \sum_{j=1}^{3} \int_{B_R} |\nabla U_{j,n}^{(l_0)}|^2 + V_{j,\infty}(U_{j,n}^{(l_0)})^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{B_R} (U_{j,n}^{(l_0)})^{p+1} \right\} \\
&\geq \frac{1}{6} \sum_{j=1}^{3} \int_{B_R} |\nabla U_{j}^{(l_0)}|^2 + V_{j,\infty}(U_{j}^{(l_0)})^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{B_R} (U_{j}^{(l_0)})^{p+1} \\
&\geq \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla U_{j}^{(l_0)}|^2 + V_{j,\infty}(U_{j}^{(l_0)})^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} (U_{j}^{(l_0)})^{p+1} - \delta \\
&= \tilde{I}^{\mathbf{V}_\infty,\alpha}(\mathbf{U}^{(l_0)}) - \delta \\
&\geq \rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha) - \delta,
\end{aligned}
$$

where $\mathbf{V}_\infty = (V_{1,\infty}, V_{2,\infty}, V_{3,\infty})$. Letting $\delta \to +0$, then we have

$$
\liminf_{n\to\infty} \frac{c_{\varepsilon_n}}{\varepsilon_n^N} \geq \rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha).
$$

From Lemma 4.10 (i), we have

$$
\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) \geq \rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha).
$$

This is a contradiction to $(\mathrm{C1})_\alpha$.

(2) and (3) From l_0 in Theorem 4.4 (1), up to a subsequence, there exists $x_{l_0,0} \in \mathbb{R}^N$ such that

$$\begin{aligned}
&\mathbf{U}_n^{(l_0)} \rightharpoonup \mathbf{U}^{(l_0)} \quad \text{weakly in } \mathbb{H},\\
&U_{j,n}^{(l_0)} \to U_j^{(l_0)} \quad \text{in } C_{\mathrm{loc}}(\mathbb{R}^N),\\
&\mathbf{U}^{(l_0)} \neq (0,0,0), \quad x_{l_0,n} \to x_{l_0,0}.
\end{aligned}$$

Recall that

$$\mathbf{U}_n^{(l_0)}(y) = \mathbf{u}_n(x_{l_0,n} + \varepsilon_n y).$$

By the same argument as in the proof of Theorem 4.4 (1),

$$\begin{aligned}
&\liminf_{n\to\infty} \tilde{c}_{\varepsilon_n}\\
&= \liminf_{n\to\infty} \left\{ \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla U_{j,n}^{(l_0)}|^2 + V_j(x_{l_0,n} + \varepsilon_n y)(U_{j,n}^{(l_0)})^2 \right.\\
&\quad \left. + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} (U_{j,n}^{(l_0)})^{p+1} \right\}\\
&\geq \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla U_j^{(l_0)}|^2 + V_j(x_{l_0,0})(U_j^{(l_0)})^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} (U_j^{(l_0)})^{p+1}\\
&= \tilde{I}^{\mathbf{V}(x_{l_0,0}),\alpha}(\mathbf{U}^{(l_0)})\\
&\geq \rho(V_1(x_{l_0,0}), V_2(x_{l_0,0}), V_3(x_{l_0,0}); \alpha)\\
&\geq \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha),
\end{aligned}$$

where

$$\mathbf{V}(x_{l_0,0}) := (V_1(x_{l_0,0}), V_2(x_{l_0,0}), V_3(x_{l_0,0})).$$

From Lemma 4.10 (i), we have

$$\begin{aligned}
&\lim_{n\to\infty} \tilde{c}_{\varepsilon_n} = \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha),\\
&\rho(V_1(x_{l_0,0}), V_2(x_{l_0,0}), V_3(x_{l_0,0}); \alpha) = \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha)\\
&\mathbf{U}^{(l_0)} \text{ is a minimizer for } \rho(V_1(x_{l_0,0}), V_2(x_{l_0,0}), V_3(x_{l_0,0}); \alpha).
\end{aligned}$$

Similar to Remark 4.1, $\mathbf{U}^{(l_0)}$ is a ground state of $(\tilde{\mathcal{P}}^{\mathbf{V}(x_{l_0,0}),\alpha})$. Moreover, we have

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}|\nabla U_{j,n}^{(l_0)}|^2+V_j(x_{l_0,n}+\varepsilon_n y)(U_{j,n}^{(l_0)})^2\geq\int_{\mathbb{R}^N}|\nabla U_j^{(l_0)}|^2+V_j(x_{l_0,0})(U_j^{(l_0)})^2,$$
$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}(U_{j,n}^{(l_0)})^{p+1}\geq\int_{\mathbb{R}^N}(U_j^{(l_0)})^{p+1}$$

and

$$\begin{aligned}&\frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla U_{j,n}^{(l_0)}|^2+V_j(x_{l_0,n}+\varepsilon_n y)(U_{j,n}^{(l_0)})^2+\frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N}(U_{j,n}^{(l_0)})^{p+1}\\&\to\frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla U_j^{(l_0)}|^2+V_j(x_{l_0,0})(U_j^{(l_0)})^2+\frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N}(U_j^{(l_0)})^{p+1}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla U_{j,n}^{(l_0)}|^2+V_j(x_{l_0,n}+\varepsilon_n y)(U_{j,n}^{(l_0)})^2&=\int_{\mathbb{R}^N}|\nabla U_j^{(l_0)}|^2+V_j(x_{l_0,0})(U_j^{(l_0)})^2,\\ \lim_{n\to\infty}\int_{\mathbb{R}^N}(U_{j,n}^{(l_0)})^{p+1}&=\int_{\mathbb{R}^N}(U_j^{(l_0)})^{p+1}.\end{aligned}\tag{4.13}$$

In addition, it follows that

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}\nabla U_{j,n}^{(l_0)}\cdot\nabla\varphi+V_j(x_{l_0,n}+\varepsilon_n y)U_{j,n}^{(l_0)}\varphi\\&=\int_{\mathbb{R}^N}\nabla U_j^{(l_0)}\cdot\nabla\varphi+V_j(x_{l_0,0})U_j^{(l_0)}\varphi\quad\text{for all }\varphi\in H^1(\mathbb{R}^N).\end{aligned}\tag{4.14}$$

From (4.13) and (4.14), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla U_{j,n}^{(l_0)}-\nabla U_j^{(l_0)}|^2+V_j(x_{l_0,n}+\varepsilon_n y)(U_{j,n}^{(l_0)}-U_j^{(l_0)})^2=0,$$

that is,

$$\|U_{j,n}^{(l_0)}-U_j^{(l_0)}\|_{H^1}\to 0.$$

□

Remark 4.7. Suppose that (V1),(V2) and $(C1)_\alpha$. If $\mathbf{U}^{(l_0)}$ is vector, then the same conclusion as in Theorem 4.5 holds. On the other hand, if $\mathbf{U}^{(l_0)}$ is scalar, then the same conclusion as in Theorem 4.6 holds. Here $\mathbf{U}^{(l_0)}$ appeared in the proof of Theorem 4.4.

Indeed, if $\mathbf{U}^{(l_0)}$ is vector, then $\mathbf{U}^{(l)}$ is also vector ($l \neq l_0$). Indeed, by the same argument as in the proof of Theorem 4.5 (1) (Step 2), $\mathbf{U}^{(l)} \neq (0,0,0)$ for all $l \in \{1,2,3\}$. Suppose that there exists $k_0 \neq l_0$ such that $\mathbf{U}^{(k_0)}$ is scalar. For simplicity, we may assume that $l_0 = 1,\ k_0 = 2$, $\mathbf{U}^{(2)} = (U_1^{(2)}, 0, 0)$, $U_1^{(2)} \neq 0$. By the same argument as in the proof of Theorem 4.4 (3), we have $\|U_{j,n}^{(2)} - U_j^{(2)}\|_{H^1} \to 0$. Since $\mathbf{U}^{(2)} = (U_1^{(2)}, 0, 0)$, we have $U_{1,n}^{(2)} \to U_1^{(2)}$, $U_{j,n}^{(2)} \to 0$ ($j = 2, 3$). Since $U_{j,n}^{(1)}(y) = U_{j,n}^{(2)}(y + (x_{1,n} - x_{2,n})/\varepsilon_n)$, we have $U_{j,n}^{(1)} \to 0$ ($j = 2, 3$). Thus we obtain $U_j^{(1)} = 0$ ($j = 2, 3$). This contradicts that $\mathbf{U}^{(1)}$ is vector. The rest of the claims of Theorem 4.5 holds by the same argument as in the proof of Theorem 4.5.

On the other hand, if $\mathbf{U}^{(l_0)}$ is scalar, then $U_l^{(l)} = 0$ ($l \neq l_0$). Indeed, from Theorem 4.4, it follows that $U_{l_0}^{(l_0)} \neq 0$ and $\|U_{j,n}^{(l_0)} - U_j^{(l_0)}\|_{H^1} \to 0$. Thus we have $U_{l,n}^{(l_0)} \to 0$ ($l \neq l_0$). Since $U_{l,n}^{(l)}(y) = U_{l,n}^{(l_0)}(y + (x_{l,n} - x_{l_0,n})/\varepsilon_n)$, we have $U_{l,n}^{(l)} \to 0$ ($l \neq l_0$). The rest of the claims of Theorem 4.6 holds by the same argument as in the proof of Theorem 4.6.

4.4 Proof of Theorem 4.5

We divide the proof of Theorem 4.5 into three parts. In subsection 4.4.1, we show Theorem 4.5 (1). Subsection 4.4.2 is devoted to the proof of Theorem 4.5 (2)–(3). Finally, Theorem 4.5 (4) is proved in subsection 4.4.3.

4.4.1 Proof of Theorem 4.5 (1)

Proof of Theorem 4.5 (1). By the same argument as in Theorem 4.4, up to a subsequence, there exist $l_0 \in \{1,2,3\}$, $x_{l_0,0} \in \mathbb{R}^N$ and $\mathbf{U}^{(l)} \in \mathbb{H}$ ($l =$

1, 2, 3) such that

$$\begin{aligned} \mathbf{U}_n^{(l)} &\rightharpoonup \mathbf{U}^{(l)} \quad \text{weakly in } \mathbb{H}, \\ U_{j,n}^{(l)} &\to U_j^{(l)} \quad \text{in } C_{\mathrm{loc}}(\mathbb{R}^N), \\ U_{l_0}^{(l_0)} &\neq 0, \\ \mathbf{U}^{(l_0)} & \text{ is a ground state of } (\tilde{\mathcal{P}}^{\mathbf{V}(x_{l_0,0}),\alpha}), \\ \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x);\alpha) &= \rho(V_1(x_{l_0,0}), V_2(x_{l_0,0}), V_3(x_{l_0,0});\alpha). \end{aligned}$$

(Step 1) $\mathbf{U}^{(l_0)}$ is vector.
If $\mathbf{U}^{(l_0)}$ is scalar, then

$$\begin{aligned} &\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x);\alpha) = \rho(V_1(x_{l_0,0}), V_2(x_{l_0,0}), V_3(x_{l_0,0});\alpha) \\ &= \tilde{I}^{\mathbf{V}(x_{l_0,0}),\alpha}(\mathbf{U}^{(l_0)}) \geq \min_{j=1,2,3} c_1^{V_j(x_{l_0,0})} \geq \min_{j=1,2,3} c_1^{V_{j,0}}. \end{aligned}$$

This contradicts $(\mathrm{C2})_\alpha$.

(Step 2) For all $l = 1, 2, 3$, $\mathbf{U}^{(l)}$ is vector.
We assume that there exists $k_0 \in \{1, 2, 3\}$ such that $\mathbf{U}^{(k_0)} = (0, 0, 0)$. Since $U_{k_0}^{(k_0)}(0) = 0$ and

$$U_{k_0,n}^{(k_0)}(0) = u_{k_0,n}(x_{k_0,n}) \geq u_{k_0,n}(x_{l,n} + \varepsilon_n y) = U_{k_0,n}^{(l)}(y) \quad (l = 1, 2, 3)$$

and $U_{k_0,n}^{(l)} \to U_{k_0}^{(l)}$ in $C_{\mathrm{loc}}(\mathbb{R}^N)$, it follows that $U_{k_0}^{(l)} = 0$ $(l = 1, 2, 3)$. In particular, we have $U_{k_0}^{(l_0)} = 0$. This is contrary to $\mathbf{U}^{(l_0)}$ being vector. Also, by the same argument as in (Step 1), there does not exist $l \in \{1, 2, 3\}$ such that $\mathbf{U}^{(l)}$ is scalar.

(Step 3) $\sup_{n\in\mathbb{N}} |x_{l,n}| < \infty$ for all $l = 1, 2, 3$.
From (Step 2), it follows that $\mathbf{U}^{(l)} \neq (0, 0, 0)$. By the same argument as in Lemma 4.4 (1), it follows that $\sup_{n\in\mathbb{N}} |x_{l,n}| < \infty$ for all $l = 1, 2, 3$. □

4.4.2 Proof of Theorem 4.5 (2) and (3)

Proof of Theorem 4.5 (2) and (3). (Step 1) From Theorem 4.5 (1), up to a subsequence, for all $l \in \{1,2,3\}$, there exists $x_{l,0} \in \mathbb{R}^N$ such that

$$\mathbf{U}^{(l)} \neq (0,0,0), \quad x_{l,n} \to x_{l,0} \quad \text{for all } l = 1,2,3.$$

By the same argument as in Theorem 4.4 (2) and (3), it follows that for all $l \in \{1,2,3\}$,

$$\begin{aligned}
&\tilde{c}_\varepsilon = \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) + o(1), \quad \text{as } \varepsilon \to +0,\\
&x_{l,n} \to x_{l,0},\\
&\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) = \rho(V_1(x_{l,0}), V_2(x_{l,0}), V_3(x_{l,0}); \alpha),\\
&\mathbf{U}^{(l)} \text{ is a ground state of } (\tilde{\mathcal{P}}^{\mathbf{V}(x_{l,0}),\alpha}),\\
&\|\mathbf{U}_n^{(l)} - \mathbf{U}^{(l)}\|_{\mathbb{H}} \to 0,
\end{aligned}$$

where $\mathbf{V}(x_{l,0}) = (V_1(x_{l,0}), V_2(x_{l,0}), V_3(x_{l,0}))$.

Now, we show that

$$\sup_{n\in\mathbb{N}} \frac{|x_{j,n} - x_{k,n}|}{\varepsilon_n} < \infty \quad \text{for all } j,k \text{ with } j \neq k. \tag{4.15}$$

We assume that

$$\sup_{n\in\mathbb{N}} \frac{|x_{j,n} - x_{k,n}|}{\varepsilon_n} = \infty.$$

For simplicity, we may assume that $j = 1$ and $k = 2$. Then, up to a subsequence,

$$\frac{|x_{1,n} - x_{2,n}|}{\varepsilon_n} \to \infty.$$

By the same argument as in the proof of Theorem 4.4 (2) and (3), we have

$$\|U_{j,n}^{(l)} - U_j^{(l)}\|_{H^1} \to 0 \quad \text{for all } j,l = 1,2,3.$$

Set

$$W_{j,n} := U_{j,n}^{(j)}, \quad W_{j,0} := U_j^{(j)}.$$

In particular,

$$\|W_{j,n} - W_{j,0}\|_{H^1} \to 0 \quad \text{for all } j = 1, 2, 3.$$

Then

$$\begin{aligned}
&c_{\varepsilon_n} = I_{\varepsilon_n}(\mathbf{u}_n) \\
&= \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} \varepsilon_n^2 |\nabla u_{j,n}|^2 + V_j(x) u_{j,n}^2 \\
&\quad - \frac{1}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} (u_{j,n})^{p+1} - \alpha \int_{\mathbb{R}^N} u_{1,n} u_{2,n} u_{3,n} \\
&= \varepsilon_n^N \left\{ \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla W_{j,n}|^2 + V_j(x_{j,n} + \varepsilon_n y)(W_{j,n})^2 - \frac{1}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} (W_{j,n})^{p+1} \right. \\
&\quad \left. -\alpha \int_{\mathbb{R}^N} W_{1,n}(y) W_{2,n}(y + x_{12,n}) W_{3,n}(y + x_{13,n}) \right\}, \qquad (4.16)
\end{aligned}$$

where

$$x_{jk,n} = \frac{x_{j,n} - x_{k,n}}{\varepsilon_n}.$$

Since $\|W_{j,n} - W_{j,0}\|_{H^1} \to 0$, we have

$$\sup_{n \in \mathbb{N}} \int_{|y| \geq R} (W_{j,n})^q \to 0, \quad \text{as } R \to \infty, \ 2 \leq q < 2^*. \qquad (4.17)$$

Indeed, since $\|W_{j,n} \to W_{j,0}\|_{H^1} \to 0$, up to a subsequence, for all $q \in [2, 2^*)$, there exists $g \in L^q(\mathbb{R}^N)$ such that

$$W_{j,n} \leq g \quad \text{a.e. in } \mathbb{R}^N.$$

Thus,

$$\sup_{n \in \mathbb{N}} \int_{|y| \geq R} (W_{j,n})^q \leq \int_{|y| \geq R} g^q \to 0, \quad \text{as } R \to \infty.$$

From (4.17), for all $\delta > 0$, there exist $R, L > 0$ such that

$$\left(\int_{|y| \geq L} W_{2,n}(y)^3 \right)^{1/3} < \delta, \quad \left(\int_{|y| \geq R} W_{1,n}(y)^3 \right)^{1/3} < \delta.$$

Since $|x_{12,n}| \to \infty$, for n sufficiently large, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} W_{1,n}(y)W_{2,n}(y+x_{12,n})W_{3,n}(y+x_{13,n}) \\
&= \int_{B_R} W_{1,n}(y)W_{2,n}(y+x_{12,n})W_{3,n}(y+x_{13,n}) \\
&\quad + \int_{\mathbb{R}^N\setminus B_R} W_{1,n}(y)W_{2,n}(y+x_{12,n})W_{3,n}(y+x_{13,n}) \\
&\le \|W_{1,n}\|_{L^3} \left(\int_{|y|\ge L} W_{2,n}(y)^3\right)^{1/3} \|W_{3,n}\|_{L^3} \\
&\quad + \left(\int_{|y|\ge R} W_{1,n}(y)^3\right)^{1/3} \|W_{2,n}\|_{L^3}\|W_{3,n}\|_{L^3} \\
&< C\delta.
\end{aligned}$$

Thus

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} W_{1,n}(y)W_{2,n}(y+x_{12,n})W_{3,n}(y+x_{13,n}) \le C\delta.$$

Letting $\delta \to +0$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W_{1,n}(y)W_{2,n}(y+x_{12,n})W_{3,n}(y+x_{13,n}) = 0. \tag{4.18}$$

$W_{j,n}$ satisfies

$$\begin{cases}
-\Delta W_{1,n} + V_1(x_{1,n}+\varepsilon_n y)W_{1,n} = (W_{1,n})^p + \alpha W_{2,n}(y+x_{12,n})W_{3,n}(y+x_{13,n}), \\
-\Delta W_{2,n} + V_2(x_{2,n}+\varepsilon_n y)W_{2,n} = (W_{2,n})^p + \alpha W_{1,n}(y+x_{21,n})W_{3,n}(y+x_{23,n}), \\
-\Delta W_{3,n} + V_3(x_{3,n}+\varepsilon_n y)W_{3,n} = (W_{3,n})^p + \alpha W_{1,n}(y+x_{31,n})W_{2,n}(y+x_{32,n}).
\end{cases} \tag{4.19}$$

Since (4.18) and $W_{j,n} \to W_{j,0}$ in $H^1(\mathbb{R}^N)$, $W_{j,0}$ satisfies

$$\begin{cases}
-\Delta W_{1,0} + V_1(x_{1,0})W_{1,0} = (W_{1,0})^p, \\
-\Delta W_{2,0} + V_2(x_{2,0})W_{2,0} = (W_{2,0})^p, \\
-\Delta W_{3,0} + V_3(x_{3,0})W_{3,0} = (W_{3,0})^p.
\end{cases} \tag{4.20}$$

Indeed, noting that $x_{21,n} = -x_{12,n}$ and $x_{32,n} - x_{31,n} = x_{12,n}$, we have

$$\begin{aligned}&|x_{21,n}| \to \infty,\\ &|x_{31,n}| \to \infty \quad \text{or} \quad |x_{32,n}| \to \infty.\end{aligned}$$

From (4.19), as in the proof of (4.18), we have (4.20). It follows that $\mathbf{W}_0 \neq (0,0,0)$ from Lemma 4.11. From (4.16) and noting that

$$\begin{aligned} c_1^{V_j(x_{j,0})} &= \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} I_1^{V_j(x_{j,0})}(tu) \\ &\geq \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} I_1^{V_{j,0}}(tu) = c_1^{V_{j,0}} \end{aligned}$$

(see for example, [3, 57]), we have

$$\begin{aligned} &\lim_{n\to\infty} \tilde{c}_{\varepsilon_n} \\ &= \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla W_{j,0}|^2 + V_j(x_{j,0})(W_{j,0})^2 - \frac{1}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N}(W_{j,0})^{p+1} \\ &\geq \min_{j=1,2,3} c_1^{V_j(x_{j,0})} \geq \min_{j=1,2,3} c_1^{V_{j,0}}. \end{aligned}$$

From Lemma 4.10 (i), we have

$$\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) \geq \min_{j=1,2,3} c_1^{V_{j,0}}.$$

This contradicts the assumption $(C2)_\alpha$. Therefore, we have proved (4.15). Hence

$$x_{1,0} = x_{2,0} = x_{3,0} =: x_0. \tag{4.21}$$

Up to a subsequence, there exists $x_{jk,0} \in \mathbb{R}^N$ such that

$$x_{jk,n} = \frac{x_{j,n} - x_{k,n}}{\varepsilon_n} \to x_{jk,0}.$$

From (4.19), for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla W_{1,n} \cdot \nabla\varphi + V_1(x_{1,n} + \varepsilon_n y) W_{1,n}\varphi$$

$$= \int_{\mathbb{R}^N} (W_{1,n})^p \varphi + \alpha \int_{\mathbb{R}^N} \varphi(y) W_{2,n}(y + x_{12,n}) W_{3,n}(y + x_{13,n}).$$

Since $\|W_{j,n} - W_{j,0}\|_{L^3} \to 0$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} \varphi(y) W_{2,n}(y + x_{12,n}) W_{3,n}(y + x_{13,n}) \right. \\
&\quad \left. - \int_{\mathbb{R}^N} \varphi(y) W_{2,0}(y + x_{12,0}) W_{3,0}(y + x_{13,0}) \right| \\
&= \left| \int_{\mathbb{R}^N} \varphi(y) W_{2,n}(y + x_{12,n}) W_{3,n}(y + x_{13,n}) \right. \\
&\quad \left. - \int_{\mathbb{R}^N} \varphi(y) W_{2,0}(y + x_{12,n}) W_{3,0}(y + x_{13,n}) \right| \\
&\quad + \left| \int_{\mathbb{R}^N} \varphi(y) W_{2,0}(y + x_{12,n}) W_{3,0}(y + x_{13,n}) \right. \\
&\quad \left. - \int_{\mathbb{R}^N} \varphi(y) W_{2,0}(y + x_{12,0}) W_{3,0}(y + x_{13,0}) \right| \\
&\to 0.
\end{aligned}$$

Then it holds that

$$-\Delta W_{1,0} + V_1(x_0) W_{1,0} = (W_{1,0})^p + \alpha W_{2,0}(y + x_{12,0}) W_{3,0}(y + x_{13,0}).$$

By the same argument as in the above, we have

$$\begin{cases} -\Delta W_{2,0} + V_2(x_0) W_{2,0} = (W_{2,0})^p + \alpha W_{1,0}(y + x_{21,0}) W_{3,0}(y + x_{23,0}), \\ -\Delta W_{3,0} + V_3(x_0) W_{3,0} = (W_{3,0})^p + \alpha W_{1,0}(y + x_{31,0}) W_{2,0}(y + x_{32,0}). \end{cases}$$

Thus $(W_{1,0}, W_{2,0}(\cdot + x_{12,0}), W_{3,0}(\cdot + x_{13,0})) \in \tilde{\mathcal{N}}^{\mathbf{V}(x_0),\alpha}$. From (4.16),

$$\begin{aligned}
&\rho(V_1(x_0), V_2(x_0), V_3(x_0); \alpha) \\
&= \lim_{n\to\infty} \tilde{c}_{\varepsilon_n} \\
&= \lim_{n\to\infty} \left\{ \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla W_{j,n}|^2 + V_j(x_{j,n} + \varepsilon_n y)(W_{j,n})^2 \right. \\
&\quad \left. - \frac{1}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} (W_{j,n})^{p+1} - \alpha \int_{\mathbb{R}^N} W_{1,n}(y) W_{2,n}(y + x_{12,n}) W_{3,n}(y + x_{13,n}) \right\}
\end{aligned}$$

$$= \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla W_{j,0}|^2 + V_j(x_0)(W_{j,0})^2$$
$$- \frac{1}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N}(W_{j,0})^{p+1} - \alpha\int_{\mathbb{R}^N} W_{1,0}(y)W_{2,0}(y+x_{12,0})W_{3,0}(y+x_{13,0}),$$

Thus

$$\begin{gathered}(W_{1,0}, W_{2,0}(\cdot + x_{12,0}), W_{3,0}(\cdot + x_{13,0})) \\ \text{is a minimizer for } \rho(V_1(x_0), V_2(x_0), V_3(x_0); \alpha).\end{gathered} \tag{4.22}$$

(Step 2) Next, we prove $x_{jk,0} = 0$ for all $j,k \in \{1,2,3\}$, $j \neq k$. This means $|x_{j,n} - x_{k,n}|/\varepsilon_n \to 0$.

Since $\mathbf{U}^{(l)}$ is a ground state of $(\tilde{\mathcal{P}}^{\mathbf{V}(x_{l,0}),\alpha})$ for all $l = 1,2,3$, then the functions $U_j^{(l)}$ $(j = 1,2,3)$ are satisfy

$$\begin{cases} -\Delta U_1^{(l)} + V_1(x_0)U_1^{(l)} = (U_1^{(l)})^p + \alpha U_2^{(l)}U_3^{(l)}, \\ -\Delta U_2^{(l)} + V_2(x_0)U_2^{(l)} = (U_2^{(l)})^p + \alpha U_1^{(l)}U_3^{(l)}, \\ -\Delta U_3^{(l)} + V_3(x_0)U_3^{(l)} = (U_3^{(l)})^p + \alpha U_1^{(l)}U_2^{(l)}, \\ U_j^{(l)} \geq 0, \quad (j = 1,2,3), \end{cases}$$

where $\mathbf{V}(x_{l,0}) = (V_1(x_{l,0}), V_2(x_{l,0}), V_3(x_{l,0}))$. From Theorem 4.5 (1), it follows that $U_j^{(l)} \not\equiv 0$ for all $j = 1,2,3$. By the strong maximum principle, we have $U_j^{(l)} > 0$ in $\mathbb{R}^N$ for all $j = 1,2,3$.

Now we claim that there exists a point $y_0 \in \mathbb{R}^N$ such that the functions $U_j^{(l)}$ are radially symmetric with respect to the origin y_0, that is $U_j^{(l)}(y) = U_j^{(l)}(|y - y_0|)$, $j = 1,2,3$. Moreover,

$$\frac{dU_j^{(l)}}{dr} < 0 \quad \text{for all } r = |y - y_0| > 0.$$

If $N \geq 2$, then it follows from Theorem 1 in Busca-Sirakov [9]. If $N = 1$, then it follows from Theorem 4.22 in Appendix. Since

$$U_{j,n}^{(j)}(0) = \max_{y\in\mathbb{R}^N} U_{j,n}^{(j)}(y), \quad U_{j,n}^{(j)} \to U_j^{(j)} \quad \text{in } C_{\mathrm{loc}}(\mathbb{R}^N),$$

we have

$$U_j^{(j)}(0) = \max_{y\in\mathbb{R}^N} U_j^{(j)}(y).$$

Thus we have $y_0 = 0$. Since $W_{j,0} = U_j^{(j)}$, $W_{j,0}$ is radially symmetric and strictly decreasing. From Burchard-Hajaiej [8], we have

$$\int_{\mathbb{R}^N} W_{1,0}(y)W_{2,0}(y+x_{12,0})W_{3,0}(y+x_{13,0}) \le \int_{\mathbb{R}^N} W_{1,0}(y)W_{2,0}(y)W_{3,0}(y).$$

Let $t_0 > 0$ be a number such that $t_0\mathbf{W}_0 \in \tilde{\mathcal{N}}^{\mathbf{V}(x_0),\alpha}$, where $\mathbf{V}(x_0) = (V_1(x_0), V_2(x_0), V_3(x_0))$. Then

$$\begin{aligned}
&\rho(V_1(x_0), V_2(x_0), V_3(x_0); \alpha)\\
&= \tilde{I}^{\mathbf{V}(x_0),\alpha}(W_{1,0}, W_{2,0}(\cdot + x_{12,0}), W_{3,0}(\cdot + x_{13,0}))\\
&\ge \tilde{I}^{\mathbf{V}(x_0),\alpha}(t_0(W_{1,0}, W_{2,0}(\cdot + x_{12,0}), W_{3,0}(\cdot + x_{13,0})))\\
&\ge \tilde{I}^{\mathbf{V}(x_0),\alpha}(t_0\mathbf{W}_0) \ge \rho(V_1(x_0), V_2(x_0), V_3(x_0); \alpha).
\end{aligned}$$

Thus we have

$$\tilde{I}^{\mathbf{V}(x_0),\alpha}(t_0(W_{1,0}, W_{2,0}(\cdot + x_{12,0}), W_{3,0}(\cdot + x_{13,0}))) = \tilde{I}^{\mathbf{V}(x_0),\alpha}(t_0\mathbf{W}_0),$$

that is,

$$\int_{\mathbb{R}^N} W_{1,0}(y)W_{2,0}(y+x_{12,0})W_{3,0}(y+x_{13,0}) = \int_{\mathbb{R}^N} W_{1,0}(y)W_{2,0}(y)W_{3,0}(y).$$

If $(x_{12,0}, x_{13,0}) \neq (0,0)$, from [8] we have

$$\int_{\mathbb{R}^N} W_{1,0}(y)W_{2,0}(y+x_{12,0})W_{3,0}(y+x_{13,0}) < \int_{\mathbb{R}^N} W_{1,0}(y)W_{2,0}(y)W_{3,0}(y).$$

This is a contradiction. Thus we have $x_{12,0} = x_{13,0} = 0$. From (4.22), $\mathbf{W}_0$ is a minimizer for $\rho(V_1(x_0), V_2(x_0), V_3(x_0); \alpha)$. That is, $\mathbf{W}_0$ is a ground state of $(\tilde{\mathcal{P}}^{\mathbf{V}(x_0),\alpha})$ and it follows that

$$\frac{|x_{j,n} - x_{k,n}|}{\varepsilon_n} \to 0, \quad \text{as } n \to \infty,\ j \neq k.$$

□

4.4.3 Proof of Theorem 4.5 (4)

Proof of Theorem 4.5 (4). Since $\mathbf{u}_n$ is a non-negative ground state of $(\mathcal{P}_{\varepsilon_n})$ and $\mathbf{U}_n^{(l)}(y) = \mathbf{u}(x_{l,n} + \varepsilon_n y)$, we have

$$\begin{cases} -\Delta U_{1,n}^{(l)} + V_1(x_{l,n} + \varepsilon_n y) U_{1,n}^{(l)} = (U_{1,n}^{(l)})^p + \alpha U_{2,n}^{(l)} U_{3,n}^{(l)}, \\ -\Delta U_{2,n}^{(l)} + V_2(x_{l,n} + \varepsilon_n y) U_{2,n}^{(l)} = (U_{2,n}^{(l)})^p + \alpha U_{1,n}^{(l)} U_{3,n}^{(l)}, \\ -\Delta U_{3,n}^{(l)} + V_3(x_{l,n} + \varepsilon_n y) U_{3,n}^{(l)} = (U_{3,n}^{(l)})^p + \alpha U_{1,n}^{(l)} U_{2,n}^{(l)}, \\ U_{j,n}^{(l)} \geq 0, \quad (l = 1, 2, 3). \end{cases}$$

Set $U_n^{(l)} := U_{1,n}^{(l)} + U_{2,n}^{(l)}$, then we have

$$-\Delta U_n^{(l)} + V_0 U_n^{(l)} \leq 2(U_n^{(l)})^p + \alpha U_{3,n}^{(l)} U_n^{(l)}.$$

Let $\eta > 0$ be a number such that $\eta < V_0$. Let $\varepsilon > 0$ be a number such that $\eta < V_0 - \varepsilon$. From Lemma 4.10 (v), it follows that

$$\sup_{n \in \mathbb{N}} U_{j,n}^{(l)}(y_0) \to 0, \quad \text{as } |y_0| \to \infty.$$

Then there exists $R > 0$ such that if $|y| \geq R$, then

$$\sup_{n \in \mathbb{N}} \{2(U_n^{(l)}(y))^{p-1} + \alpha U_{3,n}^{(l)}(y)\} < \varepsilon.$$

Then if $|y| \geq R$, we have

$$-\Delta U_n^{(l)} + (V_0 - \varepsilon) U_n^{(l)} \leq 0.$$

By the same argument as in the proof of Lemma 2.7 in [42], there exists $C_\eta > 0$ such that

$$U_{j,n}^{(l)}(y) \leq U_n^{(l)}(y) \leq C_\eta e^{-\sqrt{\eta}|y|} \quad \text{for all } y \in \mathbb{R}^N, \ n \in \mathbb{N}, \ j = 1, 2.$$

We can prove also that

$$U_{3,n}^{(l)}(y) \leq C_\eta e^{-\sqrt{\eta}|y|} \quad \text{for all } y \in \mathbb{R}^N, \ n \in \mathbb{N}.$$

Recall $U_{j,n}^{(l)}(y) = u_{j,n}(x_{l,n} + \varepsilon_n y)$. Then we have

$$u_{j,n}(x) \leq C_\eta e^{-\sqrt{\eta}|x - x_{l,n}|/\varepsilon_n} \quad \text{for all } x \in \mathbb{R}^N, \ n \in \mathbb{N}, \ j, l = 1, 2, 3.$$

In particular, we have

$$u_{j,n}(x) \leq C_\eta e^{-\sqrt{\eta}|x - x_{j,n}|/\varepsilon_n} \quad \text{for all } x \in \mathbb{R}^N, \ n \in \mathbb{N}, \ j = 1, 2, 3.$$

□

4.5 Proof of Theorem 4.6

To prove Theorem 4.6, we prove the following two lemmas needed later.

Lemma 4.12. Assume (V1),(V2) and fix α so that $(C1)_\alpha$ and $(C3)_\alpha$ hold. In addition, we assume that there exists $\alpha' > \alpha$ such that $(C3)_{\alpha'}$ holds. Let $y_0 \in \mathbb{R}^N$ be a point such that

$$\rho(V_1(y_0), V_2(y_0), V_3(y_0); \alpha) = \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha).$$

Let $\mathbf{v}$ be a ground state of $(\tilde{\mathcal{P}}^{\mathbf{V}(y_0),\alpha})$, where $\mathbf{V}(y_0) = (V_1(y_0), V_2(y_0), V_3(y_0))$. Then, there exists $j_0 \in \{1, 2, 3\}$ such that $v_{j_0} \neq 0$ and $v_j = 0$ for $j \neq j_0$.

Proof. If $v_j \neq 0$ for all $j = 1, 2, 3$, by the same argument as in Theorem 1.4 in [30], we have $\rho(V_1(y_0), V_2(y_0), V_3(y_0); \alpha) > \rho(V_1(y_0), V_2(y_0), V_3(y_0); \alpha')$. Thus

$$\begin{aligned} \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) &= \rho(V_1(y_0), V_2(y_0), V_3(y_0); \alpha) \\ &> \rho(V_1(y_0), V_2(y_0), V_3(y_0); \alpha') \\ &\geq \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha'). \end{aligned} \tag{4.23}$$

Since $(C3)_\alpha$ and $(C3)_{\alpha'}$ hold, it follows that

$$\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) = \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha').$$

This contradicts (4.23). □

Lemma 4.13. Assume (V1),(V2) and fix α so that $(C1)_\alpha$ holds. Let $\{\varepsilon_n\}_{n=1}^\infty \subset (0, \infty)$ such that $\varepsilon_n \to +0$ and let $\mathbf{u}_n$ be a ground state of $(\mathcal{P}_{\varepsilon_n,\alpha})$. Let $x_{j,n}$ be a maximum point of $u_{j,n}$. Set $U_{j,n}^{(l)}(y) = u_{j,n}(x_{l,n} + \varepsilon_n y)$. Then, if $\sup_{n\in\mathbb{N}}|x_{l,n}| = \infty$, then $U_{j,n}^{(l)} \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ for all $j = 1, 2, 3$.

Proof. Recall that from Lemma 4.10 (iii),(iv), $\{U_{j,n}^{(l)}\}_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$ and up to a subsequence, there exists $U_j^{(l)} \in H^1(\mathbb{R}^N)$ such that

$U_{j,n}^{(l)} \rightharpoonup U_j^{(l)}$ weakly in $H^1(\mathbb{R}^N)$. If $\mathbf{U}^{(l)} \neq (0,0,0)$, by the same argument as in the proof of Theorem 4.4 (2) and (3), we have

$$\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) \geq \rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha).$$

This contradicts $(\mathrm{C1})_\alpha$. Hence, we obtain $\mathbf{U}^{(l)} = (0,0,0)$. □

Proof of Theorem 4.6. From Lemma 4.10 (iv), up to a subsequence, it follows that $U_{j,n}^{(l)} \rightharpoonup U_j^{(l)}$ weakly in $H^1(\mathbb{R}^N)$. From Theorem 4.4 and $(\mathrm{C3})_\alpha$, up to a subsequence, there exist $l_0 \in \{1,2,3\}$ and $x_{l_0,0} \in \mathbb{R}^N$ such that

$$\begin{aligned}
&c_\varepsilon = \varepsilon^N \left(\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) + o(1) \right) \\
&\quad = \varepsilon^N \left(\min_{j=1,2,3} c_1^{V_{j,0}} + o(1) \right), \quad \text{as } \varepsilon \to +0, \\
&x_{l_0,n} \to x_{l_0,0}, \\
&\rho(V_1(x_{l_0,0}), V_2(x_{l_0,0}), V_3(x_{l_0,0}); \alpha) = \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha), \\
&\mathbf{U}^{(l_0)} \text{ is a ground state of } (\tilde{\mathcal{P}}^{V(x_{l_0,0}),\alpha}), \quad U_{j,n}^{(l_0)} \to U_j^{(l_0)} \text{ in } H^1(\mathbb{R}^N).
\end{aligned} \tag{4.24}$$

From $(\mathrm{C3})_\alpha$ and $(\mathrm{C3})_{\alpha'}$ and Lemma 4.12, there exists $j_0 \in \{1,2,3\}$ such that $U_j^{(l_0)} = 0$ for all $j \neq j_0$. Since $U_{l_0}^{(l_0)} \neq 0$, it follows that $j_0 = l_0$. Thus it holds that $U_{l_0}^{(l_0)} \neq 0$ and $U_j^{(l_0)} = 0$ for all $j \neq l_0$. From (4.24), we have

$$\begin{aligned}
&\|U_{l_0,n}^{(l_0)} - U_{l_0}^{(l_0)}\|_{H^1} \to 0, \\
&\|U_{j,n}^{(l_0)}\|_{H^1} \to 0 \quad \text{for all } j \neq l_0.
\end{aligned}$$

Since $U_{j,n}^{(l_0)}(y) = U_{j,n}^{(j)}(y + (x_{l_0,n} - x_{j,n})/\varepsilon_n)$, we have

$$\|U_{j,n}^{(j)}\|_{H^1} \to 0 \quad \text{for all } j \neq l_0.$$

Note that $\mathbf{U}^{(l_0)}$ satisfies the following system:

$$\begin{cases}
-\Delta U_1^{(l_0)} + V_1(x_{l_0,0}) U_1^{(l_0)} = (U_1^{(l_0)})^p + \alpha U_2^{(l_0)} U_3^{(l_0)}, \\
-\Delta U_2^{(l_0)} + V_2(x_{l_0,0}) U_2^{(l_0)} = (U_2^{(l_0)})^p + \alpha U_1^{(l_0)} U_3^{(l_0)}, \\
-\Delta U_3^{(l_0)} + V_3(x_{l_0,0}) U_3^{(l_0)} = (U_3^{(l_0)})^p + \alpha U_1^{(l_0)} U_2^{(l_0)}, \\
U_j^{(l_0)} \geq 0.
\end{cases}$$

Since $U_{l_0}^{(l_0)} \neq 0$ and $U_j^{(l_0)} = 0$ $(j \neq l_0)$, $U_{l_0}^{(l_0)}$ satisfies

$$\begin{cases} -\Delta U_{l_0}^{(l_0)} + V_{l_0}(x_{l_0,0}) U_{l_0}^{(l_0)} = (U_{l_0}^{(l_0)})^p, \\ U_{l_0}^{(l_0)} \geq 0. \end{cases}$$

Then it follows that

$$\begin{aligned} \min_{j=1,2,3} c_1^{V_{j,0}} &= \liminf_{n\to\infty} \tilde{c}_{\varepsilon_n} \\ &= \liminf_{n\to\infty} \left\{ \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla U_{j,n}^{(l_0)}|^2 + V_j(x_{l_0,n} + \varepsilon_n y)(U_{j,n}^{(l_0)})^2 \right. \\ &\qquad\qquad \left. + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} (U_{j,n}^{(l_0)})^{p+1} \right\} \\ &= \frac{1}{6} \int_{\mathbb{R}^N} |\nabla U_{l_0}^{(l_0)}|^2 + V_{l_0}(x_{l_0,0})(U_{l_0}^{(l_0)})^2 + \frac{p-2}{3(p+1)} \int_{\mathbb{R}^N} (U_{l_0}^{(l_0)})^{p+1} \\ &= I_1^{V_{l_0}(x_{l_0,0})}(U_{l_0}^{(l_0)}) \geq c_1^{V_{l_0}(x_{l_0,0})} \geq c_1^{V_{l_0,0}} \geq \min_{j=1,2,3} c_1^{V_{j,0}} = c_1^{V_0}. \end{aligned}$$

Hence we have $V_{l_0}(x_{l_0,0}) = V_{l_0,0} = V_0$ and $\min_{j=1,2,3} c_1^{V_{j,0}} = c_1^{V_{l_0,0}}$. From (4.24), we have

$$c_\varepsilon = \varepsilon^N \left(\min_{j=1,2,3} c_1^{V_{j,0}} + o(1) \right) = \varepsilon^N \left(c_1^{V_{l_0,0}} + o(1) \right), \quad \text{as } \varepsilon \to +0.$$

Moreover, since $V_{l_0}(x_{l_0,0}) = V_{l_0,0} = V_0$, $U_{l_0}^{(l_0)}$ satisfies

$$\begin{cases} -\Delta U_{l_0}^{(l_0)} + V_0 U_{l_0}^{(l_0)} = (U_{l_0}^{(l_0)})^p, \\ U_{l_0}^{(l_0)} \geq 0. \end{cases}$$

By the elliptic regularity, we have

$$U_{l_0}^{(l_0)} \in C^2(\mathbb{R}^N), \quad \lim_{|x|\to\infty} U_{l_0}^{(l_0)}(x) = 0.$$

By the strong maximum principle, we have $U_{l_0}^{(l_0)} > 0$. In addition, since

$$U_{l_0,n}^{(l_0)}(0) = \max_{y\in\mathbb{R}^N} U_{l_0,n}^{(l_0)}(y), \quad U_{l_0,n}^{(l_0)} \to U_{l_0}^{(l_0)} \quad \text{in } C_{\mathrm{loc}}(\mathbb{R}^N),$$

it follows that $U_{l_0}^{(l_0)}(0) = \max_{y\in\mathbb{R}^N} U_{l_0}^{(l_0)}(y)$. Thus from [21] and [32], it holds that $U_{l_0}^{(l_0)} = W$. □

4.6 Asymptotic behavior of $\inf_{x\in\mathbb{R}^N}\rho(V_1(x),\,V_2(x),\,V_3(x);\alpha)$ as $\alpha\to\infty$

In this section, we consider the asymptotic expansion of $\inf_{x\in\mathbb{R}}\rho(V_1(x),V_2(x),$ $V_3(x);\alpha)$ as $\alpha\to\infty$. We prove the following proposition:

Proposition 4.14. Suppose that (V1),(V2). Then it follows that

$$\begin{aligned}&\inf_{x\in\mathbb{R}^N}\rho(V_1(x),V_2(x),V_3(x);\alpha)\\ &=\inf_{x\in\mathbb{R}^N}\rho^\infty(V_1(x),V_2(x),V_3(x))/\alpha^2+o(1/\alpha^2),\quad \text{as } \alpha\to\infty,\end{aligned}$$

where

$$\begin{aligned}\lambda&:=(\lambda_1,\lambda_2,\lambda_3),\\ \rho^\infty(\lambda_1,\lambda_2,\lambda_3)&:=\inf_{\mathbf{w}\in\mathcal{M}^{\lambda,\infty}}J^{\lambda,\infty}(\mathbf{w}),\\ J^{\lambda,\infty}(\mathbf{w})&:=\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla w_j|^2+\lambda_j w_j^2-\int_{\mathbb{R}^N}w_1w_2w_3,\\ \mathcal{M}^{\lambda,\infty}&:=\{\mathbf{w}\in\mathbb{H}\setminus\{(0,0,0)\}\mid F^{\lambda,\infty}(\mathbf{w})=0\},\\ F^{\lambda,\infty}(\mathbf{w})&:=\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla w_j|^2+\lambda_j w_j^2-3\int_{\mathbb{R}^N}w_1w_2w_3.\end{aligned}$$

To prove Proposition 4.14, we prove the following lemmas needed later. The following lemma follows from Lemma 2.5 in [43].

Lemma 4.15. Suppose that (V1),(V2). Then it follows that

$$\begin{aligned}&\inf_{x\in\mathbb{R}^N}\rho(V_1(x),V_2(x),V_3(x);\alpha_1)\\ &\geq\inf_{x\in\mathbb{R}^N}\rho(V_1(x),V_2(x),V_3(x);\alpha_2)\quad\text{for all } 0<\alpha_1<\alpha_2.\end{aligned}$$

Now we Set

$$Q(x;\alpha):=\rho(V_1(x),V_2(x),V_3(x);\alpha),\quad Q_0(\alpha):=\inf_{x\in\mathbb{R}^N}Q(x;\alpha).$$

We prove the continuity of Q_0 over $[0,\infty)$.

Lemma 4.16. Suppose that (V1),(V2). Then $Q_0 : [0, \infty) \to \mathbb{R}$ is continuous.

Proof. Let $\alpha_0 \geq 0$ and let $\{\alpha_n\}_{n=1}^{\infty}$ be a positive sequence such that $\alpha_n \to \alpha_0$ as $n \to \infty$. We show the continuity of Q_0 at $\alpha = \alpha_0$. From the definition of infimum, for all $\delta > 0$, there exists $x_\delta \in \mathbb{R}^N$ such that

$$Q(x_\delta; \alpha_0) < Q_0(\alpha_0) + \delta.$$

Then there exists a non-negative minimizer $\mathbf{v}_\delta$ for $Q(x_\delta; \alpha_0)$. Moreover, there exists $t_{\delta,n} > 0$ such that $t_{\delta,n}\mathbf{v}_\delta \in \tilde{\mathcal{N}}^{\mathbf{V}(x_\delta),\alpha_n}$, where $\mathbf{V}(x_\delta) = (V_1(x_\delta), V_2(x_\delta), V_3(x_\delta))$. Since $t_{\delta,n}\mathbf{v}_\delta \in \tilde{\mathcal{N}}^{\mathbf{V}(x_\delta),\alpha_n}$, it follows that

$$\sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla v_{j,\delta}|^2 + V_j(x_\delta) v_{j,\delta}^2 = t_{\delta,n}^{p-1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |v_{j,\delta}|^{p+1} + 3t_{\delta,n}\alpha_n \int_{\mathbb{R}^N} v_{1,\delta} v_{2,\delta} v_{3,\delta}. \tag{4.25}$$

From (4.25), $\{t_{\delta,n}\}_{n=1}^{\infty}$ is bounded for all $\delta > 0$. Then it holds that

$$\begin{aligned} Q_0(\alpha_0) + \delta > Q(x_\delta; \alpha_0) &= \tilde{I}^{\mathbf{V}(x_\delta),\alpha_0}(\mathbf{v}_\delta) \geq \tilde{I}^{\mathbf{V}(x_\delta),\alpha_0}(t_{\delta,n}\mathbf{v}_\delta) \\ = \tilde{I}^{\mathbf{V}(x_\delta),\alpha_n}(t_{\delta,n}\mathbf{v}_\delta) + o(1) &\geq Q(x_\delta; \alpha_n) + o(1) \geq Q_0(\alpha_n) + o(1). \end{aligned}$$

Thus we have

$$\limsup_{n\to\infty} Q_0(\alpha_n) \leq Q_0(\alpha_0) + \delta$$

and letting $\delta \to +0$, we have

$$\limsup_{n\to\infty} Q_0(\alpha_n) \leq Q_0(\alpha_0). \tag{4.26}$$

On the other hand, there exists $z_n \in \mathbb{R}^N$ such that

$$Q(z_n; \alpha_n) < Q_0(\alpha_n) + \frac{1}{n}.$$

Let $\mathbf{v}_n$ be a non-negative minimizer for $Q(z_n; \alpha_n)$. Note that

$$Q(z_n; \alpha_n) - \frac{1}{n} < Q_0(\alpha_n) = \inf_{x \in \mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha_n)$$

$$\leq \inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); 0) < \infty.$$

Then we have

$$\begin{aligned}
&\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); 0) + \frac{1}{n} > Q(z_n;\alpha_n) = \tilde{I}^{\mathbf{V}(z_n),\alpha_n}(\mathbf{v}_n)\\
&= \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + V_j(z_n)v_{j,n}^2 + \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|v_{j,n}|^{p+1}\\
&\geq \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + V_{j,0}v_{j,n}^2.
\end{aligned}$$

Thus $\{\mathbf{v}_n\}$ is bounded in $\mathbb{H}$. Let $s_n > 0$ be a number such that $s_n\mathbf{v}_n \in \tilde{\mathcal{N}}^{\mathbf{V}(z_n),\alpha_0}$. Hence

$$\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + V_j(z_n)v_{j,n}^2 = s_n^{p-1}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|v_{j,n}|^{p+1} = 3s_n\alpha_0\int_{\mathbb{R}^N} v_{1,n}v_{2,n}v_{3,n}.$$

As in the argument in Appendix A in Kurata-Osada [30], up to a subsequence, there exists $C > 0$ such that

$$\sum_{j=1}^{3}\int_{\mathbb{R}^N}|v_{j,n}|^{p+1} \geq C \quad \text{for all } n \in \mathbb{N}.$$

Then $\{s_n\}$ is bounded. Thus we have

$$\begin{aligned}
&Q_0(\alpha_n) + \frac{1}{n} > Q(z_n;\alpha_n) = \tilde{I}^{\mathbf{V}(z_n),\alpha_n}(\mathbf{v}_n) \geq \tilde{I}^{\mathbf{V}(z_n),\alpha_n}(s_n\mathbf{v}_n)\\
&= \tilde{I}^{\mathbf{V}(z_n),\alpha_0}(s_n\mathbf{v}_n) + o(1) \geq Q(z_n;\alpha_0) + o(1) \geq Q_0(\alpha_0) + o(1).
\end{aligned}$$

Hence

$$\liminf_{n\to\infty} Q_0(\alpha_n) \geq Q_0(\alpha_0). \tag{4.27}$$

From (4.26)–(4.27), we have

$$\lim_{n\to\infty} Q_0(\alpha_n) = Q_0(\alpha_0).$$

□

We prove the upper bound for $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha)$.

Lemma 4.17. Suppose that (V1),(V2). Then it follows that

$$\begin{aligned}&\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha)\\ &\le \inf_{x\in\mathbb{R}^N} \rho^\infty(V_1(x), V_2(x), V_3(x))/\alpha^2 \quad \text{for all } \alpha > 0.\end{aligned}$$

Proof. From Proposition 2 (Step 1) in Kurata-Osada [30], we have

$$\rho(V_1(x), V_2(x), V_3(x); \alpha) \le \rho^\infty(V_1(x), V_2(x), V_3(x))/\alpha^2 \quad \text{for all } \alpha > 0,\ x \in \mathbb{R}^N.$$

Thus we have the desired inequality. □

Now, we prove Proposition 4.14.

Proof of Proposition 4.14. Let $\{\alpha_n\}_{n=1}^\infty \subset (0,\infty)$ be a sequence such that $\alpha_n \to \infty$. Let $z_n \in \mathbb{R}^N$ be a point such that

$$Q(z_n; \alpha_n) < Q_0(\alpha_n) + \frac{1}{\alpha_n^3} \quad \text{for all } n \in \mathbb{N}.$$

Let $\mathbf{v}_n$ be a minimizer for $Q(z_n; \alpha_n)$. Set $\mathbf{w}_n := \alpha_n \mathbf{v}_n$. Let $s_n > 0$ be a number such that $s_n\mathbf{w}_n \in \mathcal{M}^{\mathbf{V}(z_n),\infty}$, where $\mathbf{V}(z_n) := (V_1(z_n), V_2(z_n), V_3(z_n))$. By the same argument as in the proof of Lemma 4.16, $\{\mathbf{v}_n\}_{n=1}^\infty$ is bounded in $\mathbb{H}$ and $\{s_n\}_{n=1}^\infty$ is bounded. Then we have

$$\begin{aligned}&Q_0(\alpha_n) + \frac{1}{\alpha_n^3} > Q(z_n; \alpha_n) = \tilde{I}^{\mathbf{V}(z_n),\alpha_n}(\mathbf{v}_n) \ge \tilde{I}^{\mathbf{V}(z_n),\alpha_n}(s_n\mathbf{v}_n)\\ &= \frac{1}{\alpha_n^2} J^{\mathbf{V}(z_n),\infty}(s_n\mathbf{w}_n) + o(1/\alpha_n^2) \ge \frac{1}{\alpha_n^2}\rho^\infty(V_1(z_n), V_2(z_n), V_3(z_n)) + o(1/\alpha_n^2)\\ &\ge \frac{1}{\alpha_n^2} \inf_{x\in\mathbb{R}^N} \rho^\infty(V_1(x), V_2(x), V_3(x)) + o(1/\alpha_n^2).\end{aligned}$$

Combining with Lemma 4.17, we have

$$\begin{aligned}&\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha_n)\\ &= \inf_{x\in\mathbb{R}^N} \rho^\infty(V_1(x), V_2(x), V_3(x))/\alpha_n^2 + o(1/\alpha_n^2), \quad \text{as } \alpha_n \to \infty.\end{aligned}$$

Therefore we obtain the conclusion. □

4.7 Positivity of α^*

In this section, we show the positivity of α^* (see Proposition 4.18). Recall

$$Q(x;\alpha) := \rho(V_1(x), V_2(x), V_3(x); \alpha), \quad Q_0(\alpha) := \inf_{x\in\mathbb{R}^N} Q(x;\alpha).$$

From Lemma 4.15, Lemma 4.16 and 4.17, we can define the following threshold for $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha)$. That is,

$$\alpha^* := \max\{\alpha \geq 0 \mid Q_0(\alpha) = Q_0(0)\}.$$

Moreover, to clarify the $V_j(x)$ dependency of $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha)$, we introduce a positive number θ and a functional $I[\tilde{w}]$ as follows:

$$\theta := \frac{p+1}{p-1} - \frac{N}{2} > 0, \quad I[\tilde{w}] := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \tilde{w}|^2 + \tilde{w}^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} \tilde{w}^{p+1},$$

Moreover, let w be the unique solution of the following equation:

$$\begin{cases} -\Delta w + w = w^p \quad \text{in } \mathbb{R}^N, \\ w > 0 \quad \text{in } \mathbb{R}^N, \\ w(0) = \max_{x\in\mathbb{R}^N} w(x), \\ w(x) \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{4.28}$$

Proposition 4.18. Assume (V1),(V2),(V3). Then $\alpha^* > 0$. Moreover, it follows that

$$\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); 0) = \min_{j=1,2,3} c_1^{V_{j,0}} = \min_{j=1,2,3} \inf_{x\in\mathbb{R}^N} V_j(x)^\theta I[w],$$

Furthermore, if $0 \leq \alpha \leq \alpha^*$, then

$$\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) = \min_{j=1,2,3} c_1^{V_{j,0}} = \min_{j=1,2,3} \inf_{x\in\mathbb{R}^N} V_j(x)^\theta I[w].$$

Remark 4.8. We can prove $\alpha^* > 0$ if we assume that $(C1)_\alpha$ for α sufficiently small instead of (V3).

Proof of Proposition 4.18. (Step 1) Suppose that $\alpha^* = 0$. Then $Q_0(\alpha) < Q_0(0)$ for all $\alpha > 0$. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence such that $\alpha_n > 0$ and $\alpha_n \to 0$

as $n \to \infty$. From (V3), $(C1)_\alpha$ holds for all $\alpha \geq 0$ (see Remark 4.3). Then there exists a point $x_n \in \mathbb{R}^N$ such that $\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha_n) = \rho(V_1(x_n), V_2(x_n), V_3(x_n); \alpha_n)$. Then

$$\rho(V_1(x_n), V_2(x_n), V_3(x_n); \alpha_n) < Q_0(0) \leq \rho(V_1(x_n), V_2(x_n), V_3(x_n); 0).$$

From the proof of Theorem 1.4 in [30], $(\tilde{\mathcal{P}}^{\mathbf{V}(x_n),\alpha_n})$ has only vector ground state where $\mathbf{V}(x_n) = (V_1(x_n), V_2(x_n), V_3(x_n))$. Let $\mathbf{v}_n$ be a non-negative ground state of $(\tilde{\mathcal{P}}^{\mathbf{V}(x_n),\alpha_n})$. Then $v_{j,n} \neq 0$ for all $j = 1, 2, 3$. Since $\{V_j(x_n)\}_{n=1}^\infty$ is bounded, up to a subsequence, there exists $\lambda_{j,0}$ ($j = 1, 2, 3$) such that $V_j(x_n) \to \lambda_{j,0}$.

(Claim 1) There exist $k_0 \in \{1, 2, 3\}$ and $v_{k_0,0} \in H^1(\mathbb{R}^N)$ such that $\|v_{k_0,n} - v_{k_0,0}\|_{H^1} \to 0$ and $\|v_{j,n}\|_{H^1} \to 0$ for $j \neq k_0$.

(Step A) We first show that $\rho(V_1(x_n), V_2(x_n), V_3(x_n); \alpha_n) \leq \min_{j=1,2,3} c_1^{\lambda_{j,0}} + o(1)$, as $n \to \infty$.

Let j_0 be a number such that $\min_{j=1,2,3} c_1^{\lambda_{j,0}} = c_1^{\lambda_{j_0,0}}$. For simplicity, we assume $j_0 = 1$. Let $w_{1,0}$ be a positive ground state of $(\mathcal{P}_1^{\lambda_{1,0}})$. Set

$$\mathbf{W}_0 = (W_{1,0}, W_{2,0}, W_{3,0}),$$
$$W_{1,0} = w_{1,0}, \quad W_{2,0} = W_{3,0} = 0.$$

Let $t_n > 0$ be a number such that $t_n\mathbf{W}_0 \in \tilde{\mathcal{N}}^{\mathbf{V}(x_n),\alpha_n}$. Then

$$\int_{\mathbb{R}^N} |\nabla w_{1,0}|^2 + V_1(x_n) w_{1,0}^2 = t_n^{p-1} \int_{\mathbb{R}^N} w_{1,0}^{p+1}.$$

Thus $\{t_n\}_{n=1}^\infty$ is bounded. Hence

$$\begin{aligned}\min_{j=1,2,3} c_1^{\lambda_{j,0}} &= c_1^{\lambda_{1,0}} = I_1^{\lambda_{1,0}}(w_{1,0}) \geq I_1^{\lambda_{1,0}}(t_n w_{1,0}) = I_1^{V_1(x_n)}(t_n w_{1,0}) + o(1) \\ &= \tilde{I}^{\mathbf{V}(x_n),\alpha_n}(t_n\mathbf{W}_0) + o(1) \geq \rho(V_1(x_n), V_2(x_n), V_3(x_n); \alpha_n) + o(1).\end{aligned}$$

(Step B) Recall $\mathbf{v}_n$ is a ground state of $(\tilde{\mathcal{P}}^{\mathbf{V}(x_n),\alpha_n})$. Let $j_0 \in \{1, 2, 3\}$ be a number such that $\min_{j=1,2,3} c_1^{\lambda_{j,0}} = c_1^{\lambda_{j_0,0}}$. For simplicity, we assume $j_0 = 1$.

From (Step A), we have

$$\begin{aligned}\min_{j=1,2,3} c_1^{\lambda_{j,0}} + o(1) &\geq \rho(V_1(x_n), V_2(x_n), V_3(x_n); \alpha_n) \\ &= \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + V_j(x_n) v_{j,n}^2 + \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N} v_{j,n}^{p+1} \\ &\geq \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + V_{j,0} v_{j,n}^2.\end{aligned}$$

Thus $\{\mathbf{v}_n\}$ is bounded in $\mathbb{H}$. Then, up to a subsequence, there exists $\mathbf{v}_0 \in \mathbb{H}$ such that

$$\mathbf{v}_n \rightharpoonup \mathbf{v}_0 \quad \text{weakly in } \mathbb{H}, \tag{4.29}$$

$$v_{j,n} \to v_{j,0} \quad \text{a.e. in } \mathbb{R}^N, \tag{4.30}$$

$$v_{j,n} \to v_{j,0} \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N), \quad 1 \leq q < 2^*. \tag{4.31}$$

Since $\mathbf{v}_0 \in \mathbb{H}$, for all $\delta > 0$, there exists $R > 0$ such that

$$\frac{1}{6}\sum_{j=1}^{3}\left|\int_{B_R} |\nabla v_{j,0}|^2 + \lambda_{j,0} v_{j,0}^2 - \int_{\mathbb{R}^N} |\nabla v_{j,0}|^2 + \lambda_{j,0} v_{j,0}^2\right| + \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\left|\int_{B_R} v_{j,0}^{p+1} - \int_{\mathbb{R}^N} v_{j,0}^{p+1}\right| < \delta.$$

Since $\mathbf{v}_n$ is a ground state of $(\tilde{\mathcal{P}}^{\mathbf{V}(x_n),\alpha_n})$, $\mathbf{v}_n$ satisfies

$$\begin{cases} -\Delta v_{1,n} + V_1(x_n) v_{1,n} = v_{1,n}^p + \alpha_n v_{2,n} v_{3,n}, \\ -\Delta v_{2,n} + V_2(x_n) v_{2,n} = v_{2,n}^p + \alpha_n v_{1,n} v_{3,n}, \\ -\Delta v_{3,n} + V_3(x_n) v_{3,n} = v_{3,n}^p + \alpha_n v_{1,n} v_{2,n}. \end{cases} \tag{4.32}$$

From (4.29),(4.31) and $\alpha_n \to 0$, we have

$$\begin{cases} -\Delta v_{1,0} + \lambda_{1,0} v_{1,0} = v_{1,0}^p, \\ -\Delta v_{2,0} + \lambda_{2,0} v_{2,0} = v_{2,0}^p, \\ -\Delta v_{3,0} + \lambda_{3,0} v_{3,0} = v_{3,0}^p. \end{cases} \tag{4.33}$$

Then

$$
\begin{aligned}
&\min_{j=1,2,3} c_1^{\lambda_{j,0}} \\
&\geq \liminf_{n\to\infty} \rho(V_1(x_n), V_2(x_n), V_3(x_n); \alpha_n) \\
&\geq \liminf_{n\to\infty} \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + V_j(x_n) v_{j,n}^2 + \liminf_{n\to\infty} \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} v_{j,n}^{p+1} \\
&\geq \liminf_{n\to\infty} \frac{1}{6} \sum_{j=1}^{3} \int_{B_R} |\nabla v_{j,n}|^2 + V_j(x_n) v_{j,n}^2 + \liminf_{n\to\infty} \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{B_R} v_{j,n}^{p+1} \\
&\geq \frac{1}{6} \sum_{j=1}^{3} \int_{B_R} |\nabla v_{j,0}|^2 + \lambda_{j,0} v_{j,0}^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{B_R} v_{j,0}^{p+1} \\
&\geq \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla v_{j,0}|^2 + \lambda_{j,0} v_{j,0}^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} v_{j,0}^{p+1} - \delta.
\end{aligned}
$$

Letting $\delta \to +0$ and from (4.33), we have

$$
\begin{aligned}
\min_{j=1,2,3} c_1^{\lambda_{j,0}} &\geq \liminf_{n\to\infty} \rho(V_1(x_n), V_2(x_n), V_3(x_n); \alpha_n) \\
&\geq \liminf_{n\to\infty} \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + V_j(x_n) v_{j,n}^2 \\
&\quad + \liminf_{n\to\infty} \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} v_{j,n}^{p+1} \\
&\geq \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla v_{j,0}|^2 + \lambda_{j,0} v_{j,0}^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} v_{j,0}^{p+1} \\
&= \sum_{j=1}^{3} I_1^{\lambda_{j,0}}(v_{j,0}).
\end{aligned} \tag{4.34}
$$

If $v_{j,0} \neq 0$ for all $j = 1, 2, 3$,

$$
\sum_{j=1}^{3} I_1^{\lambda_{j,0}}(v_{j,0}) \geq \sum_{j=1}^{3} c_1^{\lambda_{j,0}} > \min_{j=1,2,3} c_1^{\lambda_{j,0}}.
$$

This contradicts (4.34). Hence there exists $k_0 \in \{1, 2, 3\}$ such that $v_{k_0,0} \neq 0$ and $v_{j,0} = 0$ for $j \neq k_0$. From (4.34), we have

$$\begin{aligned}\min_{j=1,2,3} c_1^{\lambda_{j,0}} &\geq \liminf_{n\to\infty} \rho(V_1(x_n), V_2(x_n), V_3(x_n); \alpha_n)\\ &\geq \sum_{j=1}^{3} I_1^{\lambda_{j,0}}(v_{j,0}) = I_1^{\lambda_{k_0,0}}(v_{k_0,0}) \geq c_1^{\lambda_{k_0,0}} = \min_{j=1,2,3} c_1^{\lambda_{j,0}}.\end{aligned}$$

Thus, we have

$$v_{k_0,0} \text{ is a minimizer for } c_1^{\lambda_{k_0,0}}, \tag{4.35}$$

$$\rho(V_1(x_n), V_2(x_n), V_3(x_n); \alpha_n) \to \min_{j=1,2,3} c_1^{\lambda_{j,0}}. \tag{4.36}$$

Moreover, from (4.34) and (4.35), we have

$$\begin{aligned}&\liminf_{n\to\infty} \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + V_j(x_n)v_{j,n}^2 \geq \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_{j,0}|^2 + \lambda_{j,0}v_{j,0}^2\\ &\liminf_{n\to\infty} \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N} v_{j,n}^{p+1} \geq \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N} v_{j,0}^{p+1},\\ &\lim_{n\to\infty}\left\{\frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_{j,n}|^2 + V_j(x_n)v_{j,n}^2 + \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N} v_{j,n}^{p+1}\right\}\\ &= \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_{j,0}|^2 + \lambda_{j,0}v_{j,0}^2 + \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N} v_{j,0}^{p+1}.\end{aligned}$$

By the same argument as in Lemma 3.3, we have

$$\|v_{k_0,n} - v_{k_0,0}\|_{H^1} \to 0, \quad \|v_{j,n}\|_{H^1} \to 0 \quad \text{for } j \neq k_0. \tag{4.37}$$

(Step 2) The following argument is based on Theorem 1.4 in [30].

For simplicity, we assume $k_0 = 1$. From (4.32), we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v_{2,n}|^2 + V_2(x_n)v_{2,n}^2 &= \int_{\mathbb{R}^N} v_{2,n}^{p+1} + \alpha_n \int_{\mathbb{R}^N} v_{1,n}v_{2,n}v_{3,n},\\ \int_{\mathbb{R}^N} |\nabla v_{3,n}|^2 + V_3(x_n)v_{3,n}^2 &= \int_{\mathbb{R}^N} v_{3,n}^{p+1} + \alpha_n \int_{\mathbb{R}^N} v_{1,n}v_{2,n}v_{3,n}.\end{aligned}$$

Then there exists $C_1 > 0$ such that

$$C_1(\|v_{2,n}\|_{H^1}^2 + \|v_{3,n}\|_{H^1}^2) \le \|v_{2,n}\|_{L^{p+1}}^{p+1} + \|v_{3,n}\|_{L^{p+1}}^{p+1} + 2\alpha_n \int_{\mathbb{R}^N} v_{1,n}v_{2,n}v_{3,n}. \tag{4.38}$$

Since $\{\mathbf{v}_n\}$ is bounded in $\mathbb{H}$, there exist $C_2, C_3 > 0$ such that

$$\begin{aligned} 2\alpha_n \int_{\mathbb{R}^N} v_{1,n}v_{2,n}v_{3,n} &\le \alpha_n \int_{\mathbb{R}^N} v_{1,n}(v_{2,n}^2 + v_{3,n}^2) \\ &\le C_2\alpha_n \|v_{1,n}\|_{L^3}(\|v_{2,n}\|_{H^1}^2 + \|v_{3,n}\|_{H^1}^2) \\ &\le C_3\alpha_n(\|v_{2,n}\|_{H^1}^2 + \|v_{3,n}\|_{H^1}^2). \end{aligned} \tag{4.39}$$

For n sufficiently large, $C_3\alpha_n \le C_1/2$. From (4.38) and (4.39), we have

$$\begin{aligned} &C_1(\|v_{2,n}\|_{H^1}^2 + \|v_{3,n}\|_{H^1}^2) \\ &\le \|v_{2,n}\|_{L^{p+1}}^{p+1} + \|v_{3,n}\|_{L^{p+1}}^{p+1} + C_3\alpha_n(\|v_{2,n}\|_{H^1}^2 + \|v_{3,n}\|_{H^1}^2) \\ &\le \|v_{2,n}\|_{L^{p+1}}^{p+1} + \|v_{3,n}\|_{L^{p+1}}^{p+1} + \frac{C_1}{2}(\|v_{2,n}\|_{H^1}^2 + \|v_{3,n}\|_{H^1}^2). \end{aligned}$$

Then there exists $C_4 > 0$ such that

$$\begin{aligned} \frac{C_1}{2}(\|v_{2,n}\|_{H^1}^2 + \|v_{3,n}\|_{H^1}^2) &\le \|v_{2,n}\|_{L^{p+1}}^{p+1} + \|v_{3,n}\|_{L^{p+1}}^{p+1} \\ &\le C_4(\|v_{2,n}\|_{H^1}^2 + \|v_{3,n}\|_{H^1}^2)^{(p+1)/2}. \end{aligned}$$

Then there exists $C_5 > 0$ such that

$$C_5 \le (\|v_{2,n}\|_{H^1}^2 + \|v_{3,n}\|_{H^1}^2)^{(p-1)/2}.$$

This contradicts (4.37). Thus we obtain $\alpha^* > 0$.

(Step 3) From Lemma 4.1 in [30], it follows that

$$\rho(V_1(x), V_2(x), V_3(x); 0) = \min_{j=1,2,3} c_1^{V_j(x)}.$$

In addition, it is easy to see that

$$c_1^{V_j(x)} = V_j(x)^\theta I[w].$$

Then

$$\rho(V_1(x), V_2(x), V_3(x); 0) = \min_{j=1,2,3} c_1^{V_j(x)} = \min_{j=1,2,3} V_j(x)^\theta I[w].$$

Since

$$\min_{j=1,2,3} V_j(x)^\theta I[w] \geq \min_{j=1,2,3} V_{j,0}^\theta I[w],$$

we have

$$\inf_{x\in\mathbb{R}^N} \min_{j=1,2,3} V_j(x)^\theta I[w] \geq \min_{j=1,2,3} V_{j,0}^\theta I[w]. \tag{4.40}$$

On the other hand, let $j_0 \in \{1, 2, 3\}$ be a number such that

$$V_{j_0,0} = \min_{j=1,2,3} V_{j,0}.$$

Let $z_0 \in \mathbb{R}^N$ be a point such that $V_{j_0}(z_0) = V_{j_0,0}$. Then, it follows that

$$\min_{j=1,2,3} V_j(z_0)^\theta I[w] = V_{j_0,0}^\theta I[w] = \min_{j=1,2,3} V_{j,0}^\theta I[w] \tag{4.41}$$

and

$$\min_{j=1,2,3} V_j(z_0)^\theta I[w] \geq \inf_{x\in\mathbb{R}^N} \min_{j=1,2,3} V_j(x)^\theta I[w]. \tag{4.42}$$

From (4.40)–(4.42), we have

$$\inf_{x\in\mathbb{R}^N} \min_{j=1,2,3} V_j(x)^\theta I[w] = \min_{j=1,2,3} V_{j,0}^\theta I[w].$$

Hence,

$$\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); 0) = \min_{j=1,2,3} V_{j,0}^\theta I[w] = \min_{j=1,2,3} c_1^{V_{j,0}}.$$

Furthermore, if $0 \leq \alpha \leq \alpha^*$, then $Q_0(\alpha) = Q_0(0)$. Thus, if $0 \leq \alpha \leq \alpha^*$, then

$$\inf_{x\in\mathbb{R}^N} \rho(V_1(x), V_2(x), V_3(x); \alpha) = \min_{j=1,2,3} V_{j,0}^\theta I[w] = \min_{j=1,2,3} c_1^{V_{j,0}}.$$

□

4.8 When all the ground states of $(\mathcal{P}_{\varepsilon,\alpha})$ are scalar or vector

To clear the dependence on α, we write $(\mathcal{P}_\varepsilon)$, $(\tilde{\mathcal{P}}_\varepsilon)$, c_ε, $\tilde{c}_\varepsilon$, I_ε, $\tilde{I}_\varepsilon$, $\mathcal{N}_\varepsilon$, $\tilde{\mathcal{N}}_\varepsilon$, G_ε and $\tilde{G}_\varepsilon$ as $(\mathcal{P}_{\varepsilon,\alpha})$, $(\tilde{\mathcal{P}}_{\varepsilon,\alpha})$, $c_{\varepsilon,\alpha}$, $\tilde{c}_{\varepsilon,\alpha}$, $I_{\varepsilon,\alpha}$, $\tilde{I}_{\varepsilon,\alpha}$, $\mathcal{N}_{\varepsilon,\alpha}$, $\tilde{\mathcal{N}}_{\varepsilon,\alpha}$, $G_{\varepsilon,\alpha}$ and $\tilde{G}_{\varepsilon,\alpha}$.

In this section, we consider when all the ground states of $(\mathcal{P}_{\varepsilon,\alpha})$ are scalar or vector. To state the main proposition in this section, we give thresholds Γ^*_ε, α^*_0 and α^*_1 as follows: From [30], $c_{\varepsilon,\alpha}$ is monotone decreasing with respect to the parameter α over $[0,\infty)$ and continuous. Moreover, $c_{\varepsilon,\alpha}$ converges to 0 as $\alpha \to \infty$. Thus we can define Γ^*_ε as follows:

$$\Gamma^*_\varepsilon := \max\{\alpha \geq 0 \mid c_{\varepsilon,\alpha} = c_{\varepsilon,0}\}.$$

In addition, we can define thresholds α^*_0 and α^*_1 such that

$$\alpha^*_0 := \liminf_{\varepsilon\to+0} \Gamma^*_\varepsilon := \lim_{r\to+0} \inf_{0<\varepsilon<r} \Gamma^*_\varepsilon,$$
$$\alpha^*_1 := \limsup_{\varepsilon\to+0} \Gamma^*_\varepsilon := \lim_{r\to+0} \sup_{0<\varepsilon<r} \Gamma^*_\varepsilon.$$

We can show $\alpha^*_0 > 0$ and $\alpha^*_1 < \infty$ under (V1),(V2) (see Lemmas 4.20 and 4.21).

The next proposition tells us when all the ground states of $(\mathcal{P}_{\varepsilon,\alpha})$ become scalar or vector depending on the size of α.

Proposition 4.19. Suppose that (V1),(V2). Then we can define the threshold α^*_0 and α^*_1 as above. Then the followings hold:

(i) If $0 \leq \alpha < \alpha^*_0$, then there exists $\varepsilon_0 > 0$ such that all the ground states of $(\mathcal{P}_{\varepsilon,\alpha})$ are scalar for $0 < \varepsilon < \varepsilon_0$.

(ii) If $\alpha > \alpha^*_1$, then there exists $\varepsilon_1 > 0$ such that all the ground states of $(\mathcal{P}_{\varepsilon,\alpha})$ are vector for $0 < \varepsilon < \varepsilon_1$.

(iii) If $\alpha > \alpha^*_0$, then there exists $\{\varepsilon_{0,n}\}_{n=1}^\infty \subset (0,\infty)$ such that $\varepsilon_{0,n} \to 0$ and all the ground states of $(\mathcal{P}_{\varepsilon_{0,n},\alpha})$ are vector for all $n \in \mathbb{N}$.

(iv) If $0 \leq \alpha < \alpha^*_1$, then there exists $\{\varepsilon_{1,n}\}_{n=1}^\infty \subset (0,\infty)$ such that $\varepsilon_{1,n} \to 0$ and all the ground states of $(\mathcal{P}_{\varepsilon_{1,n},\alpha})$ are scalar for all $n \in \mathbb{N}$.

Lemma 4.20. Suppose that (V1),(V2). Then, there exists $r > 0$ such that

$$(\alpha_0^* \geq) \inf_{0<\varepsilon<r} \Gamma_\varepsilon^* > 0.$$

Proof. **(Claim 1)** There exists $r_1 > 0$ such that if $0 < \varepsilon < r_1$, $\alpha \geq 0$, then $\tilde{c}_{\varepsilon,\alpha} < \min_{j=1,2,3} c_1^{V_{j,\infty}}$.

Indeed, suppose that $\min_{j=1,2,3} c_1^{V_{j,0}} = c_1^{V_{1,0}}$ for simplicity. Let $w_{1,0}$ be a positive minimizer for $c_1^{V_{1,0}}$. From (V1),(V2), let z_0 be a minimum point of $V_1(x)$. Set $\mathbf{v}_{0,\varepsilon} = (v_{1,0,\varepsilon}, 0, 0)$, $v_{1,0,\varepsilon}(y) = w_{1,0}(y - z_0/\varepsilon)$. Moreover, let $t_{0,\varepsilon}$ be a positive number such that $t_{0,\varepsilon}\mathbf{v}_{0,\varepsilon} \in \tilde{\mathcal{N}}_{\varepsilon,\alpha}$. Then it follows that

$$\int_{\mathbb{R}^N} |\nabla v_{1,0,\varepsilon}|^2 + V_1(\varepsilon y) v_{1,0,\varepsilon}^2 = t_{0,\varepsilon}^{p-1} \int_{\mathbb{R}^N} v_{1,0,\varepsilon}^{p+1},$$

that is,

$$\int_{\mathbb{R}^N} |\nabla w_{1,0}|^2 + V_1(z_0 + \varepsilon y) w_{1,0}^2 = t_{0,\varepsilon}^{p-1} \int_{\mathbb{R}^N} w_{1,0}^{p+1}.$$

Note that $t_{0,\varepsilon}$ is bounded and independent of α. Since

$$\lim_{\varepsilon\to+0} \int_{\mathbb{R}^N} V_1(z_0 + \varepsilon y) w_{1,0}^2 = \int_{\mathbb{R}^N} V_1(z_0) w_{1,0}^2,$$

there exists $r_1 > 0$ such that if $0 < \varepsilon < r_1$, then

$$\frac{t_{0,\varepsilon}^2}{2} \left| \int_{\mathbb{R}^N} V_1(z_0 + \varepsilon y) w_{1,0}^2 - \int_{\mathbb{R}^N} V_1(z_0) w_{1,0}^2 \right| < \frac{\min_{j=1,2,3} c_1^{V_{j,\infty}} - \min_{j=1,2,3} c_1^{V_{j,0}}}{2}.$$

Then if $0 < \varepsilon < r_1$,

$$\begin{aligned}
\min_{j=1,2,3} c_1^{V_{j,\infty}} &> \min_{j=1,2,3} c_1^{V_{j,0}} + \frac{\min_{j=1,2,3} c_1^{V_{j,\infty}} - \min_{j=1,2,3} c_1^{V_{j,0}}}{2} \\
&= c_1^{V_{1,0}} + \frac{\min_{j=1,2,3} c_1^{V_{j,\infty}} - \min_{j=1,2,3} c_1^{V_{j,0}}}{2} \\
&= I_1^{V_{1,0}}(w_{1,0}) + \frac{\min_{j=1,2,3} c_1^{V_{j,\infty}} - \min_{j=1,2,3} c_1^{V_{j,0}}}{2} \\
&\geq I_1^{V_{1,0}}(t_{0,\varepsilon} w_{1,0}) + \frac{\min_{j=1,2,3} c_1^{V_{j,\infty}} - \min_{j=1,2,3} c_1^{V_{j,0}}}{2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{t_{0,\varepsilon}^2}{2}\int_{\mathbb{R}^N}|\nabla w_{1,0}|^2 + V_1(z_0+\varepsilon y)w_{1,0}^2 - \frac{t_{0,\varepsilon}^{p+1}}{p+1}\int_{\mathbb{R}^N} w_{1,0}^{p+1}\\
&\quad + \frac{t_{0,\varepsilon}^2}{2}\left(\int_{\mathbb{R}^N} V_1(z_0)w_{1,0}^2 - \int_{\mathbb{R}^N} V_1(z_0+\varepsilon y)w_{1,0}^2\right)\\
&\quad + \frac{\min_{j=1,2,3} c_1^{V_{j,\infty}} - \min_{j=1,2,3} c_1^{V_{j,0}}}{2}\\
&> \tilde{I}_{\varepsilon,\alpha}(t_{0,\varepsilon}\mathbf{v}_{0,\varepsilon}) \geq \tilde{c}_{\varepsilon,\alpha}.
\end{aligned}
$$

We can prove the following claim by the similar argument as in (Claim 1):
(Claim 2) There exists $r_2 > 0$ such that if $0 < \varepsilon < r_2$, then $\tilde{c}_{2,\varepsilon}^{V_j} < \tilde{c}_2^{V_{j,\infty}}$ for all $j = 1, 2, 3$, where

$$
\begin{aligned}
\tilde{I}_{2,\varepsilon}^{V_j}(w) &:= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla w|^2 + V_j(\varepsilon y)w^2 - \frac{1}{p+1}\int_{\mathbb{R}^N}|w|^{p+1},\\
\tilde{c}_{2,\varepsilon}^{V_j} &:= \inf_{w\in\tilde{\mathcal{N}}_{2,\varepsilon}^{V_j}} \tilde{I}_{2,\varepsilon}^{V_j},\\
\tilde{\mathcal{N}}_{2,\varepsilon}^{V_j} &:= \{w \in H^1(\mathbb{R}^N)\setminus\{0\} \mid \tilde{G}_{2,\varepsilon}^{V_j}(w) = 0\},\\
\tilde{G}_{2,\varepsilon}^{V_j}(w) &:= \int_{\mathbb{R}^N}|\nabla w|^2 + V_j(\varepsilon y)w^2 - |w|^{p+1},
\end{aligned}
$$

Set $2r := \min\{r_1, r_2\}$. We now prove $\inf_{0<\varepsilon<r}\Gamma_\varepsilon^* > 0$. Suppose that $\inf_{0<\varepsilon<r}\Gamma_\varepsilon^* = 0$. Then there exists $\{\varepsilon_n\}_{n=1}^\infty \subset (0, r)$ such that $\Gamma_{\varepsilon_n}^* \to 0$. Let $\{\alpha_n\}_{n=1}^\infty \subset (0,\infty)$ be a sequence such that $\alpha_n \to 0$ and $\alpha_n > \Gamma_{\varepsilon_n}^*$. From (Claim 1), we have $\tilde{c}_{\varepsilon_n,\alpha_n} < \min_{j=1,2,3} c_1^{V_{j,\infty}}$ for all $n \in \mathbb{N}$. From Theorem 1.4 in [30], there exists $\alpha_{\mathbf{V}_\infty}^* > 0$ such that

$$\rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha) = \min_{j=1,2,3} c_1^{V_{j,\infty}} \quad \text{for all } 0 \leq \alpha \leq \alpha_{\mathbf{V}_\infty}^*.$$

Then for n sufficiently large,

$$\rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha_n) = \min_{j=1,2,3} c_1^{V_{j,\infty}}.$$

Then, for n sufficiently large,

$$\tilde{c}_{\varepsilon_n,\alpha_n} < \rho(V_{1,\infty}, V_{2,\infty}, V_{3,\infty}; \alpha_n).$$

Then by the same argument as in [43], there exists a non-negative ground state $\mathbf{v}_n$ of $(\tilde{\mathcal{P}}_{\varepsilon_n,\alpha_n})$. Set $\mathbf{u}_n(x) = \mathbf{v}_n(x/\varepsilon_n)$. Then $\mathbf{u}_n$ is a non-negative ground state of $(\mathcal{P}_{\varepsilon_n,\alpha_n})$. Since $\alpha_n > \Gamma^*_{\varepsilon_n}$, then $c_{\varepsilon_n,\alpha_n} < c_{\varepsilon_n,0}$. From Theorem 1.4 in [30], it follows that

$$u_{j,n} \neq 0 \quad \text{for all } j = 1,2,3. \tag{4.43}$$

Since $0 < \varepsilon_n < r$, up to a subsequence, there exists $\varepsilon_0 \geq 0$ such that $\varepsilon_n \to \varepsilon_0$. Now, we divided the proof into the case $\varepsilon_0 = 0$ and the case $\varepsilon_0 > 0$.

(Case 1) First, we shall prove that $\varepsilon_0 = 0$ does not occur.
Recall $U^{(l)}_{j,n}(y) = u_{j,n}(x_{l,n} + \varepsilon_n y)$. Since $\mathbf{u}_n$ satisfies

$$\begin{cases} -\varepsilon_n^2 \Delta u_{1,n} + V_1(x) u_{1,n} = u_{1,n}^p + \alpha_n u_{2,n} u_{3,n}, \\ -\varepsilon_n^2 \Delta u_{2,n} + V_2(x) u_{2,n} = u_{2,n}^p + \alpha_n u_{1,n} u_{3,n}, \\ -\varepsilon_n^2 \Delta u_{3,n} + V_3(x) u_{3,n} = u_{3,n}^p + \alpha_n u_{1,n} u_{2,n}, \end{cases}$$

$U^{(l)}_{j,n}$ satisfies

$$\begin{cases} -\Delta U^{(l)}_{1,n} + V_1(x_{l,n} + \varepsilon_n y) U^{(l)}_{1,n} = (U^{(l)}_{1,n})^p + \alpha_n U^{(l)}_{2,n} U^{(l)}_{3,n}, \\ -\Delta U^{(l)}_{2,n} + V_2(x_{l,n} + \varepsilon_n y) U^{(l)}_{2,n} = (U^{(l)}_{2,n})^p + \alpha_n U^{(l)}_{1,n} U^{(l)}_{3,n}, \\ -\Delta U^{(l)}_{3,n} + V_3(x_{l,n} + \varepsilon_n y) U^{(l)}_{3,n} = (U^{(l)}_{3,n})^p + \alpha_n U^{(l)}_{1,n} U^{(l)}_{2,n}. \end{cases}$$

Recall that from Lemma 4.10 (iii),(iv), $\{\mathbf{U}^{(l)}_n\}_{n=1}^\infty$ is bounded in $\mathbb{H}$ and up to a subsequence, there exists $\mathbf{U}^{(l)} \in \mathbb{H}$ such that

$$\begin{aligned} &\mathbf{U}^{(l)}_n \rightharpoonup \mathbf{U}^{(l)} \quad \text{weakly in } \mathbb{H}, \\ &U^{(l)}_{j,n} \to U^{(l)}_j \quad \text{in } C_{\mathrm{loc}}(\mathbb{R}^N). \end{aligned}$$

Suppose that $\sup_{n\in\mathbb{N}} |x_{l,n}| = \infty$. Then up to a subsequence, $|x_{l,n}| \to \infty$. Then from $\alpha_n \to +0$, we have

$$\begin{cases} -\Delta U^{(l)}_1 + V_{1,\infty} U^{(l)}_1 = (U^{(l)}_1)^p, \\ -\Delta U^{(l)}_2 + V_{2,\infty} U^{(l)}_2 = (U^{(l)}_2)^p, \\ -\Delta U^{(l)}_3 + V_{3,\infty} U^{(l)}_3 = (U^{(l)}_3)^p. \end{cases}$$

Suppose that $\mathbf{U}^{(l)} \neq (0,0,0)$. By the same argument as in the proof of Theorem 4.4 (2) and (3), we have

$$\min_{j=1,2,3} c_1^{V_{j,0}} \geq \limsup_{n\to\infty} \tilde{c}_{\varepsilon_n,\alpha_n} \geq \liminf_{n\to\infty} \tilde{c}_{\varepsilon_n,\alpha_n}$$

$$= \liminf_{n\to\infty} \left\{ \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla U_{j,n}^{(l)}|^2 + V_j(x_{l,n} + \varepsilon_n y)(U_{j,n}^{(l)})^2 \right.$$
$$\left. + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} (U_{j,n}^{(l)})^{p+1} \right\}$$
$$\geq \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla U_j^{(l)}|^2 + V_{j,\infty}(U_j^{(l)})^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} (U_j^{(l)})^{p+1}$$
$$\geq \min_{j=1,2,3} c_1^{V_{j,\infty}} > \min_{j=1,2,3} c_1^{V_{j,0}}.$$

This is a contradiction. Thus if $\sup_{n\in\mathbb{N}}|x_{l,n}| = \infty$, then $\mathbf{U}^{(l)} = (0,0,0)$. Moreover, there exists $l \in \{1,2,3\}$ such that $\sup_{n\in\mathbb{N}}|x_{l,n}| < \infty$. Indeed, if for all $l \in \{1,2,3\}$, $\sup_{n\in\mathbb{N}}|x_{l,n}| = \infty$, then it follows that $\mathbf{U}^{(l)} = (0,0,0)$ for all $l \in \{1,2,3\}$ by the argument as above. This contradicts Lemma 4.11. Thus, there exists $l \in \{1,2,3\}$ such that $\sup_{n\in\mathbb{N}}|x_{l,n}| < \infty$. Up to a subsequence, there exists $x_{l,0} \in \mathbb{R}^N$ such that $x_{l,n} \to x_{l,0}$. We next show there exists $j_0 \in \{1,2,3\}$ such that $\|U_{j_0,n}^{(l)} - U_{j_0}^{(l)}\|_{H^1} \to 0$ and $\|U_{j,n}^{(l)}\|_{H^1} \to 0$ for all $j \neq j_0$. If $U_j^{(l)} \neq 0$ for all $j \in \{1,2,3\}$, then $\mathbf{U}^{(l)}$ satisfies

$$\begin{cases} -\Delta U_1^{(l)} + V_1(x_{l,0})U_1^{(l)} = (U_1^{(l)})^p, \\ -\Delta U_2^{(l)} + V_2(x_{l,0})U_2^{(l)} = (U_2^{(l)})^p, \\ -\Delta U_3^{(l)} + V_3(x_{l,0})U_3^{(l)} = (U_3^{(l)})^p. \end{cases}$$

Then

$$\min_{j=1,2,3} c_1^{V_{j,0}} \geq \limsup_{n\to\infty} \tilde{c}_{\varepsilon_n,\alpha_n} \geq \liminf_{n\to\infty} \tilde{c}_{\varepsilon_n,\alpha_n}$$
$$= \liminf_{n\to\infty} \left\{ \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla U_{j,n}^{(l)}|^2 + V_j(x_{l,n} + \varepsilon_n y)(U_{j,n}^{(l)})^2 \right.$$
$$\left. + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} (U_{j,n}^{(l)})^{p+1} \right\}$$
$$\geq \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla U_j^{(l)}|^2 + V_j(x_{l,0})(U_j^{(l)})^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} (U_j^{(l)})^{p+1}$$
$$\geq \sum_{j=1}^{3} c_1^{V_j(x_{l,0})} > \min_{j=1,2,3} c_1^{V_{j,0}}.$$

This is a contradiction. Thus there exists $j_0 \in \{1,2,3\}$ such that $U_{j_0}^{(l)} \neq 0$ and $U_j^{(l)} = 0$ for all $j \neq j_0$. Then

$$\begin{aligned}
&\min_{j=1,2,3} c_1^{V_{j,0}} \geq \limsup_{n\to\infty} \tilde{c}_{\varepsilon_n,\alpha_n} \geq \liminf_{n\to\infty} \tilde{c}_{\varepsilon_n,\alpha_n} \\
&= \liminf_{n\to\infty} \left\{ \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla U_{j,n}^{(l)}|^2 + V_j(x_{l,n}+\varepsilon_n y)(U_{j,n}^{(l)})^2 \right. \\
&\qquad\qquad \left. + \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N} (U_{j,n}^{(l)})^{p+1} \right\} \\
&\geq \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla U_j^{(l)}|^2 + V_j(x_{l,0})(U_j^{(l)})^2 + \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N}(U_j^{(l)})^{p+1} \\
&\geq \min_{j=1,2,3} c_1^{V_j(x_{l,0})} \geq \min_{j=1,2,3} c_1^{V_{j,0}}.
\end{aligned}$$

By the same argument as in the proof of Theorem 4.4 (2) and (3), we have

$$\begin{aligned}
&\|U_{j_0,n}^{(l)} - U_{j_0}^{(l)}\|_{H^1} \to 0, \\
&\|U_{j,n}^{(l)}\|_{H^1} \to 0 \quad \text{for all } j \neq j_0.
\end{aligned} \tag{4.44}$$

For simplicity, we assume $j_0 = 1$. Noting (4.43), by the same argument as in (Step 2) in Proposition 4.18, there exists $C > 0$ such that

$$\|U_{2,n}^{(l)}\|_{H^1}^2 + \|U_{3,n}^{(l)}\|_{H^1}^2 \geq C.$$

This contradicts (4.44). Thus the case $\varepsilon_0 = 0$ does not occur.

(Case 2) Next, we exclude the possibility of $\varepsilon_0 > 0$.
We first show the upper bound of $c_{\varepsilon_n,\alpha_n}$. To this end, we consider the following equation and define the following minimization problem:

$$\begin{aligned}
&-\varepsilon^2\Delta u + V_j(x)u = |u|^{p-1}u, \qquad (\mathcal{P}_{2,\varepsilon}^{V_j}) \\
&c_{2,\varepsilon}^{V_j} := \inf_{u\in\mathcal{N}_{2,\varepsilon}^{V_j}} I_{2,\varepsilon}^{V_j}(u),
\end{aligned}$$

where

$$I_{2,\varepsilon}^{V_j}(u) := \frac{1}{2}\int_{\mathbb{R}^N} \varepsilon^2|\nabla u|^2 + V_j(x)u^2 - \frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1},$$

$$\mathcal{N}_{2,\varepsilon}^{V_j} := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid G_{2,\varepsilon}^{V_j}(u) = 0\},$$
$$G_{2,\varepsilon}^{V_j}(u) := \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u|^2 + V_j(x) u^2 - |u|^{p+1}.$$

Now, we prove the following:

$$c_{\varepsilon_n,\alpha_n} \le \min_{j=1,2,3} c_{2,\varepsilon_0}^{V_j} + o(1) \quad \text{as } n \to \infty. \tag{4.45}$$

Indeed, let j_0 be a number such that $c_{2,\varepsilon_0}^{V_{j_0}} = \min_{j=1,2,3} c_{2,\varepsilon_0}^{V_j}$. For simplicity, we assume $j_0 = 1$. Note that $0 < \varepsilon_0 \le r < 2r$. From (Claim 2), there exists a positive ground state u_0 of $(\mathcal{P}_{2,\varepsilon_0}^{V_1})$. Set

$$\mathbf{u}_0 = (u_{1,0}, u_{2,0}, u_{3,0}), \quad u_{1,0} = u_0, \quad u_{2,0} = u_{3,0} = 0.$$

Let $t_n > 0$ be a number such that $t_n \mathbf{u}_n \in \mathcal{N}_{\varepsilon_n,\alpha_n}$. Then it follows that

$$\sum_{j=1}^{3} \int_{\mathbb{R}^N} \varepsilon_n^2 |\nabla u_{j,0}|^2 + V_j(x) u_{j,0}^2 = t_n^{p-1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} u_{j,0}^{p+1},$$

that is,

$$\int_{\mathbb{R}^N} \varepsilon_n^2 |\nabla u_0|^2 + V_1(x) u_0^2 = t_n^{p-1} \int_{\mathbb{R}^N} u_0^{p+1}.$$

Since $\varepsilon_n \to \varepsilon_0 > 0$, $\{t_n\}_{n=1}^{\infty}$ is bounded. Then

$$\begin{aligned}
c_{2,\varepsilon_0}^{V_1} &= I_{2,\varepsilon_0}^{V_1}(u_0) \ge I_{2,\varepsilon_0}^{V_1}(t_n u_0) \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \varepsilon_0^2 |\nabla (t_n u_0)|^2 + V_1(x)(t_n u_0)^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} (t_n u_0)^{p+1} \\
&= I_{\varepsilon_n,\alpha_n}(t_n \mathbf{u}_0) + o(1) \\
&\ge c_{\varepsilon_n,\alpha_n} + o(1) \quad \text{as } n \to \infty.
\end{aligned}$$

Thus we have (4.45). For n sufficiently large,

$$\begin{aligned}
&\min_{j=1,2,3} c_{2,\varepsilon_0}^{V_j} + o(1) \ge c_{\varepsilon_n,\alpha_n} \\
&\qquad = \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} \varepsilon_n^2 |\nabla u_{j,n}|^2 + V_j(x) u_{j,n}^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} u_{j,n}^{p+1}
\end{aligned}$$

$$\geq \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N}\frac{\varepsilon_0^2}{2}|\nabla u_{j,n}|^2 + V_{j,0}u_{j,n}^2.$$

Then $\{\mathbf{u}_n\}_{n=1}^{\infty}$ is bounded in $\mathbb{H}$. Thus, up to a subsequence, there exists $\mathbf{u}_0 \in \mathbb{H}$ such that

$$\begin{aligned} \mathbf{u}_n &\rightharpoonup \mathbf{u}_0 \quad \text{weakly in } \mathbb{H}, \\ u_{j,n} &\to u_{j,0} \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N), \quad 1 \leq q < 2^*, \\ u_{j,n} &\to u_{j,0} \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

(Claim 3) $\mathbf{u}_0 \neq (0,0,0)$.
If we assume $\mathbf{u}_0 = (0,0,0)$, then it follows that

$$u_{j,n} \to 0 \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N), \quad 1 \leq q < 2^*. \tag{4.46}$$

Let $t_n > 0$ be a number such that $t_n\mathbf{u}_n \in \mathcal{N}^{\mathbf{V}_\infty}_{\varepsilon_0,\alpha_n}$, where

$$\begin{aligned} I^{\mathbf{V}_\infty}_{\varepsilon,\alpha}(\mathbf{u}) &:= \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N}\varepsilon^2|\nabla u_j|^2 + V_{j,\infty}u_j^2 - \frac{1}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|u_j|^{p+1} - \alpha\int_{\mathbb{R}^N}u_1u_2u_3, \\ c^{\mathbf{V}_\infty}_{\varepsilon,\alpha} &:= \inf_{\mathbf{u}\in\mathcal{N}^{\mathbf{V}_\infty}_{\varepsilon,\alpha}} I^{\mathbf{V}_\infty}_{\varepsilon,\alpha}(\mathbf{u}), \\ \mathcal{N}^{\mathbf{V}_\infty}_{\varepsilon,\alpha} &:= \{\mathbf{u}\in\mathbb{H}\setminus\{(0,0,0)\} \mid G^{\mathbf{V}_\infty}_{\varepsilon,\alpha}(\mathbf{u}) = 0\}, \\ G^{\mathbf{V}_\infty}_{\varepsilon,\alpha}(\mathbf{u}) &:= \sum_{j=1}^{3}\int_{\mathbb{R}^N}\varepsilon^2|\nabla u_j|^2 + V_{j,\infty}u_j^2 - |u_j|^{p+1} - 3\alpha\int_{\mathbb{R}^N}u_1u_2u_3. \end{aligned}$$

Then

$$\sum_{j=1}^{3}\int_{\mathbb{R}^N}\varepsilon_0^2|\nabla u_{j,n}|^2 + V_{j,\infty}u_{j,n}^2 = t_n^{p-1}\sum_{j=1}^{3}\int_{\mathbb{R}^N}u_{j,n}^{p+1} + 3t_n\alpha_n\int_{\mathbb{R}^N}u_{1,n}u_{2,n}u_{3,n}.$$

As in the argument in Appendix A in [30], up to a subsequence, there exists $C > 0$ such that

$$\sum_{j=1}^{3}\int_{\mathbb{R}^N}u_{j,n}^{p+1} \geq C \quad \text{for all } n \in \mathbb{N}.$$

Since $\{\mathbf{u}_n\}_{n=1}^{\infty}$ is bounded in $\mathbb{H}$ and $\varepsilon_n \to \varepsilon_0$ and $\varepsilon_0 > 0$, $\{t_n\}_{n=1}^{\infty}$ is bounded. From the argument as in Proposition 1 in [30], it follows that

$$\begin{aligned} c_{\varepsilon_n,\alpha_n} &= I_{\varepsilon_n,\alpha_n}(\mathbf{u}_n) \geq I_{\varepsilon_n,\alpha_n}(t_n\mathbf{u}_n) \\ &= I^{\mathbf{V}_\infty}_{\varepsilon_n,\alpha_n}(t_n\mathbf{u}_n) + o(1) = I^{\mathbf{V}_\infty}_{\varepsilon_0,\alpha_n}(t_n\mathbf{u}_n) + o(1) \geq c^{\mathbf{V}_\infty}_{\varepsilon_0,\alpha_n} + o(1). \end{aligned}$$

In addition, from Theorem 1.4 in [30], there exists $\alpha^*_{\varepsilon_0,\mathbf{V}_\infty} > 0$ such that

$$c^{\mathbf{V}_\infty}_{\varepsilon_0,\alpha} = c^{\mathbf{V}_\infty}_{\varepsilon_0,0} = \min_{j=1,2,3} c^{V_{j,\infty}}_{2,\varepsilon_0} \quad \text{for all } 0 \leq \alpha \leq \alpha^*_{\varepsilon_0,\mathbf{V}_\infty}.$$

Then we have for n sufficiently large,

$$\liminf_{n\to\infty} c_{\varepsilon_n,\alpha_n} \geq \liminf_{n\to\infty} c^{\mathbf{V}_\infty}_{\varepsilon_0,\alpha_n} = \min_{j=1,2,3} c^{V_{j,\infty}}_{2,\varepsilon_0}. \tag{4.47}$$

On the other hand, from (4.45) and (Claim 2), it follows that

$$\limsup_{n\to\infty} c_{\varepsilon_n,\alpha_n} \leq \min_{j=1,2,3} c^{V_j}_{2,\varepsilon_0} < \min_{j=1,2,3} c^{V_{j,\infty}}_{2,\varepsilon_0}.$$

This contradicts (4.47). Hence $\mathbf{u}_0 \neq (0,0,0)$.

Since $\mathbf{u}_n$ is a ground state of $(\mathcal{P}_{\varepsilon_n,\alpha_n})$, $\mathbf{u}_n$ satisfies

$$\begin{cases} -\varepsilon_n^2\Delta u_{1,n} + V_1(x)u_{1,n} = u_{1,n}^p + \alpha_n u_{2,n}u_{3,n}, \\ -\varepsilon_n^2\Delta u_{2,n} + V_2(x)u_{2,n} = u_{2,n}^p + \alpha_n u_{1,n}u_{3,n}, \\ -\varepsilon_n^2\Delta u_{3,n} + V_3(x)u_{3,n} = u_{3,n}^p + \alpha_n u_{1,n}u_{2,n}. \end{cases}$$

Since $\varepsilon_n \to \varepsilon_0$ and $\alpha_n \to 0$, then $\mathbf{u}_0$ satisfies

$$\begin{cases} -\varepsilon_0^2\Delta u_{1,0} + V_1(x)u_{1,0} = u_{1,0}^p, \\ -\varepsilon_0^2\Delta u_{2,0} + V_2(x)u_{2,0} = u_{2,0}^p, \\ -\varepsilon_0^2\Delta u_{3,0} + V_3(x)u_{3,0} = u_{3,0}^p. \end{cases}$$

If there exist $j_1, j_2 \in \{1,2,3\}$ such that $j_1 \neq j_2$ and $u_{j_1,0} \neq 0$ and $u_{j_2,0} \neq 0$, then

$$\begin{aligned} &\liminf_{n\to\infty} c_{\varepsilon_n,\alpha_n} \\ &= \liminf_{n\to\infty} \left\{ \frac{1}{6}\sum_{j=1}^{3}\int_{\mathbb{R}^N} \varepsilon_n^2|\nabla u_{j,n}|^2 + V_j(x)u_{j,n}^2 + \frac{p-2}{3(p+1)}\sum_{j=1}^{3}\int_{\mathbb{R}^N} u_{j,n}^{p+1} \right\} \end{aligned}$$

$$\begin{aligned}
&= \liminf_{n\to\infty} \left\{ \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} \varepsilon_0^2 |\nabla u_{j,n}|^2 + V_j(x) u_{j,n}^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} u_{j,n}^{p+1} \right\} \\
&\geq \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} \varepsilon_0^2 |\nabla u_{j,0}|^2 + V_j(x) u_{j,0}^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} u_{j,0}^{p+1} \\
&= \sum_{j=1}^{3} I_{2,\varepsilon_0}^{V_j}(u_{j,0}) \geq \sum_{j=1}^{3} c_{2,\varepsilon_0}^{V_j} > \min_{j=1,2,3} c_{2,\varepsilon_0}^{V_j}.
\end{aligned}$$

This contradicts (4.45). Thus there exists $j_0 \in \{1,2,3\}$ such that $u_{j_0,0} \neq 0$ and $u_{j,0} = 0$ for all $j \neq j_0$. For simplicity, we assume $j_0 = 1$. Then

$$\begin{aligned}
&\liminf_{n\to\infty} c_{\varepsilon_n,\alpha_n} \\
&= \liminf_{n\to\infty} \left\{ \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} \varepsilon_n^2 |\nabla u_{j,n}|^2 + V_j(x) u_{j,n}^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} u_{j,n}^{p+1} \right\} \\
&= \liminf_{n\to\infty} \left\{ \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} \varepsilon_0^2 |\nabla u_{j,n}|^2 + V_j(x) u_{j,n}^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} u_{j,n}^{p+1} \right\} \\
&\geq \frac{1}{6} \int_{\mathbb{R}^N} \varepsilon_0^2 |\nabla u_{1,0}|^2 + V_1(x) u_{1,0}^2 + \frac{p-2}{3(p+1)} \int_{\mathbb{R}^N} u_{1,0}^{p+1} \\
&\geq I_{2,\varepsilon_0}^{V_1}(u_{1,0}) \geq c_{2,\varepsilon_0}^{V_1} \geq \min_{j=1,2,3} c_{2,\varepsilon_0}^{V_j}.
\end{aligned}$$

From (4.45), we have

$$\begin{aligned}
&\lim_{n\to\infty} \left\{ \frac{1}{6} \sum_{j=1}^{3} \int_{\mathbb{R}^N} \varepsilon_0^2 |\nabla u_{j,n}|^2 + V_j(x) u_{j,n}^2 + \frac{p-2}{3(p+1)} \sum_{j=1}^{3} \int_{\mathbb{R}^N} u_{j,n}^{p+1} \right\} \\
&= \frac{1}{6} \int_{\mathbb{R}^N} \varepsilon_0^2 |\nabla u_{1,0}|^2 + V_1(x) u_{1,0}^2 + \frac{p-2}{3(p+1)} \int_{\mathbb{R}^N} u_{1,0}^{p+1}, \\
&\liminf_{n\to\infty} \int_{\mathbb{R}^N} \varepsilon_0^2 |\nabla u_{1,n}|^2 + V_1(x) u_{1,n}^2 \geq \int_{\mathbb{R}^N} \varepsilon_0^2 |\nabla u_{1,0}|^2 + V_1(x) u_{1,0}^2, \\
&\liminf_{n\to\infty} \int_{\mathbb{R}^N} \varepsilon_0^2 |\nabla u_{j,n}|^2 + V_j(x) u_{j,n}^2 \geq 0 \quad j = 2,3, \\
&\liminf_{n\to\infty} \int_{\mathbb{R}^N} u_{1,n}^{p+1} \geq \int_{\mathbb{R}^N} u_{1,0}^{p+1}, \quad \liminf_{n\to\infty} \int_{\mathbb{R}^N} u_{j,n}^{p+1} \geq 0 \quad j = 2,3.
\end{aligned}$$

Then we have

$$\|u_{1,n} - u_{1,0}\|_{H^1} \to 0, \quad \|u_{j,n}\|_{H^1} \to 0 \quad j = 2,3. \tag{4.48}$$

By the same argument as in (Step 2) in Proposition 4.18, there exists $C > 0$ such that

$$\|u_{2,n}\|_{H^1}^2 + \|u_{3,n}\|_{H^1}^2 \geq C.$$

This contradicts (4.48). Thus the case $\varepsilon_0 > 0$ does not occur.

From (Case 1) and (Case 2), we can conclude that

$$\inf_{0<\varepsilon<r} \Gamma_\varepsilon^* > 0.$$

□

Lemma 4.21. Suppose that (V1),(V2). Then it follows that

$$(\alpha_1^* \leq) \sup_{0<\varepsilon<1} \Gamma_\varepsilon^* < \infty.$$

Proof. We define the following minimization problem:

$$\begin{aligned}
\tilde{c}_\varepsilon^{\mathbf{V},\infty} &:= \inf_{\mathbf{v}\in\tilde{\mathcal{N}}_\varepsilon^{\mathbf{V},\infty}} \tilde{I}_\varepsilon^{\mathbf{V},\infty}(\mathbf{v}),\\
\tilde{I}_\varepsilon^{\mathbf{V},\infty}(\mathbf{v}) &:= \frac{1}{2}\sum_{j=1}^3 \int_{\mathbb{R}^N} |\nabla v_j|^2 + V_j(\varepsilon y)v_j^2 - \int_{\mathbb{R}^N} v_1 v_2 v_3,\\
\tilde{\mathcal{N}}_\varepsilon^{\mathbf{V},\infty} &:= \{\mathbf{v} \in \mathbb{H} \setminus \{(0,0,0)\} \mid \tilde{G}_\varepsilon^{\mathbf{V},\infty}(\mathbf{v}) = 0\},\\
\tilde{G}_\varepsilon^{\mathbf{V},\infty}(\mathbf{v}) &:= \sum_{j=1}^3 \int_{\mathbb{R}^N} |\nabla v_j|^2 + V_j(\varepsilon y)v_j^2 - \int_{\mathbb{R}^N} v_1 v_2 v_3.
\end{aligned}$$

By the same argument as in Proposition 2 (Step 1) in [30] and, we obtain

$$\tilde{c}_{\varepsilon,\alpha} \leq \tilde{c}_\varepsilon^{\mathbf{V},\infty}/\alpha^2 \quad \text{for all } \alpha > 0. \tag{4.49}$$

We have $\tilde{c}_\varepsilon^{\mathbf{V},\infty} \leq \tilde{c}_1^{\mathbf{V}_{\max},\infty}$, where

$$\begin{aligned}
\mathbf{V}_{\max} &:= (V_{1,\max}, V_{2,\max}, V_{3,\max}), \quad V_{j,\max} := \sup_{x\in\mathbb{R}^N} V_j(x),\\
\tilde{c}_1^{\mathbf{V}_{\max},\infty} &:= \inf_{\mathbf{v}\in\tilde{\mathcal{N}}_1^{\mathbf{V}_{\max},\infty}} \tilde{I}_1^{\mathbf{V}_{\max},\infty}(\mathbf{v}),
\end{aligned}$$

$$\tilde{I}_1^{\mathbf{V}_{\max},\infty}(\mathbf{v}) := \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla v_j|^2 + V_{j,\max}v_j^2 - \int_{\mathbb{R}^N} v_1v_2v_3,$$
$$\tilde{\mathcal{N}}_1^{\mathbf{V}_{\max},\infty} := \{\mathbf{v}\in\mathbb{H}\setminus\{(0,0,0)\} \mid \tilde{G}_1^{\mathbf{V}_{\max},\infty}(\mathbf{v}) = 0\},$$
$$\tilde{G}_1^{\mathbf{V}_{\max},\infty}(\mathbf{v}) := \sum_{j=1}^{3}\int_{\mathbb{R}^N}|\nabla v_j|^2 + V_{j,\max}v_j^2 - \int_{\mathbb{R}^N} v_1v_2v_3.$$

Indeed, from Proposition 1 in [30], there exists a minimizer $\mathbf{v}$ for $\tilde{c}_1^{\mathbf{V}_{\max},\infty}$. Note that $\int_{\mathbb{R}^N} v_1v_2v_3 > 0$. By the same argument as in Lemma 2.3 in [43], there exists $t_\varepsilon > 0$ such that $t_\varepsilon\mathbf{v} \in \tilde{\mathcal{N}}_\varepsilon^{\mathbf{V},\infty}$. Since $\mathbf{v} \in \tilde{\mathcal{N}}_1^{\mathbf{V}_{\max},\infty}$ and Lemma 2.3 in [43], it follows that $I_1^{\mathbf{V}_{\max},\infty}(t\mathbf{v}) \le I_1^{\mathbf{V}_{\max},\infty}(\mathbf{v})$ for all $t > 0$. Hence, we have

$$\tilde{c}_\varepsilon^{\mathbf{V},\infty} \le \tilde{I}_\varepsilon^{\mathbf{V},\infty}(t_\varepsilon\mathbf{v}) \le \tilde{I}_1^{\mathbf{V}_{\max},\infty}(t_\varepsilon\mathbf{v}) \le \tilde{I}_1^{\mathbf{V}_{\max},\infty}(\mathbf{v}) = \tilde{c}_1^{\mathbf{V}_{\max},\infty}. \tag{4.50}$$

Note that

$$\tilde{c}_{\varepsilon,0} = \min_{j=1,2,3} \tilde{c}_{2,\varepsilon}^{V_j}.$$

Then we have

$$\tilde{c}_{\varepsilon,0} = \min_{j=1,2,3} \tilde{c}_{2,\varepsilon}^{V_j} \ge \min_{j=1,2,3} \tilde{c}_1^{V_{j,0}} > 0. \tag{4.51}$$

Then there exists $\alpha_0 > 0$ (independent of ε) such that

$$\tilde{c}_1^{\mathbf{V}_{\max},\infty}/\alpha^2 < \min_{j=1,2,3} \tilde{c}_1^{V_{j,0}} \quad \text{for all } \alpha > \alpha_0. \tag{4.52}$$

From (4.49)–(4.52), we have

$$\tilde{c}_{\varepsilon,\alpha} < \tilde{c}_{\varepsilon,0} \quad \text{for all } \alpha > \alpha_0,\ 0 < \varepsilon < 1.$$

Hence we obtain

$$\Gamma_\varepsilon^* \le \alpha_0 \quad \text{for all } 0 < \varepsilon < 1.$$

□

Suppose that (V1),(V2). From Lemma 4.20 and 4.21, it follows that $\alpha_0^* > 0$ and $\alpha_1^* < \infty$. Now, we prove Proposition 4.19.

Proof of Proposition 4.19. (i) Let $0 \leq \alpha < \alpha_0^*$. By the definition of α_0^*, there exists $\varepsilon_0 > 0$ such that

$$\inf_{0<\varepsilon<\varepsilon_0} \Gamma_\varepsilon^* > \alpha.$$

Hence we have $\Gamma_\varepsilon^* > \alpha$ for all $0 < \varepsilon < \varepsilon_0$. From Theorem 1.4 in [30], all the ground states of $(\mathcal{P}_{\varepsilon,\alpha})$ are scalar.

(ii) Let $\alpha > \alpha_1^*$. By the definition of α_1^*, there exists $\varepsilon_0 > 0$ such that

$$\sup_{0<\varepsilon<\varepsilon_0} \Gamma_\varepsilon^* < \alpha.$$

Hence we have $\Gamma_\varepsilon^* < \alpha$ for all $0 < \varepsilon < \varepsilon_0$. From Theorem 1.4 in [30], all the ground states of $(\mathcal{P}_{\varepsilon,\alpha})$ are vector.

(iii) Let $\alpha > \alpha_0^*$. By the definition of α_0^*, there exists $\{\varepsilon_{0,n}\}_{n=1}^\infty \subset (0,\infty)$ such that $\varepsilon_{0,n} \to 0$ and $\Gamma_{\varepsilon_{0,n}}^* \to \alpha_0^*$. Since $\alpha_0^* < \alpha$, up to a subsequence, $\Gamma_{\varepsilon_{0,n}}^* < \alpha$. From Theorem 1.4 in [30], all the ground states of $(\mathcal{P}_{\varepsilon_{0,n},\alpha})$ are vector.

(iv) Let $0 \leq \alpha < \alpha_1^*$. By the definition of α_1^*, there exists $\{\varepsilon_{1,n}\}_{n=1}^\infty \subset (0,\infty)$ such that $\varepsilon_{1,n} \to 0$ and $\Gamma_{\varepsilon_{1,n}}^* \to \alpha_1^*$. Since $\alpha < \alpha_1^*$, up to a subsequence, $\Gamma_{\varepsilon_{1,n}}^* > \alpha$. From Theorem 1.4 in [30], all the ground states of $(\mathcal{P}_{\varepsilon_{1,n},\alpha})$ are scalar. □

4.9 Appendix

In this Appendix, we consider the radial symmetry and monotonicity of classical solutions of elliptic system of the following type:

$$\begin{cases} u_i'' + f_i(x, u_1, \ldots, u_k) = 0 & \text{in } \mathbb{R}, \quad i = 1, \ldots, k, \\ u_i > 0 & \text{in } \mathbb{R}, \\ u_i(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{4.53}$$

where $k \geq 1$.

Busca-Sirakov [9] studied the radial symmetry and monotonicity of classical solutions of elliptic systems for $N \geq 2$. Moreover, Ikoma [25] considered the symmetry and monotonicity of the solutions in the case $N = 1$ and $k = 2$.

Here, we show the symmetry result for $N = 1$ after slight modification.

Let us note $\mathbf{u} = (u_1, \ldots, u_k) \in (0, \infty)^k$ and

$$A(x, \mathbf{u}) = \left(\frac{\partial f_i}{\partial u_j}(x, \mathbf{u}) \right)_{1 \leq i,j \leq k}.$$

We suppose that $f_i \in C^1(\mathbb{R} \times (0, \infty)^k, \mathbb{R})$ for all $i = 1, \ldots, k$ and (A0)–(A4):

(A0) $f_i(-x, \mathbf{u}) = f_i(x, \mathbf{u})$ for all $x \in \mathbb{R}$, $\mathbf{u} \in (0, \infty)^k$ and $i = 1, \ldots, k$.

(A1) $(\partial f_i / \partial x)(x, \mathbf{u}) \leq 0$ for all $x \geq 0$, $\mathbf{u} \in (0, \infty)^k$ and $i = 1, \ldots, k$.

(A2) $(\partial f_i / \partial u_j)(x, \mathbf{u}) \geq 0$ for all $x \in \mathbb{R}$, $\mathbf{u} \in (0, \infty)^k$ and $i, j \in \{1, \ldots, k\}$, $i \neq j$.

(A3) there exist constants $\varepsilon > 0$ and $R_1 > 0$ such that for any $I, J \subset \{1, \ldots, k\}$, $I \cap J = \emptyset$, $I \cup J = \{1, \ldots, k\}$, there exist $i_0 \in I$ and $j_0 \in J$ such that $(\partial f_{i_0} / \partial u_{j_0})(x, \mathbf{u}) > 0$ for all $(x, \mathbf{u}) \in \mathcal{O}$, where

$$\mathcal{O} = \{(x, \mathbf{u}) \in \mathbb{R} \times (0, \infty)^k \mid |x| > R_1,\ |\mathbf{u}| < \varepsilon\}.$$

(A4) all k-principal minors of $-A(x, u_1, \ldots, u_k)$ have positive determinants, for all $(x, \mathbf{u}) \in \mathcal{O}$, $1 \leq i \leq k$. We recall that the k-principal minors of a matrix $(m_{ij})_{1 \leq i,j \leq k}$ are the submatrices $(m_{ij})_{1 \leq i,j \leq k'}$ with $1 \leq k' \leq k$.

Theorem 4.22. Suppose $f_1, \ldots, f_k$ satisfy (A0)–(A4), and $\mathbf{u} = (u_1, \ldots, u_k)$ is a classical solution of (4.53). Then there exists a point $y_0 \in \mathbb{R}$ such that the functions u_i are radially symmetric with respect to the origin y_0, that is $u_i(x) = u_i(|x - y_0|)$, $i = 1, \ldots, k$. Moreover,

$$\frac{du_i}{dr} < 0 \quad \text{for all } r = |x - y_0| > 0.$$

For $\lambda \in \mathbb{R}$, set $\Sigma_\lambda := (\lambda, \infty)$ and for $x \in \Sigma_\lambda$, we define

$$x^\lambda := 2\lambda - x,$$
$$U_i^\lambda(x) := u_i(x^\lambda) - u_i(x) = u_i(2\lambda - x) - u_i(x).$$

Outline of the proof of Theorem 4.22. We define

$$\Lambda := \inf\{\lambda > 0 \mid U_i^\mu \geq 0 \text{ in } \Sigma_\mu \text{ for } i = 1, \ldots, k, \text{ and all } \mu \geq \lambda\}.$$

(Step 1) Since $u_i(x) \to 0$ as $|x| \to \infty$, there exists $R_0 \geq R_1$ such that $u_i(x) < \varepsilon/\sqrt{k}$ if $|x| \geq R_0$ for all $i = 1, \ldots, k$. We take $\lambda^* > R_0$, for which

$$\max_{\substack{1 \leq i \leq k \\ x \in [2\lambda - R_0, 2\lambda + R_0]}} u_i(x) < \min_{\substack{1 \leq i \leq k \\ x \in [-R_0, R_0]}} u_i(x),$$

for all $\lambda > \lambda^*$. Hence $U_i^\lambda > 0$ in $[2\lambda - R_0, 2\lambda + R_0] \subset \Sigma_\lambda$ for all $\lambda > \lambda^*$. We notice that the functions U_i^λ satisfy the following system

$$(U_i^\lambda)'' + \frac{\partial f_i}{\partial x}(\eta)(|x^\lambda| - x) + \sum_{j=1}^{k} \frac{\partial f_i}{\partial u_j}(x, \xi_{i1}, \ldots, \xi_{ik}) U_j^\lambda = 0, \quad i = 1, \ldots, k,$$

where $\eta = \eta(x, \lambda) \in (0, \infty)^{k+1}$ and

$$\xi_{ij} = \xi_{ij}(x, \lambda) \in (\min\{u_j(x), u_j(x^\lambda)\}, \max\{u_j(x), u_j(x^\lambda)\}).$$

Since $|x^\lambda| < x$ for $x \in \Sigma_\lambda$, we obtain from (A1) the following systems of inequality for U_i^λ

$$(U_i^\lambda)'' + \sum_{j=1}^{k} \frac{\partial f_i}{\partial u_j}(x, \xi_{i1}, \ldots, \xi_{ik}) U_j^\lambda \leq 0, \quad i = 1, \ldots, k. \tag{4.54}$$

We want to show that $U_i^\lambda \geq 0$ in Σ_λ, for all $\lambda > \lambda^*$. We argue by contradiction. Suppose there exist $\lambda > \lambda^*$ and $i_0 \in \{1, \ldots, k\}$ such that $\inf_{\Sigma_\lambda} U_{i_0}^\lambda < 0$. We set $J = \{j \mid U_j^\lambda \geq 0 \text{ in } \Sigma_\lambda\}$, and $I = \{1, \ldots, k\} \setminus J$ (note that $i_0 \in I$).

Since all k-principal minors of $-A(x, \mathbf{u})$ have positive determinants, for all $(x, \mathbf{u}) \in \mathcal{O}$, we do not need to introduce a function g as in [9]. Note the following lemma stated as Lemma 2.2 in [20].

Lemma 4.23. Let $M = (m_{ij})_{1\le i,j\le k}$ be a matrix such that $m_{ij} \le 0$ for $i \ne j$. Assume all k-principal minors of M have positive determinants. Then

(i) all minors of M obtained by dropping lines and columns of the same order have positive determinants.

(ii) if M_{ij} is the minor of M obtained by dropping the i th line and j th column we have $(-1)^{i+j} \det M_{ij} \ge 0$.

Hence we may assume that $I = \{1, \dots, p\}$. Since $U_j^\lambda \ge 0$ in Σ_λ for $j \in J$, from (A2) and (4.54), we have

$$(U_i^\lambda)'' + \sum_{j=1}^{p} \frac{\partial f_i}{\partial u_j}(x, \xi_{i1}, \dots, \xi_{ik}) U_j^\lambda \le 0, \quad i = 1, \dots, p.$$

Since $\inf_{\Sigma_\lambda} U_i^\lambda < 0$ for all $i = 1, \dots, p$, $U_i^\lambda > 0$ in $[2\lambda - R_0, 2\lambda + R_0]$, $U_i^\lambda(\lambda) = 0$ and $U_i^\lambda(x) \to 0$ as $|x| \to \infty$, there exist $x_1, \dots, x_p \in \Sigma_\lambda \setminus [2\lambda - R_0, 2\lambda + R_0]$ such that $U_i^\lambda(x_i) = \min_{\Sigma_\lambda} U_i^\lambda < 0$. Then $(U_i^\lambda)''(x_i) \ge 0$ and $(U_i^\lambda)'(x_i) = 0$. Substituting $x = x_i$ at i-th equation and using the fact that $U_j^\lambda(x_j) \le U_j^\lambda(x_i)$, we have

$$\sum_{j=1}^{p} \frac{\partial f_i}{\partial u_j}(x_i, \xi_{i1}, \dots, \xi_{ik}) U_j^\lambda(x_j) \le 0, \quad i = 1, \dots, p. \tag{4.55}$$

(4.55) can be written as

$$M\mathbf{U} = Y,$$

where

$$\mathbf{U} := (U_1^\lambda(x_1), \dots, U_p^\lambda(x_p)), \quad M = (m_{ij})_{1\le i,j\le p}, \quad Y = (y_1, \dots, y_p),$$
$$y_i \ge 0, \quad m_{i,j} := -\frac{\partial f_i}{\partial u_j}(x_i, \xi_{i1}, \dots, \xi_{ik})$$

Since $x_i \notin [2\lambda - R_0, 2\lambda + R_0]$, then $x_i^\lambda \notin [-R_0, R_0]$, that is, $|x_i^\lambda| > R_0$. Noting that $x_i > R_0$, we have $u_j(x_i), u_j(x_i^\lambda) < \varepsilon/\sqrt{k}$. Thus we have $\xi_{i1}(x_i, \lambda)^2 + \cdots + \xi_{ik}(x_i, \lambda)^2 < \varepsilon^2$. From (A2) and (A4), we have $m_{ij} \le 0$ for $i \ne j$, and all p-principal minors of M have positive determinants. Since $\det M > 0$, it follows that $\mathbf{U} = M^{-1}Y$. From Lemma 4.23, $U_i^\lambda(x_i) \geq 0$ for all $i = 1, \ldots, p$. This contradicts the fact that $U_i(x_i) < 0$. Hence $\Lambda < \infty$.

The rest of the proof of Theorem 4.22 can be showed by the same argument as in [9]. □

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