The first variational formulae for integral invariants of degree two of the second fundamental form of a map between pseudo-Riemannian manifolds

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1 Introduction

The theory of harmonic maps and biharmonic maps is one of the important fields in differential geometry. Recall that a smooth map $\varphi : (M, g_M) \to (N, g_N)$ between Riemannian manifolds is said to be *harmonic* if it is a critical point of the energy functional

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 d\mu_{g_M}.$$

By the first variational formula, then φ is a harmonic map if and only if

$$\tau(\varphi) = \mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi) = 0, \tag{1.1}$$

where $\widetilde{\nabla} d\varphi$ is the second fundamental form and $\tau(\varphi)$ is the tension field of φ. The Euler–Lagrange equation (1.1) is a second order nonlinear PDE, therefore the theory of harmonic maps has been developed in geometric analysis, furthermore it is investigated applying methods of integrable systems. For example, geodesics, harmonic functions and minimal submanifolds are harmonic maps. As a generalization of harmonic maps, Eells and Lemaire [10] introduced the notion of *biharmonic map*, which is a critical point of the bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 d\mu_{g_M}.$$

Jiang [14] showed that φ is a biharmonic map if and only if

$$\tau_2(\varphi) = -\overline{\nabla}^* \overline{\nabla} \tau(\varphi) - \mathrm{tr}_{g_M} R^N \left(d\varphi(\cdot), \tau(\varphi)\right) d\varphi(\cdot) = 0, \tag{1.2}$$

where $-\overline{\nabla}^* \overline{\nabla}$ is the rough Laplacian and R^N is the Riemannian curvature tensor of (N, g_N). The Euler–Lagrange equation (1.2) is a fourth order nonlinear PDE. By definition, it is clear that a harmonic map is biharmonic. One of the important problems in the study of biharmonic maps is Chen's conjecture, that is, an arbitrary biharmonic submanifold of a Euclidean space must be minimal. This conjecture has been partially positively resolved, but it is still open. As a higher order energy functional, r-energy functional [19, 22], ES-r-energy functional [3, 10], F-k-energy functional [11], and so on. have been introduced, and various researchers have studied them from the viewpoint of variational problems and submanifolds (cf. Figure 1).

On the other hand, in integral geometry, Howard [12] provided integral invariants of submanifolds by using invariant polynomials of the second fundamental form, and then he formulated the kinematic formula in Riemannian homogeneous spaces (see also [15]). In his formulation, there are some notable integral invariants of submanifolds. One is integral invariants in the Chern–Federer kinematic formula. These integral invariants played significant roles in differential geometry. For example, Weyl [23] showed that the volume of a tube around a compact submanifold in a Euclidean space can be represented as a polynomial of the radius of the tube, where the coefficients are integral invariants of the second fundamental form of the submanifold. Also, Allendoerfer and Weil [2] used these integral invariants to describe the extended Gauss–Bonnet theorem, and this leads to the development of the theory of characteristic classes. Another notable one is the integral invariant defined from a certain invariant homogeneous polynomial of degree two. This invariant polynomial also appears in the definition of the Willmore–Chen invariant, which is a conformal invariant of submanifolds ([6, 7]).

In this thesis, we study variational problems for integral invariants, which are defined as integrations of invariant functions of the second fundamental form, of a smooth map between pseudo-Riemannian manifolds. We derive the first variational formulae for integral invariants defined from invariant homogeneous polynomials of degree two. Among these integral invariants, we show that the Euler–Lagrange equation of the Chern–Federer energy functional is reduced to a

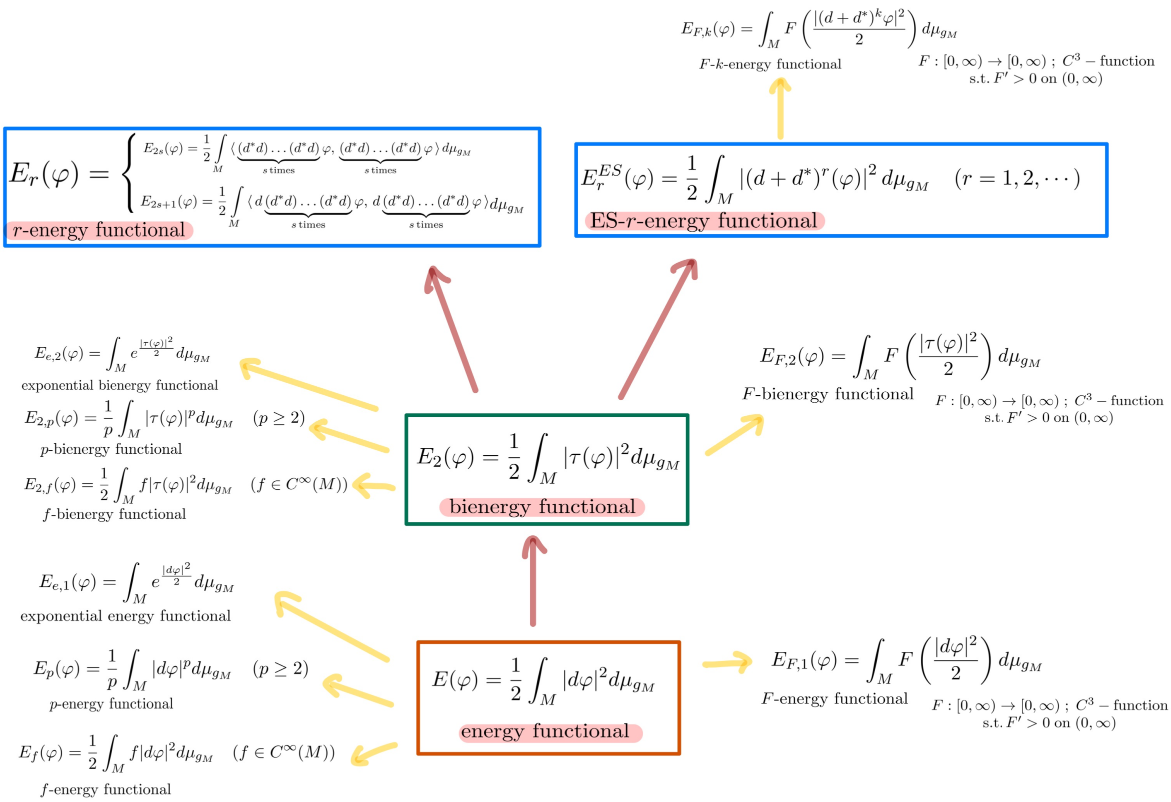

Figure 1: Correlation diagram of higher order energy functionals

second order PDE. Then we give some examples of Chern–Federer submanifolds in Riemannian space forms. The most important point of this research is that we consider the variational problem for a family of energy functionals, rather than fixing one energy functional.

In Section 2, we recall fundamental notions of pseudo-Riemannian manifolds and induced bundles. In Section 3, with an idea of integral geometry, we introduce integral invariants of a smooth map $\varphi : (M, g_M) \to (N, g_N)$ between pseudo-Riemannian manifolds by using invariant functions of the second fundamental form of φ. In particular, we focus on integral invariants of φ defined from invariant homogeneous polynomials of degree two. The space of those polynomials is spanned by the square norm of the second fundamental form and the square norm of the tension field, which are denoted by $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively. Hence, here the family of integral invariants includes the bienergy functional. In this thesis, we study variational problems for these integral invariants of φ. In Section 4, we derive the first variational formulae for $\mathcal{Q}_1$- and $\mathcal{Q}_2$-energy functionals. By the linearity, then we have the first variational formulae for all integral invariants of degree two. Note that it implies an alternative expression of the Euler–Lagrange equation of the bienergy functional. As mentioned above, from the viewpoint of integral geometry, there are two notable polynomials, called the Chern–Federer polynomial and the Willmore–Chen polynomial, in the space of invariant homogeneous polynomials of degree two. In Section 5, we discuss some properties of the Chern–Federer energy functional from the viewpoint of variational problems. The Euler–Lagrange equation of an integral invariant of degree two is a fourth order PDE in general, however, we show that the Euler–Lagrange equation of the Chern–Federer energy functional is reduced to a second order PDE. In Section 5.2, we describe a symmetry of the Euler–Lagrange equation of the Chern–Federer energy functional comparing with a symmetry of the Chern–Federer polynomial. In Section 6, we give some examples of Chern–Federer submanifolds in Riemannian space forms. Here, a Chern–Federer submanifold is the image of an isometric immersion which is a Chern–Federer map. For an isometric immersion into a Riemannian space form, a necessary and sufficient condition to be a Chern–Federer map is described in Theorem 6.1. Considering this condition, there is an obstruction for the domain manifold. In addition, as a trivial example, we can see that any isometric immersion of a Ricci-flat manifold into a Euclidean space is a Chern–Federer map. Finally, we discuss isometric immersions of flat tori into the 3-sphere and isoparametric hypersurfaces in Riemannian space forms.

This thesis includes the content of the paper [1] to be published in Osaka Journal of Mathematics.

2 Preliminaries

In this section, we explain fundamental properties of pseudo-Riemannian manifolds and induced bundles for later use ([21, 24]).

2.1 Pseudo-Riemannian manifolds

Let (M_p^m, g_M) be a pseudo-Riemannian manifold of dimension m with a nondegenerate metric with index p. Here nondegeneracy means that, at each point $x \in M$, the only vector $X \in T_xM$ satisfying $(g_M)_x(X, Y) = 0$ for all $Y \in T_xM$ is $X = 0$. Every pseudo-Riemannian manifold has the unique Levi-Civita connection ∇. Also, a local pseudo-orthonormal frame field of (M_p^m, g_M) is a set of m-local vector fields $\{e_i\}_{i=1}^m$ such that $g_M(e_i, e_j) = \varepsilon_i \delta_{ij}$ with $\varepsilon_1 = \cdots = \varepsilon_p = -1, \varepsilon_{p+1} = \cdots \varepsilon_m = 1$.

For a local pseudo-orthonormal frame field $\{e_i\}_{i=1}^m$ on a neighborhood U of (M_p^m, g_M), we have the following local expressions

$$\begin{aligned} X &= \sum_{i=1}^m \varepsilon_i g_M(X, e_i) e_i, \\ \mathrm{grad} f &= \sum_{i=1}^m \varepsilon_i \, df(e_i) e_i, \\ \mathrm{div} X &= \sum_{i=1}^m \varepsilon_i g_M(\nabla_{e_i} X, e_i), \\ \mathrm{tr}_{g_M} H &= \sum_{i=1}^m \varepsilon_i H(e_i, e_i), \end{aligned}$$

where $X \in \Gamma(TM)$, $f \in C^\infty(M)$ and H is any bilinear form.

The Riemannian curvature tensor R^M of (M_p^m, g_M) is a correspondence that assigns to every pair $X, Y \in \Gamma(TM)$ a mapping:

$$R^M(X, Y) : \Gamma(TM) \to \Gamma(TM)$$

defined by

$$R^M(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad Z \in \Gamma(TM).$$

Also, let $\{x^i\}_{i=1}^m$ be a local coordinate system on M, then we get that:

$$R^M\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k} = \left(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}}\right) \frac{\partial}{\partial x^k},$$

since $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$.

Proposition 2.1 ([21]). *Let M be a pseudo-Riemannian manifold. The curvature tensor of M satisfies the following properties:*

$$\begin{aligned}
&R^M(X,Y)Z = -R^M(Y,X)Z,\\
&\left\langle R^M(X,Y)Z, W\right\rangle = -\left\langle R^M(X,Y)W, Z\right\rangle,\\
&R^M(X,Y)Z + R^M(Y,Z)X + R^M(Z,X)Y = 0,\\
&\left\langle R^M(X,Y)Z, W\right\rangle = \left\langle R^M(Z,W)X, Y, \right\rangle
\end{aligned}$$

for any vector fields $X, Y, Z, W \in \Gamma(TM)$.

The third equation is well-known as the first Bianchi identity.

Proposition 2.2 ([21]). *Let M be a pseudo-Riemannian manifold. The curvature tensor of M satisfies the second Bianchi identity:*

$$(\nabla R^M)(X,Y,Z) + (\nabla R^M)(Y,Z,X) + (\nabla R^M)(Z,X,Y) = 0,$$

for any vector fields $X, Y, Z \in \Gamma(TM)$, where $(\nabla R^M)(X,Y,Z)$ is defined by

$$(\nabla R^M)(X,Y,Z)W := \nabla_X\big(R^M(Y,Z)W\big) - R^M(\nabla_X Y, Z)W - R^M(Y, \nabla_X Z)W - R^M(Y,Z)\nabla_X W$$

for any vector field $W \in \Gamma(TM)$.

2.2 The connections induced on the induced bunbles

Let (M^m_p, g_M) be an m-dimensional pseudo-Riemannian manifold with index p, (N^n_q, g_N) an n-dimensional pseudo-Riemannian manifold with index q, and $\varphi : M \to N$ a C^∞-map. Then, the fiber metric $(\cdot,\cdot)$ is naturally defined on the tensor product $T^*M \otimes \varphi^{-1}TN$ from the pseudo-Riemannian metrics g_M and g_N. In this section, we see that the connection is naturally introduced on the tensor product $T^*M \otimes \varphi^{-1}TN$, which is compatible with the fiber metric $(\cdot,\cdot)$.

First, we see that the fiber metric $(\cdot,\cdot)$ is defined on $T^*M \otimes \varphi^{-1}TN$. Let U be an open set of M and $\{x^i\}_{i=1}^m$ a local coordinate system on U. Then,

$$\left\{\left(\frac{\partial}{\partial x^1}\right)_x, \cdots, \left(\frac{\partial}{\partial x^m}\right)_x\right\}, \qquad x \in U$$

is the basis of the tangent space T_xM, and

$$\left\{(dx^1)_x, \cdots, (dx^m)_x\right\}$$

is the dual basis of the dual space T^*_xM. So that:

$$(dx^i)_x\left(\left(\frac{\partial}{\partial x^j}\right)_x\right) = \delta^i_j \qquad (1 \le i, j \le m).$$

Also, the pseudo-Riemannian metric g_M of M on U is expressed as

$$g_M = \sum_{i,j=1}^m (g_M)_{ij} dx^i \otimes dx^j.$$

Here, $(g_M)_{ij}$ is a C^∞-function on U defined as follows:

$$(g_M)_{ij} = g_M\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$

and $[(g_M)_{ij}]$ forms an m-order symmetric matirix at each point on U. Then, the inverse matrix of $[(g_M)_{ij}]$ is expressed as $[(g_M)^{ij}]$. Also, it is defined as

$$\sum_{k=1}^{m}[(g_M)_{ik}][(g_M)^{kj}] = \delta_i^j, \quad \sum_{k=1}^{m}[(g_M)^{ik}][(g_M)_{kj}] = \delta_j^i, \qquad (1 \leq i, j \leq m).$$

The pseudo-Riemannian metric g_M gives a natural isomorphism between T_xM and T_x^*M. That is, the following maps is obtained:

$$\flat : T_xM \to T_x^*M, \quad \sharp : T_x^*M \to T_xM.$$

In fact, for $X_x \in T_xM$ and $\omega_x \in T_x^*M$, we define

$$X_x^\flat(Y_x) = (g_M)_x(X_x, Y_x), \quad (g_M)_x(\omega_x^\sharp, Y_x) = \omega_x(Y_x), \qquad Y_x \in T_xM.$$

By using the local coordinate system, we represent

$$X_x = \sum_{i=1}^{m} X^i(x)\left(\frac{\partial}{\partial x^i}\right)_x, \quad \omega_x = \sum_{i=1}^{m}\omega_i(x)(dx^i)_x.$$

Then $X_x^\flat$ and $\omega_x^\sharp$ are denoted by

$$X_x^\flat = \sum_{i=1}^{m}\left(\sum_{j=1}^{m}(g_M)_{ij}(x)X^j(x)\right)(dx^i)_x, \quad \omega_x^\sharp = \sum_{i=1}^{m}\left(\sum_{j=1}^{m}(g_M)^{ij}(x)\omega_j(x)\right)\left(\frac{\partial}{\partial x^i}\right)_x.$$

Therefore, for $\omega_x, \theta_x \in T_x^*M$, the inner product $(g_M^*)_x$ on T_x^*M is defined by

$$(g_M^*)_x(\omega_x, \theta_x) := (g_M)_x(\omega_x^\sharp, \theta_x^\sharp).$$

Thus, we have

$$(g_M^*)_x\big((dx^i)_x, (dx^j)_x\big) = (g_M)^{ij}(x).$$

Thus, the inverse matrix $\big[(g_M)^{ij}(x)\big]$ of $\big[(g_M)_{ij}(x)\big]$ is the matrix that represents the components of the inner production $(g_M^*)_x$.

Let $y \in V$ be an open set of N and $\{y_\alpha\}_{\alpha=1}^n$ a local coordinate system on V. Then, for $x \in U$ that satisfies $\varphi(x) \in V$, the local representation of $\varphi(x)$ is expressed by the C^∞-function $\varphi^\alpha = y^\alpha \circ \varphi$ on U as follows:

$$\varphi(x) = \big(\varphi^1(x^1, \cdots, x^m), \cdots, \varphi^n(x^1, \cdots, x^m)\big).$$

Also, the pseudo-Riemannian metric g_N on V is expressed as follows:

$$g_N = \sum_{\alpha,\beta=1}^{n}(g_N)_{\alpha\beta}dy^\alpha \otimes dy^\beta.$$

Then, for $x \in U$, the differential map $d\varphi_x : T_xM \to T_{\varphi(x)}N$ is locally written as follows:

$$d\varphi_x\left(\left(\frac{\partial}{\partial x^i}\right)_x\right) = \sum_{\alpha=1}^{n}\left(\frac{\partial\varphi^\alpha}{\partial x^i}\right)(x)\left(\frac{\partial}{\partial y^\alpha}\right)_{\varphi(x)} \qquad (1 \leq i \leq m).$$

That is, $d\varphi_x$ is a linear map represented by a (n, m)-type matrix $\left(\left(\frac{\partial\varphi^\alpha}{\partial x^i}\right)(x)\right)$.

Let $\mathrm{Hom}(T_xM, T_{\varphi(x)}N)$ be

$$\mathrm{Hom}(T_xM, T_{\varphi(x)}N) = \left\{ f : T_xM \to T_{\varphi(x)}N \ ; \ \text{linear map} \right\}.$$

Then we have a linear isomorphism $\mathrm{Hom}(T_xM, T_{\varphi(x)}N) \simeq T_x^*M \otimes T_{\varphi(x)}N$. Also, since the basis of $T_x^*M \otimes T_{\varphi(x)}N$ is

$$\left\{ (dx^i)_x \otimes \left(\frac{\partial}{\partial y^\alpha} \right)_{\varphi(x)} \ \middle| \ 1 \le i \le m, 1 \le \alpha \le n \right\},$$

the differentiable map $d\varphi_x$ can be expressed by the following equation:

$$d\varphi_x = \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \left(\frac{\partial \varphi^\alpha}{\partial x^i} \right)(x) \ (dx^i)_x \otimes \left(\frac{\partial}{\partial y^\alpha} \right)_{\varphi(x)}.$$

Next, we define the inner product $(\cdot,\cdot)_x$ on the tensor product $T_x^*M \otimes T_{\varphi(x)}N$ from the inner products $(g_M^*)_x$ of T_x^*M and $(g_N)_{\varphi(x)}$ of $T_{\varphi(x)}N$. In fact, $(\cdot,\cdot)_x$ defined by the following formula:

$$\left((dx^i)_x \otimes \left(\frac{\partial}{\partial y^\alpha} \right)_{\varphi(x)}, (dx^j)_x \otimes \left(\frac{\partial}{\partial y^\beta} \right)_{\varphi(x)} \right)_x = (g_M)^{ij}(x)(g_N)_{\alpha\beta}(\varphi(x))$$

and extended bilinearly to general elements of $T_x^*M \otimes T_{\varphi(x)}N$. If we denote the norm induced this inner product $(\cdot,\cdot)_x$ as $|\cdot,\cdot|_x$, we get the following equation:

$$\begin{aligned} |d\varphi_x|_x^2 &= (d\varphi_x, d\varphi_x)_x \\ &= \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} (g_M)^{ij}(x)(g_N)_{\alpha\beta}(\varphi(x)) \left(\frac{\partial \varphi^\alpha}{\partial x^i} \right)(x) \left(\frac{\partial \varphi^\beta}{\partial x^j} \right)(x). \end{aligned}$$

These facts show the following. Let $\Gamma(T^*M \otimes \varphi^{-1}TN)$ be the space of the C^∞-sections of the vector bundle $T^*M \otimes \varphi^{-1}TN$. Here, the map $d\varphi : M \to T^*M \otimes \varphi^{-1}TN$ is defined by $d\varphi(x) = d\varphi_x$ for $x \in M$. Hence, $d\varphi \in \Gamma(T^*M \otimes \varphi^{-1}TN)$. In addition, we can naturally define the fiber metric on the vector bundle $T^*M \otimes \varphi^{-1}TN$ from the inner product $(\cdot,\cdot)_x$ of the tensor product $T_x^*M \otimes T_{\varphi(x)}N$. In fact, for $\sigma, \widetilde{\sigma} \in \Gamma(T^*M \otimes \varphi^{-1}TN)$, we define

$$(\sigma, \widetilde{\sigma})(x) = (\sigma(x), \widetilde{\sigma}(x))_x \qquad (x \in M).$$

From this, for $d\varphi \in \Gamma(T^*M \otimes \varphi^{-1}TN)$, we obtain the norm $|d\varphi|$ of this fiber metric $(\cdot,\cdot)$ as follows.

$$|d\varphi|^2 = \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} (g_M)^{ij}(g_N)_{\alpha\beta}(\varphi) \left(\frac{\partial \varphi^\alpha}{\partial x^i} \right) \left(\frac{\partial \varphi^\beta}{\partial x^j} \right).$$

Next, we show that a connection on the fiber bundle $T^*M \otimes \varphi^{-1}TN$ is naturally defined that is compatible with the fiber metric $(\cdot,\cdot)$.

On the tangent vector bundle TM, the Levi-Civita connection ∇ is defined by the pseudo-Riemannian metric g_M. In fact, let $\{x^i\}_{i=1}^m$ be a local coordinate system of M, and we define a coefficients of connection $\{\Gamma_{ij}^k\}$ with respect to $\{x_i\}$ of ∇ as

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{m} \Gamma_{ij}^k \frac{\partial}{\partial x^k},$$

then Γ_{ij}^k is given by:

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{m} (g_M)^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right),$$

where $(g_M)_{ij}$ are components of g_M with respect to $\{x^i\}$ and $(g_M)^{ij}$ are components of the inverse matrix of $[(g_M)_{ij}]$. For $Y \in \Gamma(TM)$, we define

$$\nabla Y(X) := \nabla_X Y \qquad (X \in \Gamma(TM)),$$

then we get the $(1,1)$-type tensor field:

$$\nabla Y \in \Gamma(T^*M \otimes TM) \simeq \mathrm{Hom}(TM, TM).$$

Therefore, the Levi-Civita connection ∇ of M defines the map:

$$\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM).$$

Here ∇Y is called the covariant differentiation of Y.

Next, we define the connection ∇^* on T^*M. From the linear isomorphism of $T_x^*M \simeq T_xM$, we get the following isomorphism between TM and T^*M:

$$\flat : TM \to T^*M, \quad \sharp : T^*M \to TM.$$

Using these isomorphisms, for $\omega \in \Gamma(T^*M)$ and $X \in \Gamma(TM)$, $\nabla^*_X\omega \in \Gamma(T^*M)$ is defined by the following:

$$\nabla^*_X\omega(Y) := \left(\nabla_X\omega^\sharp\right)^\flat(Y) \qquad (Y \in \Gamma(TM)).$$

Here $\nabla^*_X\omega$ is called the covariant derivative of ω with respect to X. For $\omega \in \Gamma(T^*M)$, we define

$$\nabla^*\omega(X) := \nabla^*_X\omega \qquad (X \in \Gamma(TM)),$$

then we get the $(0,2)$-type tensor field:

$$\nabla^*\omega \in \Gamma(T^*M \otimes T^*M) \simeq \mathrm{Hom}(TM, T^*M).$$

And, the map:

$$\nabla^* : \Gamma(T^*M) \to \Gamma(T^*M \otimes T^*M)$$

is called the connection of T^*M induced from ∇. Also, from the definitions of $\flat$ and $\sharp$, we can see the equation:

$$\nabla^*_X\omega(Y) = X\big(\omega(Y)\big) - \omega(\nabla_X Y).$$

Furthermore, for $X \in \Gamma(TM)$ and $\omega, \theta \in \Gamma(T^*M)$, we see that

$$X\big(g_M^*(\omega,\theta)\big) = g_M^*(\nabla^*_X\omega, \theta) + g_M^*(\omega, \nabla^*_X\theta),$$

which means that ∇^* is compatible with the fiber metric g_M^* of T^*M.

For a smooth map $\varphi : M \to N$, we can uniquely define the connection $\overline{\nabla}$ of $\varphi^{-1}TN$ from the Levi-Civita connection ∇' of N. Let U and V be coordinate neighborhoods of M and N such that $\varphi(U) \subset V$, and $\{x^i\}_{i=1}^m$ and $\{y^\alpha\}_{\alpha=1}^n$ local coordinate systems on U and V respectively. For each $1 \leq \alpha \leq n$,

$$\left(\frac{\partial}{\partial y^\alpha} \circ \varphi\right)(x) = \left(\frac{\partial}{\partial y^\alpha}\right)_{\varphi(x)} \qquad (x \in U)$$

is a C^∞-section of $\varphi^{-1}TN$ on U, and at each point $x \in U$, the set

$$\left\{\left(\frac{\partial}{\partial y^1} \circ \varphi\right)(x), \cdots, \left(\frac{\partial}{\partial y^n} \circ \varphi\right)(x)\right\}$$

is a basis of the fiber $T_{\varphi(x)}N$ on x of $\varphi^{-1}TN$. Here, for each $1 \leq i \leq m$ and $1 \leq \gamma \leq n$, we define the covariant derivative $\overline{\nabla}$ of the section of $\varphi^{-1}TN$ as:

$$\left(\overline{\nabla}_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial y^\gamma} \circ \varphi\right)\right)(x) = \nabla'_{d\varphi_x\left(\left(\frac{\partial}{\partial x^i}\right)_x\right)} \frac{\partial}{\partial y^\gamma}.$$

Here $\overline{\nabla}$ is called the induced connection on $\varphi^{-1}TN$. Notice that the induced connection $\overline{\nabla}$ is compatible with the fiber metric $\varphi^* g_N = \{(g_N)_{\varphi(x)}\}_{x \in M}$ of $\varphi^{-1}TN$.

Lastly, we explain how the connection $\widetilde{\nabla}$ on $T^*M \otimes \varphi^{-1}TN$ can be defined from the connection ∇^* of T^*M and the connection$\overline{\nabla}$ of $\varphi^{-1}TN$. In general, the C^∞-section of $T^*M \otimes \varphi^{-1}TN$ is represented by the sections of T^*M and $\varphi^{-1}TN$. Therefore, for $\omega \in \Gamma(T^*M)$ and $W \in \Gamma(\varphi^{-1}TN)$, we define the covariant derivative $\widetilde{\nabla}(\omega \otimes W) \in \Gamma(T^*M \otimes T^*M \otimes \varphi^{-1}TN)$ as follows:

$$\widetilde{\nabla}(\omega \otimes W) = (\nabla^* \omega) \otimes W + \omega \otimes (\overline{\nabla} W).$$

Under this definition, for $X \in \Gamma(TM)$, the covariant derivative $\widetilde{\nabla}_X(\omega \otimes W)$ is defined as follows:

$$\widetilde{\nabla}_X(\omega \otimes W) = (\nabla^*_X \omega) \otimes W + \omega \otimes (\overline{\nabla}_X W).$$

This induces, the map:

$$\widetilde{\nabla} : \Gamma(T^*M \otimes \varphi^{-1}TN) \to \Gamma(T^*M \otimes T^*M \otimes \varphi^{-1}TN),$$

which is called the induced connection of $T^*M \otimes \varphi^{-1}TN$. The connection $\widetilde{\nabla}$ is compatible with the fiber metric $(,)$ on $T^*M \otimes \varphi^{-1}TN$.

3 Integral invariants of a map between pseudo-Riemannian manifolds

In this section, we define integral invariants of the second fundamental form of a map between pseudo-Riemannian manifolds. An m-dimensional pseudo-Euclidean space with index p is denoted by $\mathbb{E}^m_p = (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbb{E}^m_p})$ with $\langle x, y \rangle_{\mathbb{E}^m_p} = -\sum_{i=1}^p x_i y_i + \sum_{j=p+1}^m x_j y_j$ $(x, y \in \mathbb{R}^m)$. Define $\mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$ to be

$$\mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q) := \left\{ H : \mathbb{E}^m_p \times \mathbb{E}^m_p \to \mathbb{E}^n_q \text{ ; symmetric bilinear map} \right\},$$

which is a $\frac{1}{2}nm(m+1)$-dimensional vector space. Let G be the direct product group of pseudo-orthogonal groups defined by

$$G := O(p, m-p) \times O(q, n-q).$$

The group G acts on $\mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$, that is for $g = (a, b) \in G$ and $H \in \mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$ then gH is given by

$$(gH)(u, v) := b\left(H(a^{-1}u, a^{-1}v)\right) \qquad (u, v \in \mathbb{E}^m_p).$$

Then a function $\mathcal{P}$ on $\mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$ is said to be *G-invariant* if $\mathcal{P}(gH) = \mathcal{P}(H)$ for all $g \in G$ and $H \in \mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$.

Let (M^m_p, g_M) and (N^n_q, g_N) be pseudo-Riemannian manifolds, and $\varphi : M \to N$ a C^∞-map. Thoughout this thesis, a fiber metric on a vector bundle is also denoted by $\langle \cdot, \cdot \rangle$. The second fundamental form of the map φ is the symmetric bilinear map $\widetilde{\nabla} d\varphi : \Gamma(TM) \times \Gamma(TM) \to \Gamma(\varphi^{-1}TN)$ defined by

$$(\widetilde{\nabla} d\varphi)(X, Y) := \overline{\nabla}_X (d\varphi(Y)) - d\varphi(\nabla_X Y)$$

for any vector fields $X, Y \in \Gamma(TM)$, which is a section of $\bigodot^2 T^*M \otimes \varphi^{-1}TN$. Here $\bigodot$ is the symmetric tensor product. We denote by ∇ is the Levi–Civita connection on the tangent bundle TM of (M^m_p, g_M), $\overline{\nabla}$ and $\widetilde{\nabla}$ are the induced connections on the bundles $\varphi^{-1}TN$ and $T^*M \otimes \varphi^{-1}TN$. If φ is an isometric immersion, then we have

$$(\widetilde{\nabla} d\varphi)(X, Y) = \nabla'_{d\varphi(X)} d\varphi(Y) - d\varphi(\nabla_X Y) = \nabla'_X Y - \nabla_X Y,$$

where ∇' is the Levi–Civita connection on the tangent bundle TN of (N^n_q, g_N), i.e. the second fundamental form of the isometric immersion φ agrees with the second fundamental form of the submanifold.

For each $x \in M$, we can write

$$(\widetilde{\nabla} d\varphi)_x : T_xM \times T_xM \to T_{\varphi(x)}N,$$

which is a symmetric bilinear map. Let $\{e_i\}_{i=1}^m$ be a pseudo-orthonormal basis of T_xM, $\{e^i\}_{i=1}^m$ the dual basis of $\{e_i\}$, and $\{\xi_\alpha\}_{\alpha=1}^n$ a pseudo-orthonormal basis of $T_{\varphi(x)}N$. Hence we identify T_xM and $T_{\varphi(x)}N$ with $\mathbb{E}^m_p$ and $\mathbb{E}^n_q$, respectively. Then $(\widetilde{\nabla} d\varphi)_x$ can be expressed as

$$(\widetilde{\nabla} d\varphi)_x = \sum_\alpha \varepsilon'_\alpha \sum_{i,j} h^\alpha_{ij}\, e^i \odot e^j \otimes \xi_\alpha,$$

where h^α_{ij} is defined by

$$h^\alpha_{ij} := \big\langle (\widetilde{\nabla} d\varphi)_x(e_i, e_j), \xi_\alpha \big\rangle,$$

and

$$\varepsilon'_\alpha = \begin{cases} -1 & (\alpha = 1, \cdots, q) \\ 1 & (\alpha = q+1, \cdots, n). \end{cases}$$

Thus we have a linear isomorphism between $T^*_xM \odot T^*_xM \otimes T_{\varphi(x)}N$ and $\mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$. That is, $(\widetilde{\nabla} d\varphi)_x \in T^*_xM \odot T^*_xM \otimes T_{\varphi(x)}N$ corresponds to $H_x := (h^\alpha_{ij}) \in \mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$. Therefore, for a G-invariant function $\mathcal{P}$ on $\mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$, we define an *invariant function* of the second fundamental form of φ as follows:

$$\mathcal{P}\big((\widetilde{\nabla} d\varphi)_x\big) := \mathcal{P}(H_x).$$

This definition does not depend on the choices of $\{e_i\}_{i=1}^m$ and $\{\xi_\alpha\}_{\alpha=1}^n$ since $\mathcal{P}$ is G-invariant and a change of a basis is the action of the pseudo-orthogonal group. Also, $\mathcal{P}\big((\widetilde{\nabla} d\varphi)_x\big)$ is a smooth function on M.

Definition 3.1. Let (M^m_p, g_M) be an m-dimensional compact pseudo-Riemannian manifold with index p, (N^n_q, g_N) an n-dimensional pseudo-Riemannian manifold with index q, and $\mathcal{P}$ a G-invariant function on $\mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$. Then for a smooth map $\varphi : M \to N$, we define

$$I^{\mathcal{P}}(\varphi) := \int_M \mathcal{P}\big((\widetilde{\nabla} d\varphi)_x\big) d\mu_{g_M}.$$

We call $I^{\mathcal{P}}(\varphi)$ the *integral invariant* of φ with respect to $\mathcal{P}$.

By definition, $I^{\mathcal{P}}(\varphi)$ is an invariant of a map φ between pseudo-Riemannian manifolds, that is, $I^{\mathcal{P}}(g \circ \varphi \circ f^{-1}) = I^{\mathcal{P}}(\varphi)$ holds for any $f \in \mathrm{Isom}(M)$ and $g \in \mathrm{Isom}(N)$.

We consider the following G-invariant polynomials on $\mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$. For $H = (h^\alpha_{ij}) \in \mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$, define

$$\mathcal{Q}_1(H) := \sum_\alpha \varepsilon'_\alpha \sum_{i,j} \varepsilon_i \varepsilon_j (h^\alpha_{ij})^2 \quad \text{and} \quad \mathcal{Q}_2(H) := \sum_\alpha \varepsilon'_\alpha \left(\sum_i \varepsilon_i h^\alpha_{ii} \right)^2$$

with

$$\varepsilon_i = \begin{cases} -1 & (i = 1, \cdots, p) \\ 1 & (i = p+1, \cdots, m). \end{cases}$$

$\mathcal{Q}_1(H)$ and $\mathcal{Q}_2(H)$ are G-invariant homogeneous polynomials of degree two on $\mathrm{II}(\mathbb{E}^m_p, \mathbb{E}^n_q)$.

Definition 3.2. For $\varphi \in C^\infty(M, N)$, the $\mathcal{Q}_1$-*energy functional* $I^{\mathcal{Q}_1}(\varphi)$ and the $\mathcal{Q}_2$-*energy functional* $I^{\mathcal{Q}_2}(\varphi)$ are defined by

$$I^{\mathcal{Q}_1}(\varphi) := \int_M \mathcal{Q}_1\big((\widetilde{\nabla} d\varphi)_x\big) d\mu_{g_M} = \int_M \left\langle \widetilde{\nabla} d\varphi, \widetilde{\nabla} d\varphi \right\rangle d\mu_{g_M} \tag{3.1}$$

and

$$I^{\mathcal{Q}_2}(\varphi) := \int_M \mathcal{Q}_2\big((\widetilde{\nabla} d\varphi)_x\big) d\mu_{g_M} = \int_M \left\langle \mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi), \mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi) \right\rangle d\mu_{g_M}. \tag{3.2}$$

Then φ is called a $\mathcal{Q}_1$-*map* if it is a critical point of $I^{\mathcal{Q}_1}(\varphi)$. Also, then φ is called a $\mathcal{Q}_2$-*map* if it is a critical point of $I^{\mathcal{Q}_2}(\varphi)$.

Remark 3.3. The $\mathcal{Q}_2$-energy functional $I^{\mathcal{Q}_2}(\varphi)$ is equal to two times of the bienergy functional $E_2(\varphi)$. Indeed, when φ is a smooth map between Riemannian manifolds, it holds that

$$I^{\mathcal{Q}_2}(\varphi) = \int_M \left\langle \mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi), \mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi) \right\rangle d\mu_{g_M} = \int_M \left| \mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi) \right|^2 d\mu_{g_M} = 2E_2(\varphi).$$

Remark 3.4. When $\dim M = 4$, the $\mathcal{Q}_1$-energy functional and $\mathcal{Q}_2$-energy functional are invariant under homothetic changes of the metric on the domain M.

4 The first variational formulae for $\mathcal{Q}_1$-energy and $\mathcal{Q}_2$-energy

4.1 Preliminaries

Let (M^m_p, g_M) be an m-dimensional compact pseudo-Riemannian manifold with index p, (N^n_q, g_N) an n-dimensional pseudo-Riemannian manifold with index q, and $\varphi : M \to N$ a C^∞-map. In this section, we use the following notation.

$\widetilde{\nabla}^2 d\varphi$ and $\widetilde{\nabla}^3 d\varphi$ are defined by

$$(\widetilde{\nabla}^2 d\varphi)(X, Y, Z) := \overline{\nabla}_X\big((\widetilde{\nabla} d\varphi)(Y, Z)\big) - (\widetilde{\nabla} d\varphi)(\nabla_X Y, Z) - (\widetilde{\nabla} d\varphi)(Y, \nabla_X Z)$$

and

$$\begin{aligned}(\widetilde{\nabla}^3 d\varphi)(X, Y, Z, W) := {}& \overline{\nabla}_X\big((\widetilde{\nabla}^2 d\varphi)(Y, Z, W)\big) - (\widetilde{\nabla}^2 d\varphi)(\nabla_X Y, Z, W) \\ & - (\widetilde{\nabla}^2 d\varphi)(Y, \nabla_X Z, W) - (\widetilde{\nabla}^2 d\varphi)(Y, Z, \nabla_X W)\end{aligned}$$

for any vector fields $X, Y, Z, W \in \Gamma(TM)$. $\widetilde{\nabla}^2 d\varphi$ and $\widetilde{\nabla}^3 d\varphi$ are sections of $\bigotimes^3 T^*M \otimes \varphi^{-1}TN$ and $\bigotimes^4 T^*M \otimes \varphi^{-1}TN$, respectively. By definition, $\widetilde{\nabla}^2 d\varphi$ and $\widetilde{\nabla}^3 d\varphi$ have the following symmetry

$$\begin{aligned}(\widetilde{\nabla}^2 d\varphi)(X, Y, Z) &= (\widetilde{\nabla}^2 d\varphi)(X, Z, Y), \\ (\widetilde{\nabla}^3 d\varphi)(X, Y, Z, W) &= (\widetilde{\nabla}^3 d\varphi)(X, Y, W, Z).\end{aligned}$$

The tension field $\tau(\varphi)$ of φ is defined by

$$\tau(\varphi) := \mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi) = \sum_i \varepsilon_i (\widetilde{\nabla} d\varphi)(e_i, e_i) = \sum_i \varepsilon_i \big(\widetilde{\nabla}_{e_i} d\varphi\big)(e_i).$$

If φ is an isometric immersion, then its tension field is equal to m times of the mean curvature vector field.

In general, the curvature tensor field R^E of a connection ∇^E on the bundle E over M is defined by

$$R^E(X,Y) := \nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X - \nabla^E_{[X,Y]} \quad (X, Y \in \Gamma(TM)).$$

In particular, for the curvature tensor field $\widetilde{R}$ of the induced connection $\widetilde{\nabla}$ on the bundle $T^*M \otimes \varphi^{-1}TN$, we have

$$\begin{aligned}\left(\widetilde{R}(X,Y)d\varphi\right)(Z) &= R^{\varphi^{-1}TN}(X,Y)d\varphi(Z) - d\varphi\left(R^M(X,Y)Z\right) \\ &= R^N\left(d\varphi(X), d\varphi(Y)\right)d\varphi(Z) - d\varphi\left(R^M(X,Y)Z\right) \quad (X, Y, Z \in \Gamma(TM)),\end{aligned}$$

where R^M, R^N and $R^{\varphi^{-1}TN}$ are the curvature tensor fields on TM, TN and $\varphi^{-1}TN$, respectively.

Then we derive the first variational formulae of the $\mathcal{Q}_1$-energy and $\mathcal{Q}_2$-energy separately.

4.2 The first variational formula for $\mathcal{Q}_1$-energy

We consider a smooth variation $\{\varphi_t\}_{t\in I}$ $(I := (-\varepsilon, \varepsilon))$ of φ, that is we consider a smooth map Φ given by

$$\Phi : M \times I \to N, \quad (x,t) \mapsto \Phi(x,t) =: \varphi_t(x)$$

such that $\varphi_0(x) = \varphi(x)$ for all $x \in M$, and denote by V its variational vector field, that is

$$V = d\Phi\left(\left.\frac{\partial}{\partial t}\right|_{t=0}\right) \in \Gamma(\varphi^{-1}TN).$$

We denote by $\boldsymbol{\nabla}$, $\overline{\boldsymbol{\nabla}}$ and $\widetilde{\boldsymbol{\nabla}}$ the induced connections on $T(M\times I)$, $\Phi^{-1}TN$ and $T^*(M\times I) \otimes \Phi^{-1}TN$, respectively. Let $\{e_i\}_{i=1}^m$ be a local pseudo-orthonormal frame field on a neighborhood U of $x \in M$, then $\left\{e_i, \frac{\partial}{\partial t}\right\}$ is a pseudo-orthonormal frame field on the neighborhood $U \times I$ of $(x,t) \in M \times I$, and it holds that

$$\boldsymbol{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0, \quad \boldsymbol{\nabla}_{\frac{\partial}{\partial t}} e_i = \boldsymbol{\nabla}_{e_i} \frac{\partial}{\partial t} = 0 \quad (1 \le i \le m).$$

First, we can write the formula (3.1) as

$$I^{\mathcal{Q}_1}(\varphi) = \int_M \left\langle \widetilde{\nabla} d\varphi, \widetilde{\nabla} d\varphi \right\rangle d\mu_{g_M} = \int_M \sum_{i,j} \varepsilon_i \varepsilon_j \left\langle \left(\widetilde{\nabla} d\varphi\right)(e_i, e_j), \left(\widetilde{\nabla} d\varphi\right)(e_i, e_j) \right\rangle d\mu_{g_M}.$$

For a variation $\{\varphi_t\}_{t\in I}$ of φ, it holds that

$$\begin{aligned}\frac{d}{dt} I^{\mathcal{Q}_1}(\varphi_t) &= \frac{d}{dt} \int_M \sum_{i,j} \varepsilon_i \varepsilon_j \left\langle \left(\widetilde{\boldsymbol{\nabla}} d\Phi\right)(e_i, e_j), \left(\widetilde{\boldsymbol{\nabla}} d\Phi\right)(e_i, e_j) \right\rangle d\mu_{g_M} \\ &= 2 \int_M \sum_{i,j} \varepsilon_i \varepsilon_j \left\langle \overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} \left((\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle d\mu_{g_M}. \end{aligned} \tag{4.1}$$

Then we have

$$
\begin{aligned}
&\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} \left((\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right) \\
&= \left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi \right) \left(\frac{\partial}{\partial t}, e_i, e_j \right) + \left(\widetilde{\boldsymbol{\nabla}} d\Phi \right) \left(\boldsymbol{\nabla}_{\frac{\partial}{\partial t}} e_i, e_j \right) + \left(\widetilde{\boldsymbol{\nabla}} d\Phi \right) \left(e_i, \boldsymbol{\nabla}_{\frac{\partial}{\partial t}} e_j \right) \\
&= \left(\widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} \widetilde{\boldsymbol{\nabla}}_{e_i} d\Phi \right) (e_j) \\
&= \left(\widetilde{\boldsymbol{\nabla}}_{e_i} \widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} d\Phi \right) (e_j) - \left(\widetilde{\boldsymbol{\nabla}}_{[e_i, \frac{\partial}{\partial t}]} d\Phi \right) (e_j) - \left(\widetilde{R} \left(e_i, \frac{\partial}{\partial t} \right) d\Phi \right) (e_j) \\
&= \overline{\boldsymbol{\nabla}}_{e_i} \left(\left(\widetilde{\boldsymbol{\nabla}} d\Phi \right) \left(\frac{\partial}{\partial t}, e_j \right) \right) - \left(\widetilde{\boldsymbol{\nabla}} d\Phi \right) \left(\boldsymbol{\nabla}_{e_i} \frac{\partial}{\partial t}, e_j \right) - \left(\widetilde{\boldsymbol{\nabla}} d\Phi \right) \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}_{e_i} e_j \right) \\
&\quad - R^N \left(d\Phi(e_i), d\Phi \left(\frac{\partial}{\partial t} \right) \right) d\Phi(e_j) \\
&= \overline{\boldsymbol{\nabla}}_{e_i} \left(\left(\widetilde{\boldsymbol{\nabla}} d\Phi \right) \left(e_j, \frac{\partial}{\partial t} \right) \right) - \left(\widetilde{\boldsymbol{\nabla}} d\Phi \right) \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}_{e_i} e_j \right) - R^N \left(d\Phi(e_i), d\Phi \left(\frac{\partial}{\partial t} \right) \right) d\Phi(e_j) \\
&= \left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi \right) \left(e_i, e_j, \frac{\partial}{\partial t} \right) + (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(\boldsymbol{\nabla}_{e_i} e_j, \frac{\partial}{\partial t} \right) + (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_j, \boldsymbol{\nabla}_{e_i} \frac{\partial}{\partial t} \right) \\
&\quad - (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}_{e_i} e_j \right) - R^N \left(d\Phi(e_i), d\Phi \left(\frac{\partial}{\partial t} \right) \right) d\Phi(e_j) \\
&= \left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi \right) \left(e_i, e_j, \frac{\partial}{\partial t} \right) - R^N \left(d\Phi(e_i), d\Phi \left(\frac{\partial}{\partial t} \right) \right) d\Phi(e_j).
\end{aligned} \tag{4.2}
$$

By substituting (4.2) into (4.1), we have

$$
\begin{aligned}
\frac{d}{dt} I^{\mathcal{Q}_1}(\varphi_t) &= 2 \int_M \sum_{i,j} \varepsilon_i \varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi) \left(e_i, e_j, \frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle d\mu_{g_M} \\
&\quad - 2 \int_M \sum_{i,j} \varepsilon_i \varepsilon_j \left\langle R^N \left(d\Phi(e_i), d\Phi \left(\frac{\partial}{\partial t} \right) \right) d\Phi(e_j), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle d\mu_{g_M}.
\end{aligned} \tag{4.3}
$$

We need the following lemma to calculate the first variation of $I^{\mathcal{Q}_1}(\varphi)$.

Lemma 4.1. *Under the setting above, for any variation $\{\varphi_t\}_{t \in I}$ of φ, it holds*

$$
\begin{aligned}
&\int_M \sum_{i,j} \varepsilon_i \varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi) \left(e_i, e_j, \frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle d\mu_{g_M} \\
&= \int_M \sum_{i,j} \varepsilon_i \varepsilon_j \left\langle d\Phi \left(\frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi) (e_i, e_j, e_i, e_j) \right\rangle d\mu_{g_M}.
\end{aligned} \tag{4.4}
$$

Proof. We define vector fields on M depending on $t \in I$ by

$$
\widetilde{X}_t := \sum_{i,j} \varepsilon_i \varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_j, \frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi) (e_i, e_j) \right\rangle e_i
$$

and

$$
\widetilde{Y}_t := \sum_{i,j} \varepsilon_i \varepsilon_j \left\langle d\Phi \left(\frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi) (e_j, e_i, e_j) \right\rangle e_i,
$$

where $\{e_i\}_{i=1}^m$ is a local pseudo-orthonormal frame field on a neighborhood U of M. $\widetilde{X}_t$ and $\widetilde{Y}_t$ are well-defined because of the independence of the choice of $\{e_i\}$. Hence $\widetilde{X}_t$ and $\widetilde{Y}_t$ are global vector fields on M. Indeed, let $\{f_i\}_{i=1}^m$ be another local pseudo-orthonormal frame field, then the transformation function $(p_{ij})_{1\le i,j\le m} : U \to O(p, m-p)$ is given by

$$f_i = \sum_j p_{ij} e_j \quad (1 \le i \le m).$$

Using this, we have

$$\begin{aligned}
&\sum_{i,j} \varepsilon_i \varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(f_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(f_i, f_j) \right\rangle f_i \\
&= \sum_{i,j,a,b,c,d} \varepsilon_i \varepsilon_j p_{ja} p_{ib} p_{jc} p_{id} \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_a, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_b, e_c) \right\rangle e_d \\
&= \sum_{a,b,c,d} \varepsilon_a \varepsilon_b \delta_{ac} \delta_{bd} \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_a, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_b, e_c) \right\rangle e_d \\
&= \sum_{a,b} \varepsilon_a \varepsilon_b \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_a, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_b, e_a) \right\rangle e_b.
\end{aligned}$$

In a similar way, we can check that $\widetilde{Y}_t$ is well-defined.

The divergnce of $\widetilde{X}_t$ is given by

$$\begin{aligned}
&\mathrm{div}\widetilde{X}_t \\
&= \sum_k \varepsilon_k \left\langle \nabla_{e_k} \widetilde{X}, e_k \right\rangle \\
&= \sum_k \varepsilon_k \left\langle \nabla_{e_k} \left(\sum_{i,j} \varepsilon_i \varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle e_i \right), e_k \right\rangle \\
&= \sum_{i,j,k} \varepsilon_k \varepsilon_i \varepsilon_j e_k \left(\left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle \right) \langle e_i, e_k \rangle \\
&\quad + \sum_{i,j,k} \varepsilon_k \varepsilon_i \varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle \langle \nabla_{e_k} e_i, e_k \rangle \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \left\{ \left\langle \overline{\boldsymbol{\nabla}}_{e_i} \left((\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \frac{\partial}{\partial t}\right) \right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle \right. \\
&\quad \left. + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \frac{\partial}{\partial t}\right), \overline{\boldsymbol{\nabla}}_{e_i} \left((\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right) \right\rangle \right\} - \sum_{j,k} \varepsilon_j \varepsilon_k \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(\nabla_{e_k} e_k, e_j) \right\rangle \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(e_i, e_j, \frac{\partial}{\partial t}\right) + (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\nabla_{e_i} e_j, \frac{\partial}{\partial t}\right) + (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \boldsymbol{\nabla}_{e_i} \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle \right. \\
&\quad + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_i, e_i, e_j) + (\widetilde{\boldsymbol{\nabla}} d\Phi)(\nabla_{e_i} e_i, e_j) + (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, \nabla_{e_i} e_j) \right\rangle \\
&\quad \left. - \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(\nabla_{e_i} e_i, e_j) \right\rangle \right\} \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(e_i, e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\nabla_{e_i} e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle \right. \\
&\quad \left. + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_i, e_i, e_j) \right\rangle + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, \nabla_{e_i} e_j) \right\rangle \right\}.
\end{aligned}$$

At the fourth equality, we use the following

$$
\begin{aligned}
&\sum_{i,j,k} \varepsilon_k \varepsilon_i \varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_j, \frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle \langle \nabla_{e_k} e_i, e_k \rangle \\
&= - \sum_{i,j,k} \varepsilon_k \varepsilon_i \varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_j, \frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle \langle e_i, \nabla_{e_k} e_k \rangle \\
&= - \sum_{j,k} \varepsilon_j \varepsilon_k \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_j, \frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(\sum_i \varepsilon_i \langle e_i, \nabla_{e_k} e_k \rangle e_i, e_j \right) \right\rangle \\
&= - \sum_{j,k} \varepsilon_j \varepsilon_k \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_j, \frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi) (\nabla_{e_k} e_k, e_j) \right\rangle.
\end{aligned}
$$

Now, take a neighborhood U of $x \in M$ such that the exponential map at x is injective onto U, which is called a normal neighborhood. And we construct a pseudo-orthonormal frame field $\{e_i\}_{i=1}^m$ by parallel transporting a pseudo-orthonormal basis at x along a geodesic $\gamma : [0, 1] \to M$ from $\gamma(0) = x$ to $\gamma(1) = y$ for every $y \in U$. The pseudo-orthonormal frame field $\{e_i\}_{i=1}^m$ is called a geodesic frame field. We note that a geodesic frame field $\{e_i\}_{i=1}^m$ around a point $x \in M$ satisfies

$$
\left(\nabla_{e_i} e_j\right)_x = 0, \quad [e_i, e_j]_x = 0 \qquad (1 \le i, j \le m)
$$

at x. Since $(\boldsymbol{\nabla}_{e_i} e_j)_{(x,t)} = (\nabla_{e_i} e_j)_x = 0$ for all $t \in I$, we have

$$
\begin{aligned}
&\left(\mathrm{div} \widetilde{X}_t\right)_x \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \Bigg\{ \left\langle \left((\widetilde{\boldsymbol{\nabla}}^2 d\Phi) \left(e_i, e_j, \frac{\partial}{\partial t} \right) \right)_{(x,t)}, \left((\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right)_{(x,t)} \right\rangle \\
&\quad + \left\langle \left((\widetilde{\boldsymbol{\nabla}} d\Phi) \left(\nabla_{e_i} e_j, \frac{\partial}{\partial t} \right) \right)_{(x,t)}, \left((\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right)_{(x,t)} \right\rangle \\
&\quad + \left\langle \left((\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_j, \frac{\partial}{\partial t} \right) \right)_{(x,t)}, \left((\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_i, e_i, e_j) \right)_{(x,t)} \right\rangle \\
&\quad + \left\langle \left((\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_j, \frac{\partial}{\partial t} \right) \right)_{(x,t)}, \left((\widetilde{\boldsymbol{\nabla}} d\Phi) (e_i, \nabla_{e_i} e_j) \right)_{(x,t)} \right\rangle \Bigg\} \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \Bigg\{ \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, (e_j)_{(x,t)}, \left(\frac{\partial}{\partial t} \right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, (e_j)_{(x,t)} \right) \right\rangle \\
&\quad + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)_{(x,t)} \left((\nabla_{e_i} e_j)_{(x,t)}, \left(\frac{\partial}{\partial t} \right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, (e_j)_{(x,t)} \right) \right\rangle \\
&\quad + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)_{(x,t)} \left((e_j)_{(x,t)}, \left(\frac{\partial}{\partial t} \right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, (e_i)_{(x,t)}, (e_j)_{(x,t)} \right) \right\rangle \\
&\quad + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)_{(x,t)} \left((e_j)_{(x,t)}, \left(\frac{\partial}{\partial t} \right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, (\nabla_{e_i} e_j)_{(x,t)} \right) \right\rangle \Bigg\} \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \Bigg\{ \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, (e_j)_{(x,t)}, \left(\frac{\partial}{\partial t} \right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, (e_j)_{(x,t)} \right) \right\rangle \\
&\quad + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)_{(x,t)} \left((e_j)_{(x,t)}, \left(\frac{\partial}{\partial t} \right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, (e_i)_{(x,t)}, (e_j)_{(x,t)} \right) \right\rangle \Bigg\}. \qquad (4.5)
\end{aligned}
$$

Each term of the last formula of (4.5) is a tensor, so we have

$$\mathrm{div}\widetilde{X}_t = \sum_{i,j} \varepsilon_i\varepsilon_j \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(e_i, e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle \right.$$
$$\left. + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_i, e_i, e_j) \right\rangle \right\}, \tag{4.6}$$

where $\{e_i\}_{i=1}^m$ is an arbitrary local pseudo-orthonormal frame field.

In a similar way, we calculate the divergence of $\widetilde{Y}_t$. We have

$$\begin{aligned}
&\mathrm{div}\widetilde{Y}_t \\
&= \sum_k \varepsilon_k \left\langle \nabla_{e_k}\widetilde{Y}_t, e_k \right\rangle \\
&= \sum_k \varepsilon_k \left\langle \nabla_{e_k}\left(\sum_{i,j} \varepsilon_i\varepsilon_j \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), \left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi\right)(e_j, e_i, e_j) \right\rangle e_i \right), e_k \right\rangle \\
&= \sum_{i,j,k} \varepsilon_k\varepsilon_i\varepsilon_j e_k \left(\left\langle d\Phi\left(\frac{\partial}{\partial t}\right), \left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi\right)(e_j, e_i, e_j) \right\rangle \right) \langle e_i, e_k\rangle \\
&\quad + \sum_{i,j,k} \varepsilon_k\varepsilon_i\varepsilon_j \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), \left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi\right)(e_j, e_i, e_j) \right\rangle \langle \nabla_{e_k} e_i, e_k\rangle \\
&= \sum_{i,j} \varepsilon_i\varepsilon_j \left\{ \left\langle \overline{\boldsymbol{\nabla}}_{e_i}\left(d\Phi\left(\frac{\partial}{\partial t}\right)\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_j, e_i, e_j) \right\rangle + \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), \overline{\boldsymbol{\nabla}}_{e_i}\left((\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_j, e_i, e_j)\right) \right\rangle \right\} \\
&\quad - \sum_{j,k} \varepsilon_k\varepsilon_j \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_j, \nabla_{e_k}e_k, e_j) \right\rangle \\
&= \sum_{i,j} \varepsilon_i\varepsilon_j \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right) + d\Phi\left(\nabla_{e_i}\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_j, e_i, e_j) \right\rangle \right. \\
&\quad + \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)(e_i, e_j, e_i, e_j) + (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(\nabla_{e_i}e_j, e_i, e_j) + (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_j, \nabla_{e_i}e_i, e_j) \right. \\
&\qquad \left. + (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_j, e_i, \nabla_{e_i}e_j) \right\rangle \\
&\quad \left. - \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_j, \nabla_{e_i}e_i, e_j) \right\rangle \right\} \\
&= \sum_{i,j} \varepsilon_i\varepsilon_j \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_j, e_i, e_j) \right\rangle + \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)(e_i, e_j, e_i, e_j) \right\rangle \right. \\
&\quad \left. + \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(\nabla_{e_i}e_j, e_i, e_j) \right\rangle + \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_j, e_i, \nabla_{e_i}e_j) \right\rangle \right\}.
\end{aligned}$$

Then, assuming that $\{e_i\}$ is a geodesic frame field around a point $x \in M$, we have

$$\begin{aligned}
&\left(\mathrm{div}\widetilde{Y}_t\right)_x \\
&= \sum_{i,j} \varepsilon_i\varepsilon_j \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)_{(x,t)}\left((e_i)_{(x,t)}, \left(\frac{\partial}{\partial t}\right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)_{(x,t)}\left((e_j)_{(x,t)}, (e_i)_{(x,t)}, (e_j)_{(x,t)} \right) \right\rangle \right. \\
&\quad \left. + \left\langle (d\Phi)_{(x,t)}\left(\left(\frac{\partial}{\partial t}\right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)_{(x,t)}\left((e_i)_{(x,t)}, (e_j)_{(x,t)}, (e_i)_{(x,t)}, (e_j)_{(x,t)} \right) \right\rangle \right\}.
\end{aligned} \tag{4.7}$$

Each term of the right hand side of (4.7) is a tensor, so we have

$$\begin{aligned}\mathrm{div}\widetilde{Y}_t = \sum_{i,j} \varepsilon_i\varepsilon_j &\left\{ \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_j, e_i, e_j) \right\rangle \right. \\ &\left. + \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)(e_i, e_j, e_i, e_j) \right\rangle \right\}, \end{aligned} \tag{4.8}$$

where $\{e_i\}_{i=1}^m$ is an arbitrary local pseudo-orthonormal frame field.

By Green's theorem, we have

$$\int_M \mathrm{div}\widetilde{X}_t \, d\mu_{g_M} = 0 = \int_M \mathrm{div}\widetilde{Y}_t \, d\mu_{g_M},$$

and together with (4.6) and (4.8), we have

$$\begin{aligned}&\int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(e_i, e_j, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j) \right\rangle d\mu_{g_M} \\ &= \int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)(e_i, e_j, e_i, e_j) \right\rangle d\mu_{g_M}.\end{aligned}$$

Here we use the symmetry of $\widetilde{\nabla}^2 d\Phi$. □

Substituting (4.4) into (4.3), we have

$$\begin{aligned}&\frac{d}{dt} I^{\mathcal{Q}_1}(\varphi_t) \\ &= 2\int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)(e_i, e_j, e_i, e_j) - R^N\left(d\Phi(e_j), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i, e_j)\right) d\Phi(e_i), d\Phi\left(\frac{\partial}{\partial t}\right) \right\rangle d\mu_{g_M}.\end{aligned}$$

Therefore we obtain the following theorem.

Theorem 4.2. *Let (M_p^m, g_M) be a compact pseudo-Riemannian manifold, (N_q^n, g_N) a pseudo-Riemannian manifold and $\varphi : M \to N$ a C^∞-map. Consider a C^∞-variation $\{\varphi_t\}_{t\in I}$ of φ with variational vector field V. Then the following formula holds*

$$\begin{aligned}&\left.\frac{d}{dt} I^{\mathcal{Q}_1}(\varphi_t)\right|_{t=0} \\ &= 2\int_M \left\langle \sum_{i,j} \varepsilon_i\varepsilon_j \left\{ \left(\widetilde{\nabla}^3 d\varphi\right)(e_i, e_j, e_i, e_j) + R^N\left((\widetilde{\nabla} d\varphi)(e_i, e_j), d\varphi(e_i)\right) d\varphi(e_j) \right\}, V \right\rangle d\mu_{g_M},\end{aligned}$$

where $\{e_i\}_{i=1}^m$ is a local pseudo-orthonormal frame field of (M_p^m, g_M) with $g_M(e_i, e_j) = \varepsilon_i\delta_{ij}$, $\varepsilon_1 = \cdots = \varepsilon_p = -1$, $\varepsilon_{p+1} = \cdots = \varepsilon_m = 1$.

For a map $\varphi \in C^\infty(M, N)$, we define $W_1(\varphi) \in \Gamma(\varphi^{-1}TN)$ by

$$W_1(\varphi) := \sum_{i,j} \varepsilon_i\varepsilon_j \left\{ (\widetilde{\nabla}^3 d\varphi)(e_i, e_j, e_i, e_j) + R^N\left((\widetilde{\nabla} d\varphi)(e_i, e_j), d\varphi(e_i)\right) d\varphi(e_j) \right\}.$$

Hence φ is a $\mathcal{Q}_1$-map if and only if $W_1(\varphi) = 0$. We can adopt the Euler–Lagrange equation $W_1(\varphi) = 0$ as the definition of a $\mathcal{Q}_1$-map. Then the domain M of φ is not nesessarily compact.

Remark 4.3. In an analytical setting, Moser [20] studied a variational problem for the $\mathcal{Q}_1$-energy functional $I^{\mathcal{Q}_1}(\varphi) = \int_M |\widetilde{\nabla} d\varphi|^2 d\mu_{g_M}$.

4.3 The first variational formula for $\mathcal{Q}_2$-energy

In a similar way, we show the first variational formula of the $\mathcal{Q}_2$-energy. Let $\{\varphi_t\}_{t\in I}$ be a C^∞-variation of φ with variational vector field V and $\{e_i\}$ a local pseudo-orthonormal frame field on a neighborhood U.

First, we can write (3.2) as

$$\begin{aligned} I^{\mathcal{Q}_2}(\varphi) &= \int_M \left\langle \mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi), \mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi) \right\rangle d\mu_{g_M} \\ &= \int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle (\widetilde{\nabla} d\varphi)(e_i,e_i), (\widetilde{\nabla} d\varphi)(e_j,e_j) \right\rangle d\mu_{g_M}. \end{aligned}$$

For a variation $\{\varphi_t\}_{t\in I}$ of φ, it holds that

$$\begin{aligned} \frac{d}{dt} I^{\mathcal{Q}_2}(\varphi_t) &= \frac{d}{dt}\int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i,e_i), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_j,e_j) \right\rangle d\mu_{g_M} \\ &= 2\int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle \overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}\big((\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i,e_i)\big), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_j,e_j) \right\rangle d\mu_{g_M}. \end{aligned} \tag{4.9}$$

Then we have

$$\begin{aligned} &\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}\left((\widetilde{\boldsymbol{\nabla}} d\Phi)(e_i,e_i)\right) \\ &= (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(\frac{\partial}{\partial t}, e_i, e_i\right) + 2(\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\nabla_{\frac{\partial}{\partial t}} e_i, e_i\right) \\ &= \left(\widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}\widetilde{\boldsymbol{\nabla}}_{e_i} d\Phi\right)(e_i) \\ &= \left(\widetilde{\boldsymbol{\nabla}}_{e_i}\widetilde{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} d\Phi\right)(e_i) - \left(\widetilde{\boldsymbol{\nabla}}_{[e_i,\frac{\partial}{\partial t}]} d\Phi\right)(e_i) - \left(\widetilde{R}\left(e_i, \frac{\partial}{\partial t}\right) d\Phi\right)(e_i) \\ &= \overline{\boldsymbol{\nabla}}_{e_i}\left((\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\frac{\partial}{\partial t}, e_i\right)\right) - (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\boldsymbol{\nabla}_{e_i}\frac{\partial}{\partial t}, e_i\right) - (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\frac{\partial}{\partial t}, \nabla_{e_i} e_i\right) \\ &\quad - R^N\left(d\Phi(e_i), d\Phi\left(\frac{\partial}{\partial t}\right)\right) d\Phi(e_i) \\ &= \overline{\boldsymbol{\nabla}}_{e_i}\left((\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right)\right) - (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\frac{\partial}{\partial t}, \nabla_{e_i} e_i\right) - R^N\left(d\Phi(e_i), d\Phi\left(\frac{\partial}{\partial t}\right)\right) d\Phi(e_i) \\ &= (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right) + (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\nabla_{e_i} e_i, \frac{\partial}{\partial t}\right) + (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \boldsymbol{\nabla}_{e_i}\frac{\partial}{\partial t}\right) - (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\frac{\partial}{\partial t}, \nabla_{e_i} e_i\right) \\ &\quad - R^N\left(d\Phi(e_i), d\Phi\left(\frac{\partial}{\partial t}\right)\right) d\Phi(e_i) \\ &= (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right) - R^N\left(d\Phi(e_i), d\Phi\left(\frac{\partial}{\partial t}\right)\right) d\Phi(e_i). \end{aligned} \tag{4.10}$$

By substituting (4.10) into (4.9), we have

$$\begin{aligned} \frac{d}{dt} I^{\mathcal{Q}_2}(\varphi_t) &= 2\int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_j,e_j) \right\rangle d\mu_{g_M} \\ &\quad - 2\int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle R^N\left(d\Phi(e_i), d\Phi\left(\frac{\partial}{\partial t}\right)\right) d\Phi(e_i), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_j,e_j) \right\rangle d\mu_{g_M}. \end{aligned} \tag{4.11}$$

Lemma 4.4. *Under the setting above, for any variation $\{\varphi_t\}_{t\in I}$ of φ, it holds*

$$\int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_j, e_j)\right\rangle d\mu_{g_M}$$
$$= \int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)(e_i, e_i, e_j, e_j)\right\rangle d\mu_{g_M}. \tag{4.12}$$

Proof. For each $t \in I$, we define vector fields on M by

$$\widehat{X}_t := \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)\,(e_j, e_j)\right\rangle e_i$$

and

$$\widehat{Y}_t := \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\,(e_i, e_j, e_j)\right\rangle e_i,$$

where $\{e_i\}_{i=1}^m$ is a pseudo-orthonormal frame field on a neighborhood U of M. Note that $\widehat{X}_t$ and $\widehat{Y}_t$ are globally defined vector fields on M.

The divergence of $\widehat{X}_t$ is given by

$$\begin{aligned}
&\mathrm{div}\widehat{X}_t \\
&= \sum_k \varepsilon_k \left\langle \nabla_{e_k}\widehat{X}_t, e_k\right\rangle \\
&= \sum_k \varepsilon_k \left\langle \nabla_{e_k}\left(\sum_{i,j}\varepsilon_i\varepsilon_j\left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)\,(e_j, e_j)\right\rangle e_i\right), e_k\right\rangle \\
&= \sum_{i,j,k} \varepsilon_k\varepsilon_i\varepsilon_j e_k\left(\left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)\,(e_j, e_j)\right\rangle\right)\langle e_i, e_k\rangle \\
&\quad + \sum_{i,j,k} \varepsilon_k\varepsilon_i\varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)\,(e_j, e_j)\right\rangle \langle \nabla_{e_k} e_i, e_k\rangle \\
&= \sum_{i,j} \varepsilon_i\varepsilon_j \left\{\left\langle \overline{\boldsymbol{\nabla}}_{e_i}\left((\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right)\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)\,(e_j, e_j)\right\rangle\right. \\
&\quad \left. + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right), \overline{\boldsymbol{\nabla}}_{e_i}\left((\widetilde{\boldsymbol{\nabla}} d\Phi)\,(e_j, e_j)\right)\right\rangle\right\} - \sum_{j,k}\varepsilon_k\varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\nabla_{e_k} e_k, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_j, e_j)\right\rangle \\
&= \sum_{i,j} \varepsilon_i\varepsilon_j \left\{\left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right) + (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\nabla_{e_i} e_i, \frac{\partial}{\partial t}\right) + (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \boldsymbol{\nabla}_{e_i}\frac{\partial}{\partial t}\right), (\widetilde{\nabla} d\Phi)(e_j, e_j)\right\rangle\right. \\
&\quad + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\,(e_i, e_j, e_j) + 2(\widetilde{\boldsymbol{\nabla}} d\Phi)\,(\nabla_{e_i} e_j, e_j)\right\rangle \\
&\quad \left. - \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(\nabla_{e_i} e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_j, e_j)\right\rangle\right\} \\
&= \sum_{i,j} \varepsilon_i\varepsilon_j \left\{\left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_j, e_j)\right\rangle + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_i, e_j, e_j)\right\rangle\right. \\
&\quad \left. +2\left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)\left(e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)\,(\nabla_{e_i} e_j, e_j)\right\rangle\right\}.
\end{aligned}$$

Then, assuming that $\{e_i\}$ is a geodesic frame field around a point $x \in M$, we have

$$\begin{aligned}
&\left(\mathrm{div}\widehat{X}_t\right)_x \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi) \left((e_i)_{(x,t)}, (e_i)_{(x,t)}, \left(\frac{\partial}{\partial t}\right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi) \left((e_j)_{(x,t)}, (e_j)_{(x,t)} \right) \right\rangle \right. \\
&\quad \left. + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi) \left((e_i)_{(x,t)}, \left(\frac{\partial}{\partial t}\right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, (e_j)_{(x,t)}, (e_j)_{(x,t)} \right) \right\rangle \right\}. \qquad (4.13)
\end{aligned}$$

Each term of the right hand side of (4.13) is a tensor, so we have

$$\begin{aligned}
\mathrm{div}\widehat{X}_t = \sum_{i,j} \varepsilon_i \varepsilon_j & \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi) \left(e_i, e_i, \frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_j, e_j) \right\rangle \right. \\
& \left. + \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_i, \frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_i, e_j, e_j) \right\rangle \right\}, \qquad (4.14)
\end{aligned}$$

where $\{e_i\}_{i=1}^m$ is an arbitrary local pseudo-orthonormal frame field.

In a similar way, we calculate the divergence of $\widehat{Y}_t$. We have

$$\begin{aligned}
&\mathrm{div}\widehat{Y}_t \\
&= \sum_k \varepsilon_k \left\langle \nabla_{e_k} \widehat{Y}_t, e_k \right\rangle \\
&= \sum_k \varepsilon_k \left\langle \nabla_{e_k} \left(\sum_{i,j} \varepsilon_i \varepsilon_j \left\langle d\Phi \left(\frac{\partial}{\partial t} \right), \left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi \right) (e_i, e_j, e_j) \right\rangle e_i \right), e_k \right\rangle \\
&= \sum_{i,j,k} \varepsilon_k \varepsilon_i \varepsilon_j e_k \left(\left\langle d\Phi \left(\frac{\partial}{\partial t} \right), \left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi \right) (e_i, e_j, e_j) \right\rangle \right) \langle e_i, e_k \rangle \\
&\quad + \sum_{i,j,k} \varepsilon_k \varepsilon_i \varepsilon_j \left\langle d\Phi \left(\frac{\partial}{\partial t} \right), \left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi \right) (e_i, e_j, e_j) \right\rangle \langle \nabla_{e_k} e_i, e_k \rangle \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \left\{ \left\langle \overline{\boldsymbol{\nabla}}_{e_i} \left(d\Phi \left(\frac{\partial}{\partial t} \right) \right), \left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi \right) (e_i, e_j, e_j) \right\rangle + \left\langle d\Phi \left(\frac{\partial}{\partial t} \right), \overline{\boldsymbol{\nabla}}_{e_i} \left(\left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi \right) (e_i, e_j, e_j) \right) \right\rangle \right\} \\
&\quad - \sum_{j,k} \varepsilon_k \varepsilon_j \left\langle d\Phi \left(\frac{\partial}{\partial t} \right), \left(\widetilde{\boldsymbol{\nabla}}^2 d\Phi \right) (\nabla_{e_k} e_k, e_j, e_j) \right\rangle \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_i, \frac{\partial}{\partial t} \right) + d\Phi \left(\boldsymbol{\nabla}_{e_i} \frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_i, e_j, e_j) \right\rangle \right. \\
&\quad + \left\langle d\Phi \left(\frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)(e_i, e_i, e_j, e_j) + (\widetilde{\boldsymbol{\nabla}}^2 d\Phi) (\nabla_{e_i} e_i, e_j, e_j) + 2 (\widetilde{\boldsymbol{\nabla}}^2 d\Phi) (e_i, \nabla_{e_i} e_j, e_j) \right\rangle \\
&\quad \left. - \left\langle d\Phi \left(\frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi) (\nabla_{e_i} e_i, e_j, e_j) \right\rangle \right\} \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_i, \frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_i, e_j, e_j) \right\rangle + \left\langle d\Phi \left(\frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)(e_i, e_i, e_j, e_j) \right\rangle \right. \\
&\quad \left. + 2 \left\langle d\Phi \left(\frac{\partial}{\partial t} \right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi) (e_i, \nabla_{e_i} e_j, e_j) \right\rangle \right\}.
\end{aligned}$$

Then, assuming that $\{e_i\}$ is a geodesic frame field around a point $x \in M$, we have

$$
\begin{aligned}
&\left(\mathrm{div}\widehat{Y}_t\right)_x \\
&= \sum_{i,j} \varepsilon_i\varepsilon_j \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, \left(\frac{\partial}{\partial t}\right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, (e_j)_{(x,t)}, (e_j)_{(x,t)}\right) \right\rangle \right. \\
&\quad \left. + \left\langle (d\Phi)_{(x,t)} \left(\left(\frac{\partial}{\partial t}\right)_{(x,t)} \right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)_{(x,t)} \left((e_i)_{(x,t)}, (e_i)_{(x,t)}, (e_j)_{(x,t)}, (e_j)_{(x,t)}\right) \right\rangle \right\}. \qquad (4.15)
\end{aligned}
$$

Each term of the right hand side of (4.15) is a tensor, so we have

$$
\begin{aligned}
\mathrm{div}\widehat{Y}_t = \sum_{i,j} \varepsilon_i\varepsilon_j & \left\{ \left\langle (\widetilde{\boldsymbol{\nabla}} d\Phi) \left(e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)(e_i, e_j, e_j) \right\rangle \right. \\
& \left. + \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)(e_i, e_i, e_j, e_j) \right\rangle \right\}, \qquad (4.16)
\end{aligned}
$$

where $\{e_i\}_{i=1}^m$ is an arbitrary local pseudo-orthonormal frame field.

By Green's theorem, we have

$$
\int_M \mathrm{div}\widehat{X}_t \, d\mu_{g_M} = 0 = \int_M \mathrm{div}\widehat{Y}_t \, d\mu_{g_M},
$$

and together with (4.14) and (4.16), we have

$$
\begin{aligned}
&\int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}}^2 d\Phi)\left(e_i, e_i, \frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_j, e_j) \right\rangle d\mu_{g_M} \\
&= \int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle d\Phi\left(\frac{\partial}{\partial t}\right), (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)(e_i, e_i, e_j, e_j) \right\rangle d\mu_{g_M}.
\end{aligned}
$$

□

Substituting (4.12) into (4.11), we have

$$
\begin{aligned}
&\frac{d}{dt} I^{\mathcal{Q}_2}(\varphi_t) \\
&= 2\int_M \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle (\widetilde{\boldsymbol{\nabla}}^3 d\Phi)(e_i, e_i, e_j, e_j) - R^N\left(d\Phi(e_i), (\widetilde{\boldsymbol{\nabla}} d\Phi)(e_j, e_j)\right) d\Phi(e_i), d\Phi\left(\frac{\partial}{\partial t}\right) \right\rangle d\mu_{g_M}.
\end{aligned}
$$

Therefore we obtain the following theorem.

Theorem 4.5. *Let (M_p^m, g_M) be a compact pseudo-Riemannian manifold, (N_q^n, g_N) a pseudo-Riemannian manifold and $\varphi : M \to N$ a C^∞-map. Consider a C^∞-variation $\{\varphi_t\}_{t\in I}$ of φ with variational vector field V. Then the following formula holds*

$$
\begin{aligned}
&\left.\frac{d}{dt} I^{\mathcal{Q}_2}(\varphi_t)\right|_{t=0} \\
&= 2\int_M \left\langle \sum_{i,j} \varepsilon_i\varepsilon_j \left\{ \left(\widetilde{\nabla}^3 d\varphi\right)(e_i, e_i, e_j, e_j) + R^N\left((\widetilde{\nabla} d\varphi)(e_i, e_i), d\varphi(e_j)\right) d\varphi(e_j) \right\}, V \right\rangle d\mu_{g_M},
\end{aligned}
$$

where $\{e_i\}_{=1}^m$ is a local pseudo-orthonormal frame field of (M_p^m, g_M) with $g_M(e_i, e_j) = \varepsilon_i\delta_{ij}$, $\varepsilon_1 = \cdots = \varepsilon_p = -1$, $\varepsilon_{p+1} = \cdots = \varepsilon_m = 1$.

For a map $\varphi \in C^\infty(M,N)$, we define $W_2(\varphi) \in \Gamma(\varphi^{-1}TN)$ by

$$W_2(\varphi) := \sum_{i,j} \varepsilon_i\varepsilon_j \left\{ \left(\widetilde{\nabla}^3 d\varphi\right)(e_i,e_i,e_j,e_j) + R^N\big((\widetilde{\nabla} d\varphi)(e_i,e_i), d\varphi(e_j)\big)d\varphi(e_j) \right\}.$$

Hence φ is a $\mathcal{Q}_2$-map if and only if $W_2(\varphi)=0$.

Remark 4.6. For a pseudo-Riemannian manifold (M_p^m, g_M), if the index $p=0$ then (M_0^m, g_M) is a Riemannian manifold. Therefore a map $\varphi : (M_0^m, g_M) \to (N_0^n, g_N)$ between Riemannian manifolds is a $\mathcal{Q}_1$-map if and only if

$$\sum_{i,j} \left\{ \left(\widetilde{\nabla}^3 d\varphi\right)(e_i,e_j,e_i,e_j) + R^N\big((\widetilde{\nabla} d\varphi)(e_i,e_j), d\varphi(e_i)\big)d\varphi(e_j) \right\} = 0,$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of (M^m, g_M). Similarly, we have that a map $\varphi : (M_0^m, g_M) \to (N_0^n, g_N)$ between Riemannian manifolds is a $\mathcal{Q}_2$-map if and only if

$$\sum_{i,j} \left\{ \left(\widetilde{\nabla}^3 d\varphi\right)(e_i,e_i,e_j,e_j) + R^N\big((\widetilde{\nabla} d\varphi)(e_i,e_i), d\varphi(e_j)\big)d\varphi(e_j) \right\} = 0.$$

By Theorem 4.2 and Theorem 4.5, we obtain all the first variational formulae of the integral invariants which belong to the space spanned by the $\mathcal{Q}_1$-energy and $\mathcal{Q}_2$-energy.

By comparing the first variational formula of the bienergy (cf. [14]) and that of $\mathcal{Q}_2$-energy (Theorem 4.5), we have the following proposition.

Proposition 4.7. *Let $\varphi : M \to N$ be a C^∞-map between pseudo-Riemannian manifolds (M_p^m, g_M) and (N_q^n, g_N). Then the following formula holds*

$$-\overline{\nabla}^*\overline{\nabla}\tau(\varphi) = \sum_{i,j} \varepsilon_i\varepsilon_j (\widetilde{\nabla}^3 d\varphi)(e_i,e_i,e_j,e_j),$$

where $-\overline{\nabla}^*\overline{\nabla}$ is the rough Laplacian and $\{e_i\}_{i=1}^m$ is a local pseudo-orthonormal frame field of (M_p^m, g_M).

Proof. For any $V \in \Gamma(\varphi^{-1}TN)$, we define vector fields on M by

$$W := \sum_{i,j}^m \varepsilon_i\varepsilon_j \left\langle V, \overline{\nabla}_{e_i}\big((\widetilde{\nabla} d\varphi)(e_j,e_j)\big) \right\rangle e_i$$

and

$$\widetilde{W} := \sum_{i,j}^m \varepsilon_i\varepsilon_j \left\langle V, (\widetilde{\nabla}^2 d\varphi)(e_i,e_j,e_j) \right\rangle e_i,$$

where $\{e_i\}_{i=1}^m$ is a local pseudo-orthonormal frame field of (M_p^m, g_M). Then, assuming that $\{e_i\}$ is a geodesic frame field around a point $x \in M$, we have

$$\begin{aligned}
\widetilde{W}_x &= \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle V_x, \big(\overline{\nabla}_{e_i}\big((\widetilde{\nabla} d\varphi)(e_j,e_j)\big)\big)_x - 2\big((\widetilde{\nabla} d\varphi)(\nabla_{e_i}e_j,e_j)\big)_x \right\rangle (e_i)_x \\
&= \sum_{i,j} \varepsilon_i\varepsilon_j \left\langle V_x, \big(\overline{\nabla}_{e_i}\big((\widetilde{\nabla} d\varphi)(e_j,e_j)\big)\big)_x \right\rangle (e_i)_x \\
&= W_x.
\end{aligned}$$

Therefore $W = \widetilde{W}$. Thus,

$$\begin{aligned}
0 &= \operatorname{div}(W - \widetilde{W})_x \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \left\{ \left\langle \left(\overline{\nabla}_{e_i} V\right)_x, \left(\overline{\nabla}_{e_i}\left((\widetilde{\nabla} d\varphi)(e_j, e_j)\right)\right)_x \right\rangle + \left\langle V_x, \left(\overline{\nabla}_{e_i}\left(\overline{\nabla}_{e_i}\left((\widetilde{\nabla} d\varphi)(e_j, e_j)\right)\right)\right)_x \right\rangle \right\} \\
&\quad - \sum_{i,j} \varepsilon_i \varepsilon_j \left\{ \left\langle \left(\overline{\nabla}_{e_i} V\right)_x, \left((\widetilde{\nabla}^2 d\varphi)(e_i, e_j, e_j)\right)_x \right\rangle + \left\langle V_x, \left(\overline{\nabla}_{e_i}\left((\widetilde{\nabla}^2 d\varphi)(e_i, e_j, e_j)\right)\right)_x \right\rangle \right\} \\
&= \left\langle V_x, \left(-\overline{\nabla}^* \overline{\nabla} \tau(\varphi)\right)_x - \sum_{i,j} \varepsilon_i \varepsilon_j \left((\widetilde{\nabla}^3 d\varphi)(e_i, e_i, e_j, e_j)\right)_x \right\rangle,
\end{aligned}$$

where $\{e_i\}_{i=1}^m$ is a geodesic frame field around a point $x \in M$. So we have

$$-\overline{\nabla}^* \overline{\nabla} \tau(\varphi) = \sum_{i,j} \varepsilon_i \varepsilon_j (\widetilde{\nabla}^3 d\varphi)(e_i, e_i, e_j, e_j),$$

where $\{e_i\}_{i=1}^m$ is an arbitrary local pseudo-orthonormal frame field. □

5 The Euler–Lagrange equation of the Chern–Federer energy

We inherit the settings in the previous section. In this section, we introduce the Chern–Federer energy functional for a map $\varphi : (M_p^m, g_M) \to (N_q^n, g_N)$ between pseudo-Riemannian manifolds, which is an integral invariant defined by a homogeneous polynomial of degree two on $\mathrm{II}(\mathbb{E}_p^m, \mathbb{E}_q^n)$ called the Chern–Federer polynomial. Then we verify the Euler–Lagrange equation of the Chern–Federer energy functional.

For $H = (h_{ij}^\alpha) \in \mathrm{II}(\mathbb{E}_p^m, \mathbb{E}_q^n)$, the *Chern–Federer polynomial* $\mathrm{CF}(H)$ is defined by

$$\mathrm{CF}(H) := \mathcal{Q}_2(H) - \mathcal{Q}_1(H). \tag{5.1}$$

From Theorems 4.2 and 4.5, the Euler–Lagrange equation of the *Chern–Federer energy functional* $I^{\mathrm{CF}}(\varphi)$ is

$$\begin{aligned}
0 &= W_2(\varphi) - W_1(\varphi) \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \Big\{ (\widetilde{\nabla}^3 d\varphi)(e_i, e_i, e_j, e_j) - (\widetilde{\nabla}^3 d\varphi)(e_i, e_j, e_i, e_j) \\
&\quad + R^N\left((\widetilde{\nabla} d\varphi)(e_i, e_i), d\varphi(e_j)\right) d\varphi(e_j) - R^N\left((\widetilde{\nabla} d\varphi)(e_i, e_j), d\varphi(e_i)\right) d\varphi(e_j) \Big\},
\end{aligned} \tag{5.2}$$

where $\{e_i\}_{i=1}^m$ is a local pseudo-orthonormal frame field of (M_p^m, g_M). In this section, we give alternative expressions of the Euler–Lagrange equation of the Chern–Federer energy functional. In particular, the Euler–Lagrange equation of $I^{\mathrm{CF}}(\varphi)$ is a second-order partial differential equation for φ. Moreover, we describe the symmetry of the Euler–Lagrange equation of the Chern–Federer energy functional and that of the Chern–Federer polynomial.

We also introduce the Willmore–Chen energy functional, which is an integral invariant defined by the homogeneous polynomial of degree two called the Willmore–Chen polynomial. For $H = (h_{ij}^\alpha) \in \mathrm{II}(\mathbb{E}_p^m, \mathbb{E}_q^n)$, the *Willmore–Chen polynomial* $\mathrm{WC}(H)$ is defined by

$$\mathrm{WC}(H) := m\mathcal{Q}_1(H) - \mathcal{Q}_2(H).$$

Let α and β be constant numbers such that $\alpha^2 + \beta^2 \neq 0$. A C^∞-map $\varphi : M \to N$ is called an *$(\alpha\mathcal{Q}_1 + \beta\mathcal{Q}_2)$-map* if it satisfies

$$\alpha W_1(\varphi) + \beta W_2(\varphi) = 0.$$

By definition, an $(\alpha\mathcal{Q}_1 + \beta\mathcal{Q}_2)$-map φ is

- a $\mathcal{Q}_1$-map when $(\alpha, \beta) = (1, 0)$;
- a $\mathcal{Q}_2$-map, that is, a biharmonic map, when $(\alpha, \beta) = (0, 1)$;
- a Chern–Federer map when $(\alpha, \beta) = (-1, 1)$;
- a Willmore–Chen map when $(\alpha, \beta) = (m, -1)$.

In Section 6, we construct some examples of these maps. In previous research, the mainstream was to increase the order of the energy functional. In contrast, we extended the energy functionals with a second order class (cf. Figure 2).

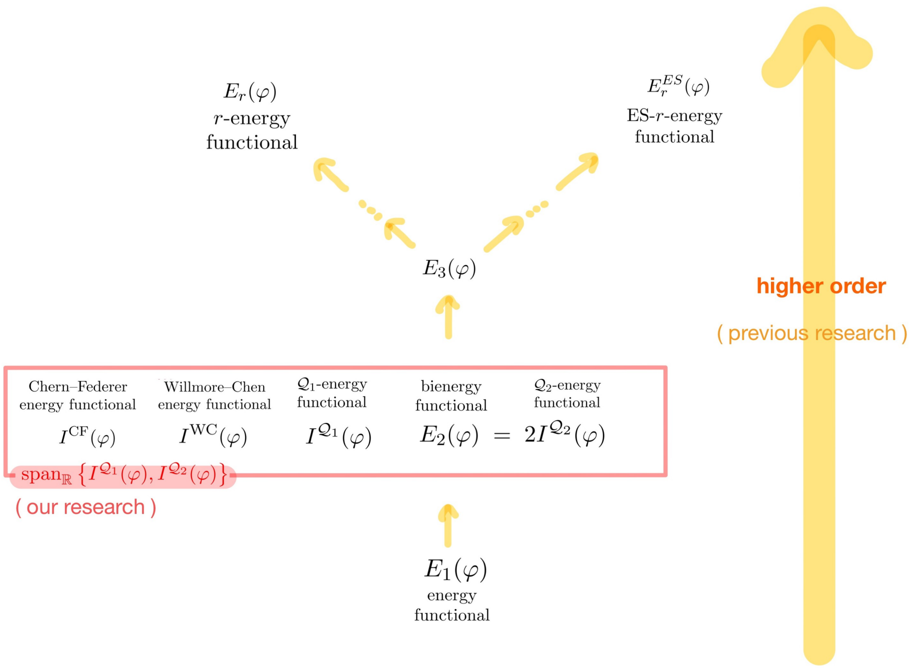

Figure 2: Correlation diagram of energies used in previous research and our research

5.1 Alternative expression of the Euler–Lagrange equation of the Chern–Federer energy functional I

First, we prepare the following lemmas.

Lemma 5.1. *For a smooth map $\varphi : M \to N$ and $X, Y, Z \in \Gamma(TM)$, the following equation holds:*

$$(\widetilde{\nabla}^2 d\varphi)(X, Y, Z) - (\widetilde{\nabla}^2 d\varphi)(Y, X, Z) = R^N(d\varphi(X), d\varphi(Y))\, d\varphi(Z) - d\varphi\left(R^M(X, Y)Z\right).$$

Proof. Let $\{e_i\}_{i=1}^m$ be a geodesic frame field of (M_p^m, g_M) around $x \in M$. At x, we have

$$\begin{aligned}
&(\widetilde{\nabla}^2 d\varphi)(e_i, e_j, e_k) - (\widetilde{\nabla}^2 d\varphi)(e_j, e_i, e_k) \\
&= \overline{\nabla}_{e_i}\left((\widetilde{\nabla} d\varphi)(e_j, e_k)\right) - (\widetilde{\nabla} d\varphi)(\nabla_{e_i} e_j, e_k) - (\widetilde{\nabla} d\varphi)(e_j, \nabla_{e_i} e_k) - \overline{\nabla}_{e_j}\left((\widetilde{\nabla} d\varphi)(e_i, e_k)\right) \\
&\quad + (\widetilde{\nabla} d\varphi)(\nabla_{e_j} e_i, e_k) + (\widetilde{\nabla} d\varphi)(e_i, \nabla_{e_j} e_k) \\
&= \overline{\nabla}_{e_i}\left((\widetilde{\nabla} d\varphi)(e_j, e_k)\right) - \overline{\nabla}_{e_j}\left((\widetilde{\nabla} d\varphi)(e_i, e_k)\right) \\
&= \overline{\nabla}_{e_i}\left(\overline{\nabla}_{e_j}(d\varphi(e_k)) - d\varphi(\nabla_{e_j} e_k)\right) - \overline{\nabla}_{e_j}\left(\overline{\nabla}_{e_i}(d\varphi(e_k) - d\varphi(\nabla_{e_i} e_k))\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j} - \overline{\nabla}_{e_j}\overline{\nabla}_{e_i}\right) d\varphi(e_k) - \overline{\nabla}_{e_i}\left(d\varphi(\nabla_{e_j}e_k)\right) + \overline{\nabla}_{e_j}\left(d\varphi(\nabla_{e_i}e_k)\right)\\
&= \left(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j} - \overline{\nabla}_{e_j}\overline{\nabla}_{e_i} - \overline{\nabla}_{[e_i,e_j]}\right) d\varphi(e_k)\\
&\quad - \left\{(\widetilde{\nabla}d\varphi)\left(e_i, \nabla_{e_j}e_k\right) + d\varphi\left(\nabla_{e_i}\nabla_{e_j}e_k\right)\right\} + \left\{(\widetilde{\nabla}d\varphi)(e_j, \nabla_{e_i}e_k) + d\varphi\left(\nabla_{e_j}\nabla_{e_i}e_k\right)\right\}\\
&= R^{\varphi^{-1}TN}(e_i, e_j)d\varphi(e_k) - d\varphi\left(\nabla_{e_i}\nabla_{e_j}e_k - \nabla_{e_j}\nabla_{e_i}e_k - \nabla_{[e_i,e_j]}e_k\right)\\
&= R^N\left(d\varphi(e_i), d\varphi(e_j)\right) d\varphi(e_k) - d\varphi\left(R^M(e_i, e_j)e_k\right).
\end{aligned}
$$

Since all terms of the first and last formulae are tensors, we have the lemma. □

Lemma 5.2. *For a smooth map $\varphi : M \to N$ and $X, Y, Z, W \in \Gamma(TM)$, the following equation holds:*

$$
\begin{aligned}
&(\widetilde{\nabla}^3 d\varphi)(X, Y, Z, W) - (\widetilde{\nabla}^3 d\varphi)(X, Z, Y, W)\\
&= \left(\nabla R^N\right)\left(d\varphi(X), d\varphi(Y), d\varphi(Z)\right) d\varphi(W) + R^N\big((\widetilde{\nabla}d\varphi)(X, Y), d\varphi(Z)\big)d\varphi(W)\\
&\quad + R^N\big(d\varphi(Y), (\widetilde{\nabla}d\varphi)(X, Z)\big)d\varphi(W) + R^N\left(d\varphi(Y), d\varphi(Z)\right)(\widetilde{\nabla}d\varphi)(X, W)\\
&\quad - (\widetilde{\nabla}d\varphi)\left(X, R^M(Y, Z)W\right) - d\varphi\left(\left(\nabla R^M\right)(X, Y, Z)W\right).
\end{aligned}
$$

Proof. First, we show that the following equation:

$$
\begin{aligned}
&(\widetilde{\nabla}^3 d\varphi)(X, Y, Z, W) - (\widetilde{\nabla}^3 d\varphi)(X, Z, Y, W)\\
&= \big(\nabla R^{\varphi^{-1}TN}\big)(X, Y, Z)d\varphi(W) + R^{\varphi^{-1}TN}(Y, Z)(\widetilde{\nabla}d\varphi)(X, W)\\
&\quad - (\widetilde{\nabla}d\varphi)(X, R^M(Y, Z)W) - d\varphi\left(\left(\nabla R^M\right)(X, Y, Z)W\right),
\end{aligned}
\tag{5.3}
$$

where $X, Y, Z, W \in \Gamma(TM)$. Let $\{e_i\}_{i=1}^m$ be a geodesic frame field of (M_p^m, g_M) around $x \in M$. At x, we have

$$
\begin{aligned}
&(\widetilde{\nabla}^3 d\varphi)(e_i, e_j, e_k, e_l) - (\widetilde{\nabla}^3 d\varphi)(e_i, e_k, e_j, e_l)\\
&= \overline{\nabla}_{e_i}\left((\widetilde{\nabla}^2 d\varphi)(e_j, e_k, e_l)\right) - \overline{\nabla}_{e_i}\left((\widetilde{\nabla}^2 d\varphi)(e_k, e_j, e_l)\right)\\
&= \overline{\nabla}_{e_i}\left(R^N\left(d\varphi(e_j), d\varphi(e_k)\right) d\varphi(e_l) - d\varphi\left(R^M(e_j, e_k)e_l\right)\right)\\
&= \overline{\nabla}_{e_i}\left(R^N\left(d\varphi(e_j), d\varphi(e_k)\right) d\varphi(e_l)\right) - (\widetilde{\nabla}d\varphi)\left(e_i, R^M(e_j, e_k)e_l\right) - d\varphi\left(\nabla_{e_i}\left(R^M(e_j, e_k)e_l\right)\right)\\
&= \overline{\nabla}_{e_i}\left(R^{\varphi^{-1}TN}(e_j, e_k)d\varphi(e_l)\right) - (\widetilde{\nabla}d\varphi)\left(e_i, R^M(e_j, e_k)e_l\right) - d\varphi\left(\nabla_{e_i}\left(R^M(e_j, e_k)e_l\right)\right)\\
&= \left(\nabla R^{\varphi^{-1}TN}\right)(e_i, e_j, e_k)d\varphi(e_l) + R^{\varphi^{-1}TN}\left(\nabla_{e_i}e_j, e_k\right) d\varphi(e_l) + R^{\varphi^{-1}TN}\left(e_j, \nabla_{e_i}e_k\right) d\varphi(e_l)\\
&\quad + R^{\varphi^{-1}TN}(e_j, e_k)\overline{\nabla}_{e_i}\left(d\varphi(e_l)\right) - (\widetilde{\nabla}d\varphi)\left(e_i, R^M(e_j, e_k)e_l\right)\\
&\quad - d\varphi\left(\left(\nabla R^M\right)(e_i, e_j, e_k, e_l) + R^M(\nabla_{e_i}e_j, e_k)e_l + R^M(e_j, \nabla_{e_i}e_k)e_l + R^M(e_j, e_k)\nabla_{e_i}e_l\right)\\
&= \left(\nabla R^{\varphi^{-1}TN}\right)(e_i, e_j, e_k)d\varphi(e_l) + R^{\varphi^{-1}TN}(e_j, e_k)(\widetilde{\nabla}d\varphi)(e_i, e_l) - (\widetilde{\nabla}d\varphi)\left(e_i, R^M(e_j, e_k)e_l\right)\\
&\quad - d\varphi\left(\left(\nabla R^M\right)(e_i, e_j, e_k, e_l)\right).
\end{aligned}
$$

Here, the second equality holds because of Lemma 5.1. Since all terms of the first and last formulae are tensors, we have (5.3). Then, on $\mathrm{End}\varphi^{-1}TN$, we have

$$
R^{\varphi^{-1}TN}(Y, Z)\big((\widetilde{\nabla}d\varphi)(X, W)\big) = R^N(d\varphi(Y), d\varphi(Z))\big((\widetilde{\nabla}d\varphi)(X, W)\big),
\tag{5.4}
$$

where $X, Y, Z, W \in \Gamma(TM)$, since the following equation holds:

$$
R^{\varphi^{-1}TN}(X, Y) = (\varphi^{-1}R^N)(X, Y) = R^N(d\varphi(X), d\varphi(Y)).
$$

Also, we can verify the following equation:

$$\begin{aligned}\big(\nabla R^{\varphi^{-1}TN}\big)(X,Y,Z) &= (\nabla R^N)(d\varphi(X),d\varphi(Y),d\varphi(Z))\\ &\quad + R^N(d\varphi(Y),(\widetilde{\nabla} d\varphi)(X,Z)) + R^N((\widetilde{\nabla} d\varphi)(X,Y),d\varphi(Z)).\end{aligned} \tag{5.5}$$

Thus we have

$$\begin{aligned}&\big(\nabla R^{\varphi^{-1}TN}\big)(X,Y,Z)\\ &= \overline{\nabla}_X(R^{\varphi^{-1}TN}(Y,Z)) - R^{\varphi^{-1}TN}(\nabla_X Y,Z) - R^{\varphi^{-1}TN}(Y,\nabla_X Z)\\ &= \overline{\nabla}_X(R^N(d\varphi(Y),d\varphi(Z))) - R^N(d\varphi(\nabla_X Y),d\varphi(Z)) - R^N(d\varphi(Y),d\varphi(\nabla_X Z))\\ &= \overline{\nabla}_X(R^N(d\varphi(Y),d\varphi(Z))) + R^N((\widetilde{\nabla} d\varphi)(X,Y),d\varphi(Z)) + R^N(d\varphi(Y),(\widetilde{\nabla} d\varphi)(X,Z))\\ &\quad - R^N(\overline{\nabla}_X(d\varphi(Y)),d\varphi(Z)) - R^N(d\varphi(Y),\overline{\nabla}_X(d\varphi(Z)))\\ &= R^N((\widetilde{\nabla} d\varphi)(X,Y),d\varphi(Z)) + R^N(d\varphi(Y),(\widetilde{\nabla} d\varphi)(X,Z)) + (\nabla R^N)(d\varphi(X),d\varphi(Y),d\varphi(Z)).\end{aligned}$$

Here, by taking local frame fields of (M,g_M) and (N,g_N) and calculating locally, we verify the last equality. Therefore the assertion holds from (5.3), (5.4) and (5.5). $\square$

Remark 5.3. Recall that $\nabla R^{\varphi^{-1}TN}$ is the derivative of the curvature tensor field $R^{\varphi^{-1}TN}$, defined by

$$\begin{aligned}\big(\nabla R^{\varphi^{-1}TN}\big)(X,Y,Z)s := \overline{\nabla}_X\big(R^{\varphi^{-1}TN}(Y,Z)s\big) - R^{\varphi^{-1}TN}(\nabla_X Y,Z)\,s\\ - R^{\varphi^{-1}TN}(Y,\nabla_X Z)\,s - R^{\varphi^{-1}TN}(Y,Z)\overline{\nabla}_X s,\end{aligned}$$

where $X,Y,Z\in\Gamma(TM)$ and $s\in\Gamma(\varphi^{-1}TN)$.

Using Lemma 5.2, we obtain the following proposition.

Proposition 5.4. *A smooth map $\varphi: M\to N$ is a Chern–Federer map if and only if*

$$\begin{aligned}0 = \sum_{i,j}\varepsilon_i\varepsilon_j\Big\{&\left(\nabla R^N\right)(d\varphi(e_i),d\varphi(e_i),d\varphi(e_j))\,d\varphi(e_j) - (\widetilde{\nabla} d\varphi)\left(e_i,R^M(e_i,e_j)e_j\right)\\ &- d\varphi\left(\left(\nabla R^M\right)(e_i,e_i,e_j)e_j\right) + 2R^N\big((\widetilde{\nabla} d\varphi)(e_i,e_i),d\varphi(e_j)\big)d\varphi(e_j)\\ &+2R^N\big(d\varphi(e_i),(\widetilde{\nabla} d\varphi)(e_i,e_j)\big)d\varphi(e_j)\Big\},\end{aligned} \tag{5.6}$$

where $\{e_i\}_{i=1}^m$ is a local pseudo-orthonormal frame field of (M_p^m,g_M).

By the equation (5.6), it can be seen that the Euler–Lagrange equation of the Chern–Federer energy functional for a map φ is a second-order partial differential equation for φ.

5.2 Alternative expression of the Euler–Lagrange equation of the Chern–Federer energy functional II

Here, we express the Chern–Federer polynomial as follows:

$$\begin{aligned}\mathrm{CF}(H) &= \mathcal{Q}_2(H) - \mathcal{Q}_1(H)\\ &= \sum_\alpha \varepsilon'_\alpha\left(\sum_i \varepsilon_i h^\alpha_{ii}\right)^2 - \sum_\alpha\varepsilon'_\alpha\sum_{i,j}\varepsilon_i\varepsilon_j(h^\alpha_{ij})^2\\ &= \sum_\alpha\varepsilon'_\alpha\sum_{i,j}\varepsilon_i\varepsilon_j \det\begin{pmatrix} h^\alpha_{ii} & h^\alpha_{ij}\\ h^\alpha_{ij} & h^\alpha_{jj}\end{pmatrix}.\end{aligned}$$

Then we have the following alternative expression of the Euler–Lagrange equation of the Chern–Federer energy functional.

Theorem 5.5. *Let $\varphi : M \to N$ be a C^∞-map between pseudo-Riemannian manifolds. We define $(0,4)$-type tensor fields μ and ν valued on $\varphi^{-1}TN$ by*

$$\mu(X_1, X_2, X_3, X_4) := (\widetilde{\nabla}^3 d\varphi)(X_1, X_2, X_3, X_4)$$

and

$$\nu(X_1, X_2, X_3, X_4) := R^N\big((\widetilde{\nabla} d\varphi)(X_3, X_4), d\varphi(X_1)\big)d\varphi(X_2),$$

where $X_1, X_2, X_3, X_4 \in \Gamma(TM)$. Then φ is a Chern–Federer map if and only if

$$C(\mu + \nu) = 0. \tag{5.7}$$

Here C is the contraction of a $(0,4)$-tensor field on M defined by

$$C := \det \begin{pmatrix} C_{12} & C_{13} \\ C_{24} & C_{34} \end{pmatrix},$$

where C_{ij} is the contraction of the i-th and j-th variables.

Proof. From the definition of μ and ν, we have $\mu, \nu \in \Gamma(T^*M \otimes T^*M \otimes (T^*M \odot T^*M) \otimes \varphi^{-1}TN)$. For simplicity, we set

$$\mu_{ijkl} := \mu(e_i, e_j, e_k, e_l)$$

and

$$\nu_{ijkl} := \nu(e_i, e_j, e_k, e_l),$$

where $\{e_i\}_{i=1}^m$ is a local pseudo-orthonormal frame field of M. Note that, by the pseudo-Riemannian metric g_M, there is a natural correspondence between a covariant tensor and a contravariant tensor on M. Hence we can consider a contraction of $(0,4)$-tensor field on M. Then we have

$$\begin{aligned}
&\sum_{i,j} \varepsilon_i \varepsilon_j \Big\{ (\widetilde{\nabla}^3 d\varphi)(e_i, e_i, e_j, e_j) - (\widetilde{\nabla}^3 d\varphi)(e_i, e_j, e_i, e_j) \Big\} \\
&= \sum_{i,j} \varepsilon_i \varepsilon_j \left(\mu_{iijj} - \mu_{ijij} \right) = \sum_{i,j} \left(\mu_i{}^i{}_j{}^j - \mu_{ij}{}^{ij} \right) \\
&= C_{12} C_{34} \mu - C_{13} C_{24} \mu = \det \begin{pmatrix} C_{12} & C_{13} \\ C_{24} & C_{34} \end{pmatrix} \mu.
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
&\sum_{i,j} \varepsilon_i \varepsilon_j \Big\{ R^N\big((\widetilde{\nabla} d\varphi)(e_i, e_i), d\varphi(e_j)\big)d\varphi(e_j) - R^N\big((\widetilde{\nabla} d\varphi)(e_i, e_j), d\varphi(e_i)\big)d\varphi(e_j) \Big\} \\
&= \det \begin{pmatrix} C_{12} & C_{13} \\ C_{24} & C_{34} \end{pmatrix} \nu.
\end{aligned}$$

Therefore the Euler–Lagrange equation (5.2) of the Chern–Federer energy functional can be expressed as the following equation:

$$\det \begin{pmatrix} C_{12} & C_{13} \\ C_{24} & C_{34} \end{pmatrix} (\mu + \nu) = 0.$$

□

In addition, we observe symmetry of the equation (5.7) and the Chern–Federer polynomial (5.1). Let $\mathcal{U}$ be the space of $O(p,m-p)\times O(q,n-q)$-invariant homogeneous polynomials of degree two on $\mathrm{II}(\mathbb{E}^m_p,\mathbb{E}^n_q)$, which is spanned by $\mathcal{Q}_1$ and $\mathcal{Q}_2$:

$$\mathcal{U} := \mathrm{span}_{\mathbb{R}}\left\{\mathcal{Q}_1,\mathcal{Q}_2\right\}.$$

Also we denote by $\mathcal{V}$ the space of sections of $\varphi^{-1}TN$ spanned by $v_1 := C_{13}C_{24}(\mu+\nu)$ and $v_2 := C_{12}C_{34}(\mu+\nu)$:

$$\mathcal{V} := \mathrm{span}_{\mathbb{R}}\left\{v_1,v_2\right\}.$$

Then, by the first variational formula of the $(\alpha\mathcal{Q}_1+\beta\mathcal{Q}_2)$-energy functional, we have a linear isomorphism between $\mathcal{U}$ and $\mathcal{V}$. From the first variational formula (5.7) of the Chern–Federer energy functional, we observe the invariance of v_2-v_1 under the symmetric group S_4 of degree four acting on $\mathcal{V}$ as the permutation of the variables. The symmetric group S_4 is generated by transpositions (1 2), (1 3) and (1 4). Here, we set

$$\sigma_1 := \begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix},\quad \sigma_2 := \begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix},\quad \sigma_3 := \begin{pmatrix}1&2&3&4\\4&2&3&1\end{pmatrix}.$$

By the symmetry of the third and fourth variables of μ and ν, we have the following relations:

$$\sigma_1(v_1)=v_1,\quad \sigma_1(v_2)=v_2,\quad \sigma_2(v_1)=v_1,\quad \sigma_2(v_2)=v_1,\quad \sigma_3(v_1)=v_2,\quad \sigma_3(v_2)=v_1.$$

From these, it can be seen that v_1 and v_2 are symmetric by the transposition $(1\ 2)=\sigma_1$, and v_2-v_1 is antisymmetric by the permutation σ_3. There are totally 24 elements in S_4, however, due to the invariance by the permutation σ_1 and the symmetry for the third and fourth variables of μ and ν, the action of S_4 on $\mathcal{V}$ is reduced to the following six permutations:

$$\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4 := \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix},\ \sigma_5 := \begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix},\ \sigma_6 := \begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}.$$

Then we can verify that v_2-v_1 is antisymmetric by the permutations σ_3 and σ_6. Furthermore, an element of $\mathcal{V}$ is antisymmetric by σ_3 and σ_6 if and only if it is a scalar multiple of v_2-v_1.

In a similar way, we observe the invariance of the Chern–Federer polynomial under the symmetric group S_4. First, we rewrite the Chern–Federer polynomial as follows. For $H = (h^\alpha_{ij})\in \mathrm{II}(\mathbb{E}^m_p,\mathbb{E}^n_q)$, we define $\rho\in\otimes^4(\mathbb{E}^m_p)^*$ as follows

$$\rho := \sum_{i,j,k,l}\rho_{ijkl}\, e^i\otimes e^j\otimes e^k\otimes e^l,$$

where ρ_{ijkl} is defined by

$$\rho_{ijkl} := \sum_\alpha \varepsilon'_\alpha h^\alpha_{ij}h^\alpha_{kl}$$

and $\{e^i\}_{i=1}^m$ is the dual basis of the standard basis of $\mathbb{E}^m_p$. Then we have

$$\mathcal{Q}_1(H) = \sum_\alpha \varepsilon'_\alpha \sum_{i,j}\varepsilon_i\varepsilon_j h^\alpha_{ij}h^\alpha_{ij} = \sum_{i,,j}\varepsilon_i\varepsilon_j\rho_{ijij} = \sum_{i,j}\rho_{ij}{}^{ij} = C_{13}C_{24}\rho$$

and

$$\mathcal{Q}_2(H) = \sum_\alpha \varepsilon'_\alpha \sum_{i,j}\varepsilon_i\varepsilon_j h^\alpha_{ii}h^\alpha_{jj} = \sum_{i,j}\varepsilon_i\varepsilon_j\rho_{iijj} = C_{12}C_{34}\rho.$$

Therefore we can rewrite the Chern–Federer polynomial $\mathrm{CF}(H)$ as follows:

$$\mathrm{CF}(H) = \mathcal{Q}_2(H) - \mathcal{Q}_1(H) = \det \begin{pmatrix} C_{12} & C_{13} \\ C_{24} & C_{34} \end{pmatrix} \rho.$$

As in the case of $\mathcal{V}$, the action of S_4 on $\mathcal{U}$ is reduced to six elements σ_i $(i = 1, 2, \cdots, 6)$. Then we can verify that an element of $\mathcal{U}$ is antisymmetric by σ_3 and σ_6 if and only if it is a scalar multiple of the Chern–Federer polynomial $\mathrm{CF}(H)$. Consequently, we find that $\mathrm{CF}(H)$ and $v_2 - v_1$ have the same symmetry via the first variational formula and the actions of S_4 on $\mathcal{U}$ and $\mathcal{V}$.

6 Chern–Federer submanifolds in Riemannian space forms

Let (M^m, g_M), (N^n, g_N) be two Riemannian manifolds. From now on, we deal with isometric immersions $\varphi : (M^m, g_M) \to (N^n, g_N)$. In this section, we firstly derive the Euler–Lagrange equation for an isometric immersion from a Riemannian manifold into a Riemannian space form. Secondly, we construct examples in the case of curves or surfaces. Finally, we consider Chern–Federer isoparametric hypersurfaces in Riemannian space forms.

6.1 Euler–Lagrange equations for isometric immersions

For an isometric immersion $\varphi : (M^m, g_M) \to (N^n, g_N)$, we denote the shape operator and the mean curvature vector field by A and $\mathcal{H}$, respectively. Namely, they are defined by

$$\langle A_\xi(X), Y\rangle = \langle(\widetilde{\nabla} d\varphi)(X, Y), \xi\rangle, \quad \mathcal{H} = \frac{1}{m}\mathrm{tr}_{g_M}(\widetilde{\nabla} d\varphi) = \frac{1}{m}\tau(\varphi)$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(T^\perp M)$, where $T^\perp M$ is the normal bundle over M of φ. In addition, we simply denote by h the second fundamental form $\widetilde{\nabla} d\varphi$ in this section.

We denote a Riemannian space form of constant curvature $c \in \mathbb{R}$ by $N^n(c)$. Namely, it is locally isometric to one of a Euclidean space $(c = 0)$, a round sphere $(c > 0)$ and a hyperbolic space $(c < 0)$.

When we denote the Ricci operator of (M^m, g_M) by Q, we obtain the Euler–Lagrange equation for an isometric immersion into a Riemannian space form.

Theorem 6.1. *Let $\varphi : (M^m, g_M) \to N^n(c)$ be an isometric immersion. Then φ is a Chern–Federer map if and only if it satisfies that*

$$\mathrm{CF}(\varphi) = -d\varphi(\mathrm{tr}_{g_M}(\nabla Q)) + 2cm(m-1)\mathcal{H} - \mathrm{tr}_{g_M} h(Q(\cdot), \cdot) = 0, \tag{6.1}$$

equivalently,

$$(\top) : \mathrm{tr}_{g_M}(\nabla Q) = 0, \quad (\perp) : 2cm(m-1)\mathcal{H} - \mathrm{tr}_{g_M} h(Q(\cdot), \cdot) = 0, \tag{6.2}$$

where $(\top)$ and $(\perp)$ denote the tangent component and the normal component of (6.1)*, respectively.*

Remark 6.2. We define two $(1,1)$-type tensor fields A^C and Ξ on M^m as

$$A^C(X) := \sum_{\alpha=1}^{k} A^2_{\xi_\alpha}(X), \quad \Xi(X) := A_{\tau(\varphi)}(X) - A^C(X) = mA_{\mathcal{H}}(X) - A^C(X),$$

where $k = n - m$ and $\{\xi_\alpha\}_{\alpha=1}^k$ is a local orthonormal frame of $T^\perp M$. The operator A^C is called the *Casorati operator* (cf. [8, 9]). Then, from the Gauss equation, we have

$$Q(X) = c(m-1)X + \Xi(X).$$

From this, we can also describe the formula (6.2) as

$$(\top) : \mathrm{tr}_{g_M}(\nabla \Xi) = 0, \quad (\perp) : cm(m-1)\mathcal{H} - \mathrm{tr}_{g_M} h(\Xi(\cdot), \cdot) = 0. \tag{6.3}$$

Proof of Theorem 6.1

Since the target space $N^n(c)$ is of constant curvature c and $\varphi^* g_N = g_M$, by using Proposition 5.4, we compute

$$\begin{aligned}&\sum_{i,j=1}^{m} \big\{ \left(\nabla R^N\right)(d\varphi(e_i), d\varphi(e_i), d\varphi(e_j), d\varphi(e_j)) - d\varphi((\nabla R^M)(e_i, e_i, e_j, e_j)) \\ &\quad - h(e_i, R^M(e_i, e_j)e_j) + 2R^N(h(e_i, e_i), d\varphi(e_j))d\varphi(e_j) + 2R^N(d\varphi(e_i), h(e_i, e_j))d\varphi(e_j) \big\} \\ &= -d\varphi(\mathrm{tr}_{g_M}(\nabla Q)) - \mathrm{tr}_{g_M} h(Q(\cdot, \cdot) + 2c(m-1)\tau(\varphi).\end{aligned}$$

Therefore, the proof is completed since $\tau(\varphi) = m\mathcal{H}$. $\square$

6.2 Examples of Chern–Federer submanifolds

Here, we construct some examples of Chern–Federer maps in the case of isometric immersions. When an isometric immersion $\varphi : (M^m, g_M) \to (N^n, g_N)$ is a Chern–Federer map, we call the image a *Chern–Federer submanifold* in (N^n, g_N).

Let $I \subset \mathbb{R}$ be an open interval. Then an arbitrary curve $\gamma : I \to (N^n, g_N)$ is a Chern–Federer map. Actually, we have $W_1(\gamma) = W_2(\gamma)$ from Theorems 4.2 and 4.5. Therefore, it is trivial that

$$\mathrm{CF}(\gamma) = W_2(\gamma) - W_1(\gamma) = 0.$$

There are other obvious examples in the following way. We consider a Euclidean n-space $\mathbb{E}^n$ as a target space (N^n, g_M), which is a flat Riemannian space form. If (M^m, g_M) is a Ricci-flat Riemannian manifold, then an arbitrary isometric immersion $\varphi : (M^m, g_M) \to \mathbb{E}^n$ is Chern–Federer. For example, Calabi–Yau manifolds, Hyperkähler manifolds and G_2-manifolds are all Ricci-flat. Moreover, for any Riemannian manifold (M^m, g_M), there exists an isometric immersion into a Euclidean space by Nash's theorem.

Next, we consider the two-dimensional case ($m = 2$).

Proposition 6.3. *Let $\varphi : (M^2, g_M) \to N^n(c)$ be an isometric immersion and K the sectional curvature of (M^2, g_M). Then φ is Chern–Federer if and only if*

(i) *K is constant and φ is minimal, or*

(ii) *$K = 2c$ and φ is arbitrary, that is, unconditional on φ.*

Proof. In the two-dimensional case, we have, for any $X \in \Gamma(TM)$,

$$Q(X) = KX.$$

Thus, since φ is Chern–Federer if and only if

$$\begin{aligned}(\top) &: \mathrm{tr}_{g_M}(\nabla Q) = \mathrm{grad}\, K = 0, \\ (\perp) &: 4c\mathcal{H} - K\mathrm{tr}_{g_M} h(\cdot, \cdot) = 2(2c - K)\mathcal{H} = 0,\end{aligned}$$

we have the conclusion. $\square$

Let $M^2(K)$ be a two-dimensional Riemannian space form of constant curvature K. For minimal isometric immersions $\varphi : M^2(K) \to N^n(c)$, the research has already completed. In fact,

- when $c = 0$, it implies that $K = 0$ and φ is totally geodesic;

- when $c = -1$, it implies that $K = -1$ and φ is totally geodesic;
- when $c = 1$, it implies that $K \geq 0$. In addition, if $N^n(1)$ is isometric to a round sphere

  $$\mathbb{S}^n(1) := \{(x_1, \cdots, x_{n+1}) \in \mathbb{E}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

  then φ is locally congruent to generalized Clifford tori, or Borůvka spheres ψ_k $(k \geq 1)$. Here a generalized Clifford torus is a minimal 2-torus in $\mathbb{S}^n(1)$ which is an orbit of an abelian closed Lie subgroup of $SO(n+1)$, and a Borůvka sphere is a minimal 2-sphere in $\mathbb{S}^n(1)$ which is an orbit of an irreducible representation of $SO(3)$. See [4, 16] in detail.

At the end of Subsection 6.2, we consider flat tori in the unit 3-sphere $\mathbb{S}^3(1)$.

Let T^2 be a flat torus, $\varphi : T^2 \to \mathbb{S}^3(1)$ an isometric immersion. Then the flat torus T^2 admits an asymptotic Chebyshev net (s_1, s_2), that is, by using the asymptotic Chebyshev net (s_1, s_2), we can express

$$g_T = ds_1^2 + 2\cos\omega\, ds_1 ds_2 + ds_2^2, \quad h_T = 2\sin\omega\, ds_1 ds_2,$$

where $\omega = \omega(s_1, s_2)$ is some smooth function, and g_T, h_T are the induced metric and the second fundamental form of φ, respectively. Moreover, we compute the mean curvature function $\mathcal{H}$ of φ from this as

$$\mathcal{H}(s_1, s_2) = -\cot[\omega(s_1, s_2)].$$

See [17] in more precise details regarding an asymptotic Chebyshev net of a flat torus.

Theorem 6.4. *Let T^2 be a flat torus, $\varphi : T^2 \to \mathbb{S}^3(1)$ an isometric immersion with constant mean curvature $\mathcal{H}$. Then φ is an $(\alpha\mathcal{Q}_1 + \beta\mathcal{Q}_2)$-map if and only if*

(i) $\mathcal{H} = 0 \quad$ *(when $\alpha + \beta = 0$),*

(ii) $\mathcal{H} = 0 \quad \left(\text{when } \alpha + \beta \neq 0,\ \dfrac{\alpha}{\alpha + \beta} \geq 0\right),$

(iii) $\mathcal{H} = 0$, *or* $\mathcal{H}^2 = -\dfrac{\alpha}{2(\alpha + \beta)} \quad \left(\text{when } \alpha + \beta \neq 0,\ \dfrac{\alpha}{\alpha + \beta} < 0\right).$

Moreover, in the case of (iii), $\mathcal{H}^2$ runs across the whole range of $(0, \infty)$.

In [18], Kitagawa showed that any isometric embedding $\varphi : T^2 \to \mathbb{S}^3(1)$ with constant mean curvature are congruent to Clifford tori. Therefore, we have the following classification theorem.

Corollary 6.5. *Let T^2 be a flat torus, $\varphi : T^2 \to \mathbb{S}^3(1)$ an isometric embedding with constant mean curvature $\mathcal{H}$. Then φ is an $(\alpha\mathcal{Q}_1 + \beta\mathcal{Q}_2)$-map if and only if it is congruent to one of the following Clifford tori*

(i) *a minimal Clifford torus defined by*

$$\mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \hookrightarrow \mathbb{S}^3(1) \quad (\text{when } \alpha + \beta = 0),$$

(ii) *a minimal Clifford torus defined by*

$$\mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \hookrightarrow \mathbb{S}^3(1) \quad \left(\text{when } \alpha + \beta \neq 0,\ \frac{\alpha}{\alpha + \beta} \geq 0\right),$$

(iii) *a minimal Clifford torus defined by*

$$\mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right)\times\mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right)\hookrightarrow\mathbb{S}^3(1),$$

or a non-minimal Clifford torus defined by

$$\mathbb{S}^1(r_1)\times\mathbb{S}^1(r_2)\hookrightarrow\mathbb{S}^3(1)\qquad\left(\textit{when } \alpha+\beta\neq 0,\ \frac{\alpha}{\alpha+\beta}<0\right),$$

where r_1, r_2 are defined by

$$r_1=\frac{1}{2}\left[\sqrt{1+\sqrt{\frac{2(\alpha+\beta)}{\alpha+2\beta}}}-\sqrt{1-\sqrt{\frac{2(\alpha+\beta)}{\alpha+2\beta}}}\right],$$

$$r_2=\frac{1}{2}\left[\sqrt{1+\sqrt{\frac{2(\alpha+\beta)}{\alpha+2\beta}}}+\sqrt{1-\sqrt{\frac{2(\alpha+\beta)}{\alpha+2\beta}}}\right],$$

and the mean curvature of the Clifford torus $\mathbb{S}^1(r_1)\times\mathbb{S}^1(r_2)\hookrightarrow\mathbb{S}^3(1)$ satisfies that

$$\mathcal{H}^2=-\frac{\alpha}{2(\alpha+\beta)}.$$

Proof of Theorem 6.4

Let (s_1, s_2) be an asymptotic Chebyshev net for T^2. We define a frame field by using this coordinates

$$e_1=\frac{\partial}{\partial s_1},\quad e_2=\mathcal{H}\frac{\partial}{\partial s_1}+\sqrt{1+\mathcal{H}^2}\frac{\partial}{\partial s_2}.$$

Then $\{e_1, e_2\}$ defines a geodesic frame. By using this, we compute

$$W_1(\varphi)=-4\mathcal{H}(1+2\mathcal{H}^2)\xi,\quad W_2(\varphi)=-8\mathcal{H}^3\xi,$$

where ξ is a unit normal vector along φ. Namely, we have

$$\alpha W_1(\varphi)+\beta W_2(\varphi)=-4\mathcal{H}\{\alpha+2(\alpha+\beta)\mathcal{H}^2\}\xi.$$

This completes the proof. □

Remark 6.6. Regarding the following hypersurfaces in unit spheres

- $\mathbb{S}^m\left(\frac{1}{\sqrt{2}}\right)\subset\mathbb{S}^{m+1}(1)$ (a totally umbilical small sphere),
- $\mathbb{S}^m\left(\frac{1}{\sqrt{2}}\right)\times\mathbb{S}^m\left(\frac{1}{\sqrt{2}}\right)\subset\mathbb{S}^{2m+1}(1)$ (a minimal generalized Clifford torus),

these inclusion maps are both $(\alpha\mathcal{Q}_1+\beta\mathcal{Q}_2)$-maps for any $\alpha,\beta\in\mathbb{R}$ such that $\alpha^2+\beta^2\neq 0$.

6.3 Chern–Federer isoparametric hypersurfaces in space forms

We remark that for a hypersurface $M^m \subset N^{m+1}$ with a unit normal vector field ξ, it holds that

$$h(X,Y) = \langle A_\xi(X), Y\rangle \xi \tag{6.4}$$

for any $X, Y \in \Gamma(TM)$, and we may denote the shape operator A_ξ by A for simplicity.

Let $M^m \subset N^{m+1}(c)$ be an isoparametric hypersurface, that is, a hypersurface with constant principal curvatures. Then the inclusion map $\iota : M^m \hookrightarrow N^{m+1}(c)$ gives an isometric immersion by considering the induced metric g_M by ι, and we have an orthogonal direct sum decomposition as vector bundles

$$TM = \bigoplus_{\alpha=1}^{g} E_\alpha,$$

where g denotes the number of distinct principal curvatures and E_α are the principal (curvature) distributions. We remark that each E_α is auto-parallel, that is, the following holds

$$\nabla_X Y \in \Gamma(E_\alpha) \quad (X, Y \in \Gamma(E_\alpha)),$$

where ∇ denotes the Levi–Civita connection of (M^m, g_M). In particular, each E_α is integrable. More precisely, see [5, Lemma 3.9] in detail.

Theorem 6.7. *Let $M^m \subset N^{m+1}(c)$ be an isoparametric hypersurface in a Riemannian space form. Then M^m is Chern–Federer if and only if it satisfies that*

$$c(m-1)(\operatorname{tr} A) - (\operatorname{tr} A)(\operatorname{tr} A^2) + (\operatorname{tr} A^3) = 0.$$

We give a proof of Theorem 6.7 after Lemmas 6.9 and 6.10 stated below.

Remark 6.8. Let $M^m \subset N^{m+1}(c)$ be an isoparametric hypersurface. Then the inclusion map ι is a $\mathcal{Q}_1$-map if and only if

$$W_1(\iota) = c(\operatorname{tr} A) - (\operatorname{tr} A^3) = 0,$$

the inclusion map is a $\mathcal{Q}_2$-map (that is, a biharmonic map) if and only if

$$W_2(\iota) = \left(mc - (\operatorname{tr} A^2)\right)(\operatorname{tr} A) = 0,$$

and the inclusion map is a Willmore–Chen map if and only if

$$\mathrm{WC}(\iota) = mW_1(\iota) - W_2(\iota) = (\operatorname{tr} A)(\operatorname{tr} A^2) - m(\operatorname{tr} A^3) = 0.$$

Lemma 6.9. *Let $M^m \subset N^{m+1}(c)$ be an isoparametric hypersurface, $\iota : M^m \hookrightarrow N^{m+1}(c)$ the inclusion map and g_M the induced metric of M^m by ι. Then it holds that*

$$\operatorname{tr}_{g_M}(\nabla \Xi) = 0.$$

Proof. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame of M^m such that

$$A(e_i) = \lambda_i e_i,$$

where λ_i's are principal curvatures, which are constant. Then we have by using (6.4)

$$\begin{aligned}
\operatorname{tr}_{g_M}(\nabla \Xi) &= \sum_{k=1}^m \langle \operatorname{tr}_{g_M}(\nabla \Xi), e_k\rangle e_k \\
&= \sum_{i,j,k=1}^m \Big[\Big\langle \nabla_{e_i}(A_{h(e_j,e_j)} e_i) - (A_{h(e_j,e_j)} \nabla_{e_i} e_i), e_k \Big\rangle \\
&\qquad\qquad - \Big\langle \nabla_{e_i}(A_{h(e_i,e_j)} e_j) - (A_{h(\nabla_{e_i} e_i, e_j)} e_j), e_k \Big\rangle \Big] e_k
\end{aligned}$$

$$= \sum_{i,j,k=1}^{m} [-\lambda_i\lambda_j\delta_{ij}\langle\nabla_{e_i}e_j, e_k\rangle + \lambda_i\lambda_j\delta_{jk}\langle\nabla_{e_i}e_i, e_j\rangle]\, e_k$$
$$= \sum_{i,j=1}^{m} (\lambda_i\lambda_j - \lambda_i^2)\langle\nabla_{e_i}e_i, e_j\rangle e_j.$$

From the last formula, we can claim the following statements for $e_i \in \Gamma(E_\alpha)$, $e_j \in \Gamma(E_\beta)$: When $\alpha = \beta$, we have

$$\lambda_i\lambda_j - \lambda_i^2 = 0$$

since $\lambda_i = \lambda_j$. When $\alpha \neq \beta$, we have

$$\langle\nabla_{e_i}e_i, e_j\rangle = 0$$

since $\nabla_{e_i}e_i \in \Gamma(E_\alpha)$ and E_α is orthogonal to E_β. Therefore, we complete the proof. □

Lemma 6.10. *Under the assumption of Lemma 6.9, it holds that*

$$\mathrm{tr}_{g_M} h(\Xi(\cdot), \cdot) = [(\mathrm{tr}\, A)(\mathrm{tr}\, A^2) - (\mathrm{tr}\, A^3)]\xi,$$

where ξ is a unit normal vector field of M^m.

Proof. Taking an orthonormal frame $\{e_i\}_{i=1}^m$ of M^m such that $A(e_i) = \lambda_i e_i$, we compute by using (6.4) that

$$\begin{aligned}\mathrm{tr}_{g_M} h(\Xi(\cdot), \cdot) &= \sum_{i,j=1}^{m} h(A_{h(e_j,e_j)}e_i - A_{h(e_i,e_j)}e_j, e_i)\\ &= \sum_{i,j=1}^{m} h(e_i, \langle A(e_j), e_j\rangle A(e_i) - \langle A(e_i), e_j\rangle A(e_j))\\ &= \sum_{i,j=1}^{m} \left[\lambda_i^2\lambda_j - \lambda_i^2\lambda_j\delta_{ij}^2\right] \xi = [(\mathrm{tr}\, A)(\mathrm{tr}\, A^2) - (\mathrm{tr}\, A^3)]\xi.\end{aligned}$$

Thus, the proof is completed. □

Proof of Theorem 6.7

From Lemma 6.9 and Lemma 6.10, we can see that an isoparametric hypersurface $M^m \subset N^{m+1}(c)$ is Chern–Federer if and only if it holds that

$$\begin{aligned}&(\top) \;:\; \mathrm{tr}_{g_M}(\nabla\Xi) = 0 \quad \text{(trivially holds)},\\ &(\perp) \;:\; cm(m-1)\mathcal{H} - \mathrm{tr}_{g_M} h(\Xi(\cdot), \cdot) = \left[c(m-1)(\mathrm{tr}\, A) - (\mathrm{tr}\, A)(\mathrm{tr}\, A^2) + (\mathrm{tr}\, A^3)\right] \xi = 0.\end{aligned}$$

Thus, we obtain the conclusion. □

Let $\mathbb{L}^n$ be a Minkowski n-space. By using the classification [5, Theorem 3.12, Theorem 3.14] of isoparametric hypersurfaces in a Euclidean space $\mathbb{E}^{m+1}$ and a hyperbolic space

$$\mathbb{H}^{m+1}(-1) := \{(x_1, \cdots, x_{m+2}) \in \mathbb{L}^{m+2} \mid -x_1^2 + x_2^2 + \cdots + x_{m+2}^2 = -1,\ x_1 > 0\},$$

we have the following results:

Theorem 6.11. *Let $M^m \subset \mathbb{E}^{m+1}$ be an isoparametric hypersurface. Then M^m is Chern–Federer if and only if it is congruent to an open portion of one of the following hypersurfaces*

[$g = 1$] $\mathbb{E}^m \subset \mathbb{E}^{m+1}$ (*a totally geodesic hyperplane*),

[$g = 2$] $\mathbb{S}^1(r) \times \mathbb{E}^{m-1} \subset \mathbb{E}^{m+1}$ (*a generalized right circular cylinder*).

Theorem 6.12. *Let $M^m \subset \mathbb{H}^{m+1}(-1)$ be an isoparametric hypersurface. Then M^m is Chern–Federer if and only if it is totally geodesic.*

In the case of a unit sphere $\mathbb{S}^{m+1}(1)$, there exist fruitfully Chern–Federer isoparametric hypersurfaces which is not minimal. This is a different situation from that of biharmonic isoparametric hypersurfaces in a unit sphere. See [13] on the classification of biharmonic isoparametric hypersurfaces. In this thesis, we do not classify Chern–Federer isoparametric hypersurfaces in $\mathbb{S}^{m+1}(1)$. However, we show some examples of Chern–Federer homogeneous hypersurfaces, which are also isoparametric. Since all of their proofs are done by direct calculations by using Theorem 6.7, detailed calculations are omitted. We again remark that g denotes the number of distinct principal curvatures of isoparametric hypersurfaces.

- [$g = 1$]

The classification is the following totally umbilical hypersurfaces

$$\mathbb{S}^m(r) = \left\{(x, \sqrt{1-r^2}) \in \mathbb{E}^{m+2} \mid ||x||^2 = r^2\right\} \subset \mathbb{S}^{m+1}(1) \quad (0 < r \leq 1), \tag{6.5}$$

where $||\cdot||$ denotes the canonical Euclidean norm of $\mathbb{E}^{m+1}$. From this, we obtain:

Proposition 6.13. *The isoparametric hypersurface (6.5) is Chern–Federer if and only if $r = 1$ (totally geodesic one), or $r = 1/\sqrt{2}$ (proper biharmonic one).*

- [$g = 2$]

The classification is the following Clifford hypersurfaces

$$\mathbb{S}^p(r_1) \times \mathbb{S}^{m-p}(r_2) \subset \mathbb{S}^{m+1}(1) \quad (r_1^2 + r_2^2 = 1). \tag{6.6}$$

We denote the distinct principal curvatures of (6.6) by λ_1, λ_2. Then by setting

$$\lambda := \lambda_1 = \cot t \quad \left(0 < t < \frac{\pi}{2}\right),$$

we have

$$\lambda_2 = \cot\left(t + \frac{\pi}{2}\right) = -\frac{1}{\cot t} = -\frac{1}{\lambda}.$$

From this, we obtain:

Proposition 6.14. *The isoparametric hypersurface (6.6) is Chern–Federer if and only if λ satisfies that*

$$p(p-1)\lambda^6 - p(2m-p-1)\lambda^4 + (m-p)(m+p-1)\lambda^2 - (m-p)(m-p-1) = 0.$$

- [$g = 3$]

The classification is the following four Cartan hypersurfaces

$$M^3 = SO(3)/\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{S}^4(1), \tag{6.7}$$
$$M^6 = SU(3)/T^2 \to \mathbb{S}^7(1), \tag{6.8}$$
$$M^{12} = Sp(3)/Sp(1)^3 \to \mathbb{S}^{13}(1), \tag{6.9}$$
$$M^{24} = F_4/Spin(8) \to \mathbb{S}^{25}(1). \tag{6.10}$$

We denote the distinct principal curvatures of (6.7–6.10) by $\lambda_1, \lambda_2, \lambda_3$. Then by setting

$$\lambda := \lambda_1 = \cot t \quad \left(0 < t < \frac{\pi}{3}\right),$$

we have

$$\lambda_2 = \frac{\lambda - \sqrt{3}}{\sqrt{3}\lambda + 1}, \quad \lambda_3 = -\frac{\lambda + \sqrt{3}}{\sqrt{3}\lambda - 1}.$$

From this, we obtain:

Proposition 6.15. *The isoparametric hypersurfaces (6.7), (6.9) or (6.10) are Chern–Federer if and only if $\lambda = \sqrt{3}$ (the only minimal one).*

The isoparametric hypersurface (6.8) is Chern–Federer if and only if λ satisfies that

$$(\lambda^2 - 3)(3\lambda^3 - 3\lambda^2 - 9\lambda + 1)(3\lambda^3 + 3\lambda^2 - 9\lambda - 1) = 0.$$

Namely, there are non-minimal ones in the case.

- $[g = 4]$

In this case, we deal with homogeneous hypersurfaces. Non-homogeneous isoparametric ones are called to be of *OT–FKM type*. The classification of homogeneous hypersurfaces is the following ones

$$M^8 = SO(5)/T^2 \to \mathbb{S}^9(1), \tag{6.11}$$
$$M^{18} = U(5)/SU(2) \times SU(2) \times U(1) \to \mathbb{S}^{19}(1), \tag{6.12}$$
$$M^{30} = U(1) \cdot Spin(10)/S^1 \cdot Spin(6) \to \mathbb{S}^{31}(1), \tag{6.13}$$
$$M^{4m-2} = S(U(2) \times U(m))/S(U(1) \times U(1) \times U(m-2)) \to \mathbb{S}^{4m-1}(1) \quad (m \geq 2), \tag{6.14}$$
$$M^{2m-2} = SO(2) \times SO(m)/\mathbb{Z}_2 \times SO(m-2) \to \mathbb{S}^{2m-1}(1) \quad (m \geq 3), \tag{6.15}$$
$$M^{8m-2} = Sp(2) \times Sp(m)/Sp(1) \times Sp(1) \times Sp(m-2) \to \mathbb{S}^{8m-1}(1) \quad (m \geq 2). \tag{6.16}$$

We denote the distinct principal curvatures of (6.11–6.16) by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Then by setting

$$\lambda := \lambda_1 = \cot t \quad \left(0 < t < \frac{\pi}{4}\right),$$

we have

$$\lambda_2 = \frac{\lambda - 1}{\lambda + 1}, \quad \lambda_3 = -\frac{1}{\lambda}, \quad \lambda_4 = -\frac{\lambda + 1}{\lambda - 1}.$$

From this, we obtain:

Proposition 6.16. *The isoparametric hypersurface (6.11) is Chern–Federer if and only if $\lambda = 1 + \sqrt{2}$ (the only minimal one).*

The isoparametric hypersurface (6.12) is Chern–Federer if and only if λ satisfies that

$$3\lambda^{12} - 40\lambda^{10} + 223\lambda^8 - 692\lambda^6 + 223\lambda^4 - 40\lambda^2 + 3 = 0,$$

which is not minimal.

The isoparametric hypersurface (6.13) is Chern–Federer if and only if λ satisfies that

$$12\lambda^{12} - 111\lambda^{10} + 488\lambda^8 - 1098\lambda^6 + 488\lambda^4 - 111\lambda^2 + 12 = 0,$$

which is not minimal.

The isoparametric hypersurface (6.14) is Chern–Federer if and only if λ satisfies that

$$\lambda^{12} - 4(2m-1)\lambda^{10} + (72m-85)\lambda^{8} - 32(4m^2 - 10m + 7)\lambda^{6} + (72m-85)\lambda^{4} - 4(2m-1)\lambda^{2} + 1 = 0.$$

The isoparametric hypersurface (6.15) is Chern–Federer if and only if λ satisfies that

$$(2m-3)\lambda^{8} - 4(5m-9)\lambda^{6} + 2(16m^2 - 62m + 63)\lambda^{4} - 4(5m-9)\lambda^{2} + 2m - 3 = 0.$$

The isoparametric hypersurface (6.16) is Chern–Federer if and only if λ satisfies that

$$3\lambda^{12} - 16m\lambda^{10} + (136m-117)\lambda^{8} - 4(64m^2 - 116m + 63)\lambda^{6} + (136m-117)\lambda^{4} - 16m\lambda^{2} + 3 = 0.$$

- $[g = 6]$

The classification is the following two homogeneous hypersurfaces

$$M^6 = SO(4)/\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{S}^7(1), \tag{6.17}$$
$$M^{12} = G_2/T^2 \to \mathbb{S}^{13}(1). \tag{6.18}$$

We denote the distinct principal curvatures of (6.17), (6.18) by $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$. Then by setting

$$\lambda := \lambda_1 = \cot t \quad \left(0 < t < \frac{\pi}{6}\right),$$

we have

$$\lambda_2 = \frac{\sqrt{3}\lambda - 1}{\lambda + \sqrt{3}}, \ \lambda_3 = \frac{\lambda - \sqrt{3}}{\sqrt{3}\lambda + 1}, \ \lambda_4 = -\frac{1}{\lambda}, \ \lambda_5 = -\frac{\lambda + \sqrt{3}}{\sqrt{3}\lambda - 1}, \ \lambda_6 = -\frac{\sqrt{3}\lambda + 1}{\lambda - \sqrt{3}}.$$

From this, we obtain:

Proposition 6.17. *The isoparametric hypersurfaces (6.17) or (6.18) are Chern–Federer if and only if $\lambda = 2 + \sqrt{3}$ (the only minimal one).*

From the configuration of the examples of the Chern–Federer map so far, we obtained the examples of the star parts in Figure 3.

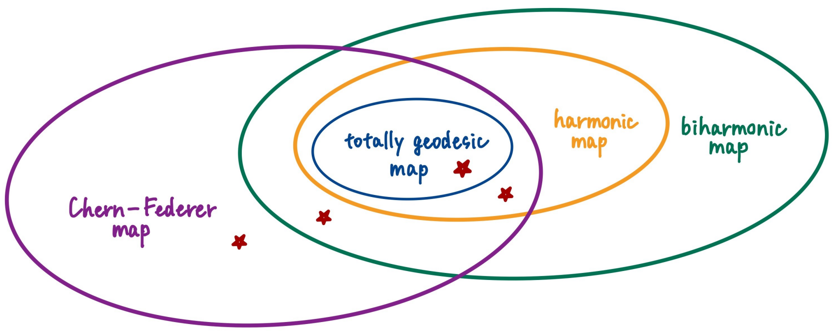

Figure 3: Correlation diagram of maps

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