# ON THE CONVERGENCE OF THE CHERN-RICCI FLOW ON COMPLEX SURFACES

**MASAYA KAWAMURA**
DEPARTMENT OF MATHEMATICS
AND
INFORMATION SCIENCES
TOKYO METROPOLITAN UNIVERSITY

2016

# Contents

# Acknowledgment

First, I would like to express my gratitude to my advisor Professor Shoichiro Takakuwa for helpful advices, encouragements and kind supports. I would like to thank sincerely Professor Masanori Kobayashi, Professor Takashi Sakai and Professor Kazuo Akutagawa for agreeing to be my thesis committe and for their valuable comments on my research.

# Chapter 1

# Introduction

## 1.1 Overview

Recently, many geometric flows have been investigated energetically and they give us some applications not only to differential geometry but also to other mathematical fields. Especially, we would like to focus on the recent study on the Ricci flow. This flow suddenly became famous after Perelman completed Hamilton's program and proved Poincaré and Thurston's Geometrization conjecture. The biggest problem for completing Hamilton's program was the clarification of the structure of a neighborhood around a point where the curvature is big just before showing up the singularity of the infinite curvature at finite singular time for the Ricci flow. Perelman introduced the idea of the entropy $\mathcal{W}$-functional and showed the local non-collapsing theorem, which implies that we can positively solve the non-appearance of the cigar soliton as Hamilton conjectured.

The Ricci flow's first appearance was in Hamilton's paper on 3-manifolds with positive Ricci curvature in 1982 [31]. In the paper, he introduced the Ricci flow and showed the short-time existence and its uniqueness on closed Riemannian manifolds. Hamilton developed powerful techniques such as the maximum principle for tensors and applied it to the evolution equation which the curvature tensors of the Ricci flow satisfies. And by applying this fundamental method for the Ricci flow, he proved that a closed 3-manifold equipped with a Riemannian metric whose Ricci curvature is strictly positive is diffeomorphic to a smooth quatient of 3-sphere. Hamilton established the foundation of the study of the Ricci flow and which became a breakthrough of the differentiable sphere theorem. In 2007, Brendle and Schoen finally proved the differentiable pointwise 1/4-pinching sphere theorem with using the Ricci flow (cf. [2]).

On compact Kähler manifolds, the Ricci flow starting at a Kähler metric is called the Kähler-Ricci flow, which reduces to the parabolic complex Monge-Ampère equation. The theory of the Kähler-Ricci flow has been developed drastically and it is known that the behavior of the Kähler-Ricci flow reflects the complex structure of manifolds. Cao [15] gave an alternative aproach to prove the existence of Kähler-Einstein metrics on closed Kähler manifolds with negative or vanishing first Chern class by studying on the convergence of the normalized Kähler-Ricci flow. On real 3-manifolds, Perelman and Hamilton showed that we can use the Ricci flow with surgery to break up the manifold into pieces. Since there exists a connection between Kähler manifolds and projective algebraic varieties, then naturally the similar question comes up for the Kähler-Ricci flow on a projective algebraic variety, which is the one that whether the Kähler-Ricci flow will give a geometric classification of algebraic varieties or not.

A minimal surface is a compact complex surface which has no special holomorphic sphere called $(-1)$-curve. When considering a projective surface, we remove irreducible disjoint finitely many $(-1)$-curves by blowing down. After blowing down finite times, the surface reaches a minimal surface. Otherwise, it is minimal from the first, or classified into a ruled surface or a rational surafce, whose Kodaira dimensions are negative. This process is understood along the Kähler-Ricci flow analytically. In the case when the singular time is infinity, it is minimal. When the solution of the Käler-Ricci flow is collapsing at the finite singular time, it is classified into a ruled surface or a rational surface.

The Minimal Model Program (MMP) is known as a process of simplifying algebraic varieties through algebraic surgeries in biratinal geometry. Before the appearance of BCHM [7], Tsuji had advocated that MMP could be understood via Kähler-Einstein geometry. After BCHM, Tian and Song discovered a complex analogue of Perelman's approach to Thurston's Geometrization conjecture (cf. [55]). For instance, algebraic operations such as flips and divisorial contractions assume the role of Perelman's idea "Surgery" and the Kähler-Ricci flow is considered to be the one of the few tools could be used for the analytification of MMP. BCHM introduced the idea of the MMP with Scaling. This idea describes a particular sequence of algebraic operations and takes a variety with a polarization to a minimal model or a Mori fiber space (cf. [55]). This process actually closely related to the Kähler-Ricci flow. The polarization corresponds to a choice of initial Kähler metric. Song and Tian showed that the Kähler-Ricci flow starting at a Kähler current can be continued through singularities in the weak sense related to the MMP with Scaling [55]. After that, Song and Weinkove [59] showed that in the case of complex dimension two, the algebraic procedure of blowing down $(-1)$-curves is corresponding to a geometric canonical surgical contraction for the Kähler-Ricci flow. Our one of main interests is that whether this correspondence is true also in the non-Kähler case.

The Chern-Ricci flow is analogue of the Kähler Ricci flow and starting at a Hermitian metric. If the initial metric is Kähler, the Chern-Ricci flow coincides with the Kähler-Ricci flow. Its study was started by Gill [26] in the setting of compact Hermitian manifolds with vanishing first Bott-Chern class. He showed that a solution of the Chern-Ricci flow converges smoothly to a unique Chern-Ricci flat metric, which can be said that this is a generalization of Cao's results in 1985 for the case of vanishing first Chern class. Tosatti and Weinkove investigated the Chern-Ricci flow in more general cases and studied the behavior of the solution on some compact complex surfaces such as Hopf surfaces, Inoue surfaces, non-Kähler properly elliptic surfaces (cf. [70], [71], [72]). They showed that for Hopf surfaces, there exists an explicit solution of the Chern-Ricci flow which collapse to a circle in the Gromov-Hausdorff sense in finite time. For Inoue surfaces, they also discovered that there exists an explicit solution of the Chern-Ricci flow and the solution devided by $t$ collapses in infinite time to a circle in the Gromov-Hausdorff sense and for non-Kähler properly elliptic surfaces, there also exists an explicit solution of the Chern-Ricci flow and the solution devided by $t$ collapses in infinite time to a compact Riemann surface with the distance function induced by an orbifold Kähler-Einstein metric on the surface in the Gromov-Hausdorff sense, and moreover, the solution devided by $t$ converges smoothly to the pullback of the orbifold Kähler-Einstein metric. These investigations tell us that the Chern-Ricci flow is a natural geometric flow whose behavior reflects the underlying geometry of manifolds. By investigating the behavior of the Chern-Ricci flow on compact complex surfaces, we may expect that we can extract some fresh topological or complex-geometric information.

Especially, the Class $VII$ surfaces are interesting objects since their classification has not yet completely done. Note that the Class $VII$ surfaces are compact complex surfaces with the Kodaira dimension $-\infty$ and the first Betti number one. Fang and Zheng analyzed the behavior of the Chern-Ricci flow on Inoue surfaces [22], well-known Class $VII$ surfaces, which come in three families. Tricerri and Vaisman constructed an explicit homogeneous Gauduchon metric $\omega_{TV}$ on each Inoue surfaces, which is strongly flat along the leaves. Fang and Zheng proved that the solution of the Chern-Ricci flow starting the initial metric in the $\partial\bar{\partial}$-class of $\omega_{TV}$ converges in the $C^\alpha$-topology for every $0 < \alpha < 1$. We focus on the convergence of a solution of the normalized Chern-Ricci flow on minimal non-Kähler properly elliptic surfaces. In the case of the unnormalized Chern-Ricci flow on minimal non-Kähler properly elliptic surfaces, a smooth solution of the flow divided by $t$ converges to an orbifold Kähler-Einstein metric smoothly as $t$ goes to infinity [67]. It also has been shown that the solution of the normalized Chern-Ricci flow converges to a Kähler-Einstein metric in $C^0$-topology on minimal non-Kähler properly elliptic surfaces [68].

## 1.2 Motivations

### 1.2.1 Canonical surgical contraction and blow-down of $(-1)$-curves

There are some investigations on the relationship between the Kähler-Ricci flow and algebraic geometry, especially MMP with Scaling. The definition is stated formally as follows:

**Definition 1.2.1.** (MMP with Scaling (cf. [55, Definition 5.2]))

(1) We start with a pair $(X, H)$, where $X$ is a normal $\mathbb{Q}$-factorial projective variety $X$ with log terminal singularities and $H$ is a big and semi-ample $\mathbb{Q}$-divisor on $X$.

(2) Let $\lambda_0 := \inf\{\lambda > 0 | \lambda H + K_X \text{ is nef}\}$ be the nef threshold. If $\lambda_0 = 0$, then we stop since the canonical divisor $K_X$ is already nef.

(3) Otherwise, there is an extremal ray $R$ of the cone of curves $\overline{\mathrm{NE}}(X)$ on which $K_X$ is negative and $\lambda_0 H + K_X$ is zero. So there exists a contraction $\pi : X \to Y$ of $R$:

   (a) If $\pi$ is a divisorial contraction, we replace $X$ by $Y$ and $H_Y$ be the strict transformation of $\lambda_0 H + K_X$ by $\pi$. Then we return to (1) with $(Y, H_Y)$.

   (b) If $\pi$ is a small contraction, we replace $X$ by its flip $X^+$ and let $H_{X^+}$ be the strict transformation of $\lambda_0 H + K_X$ by $\pi$. Then we return to (1) with $(X^+, H_{X^+})$.

   (c) If $\dim Y < \dim X$, then $X$ is a Mori fibre space, i.e., the fibers of $\pi$ are Fano. Then we stop.

A variety $X$ is called normal if a local ring $\mathcal{O}_{X,x}$ is a normal ring for each $x \in X$ and $X$ is said $\mathbb{Q}$-factorial if any $\mathbb{Q}$-divisor on $X$ is $\mathbb{Q}$-Cartier. It is known that $\mathbb{Q}$-factoriality is preserved after divisorial contractions and flips. A normal $\mathbb{Q}$-factorial projective variety $X$ is said to have log terminal singularities if $a_i > -1$ for all $i$, where $a_i \in \mathbb{Q}$ is a unique collection satisfying

$$K_{\tilde{X}} = \pi^* K_X + \sum_{i=1}^{p} a_i E_i,$$

$\pi : \tilde{X} \to X$ is a resolution and $\{E_i\}_{i=1}^{p}$ is the irreducible components of the exceptional locus $Exc(\pi)$ of $\pi$, where $\pi$ is not isomorphic.

Let $X$ be a normal projective variety and $H$ be a Cartier divisor on $X$. For $m \in \mathbb{Z}_{>0}$, if $H^0(X, mH) \neq 0$, then there exists a rational map

$$\Phi_{|mH|} : X - - \to \mathbb{P}(H^0(X, mH))$$

associated to the linear system $|mH|$. We define the Iitaka-Kodaira dimension of $(X, H)$ in the following:

$$\kappa(X, H) := \max_{m \in \mathbb{Z}_{>0}} \{\dim \operatorname{Im}(\Phi_{|mH|})\}.$$

A Cartier divisor $H$ is called big when $\kappa(X, H) = \dim X$ and $H$ is said nef if the intersection number $(H \cdot C) \geq 0$ for any curve $C$ on $X$. We say that $X$ is of general type if the canonical divisor $K_X$ is big. When $K_X$ is nef, $X$ is called a minimal model. A Cartier divisor $H$ is called semi-ample if the associated invertible sheaf $\mathcal{O}_X(H)$ satisfies that $\mathcal{O}_X(H)^{\otimes m}$ is globally generated for some $m \in \mathbb{Z}_{>0}$.

In Definition 1.2.1, $\mathrm{NE}(X)$ is the set of classes of effective 1-cycles in $N_1(X)_{\mathbb{R}}$, where $N_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_1(X)_{\mathbb{Z}}$ is the group of numerically equivalent 1-cycles, $\overline{\mathrm{NE}}(X)$ is the closure of $\mathrm{NE}(X)$ in the Euclidean topology. Two 1-cycles are said numerically equivalent if they have the same intersection number with every Cartier divisor (i.e. the same intersection number with every invertible sheaf associated to Cartier divisor). Let $L$ be a nef $\mathbb{Q}$-Cartier divisor but not ample, with $L - aK_X$ ample for some $a \in \mathbb{R}_{>0}$. Then the divisor $L$ is called a supporting divisor. We define an extremal face $F$ by

$$F = \{[C] \in \overline{\mathrm{NE}}(X) | (L \cdot C) = 0\},$$

where $(L \cdot C) = (\mathcal{O}_X(L) \cdot C)$ is the intersection number with the invertible sheaf $\mathcal{O}_X(L)$ associated to the supporting divisor $L$. When $[L] = 0 \in N^1(X)_{\mathbb{R}}$, where $N^1(X)_{\mathbb{R}}$ is the set of numerically equivalent classes of $\mathbb{R}$-invertible sheves, then we have $F = \overline{\mathrm{NE}}(X)$. Additionally when $F$ is a ray, which is called an extremal ray and written by $R$. By applying the base point free theorem, we see that there exists a contraction morphism $\phi_F : X \to Y$ associated to an extremal face $F$. Note that $\phi_F(C) = \{1pt\}$ for any curve $C$ if and only if we have $[C] \in F$. For the contraction $\phi_F$ associated to $F$, $-K_X$ is $\phi_F$-ample. Especially, a contraction morphism associated to an extremal ray is called an elementary contraction. Notice that a contraction morphism is determined by only an extremal face, that is to say, it is independent of a suppoting divisor.

Note that projective varieties $X$, $Y$ and $X^+$ are bimeromorphic (equivalently we can say "birational" since they are projective) each other. When a morphism $\mu : V_1 \to V_2$ is bimeromorphic between analytic spaces $V_1$. $V_2$, then there exist closed subsets $W_1 \subset V_1$, $W_2 \subset V_2$ with codim$W_2 \geq 2$ such that

$$\mu|_{V_1 \setminus W_1} : V_1 \setminus W_1 \overset{\cong}{\to} V_2 \setminus W_2$$

is biholomorphic (cf. [3, p.89]).

Let $D$ be a Cartier divisor on $V_2$ and let $\mu : V_1 \to V_2$ be a morphism between analytic spaces. Since $D$ is Cartier, there exist an open set $U \subset V_2$ and a meromorphic function $f_U$ on $U$ such that $D \cap U = (f_U)$, where $(f_U)$ is a divisor of a meromorphic function. On $\mu^{-1}(U)$, we define $\mu^* D \cap \mu^{-1}(U) = (\mu^* f_U)$. Then we can define a Cartier divisor $\mu^* D$ on $V_1$ by varying $U$. The divisor $\mu^* D$ is called a total transform by $\mu$ of $D$. On the other hand, when $\mu$ is bimeromorphic, there exist closed subsets $W_1 \subset V_1$, $W_2 \subset V_2$ with codim$W_2 \geq 2$ such that $\mu|_{V_1 \setminus W_1} : V_1 \setminus W_1 \to V_2 \setminus W_2$ is biholomorphic. So then we define a divisor $(\mu^{-1})_*(D)$ on $V_1$ by the closure of $(\mu|_{V_1 \setminus W_1})^*(D \cap (V_2 \setminus W_2))$ in $V_1$. The divisor $(\mu^{-1})_*(D)$ is called a strict transform of $D$ (cf. [3, p.75]). In this sense, the strict transformations $H_Y$, $H_{X^+}$ in Definition 1.2.1 are given by

$$H_Y = \pi_*(\lambda_0 H + K_X), \quad H_{X^+} = ((\pi^+)^{-1} \circ \pi)_*(\lambda_0 H + K_X).$$

Note that the divisor $H_Y$ is ample, the divisor $H_{X^+}$ is semi-ample and big and $H_{X^+} + \varepsilon K_{X^+}$ is ample for sufficiently small $\varepsilon > 0$ since $K_{X^+}$ is $\pi^+$-ample, hence we can go back to the first step (1) in MMP with Scaling. In the notations of Definition 1.2.1, we say that $\pi$ is a divisorial contraction in the case when the exceptional locus $Exc(\pi)$ is a divisor whose image of $\pi$ has codimension at least 2. In this case, $Y$ is still $\mathbb{Q}$-factorial and has at worst log terminal singularities. We say that $\pi$ is a small contraction in the case when $Exc(\pi)$ has codimension at least 2. In this case, $Y$ have rather bad singularities and the canonical divisor $K_Y$ is no longer a $\mathbb{Q}$-Cartier divisor. Hence we need to repalce $X$ by a birationally equivalent variety which is called a flip, with singularities milder than those of $Y$. The definition of a flip is as follows (cf. [55, Definition 5.4]):

**Definition 1.2.2.** Let $X$ be a normal $\mathbb{Q}$-factorial projective variety with log terminal singularities and let $\pi : X \to Y$ be a small contraction such that $-K_X$ is $\pi$-ample. A variety $X^+$ together with a proper bimeromorphic morphism $\pi^+ : X^+ \to Y$ is called a flip of $\pi$ if $\pi^+$ is also a small contraction and $K_{X^+}$ is $\pi^+$-ample. The morphism $(\pi^+)^{-1} \circ \pi : X \to X^+$ is bimeromorphic. The variety $X^+$ is $\mathbb{Q}$-factorial and has at worst log terminal singularities.

Notice that since the small contraction $\pi : X \to Y$ is a contraction of the extremal ray $R$ in the case (3)-($b$) in Definision 1.2.1, $-K_X$ is then $\pi$-ample.

In 2006, there was a breakthrough in algebraic geometry:

**Theorem 1.2.1.** (cf. [7], [55, Theorem 5.1])If $X$ is a normal $\mathbb{Q}$-factorial projective variety of general type with log terminal singularities, then the MMP with Scaling terminates in finite steps.

Theorem 1.2.1 means that there exist some flips needed, and does not exist infinite sequence of flips.

Let $H$ be a big and semi-ample $\mathbb{Q}$-divisor on $X$ and $\Omega$ be a smooth volume form on $X$. Then we define

$$\mathrm{PSH}_p(X,\omega_0,\Omega) := \{\varphi \in \mathrm{PSH}(X,\omega_0)\cap L^\infty(X) | \frac{(\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} \in L^p(X,\Omega)\}$$

and

$$\mathcal{K}_{H,p}(X) := \{\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi | \varphi \in \mathrm{PSH}_p(X,\omega_0,\Omega)\}$$

for $p \in (0,\infty]$, $\omega_0 \in c_1([H])$ a smooth closed $(1,1)$-form, where $[H]$ is the associated holomorphic line bundle, $c_1([H])$ is the first Chern class and $\mathrm{PSH}(X,\omega_0)$ denotes the set of all upper semi-continuous functions $\varphi : X \to [-\infty,\infty)$ such that $\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi \geq 0$ as a current.

We introduce the definition of the weak Kähler-Ricci flow on projective varieties with singularities:

**Definition 1.2.3.** (Weak Kähler-Ricci flow (cf. [55, Definition 4.3]))Let $X$ be a normal $\mathbb{Q}$-factorial projective variety with log terminal singularities and $\omega_0 \in c_1([H])$ be a smooth closed $(1,1)$-form on $X$ associated to a big and semi-ample $\mathbb{Q}$-divisor $H$ on $X$. Suppose that

$$T_0 = \sup\{t > 0 | H + tK_X \text{ is nef}\}.$$

A family of closed positive $(1,1)$-current $\omega(t,\cdot)$ on $X$ for $t \in [0,T_0)$ is called a solution of the unnormalized weak Kähler-Ricci flow if the following conditions hold.

(1) $\omega \in C^\infty((0,T_0)\times(X\setminus D))$, where $D$ is a subvariety of $X$. Let $\hat{\omega}_t \in c_1([H+tK_X])$ be a smooth family of smooth closed $(1,1)$-forms on $X$ for $t \in [0,T_0)$ such that

$$\hat{\omega}_0 = \omega_0 \in c_1([H]).$$

Then

$$\omega = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$$

for $\varphi \in C^0([0,T_0)\times(X\setminus D))\cap C^\infty((0,T_0)\times(X\setminus D))$ and $\varphi(t,\cdot) \in \mathrm{PSH}(X,\hat{\omega}_t)\cap L^\infty(X)$ for all $t \in [0,T_0)$ with $\varphi(0,\cdot) = \varphi_0(\cdot) \in \mathrm{PSH}(X,\omega_0)\cap L^\infty(X)$. Especially,

$$\omega_0' := \omega(0) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0$$

is a closed positive $(1,1)$-current on $X$.

(2)

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)), & \text{on } (0,T_0)\times(X\setminus D), \\ \omega(t)|_{t=0} = \omega_0', & \text{on } X. \end{cases}$$

In the case when the $\mathbb{Q}$-divisor $H$ is ample, $T_0$ is always positive and $X\setminus D = X_{\mathrm{reg}}$.

Since the contraction of the extremal ray and the contraction induced by the semi-ample divisor $\lambda_0 H + K_X$ might be different, we need to choose a special ample divisor called a *good initial divisor*, so that at each step, there is only one extremal ray contracted by the morphism induced by $\lambda_0 H + K_X$. The definition of a good initial divisor is as follows:

**Definition 1.2.4.** ([55, Definition 5.3])Let $X$ be a normal $\mathbb{Q}$-factorial projective variety with log terminal singularities. An ample $\mathbb{Q}$-divisor $H$ on $X$ is called a good initial divisor $H$ if the following conditions are satisfied.

(1) Let $X_0 = X$ and $H_0 = H$. The MMP with scaling terminates in finite steps by replacing $(X_0, H_0)$ by $(X_1, H_1), \ldots, (X_m, H_m)$ until $X_{m+1}$ is a minimal model or $X_m$ is a Mori fiber space.

(2) Let $\lambda_i$ be the nef threshold for each pair $(X_i, H_i)$ for $i = 1, \ldots, m$. Then the contraction induced by the semi-ample divisor $\lambda_i H_i + K_{X_i}$ contracts exactly one extremal ray.

Note that a good initial divisor always exists if $\dim X = 2$ and $\mathrm{Kod}(X) \geq 0$, and then the normalized Kähler-Ricci flow with a good initial divisor converges to the canonical model or the minimal model of $X$ coupled with a generalized Kähler-Einstein metric.

From the important result in Theorem 1.2.1, Song and Tian established the following analytification of MMP as a Kähler analogue of Perelman's approach to Thurston's Geometrization Conjecture.

**Theorem 1.2.2.** ([55, Theorem 5.7]) Let $X$ be a normal $\mathbb{Q}$-factorial projective variety with log terminal singularities. If there exists a good initial divisor $H$ on $X$, then either $X$ does not admit a minimal model or the unnormalized weak Kähler-Ricci flow has long time solution for any Kähler current $\omega_0 \in \mathcal{K}_{H,p}(X)$ with $p > 1$, after finitely many surgeries through divisorial contractions and flips.

Importantly, Song and Tian showed the smoothing property of the Kähler-Ricci flow with rough initial data away from singularities. That is, the associated parabolic Monge-Ampère flow is starting at a bounded plurisubharmonic function. Since the flow goes to a degenerate positive $(1,1)$-current as time goes to a finite singular time through such as flips and divisorial contractions, so it is inevitable to start with a Kähler current. But thanks to this smoothing effect, the flow becomes smooth all at once away from singularities and if a given variety has a minimal model, in this sense we see that the flow has a long time solution through finitely many flips and divisorial contractions.

It is conjectured in [55] that the Kähler-Ricci flow will either deform a projective variety $X$ to its minimal model via finitely many divisorial contractions and flips in the Gromov-Hausdorff sense, and then converge to a generalized Kähler-Einstein metric on the canonical model of $X$, or collapse in finite time. This process is the analytic analogue of Mori's minimal model program. Although the existence and uniqueness was proven for the weak Kähler-Ricci flow through divisorial contractions and flips in [55], the convergence in the Gromov-Hausdorff sense at the finite singular time was still largely open. After that, Song and Weinkove [59] showed that on a smooth projective algebraic surface $X$ with a Kähler metric $\omega_0$ satisfying $[\omega_0] \in H^{1,1}(X, \mathbb{Q})$, which indicates that there exists an ample holomorphic line bundle such that $c_1(L) = [\omega_0]$, there exists a unique maximal Kähler-Ricci flow $\omega(t)$ with canonical surgical contractions starting at $(X, \omega_0)$ on $X_0 = X, X_1, \ldots, X_k$ on maximal intervals $[0, T = T_0)$, $(T_0, T_1), \ldots, (T_{k-1}, T_k)$ such that $\omega(t)$ performs a canonical surgical contraction at $T_0, T_1 \ldots, T_{k-1}$ but not at $T_k$ (possibly $T_k = \infty$), and each canonical surgical contraction corresponds to a blow-down map $\pi_i : X_i \to X_{i+1}$ of a finite number of disjoint exceptional curves on $X_i$. Then, along the flow we see that either $T_k < \infty$ and then $X_k$ is $\mathbb{CP}^2$ or a ruled surface, or $T_k = \infty$ and $X_k$ has no exceptional curves. We state this formally in the following:

**Theorem 1.2.3.** ([59, Theorem 1.2])Let $X$ be a projective algebraic surface and $\omega_0$ a Kähler metric with $[\omega_0] \in H^{1,1}(X, \mathbb{Q})$. Then there exists a unique maximal Kähler-Ricci flow $\omega(t)$ on $X_0, X_1, \ldots, X_k$ contraction corresponds to a blow-down $\pi : X_i \to X_{i+1}$ of a finite number of disjoint exceptional curves on $X_i$. In addition we have:

(1) Either $T_k < \infty$ and the Kähler-Ricci flow $\omega(t)$ collapses $X_k$, in the sense that the volume of $X_k$ with respect to $\omega(t)$ tends to zero as $t \to T_k^-$:

$$\mathrm{Vol}_{\omega(t)} X_k \to 0, \quad \text{as } t \to T_k^-.$$

in this case $X_k$ is a Fano surface or a ruled surface.

(2) Or $T_k = \infty$ and $X_k$ has no exceptional curves of the first kind.

We expect that this process can be proceeded along also the Chern-Ricci flow, that is, if the Chern-Ricci flow is non-collapsing in finite time, then it blows down finitely many $(-1)$-curves and continues in a unique way on a new complex surface. Then we need global Gromov-Hausdorff convergence of the metrics and smooth convergence away from the $(-1)$-curves. With the terminology of the Kähler case, we say the solution $g(t)$ of the Chern-Ricci flow performs a canonical surgical contraction if the following occurs:

**Definition 1.2.5.** (Canonical surgical contraction (cf. [59, Definition 1.1])) Let $M$ be a compact complex surface, and let $g_0$ be a Gauduchon metric on $M$. Suppose that the Chern-Ricci flow is non-collapsing at time $T < \infty$, that is, the volume of $M$ with respect to the smooth solution of the Chern-Ricci flow $\omega(t) = g(t)$ starting at the metric $g_0$ stays positive as $t \to T^-$. Then there exist finitely many disjoint $(-1)$-curves $E_1, \ldots, E_k$ on a compact complex surface $M$ giving rise to a surjective holomorhic map $\pi : M \to N$ on to a compact complex surface $N$ blowing down each $E_i$ to a point $\pi(E_i) = y_i \in N$ and $\pi|_{M \setminus \cup_{i=1}^k E_i}$ a biholomorphic onto $N' := N \setminus \{y_1, \ldots, y_k\}$ such that

(1) As $t \to T^-$, on $M' := M \setminus \cup_{i=1}^k E_i$, the metrics $g(t)$ converge to a smooth Gauduchon metric $g_T$ in $C^\infty_{\mathrm{loc}}(M')$. Using $\pi$, we may regard $g_T$ as a Gauduchon metric on $N'$.

(2) Let $d_{g_T}$ be the distance function on $N'$ given by $g_T$. Then there exists a unique metric $d_T$ on $N$ extending $d_{g_T}$ such that $(N, d_T)$ is a compact metric space homeomorphic to $N$.

(3) $(M, g(t))$ converges to $(N, d_T)$ in the Gromov-Hausdorff sense as $t \to T^-$.

(4) There exists a smooth maximal solution $g(t)$ of the Chern-Ricci flow on $N$ for $t \in (T, T_N)$ with $T < T_N \leq \infty$ such that $g(t)$ converges to $g_T$ as $t \to T^+$ in $C^\infty_{\mathrm{loc}}(N')$.

(5) $(N, g(t))$ converges to $(N, d_T)$ in the Gromov-Hausdorff sense as $t \to T^+$.

We extend $g_T$ to a nonnegative $(1,1)$-tensor $\tilde{g}_T$ on the whole space $Y$ by setting

$$\tilde{g}_T|_{y_i}(\cdot,\cdot) = 0 \quad \text{for } i = 1,\ldots,k.$$

Notice that $\tilde{g}_T$ may be discontinuous at $y_1,\ldots,y_k$. Then we define the distance function $d_T$ appeared in the definition above with using $\tilde{g}_T$:

**Definition 1.2.6.** ([59, Definition 3.1])Define a distance function $d_T : Y \times Y \to \mathbb{R}$ by

$$d_T(y_1,y_2) := \inf_{\gamma} \int_0^1 \sqrt{\tilde{g}_T(\gamma'(s),\gamma'(s))}ds,$$

where the infimum is taken over all piecewise smooth paths $\gamma : [0,1] \to Y$ with $\gamma(0) = y_1$, $\gamma(1) = y_2$ for $y_1, y_2 \in Y$.

In the Kähler case, a smooth solution of the Kähler-Ricci flow performs a canonical surgical contraction.

**Theorem 1.2.4.** ([57, Theorem 1.1])Let $\omega(t)$ be a smooth solution of the Kähler-Ricci flow starting at an arbitrary fixed Kähler metric $\omega_0$ on a compact Kähler manifold for $t \in [0,T)$ and assume $T < \infty$. Suppose there exists a blow-down map $\pi : X \to Y$ contracting disjoint irreducible exceptional divisors $E_1,\ldots,E_k$ on $X$ with $\pi(E_i) = y_i \in Y$, for a smooth compact Kähler manifold $(Y,\omega_Y)$ such that the limiting Kähler class satisfies

$$[\omega_0] + Tc_1(K_X) = [\pi^*\omega_Y].$$

Then the Kähler-Ricci flow $\omega(t)$ performs a canonical surgical contarction with respect to the data $E_1,\ldots,E_k, Y$ and $\pi$.

This holds also for a map $\pi : X \to Y$ blowing down the $(-k)$-exceptional divisors of the $\mathbb{Z}_k$-orbifold points under the same cohomology condition for some smooth orbifold Kähler metric $\omega_Y$ on $Y$ [60].

Recently, Guo, Song and Weinkove [30] established the global geometric convergence for the normalized Kähler-Ricci flow on all minimal surfaces of general type, not only the ones include only distinct irreducible $(-2)$-curves, starting with any initial Kähler metric. By definition, a minimal surface of general type is a smooth complex surface $X$ whose canonical bundle $K_X$ is nef and big, and then $X$ is projective. So by the base point free theorem, $K_X$ is actually semi-ample and then $K_X^m$ is globally generated for sufficiently large positive integer $m$, so given an ordered basis $(s_0,\ldots,s_N)$ of the holomorphic sections of $K_X^m$ induce a well-defined holomorphic map $\Phi : X \to \mathbb{P}^N$ by $\Phi(x) = [s_0(x),\ldots,s_N(x)]$ for $x \in X$ with image $X_{\text{can}}$, the canonical model of $X$, which is an algebraic surface with at worst finitely many orbifold $A$-$D$-$E$-singularities and which admits a unique orbifold Kähler-Einstein metric since $K_{X_{\text{can}}}$ is ample. The map $\Phi$ contracts $(-2)$-curves on $X$ to orbifold points on $X_{\text{can}}$. A surface of general type is a complex surface whose minimal model is a minimal surface of general type, which means that a surface of general type can be obtaind by finitely many blow-ups of a minimal model of general type. By putting all together, Theorem 1.2.3 and the contraction of $(-2)$-curves along the normalized Kähler-Ricci flow in the Gromov-Hausdorff sense, we obtain the following convergence result:

**Theorem 1.2.5.** ([30, Corollary 1.1])Let $X$ be a compact complex surface of general type. Then the normalized Kähler-Ricci flow on $X$ starting with any initial Kähler metric $g_0$ is continuous through finitely many contraction surgeries in the Gromov-Hausdorff topology for $t \in [0, \infty)$ and converges in the Gromov-Hausdorff topology to $(X_{\text{can}}, g_{\text{KE}})$. The convergence is smooth away from the $(-2)$-curves, where $g_{\text{KE}}$ is the unique orbifold Kähler-Einstein metric on $X_{\text{can}}$.

The condition (1) in the conditions of the canonical surgical contraction in Definition 1.2.5 has proven by Tosatti and Weinkove in the non-Kähler case:

**Theorem 1.2.6.** ([70, Theorem 1.1])Let $M$ be a compact complex surface and let $\omega_0$ be a Gauduchon metric on $M$. Suppose that the Chern-Ricci flow $\omega(t)$ starting at $\omega_0$ is non-collapsing at time $T < \infty$. Then there exist finitely many disjoint $(-1)$-curve $E_1, \ldots, E_k$ on $M$ giving rise to a map $\pi : M \to N$ onto a complex surface $N$ blowing down each $E_i$ to a point $y_i \in N$ for $i = 1, \ldots, k$. Write $M' = M \setminus \bigcup_{i=1}^{k} E_i$ and $N' = N \setminus \{y_1, \ldots, y_k\}$. Then the map $\pi$ gives an isomorphism from $M'$ to $N'$. As $t \to T^-$, the metrics $\omega(t)$ converge to a smooth Gauduchon metric $\omega_T$ on $M'$ in $C^\infty_{\text{loc}}(M')$.

Notice that the finite time non-collapsing for the Chern-Ricci flow occurs commonly. For instance, whenever $M$ is a non-minimal compact complex surface with the Kodaira dimension $\text{Kod}(M) \neq -\infty$, there will be the finite time non-collapsing for any initial Gauduchon metric $\omega_0$. Remark that before that Theorem 1.2.6 was proved by applying the Buchdahl's Nakai-Moishezon criterion, this result in general dimensions above had been proved under the condition (1.5) in [72, Theorem 1.6].

If we impose the condition $(*)$ in Theorem 1.3 in [70]: $(*)$ there exist a smooth function $f$ and a smooth real $(1,1)$-form $\beta$ with

$$\omega_0 - T\,\text{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f = \pi^*\beta,$$

we have already known that we have (2) and (3) in the definition of the canonical surgical contraction. Note that after replacing $f$ by another smooth function, we may assume that $\beta$ is a Gauduchon metric by applying Buchdahl's Nakai-Moishezon criterion (cf. [70, Lemma 3.2]). We will observe that (4) and (5) in Definition 1.2.5 hold under the assumption $(*)$ in Chapter 4. When it comes to the Kähler case, as considering the contraction of $(-1)$-curves on a Kähler surfaces, such a surface has the Kodaira dimension $\text{Kod} = 2$ and then its algebraic dimension is equal to 2, which is equivalent to that the surface is projective. Since we see that for a projective Kähler surface by choosing a initial Kähler metric, the condition $(*)$ holds automatically. For this reason, we can repeatedly observe that the contraction of $(-1)$-curves can be understood by the canonical surgical contraction for the Kähler-Ricci flow analytically. Although we can construct an initial Gauduchon metric satisfying the condition $(*)$ artificially for the Chern-Ricci flow, it is not enough to interpret the contraction of $(-1)$-curves repeatedly as in the Kähler case. For these reason, removing the assumption $(*)$ is essential for improving the results in the case of the Chern-Ricci flow as in the Kähler case. We will observe that even a compact complex surface is non-Kähler, the condition $(*)$ can be actually removed and we can show the convergence in the Gromov-Hausdorff sense alonf the Chern-Ricci flow without any special assumptions in Chapter 3.

### 1.2.2 Hölder convergence of the Chern-Ricci flow on elliptic surfaces

Gill [28] showed that a suitably normalized solution of the parabolic Monge-Ampère flow converges to Hermitian metrics with vanishing Chern -Ricci form in the $C^\infty$-topology on a compact Hermitian manifold with its first Bott-Chern class is equal to zero. It was the beginning of the investigation of the Chern-Ricci flow. After that, Tosatti and Weinkove (cf. [70], [71], [72]) started to study on the Chern-Ricci flow on some complex surfaces such as properly elliptic surfaces, Hopf surfaces and Inoue surfaces. We would like to especially focus on the convergence of a solution of the normalized Chern-Ricci flow on minimal non-Kähler properly elliptic surfaces.

In the Kähler case, Song and Tian [56] investigated the Kähler-Ricci flow on a general minimal Kähler elliptic surface, and they showed that the flow converges at the level of potentials to a generalized Kähler-Einstein metric on the base Riemannian surface. Since generally, the fibration structure on a Kähler elliptic surface is not locally trivial and may have singular fibers, the generalized Kähler-Einstein equation involves the Weil-Petersson metric and singular currents. It has been studied on the behavior of the Kähler-Ricci flow in the case of a product elliptic surface $M = E \times S$, where $E$ is an elliptic curve and $S$ is a compact Riemann surface of genus at least 2 by Song and Weinkove [61]. In this case, the solution of the normalized Kähler-Ricci flow on $E \times S$ conveges to a Kähler-Einstein metric on $S$ in $C^\alpha$-topology for any $0 < \alpha < 1$ and Gill developed this result into the $C^\infty$-convergence [27]. Fong and Zhang [23] showed the $C^\infty$ convergence result for the Kähler-Ricci flow on more general elliptic bundles with using the idea established by Gross, Tosatti and Zhang.

In the case of (unnormalized) Chern-Ricci flow on a minimal non-Kähler properly elliptic surface $\pi : M \to S$, there exists an explicit solution $\omega(t)$ of the Chern-Ricci flow on $M$ for $t \in [0.\infty)$ and the solution $\omega(t)$ divided by $t$ converges smoothly to $\pi^*\omega_{KE}$ on $M$ as $t \to \infty$, where $\omega_{KE}$ is an orbifold Kähler-Einstein metric on $S$. And also, with the normalized metrics $\frac{\omega(t)}{t}$, we have that

$$\left(M, \frac{\omega(t)}{t}\right) \overset{GH}{\to} (S, d_{\mathrm{KE}}), \quad \text{as } t \to \infty$$

in the Gromov-Hausdorff sense, where $d_{\mathrm{KE}}$ is the distance function induced by $\omega_{KE}$ (cf. [70]). And also for an elliptic bundle over a compact Riemann surface $S$ of genus at least 2 with fiber an elliptic curve, it has shown that the solution of the normalized Chern-Ricci flow converges to a pull-backed Kähler-Einstein metric on $S$ exponentially fast in $C^0$-topology [71]. By essentially using the fact that any minimal non-Kähler properly elliptic surface is covered by an elliptic fiber bundle, this convergence result for an elliptic fiber bundle can be applied to the case considering a minimal non-Kähler properly elliptic surface. Formally, which is stated as follows: Let $\pi : M \to S$ be firstly an elliptic bundle over a compact Riemann surface $S$ of genus at least 2, with fiber an elliptic curve $E$. And let $\omega_{\mathrm{flat},y}$ be the unique flat metric on the fiber $\pi^{-1}(y)$ for each point $y \in S$ in the Kähler classs $[\omega_0|_{\pi^{-1}(y)}]$. Let $\omega_S$ be the unique Kähler-Einstein metric on $S$ with $\mathrm{Ric}(\omega_S) = -\omega_S$ and $\omega_0$ be a Gauduchon metric on $M$.

Then we investigate the normalized Chern-Ricci flow

$$\frac{\partial}{\partial t}\omega(t) = -\operatorname{Ric}(\omega(t)) - \omega(t), \quad \omega(t)|_{t=0} = \omega_0.$$

With this flow above, we can observe that elliptic fibers collapse along the flow, on the other hand, the volume of the base surface $S$ remains positive and bounded. Under the setting above, the convergence result as $t \to \infty$ can be shown:

**Theorem 1.2.7.** ([71, Theorem 1.1])Let $\omega(t)$ be a solution of the normalized Chern-Ricci flow on $M$ starting at $\omega_0$. Then as $t \to \infty$, $\omega(t)$ converges to $\pi^*\omega_S$ exponentially fast in the $C^0(M, g_0)$ topology. In particular, the diameter of each elliptic fiber tends to zero uniformly exponentially fast and $(M, \omega(t))$ converges to $(S, \omega_S)$ in the Gromov-Hausdorff topology. Furthermore, $e^t\omega(t)|_{\pi^{-1}(y)}$ converges to the flat metric $\omega_{\text{flat},y}$ exponentially fast in the $C^1(\pi^{-1}(y), g_0)$ topology, uniformly in $y \in S$.

Then we apply the key fact that for any minimal non-Kähler properly elliptic surface $M$, there always exists a finite unramified covering $p : M' \to M$, which is also a minimal properly elliptic surface, $\pi' : M' \to S'$ is an elliptic fiber bundle with $S'$ a compact Riemann surface of genus at least 2, and obtain the following convergence result:

**Theorem 1.2.8.** ([71, Corollary 1.2]) Let $\pi : M \to S$ be any minimal non-Kähler properly elliptic surface. Then given any initial Gauduchon metric $\omega_0$ on $M$ we have that $(M, \omega(t))$ converges to $(S, d_S)$ in the Gromov-Hausdorff topology. Here $d_S$ is the distance function induced by an orbifold Kähler-Einstein metric $\omega_S$ on $S$, whose set $Z$ of orbifold points is precisely the image of the multiple fibers of $\pi$. Furthermore, $\omega(t)$ converges to $\pi^*\omega_S$ in the $C^0(M, g_0)$ topology, and for any $y \in S \setminus Z$ the metrics $e^t\omega(t)|_{\pi^{-1}(y)}$ converge exponentially fast in the $C^1(\pi^{-1}(y), g_0)$ topology and uniformly as $y$ varies in a compact set of $S \setminus Z$ to the flat Kähler metric $\pi^{-1}(y)$ cohomologous to $[\omega|_{\pi^{-1}(y)}]$.

Our aim is to show that the smooth solution the normalized Chern-Ricci flow $\omega(t)$ converges to $\pi^*\omega_S$ as $t \to \infty$ for some orbifold Kähler-Einstein metric $\omega_S$ is possible in $C^\alpha$-topology for any $0 < \alpha < 1$. We will observe that this $C^\alpha$-convergence can be realized by choosing an initial Gauduchon metric from the $\partial\bar{\partial}$-class of the Vaisman metric in Chapter 5. If we let $z \in H$ be the variable in the upper half plane $H$ in $\mathbb{C}$, $w \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and $y = \operatorname{Im} z$, then we observe that the form $\pi^*\omega_S$ is induced from the form $\frac{\sqrt{-1}}{2y^2} dz \wedge d\bar{z}$ on $\mathbb{C}^* \times H$. This study is the one which was stimulated by the investigation of the normalized Chern-Ricci flow on Inoue surfaces (cf. [22]).

## 1.3 Summary of new results

In Chapter 3, we will show that we can remove the condition (†) with using some tools in pluripotential theory. We consider a map $\pi : M \to N$ blows down the only one $(-1)$-curve $E$ on $M$ to the point $y_0 \in N$ for simplicity.

**Theorem 1.3.1.** ([35, Theorem 1.1])Let $M$ be a non-Kähler compact complex surface and $\pi : M \to N$ be a blow-down map of the $(-1)$-curve $E$ on $M$ to the point $y_0 \in N$, where $N$ is a compact complex surface. Let $\omega_0$ be a Gauduchon metric on $M$. Suppose that we have

$$\int_M (\omega_0 - T\,\mathrm{Ric}(\omega_0))^2 > 0, \quad \text{and} \quad \int_D (\omega_0 - T\,\mathrm{Ric}(\omega_0)) > 0$$

for all irreducible curves $D$ on $M$ with $D^2 = (D \cdot D) < 0$ different from $E$, where $T$ is a finite singular time of the Chern-Ricci flow $\omega(t)$ starting at $\omega_0$ for $t \in [0, T)$, $0 < T < \infty$. Then there exist a smooth real function $u_0'$ on $M$ and a Gauduchon metric $\hat{\omega}_N$ on $N$ such that

$$\omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}u_0' = \pi^*\hat{\omega}_N.$$

From the result of Theorem 1.3.1, we can show that the convergence in the Gromov-Hausdorff sense holds without the cohomology condition (†):

**Theorem 1.3.2.** ([35, Theorem 1.2])Let $M$ be a non-Kähler compact complex surface and $\pi : M \to N$ be a blow-down map of finitely many disjoint $(-1)$-curves on $M$ onto a complex surface $N$. Let $\omega_0$ be a Gauduchon metric on $M$. We assume that the Chern-Ricci flow $\omega(t)$ starting at $\omega_0$ is non-collapsing at a singular time $T < \infty$. Then there exists a distance function $d_T$ on $N$ such that $(N, d_T)$ is a compact metric space and $(M, d_{\omega(t)})$ converges in the Gromov-Hausdorff sense to $(N, d_T)$ as $t \to T^-$, where $d_{\omega(t)}$ are distance functions induced from the metrics $\omega(t)$.

Under the assumption that the theorem above holds, we will prove the solution of the Chern-Ricci flow peforms a canonical surgical contraction (Definition 1.2.5) in Chapter 4.

**Theorem 1.3.3.** ([36, Theorem 1.1])Let $M$ be a non-Kähler compact complex surface and $\pi : M \to N$ be a blow-down map of the $(-1)$-curve $E$ on $M$ to the point $y_0 \in N$, where $N$ is a compact complex surface. Let $\omega_0$ be a Gauduchon metric on $M$. Suppose that we have

$$\int_M (\omega_0 - T\,\mathrm{Ric}(\omega_0))^2 > 0, \quad \text{and} \quad \int_D (\omega_0 - T\,\mathrm{Ric}(\omega_0)) > 0$$

for all irreducible curves $D$ on $M$ with $D^2 = (D \cdot D) < 0$ different from $E$, where $T$ is a finite singular time of the Chern-Ricci flow $\omega(t)$ starting at $\omega_0$ for $t \in [0, T)$, $0 < T < \infty$. Then the Chern-Ricci flow $\omega(t)$ performs a canonical surgical contraction with respect to the data $E$, $N$ and $\pi$.

In Chapter 5, we will observe $C^\alpha$-convergence of the solution of the normalized Chern-Ricci flow starting at the initial metric in the $\partial\bar{\partial}$-class of the Vaisman metric $\omega_V$ [74] on a minimal non-Kähler properly elliptic surface.

**Theorem 1.3.4.** ([37, Theorem 1.1])Let $M$ be a minimal non-Kähler properly elliptic surface and let $\omega(t)$ be the solution of the normalized Chern-Ricci flow starting at a Hermitian metric of the form

$$\omega_0 = \omega_V + \sqrt{-1}\partial\bar{\partial}\psi > 0,$$

where $\omega_V$ is the Vaisman metric and $\psi$ is a smooth function on $M$. Then the metrics $\omega(t)$ are uniformly bounded in the $C^1$-topology, and as $t \to \infty$,

$$\omega(t) \to \pi^*\omega_S,$$

in the $C^\alpha$-topology, for every $0 < \alpha < 1$, where $\omega_S$ is the orbifold Kähler-Einstein metric on $S$ with $\mathrm{Ric}(\omega_S) = -\omega_S$ away from finitely many orbifold points induced by the form $\frac{\sqrt{-1}}{2y^2} dz \wedge d\bar{z}$ on $\mathbb{C}^* \times H$, $H$ is the upper half palne in $\mathbb{C}$, $z \in H$ is the variable, $y = \mathrm{Im} z$.

# Chapter 2

# Background

## 2.1 Notations

Let $M$ be a differentiable manifold and $g$ be a Riemannian metric on $M$. Let $J \in \Gamma(\mathrm{End}(TM))$ be the endomorphism $J$ satisfies $J^2 = -\mathrm{id}_{TM}$, where $\Gamma(\mathrm{End}(TM))$ is the space of all sections of $\mathrm{End}(TM) = T^*M \otimes TM$. Then $J$ is called the almost complex structure and $(M, J)$ is called an almost complex manifold. Additionally, if $J$ is integrable, $J$ is called the complex structure and then $(M, J)$ is a complex manifold. That the almost complex structure $J$ is integrable is equivalent to that the Nijenhuis tensor $\equiv 0$.

Let $(M, J)$ be an almost complex manifold. A Riemannian metric $g$ on $M$ is called $J$-invariant if $J$ is compatible with $g$, i.e., for any $X, Y \in \Gamma(TM)$,

$$g(X, Y) = g(JX, JY).$$

The fundamental 2-form $\omega$ associated to a $J$-invariant Riemannian metric $g$ is determined by, for $X, Y \in \Gamma(TM)$,

$$\omega(X, Y) = g(JX, Y).$$

Indeed we have, for any $X, Y \in \Gamma(TM)$,

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y)$$

and $\omega \in \Gamma(\bigwedge^2 T^*M)$. A $J$-invariant Riemannian metric $g$ on a complex manifold $(M, J)$ is called Kähler if the fundamental 2-form $\omega$ associated to $g$ is $d$-closed and then $\omega$ is called a Kähler form.

We write $T^{\mathbb{R}}M$ for the real tangent space of $M$. Then its complexifieid tangent space is given by

$$T^{\mathbb{C}}M = T^{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}.$$

By extending $J$ linearly in $\mathbb{C}$ and $g$, $\omega$ bilinearly in $\mathbb{C}$ to $T^{\mathbb{C}}M$, they are also defined on $T^{\mathbb{C}}M$ and we observe that the complexified tangent space $T^{\mathbb{C}}M$ can be decomposed as

$$T^{\mathbb{C}}M = T'M \oplus T''M,$$

where $T'M$ and $T''M$ are the eigenspaces of $J$ corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Extending $J$ to forms, we can uniquely decompose $m$-forms into $(p, q)$-forms for each $p, q$ with $p + q = m$.

Now, let $(M, J)$ be a complex manifold of dimension $n$ and let $g$ be a $J$-invariant Riemannian metric on $M$. Then we define a Hermitian metric $h$ by

$$h(X, Y) = g(X, \overline{Y})$$

for $X, Y \in \Gamma(T^{\mathbb{C}}M)$. The decomposition $T^{\mathbb{C}}M = T'M \oplus T''M$ is orthogonal with respect to $h$. Indeed, for $X \in \Gamma(T'M)$, $Y \in \Gamma(T''M)$, we have $\overline{Y} \in \Gamma(T'M)$ and

$$h(X, Y) = g(X, \overline{Y}) = g(JX, J\overline{Y}) = -h(X, Y).$$

It follows that we have $h(X, Y) = 0$ for any $X \in \Gamma(T'M)$, $Y \in \Gamma(T''M)$.

Let $\nabla$ be the Chern connection of $h$, which satisfies for any $X, Y, Z \in \Gamma(T'M)$,

$$\nabla_X(h(Y, Z)) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z).$$

The torsion $T$ and curvature $R$ of $\nabla$ are defined by, for $X, Y, Z \in \Gamma(T^{\mathbb{C}}M)$,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Since $\nabla J = 0$, we obtain for $X, Y, Z, W \in \Gamma(T^{\mathbb{C}}M)$,

$$T(JX, JY) = JT(X, Y), \quad R(X, Y)JZ = JR(X, Y)Z$$

and it follows that

$$g(T(JX, JY), JZ) = g(T(X, Y), Z) =: T(X, Y, Z)$$

and

$$g(R(X, Y)JZ, JW) = g(R(X, Y)Z, W) =: R(X, Y, Z, W).$$

Hence we have $R(X, Y, Z, W) = 0$ unless $Z, W$ are of different type. In local coordinates $(z_1, \dots, z_n)$, we have

$$g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) = 0, \quad g(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}) = 0$$

since we have

$$J\Big(\frac{\partial}{\partial z_i}\Big) = \sqrt{-1}\frac{\partial}{\partial z_i}, \quad J\Big(\frac{\partial}{\partial \bar{z}_i}\Big) = -\sqrt{-1}\frac{\partial}{\partial \bar{z}_i}$$

and we write

$$g_{i\bar{j}} = g\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\Big),$$

$(g_{i\bar{j}})^{-1} = (g^{i\bar{j}})$, which denotes its inverse matrix, i.e., we have $g^{i\bar{j}} g_{k\bar{j}} = \delta_{ik}$. The Christoffel symbols $\Gamma^k_{ij}$, torsion tensor $T$ and Chern curvature tensor $R$ of $g$ are defined by

$$\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} = \Gamma^k_{ij} \frac{\partial}{\partial z_k},$$

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} = g^{k\bar{l}}\Big(\frac{\partial g_{j\bar{l}}}{\partial z_i} - \frac{\partial g_{i\bar{l}}}{\partial z_j}\Big),$$

$$R_{i\bar{j}k\bar{l}} \equiv R\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\Big) = -g_{m\bar{l}}\frac{\partial \Gamma^m_{ik}}{\partial \bar{z}_j} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{p\bar{q}}\frac{\partial g_{k\bar{q}}}{\partial z_i}\frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j},$$

$$R_{ijk\bar{l}} = g_{m\bar{l}}\Big(\frac{\partial \Gamma^m_{jk}}{\partial z_i} - \frac{\partial \Gamma^m_{ik}}{\partial z_j} + \Gamma^m_{iq}\Gamma^q_{jk} - \Gamma^m_{jq}\Gamma^q_{ik}\Big),$$

$$R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} = g_{m\bar{l}}\frac{\partial T^m_{ki}}{\partial \bar{z}_j} = g_{m\bar{l}}\nabla_{\bar{j}}\Gamma^m_{ki}$$

and the traces of the curvature tensor

$$R_{k\bar{l}} = g^{i\bar{j}}R_{i\bar{j}k\bar{l}}, \quad \mathrm{Ric}_{i\bar{j}} = g^{k\bar{l}}R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log\det g_{k\bar{l}}$$

are called the first and second Ricci tensors, respectively. The second Ricci tensor $\mathrm{Ric}_{i\bar{j}}$ is often called the Chern-Ricci tensor. We also define the scalar curvature

$$R = g^{i\bar{j}}g^{k\bar{l}}R_{i\bar{j}k\bar{l}}.$$

The covariant derivatives of $a = a_j dz^j$ and $X = X^j\frac{\partial}{\partial z^j}$ are defined in components as

$$\nabla_i a_j = \partial_i a_j - \Gamma^k_{ij}a_k, \quad \nabla_i X^j = \partial_i X^j + \Gamma^j_{ik}X^k.$$

Then the Chern connection $\nabla$ can be extended naturally to any tensors.

We can choose the following special local coordinates (cf. [29]):

**Lemma 2.1.1.** Around a point $p \in M$, there exist local coordinates such that, for any $i, j$,

$$g_{i\bar{j}}(p) = \delta_{ij}, \quad \frac{\partial g_{i\bar{i}}}{\partial z_j}(p) = 0.$$

**Lemma 2.1.2.** Around a point $p \in M$, there exist local coordinates such that, for any $i, j, k$,

$$g_{i\bar{j}}(p) = \delta_{ij}, \quad \frac{\partial g_{i\bar{j}}}{\partial z_k}(p) + \frac{\partial g_{k\bar{j}}}{\partial z_i}(p) = 0.$$

Especially we have

$$T^k_{ij}(p) = 2\frac{\partial g_{j\bar{k}}}{\partial z_i}(p).$$

**Remark 2.1.1.** It is impossible to choose local coordinates satisfying both in Lemma 2.1 and Lemma 2.2 simultaneously in general.

Let $\Lambda^{p,q}$ denote differential $(p,q)$-forms on $M$. The exterior differential $d$ has a decomposition $d = \partial + \bar{\partial}$ where

$$\partial : \Lambda^{p,q} \to \Lambda^{p+1,q}, \quad \bar{\partial} : \Lambda^{p,q} \to \Lambda^{p,q+1}.$$

Note that we have $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$ and by the Stokes theorem,

$$\int_M \partial\alpha = \int_{\partial M}\alpha$$

for any $\alpha \in \Lambda^{n-1,n}$.

In local coordinates, $\partial\bar{\partial}u$ for a function $u \in C^2(M)$ is locally given by

$$\partial\bar{\partial}u = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}dz_i \wedge d\bar{z}_j.$$

## 2.2 Holomorphic line bundles and divisors

Let $X$ be a compact complex manifold. A holomorphic line bundle $L$ over $X$ is given by an open cover $\{U_\alpha\}$ of $X$ with collection of transition functions $\{t_{\alpha\beta}\}$ which are holomorphic maps $t_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ satisfying

$$(T) \qquad t_{\alpha\beta}t_{\beta\alpha} = 1, \quad t_{\alpha\beta}t_{\beta\gamma} = t_{\alpha\gamma}.$$

If there exist holomorphic functions $f_\alpha : U_\alpha \to \mathbb{C}^*$ such that $t'_{\alpha\beta} = \frac{f_\alpha}{f_\beta}t_{\alpha\beta}$, we identify collections of transition functions $\{t_{\alpha\beta}\}$ and $\{t'_{\alpha\beta}\}$. Given two holomorphic line bundles $L$ and $L'$ with transition functions $\{t_{\alpha\beta}\}$ and $\{t'_{\alpha\beta}\}$ respectively, we write $LL'$ for the new holomorphic line bundle with transition functions $\{t_{\alpha\beta}t'_{\alpha\beta}\}$. We define holomorphic line bundles $L^m = mL$ by $\{t^m_{\alpha\beta}\}$ for $m \in \mathbb{Z}$. Let $L$ be a holomorphic line bundle over $X$. A holomorphic section $s$ of $L$ is a collection $\{s_\alpha\}$ of holomorphic maps $s_\alpha : U_\alpha \to \mathbb{C}$ satisfying the transformation rule

$$s_\alpha = t_{\alpha\beta}s_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

A Hermitian metric $h$ on $L$ is a collection $\{h_\alpha\}$ of smooth positive functions $h_\alpha : U_\alpha \to \mathbb{R}$ satisfying the transformation rule

$$h_\alpha = t_{\beta\alpha}\bar{t}_{\beta\alpha}h_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

We define the curvature $R_h$ of a Hermitian metric $h$ on $L$ to be the closed $(1,1)$-form on $X$ locally given by

$$R_h = -\sqrt{-1}\partial\bar{\partial}\log h_\alpha$$

on $U_\alpha$, which is well-defined. Note that we omit a factor of $2\pi$. We also define the first Chern class $c_1(L)$ of $L$ to be the cohomology class $c_1(L) = [R_h]$. Since any two Hermitian metrics $h$, $h'$ on $L$ are related by $h' = e^{-\varphi}h$ for some smooth function $\varphi$, we have

$$R_{h'} = R_h + \sqrt{-1}\partial\bar{\partial}\varphi$$

and hence $c_1(L)$ is well-defined. If $h$ is a Hermitian metric on $L$, then $h^m$ is a Hermitian metric on $L^m$ and we have $c_1(L^m) = mc_1(L)$.

We say that $L$ is positive if $c_1(L) > 0$. We write $H^0(X, L)$ for the vector space of holomorphic sections of $L$, whose dimension is finite if it is not empty. We say that $L$ is very ample if for any ordered basis $(s_0, \ldots, s_N)$ of $H^0(X, L)$, the map $\iota : X \to \mathbb{P}^N$ given by

$$\iota(x) = [s_0(x), \ldots, s_N(x)],$$

well-defined and an embedding. We say that $L$ is ample if there exists a positive integer $m_0$ such that $L^m$ is very ample for all integer $m \geq m_0$. The Kodaira Embedding Theorem states as follows:

**Theorem 2.2.1.** *Let $X$ be a compact complex manifold and let $L$ be a positive holomorphic line bundle over $X$. Then there exists a positive integer $m_0$ such that for all integer $m \geq m_0$, $L^m$ is very ample.*

A holomorphic line bundle $L$ is called globally generated if for each $x \in X$, there exists a holomorphic section $s$ of $L$ such that $s(x) \neq 0$. We say that $L$ is semi-ample if there exists a positive integer $m_0$ such that $L^{m_0}$ is globally generated.

A subset $D \subset X$ is called an analytic hypersurface if $D$ is locally given as the zero set $\{f = 0\}$ of a locally defined one holomorphic function vanishing of order 1. Denote $D_{\mathrm{reg}}$ the set of points $p \in D$ for which $D$ is a submanifold of $X$ near $p$. An analytic hypersurface $D$ is called irreducible if $D_{\mathrm{reg}}$ is connected. A divisor $D$ on $X$ is a formal finite sum $\sum_i d_i D_i$ where $d_i \in \mathbb{Z}$ and each $D_i$ is an irreducible analytic hypersurface of $X$. Suppose that a given divisor $D = \sum_{i=1}^k d_i D_i$, each irreducible analytic hypersurface $D_i \cap U_\alpha$ is given by a holomorphic function $f_{i\alpha} = 0$ vanishing on $D$ to order 1 over a sufficiently small open cover $U_\alpha$. The support of $D$ Supp$(D)$ is the union of the $D_i$ for each $i$ with $d_i \neq 0$. Then the divisor $D$ is given by a meromorphic function

$$f_\alpha = \prod_{i=1}^k f_{i\alpha}^{d_i} \quad \text{in } U_\alpha.$$

Define transition functions $t_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ on $U_\alpha \cap U_\beta$, which are holomorphic functions and nonvanishing on $U_\alpha \cap U_\beta$ and satisfy the conditions $(T)$. We write $[D]$ for the associated holomorphic line bundle, which is independent of choice of local defining functions. Suppose that $f'_{i\alpha}$ is another locally defined holomorphic function which gives $D_i \cap U_\alpha$. Then there exists a holomorphic function $h_{i\alpha}$ which does not have any zero in $U_\alpha$ such that $f'_{i\alpha} = f_{i\alpha} h_{i\alpha}$. By defining

$$f'_\alpha = \prod_{i=1}^k f'_{i\alpha}, \quad h_\alpha = \prod_{i=1}^k h_{i\alpha} \quad \text{in } U_\alpha,$$

we have $f'_\alpha = f_\alpha h_\alpha$ and the transition function $t'_{\alpha\beta} = \frac{f'_\alpha}{f'_\beta}$ on $U_\alpha \cap U_\beta$ is related to $t_{\alpha\beta}$ by $t'_{\alpha\beta} = h_\alpha t_{\alpha\beta} h_\beta^{-1}$, which means that two transition functions $\{t_{\alpha\beta}\}$, $\{t'_{\alpha\beta}\}$ define an equivalent holomorphic line bundle, so the associated holomorphic line bundle $[D]$ is well-defined independent of choice of local defining functions.

Let $f$ be a meromorphic function on a complex manifold $X$. Write Zero$(f)$ as the set of zeros of $f$, where zeros of $f$ means zeros of $g$ locally given by $f = \frac{g}{h}$ for relatively prime holomorphic functions $g$, $h$. And we define Pole$(f) = $ Zero$(\frac{1}{f})$. For an irreducible analytic hypersurface $D$ in a complex manifold $X$, we choose local coordinate chart $(U, (z_1, \ldots, z_n))$ around a non-singular point $p \in D$ with $D \cap U = \{z_n = 0\}$. In the case of $D \cap U \subset$ Zero$(f)$, we define an integer $\nu_D(f)$ by choosing maximum of a positive integer $m$ satisfying

$$g(z_1, \ldots, z_n) = z_n^m g'(z_1, \ldots, z_n),$$

where $g$ is the holomorphic function appeared in $f = \frac{g}{h}$ above and $g'$ is another holomorphic function. Here, note that the definition of $\nu_D(f)$ is independent of choice of a point $p$ since $\nu_D(f)$ is constant in a neighborhood of $p$ and the set of non-singular points is connected. In the case of $D \cap U \subset$ Pole$(f)$, we define $\nu_D(f) = -\nu_D(\frac{1}{f})$.

Define the following two effective divisors

$$(f)_0 = \sum_{D \subset \mathrm{Zero}(f)} \nu_D(f) D, \quad (f)_\infty = \sum_{D \subset \mathrm{Pole}(f)} (-\nu_D(f)) D,$$

and then we define a principal divisor $(f)$ of $f$ by

$$(f) = (f)_0 - (f)_\infty.$$

A divisor $D$ is called a Cartier divisor if there exists for any $x \in \mathrm{Supp}(D)$, a open neighborhood $U$ of $x$ and a meromorphic function $f$ such that

$$D|_U = (f).$$

Two divisors $D, D' \in \mathrm{Div}(X)$, $\mathrm{Div}(X)$ is an Abelian group called a divisor group, are called linearly equivalent if there exists a meromorphic function $f \neq 0$ such that

$$D - D' = (f).$$

The set of all principal divisors is subgroup of $\mathrm{Div}(X)$ and we write it $\mathrm{Div}_l(X)$. All equivalence classes of holomorphic line bundles on a complex manifold $X$ is an Abelian group with tensor product as an operation. We write it $\mathrm{Pic}(X)$ and call a Picard group. By applying Theorem 2.1.1, we have the following result:

**Theorem 2.2.2.** Let $X$ be a compact complex manifold. If there exists a positive holomorphic line bundle $L$ over $X$, we have the isomorphism

$$\mathrm{Div}(X)/\mathrm{Div}_l(X) \overset{\cong}{\to} \mathrm{Pic}(X).$$

Let $C$ be a curve on $X$, which means that it is an analytic subvariety of dimension 1. If $C$ is smooth, then we define the intersection number

$$(L \cdot C) = \int_C R_h,$$

where $h$ is any Hermitian metric on $L$. By Stokes' Theorem, $(L \cdot C)$ is independent of choice of $h$. Since Stokes' Theorem still holds for analytic subvarieties (cf. [28, p.33]), even if $C$ is not smooth, we integrate over $C_{\mathrm{reg}}$ and we can define $(L \cdot C)$ as well. We say that a holomorphic line bundle $L$ is nef if $(L \cdot C) \geq 0$ for all curves $C$ on $X$. For a divisor $D$, we define the intersection number by $(D \cdot C) = ([D] \cdot C)$.

## 2.3 Minimal non-Kähler compact complex surfaces

The Kodaira-Enriques classification (cf. [3, p.244]) tells us that minimal non-Kähler compact complex surfaces fall into one of the followings: When the Kodaira dimension Kod $= 1$, they are minimal non-Kähler properly elliptic surfaces. If the Kodaira dimension Kod $= 0$, then they are primary or secondary Kodaira surfaces. Compact complex surfaces with Kod $= -\infty$ and the first Betti number $b_1 = 1$, they are called class $VII$ surfces. Minimal surfaces in this class are called class $VII_0$ surfaces. In the case of Kod $= -\infty$, then they are of class $VII_0$. Class $VII_0$ surfaces are classified into three cases by the second Betti number $b_2$ as follows.

(1) $b_2 = 0$ : Hopf surfaces or Inoue surfaces (cf. [10], [32], [62]).

(2) $b_2 = 1$ : These are classified into Kato surfaces (cf. [63]).

(3) $b_2 > 1$ : Still unclassified (cf. [20]).

A properly elliptic surface is an elliptic surface with its Kodaira dimension 1. A simple example is the product of two curves, one elliptic and the other of genus at least 2. A primary Kodaira surface is a surface with $b_1 = 3$, admitting a holomorphic locally trivial fibration over an elliptic curve with an elliptic curve as typical fibre. A secondary Kodaira surface is a surface admitting a primary Kodaira surface as unramified covering. They are elliptic fibre spaces over rational curves with $b_1 = 1$.

Speaking of the classification of class $VII_0$ surfaces, the only compact complex surfaces known with Kod $= -\infty$, $b_1 = 1$ were the Hopf surface for ages. In 1972, Inoue introduced the example which now called Inoue surfaces [32], whose second Betti number vanish. In 1976, Bogomolov claimed that class $VII_0$ surfaces with $b_2 = 0$ are completely classified under the additional condition that they do not contain curves [10]. After that, finally Teleman completed the classification in the case $b_2 = 0$ [62]. In 1974, Inoue constructed examples with $b_2 > 0$ in [33], which are called the Inoue-Hirzebruch surfaces.

Hopf surfaces are defined by $H = \mathbb{C}^2 \setminus \{0\}/\sim$, where $(z_1, z_2) \sim (\alpha z_1, \beta z_2)$ for $\alpha, \beta \in \mathbb{C}^*$ with $|\alpha| = |\beta| \neq 1$. The Hopf surface $H$ is diffeomorphic to $S^1 \times S^3$. The diffeomorphism $H \overset{\cong}{\to} S^1 \times S^3$ is realized by sending a representative $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$ to $(r, \frac{\mathbf{z}}{r})$, where $r = \sqrt{|z_1|^2 + |z_2|^2}$ and note that $S^1 \cong \mathbb{R}^+/(r \sim |\alpha| r)$.

Inoue surfaces were firstly introduced by Inoue in [32]. They form three families, $S_M$, $S^+_{N,p,q,r;\mathbf{t}}$ and $S^-_{N,p,q,r}$. First of all, we construct the Inoue surface $S_M$. Let $M \in \mathrm{SL}(3, \mathbb{Z})$ be a matrix with one real eigenvalue $\lambda > 1$ and two complex conjugate eigenvalues $\mu \neq \bar{\mu}$. Let $(l_1, l_2, l_3)$ be a real eigenvector for $M$ with eigenvalue $\lambda$ and $(m_1, m_2, m_3)$ be an eigenvector with eigenvalue $\mu$. Let $G_M$ be the group of automorphisms of $\mathbb{C} \times H$, where $H$ is the upper half plane in $\mathbb{C}$ generated by

$$f_0(z_1, z_2) = (\mu z_1, \lambda z_2), \quad f_i(z_1, z_2) = (z_1 + m_i, z_2 + l_i)$$

for $(z_1, z_2) \in \mathbb{C} \times H$, $1 \leq i \leq 3$. We define $S_M$ to be the quatient surafce $(\mathbb{C} \times H)/G_M$, which is a $T^3$-torus bundle over a circle. We consider the subgroup $\tilde{G}_M \subset G_M$ generated by $f_1$, $f_2$ and $f_3$, which is isomorphic to $\mathbb{Z}^3$ and acts on $\mathbb{C} \times H$ properly discontinuous and freely, with quotient the product $T^3 \times \mathbb{R}_{>0}$. The projection $\pi : T^3 \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is induced by $(z_1, z_2) \mapsto \mathrm{Im} z_2$ for $(z_1, z_2) \in \mathbb{C} \times H$. Since $f_0$ descends to a map $T^3 \times \mathbb{R}_{>0} \to T^3 \times \mathbb{R}_{>0}$, we obtain that

$$S_M = (T^3 \times \mathbb{R}_{>0})/\langle f_0 \rangle,$$

and since $\alpha \in \mathbb{R}_{>1}$, $f_0$ maps $T_y = \pi^{-1}(y)$ to $T_{\alpha y} = \pi^{-1}(\alpha y)$. Especially, we have a diffeomorphism $F_0 : T_1 \to T_\alpha$ induced by $f_0$. Then we have that $S_M$ is diffeomorphic to the quotient space $(T^3 \times [1, \alpha])/\sim$, where $(p, 1) \sim (F_0(p), \alpha)$.

We next construct $S^+_{N,p,q,r;\mathbf{t}}$. Let $N = (n_{ij}) \in \mathrm{SL}(2, \mathbb{Z})$ with two real eigenvalues $\alpha > 1$ and $\frac{1}{\alpha}$. Let $(a_1, a_2)$ and $(b_1, b_2)$ be two real eigenvectors for $N$ with eigenvalues $\alpha$ and $\frac{1}{\alpha}$, respectively. Fix integers $p, q, r \in \mathbb{Z}$ with $r \neq 0$ and a complex number $\mathbf{t} \in \mathbb{C}$. Define

$$e_i := \frac{1}{2} n_{i1}(n_{i1} - 1) a_1 b_1 + \frac{1}{2} n_{i2}(n_{i2} - 1) a_2 b_2 + n_{i1} n_{i2} b_1 a_2$$

for $i = 1, 2$. Using $N$, $a_i$, $b_i$, $p$, $q$, $r$, one gets two real numbers $(c_1, c_2)$ as solutions of the linear equation

$$(c_1, c_2) = (c_1, c_2) \cdot N^t + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q).$$

Let $G^+_N$ be the group of automorphism of $\mathbb{C} \times H$ generated by

$$f_0(z_1, z_2) = (z_1 + \mathbf{t}, \alpha z_2), \quad f_i(z_1, z_2) = (z_1 + b_i z_2 + c_i, z_2 + a_i)$$

for $i = 1, 2$ and

$$f_3(z_1, z_2) = (z_1 + \frac{b_1 a_2 - b_2 a_1}{r}, z_2)$$

for $(z_1, z_2) \in \mathbb{C} \times H$. We define $S^+_{N,p,q,r;\mathbf{t}}$ to be the quatient surface $(\mathbb{C} \times H)/G^+_N$, which is diffeomorphic to a bundle over a circle with fiber a compact 3-manifold $X$. We consider the subgroup $\tilde{G}^+_N \subset G^+_N$ generated by $f_1$, $f_2$ and $f_3$. Write $z_i = x_i + \sqrt{-1} y_i$ for $i = 1, 2$. For fixed $y_2 = \mathrm{Im} z_2$, the group $\tilde{G}^+_N$ acts on $\{(x_2, y_2, z_1) | x_2 \in \mathbb{R}, z_1 \in \mathbb{C}\} \cong \mathbb{R}^3$ properly discontinuous and freely, with quotient a compact 3-manifold $X_{y_2}$. Compact 3-manifolds $X_y$ for different values of $y$ are all diffeomorphic to a fixed compact 3-manifold $X$. We may consider that the group $\tilde{G}^+_N$ acts on $\mathbb{C} \times H$ with the quotient diffeomorphic to the product $X \times \mathbb{R}_{>0}$ with the projection $\pi : X \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ induced by $(z_1, z_2) \mapsto y_2$ and with $X_{y_2} = \pi^{-1}(y_2)$. Since $f_0$ descends to a map $X \times \mathbb{R}_{>0} \to X \times \mathbb{R}_{>0}$, we have that

$$S^+_{N,p,q,r;\mathbf{t}} = (X \times \mathbb{R}_{>0})/\langle f_0 \rangle.$$

Since $\alpha \in \mathbb{R}_{>1}$, $f_0$ maps $X_1$ to $X_\alpha$ and then induces a diffeomorphism $F_0$ of $X$ such that $S^+_{N,p,q,r;\mathbf{t}}$ is diffeomorphic to the quotient space $(X \times [1, \alpha])/\sim$, where $(p, 1) \sim (F_0(p), \alpha)$.

We finally construct $S^-_{N,p,q,r}$. Let $N = (n_{ij}) \in \mathrm{GL}(2, \mathbb{Z})$ with $\det N = -1$ and with two real eigenvalues $\alpha > 1$ and $-\frac{1}{\alpha}$. Let $(a_1, a_2)$ and $(b_1, b_2)$ be two real eigenvectors for $N$ with eigenvalues $\alpha$ and $-\frac{1}{\alpha}$, respectively. Fix integers $p, q, r \in \mathbb{Z}$ with $r \neq 0$. One gets two real numbers $(c_1, c_2)$ as solutions of the following linear equation

$$-(c_1, c_2) = (c_1, c_2) \cdot N^t + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q),$$

where $e_i$ for each $i = 1, 2$ is defined as in the case $S^+_{N,p,q,r;\mathbf{t}}$. Let $G^-_N$ be the group of automorphism of $\mathbb{C} \times H$ generated by

$$f_0(z_1, z_2) = (-z_1, \alpha z_2), \quad f_i(z_1, z_2) = (z_1 + b_i z_2 + c_i, z_2 + a_i)$$

for $i = 1, 2$ and

$$f_3(z_1, z_2) = (z_1 + \frac{b_1 a_2 - b_2 a_1}{r}, z_2)$$

for $(z_1, z_2) \in \mathbb{C} \times H$. We define $S^-_{N,p,q,r}$ to be the quatient surface $(\mathbb{C} \times H)/G^-_N$. Note that every surface $S^-_{N,p,q,r}$ has as an unramified double cover an Inoue surface $S^+_{N^2,p',q',r;0}$ for suitable integers $p', q'$. In fact, we have the involution of $S^+_{N^2,p',q',r;0}$: $\iota(z_1, z_2) = (-z_1, \alpha z_2)$ satisfies $\iota^2 = \mathrm{Id}$ and

$$S^-_{N,p,q,r} = S^+_{N^2,p',q',r;0}/\iota.$$

A Kato surface is a minimal compact complex surface $S$ with $b_2(S) > 0$ containing a global spherical shell. Kato showed that Kato surfaces have small analytic deformations that the blow-ups of primary Hopf surfaces at a finite number of points. Note that a compact complex surface $S$ is said to be a primary Hopf surface if and only if its fundamental group $\pi_1(S) \cong \mathbb{Z}$ and $b_2(S) = 0$ (cf. [3, (18.4) Theorem.]). In particular, they have an infinite cyclic fundamental group, and are non-Kähler [34]. Note that they are not always a modification of a Hopf surface. Indeed, none of compact complex sufaces constructed by Inoue in [33], which are class $VII_0$ surfaces with $b_2 > 0$ containing global spherical shells is a modification of a Hopf surface. Cosequently, we have that all compact complex surfaces constructed by Inoue in [33] are deformations of modification of primary Hopf surfaces. Examples of Kato surfaces include Inoue-Hirzebruch surfaces and Enoki surfaces. Kato surfaces always admit exactly $b_2$-rational curves.

A spherical shell in a compact complex surface $S$ is an open subset $V \subset S$ biholomorphic to a neighborhood $U$ of 3-sphere $S^3 \subset \mathbb{C}^2$. A spherical shell $V$ in $S$ is said to be global if $S \setminus V$ is connected. Otherwise, $V$ is said to be local. Any complex manifolds contain local spherical shells. But global spherical shells can be contained in only special types of manifolds. A class $VII_0$ surface $S$ with $b_2(S) > 0$ has at most $b_2(S)$-rational curves. All compact complex surface containing a global spherical shell may be constructed by a procedure due to Kato [34]. As a result, if a class $VII_0$ surface $S$ with $b_2(S) > 0$ admits a global spherical shell exactly $b_2(S)$-rational curves. In the classification of class $VII_0$ surfaces with $b_2 = 1$ above (2), they are classified into Kato surfaces since the global spherical shell conjecture was proven by Teleman in the case $b_2 = 1$ [63, Corollary 1.3]. The global spherical shell conjecture claims that all class $VII_0$ surfaces with $b_2 > 0$ have a global spherical shell. Kato surfaces are reasonably well understood, therefore a proof of this conjecture lead to a classification of the class $VII_0$ surfaces.

Since all known examples of class $VII$ surfaces with $b_2 > 0$ have global spherical shells, Kato conjectured that any class $VII_0$ surface with $b_2 > 0$ which has $b_2$-rational curves, contains a global spherical shell. By Doloussky-Oeljeklaus-Toma, the proof of Kato's conjecture was given in [20]. It follows that it is classified into Kato surfaces. Hence the classification problem for class $VII_0$ surfaces reduces to the existence of sufficiently many rational curves.

## 2.4 The Chern-Ricci flow

Let $M$ be a compact complex surface. We introduce the definition of the Gauduchon metric in the following.

**Definition 2.4.1.** A metric $g_0$ is called a Gauduchon metric on a compact complex manifold of complex dimension $n$ if $g_0$ is a Hermitian metric whose associated $(1,1)$-form $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}}dz_i \wedge d\bar{z}_j$ satisfies

$$\partial\bar{\partial}(\omega_0^{n-1}) = 0.$$

We will also refer to the associated $(1,1)$-form $\omega_0$ as a Gauduchon metric. The following states that there are lot of Gauduchon metrics on any compact complex manifold $X$ of complex dimension $n$.

**Proposition 2.4.1.** (cf. [24], [62, Proposition 1.1])Any Hermitian metric on $X$ is conformally equivalent to a Gauduchon metric. If $n \geq 2$, then this Gauduchon metric is unique up to a positive factor.

Now let $\omega_0$ be a Gauduchon metric on $M$. The Chern-Ricci flow $\omega(t)$ starting at $\omega_0$ is the flow of Gauduchon metrics

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = -\operatorname{Ric}(\omega(t)), \\ \\ \omega(t)|_{t=0} = \omega_0, \end{cases}$$

for $t \in [0, T)$ where $T = T(\omega_0)$ is a finite singular time with $0 < T \leq \infty$ stated by

$$T = \sup\{t \geq 0 | \exists \psi \in C^\infty(M, \mathbb{R}) \text{ with } \omega_0 - t\operatorname{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi > 0\}$$

and $\operatorname{Ric}(\omega_0)$ is the Chern-Ricci curvature of $\omega_0$, given locally by

$$\operatorname{Ric}(\omega_0) = -\sqrt{-1}\partial\bar{\partial}\log\omega_0^2,$$

which determines the Bott-Chern cohomology class denoted by $c_1^{BC}(M) \in H_{BC}^{1,1}(M, \mathbb{R})$, where

$$H_{BC}^{1,1}(M, \mathbb{R}) = \frac{\{d\text{-closed real } (1,1)\text{-forms}\}}{\{\sqrt{-1}\partial\bar{\partial}\psi | \psi \in C^\infty(M, \mathbb{R})\}}.$$

. We call the cohomology class $c_1^{BC}(M)$ the first Bott-Chern class of $M$. It is independent of the choice of Hermitian metrics. Here we have omitted a factor of $2\pi$.

Note that a compact complex manifold is said be in Fujiki's class $\mathcal{C}$ if it is bimeromorphic to a Kähler manifold. Class $\mathcal{C}$ includes all Moishezon manifolds since they are bimeromorphic to projective manifolds. If a compact complex manifold $M$ is in $\mathcal{C}$, then the first Bott-Chern class $c_1^{BC}(M) = 0$ if and only if the first Chern class $c_1(M) = 0$ in $H^2(M, \mathbb{R})$ (cf. [66]).

According to [72, Theorem 1.2], there exists a unique maximal solution to the Chern-Ricci flow $\omega(t)$ for $t \in [0, T)$. Note that if $\omega_0$ is Kähler, this flow is exactly a Kähler Ricci flow. Since the Kähler-Ricci flow preserves the Kähler condition, a solution of the Kähler Ricci flow starting at a Kähler metric is a family of Kähler metrics. If the volume of $M$

with respect to $\omega(t)$ tends to zero as $t \to T$, we say that the Chern-Ricci flow $\omega(t)$ is collapsing at $T$. Otherwise, we say that the Chern-Ricci flow $\omega(t)$ is non-collapsing at $T$. And [72, Theorem 1.3] tells us that on compact complex surface $M$ equipped with a Gauduchon metric $\omega_0$, $T$ can be rewritten as

$$T = \sup\Big\{T_0 \geq 0 \ \Big|\ \forall t \in [0, T_0],\ \int_M (\omega_0 - t\,\mathrm{Ric}(\omega_0))^2 > 0,\ \int_D (\omega_0 - t\,\mathrm{Ric}(\omega_0)) > 0,$$
$$\text{for all irreducible effective divisors } D \text{ with } D^2 < 0\Big\}.$$

Notice that for $t \in [0, T)$, the quantity $\int_M (\omega_0 - t\,\mathrm{Ric}(\omega_0))^2 = \int_M \omega(t)^2$ is the volume of $M$ with respect to $\omega(t)$ and $\int_D (\omega_0 - t\,\mathrm{Ric}(\omega_0)) = \int_D \omega(t)$ is the volume of $D$. Note that since a curve $C$ is an analytic subset with its codimension is 1, $C$ is given locally by the set of zero points of one holomorphic function. So there is a natural $1:1$ correspondence between curves and effective divisors and which tells us that saying that "irreducible effective divisor $D$ with $D^2 < 0$" is the same as that "irreducible curves $C$ with $C^2 < 0$".

The behavior of the Chern-Ricci flow on Hopf surfaces, Inoue surfaces and properly elliptic surfaces that Weinkove and Tosatti found is similar to the behavior of the Ricci flow on geometric 3-manifolds. We introduce thier discovery in the following:

**Theorem 2.4.1.** ([70, Theorem 1.6])We have

(1) Let $H$ be the Hopf surface. Then there exists an explicit solution $\omega(t)$ of the Chern-Ricci flow on $H$ for $t \in [0, \frac{1}{2})$ with

$$(H, \omega(t)) \overset{GH}{\to} (S^1, d), \quad \text{as } t \to \frac{1}{2},$$

where $d$ is the standard distance function on the unit circle $S^1 \subset \mathbb{R}$.

(2) Let $S$ be any Inoue surface. Then there exists an explicit solution $\omega(t)$ of the Chern-Ricci flow on $S$ for $t \in [0, \infty)$ with

$$\left(S, \frac{\omega(t)}{t}\right) \overset{GH}{\to} (S^1, d), \quad \text{as } t \to \infty,$$

where $d$ is the standard distance function on the unit circle $S^1 \subset \mathbb{R}$.

(3) Let $\pi : S \to C$ be any non-Kähler minimal properly elliptic surface. Then there exists an explicit solution $\omega(t)$ of the Chern-Ricci flow on $S$ for $t \in [0, \infty)$ with

$$\left(S, \frac{\omega(t)}{t}\right) \overset{GH}{\to} (C, d_{KE}), \quad \text{as } t \to \infty,$$

where $d_{KE}$ is the distance function on the Riemann surface $C$ induced by an orbifold Kähler-Einstein metric $\omega_{KE}$ on $C$ which satisfies $\mathrm{Ric}(\omega_{KE}) = -\omega_{KE}$ away from the images of the multiple fibers of $\pi$. We also have that $\pi^*\omega_{KE}$ is a smooth form on $S$ and $\frac{\omega(t)}{t} \to \pi^*\omega_{KE}$ smoothly on $S$.

Remark that it is not difficult to write down explicit solutions of the Chern-Ricci flow on also the Kodaira surfaces. In fact, there are explicit Chern-Ricci flat Gauduchon metrics on all these manifolds [74, (1.3)] and these give trivial solutions to the Chern-Ricci flow. Generally, Gill showed that on a compact Hermitian manifold $M$, whenever the first Bott-Chern class $c_1^{BC}(M) = 0$ (if $M$ is in Fujiki's class $\mathcal{C}$, equivalently $c_1(M) = 0$ in $H^2(M, \mathbb{R})$), the Chern-Ricci flow converges to a Chern-Ricci flat metric in any dimension (cf. [26]).

## 2.5 Pluripotential theory

Recent years, by mainly Kolodziej, the Pluripotential theory has been developed on Hermitian manifolds. The important tool in this theory, so called *modified comparison principle*, is a generalized version of the comparison principle of Bedford and Taylor. Let $(X, \omega)$ be a compact Hermitian manifold of complex dimension $n$. We set $d^c = \frac{\sqrt{-1}}{2\pi}(\bar{\partial} - \partial)$, $dd^c = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}$. We consider the "curvature" constant of the metric $\omega$ denoted by $B = B(\omega) > 0$ and it satisfies

$$-B\omega^2 \leq 2n dd^c\omega \leq B\omega^2, \quad = B\omega^3 \leq 4n^2 d\omega \wedge d^c\omega \leq B\omega^3.$$

**Definition 2.5.1.** A function $u : X \to [-\infty, +\infty)$ is $\omega$-plurisubharmonic ($\omega$-psh for short) if it is upper semi-continuous, $u \in L^1(X, \omega^n)$ and $\omega + dd^c u \geq 0$ on $X$ as a current. The set of all $\omega$-psh functions on $X$ is denoted by $\mathrm{PSH}(\omega)$.

With using partition of unity, we can define the Monge-Ampère operators $\omega_u^n$ for $u \in \mathrm{PSH}(\omega) \cap L^\infty(X)$ by applying the local argument in $\mathbb{C}^n$. We start with a local argument in a open set $\Omega \subset \mathbb{C}^n$.

**Definition 2.5.2.** Let $\omega$ be a Hermitian metric in $\mathbb{C}^n$ and $u : \Omega \to [-\infty, +\infty)$ be a upper semi-continuous function. Then $u$ is called $\omega$-psh if $u \in L^1_{\mathrm{loc}}(\Omega, \omega^n)$ and $\omega + dd^c u \geq 0$ in $\Omega$ as a current. We denote the set of these functions on $\Omega$ by $\mathrm{PSH}(\Omega, \omega)$.

According to Bedford and Taylor, we can define $\omega_{v_1} \wedge \cdots \wedge \omega_{v_k}$ for $v_1, \ldots, v_k \in \mathrm{PSH}(\Omega, \omega) \cap L^\infty(\Omega)$, $1 \leq k \leq n-1$. This is shown by proceeding induction over $k$. When $k = 1$, the definition is given by classical distribution theory. Suppose that for $1 \leq k \leq n-1$, the current $T := \omega_{v_1} \wedge \cdots \wedge \omega_{v_k}$ is well defined. We fix a small ball $\mathbb{B} \subset \Omega$ and a strictly psh function $\rho$ such that $dd^c\rho \geq 2\omega$ in $\mathbb{B}$. Set $\gamma := dd^c\rho - \omega$ and $u_l := \rho + v_l \in \mathrm{PSH}(\mathbb{B}) \cap L^\infty(\mathbb{B})$, then $T$ can be written in $\mathbb{B}$ as a linear combination of positive currents

$$(\spadesuit) \qquad dd^c u_{j_1} \wedge \cdots \wedge dd^c u_{j_l} \wedge \gamma^{k-l},$$

for $1 \leq j_1 < \cdots < j_l \leq k$, $1 \leq l \leq k$. By Demailly's regularization theorem for quasi-psh functions (cf. [11, Theorem 2.3), there are sequences of smooth $\omega$-psh function $\{v_l^j\}_{j=1}^\infty$ which decrease to $v_l$ for $1 \leq l \leq k$. Since $T$ is a linear combination of positive currents of the form ($\spadesuit$), we obtain from the result of Bedford and Taylor,

$$T = \lim_{j\to\infty} T_j = \lim_{j\to\infty} \omega_{v_1^j} \wedge \cdots \wedge \omega_{v_k^j} \quad \text{weakly}.$$

It follows that $T$ is a positive current and we obtain the following well defined formulas;

$$dT = \sum_{l=1}^{k} d\omega \wedge \omega_{v_1} \wedge \cdots \wedge \widehat{\omega}_{v_l} \wedge \cdots \wedge \omega_{v_k};$$

$$\begin{aligned} dd^c T &= 2 \sum_{1 \le l \le m \le k} d\omega \wedge d^c\omega \wedge \omega_{v_1} \wedge \cdots \wedge \widehat{\omega}_{v_l} \wedge \cdots \wedge \widehat{\omega}_{v_m} \wedge \cdots \wedge \omega_{v_k} \\ &\quad + \sum_{l=1}^{k} dd^c\omega \wedge \omega_{v_1} \wedge \omega_{v_1} \wedge \cdots \wedge \widehat{\omega}_{v_l} \wedge \cdots \wedge \omega_{v_k}, \end{aligned}$$

where $\widehat{\omega}_{v_l}$ implies that the term does not appear in the wedge product. So now we can define

$$dd^c u \wedge T := dd^c(uT) - du \wedge d^c T + d^c u \wedge dT - u dd^c T$$

for $u \in \mathrm{PSH}(\Omega, \omega) \cap L^\infty(\Omega)$. Let $\{u^j\}_{j=1}^\infty$ be a sequence of smooth $\omega$-psh functions decreasing to $u$. Then we have $dd^c u^j \wedge T_j$ converges weakly to $dd^c u \wedge T$ as $j \to \infty$. For any test form $\varphi$ of bidegree $(n-k-1, n-k-1)$, we have

$$du \wedge d^c T \wedge \varphi = -d^c u \wedge dT \wedge \varphi.$$

Hence

$$\omega_u \wedge T := \omega \wedge T + dd^c(uT) - 2du \wedge d^c T - u dd^c T$$

is a positive current of bidegree $(k+1, k+1)$. When $v_1 = \cdots = v_n = v \in \mathrm{PSH}(\Omega, \omega) \cap L^\infty(\Omega)$, we obtain the definition of the Monge-Ampère operator $\omega_v := \omega_v \wedge \cdots \wedge \omega_v$. Then the Bedford-Taylor convergence theorem on $\Omega$ can be stated as follows:

**Theorem 2.5.1.** (Bedford-Taylor [5])Let $v_1, \ldots, v_k \in \mathrm{PSH}(\Omega, \omega) \cap L^\infty(\Omega)$, $1 \le k \le n$. Suppose that the sequences of bounded $\omega$-psh functions $\{v_1^j\}_{j=1}^\infty, \ldots, \{v_k^j\}_{j=1}^\infty$ decrease (or uniformly converge) to $v_1, \ldots, v_k$ respectively. Then

$$\lim_{j \to \infty} \omega_{v_1^j} \wedge \cdots \wedge \omega_{v_k^j} = \omega_{v_1} \wedge \cdots \wedge \omega_{v_k} \quad \text{weakly.}$$

In particular, if $\{u_j\}_{j=1}^\infty \subset \mathrm{PSH}(\Omega, \omega) \cap L^\infty(\Omega)$ decreases (or uniformly converges) to $u \in \mathrm{PSH}(\Omega, \omega) \cap L^\infty(\Omega)$, then

$$\lim_{j \to \infty} \omega_{u_j}^n = \omega_u^n \quad \text{weakly.}$$

The same statement holds for functions in $\mathrm{PSH}(\omega) \cap L^\infty(X)$ on a compact Hermitian manifolds with arbitrary fixed Hermitian metric $\omega$. Note that $u \in \mathrm{PSH}(\omega)$ if and only if $u \in \mathrm{PSH}(\Omega, \omega)$ for any coordinate chart $\Omega \subset\subset X$.

We introduce the $L^1$-Chern-Levine-Nirenberg (CLN) inequality:

**Proposition 2.5.1.** ($L^1$-CLN inequality (cf. [17, Proposition 3.11])) Let $K, L \subset X$ be compact subsets with $L \subset K^\circ$. For any plurisubharmonic functions $V, u_1, \ldots, u_q$ on $X$ such that $u_1, \ldots, u_q$ are locally bounded, there is an inequality

$$||V dd^c u_1 \wedge \cdots \wedge dd^c u_q||_L \le C_{K,L} ||V||_{L^1(K)} ||u_1||_{L^\infty(K)} \cdots ||u_q||_{L^\infty(K)}.$$

We notice that all functions $u$ in $\mathrm{PSH}(\omega)$ normalized by the condition $\sup_X u = 0$ are uniformly integrable.

**Proposition 2.5.2.** ([19, Proposition 2.1])Let $u \in \mathrm{PSH}(\omega)$ be a function with $\sup_X u = 0$. Then there exists a constant $C$ dependent only on $X$, $\omega$ such that

$$\int_X |u|\omega^n \leq C.$$

We need the following two lemmata, which can be given by the proof in [2, Theorem 3.1] and the regularization result in [6], for proving the modified comparison principle:

**Lemma 2.5.1.** For $T := \omega_{v_1} \wedge \cdots \wedge \omega_{v_{n-1}}$, where $v_1 \ldots, v_{n-1} \in \mathrm{PSH}(\omega) \cap L^\infty(X)$ and for $\varphi, \psi \in \mathrm{PSH}(\omega) \cap L^\infty(X)$ we have

$$\int_{\{\varphi<\psi\}} dd^c\psi \wedge T \leq \int_{\{\varphi<\psi\}} dd^c\varphi \wedge T + \int_{\{\varphi<\psi\}} (\psi - \varphi) dd^c T.$$

The following is a weaker version of the comprison principle.

**Lemma 2.5.2.** Let $\varphi, \psi \in \mathrm{PSH}(\omega) \cap L^\infty(X)$. Then there exists a constant $C_n = C(n) > 0$ such that, for $B \sup_{\{\varphi<\psi\}}(\psi - \varphi) \leq 1$,

$$\int_{\{\varphi<\psi\}} \omega_\psi^n \leq \int_{\{\varphi<\psi\}} \omega_\varphi^n + C_n B \sup_{\{\varphi<\psi\}} (\psi - \varphi) \sum_{k=0}^{n-1} \int_{\{\varphi<\psi\}} \omega_\varphi^k \wedge \omega^{n-k}.$$

**Theorem 2.5.2.** (Modified comparison principle (cf. [48, Theorem 2.3])) Let $(X, \omega)$ be a compact Hermitian manifold and suppose that $\varphi, \psi \in \mathrm{PSH}(\omega) \cap L^\infty(X)$. Fix $0 < \delta < 1$ and set $m(\delta) = \inf_X(\varphi - (1 - \delta)\psi)$. Then, for any $0 < s < \frac{\delta^3}{16B}$, we have

$$\int_{\{\varphi<(1-\delta)\psi+m(\delta)+s\}} \omega_{(1-\delta)\psi}^n \leq \left(1 + \frac{Cs}{\delta^n}\right) \int_{\{\varphi<(1-\delta)\psi+m(\delta)+s\}} \omega_\varphi^n,$$

where $C$ is a uniform constant depending only on $n$, $B$.

We use the notation $\mathrm{Vol}_\omega(E) := \int_E \omega^n$ for any Borel set $E \subset X$, and we write $L^p(\omega^n)$ for $L^p(X, \omega^n)$. We denote for a Borel set $E$,

$$\mathrm{cap}_\omega(E) := \sup\left\{ \int_E (\omega + dd^c\rho)^n : \rho \in \mathrm{PSH}(\omega), 0 \leq \rho \leq 1 \right\}.$$

From the argument in [41, Lemma 4.], [42, Lemma 4.3], we obtain the following result:

**Proposition 2.5.3.** ([19, Corollary 2.4])There are a univarsal number $0 < \alpha = \alpha(X, \omega)$ and a uniform constant $0 < C = C(X, \omega)$ such that for any Borel subset $E \subset X$

$$\mathrm{Vol}_\omega(E) \leq C \exp\left(\frac{-\alpha}{\mathrm{cap}_\omega^{\frac{1}{n}}(E)}\right).$$

Let $h : \mathbb{R}_+ \to (0, \infty)$ be an increasing function such that

$$\int_1^\infty \frac{dx}{x h(x)^{\frac{1}{n}}} < +\infty.$$

In particular, $\lim_{x\to\infty} h(x) = +\infty$. We call such a function $h$ admissible. If $h$ is admissible, then so is $Ah$ for any number $A > 0$. Define

$$F_h(x) := \frac{x}{h(x^{-\frac{1}{n}})}.$$

For such $F_h$, we consider the family of bounded $\omega$-psh functions such that their Monge-Ampère measures satisfy

$$(\clubsuit)_\omega \qquad \int_E \omega_\varphi^n \le F_h(\mathrm{cap}_\omega(E)),$$

for any Borel set $E \subset X$, where $\omega_\varphi = \omega + dd^c\varphi$. From Proposition 2.4.3, it follows that

**Corollary 2.5.1.** Let $\varphi \in \mathrm{PSH}(\omega) \cap L^\infty(X)$. If $\omega_\varphi^n = f\omega^n$ for $0 \le f \in L^p(\omega^n)$, $p > 1$, then $\omega_\varphi^n$ satisfies $(\clubsuit)_\omega$ for the admissible function $h_p(x) = C||f||^{-1}_{L^p(\omega^n)} \exp(ax)$ with some universal number $a > 0$.

Thanks to the modified comparison principle (Theorem 2.4.2), we can prove the following crucial lemma:

**Lemma 2.5.3.** ([48, Lemma 5.4])Fix $0 < \delta < 1$. Let $\varphi, \psi \in \mathrm{PSH}(\omega) \cap L^\infty(X)$ be such that $-1 \le \psi \le 0$. Set $m(\delta) = \inf_X(\varphi - (1-\delta)\psi)$ and $U(\delta, s) = \{\varphi < (1-\delta)\psi + m(\delta) + s\}$. For any $0 < s, t \le \frac{1}{3}\min\{\delta^n, \frac{\delta^3}{16B}\}$, one has

$$(1-\delta)^n t^n \mathrm{cap}_\omega(U(\delta, s)) \le (1 + C) \int_{U(\delta, s+4(1-\delta)t)} \omega_\varphi^n.$$

**Remark 2.5.1.** By rescalimg $t$, the statement above can be restated in the following way: For any $0 < s \le \frac{1}{3}\min\{\delta^n, \frac{\delta^3}{16B}\}$, $0 < t \le \frac{4}{3}(1-\delta)\min\{\delta^n, \frac{\delta^3}{16B}\}$, we have

$$t^n \mathrm{cap}_\omega(U(\delta, s)) \le 4^n C \int_{U(\delta, s+t)} \omega_\varphi^n,$$

where $C$ is a dimensional constant.

Then the next essential statement can be proven with using the result in Remark 2.4.1.

**Proposition 2.5.4.** ([48, Theorem 5.3])Fix $0 < \delta < 1$. Let $\varphi, \psi \in \mathrm{PSH}(\omega) \cap L^\infty(X)$ be such that $\varphi \le 0$, and $-1 \le \psi \le 0$. Set $m(\delta) = \inf_X(\varphi - (1-\delta)\psi)$, and

$$\delta_0 := \frac{1}{3}\min\{\delta^n, \frac{\delta^3}{16B}, 4(1-\delta)\delta^n, 4(1-\delta)\frac{\delta^3}{16B}\}.$$

Suppose that $\omega_\varphi^n$ satisfies $(\clubsuit)$ for an admissible function $h$. Then, for $0 < D < \delta_0$,

$$D \le \kappa(\mathrm{cap}_\omega(U(\delta, D))),$$

where $U(\delta, D) = \{\varphi < (1-\delta)\psi + m(\delta) + D\}$, and the function $\kappa$ is defined on the interval $(0, \mathrm{cap}_\omega(X))$ by the formula

$$\kappa(s^{\frac{1}{n}}) := 4C_n\Big(\frac{1}{h(s)^{\frac{1}{n}}} + \int_s^\infty \frac{dx}{xh(x)^{\frac{1}{n}}}\Big),$$

with a dimensional constant $C_n$.

Then we obtain the following a *priori* estimate.

**Corollary 2.5.2.** ([48, Corollary 5.6])Suppose that $\varphi \in \mathrm{PSH}(\omega) \cap L^\infty(X)$, $\sup_X \varphi = 0$ satisfies

$$\omega_\varphi^n = f\omega,$$

where $0 \leq f \in L^p(\omega^n)$, $p > 1$. Then there exists a constant $0 < H = H(h)$, depending only on $h$, $X$, and $\omega$ such that

$$-H \leq \varphi \leq 0.$$

We finally obtain the existence of continuous solutions to the complex Monge-Ampère equation $\omega_\varphi^n = f\omega^n$, where $0 \leq f \in L^p(\omega^n)$, $p > 1$, and understood in the weak sense of currents.

**Theorem 2.5.3.** ([48, Theorem 0.1])Let $0 \leq f \in L^p(\omega^n)$, $p > 1$, be such that $\int_X f\omega^n > 0$. There exist a constant $c > 0$ and a function $\mathrm{PSH}(\omega) \cap C^0(X)$ satisfying the equation

$$\omega_u^n = cf\omega^n,$$

in the weak sense of currents.

Notice that one can get a weak stability statement from the argument in the proof of the theorem above:

**Corollary 2.5.3.** ([48, Corollary 5.10])Let $\{u_j\}_{j=1}^\infty \subset \mathrm{PSH}(\omega) \cap C^0(X)$ be such that $\sup_X u_j = 0$. Suppose that for every $j \geq 1$,

$$\omega_{u_j}^n = f_j\omega^n,$$

where $f_j$'s are uniformly bounded in $L^p(\omega^n)$, $p > 1$. If $\{u_j\}$ is Cauchy in $L^1(\omega^n)$, then it is Cauchy in $\mathrm{PSH}(\omega) \cap C^0(X)$.

We can obtain the stability theorem for strictly positive $L^p$-function $f$:

**Theorem 2.5.4.** ([50, Theorem A.])Let $0 \leq f, g \in L^p(\omega^n)$, $p > 1$, be such that $\int_X f\omega^n > 0$, $\int_X g\omega^n > 0$. Consider two continuous $\omega$-psh solutions of the Monge-Ampère equation

$$\omega_u^n = f\omega^n, \quad \omega_v^n = g\omega^n,$$

with $\sup_X u = \sup_X v = 0$. Assume that

$$f \geq c_0 > 0$$

for some constant $c_0 > 0$. Fix $0 < \alpha < \frac{1}{n+1}$. Then, there exists $C = C(c_0, \alpha, ||f||_{L^p}, ||g||_{L^p})$ such that

$$||u - v||_{L^\infty} \leq C||f - g||_{L^p}^\alpha.$$

Then we can develop the statement in Theorem 2.4.3 as follows:

**Corollary 2.5.4.** ([50, Corollary 3.9])Suppose that $0 < c_0 \leq f \in L^p(\omega^n)$, $p > 1$. Then there is a unique $u \in \mathrm{PSH}(\omega) \cap C^0(X)$, $\sup_X u = 0$, and unique $c > 0$ such that

$$\omega_u^n = cf\omega^n.$$

At the last of this section, we introduce that in the case of the right hand side of the Monge-Ampère equation is smooth, Weinkove and Tosatti proved the following theorem:

**Theorem 2.5.5.** (Weinkove, Tosatti [68, Corollary 1.])Let $(X, \omega)$ be a compact Hermitian manifold of complex dimension $n \geq 2$. For every smooth real-valued function $F$ on $X$, there exist a unique real number $b$ and a unique real valued function $u$ on $X$ solving

$$(\omega + dd^c u)^n = e^{F+b}\omega^n, \quad \text{with}$$

$$\omega + dd^c u > 0, \quad \sup_X u = 0.$$

## 2.6 Orbifolds

An orbifold is a space locally modelled on the quotients of Euclidean space by finite groups. These local models are glued together by maps compatible with the finite group actions. Let $X$ be a Hausdorff topological space. For an open set $U \subset X$, we define an $n$-dimensional orbifold chart on $X$ (cf. [1], [16]).

**Definition 2.6.1.** An $n$-dimensional orbifold chart is a 3-tuple $(\tilde{U}, \Gamma, \pi)$, where

(1) $\tilde{U}$ is a connected open subset of $\mathbb{R}^n$,

(2) $\Gamma$ is a finite group of homeomorphisms of $\tilde{U}$,

(3) $\pi : \tilde{U} \to U$ is a map defined by $\pi = \bar{\pi} \circ p$, where $p : \tilde{U} \to \tilde{U}/\Gamma$ is the orbit map and $\bar{\pi} : \tilde{U}/\Gamma \to X$ is a map that induces a homeomorphism of $\tilde{U}/\Gamma$ onto an open subset $U \subset X$.

Define an embedding $\lambda : (\tilde{U}, \Gamma_1, \pi_1) \hookrightarrow (\tilde{V}, \Gamma_2, \pi_2)$ between such orbifold charts is a smooth embedding $\lambda : \tilde{U} \hookrightarrow \tilde{V}$ with $\pi_1 = \pi_2 \circ \lambda$. Note that given two embeddings of orbifold charts $\lambda, \mu : (\tilde{U}, \Gamma_1, \pi_1) \hookrightarrow (\tilde{V}, \Gamma_2, \pi_2)$, there exists a unique $\gamma \in \Gamma_2$ such that $\mu = \gamma \cdot \lambda$. As a result, an embedding of orbifold charts $\lambda : (\tilde{U}, \Gamma_1, \pi_1) \hookrightarrow (\tilde{V}, \Gamma_2, \pi_2)$ induces an injective group homomorphism $\lambda : \Gamma_1 \hookrightarrow \Gamma_2$.

Soppose that $(\tilde{U}_i, \Gamma_i, \pi_i)$ with $\pi_i(\tilde{U}_i) = U_i$ for open sets $U_i \subset X$ are orbifold charts on $X$ for $i = 1, 2$. We say the charts are compatible if given a point $x \in U_1 \cap U_2$, there exist an open neighborhood $U_3 \subset U_1 \cap U_2$ of the point $x$ and an orbifold chart $(\tilde{U}_3, \Gamma_3, \pi_3)$ with $\pi_3(\tilde{U}_3) = U_3$ such that there are two embeddings $\lambda_i : (\tilde{U}_3, \Gamma_3, \pi_3) \hookrightarrow (\tilde{U}_i, \Gamma_i, \pi_i)$ for $i = 1, 2$.

Now we define an $n$-dimensional orbifold atlas $\mathcal{U}$ on a Hausdorff topological space $X$.

**Definition 2.6.2.** An $n$-dimensional orbifold atlas on $X$ is a collection $\mathcal{U} = \{(\tilde{U}_j, \Gamma_j, \pi_j)\}_{j \in J}$ of compatible $n$-dimensional orbifold chart which cover $X$.

**Definition 2.6.3.** An $n$-dimensional orbifold $Q$ consists of a paracompact Hausdorff topological space $X_Q$ together with an $n$-dimensional orbifold atlas of charts $\mathcal{U}_Q$.

Every orbifold atlas for $X$ is contained in a unique maximal atlas, and two orbifold atlases are equivalent if and only if they are contained in the same maximal atlas.

An orbifold atlas $\mathcal{U}$ is said to refine another orbifold atlas $\mathcal{V}$ if for every orbifold chart in $\mathcal{U}$ there exists an embedding into some orbifold chart of $\mathcal{V}$. Two orbifold atlasses are said to be equivalent if they have a common refinement. A paracompact Hausdorff space equipped with an equivalence class $[\mathcal{U}]$ of $n$-dimensional orbifold atlases is called an effective orbifold.

Remark that an orbifold is smooth if the finite groups $\Gamma$ act via diffeomorphisms and the charts are compatible via diffeomorphisms $h$.

**Example 2.6.1.** A manifold $X$ is an orbifold where each $\Gamma_j$ is the trivial group, so that we have that $\pi_j : \tilde{U}_j \to U_j$ is homeomorphic.

**Definition 2.6.4.** Let $Q_1 = (X_{Q_1}, \mathcal{U}_{Q_1})$, $Q_2 = (X_{Q_2}, \mathcal{U}_{Q_2})$ be two orbifolds. A map $f : X_{Q_1} \to X_{Q_2}$ is a smooth map between orbifolds if for any point $x \in X_{Q_1}$, there are charts $(\tilde{U}_1, \Gamma_1, \pi_1)$ around $x$ and $(\tilde{U}_2, \Gamma_2, \pi_2)$ around $f(x)$ such that $f$ maps $\pi_1(\tilde{U}_1)$ into $\pi_2(\tilde{U}_2)$ and $f$ can be lifted to a smooth map $\tilde{f} : \tilde{U}_1 \to \tilde{U}_2$ such that $\pi_2 \circ \tilde{f} = f \circ \pi_1$.

**Definition 2.6.5.** Two orbifolds $Q_1 = (X_{Q_1}, \mathcal{U}_{Q_1})$ and $Q_2 = (X_{Q_2}, \mathcal{U}_{Q_2})$ are diffeomorphic if there are smooth maps of orbifolds $f_1 : X_{Q_1} \to X_{Q_2}$ and $f_2 : X_{Q_2} \to X_{Q_1}$ with $f_1 \circ f_2 = \mathrm{id}_{X_{Q_2}}$ and $f_2 \circ f_1 = \mathrm{id}_{X_{Q_1}}$.

If $\Gamma$ is a discrete group and $X$ is a Hausdorff topological space such that $\Gamma$ acts on $X$, we say that this action is properly discontinuously if given two points $x, y \in X$, there are open neighborhoods $U_x$ of $x$ and $U_y$ of $y$ for which $(\gamma U_x) \cap U_y \neq \emptyset$ for only finitely many $\gamma \in \Gamma$, which is equivalent to that $X/\Gamma$ is Hausdorff, or equivalent to that for given $x \in X$, each isotropy subgroup $\Gamma_x = \{\gamma \in \Gamma | \gamma x = x\}$ is finite. We have the following propositions for a group acting properly discontinuously (cf. [16]):

**Proposition 2.6.1.** If $X$ is a manifold and $\Gamma$ is a group acting properly discontinuously on $X$, then $X/\Gamma$ has the structure of an orbifold.

**Proposition 2.6.2.** If a group $\Gamma$ acts properly discontinuously on a manifold $X$ and $\Gamma' \subset \Gamma$ is a subgroup, then $X/\Gamma' \to X/\Gamma$ is an orbifold covering projection.

The condition that an orbifold $Q$ is covered by a manifold is equivalent to that $Q$ is the quotient of a group acting properly discontinuously on a manifold. Then $Q$ is also said a good orbifold. Notice that not every orbifold is covered by a manifold.

**Example 2.6.2.** Let $H$ be the complex upper half plane. The projective special linear group $PSL(2, \mathbb{Z})$ is the quotient of $SL(2, \mathbb{Z})$ by its center $\{I, -I\}$. The group $PSL(2, \mathbb{Z})$ is isomorphic to the group of linear fractional transformations of $H$ of the form $z \mapsto \frac{az+b}{cz+d}$ with $ad - bc = 1$ for $z \in H$, $a, b, c, d \in \mathbb{Z}$. Let $\Gamma_i$ be the isotropy group of $i \in H$. Then we have $\Gamma_i = SO(2)$. By considering the map $\xi : SL(2, \mathbb{R})/\Gamma_i \to H$ defined by $\xi(\gamma) = \gamma i$, we have that $H$ is homeomorphic to $SL(2, \mathbb{R})/SO(2)$. In addition to it, since $PSL(2, \mathbb{Z})$ is a discrete subgroup of $SL(2, \mathbb{R})$, we have that $PSL(2, \mathbb{Z})$ acts properly discontinuously on $H$. Then by applying Proposition 2.6.1, we have that $H/PSL(2, \mathbb{Z})$ has the structure of an orbifold.

# Chapter 3

# Convergence in the Gromov-Hausdorff sense and the Chern-Ricci flow on complex surfaces

## 3.1 On the Kähler case and other classifications

In [70], Weinkove and Tosatti showed the convergence result in the Gromov-Hausdorff sense for the Chern-Ricci flow $\omega(t)$ starting at a Gauduchon metric $\omega_0$ on a compact complex surface $M$ when $\omega(t)$ is non-collapsing at a singular time $T < \infty$ with the assumption that

(†) there exists $f \in C^\infty(M, \mathbb{R})$ and a smooth real $(1,1)$ form $\beta$ on $N$ such that
$$\omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f = \pi^*\beta,$$

where $\pi : M \to N$ is the blow-down map of finitely many disjoint $(-1)$ curves, $N$ is a complex surface. The non-collapsing condition to $\omega(t)$ at a singular time $T < \infty$ is equivalent to the condition $\int_M (\omega_0 - T\,\mathrm{Ric}(\omega_0))^2 > 0$. As we see in [70, Remark 1.4], the condition (†) holds automatically in the projective Kähler case with a Kähler metric $\omega_0$: Denote by $X$ the blow-up of $\mathbb{CP}^n$ at the point $y_0$. Let $\pi : X \to \mathbb{CP}^n$ be the blow-down map, which sends the exceptional divisor $E$ to $y_0 \in \mathbb{CP}^n$. Kähler classes on $X$ can be written as $\alpha = b\pi^*[H] - a[E]$ for $0 < a < b$, where $H$ is a hyperplane in $\mathbb{CP}^n$ and $[H] = c_1(\mathcal{O}_{\mathbb{CP}^n}(1))$. We consider a solution of the Kähler-Ricci flow starting at a form $\omega_0$ in a Kähler class $\alpha_0 = b_0\pi^*[H] - a_0[E]$ where $a_0$ and $b_0$ satisfy the condition $a_0(n+1) < b_0(n-1)$. Since we have $K_X \cong \pi^* K_{\mathbb{CP}^n} \otimes L^{n-1}$, where $K_X$, $K_{\mathbb{CP}^n}$ are the canonical line bundles on $X$ and $\mathbb{CP}^n$ respectively and $L$ is the holomorphic line bundle associated to the divisor $E$, we have $c_1(K_X) = (n-1)c_1(L) + c_1(K_{\mathbb{CP}^n}) = (n-1)[E] - (n+1)\pi^*[H]$, where we used that $c_1(L) = [E]$ and $c_1(K_{\mathbb{CP}^n}) = -c_1(\mathbb{CP}^n) = -(n+1)c_1(\mathcal{O}_{\mathbb{CP}^n}(1))$. Then the singular time of the Kähler-Ricci flow $T$ is equal to $\frac{a_0}{n-1}$, and for $\kappa := b_0 - \frac{n+1}{n-1}a_0 > 0$, we obtain $[\omega_0] + Tc_1(K_X) = [\kappa\pi^*\omega_{FS}]$, where $\omega_{FS}$ is the Fubini-Study metric on $\mathbb{CP}^n$ (cf. [59, Example 1.1.2]). If $M$ is projective, the complex surface $N$ in (†) is also projective (cf.

[3, IV(6.7)Corollary]) and imbedded into $\mathbb{CP}^l$ for some integer $l > 0$. Then, from the observation above, the condition (†) holds with $\beta = \omega_{FS}|_N$ multiplied by some positive constant, where $\omega_{FS}|_N$ is the Fubini-Study metric on $\mathbb{CP}^l$ restricted to $N$. On the other hand, the condition (†) does not hold automatically in the non-Kähler case in general. But in the special case, we do not need to assume the condition (†). Actually, we can artificially construct such initial data $\omega_0$ on $M$ satisfying the condition (†) (cf. [70, Remark 3.1]): For a constant $C > 0$ sufficiently large, there exists a smooth real function $\tilde{f}$ on $M$ so that $\omega_0 := C\pi^*\omega_N + T\operatorname{Ric}(\omega_M) + \sqrt{-1}\partial\bar{\partial}\tilde{f}$ is Gauduchon, where $\omega_M$, $\omega_N$ are Gauduchon metrics on $M$ and $N$ respectively. We can check that the Chern-Ricci flow starting at $\omega_0$ is non-collapsing at $T$ and satisfying the condition (†) with $\beta = C\omega_N$. However, it is not enough since we would like to continue the Chern-Ricci flow on new surfaces and repeat contractions of exceptional divisors along the flow until we reach to the minimal model. Hence, it is necessary for us to remove the condition (†).

We hope that the condition (†) holds automatically on any non-Kähler compact complex surfaces contain some disjoint $(-1)$-curves. For simplicity, we consider a map $\pi$ blows down the only one $(-1)$-curve $E$ on $M$ to a point $y_0 \in N$. Then we have a biholomorphism $\pi|_{M\setminus E} : M \setminus E \overset{\cong}{\to} N \setminus \{y_0\}$. Our main results are as follows:

**Theorem 3.1.1.** Let $M$ be a non-Kähler compact complex surface and $\pi : M \to N$ be a blow-down map of the $(-1)$-curve $E$ on $M$ to the point $y_0 \in N$, where $N$ is a compact complex surface. Let $\omega_0$ be a Gauduchon metric on $M$. Suppose that we have

$$\int_M (\omega_0 - T\operatorname{Ric}(\omega_0))^2 > 0, \quad \text{and} \quad \int_D (\omega_0 - T\operatorname{Ric}(\omega_0)) > 0$$

for all irreducible curves $D$ on $M$ with $D^2 = (D \cdot D) < 0$ different from $E$, where $T$ is a finite singular time of the Chern-Ricci flow $\omega(t)$ starting at $\omega_0$ for $t \in [0, T)$, $0 < T < \infty$. Then, there exist a smooth real function $u_0'$ on $M$ and a Gauduchon metric $\hat{\omega}_N$ on $N$ such that

$$\omega_0 - T\operatorname{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}u_0' = \pi^*\hat{\omega}_N.$$

Note that the first condition $\int_M (\omega_0 - T\operatorname{Ric}(\omega_0))^2 > 0$ implies that the volume of $M$ with respect to the Chern-Ricci flow $\omega(t)$ stay strictly positive as $t \to T$, that is, $\omega(t)$ is non-collapsing at $T$, and that the second condition $\int_D (\omega_0 - T\operatorname{Ric}(\omega_0)) > 0$ for all irreducible curves $D$ on $M$ with $D^2 = (D \cdot D) < 0$ different from $E$ means that there is no other $(-1)$-curve on $M$ except for $E$. This is from the following proposition:

**Proposition 3.1.1.** (cf. [3, (2.2)Proposition])Let $X$ be a compact complex surface. An irreducible curve $D \subset X$ is a $(-1)$-curve if and only if

$$D^2 < 0 \quad \text{and} \quad (K_X \cdot D) < 0.$$

If there exists an irreducible curve $D$ with $D^2 < 0$ such that $\int_D (\omega_0 - T\operatorname{Ric}(\omega_0)) = 0$, then we have $(K_X \cdot D) < 0$. Then $D$ must be another $(-1)$-curve on $M$, which contradicts to our assumption that $E$ is the only one $(-1)$-curve on $M$. Hence, under the condition that there is an only one $(-1)$-curve, the second condition always holds. Here, "$M$ is non-Kähler" means that there is no Kähler metric on $M$. The condition (†) is not always true when only $M$ is Kähler. The condition (†) holds if $M$ is Kähler and the initial metric $\omega_0$ is Kähler. Note that there are examples when $M$ is projective, $\omega_0$ is not Kähler, and (†) fails (cf. [70, Remark 3.7]). That is why we assume that $M$ is non-Kähler in the statements above.

In the case of $\mathrm{Kod}(M) = 1$ with the first Betti number $b_1(M) =$ even, or the case of $\mathrm{Kod}(M) = 2$, then the surface $M$ is projective and Kähler. So the condition (†) holds automatically with $\omega_0$ Kähler as we confirmed. Plus, the surface $M$ with $\mathrm{Kod}(M) = 0$, or with $\mathrm{Kod}(M) = -\infty$ and $b_1(M) \neq 1$, and the surface of class $VII$ with the second Betti number $b_2(M) \geq 0$ can be excluded as well. Hence our interest inclines only to the surface with $\mathrm{Kod}(M) = 1$ and $b_1(M) =$ odd. Remark that their first Betti numbers are odd. We will see more specific reasons in Remark 3.1.2.

We here note the definition of the Kodaira dimension. In our case, it is stated in the following way:

$$\mathrm{Kod}(M) := \limsup_{m\to\infty} \frac{\log \dim H^0(M, mK_M)}{\log m} \in \{-\infty, 0, 1, 2\},$$

where $K_M = \bigwedge^n T^*M$ is the canonical line bundle of $M$ and $H^0(M, mK_M)$ is the vector space of holomorphic sections of the holomorphic line bundle $K_M^m = mK_M$.

We introduce the Buchdahl's Nakai-Moishezon criterion.

**Lemma 3.1.1.** (Buchdahl's Nakai-Moishezon criterion [14, Theorem.])Let $M$ be a compact complex surface equipped with a Gauduchon metric $\omega_G$ and let $\psi$ be a smooth real $\partial\bar{\partial}$-closed $(1,1)$-form satisfying

$$\int_M \psi^2 > 0, \quad \int_M \psi \wedge \omega_G > 0, \quad \int_D \psi > 0$$

for every irreducible effective divisor $D \subset M$ with $D^2 = (D \cdot D) < 0$. Then there exists a smooth real function $f$ on $M$ such that

$$\psi + \sqrt{-1}\partial\bar{\partial} f > 0.$$

**Remark 3.1.1.** In the condition (†), a smooth real $(1,1)$ form $\beta$ is not supposesed to be positive definite. But since $\beta$ is then $\partial\bar{\partial}$-closed and we actually may apply the Buchdahl's Nakai-Moishezon criterion. We obviously have $\int_N \beta^2 = \int_M (\omega_0 - T\,\mathrm{Ric}(\omega_0))^2 > 0$ and for any irreducible curve $C \subset N$ with $C^2 < 0$, we have $\int_C \beta = \int_{\pi^* C} (\omega_0 - T\,\mathrm{Ric}(\omega_0)) > 0$, where note that we have $\pi^* C \neq E$ and $(\pi^* C)^2 < 0$ for curves $C$ with $C^2 < 0$. Let $\omega_G$ be a Gauduchon metric on $N$. Then we have

$$\int_N \beta \wedge \omega_G = \int_M \pi^*\beta \wedge \pi^*\omega_G = \int_M (\omega_0 - T\,\mathrm{Ric}(\omega_0)) \wedge \pi^*\omega_G = \lim_{t\to T^-} \int_M \omega(t) \wedge \pi^*\omega_G \geq 0.$$

For $\delta > 0$ sufficiently small, $\omega_G + \delta\beta$ is positive definite and becomes Gauduchon, then

$$\int_N \beta \wedge (\omega_G + \delta\beta) \geq \delta \int_N \beta^2 > 0.$$

Therefore, all assumptions in Lemma 1.1 are satisfied and there exists a smooth real function $h_N$ on $N$ such that $\omega_N := \beta + \sqrt{-1}\partial\bar{\partial}h_N > 0$, which is a Gauduchon metric on $N$, and then we have

$$\omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}f = \pi^*\omega_N$$

where $f := f_0' + \pi^*h_N$ is a smooth real function on $M$.

Then we see that $\omega_0 - \pi^*\omega_N$ is $d$-closed and which tells us that we obtain

$$(T_0)^p_{jl}(\omega_0)_{p\bar{k}} = (\pi^*T_N)^p_{jl}(\pi^*\omega_N)_{p\bar{k}},$$

where $T_0$ and $\pi^*T_N$ are torsion tensors with respect to $\omega_0$ and $\pi^*\omega_N$ respectively. This is used crucially in the argument of [72, Proposition 3.1].

**Remark 3.1.2.** The followings are the reasons that we may exclude the cases of the surface $M$ with $\mathrm{Kod}(M) = 2$, $0$ and $-\infty$ from our concern. All notations and settings are the same as in Proposition 3.1.1.

(1) There is a possibility that the surface $M$ with $\mathrm{Kod}(M) \geq 0$ has some $(-1)$-curves since $M$ with $\mathrm{Kod}(M) \geq 0$ and $(K_M \cdot D) < 0$ for the canonical divisor $K_M$ and an effective divisor $D \subset M$, where $(K_M \cdot D)$ indicates their intersection number, then $D$ contains a $(-1)$-curves (cf. [3, III(2.3)Proposition.]).

  (a) In the case of $\mathrm{Kod}(M) = 2$, if the surface $M$ does not have any $(-1)$-curves, then $M$ is projective since its algebraic dimension is equal to 2; the complex dimension of $M$ (cf. [3, IV(6.5)Corollary]). And its blow-ups are also projective (cf. [3, IV(6.7)Corollary]). Hence the surface $M$ with $\mathrm{Kod}(M) = 2$ can be excluded from our concern since the condition (†) is satisfied automatically with $\omega_0$ Kähler.

  (b) In the case of $\mathrm{Kod}(M) = 0$, they are divided into five cases (cf. [3, p.244]): Enriques surfaces, bi-elliptic surfaces, Kodaira surfaces, K 3-surfaces and tori. Firstly, Enriques surfaces and bi-elliptic surfaces are projective Kähler since their algebraic dimensions are equal to 2. Then, they can be excluded from our concern as in the case (a) above. If the surface $M$ is a primary or secondary Kodaira surface, it has torsion canonical bundle, which means that some power $lK_M$, $l \geq 1$ is holomorphically trivial. Then we have $c_1^{BC}(M) = 0$ and $T = \infty$ (cf. [28, Theorem 1.1], [72]). And if the surface $M$ is a K 3-surface or torus, we have $c_1(M) = 0$ and $T = \infty$. Note that a complex torus does not contain any rational curves. These are in the case (1) of Proposition 3.2.1.

(2) If $\mathrm{Kod}(M) = -\infty$ and $b_1(M) = 1$, then the surface $M$ is called a surface of class $VII$. Surfaces of class $VII$ with $b_2(M) = 0$ are completely classified and they are either Inoue surfaces or Hopf surfaces.

  (a) An Inoue surface, which is a $T^3$-torus bundle over $S^1$, does not have any curves and then we have $T = \infty$.

(b) On a Hopf surface $H$, diffeomorphic to $S^1 \times S^3$, we have $\int_H \omega_0 \wedge c_1^{BC}(H) > 0$ for any Gauduchon metric $\omega_0$ on $H$. It implies that $\int_H (\omega_0 - T\,\mathrm{Ric}(\omega_0))^2 = 0$. Note that $(H, \omega(t))$ converges to $S^1$ in the Gromov-Hausdorff sense, where $\omega(t)$ is a solution of the Chern-Ricci flow on $H$. Since every curve on $H$ is homologous to zero, the flow exists precisely as long as the volume stays positive and then it collapses. Hence this is included in the case (2) of Proposition 3.2.1 (cf. [70, Theorem 1.6], [72]).

(c) If the surface $M$ of class $VII$ has $b_2(M) =: n > 0$, then we can observe that we have $\int_M c_1^{BC}(M)^2 = -n$ (cf. [63, p.494]) and then we obtain $\int_M \omega(t)^2 \to 0$ as $t \to T$ for some $0 < T < \infty$, where $\omega(t)$ is the solution of the Chern-Ricci flow starting from a Gauduchon metric. There might be some $(-1)$-curves on $M$ but as we see that $\omega(t)$ is collapsing, we may exclude this case from our concern. There are lots of examples of minimal surfaces in this case and a complete classification has not been done yet except for the case $b_2(M) = 1$ (cf. [63]).

(3) When $\mathrm{Kod}(M) = -\infty$ with $b_1(M) \neq 1$, and if $M$ is additionally minimal, it is limited to be a ruled surface of genus $g \geq 1$ or a minimal rational surface. They both are projective Kähler since their algebraic dimensions are equal to 2 (cf. [3, p.244]).

Here we recall the definition of the convergence in the sense of Gromov-Hausdorff. Then we need to define the Gromov-Hausdorff distance $d_{GH}((M, d_M), (N, d_N))$ between two metric spaces $(M, d_M)$, $(N, d_N)$. Which is defined to be the infimum of all $\epsilon > 0$ such that the following holds: there exist maps $F : M \to N$ and $G : N \to M$ such that

$$|d_M(x_1, x_2) - d_N(F(x_1), F(x_2))| \leq \epsilon, \quad \text{for all } x_1, x_2 \in M$$

and

$$d_M(x, G \circ F(x)) \leq \epsilon, \quad \text{for all } x \in M$$

and the two symmetric properties for $N$ also hold. We do not require the maps $F$ and $G$ to be continuous. In this sense, we say that $(M, d_{\omega(t)})$ converges to $(N, d_T)$ as $t \to T^-$ in the Gromov-Hausdorff sense if we have

$$d_{GH}((M, d_{\omega(t)}), (N, d_T)) \to 0$$

as $t \to T^-$, where $d_{\omega(t)}$, $d_T$ are distance functions induced from $\omega(t)$ and $\omega(T)$ respectively.

From the result of Theorem 3.1.1, we can restate [70, Theorem 1.3] as follows:

**Theorem 3.1.2.** Let $M$ be a non-Kähler compact complex surface and $\pi : M \to N$ be a blow-down map of finitely many disjoint $(-1)$-curves on $M$ onto a complex surface $N$. Let $\omega_0$ be a Gauduchon metric on $M$. We assume that the Chern-Ricci flow $\omega(t)$ starting from $\omega_0$ is non-collapsing at a singular time $T < \infty$. Then there exists a distance function $d_T$ on $N$ such that $(N, d_T)$ is a compact metric space and $(M, d_{\omega(t)})$ converges in the Gromov-Hausdorff sense to $(N, d_T)$ as $t \to T^-$, where $d_{\omega(t)}$ are distance functions induced from the metrics $\omega(t)$.

The way of the proof for Theorem 3.1.2 is totally the same as in [70] except for the results in Proposition 3.1.1 and Theorem 3.1.1.

## 3.2 The Chern-Ricci flow and some convergence results

Let $M$ be a compact complex surface and $g_0$ be a Gauduchon metric on $M$. In local complex coordinates, the associated $(1,1)$-form is given by $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}}dz_i \wedge d\bar{z}_j$, which we will also often refer to as a Gauduchon metric. A Gauduchon metric on $M$ is a Hermitian metric $g$ whose associated $(1,1)$-form $\omega$ satisfies $\partial\bar{\partial}\omega = 0$. Note that Gauduchon showed that every Hermitian metric on a compact complex surface is conformal to a unique Gauduchon metric [24].

The Chern-Ricci flow $\omega(t)$ starting at $\omega_0$ is the flow of Gauduchon metrics

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = -\operatorname{Ric}(\omega(t)), \\ \\ \omega(t)|_{t=0} = \omega_0, \end{cases}$$

for $t \in [0,T)$ where $T = T(\omega_0)$ is a finite singular time with $0 < T \leq \infty$ stated by

$$T = \sup\{t \geq 0 | \exists \psi \in C^\infty(M,\mathbb{R}) \text{ with } \omega_0 - t\operatorname{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi > 0\}$$

and $\operatorname{Ric}(\omega_0)$ is the Chern-Ricci curvature of $\omega_0$, given locally by $\operatorname{Ric}(\omega_0) = -\sqrt{-1}\partial\bar{\partial}\log\omega_0^2$, which determines the Bott-Chern cohomology class denoted by $c_1^{BC}(M) \in H^{1,1}_{BC}(M,\mathbb{R})$. We call it the first Bott-Chern class of $M$. Note that we omit a factor of $2\pi$ in this paper, and it is independent of the choice of Hermitian metrics. There exists a unique maximal solution to the Chern-Ricci flow on $[0,T)$ [72, Theorem 1.2].

There is a strong relationship between the Kähler-Ricci flow and the minimal model program. A minimal surface is a surface with no $(-1)$-curves. A $(-1)$-curve is defined to be smooth rational curves with self-intersection $-1$. In [59, Theorem 1.2], Song and Weinkove showed that along the Kähler-Ricci flow starting at a Kähler metric $\omega_0$ with $[\omega_0] \in H^{1,1}(X,\mathbb{Q})$ on projective algebraic surfaces $X = X_0, X_1, \ldots, X_k$, algebraic contractions can be proceeded along the flow and in the end of this process, which tells us that $X_k$ is Fano or ruled surface, or the singular time $T_k$ of the Kähler-Ricci flow on $X_k$ is infinite and $X_k$ has no exceptional curves of the first kind, i.e, no $(-1)$-curves. As we confirmed, we do not need to assume the condition (†) in this case since it automatically holds in the projective Kähler case with an initial Kähler metric $\omega_0$. In [72], Weinkove and Tosatti conjectured that this algebraic procedure can be proceeded along also the Chern-Ricci flow on any compact comple surfaces. They showed the following result:

**Proposition 3.2.1.** ([72, Theorem 1.5])Let $M$ be a compact complex surface with a Gauduchon metric $\omega_0$, and let $[0,T)$ be the maximal existence time of the Chern-Ricci flow starting from $\omega_0$. Then

(1) If $T = \infty$ then $M$ is minimal

(2) If $T < \infty$ and $\int_M(\omega_0 - T\operatorname{Ric}(\omega_0))^2 = 0$, then $M$ is either birational to a ruled surface or it is a surface of class $VII$ (and in this case it cannot be an Inoue surface)

(3) If $T < \infty$ and $\int_M(\omega_0 - T\operatorname{Ric}(\omega_0))^2 > 0$, then $M$ contains $(-1)$-curves.

Furthermore, if $M$ is minimal then $T = \infty$ unless $M$ is $\mathbb{CP}^2$, a ruled surface, a Hopf surface or a surface of class $VII$ with $b_2(M) > 0$, in which cases (2) holds.

When $M$ is not minimal and (3) occurs, we expect that the Chern-Ricci flow will contract finitely many $(-1)$-curves and can be uniquely continued on a new surface. They conjectured that this process can be repeated until one obtains a minimal surface, or ends up in the case (2). It is crucial to show that we can remove the condition (†) for proving this conjecture. Additionally, they have proved in any complex dimension, the following result can be realized under the condition (†):

**Proposition 3.2.2.** ([72, Theorem 1.6])Assume that there exists a holomorphic map between compact Hermitian manifolds $\pi : (M, \omega_0) \to (N, \omega_N)$ blowing down the exceptional divisor $E$ on $M$ to a point $y_0 \in N$. In addition, we suppose the condition (†) with $T < \infty$. Then the solution $\omega(t)$ to the Chern-Ricci flow starting at $\omega_0$ converges in $C^\infty$ on compact subsets of $M \setminus E$ to a smooth Hermitian metric $\omega_T$ on $M \setminus E$.

They also showed the convergence in the Gromov-Hausdorff sense under the assumption (†) on a compact complex surface with a Gauduchon metric $\omega_0$ with using some arguments in [59] (cf. [70, Theorem 1.3]).

As we stated "finitely many disjoint $(-1)$-curves" in Theorem 3.1.1 and other parts, we can check that $(-1)$-curves $E_1, \ldots, E_k$ on $M$ are finite and disjoint each other, giving rise to a map $\pi : M \to N$ onto a complex surface $N$, blowing down each $E_i$ to a point $y_i \in N$. Now we assume that $E_1, E_2$ are irreducible distinct $(-1)$-curves with $(E_1 \cdot E_2) \geq 0$ and $\int_{E_1}(\omega_0 - T\,\mathrm{Ric}(\omega_0)) = \int_{E_2}(\omega_0 - T\,\mathrm{Ric}(\omega_0)) = 0$, then we show that they are disjoint. The Poincaré-Lelong formula tells us that we have an expression of the divisor $E_1 + E_2$ in the sence of currents:

$$(s_{E_1+E_2}) = \eta + \sqrt{-1}\partial\bar{\partial}\log|s_{E_1+E_2}|^2_{h_{E_1+E_2}},$$

where $s_{E_1+E_2}$ is a holomorphic defining section of holomorphic line bundle $[E_1 + E_2]$ associated to the divisor $E_1 + E_2$. $s_{E_1+E_2}$ goes to zero of order 1 along $E_1 + E_2$. $(s_{E_1+E_2})$ denotes its principal divisor corresponding to $E_1 + E_2$. $h_{E_1+E_2}$ is a smooth Hermitian metric on $[E_1 + E_2]$ and $\eta := c_{h_{E_1+E_2}}$ is the Chern form, which is a smooth $d$-closed real $(1,1)$-form represents $c_1([E_1 + E_2])$. We here introduce an important lemma for our argument:

**Lemma 3.2.1.** ([13, Lemma 4.])Let $M$ be a compact complex surface and let $\psi$, $\omega$ be smooth real $\partial\bar{\partial}$-closed $(1,1)$-forms on $M$. Assume that $\int_M \omega^2 > 0$.Then we have

$$\Big(\int_M \omega \wedge \psi\Big)^2 \geq \Big(\int_M \omega^2\Big)\Big(\int_M \psi^2\Big)$$

with equality if and only if $\psi = c\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for some constant $c$ and a smooth real function $\varphi$ on $M$.

Since $\omega_0 - T\operatorname{Ric}(\omega_0)$, $\eta$ are $\partial\bar{\partial}$-closed and $\int_M (\omega_0 - T\operatorname{Ric}(\omega_0))^2 > 0$, we may apply Lemma 2.1 and obtain

$$\begin{aligned} 0 = \Big(\int_{E_1} (\omega_0 - T\operatorname{Ric}(\omega_0)) + \int_{E_2} (\omega_0 - T\operatorname{Ric}(\omega_0))\Big)^2 \;&=\; \Big(\int_{(s_{E_1+E_2})} (\omega_0 - T\operatorname{Ric}(\omega_0))\Big)^2 \\ &=\; \Big(\int_M \eta \wedge (\omega_0 - T\operatorname{Ric}(\omega_0))\Big)^2 \\ &\geq\; \Big(\int_M \eta^2\Big)\Big(\int_M (\omega_0 - T\operatorname{Ric}(\omega_0))^2\Big). \end{aligned}$$

Then, since $\int_M (\omega_0 - T\operatorname{Ric}(\omega_0))^2 > 0$, we have

$$(E_1 + E_2)^2 = \int_M \eta^2 \leq 0.$$

If $\int_M \eta^2 = 0$ holds, then we have

$$\Big(\int_M \eta \wedge (\omega_0 - T\operatorname{Ric}(\omega_0))\Big)^2 = \Big(\int_M \eta^2\Big)\Big(\int_M (\omega_0 - T\operatorname{Ric}(\omega_0))^2\Big) = 0$$

and $\eta = \sqrt{-1}\partial\bar{\partial}\varphi$ for some smooth real function $\varphi$.

Recall the fact that an irreducible curve $C$ on $M$ is a $(-1)$-curve if and only if $C^2 < 0$ and $(K_M \cdot C) < 0$ (cf. [1, III(2.2)Proposition.]). Combining these, we have

$$0 > (K_M \cdot (E_1 + E_2)) = -\int_{(s_{E_1+E_2})} c_1^{BC}(M) = -\int_M \eta \wedge c_1^{BC}(M) = 0,$$

which is a contradiction. Hence, we conclude that $(E_1 + E_2)^2 < 0$, which gives us that $0 \leq (E_1 \cdot E_2) < 1$. Therefore, we have $(E_1 \cdot E_2) = 0$, that is to say $E_1, E_2$ are disjoint each other. Additionally, the set of all these $(-1)$-curves is finite, for instance $E_1, \ldots, E_k$, because they give linearly independent classes in homology.

## 3.3 Proof of Theorem 3.1.1

Since the condition (†) holds automatically in the projective Kähler case with an initial Kähler metric $\omega_0$ as we see in Section 1, our concern is only for non-Kähler surfaces contain some $(-1)$-curves. For giving a proof in the non-Kähler case, we crucially use [64, Remark 3.3], which says that a complex surface with its first Betti number $b_1 = \text{odd}$ has only finitely many irreducible curves with negative self-intersection. We can confirm that the non-Kähler compact complex surfaces $M$ contain some $(-1)$-curves is only the case of the surfaces $M$ with $\mathrm{Kod}(M) = 1$ and $b_1(M) = \text{odd}$. The reason is as follows: In [13], Buchdahl showed that a compact complex surface with its first Betti number $b_1 \equiv 0 \pmod 2$ admits a Kähler metric. Moishezon showed that a smooth Moishezon variety is projective if and only if it admits a Kähler metric (cf. [47]). By applying Riemann-Roch theorem formula and Grauert's ampleness criterion for surfaces, we obtain the fact that a compact complex surface is projective if and only if there exists a line bundle $L$ with $c_1(L)^2 > 0$ (cf. [3, IV(6.2)Theorem.]), which indicates that a compact complex surface is projective if and only if it has algebraic dimension 2, i.e., it is Moishezon (cf. [3, IV(6.5)Corollary]). When the dimension is more than 2, it does not hold in general.

Note that a Moishezon manifold is a compact complex manifold which is bimeromorphic to a projective manifold. Equivalently, it is defined to be that it is a compact complex manifold admitting a big line bundle (cf. [69]). This gives us that if $M$ is a minimal complex surface with $\mathrm{Kod}(M) = 2$, then it is a Moishezon surface whose first Betti number is even, and the surface $M$ is a projective Kähler surface. So, its blow-ups are also projective (cf. [3, IV(6.7)Corollary]). When $\mathrm{Kod}(M) = 1$ with $b_1(M) = \text{even}$, then the surface $M$ admits a Kähler metric. In this case, its minimal model is a minimal properly elliptic surface and whose algebraic dimension must be equal to 2 since it also has a Kähler metric, which means that it is Moishezon and then it is projective. So its blow-ups are also projective. As we see in Remark 3.1.2, we do not need to consider the case of $\mathrm{Kod}(M) = 0$, an Inoue surface, a Hopf surface or the case of $\mathrm{Kod}(M) = -\infty$ with $b_1(M) \neq 1$, the case of $\mathrm{Kod}(M) = -\infty$ with $b_1(M) = 1$ and $b_2(M) > 0$.

For these reasons, the remaining case is of the surfaces $M$ with $\mathrm{Kod}(M) = 1$ and $b_1(M) = \text{odd}$. Hence if it is additionally minimal, the surface $M$ is limited to be a non-Kähler minimal properly elliptic surface. Therefore, we may assume that $M$ has only finitely many irreducible curves with negative self-intersection. Let $C$ be any such curve. The notations and settings are the same as in the previous sections such as that $E$ is the only one $(-1)$-curve contained in $M$ which is blown down to the point $y_0 \in N$. Then we have either $C = E$ or $\int_C (\omega_0 - T\,\mathrm{Ric}(\omega_0)) > 0$, since $E$ is the only curve whose intersection with $\omega_0 - T\,\mathrm{Ric}(\omega_0)$ is zero. Let $h$ be a smooth Hermitian metric on the holomorphic line bundle $[E]$. Since $[E]$ has self-intersection $-1$, its curvature $R_h$, locally given by $R_h = -\sqrt{-1}\partial\bar\partial \log h$, satisfies $\int_E R_h = -1$.

When we take $\varepsilon > 0$ sufficiently small, then we claim that we have

$$\int_M (\omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon R_h)^2 > 0, \quad \int_M (\omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon R_h) \wedge \omega_G > 0$$

for any Gauduchon metric $\omega_G$. The first one is easy because we have assumed that

$\int_M(\omega_0 - TRic(\omega_0))^2 > 0$. The second can be showed since we have

$$\int_M (\omega_0 - T\,\mathrm{Ric}(\omega_0)) \wedge \omega_G = \lim_{t\to T^-} \int_M \omega(t) \wedge \omega_G \geq 0$$

and if we have $\int_M(\omega_0 - T\,\mathrm{Ric}(\omega_0)) \wedge \omega_G = 0$, then we have $\int_M \omega_G^2 \leq 0$ from Lemma 2.1, which is a contradiction. Therefore we have

$$\int_M (\omega_0 - T\,\mathrm{Ric}(\omega_0)) \wedge \omega_G > 0$$

for any Gauduchon metric $\omega_G$ and we obtain

$$\int_M (\omega_0 - T\,\mathrm{Ric}(\omega_0)) \wedge \omega_G - \varepsilon \int_M R_h \wedge \omega_G > 0$$

for sufficiently small $\varepsilon > 0$.

In the case of $C = E$, we have

$$\int_E (\omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon R_h) = -\varepsilon(E \cdot E) = \varepsilon > 0,$$

and if $C$ is different from $E$ then

$$\int_C (\omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon R_h) = \int_C (\omega_0 - T\,\mathrm{Ric}(\omega_0)) - \varepsilon(C \cdot E).$$

Since there are only finitely many such curves $C$, it follows that we can choose $\varepsilon > 0$ sufficiently small so that

$$\int_C (\omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon R_h) > 0,$$

for all such $C$.

Therefore, we can apply the Buchdahl's Nakai-Moishezon criterion (Lemma 3.1.1) to $\omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon R_h$ for sufficiently small $\varepsilon > 0$ and then we obtain the following result: For a smooth Hermitian metric $h'$ on $[E]$ and for each sufficiently small $\varepsilon > 0$, there exists a smooth function $f'_\varepsilon$ on $M$ such that

$$\omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon R_{h'} + \sqrt{-1}\partial\bar{\partial} f'_\varepsilon > 0$$

where $R_{h'}$ is the curvature of $h'$.

Additonally, we need the following Lemma for proving our result.

**Lemma 3.3.1.** (cf. [28, p.187]) Let $\pi : M \to N$ be a blow-down map of the $(-1)$-curve $E$ on $M$ and let $\omega_N$ be a Hermitian metric on $N$. We can choose a smooth Hermitian metric $h$ on the holomorphic line bundle $[E]$ associated to $E$ with its curvature $R_h$ such that

$$\pi^*\omega_N - \varepsilon R_h > 0$$

for any sufficiently small $\varepsilon > 0$.

From these results, for our Hermitian metric $\omega_N$ on $N$ and for any sufficiently small $\varepsilon > 0$, we have the equivalence depends on $\varepsilon$ between the metrics

$$\pi^*\omega_N - \varepsilon R_h > 0$$

and

$$\omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon R_{h'} + \sqrt{-1}\partial\bar{\partial} f'_\varepsilon > 0.$$

Hence, there exists a positive constant $C_\varepsilon > 1$ depends on $\varepsilon$ such that

$$(\ddagger) \quad \frac{1}{C_\varepsilon}(\pi^*\omega_N - \varepsilon R_h) \leq \omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon R_{h'} + \sqrt{-1}\partial\bar{\partial} f'_\varepsilon \leq C_\varepsilon(\pi^*\omega_N - \varepsilon R_h)$$

for any $\varepsilon > 0$ sufficiently small.

We will choose a sequence $\{\varepsilon_j\}_{j=1}^\infty$ such that $\varepsilon_j \to 0$ as $j \to \infty$. The inequality ($\ddagger$) replaced $\varepsilon$ with $\varepsilon_j$ holds for $j$ chosen sufficiently large since sufficiently small $\varepsilon$ was chosen arbitrary.

Set $\tilde{\omega}_{\varepsilon_j} := \omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon_j R_{h'} + \sqrt{-1}\partial\bar{\partial} f'_{\varepsilon_j}$ and then it is a Hermitian metric for each $j \geq j_0$ for some sufficiently large $j_0$. We fix such a large number $j_0$. By applying the Tosatti-Weinkove theorem (Theorem 2.5.5 [68, Corollary 1]), the Hermitian version of Yau's theorem, for each $j \geq j_0$, there exist a unique smooth function $u_{\varepsilon_j}$ on $M$ and a unique positive constant $c_{\varepsilon_j}$ such that

$$(\flat)_j \qquad (\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial} u_{\varepsilon_j})^2 = c_{\varepsilon_j}(\pi^*\omega_N - \varepsilon_j R_h)^2$$

with $\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial} u_{\varepsilon_j} > 0$ and $\sup_M(f'_{\varepsilon_j} + u_{\varepsilon_j}) = 0$ (cf. [67, Section 2], [69, Section 3]). Set $u'_{\varepsilon_j} := f'_{\varepsilon_j} + u_{\varepsilon_j}$. By applying Proposition 2.5.2, we see that the set

$$\{u'_{\varepsilon_j} \in \mathrm{PSH}(\omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon_j R_{h'}); \sup_M u'_{\varepsilon_j} = 0\}$$

is compact in $L^1(M, \omega_0^2)$, since $u'_{\varepsilon_j} \in \mathrm{PSH}(C\omega_0)$ for some uniform constant $C > 0$. Hence, after passing a subsequence, still writing $u_{\varepsilon_j}$ and $\varepsilon_j \to 0$ as $j \to \infty$, we may assume that $\{u'_{\varepsilon_j}\}_j$ is Cauchy in $L^1(M, \omega_0^2)$, that is, we have that

$$u'_{\varepsilon_j} \to u'_0 \in L^1(M, \omega_0^2)$$

in $L^1(M, \omega_0^2)$-toplology as $j \to \infty$.

We may normalize $f'_{\varepsilon_j}$ by $\sup_M f'_{\varepsilon_j} = 0$ after subtraction of fixed constants for each $j \geq j_0$. Since we have $f'_{\varepsilon_j} \in \mathrm{PSH}(C\omega_0)$ for some uniform constant $C > 0$, thanks to Proposition 2.5.2, after passing a subsequence, $f'_{\varepsilon_j}$ converges to $f'_0$ in $L^1(M, \omega_0^2)$-topology as $j \to \infty$. So we have $f'_0 \in L^1(M, \omega_0^2)$ and then also we have $u_0 := \lim_{j\to\infty} u_{\varepsilon_j} \in L^1(M, \omega_0^2)$ since $u'_0 \in L^1(M, \omega_0^2)$. The following lemma will be used crucially in our argument.

**Lemma 3.3.2.** For any Borel set $D \subset M$ and any $j \geq \tilde{j}_0$ for some sufficiently large number $\tilde{j}_0 > 0$, we have

$$\mathrm{cap}_{\omega_0}(D) \leq \tilde{A}_0 \mathrm{cap}_{\tilde{\omega}_{\varepsilon_j}}(D)$$

for some sufficiently large constant $\tilde{A}_0 > 0$ depends only on $\omega_0$ independent of $\varepsilon_j$.

Proof. We arbitrary fix a function $v \in \mathrm{PSH}(M, \omega_0)$, $0 \leq v \leq 1$.

For proving the lefthand side of the inequality, we use the fact that for any Borel set $D \subset M$, we have for any $j \geq j_0$,

$$\int_D (\omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f'_{\varepsilon_j})^2 > 0.$$

Indeed, if there exists a Borel set $D \subset M$ such that $\int_D (\omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f'_{\varepsilon_j})^2 = 0$, then for any open set $U \subset D$ we have

$$\int_U (\omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f'_{\varepsilon_j})^2 = \int_U (\omega_0 - T\,\mathrm{Ric}(\omega_0))^2 = 0$$

and $U$ is birational to a ruled surface or it is a surface of class $VII$ (Proposition 3.2.1). Then we must have $\mathrm{Kod}(U) = -\infty$, which contradicts to that surfaces in our concern are limited to the blow-ups of non-Kähler minimal properly elliptic surfaces: Since the Kodaira dimension is biholomorphic invariant, we may assume that $M$ is a non-Kähler minimal properly elliptic surface by choosing sufficiently small open set $U \subset D$ which does not intersect any finitely many $(-1)$-curves. Then, there always exists a finite unramified covering $p : M' \to M$ which is also a minimal properly elliptic surface $\pi' : M' \to S'$ and $\pi'$ is an elliptic fiber bundle over a compact Riemann surface $S'$ of genus at least 2, with fiber an elliptic curve $E$ (cf. [12, Lemmas 1, 2]). If needed, by choosing sufficiently small open set $U \subset D$, we have that $p^{-1}(U)$ is a disjoint union of finitly many copies $U_j$ of $U$. Then $p : U_j \to U$ is a biholomorphism for each $j$. Since $\pi'$ is an elliptic bundle, we can choose a sufficiently small open set $U' \subset S'$ satisfying $\pi'^{-1}(U')$ is inclueded in $U_j$ for some $j$ and that we have the biholomorphism

$$U' \times E \cong \pi'^{-1}(U') \subset U_j$$

at the same time, where $E$ is an elliptic curve, i.e., 1-dimensional complex torus. Then we obtain

$$\mathrm{Kod}(U') = \mathrm{Kod}(U') + \mathrm{Kod}(E) = \mathrm{Kod}(U' \times E) = \mathrm{Kod}(\pi'^{-1}(U')) \leq \mathrm{Kod}(U_j) = \mathrm{Kod}(U),$$

where we used that $\mathrm{Kod}(E) = 0$, that the Kodaira dimension is a biholomorphic invariant and additionally it requires the following two lemmas:

**Lemma 3.3.3.** ([3, (7.3)Theorem.])If $X_1$ and $X_2$ are connected compact complex manifolds, then

$$\mathrm{Kod}(X_1 \times X_2) = \mathrm{Kod}(X_1) + \mathrm{Kod}(X_2).$$

**Lemma 3.3.4.** ([3, (7.4)Theorem.])Let $X$ and $Y$ be compact, connected complex manifolds of the same dimension. If there exists a generically finite holomorphic map from $X$ onto $Y$, then $h^0(\mathcal{O}_X(K_X)^{\otimes n}) \geq h^0(\mathcal{O}_Y(K_Y)^{\otimes n})$ for $n \geq 1$, hence $\mathrm{Kod}(X) \geq \mathrm{Kod}(Y)$. If the map is an unramified covering, then $\mathrm{Kod}(X) = \mathrm{Kod}(Y)$.

Hence we have $\mathrm{Kod}(U') = -\infty$ since $\mathrm{Kod}(U) = -\infty$. But on the other hand, since the genus of $S'$ is at least 2, there exists a metric with negative constant curvature, which is a Kähler-Einstein metric $\omega_{S'}$ induced by the Poincaré metric on the upper half plane in $\mathbb{C}$ such that $\mathrm{Ric}(\omega_{S'}) = -\omega_{S'}$ and we have $c_1(K_{S'}) > 0$. Then for the canonical bundle $K_{S'}$ restricted to $U'$, we obtain $c_1(K_{S'}|_{U'}) > 0$, which means that $K_{S'}|_{U'}$ is positive. By applying the Kodaira Embedding Theorem (Theorem 2.2.1), we have that $K_{S'}|_{U'}$ is ample. It follows from the Riemann-Roch Theorem that a nef holomorphic line bundle $L$ over a smooth projective variety $X$ is big if and only if

$$c_1(L)^n = \int_X (R_h)^n > 0,$$

where $h$ is a Hermitian metric on $L$, $R_h$ is the curvature of $h$ and $n$ is the complex dimension of $X$. It follows that since the restricted canonical divisor $K_{S'}|_{U'}$ is ample, it is then nef and big. It follows that we must have $\mathrm{Kod}(U') = 1$, which leads a contradiction.

So we have for some sufficiently large $j_0' > 0$, we have for any $j \geq j_0'$,

$$\int_D (\omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f'_{\varepsilon_j} - \varepsilon_j R_{h'})^2 = \int_D \tilde{\omega}^2_{\varepsilon_j} > \rho > 0$$

for some uniform constant $\rho > 0$.

We then set $\tilde{j}_0 := \max\{j_0, j_0'\}$. Hence we have for any $j \geq \tilde{j}_0$,

$$\begin{aligned} \int_D (\omega_0 + \sqrt{-1}\partial\bar{\partial} v)^2 &\leq \tilde{A}_0 \int_D \tilde{\omega}^2_{\varepsilon_j} \\ &\leq \tilde{A}_0 \mathrm{cap}_{\tilde{\omega}_{\varepsilon_j}}(D) \end{aligned}$$

for some uniform sufficiently large constant $\tilde{A}_0 > 0$ depending on $\omega_0$ and $\tilde{j}_0$. Taking supremum over $v$, then we obtain

$$\mathrm{cap}_{\omega_0}(D) \leq \tilde{A}_0 \mathrm{cap}_{\tilde{\omega}_{\varepsilon_j}}(D).$$

$\square$

**Remark 3.3.1.** (cf. [53]) Recall that the following conditions are equivalent: Let $X$ be a compact complex manifold with $\dim_{\mathbb{C}} X = n$.

$(H)$ there exists a Hermitian metric $\omega$ on $X$ such that

$$\partial\bar{\partial}\omega^k = 0 \quad \text{for all } k = 1, 2, \ldots, n-1.$$

The condition $(H)$ is equivalent to either of the following two equivalent conditions:

$$\partial\bar{\partial}\omega = 0 \quad \text{and} \quad \partial\bar{\partial}\omega^2 = 0 \Longleftrightarrow \partial\bar{\partial}\omega = 0 \quad \text{and} \quad \partial\omega \wedge \bar{\partial}\omega = 0.$$

In Chapter 2, we defined the so called "curvature" constant $B_\omega$. Under consideration of the condition $(H)$, when the cases $\omega = \omega_0$ or $\omega = \tilde{\omega}_{\varepsilon_j}$, the curvature constants $B_{\omega_0}$ and $B_{\tilde{\omega}_{\varepsilon_j}}$ with respect to $\omega_0$, $\tilde{\omega}_{\varepsilon_j}$ respectively can be chosen equal to 0 since we have $\partial\bar{\partial}\omega_0 = 0$, $\partial\omega_0 \wedge \bar{\partial}\omega_0 = 0$ and that the forms $-T\,\mathrm{Ric}(\omega_0) - \varepsilon_j R_{h'} + \sqrt{-1}\partial\bar{\partial} f'_{\varepsilon_j}$ are $d$-closed. Note that we have the equivalence that

$$d\text{-closed} \Leftrightarrow \partial\text{-closed} \Leftrightarrow \bar{\partial}\text{-closed}.$$

Then we can choose uniform constant $C > 0$ independent $\varepsilon_j$ in the inequality appeared in Remark 2.5.1.

The equation $(\flat)_j$ for each $j \geq j_0$ can be rewritten by

$$(\sharp)_j \qquad (\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial}u_{\varepsilon_j})^2 = c_{\varepsilon_j} F_{\varepsilon_j}\omega_0^2,$$

where we put

$$F_{\varepsilon_j} := \frac{(\pi^*\omega_N - \varepsilon_j R_h)^2}{\omega_0^2} > 0.$$

We observe the following lemma:

**Lemma 3.3.5.** For any $p > 1$ sufficiently close to 1 and for any $j \geq j_0$, the functions $F_{\varepsilon_j}$'s are uniformly bounded in $L^p(M, \omega_0^2)$.

PROOF. We may assume that $p' := \frac{1}{p-1} > 1$. By the Hölder inequality for $\frac{1}{p'} + \frac{1}{q'} = 1$,

$$\begin{aligned}
\int_M F_{\varepsilon_j}^p \omega_0^2 &= \int_M F_{\varepsilon_j}^{\frac{1}{p'}} (\pi^*\omega_N - \varepsilon_j R_h)^2 \\
&\leq \Big(\int_M F_{\varepsilon_j}(\pi^*\omega_N - \varepsilon_j R_h)^2\Big)^{\frac{1}{p'}} \Big(\int_M (\pi^*\omega_N - \varepsilon_j R_h)^2\Big)^{\frac{1}{q'}} \\
&\leq \Big(\int_M F_{\varepsilon_j}(\pi^*\omega_N - \varepsilon_j R_h)^2\Big)^{\frac{1}{p'}} A_0^{\frac{1}{q'}} \Big(\int_M \omega_0^2\Big)^{\frac{1}{q'}}
\end{aligned}$$

for some sufficiently large uniform constant $A_0 > 0$ depending only on $\omega_0$.

Since $F_{\varepsilon_j} > 0$ for any sufficiently large $j \geq j_0$,

$$\begin{aligned}
\int_M F_{\varepsilon_j}(\pi^*\omega_N - \varepsilon_j R_h)^2 &\leq A_0 \int_M F_{\varepsilon_j}\omega_0^2 \\
&= A_0 \int_M (\pi^*\omega_N - \varepsilon_j R_h)^2 \\
&\leq A_0^2 \int_M \omega_0^2
\end{aligned}$$

for some sufficiently large uniform constant $A_0 > 0$.

Combining these estimates, we obtain

$$\int_M F_{\varepsilon_j}^p \omega_0^2 \leq A_0^p \int_M \omega_0^2$$

since $\frac{p'+1}{p'} = p$. □

Hereafter, we consider $p > 1$ sufficiently close to 1 such that $F_{\varepsilon_j}$'s are uniformly bounded in $L^p(M, \omega_0^2)$. We note here that by defining the admissible functioon $h_j$ for each $j \geq j_0$ by

$$(\heartsuit)_j \qquad h_j(x) := C c_{\varepsilon_j}^{-1} ||F_{\varepsilon_j}||^{-1}_{L^p(M,\omega_0^2)} \exp(ax)$$

for some constant $C > 0$ and some number $a > 0$ depending only on $M$, $\omega_0$, and also defining

$$F_{h_j}(x) := \frac{x}{h_j(x^{-\frac{1}{2}})},$$

then from Corollary 2.5.1, $(\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial}u_{\varepsilon_j})^2$ satisfies the inequality $(\clubsuit)_{\tilde{\omega}_{\varepsilon_j}}$:

$$\int_D (\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial}u_{\varepsilon_j})^2 \leq F_{h_j}(\mathrm{cap}_{\tilde{\omega}_{\varepsilon_j}}(D))$$

for any Borel set $D \subset M$. Indeed, we have for any Borel set $D \subset M$,

$$(\star) \qquad \begin{aligned} \int_D (\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial}u_{\varepsilon_j})^2 &= c_{\varepsilon_j} \int_D F_{\varepsilon_j}\omega_0^2 \\ &\leq c_{\varepsilon_j} ||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)} \Big( \int_D \omega_0^2 \Big)^{\frac{1}{q}} \\ &\leq C c_{\varepsilon_j} ||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)} \exp\Big( \frac{-\frac{1}{q}\alpha}{\mathrm{cap}_{\omega_0}^{\frac{1}{2}}(D)} \Big) \\ &\leq C c_{\varepsilon_j} ||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)} \exp\Big( \frac{-\tilde{\alpha}}{\mathrm{cap}_{\tilde{\omega}_{\varepsilon_j}}^{\frac{1}{2}}(D)} \Big) \end{aligned}$$

for a number $\alpha = \alpha(M, \omega_0) > 0$ and a constant $C = C(M, \omega_0) > 0$, where we put $\tilde{\alpha} := \tilde{A}_0^{-\frac{1}{2}} \frac{\alpha}{q}$ and we used the Hölder inequality for $\frac{1}{p} + \frac{1}{q} = 1$ at the second line, the result in Proposition 2.5.3 at the third line and Lemma 3.3.2 at the forth line for each $j \geq \tilde{j}_0$. Hence, from the estimate $(\star)$, we can apply Proposition 2.5.4 to $(\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial}u_{\varepsilon_j})^2$ with the equations $(\sharp)_j$, $\omega = \frac{1}{s}\tilde{\omega}_{\varepsilon_j}$, $\varphi = u_{\varepsilon_j}$ and $\psi = 0$ for each $j \geq \tilde{j}_0$. Recall the definition of the function $\kappa$ in Proposition 2.5.4, we define

$$\kappa_j(s^{\frac{1}{2}}) := 4C_2 \Big( \frac{1}{h_j(s)^{\frac{1}{2}}} + \int_s^\infty \frac{dx}{x h_j(x)^{\frac{1}{2}}} \Big),$$

with a dimensional constant $C_2$. By the definition of the admissible function $h_j$ in $(\heartsuit)_j$, we compute and obtain that

$$\kappa_j(x) \leq \tilde{C} c_{\varepsilon_j}^{\frac{1}{2}} ||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)}^{\frac{1}{2}} \exp(-\tilde{a} x^{-\frac{1}{2}})$$

for some uniform constants $\tilde{C}, \tilde{a} > 0$ independent of $\varepsilon_j$. As $\kappa_j$ is an increasing function, its inverse function $\hbar_j$ satisfies

$$\hbar_j(x) \geq \Big( \frac{1}{\tilde{a}} \log \Big( \frac{\tilde{C} c_{\varepsilon_j}^{\frac{1}{2}} ||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)}^{\frac{1}{2}}}{x} \Big) \Big)^{-2}.$$

We will use Proposition 2.5.4 to prove the following lemma which is used for showing the uniform convergence.

**Lemma 3.3.6.** There exists a large number $\tilde{j}_0' > 0$ such that for any $j \geq \tilde{j}_0'$, we have

$$c_0 \leq c_{\varepsilon_j} \leq C_{0,N}$$

for some unform constant $C_{0,N}, c_0 > 0$ independent of $\varepsilon_j$.

PROOF. Fix $0 < \delta < 1$. Define $S_{\varepsilon_j} := \inf_M u_{\varepsilon_j}$ and $\delta_0$ is the positive number defined in Proposition 2.5.4. Then for any $0 < s, t < \delta_0$, we have by applying Remark 2.5.1,

$$\begin{aligned} t^2 \mathrm{cap}_{\tilde{\omega}_{\varepsilon_j}}(\{u_{\varepsilon_j} < S_{\varepsilon_j} + s\}) &\leq C \int_{\{u_{\varepsilon_j} < S_{\varepsilon_j} + s + t\}} (\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial} u_{\varepsilon_j})^2 \\ &= C \int_{\{u_{\varepsilon_j} < S_{\varepsilon_j} + s + t\}} c_{\varepsilon_j} F_{\varepsilon_j} \omega_0^2 \\ &\leq C c_{\varepsilon_j} ||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)} \mathrm{Vol}_{\omega_0}(\{u_{\varepsilon_j} < S_{\varepsilon_j} + s + t\})^{\frac{1}{q}} \end{aligned}$$

for some uniform constant $C > 0$ independent of $\varepsilon_j$ (Remark 3.3.1), where $\frac{1}{p} + \frac{1}{q} = 1$.

Hence for fixed $0 < s = t < \delta_0$, we obtain

$$\begin{aligned} \mathrm{cap}_{\tilde{\omega}_{\varepsilon_j}}(\{u_{\varepsilon_j} < S_{\varepsilon_j} + s\}) &\leq \frac{C c_{\varepsilon_j}}{s^2} ||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)} \mathrm{Vol}_{\omega_0}(\{u_{\varepsilon_j} < S_{\varepsilon_j} + 2s\})^{\frac{1}{q}} \\ &\leq \frac{C' c_{\varepsilon_j}}{s^2} \mathrm{Vol}_{\omega_0}(M)^{\frac{1}{q}} =: C_0(M) c_{\varepsilon_j} s^{-2} \end{aligned}$$

for some uniform constant $C' > 0$, where we used that $||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)}$ is uniformy bounded from above (lemma 3.3.5) and we put $C_0(M) := C'\mathrm{Vol}_{\omega_0}(M)^{\frac{1}{q}} > 0$.

Then from Proposition 2.5.4, for any $j \geq \tilde{j}_0$,

$$\begin{aligned} 0 < s &\leq \kappa_j(\mathrm{cap}_{\tilde{\omega}_{\varepsilon_j}}(\{u_{\varepsilon_j} < S_{\varepsilon_j} + s\})) \\ &\leq \kappa_j(C_0(M) c_{\varepsilon_j} s^{-2}) \\ &\leq \tilde{C} c_{\varepsilon_j}^{\frac{1}{2}} ||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)}^{\frac{1}{2}} \exp\Big(\frac{-\tilde{a}s}{C_0(M)^{\frac{1}{2}} c_{\varepsilon_j}^{\frac{1}{2}}}\Big) \\ &\leq \tilde{C}' c_{\varepsilon_j}^{\frac{1}{2}} \exp\Big(\frac{-\tilde{a}s}{C_0(M)^{\frac{1}{2}} c_{\varepsilon_j}^{\frac{1}{2}}}\Big) \end{aligned}$$

for some uniform positive constants $\tilde{C}, \tilde{a}$ and $\tilde{C}'$, where we used that $||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)}^{\frac{1}{2}}$ is uniformy bounded from above. If $c_{\varepsilon_j} \to 0$ as $j \to \infty$, then

$$0 < s \leq \tilde{C}' c_{\varepsilon_j}^{\frac{1}{2}} \exp\Big(\frac{-\tilde{a}s}{C_0(M)^{\frac{1}{2}} c_{\varepsilon_j}^{\frac{1}{2}}}\Big) \to 0.$$

This is a contradiction, and therefore $c_{\varepsilon_j}$ must be uniformly bounded away from 0.

For the uniform upper bound, we use the pointwise arithmetic-geometric means inequality and which implies that we have

$$\begin{aligned} (\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial} u_{\varepsilon_j}) \wedge (\pi^*\omega_N - \varepsilon_j R_h) &\geq \Big(\frac{(\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial} u_{\varepsilon_j})^2}{(\pi^*\omega_N - \varepsilon_j R_h)^2}\Big)^{\frac{1}{2}} (\pi^*\omega_N - \varepsilon_j R_h)^2 \\ &= c_{\varepsilon_j}^{\frac{1}{2}} (\pi^*\omega_N - \varepsilon_j R_h)^2. \end{aligned}$$

Since we have that $\int_M (\pi^*\omega_N - \varepsilon_j R_h)^2 > 0$ for sufficiently large $j$, there exists a large number $j_0'' > 0$ such that for any $j \geq j_0''$,

$$\int_M (\pi^*\omega_N - \varepsilon_j R_h)^2 > \rho_N > 0$$

for some uniform constant $\rho_N > 0$ depending on $\omega_N$ and $j_0''$. We put $\tilde{j}_0' := \max\{j_0, j_0''\}$.

It follows that for any $j \geq \tilde{j}_0'$,

$$\begin{aligned} c_{\varepsilon_j}^{\frac{1}{2}} &\leq \Big(\int_M (\pi^*\omega_N - \varepsilon_j R_h)^2\Big)^{-1} \int_M (\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial} u_{\varepsilon_j}) \wedge (\pi^*\omega_N - \varepsilon_j R_h) \\ &= \Big(\int_M (\pi^*\omega_N - \varepsilon_j R_h)^2\Big)^{-1} \int_M (\omega_0 - T\,\mathrm{Ric}(\omega_0) - \varepsilon_j R_{h'}) \wedge (\pi^*\omega_N - \varepsilon_j R_h) \\ &\leq \frac{\tilde{A}_0'}{\rho_N} \int_M \omega_0^2 =: C_{0,N}^{\frac{1}{2}} \end{aligned}$$

for some sufficiently large $\tilde{A}_0' > 0$ depending on $\omega_0$ and $\tilde{j}_0'$, where we used that $\pi^*\omega_N - \varepsilon_j R_h$ are $\partial\bar{\partial}$-closed. $\square$

We now arbitrary choose a sufficiently small open set $U \subset M$ such that we have

$$\sqrt{-1}\partial\bar{\partial} u_{\varepsilon_j}'' = -T\,\mathrm{Ric}(\omega_0) - \varepsilon_j R_{h'} + \sqrt{-1}\partial\bar{\partial} u_{\varepsilon_j}'$$

for the smooth function $u_{\varepsilon_j}'' = T \log \omega_0^2 + \varepsilon_j \log h' + u_{\varepsilon_j}'$ on $U$. Then the equation $(\flat)_j$ for each $j \geq j_0$ on $U$ can be rewritten by

$$(\natural)_j \qquad (\omega_0 + \sqrt{-1}\partial\bar{\partial} u_{\varepsilon_j}'')^2 = c_{\varepsilon_j} F_{\varepsilon_j} \omega_0^2.$$

Since $u_{\varepsilon_j}'$ converges to $u_0'$ in $L^1(M, \omega_0^2)$, we have $u_{\varepsilon_j}' \to u_0'$ in $L^1(U, \omega_0^2)$. Hence we have that $\{u_{\varepsilon_j}''\}_j$ is a Cauchy sequence in $L^1(U, \omega_0^2)$. Since the righthand side $c_{\varepsilon_j} F_{\varepsilon_j}$'s of the equations $(\natural)_j$ for any $j \geq \tilde{j}_0'$ are uniformly bounded in $L^p(M, \omega_0^2)$, $\{u_{\varepsilon_j}''\}_j$ are uniformly bounded (Corollary 2.5.2) and the sequence $\{u_{\varepsilon_j}''\}_j$ is Cauchy in $C^0(U)$ (Corollary 2.5.3). Then we have

$$u_{\varepsilon_j}'' \to u_0'' = T \log \omega_0^2 + u_0' \in \mathrm{PSH}(U, \omega_0) \cap C^0(U)$$

uniformly on $U$ as $j \to \infty$, which implies that $u_{\varepsilon_j}'$ converges to $u_0'$ uniformly in $C^0(U)$-topology as $j \to \infty$ on $U$. Since $M$ is compact, we can cover $M$ with finitly many sufficiently small open sets. Therefore, we conclude that, on whole $M$, as $j \to \infty$ uniformly,

$$(\diamondsuit) \qquad u_{\varepsilon_j}' \to u_0' = f_0' + u_0 \in \mathrm{PSH}(\omega_0 - T\,\mathrm{Ric}(\omega_0)) \cap C^0(M).$$

We may normalize $u_{\varepsilon_j}$ by $\sup_M u_{\varepsilon_j} = 0$. Then, since the righthand side $c_{\varepsilon_j} F_{\varepsilon_j}$'s of the equations $(\sharp)_j$ are uniformly bounded in $L^p(M, \omega_0^2)$ for any $j \geq \tilde{j}_0'$, from Corollary 2.5.2 (cf. [30, Corollary 5.6]), there exists a uniform constant $H > 0$ such that $-H \leq u_{\varepsilon_j} \leq 0$ for $j \geq \tilde{j}_0$. Indeed, as we see in the proof of [48, Corollary 5.6], by applying the $L^1$-CLN inequality (Proposition 2.5.1) (cf. [17, Proposition 3.11], [44, p.8]) and the capacity estimate of sublevel sets ([19, Proposition 2.5]), we have

$$|\inf_M u_{\varepsilon_j}| \leq s + \frac{C}{\hbar_j(s)} \sum_{\mathbb{B}} \Big(\int_M |u_{\varepsilon_j}| \omega_0^2 + \int_{\mathbb{B}} |\psi_{\varepsilon_j}| \omega_0^2\Big)$$

for any $0 < s < \delta_0$ for some uniform positive constant $C$ independent of $\varepsilon_j$, where $\hbar_j$ is the inverse function of the function $\kappa_j$, and $\psi_{\varepsilon_j}$ are the strictly plurisubharmonic functions can be chosen locally on a sufficiently small ball $\mathbb{B}$ in $M$ such that for each $j \geq j_0$ they are smooth, $\sup_{\mathbb{B}} \psi_{\varepsilon_j} = 0$ and satisfy on the small ball $\mathbb{B}$,

$$\sqrt{-1}\partial\bar{\partial}\psi_{\varepsilon_j} \geq \tilde{\omega}_{\varepsilon_j}.$$

Since we have that $\psi_{\varepsilon_j}$ are plurisubharmonic on the sufficiently small ball $\mathbb{B}$, by applying Proposition 2.5.2, the functions $\psi_{\varepsilon_j}$ are uniformly integrable on $\mathbb{B}$. Since $u_{\varepsilon_j}$ are uniformly integrable with respect to $\omega_0^2$, then by combining with the lower bound of $\hbar_j$ as we observed before, we obtain the uniform bound for $u_{\varepsilon_j}$.

We observe this argument for the uniform bound of $u_{\varepsilon_j}$ more specifically below: Let $\{B_i(r)\}_{i=1}^I$ be a finite covering of $M$ for $i = 1, 2, \ldots, I$, where $B_i(r) = B(x_i, r)$ is the ball centered at $x_i \in M$ of radius $r > 0$ with $B_i(r) \subset\subset B_i(2r)$. We may choose $r > 0$ small enough such that for all $i = 1, \ldots, I$, each $j \geq j_0$, there exist smooth negative strictly plurisubharmonic functions $\psi_{\varepsilon_j,i}$ on $B_i(3r)$ and $\rho_i$ on $B_i(2r)$ satisfying that

$$\sup_{B_i(2r)} \psi_{\varepsilon_j,i} = 0, \quad \sqrt{-1}\partial\bar{\partial}\psi_{\varepsilon_j,i} \geq \tilde{\omega}_{\varepsilon_j} \quad \text{on } B_i(2r),$$

and

$$\rho_i|_{\partial B_i(2r)} = 0, \quad \inf_{B_i(2r)} \rho_i \geq -C_1, \quad \sqrt{-1}\partial\bar{\partial}\rho_i \geq \omega_0 \quad \text{on } B_i(2r),$$

where $C_1 > 0$ is a constant depending only on the covering and $\omega_0$.

Then, since $\psi_{\varepsilon_j,i} \in \mathrm{PSH}(B_i(2r), \omega_0)$, thanks to Proposition 2.5.2, we have

$$\int_{B_i(2r)} |\psi_{\varepsilon_j,i}|\omega_0^2 \leq C_{i,r}$$

for some constant $C_{i,r} > 0$ independent of $\varepsilon_j$. Fix a function $v \in \mathrm{PSH}(M, \omega_0)$, $0 \leq v \leq 1$, then we have for sufficiently small $s > 0$, $S_{\varepsilon_j} = \inf_M u_{\varepsilon_j}$,

$$\begin{aligned}
\int_{\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j} + s\}} (\omega_0 + \sqrt{-1}\partial\bar{\partial}v)^2 \quad &\leq \quad \frac{1}{|\frac{1}{s}S_{\varepsilon_j} + s|} \int_M |\frac{1}{s}u_{\varepsilon_j}|(\omega_0 + \sqrt{-1}\partial\bar{\partial}v)^2 \\
&\leq \quad \frac{1}{|\frac{1}{s}S_{\varepsilon_j} + s|} \Big( \sum_{i=1}^I \int_{B_i(r)} |\frac{1}{s}u_{\varepsilon_j}|(\sqrt{-1}\partial\bar{\partial}(\rho_i + v))^2 \Big) \\
&\leq \quad \frac{1}{|\frac{1}{s}S_{\varepsilon_j} + s|} \Big( \sum_{i=1}^I \int_{B_i(r)} |\frac{1}{s}u_{\varepsilon_j} + \frac{1}{s}\psi_{\varepsilon_j,i}|(\sqrt{-1}\partial\bar{\partial}(\rho_i + v))^2 \Big) \\
&\leq \quad \sum_{i=1}^I \frac{C_{B_i(r),B_i(2r)}}{|\frac{1}{s}S_{\varepsilon_j} + s|} ||\frac{1}{s}u_{\varepsilon_j} + \frac{1}{s}\psi_{\varepsilon_j,i}||_{L^1(B_i(2r))} ||\rho_i + v||^2_{L^\infty(B_i(2r))},
\end{aligned}$$

where notice that $\frac{1}{s}u_{\varepsilon_j} + \frac{1}{s}\psi_{\varepsilon_j,i}$, $\rho_i + v$ belongs to $\mathrm{PSH}(B_i(2r))$, so we applied the $L^1$-CLN inequality (Proposition 2.5.1) at the last line. Since $\rho_i$, $v$ are uniformly bounded,

$$||\rho_i + v||^2_{L^\infty(B_i(2r))} \leq C_{B_i(2r)}$$

for some constant $C_{B_i(2r)} > 0$ depends only on $\omega_0$ and $B_i(2r)$. Then we have

$$\int_{\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j}+s\}} (\omega_0 + \sqrt{-1}\partial\bar{\partial}v)^2 \leq \sum_{i=1}^{I} \frac{C'_{B_i(r),B_i(2r)}}{|\frac{1}{s}S_{\varepsilon_j}+s|} ||\frac{1}{s}u_{\varepsilon_j} + \frac{1}{s}\psi_{\varepsilon_j,i}||_{L^1(B_i(2r))},$$

where we put $C'_{B_i(r),B_i(2r)} := C_{B_i(r),B_i(2r)}C_{B_i(2r)}$. Taking supremum over $v$, we obtain

$$\mathrm{cap}_{\omega_0}(\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j}+s\}) \leq \sum_{i=1}^{I} \frac{C'_{B_i(r),B_i(2r)}}{|\frac{1}{s}S_{\varepsilon_j}+s|} ||\frac{1}{s}u_{\varepsilon_j} + \frac{1}{s}\psi_{\varepsilon_j,i}||_{L^1(B_i(2r))}.$$

We compute for $0 < s < \delta_0$,

$$\begin{aligned} s^2 \int_{\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j}+s\}} \Big(\frac{1}{s}\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial}\Big(\frac{u_{\varepsilon_j} - S_{\varepsilon_j}}{s}\Big)\Big)^2 &= \int_{\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j}+s\}} (\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial}u_{\varepsilon_j})^2 \\ &= c_{\varepsilon_j} \int_{\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j}+s\}} (\pi^*\omega_N - \varepsilon_j R_h)^2 \\ &\leq \hat{A}_0 C_{0,N} \int_{\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j}+s\}} \omega_0^2 \\ &\leq \hat{A}_0 C_{0,N} \mathrm{cap}_{\omega_0}(\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j}+s\}) \end{aligned}$$

for some large constant $\hat{A}_0 > 0$ depending on $\omega_0$ and $j_0$, where we used that $c_{\varepsilon_j} \leq C_{0,N}$.

Since $0 \leq \frac{u_{\varepsilon_j} - S_{\varepsilon_j}}{s} < s < \delta_0 < 1$ on the set $\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j}+s\}$ and $\frac{u_{\varepsilon_j} - S_{\varepsilon_j}}{s} \in \mathrm{PSH}(\frac{\tilde{\omega}_{\varepsilon_j}}{s})$, by taking supremum, we obtain

$$\mathrm{cap}_{\frac{1}{s}\tilde{\omega}_{\varepsilon_j}}(\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j}+s\}) \leq \frac{\hat{A}_0 C_{0,N}}{s^2} \mathrm{cap}_{\omega_0}(\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j}+s\}).$$

We note that by defining the admissible functioon $h_{j,s}$ for each $j \geq j_0$ by

$$(\heartsuit)_{j,s} \qquad h_{j,s}(x) := Cs^2 c_{\varepsilon_j}^{-1} ||F_{\varepsilon_j}||^{-1}_{L^p(M,\omega_0^2)} \exp(ax)$$

for some constant $C > 0$ and some number $a > 0$ depending only on $M$, $\omega_0$, and also defining

$$F_{h_{j,s}}(x) := \frac{x}{h_{j,s}(x^{-\frac{1}{2}})},$$

then from Corollary 2.5.1,

$$(\sharp)_{j,s} \qquad (\frac{1}{s}\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial}(\frac{1}{s}u_{\varepsilon_j}))^2 = c_{\varepsilon_j} F_{\varepsilon_j} s^{-2} \omega_0^2$$

satisfies the inequality $(\clubsuit)_{\frac{\tilde{\omega}_{\varepsilon_j}}{s}}$:

$$\int_D (\frac{1}{s}\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial}(\frac{1}{s}u_{\varepsilon_j}))^2 \leq F_{h_{j,s}}(\mathrm{cap}_{\frac{1}{s}\tilde{\omega}_{\varepsilon_j}}(D))$$

for any Borel set $D \subset M$ from the estimate in $(\star)$. We then define

$$\kappa_{j,s}(s^{\frac{1}{2}}) := 4C_2\Big(\frac{1}{h_{j,s}(s)^{\frac{1}{2}}} + \int_s^\infty \frac{dx}{x h_{j,s}(x)^{\frac{1}{2}}}\Big),$$

with a dimensional constant $C_2$. By the definition of the admissible function $h_{j,s}$ in $(\heartsuit)_{j,s}$, we compute and obtain that

$$\kappa_{j,s}(x) \leq \tilde{C} s^{-1} c_{\varepsilon_j}^{\frac{1}{2}} ||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)}^{\frac{1}{2}} \exp(-\tilde{a} x^{-\frac{1}{2}})$$

for some uniform constants $\tilde{C}, \tilde{a} > 0$ independent of $\varepsilon_j$. As $\kappa_{j,s}$ is an increasing function, its inverse function $\hbar_{j,s}$ satisfies

$$\hbar_{j,s}(x) \geq \Big(\frac{1}{\tilde{a}} \log\Big(\frac{\tilde{C} s^{-1} c_{\varepsilon_j}^{\frac{1}{2}} ||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)}^{\frac{1}{2}}}{x}\Big)\Big)^{-2}.$$

Therefore, since we may apply Proposition 2.5.4 to $(\frac{1}{s}\tilde{\omega}_{\varepsilon_j} + \sqrt{-1}\partial\bar{\partial}(\frac{1}{s}u_{\varepsilon_j}))^2$ for $j \geq \tilde{j}_0$ with the equations $(\sharp)_{j,s}$, $\omega = \frac{1}{s}\tilde{\omega}_{\varepsilon_j}$, $\varphi = \frac{1}{s}u_{\varepsilon_j}$ and $\psi = 0$, then we have

$$\begin{aligned}
\hbar_{j,s}(s) &\leq \mathrm{cap}_{\frac{1}{s}\tilde{\omega}_{\varepsilon_j}}(\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j} + s\}) \\
&\leq \frac{\hat{A}_0 C_{0,N}}{s^2} \mathrm{cap}_{\omega_0}(\{\frac{1}{s}u_{\varepsilon_j} < \frac{1}{s}S_{\varepsilon_j} + s\}) \\
&\leq \frac{\hat{A}_0 C_{0,N}}{s^2|\frac{1}{s}S_{\varepsilon_j} + s|} \sum_{i=1}^{I} C'_{B_i(r),B_i(2r)}\Big(\int_{B_i(2r)} |\frac{1}{s}u_{\varepsilon_j}|\omega_0^2 + \int_{B_i(2r)} |\frac{1}{s}\psi_{\varepsilon_j,i}|\omega_0^2\Big) \\
&\leq \frac{\hat{A}_0 C_{0,N}}{s^2|\frac{1}{s}S_{\varepsilon_j} + s|} I C'_{B(r),B(2r)} \frac{1}{s}\Big(\int_M |u_{\varepsilon_j}|\omega_0^2 + C_r\Big),
\end{aligned}$$

where $C'_{B(r),B(2r)} := \max_{1\leq i\leq I} C'_{B_i(r),B_i(2r)}$ and $C_r := \max_{1\leq i\leq I} C_{i,r}$. Since we have that $\int_M |u_{\varepsilon_j}|\omega_0^2 \leq \hat{C}$ for some uniform constant $\hat{C} > 0$, and that $c_0 \leq c_{\varepsilon_j} \leq C_{0,N}$ for $j \geq \tilde{j}'_0$, we finally obtain for any $j \geq \tilde{j}''_0 := \max\{\tilde{j}_0, \tilde{j}'_0\}$,

$$\begin{aligned}
|S_{\varepsilon_j}| &\leq s^2 + \frac{\hat{A}_0 C_{0,N} I C'_{B(r),B(2r)}}{s^2 \hbar_{j,s}(s)} (\hat{C} + C_r) \\
&\leq \delta_0^2 + \frac{1}{s^2} \hat{A}_0 C_{0,N} I C'_{B(r),B(2r)} (\hat{C} + C_r) \Big(\frac{1}{\tilde{a}} \log\Big(\frac{\tilde{C}'}{s^2}\Big)\Big)^2 < +\infty,
\end{aligned}$$

uniformly bounded independent of $\varepsilon_j$, where we used the following estimate:

$$\frac{1}{\hbar_{j,s}(s)} \leq \Big(\frac{1}{\tilde{a}} \log\Big(\frac{\tilde{C} s^{-1} c_{\varepsilon_j}^{\frac{1}{2}} ||F_{\varepsilon_j}||_{L^p(M,\omega_0^2)}^{\frac{1}{2}}}{s}\Big)\Big)^2 \leq \Big(\frac{1}{\tilde{a}} \log\Big(\frac{\tilde{C}'}{s^2}\Big)\Big)^2$$

for some uniform positive constants $\tilde{C}, \tilde{a}$ and $\tilde{C}'$.

Hence, we conclude that $u_{\varepsilon_j}$ for $j \geq \tilde{j}_0''$ are uniformly bounded and so by rescaling, we may assume that $-1 \leq u_{\varepsilon_j} \leq 0$. We define

$$U_{kj} := \inf_M (u_{\varepsilon_k} - u_{\varepsilon_j}) \leq 0.$$

Suppose that $U_{kj}$ does not converge to 0 as $k, j \to \infty$. Then there exists $0 < \tau < 1$ such that

$$U_{kj} \leq -4\tau$$

for arbitrary chosen large $k \neq j$. We choose sufficiently large numbers $\tilde{k}_0$, $\tilde{k}_0'$ and $\tilde{k}_0''$ in the same way as the numbers $\tilde{j}_0$, $\tilde{j}_0'$ and $\tilde{j}_0''$ in Lemma 3.3.2 and in Lemma 3.3.6 and the argument above respectively. We define $m(\tau) := \inf_M (u_{\varepsilon_k} - (1-\tau)u_{\varepsilon_j})$,

$$U(\tau, s) := \{u_{\varepsilon_k} < (1-\tau)u_{\varepsilon_j} + m(\tau) + s\}$$

and $\tau_0 := \frac{1}{3}\min\{\tau^2, \frac{\tau^3}{16B}, 4(1-\tau)\tau^2, 4(1-\tau)\frac{\tau^3}{16B}\}$. Obviously we have $m(\tau) \leq U_{kj}$. From Remark 2.5.1, we have for any $0 < s, t < \tau_0$ and $k \geq \tilde{k}_0'$,

$$\begin{aligned} t^2 \mathrm{cap}_{\tilde{\omega}_{\varepsilon_k}}(U(\tau, s)) &\leq C \int_{U(\tau, s+t)} (\tilde{\omega}_{\varepsilon_k} + \sqrt{-1}\partial\bar{\partial} u_{\varepsilon_k})^2 \\ &= C c_{\varepsilon_k} \int_{U(\tau, s+t)} F_{\varepsilon_k} \omega_0^2 \\ &\leq C C_{0,N} ||F_{\varepsilon_k}||_{L^p(M, \omega_0^2)} \Big( \int_{U(\tau, s+t)} \omega_0^2 \Big)^{\frac{1}{q}} \end{aligned}$$

for some uniform constant $C > 0$ independent of $\varepsilon_j$ (Remark 3.3.1), where $\frac{1}{p} + \frac{1}{q} = 1$ and we used that $c_{\varepsilon_k} \leq C_0$ for $k \geq \tilde{k}_0'$.

We can observe the following inclusions hold:

$$U(\tau, s+t) \subset \{u_{\varepsilon_k} < u_{\varepsilon_j} + U_{kj} + \tau + s + t\} \subset \{u_{\varepsilon_k} < u_{\varepsilon_j} - \tau\} \subset \{|u_{\varepsilon_k} - u_{\varepsilon_j}| > \tau\}.$$

Then we obtain

$$\begin{aligned} t^2 \mathrm{cap}_{\tilde{\omega}_{\varepsilon_k}}(U(\tau, s)) &\leq C C_{0,N} ||F_{\varepsilon_k}||_{L^p(M, \omega_0^2)} \Big( \int_{\{|u_{\varepsilon_k} - u_{\varepsilon_j}| > \tau\}} \omega_0^2 \Big)^{\frac{1}{q}} \\ &\leq \frac{C C_{0,N}}{\tau^{\frac{1}{q}}} ||F_{\varepsilon_k}||_{L^p(M, \omega_0^2)} \Big( \int_M |u_{\varepsilon_k} - u_{\varepsilon_j}| \omega_0^2 \Big)^{\frac{1}{q}}. \end{aligned}$$

For fixed $0 < s = t = s_0 < \tau_0$, from Proposition 2.5.4, we have for $k \geq \tilde{k}_0''$,

$$\begin{aligned} s_0 &\leq \kappa_k (\mathrm{cap}_{\tilde{\omega}_{\varepsilon_k}}(U(\tau, s))) \\ &\leq \kappa_k \Big( \frac{C C_{0,N}}{s_0^2 \tau^{\frac{1}{q}}} ||F_{\varepsilon_k}||_{L^p(M, \omega_0^2)} \Big( \int_M |u_{\varepsilon_k} - u_{\varepsilon_j}| \omega_0^2 \Big)^{\frac{1}{q}} \Big) \\ &\leq \kappa_k \Big( \frac{C'}{s_0^2 \tau^{\frac{1}{q}}} \Big( \int_M |u_{\varepsilon_k} - u_{\varepsilon_j}| \omega_0^2 \Big)^{\frac{1}{q}} \Big) \\ &\leq \tilde{C} c_{\varepsilon_k}^{\frac{1}{2}} ||F_{\varepsilon_k}||^{\frac{1}{2}}_{L^p(M, \omega_0^2)} \exp\Big( - \frac{\tilde{a} s_0 \tau^{\frac{1}{2q}}}{(C')^{\frac{1}{2}}} \Big( \int_M |u_{\varepsilon_k} - u_{\varepsilon_j}| \omega_0^2 \Big)^{-\frac{1}{2q}} \Big) \end{aligned}$$

for some uniform positive constants $\tilde{C}, \tilde{a}, C'$, where we used that $||F_{\varepsilon_k}||_{L^p(M,\omega_0^2)}$ is uniformly bounded from above and the functions $\kappa_k$ are increasing.

Recall that the sequence $\{u_{\varepsilon_j}\}_j$ is Cauchy in $L^1(M, \omega_0^2)$. Since we have that $||F_{\varepsilon_k}||_{L^p(M,\omega_0^2)}$ is uniformly bounded from above and that $c_{\varepsilon_k}^{\frac{1}{2}} \leq C_{0,N}^{\frac{1}{2}}$ for $k \geq \tilde{k}_0'$, then we obtain for some uniform constant $\tilde{C}' > 0$,

$$0 < s_0 \leq \tilde{C}' \exp\Big( - \frac{\tilde{a} s_0 \tau^{\frac{1}{2}}}{(C')^{\frac{1}{2}}} \Big( \int_M |u_{\varepsilon_k} - u_{\varepsilon_j}| \omega_0^2 \Big)^{-\frac{1}{2q}} \Big) \to 0$$

as $k, j \to \infty$, which is obviously a contradiction. Hence we have $U_{kj} \to 0$ as $k, j \to \infty$. Therefore we obtain

$$|u_{\varepsilon_k} - u_{\varepsilon_j}| \leq 2|U_{kj}| \to 0$$

as $k, j \to \infty$, which indicates that the sequence $\{u_{\varepsilon_j}\}_j$ is Cauchy in $C^0(M)$ and $u_0 \in C^0(M)$. Thus, from the convergence result $(\diamondsuit)$, we have $f_0' \in \mathrm{PSH}(\omega_0 - T\,\mathrm{Ric}(\omega_0)) \cap C^0(M)$ and then we obtain

$$||f_0'||_{C^0(M)} \leq C$$

for some constant $C > 0$. Therefore, we obtain the following result under the assumptions in Theorem 3.1.1:

**Proposition 3.3.1.** We can choose a uniform positive constant $C$ such that

$$(\ddagger)' \quad \frac{1}{C}(\pi^*\omega_N - \varepsilon_j R_h) \leq \tilde{\omega}_{\varepsilon_j} \leq C(\pi^*\omega_N - \varepsilon_j R_h)$$

holds in the weak sense of currents on $M$.

From the inequality $(\ddagger)'$, by restricting on $E$, we have

$$\frac{1}{C}\varepsilon_j(-R_h)|_E = \frac{1}{C}\varepsilon_j \omega_{FS} \leq \tilde{\omega}_{\varepsilon_j}|_E \leq C\varepsilon_j \omega_{FS} = C\varepsilon_j(-R_h)|_E$$

in the weak sense of currents. Now we define

$$\tilde{\omega}_0 := \omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f_0'$$

as a positive current by the clasical distribution theory. Then we must have

$$\int_E \varphi \tilde{\omega}_{\varepsilon_j} \to \int_E \varphi \tilde{\omega}_0 = 0$$

as $j \to \infty$ for any test function $\varphi \in C_0^\infty(E)$. Hence, we have

$$\tilde{\omega}_0|_E = (\omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f_0')|_E = 0$$

in the weak sense of currrents on $E$.

After passing a subsequence $\{\varepsilon_{j_i}\}_i$, by letting $i \to \infty$ in $(\flat)_{j_i}$, since $u_0, f_0' \in C^0(M)$ and we have that $c_{\varepsilon_{j_i}} \to c^2$ for some constant $c > 0$ from the uniform estimate in Lemma 3.3.6, we obtain

$$(\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial} u_0)^2 = (c\pi^*\omega_N)^2$$

on $M$ as currents. Then we obtain $(\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0)|_E = 0$ on $E$ as a current. Since we have $\tilde{\omega}_0|_E = 0$ as a current, we obtain

$$\sqrt{-1}\partial\bar{\partial}u_0|_E = 0$$

on $E$ as a current.

Notice that since we have assumed that $E$ is the only one $(-1)$ curve on $M$ and we have a biholomorphism $\pi|_{M\setminus E} : M \setminus E \overset{\cong}{\to} N \setminus \{y_0\}$, we may identify forms, metrics and functions on $M \setminus E$ and $N \setminus \{y_0\}$. Then we have that $(\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0)^2 = (c\omega_N)^2$ on $M \setminus E$ as currents.

For an arbitrary chosen point $p \in M \setminus E$, we choose sufficiently small open neighborhood $U$ of $p$. We may assume that $c\omega_N - (\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0)$ is a positive current on $U$ (If it is not possible for any sufficiently small $U$, we consider $\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0 - c\omega_N$ and choose a sufficiently small open neighborhood $U$ so that $\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0 - c\omega_N$ is a positive current on $U$). Then (in either case), $(c\omega_N - (\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0))^2$ is also a positive current on $U$ and we have for any $\varphi \in C_0^\infty(U)$ with $\varphi \geq 0$ on $U$,

$$\int_U \varphi(c\omega_N - (\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0))^2 \geq 0.$$

On the other hand, using the equality $(\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0)^2 = (c\omega_N)^2$ on $N \setminus \{y_0\}$, we can find a unitary frame $\theta_1$ and $\theta_2$ with respect to $(c\omega_N, J)$, where $J$ is the complex structure, at a fixed point $p_0 \in U$, so that

$$c\omega_N = \sqrt{-1}\theta_1 \wedge \bar{\theta}_1 + \sqrt{-1}\theta_2 \wedge \bar{\theta}_2, \quad \tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0 = \sqrt{-1}\lambda\theta_1 \wedge \bar{\theta}_1 + \frac{\sqrt{-1}}{\lambda}\theta_2 \wedge \bar{\theta}_2$$

for some positive constant $\lambda$. Additionally, we have

$$(c\omega_N - (\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0))^2 = (c\omega_N)^2\Big(2 - \Big(\lambda + \frac{1}{\lambda}\Big)\Big) \leq 0,$$

with equality if and only if $\lambda = 1$.

Then by combining these, we must have $\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0 = c\omega_N$ as currents on $U$. Since the choice of a point $p \in M \setminus E$ was arbitrary, we obtain

$$\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial}u_0 = c\omega_N$$

as currents on whole $M \setminus E$. The similar argument can be seen in [67].

We compute that for an arbitrary chosen open set $U \subset M \setminus E$, for an arbitrary chosen test function $\varphi \in C_0^\infty(U)$ and the function $u_0' = f_0' + u_0 \in C^0(M)$,

$$\begin{aligned}
\Big|\int_U \varphi\sqrt{-1}\partial u_0' \wedge \bar\partial u_0'\Big| &= \Big|-\int_U u_0'\sqrt{-1}\partial\varphi \wedge \bar\partial u_0' - \int_U \varphi u_0'\sqrt{-1}\partial\bar\partial u_0'\Big| \\
&= \Big|-\frac{1}{2}\int_U \sqrt{-1}\partial\varphi \wedge \bar\partial(u_0')^2 - \int_U \varphi u_0'\sqrt{-1}\partial\bar\partial u_0'\Big| \\
&= \Big|\frac{1}{2}\int_U (u_0')^2\sqrt{-1}\partial\bar\partial\varphi - \int_U \varphi u_0'\sqrt{-1}\partial\bar\partial u_0'\Big| \\
&\leq \frac{1}{2}||u_0'||^2_{C^0(M)}\Big|\int_U \sqrt{-1}\partial\bar\partial\varphi\Big| + ||u_0'||_{C^0(M)}\Big|\int_U \varphi\sqrt{-1}\partial\bar\partial u_0'\Big| \\
&= \frac{1}{2}||u_0'||^2_{C^0(M)}\Big|\int_U \sqrt{-1}\partial\bar\partial\varphi\Big| \\
&\quad +||u_0'||_{C^0(M)}\Big|\int_U \varphi(T\,\mathrm{Ric}(\omega_0) - \omega_0 + c\omega_N)\Big| \\
&\leq C_U(||u_0'||^2_{C^0(M)} + ||u_0'||_{C^0(M)}) < \infty
\end{aligned}$$

for some positive constant $C_U = C(U, \omega_0, \omega_N)$, where we used that we have

$$\sqrt{-1}\partial\bar\partial u_0' = T\,\mathrm{Ric}(\omega_0) - \omega_0 + c\omega_N$$

as currents on $M\backslash E$. It follows that we have $u_0' \in W^{1,2}(M\backslash E)$ since $U$ is chosen arbitrary.

From the equality $\tilde\omega_0 + \sqrt{-1}\partial\bar\partial u_0 = c\omega_N$,

$$\Delta_0 u_0' = -\mathrm{tr}_{\omega_0}(\omega_0 - T\,\mathrm{Ric}(\omega_0) - c\omega_N) =: F_{M\backslash E}$$

holds in the weak sense of currents on $M \setminus E$ for $(g_0)^{i\bar j}|_{M\backslash E}, F_{M\backslash E} \in C^\infty(M \setminus E)$ and $u_0' \in W^{1,2}(M \setminus E)$, where $\Delta_0$ is the Laplacian of $\omega_0$, and $\omega_0 = \sqrt{-1}\sum_{i,j}(g_0)_{i\bar j}dz^i \wedge d\bar z^j$ in local coordinates. Then, by applying the regularity theory for weak solutions (cf. [25, Theorem 8.10]), we have $u_0' \in W^{m,2}(M\setminus E)$ for any $m \in \mathbb{N}$, and by the Sobolev imbedding theorem (cf. [25, Corollary 7.11, Corollary 8.11]), we have $u_0' \in C^\infty(M \setminus E)$.

We similarly compute for arbitrary chosen open set $V \subset E$, for an arbitrary chosen test function $\varphi \in C_0^\infty(V)$

$$\begin{aligned}
\Big|\int_V \varphi\sqrt{-1}\partial f_0' \wedge \bar\partial f_0'\Big| &= \Big|\frac{1}{2}\int_V (f_0')^2\sqrt{-1}\partial\bar\partial\varphi - \int_V \varphi f_0'\sqrt{-1}\partial\bar\partial f_0'\Big| \\
&\leq \frac{1}{2}||f_0'||^2_{C^0(M)}\Big|\int_V \sqrt{-1}\partial\bar\partial\varphi\Big| + ||f_0'||_{C^0(M)}\Big|\int_V \varphi\sqrt{-1}\partial\bar\partial f_0'\Big| \\
&= \frac{1}{2}||f_0'||^2_{C^0(M)}\Big|\int_V \sqrt{-1}\partial\bar\partial\varphi\Big| + ||f_0'||_{C^0(M)}\Big|\int_V \varphi(T\,\mathrm{Ric}(\omega_0) - \omega_0)\Big| \\
&\leq C_V(||f_0'||^2_{C^0(M)} + ||f_0'||_{C^0(M)}) < \infty
\end{aligned}$$

for some positive constant $C_V = C(V, \omega_0)$, where we used that we have

$$\sqrt{-1}\partial\bar\partial f_0' = T\,\mathrm{Ric}(\omega_0) - \omega_0$$

as currents on $E$. It follows that we have $f_0' \in W^{1,2}(E)$ since $V$ is chosen arbitrary. Symmetrically, with using that we have as currents on $E$, $\sqrt{-1}\partial\bar{\partial}u_0|_E = 0$, we obtain the following estimate for any open set $V \subset E$:

$$\Big|\int_V \varphi\sqrt{-1}\partial u_0 \wedge \bar{\partial}u_0\Big| \leq C_V' ||u_0||^2_{C^0(M)} < \infty$$

for some positive constant $C_V'$. Hence, we also have $u_0 \in W^{1,2}(E)$.

From $\tilde{\omega}_0|_E = 0$ in the weak sense on $E$, the following equation

$$\Delta_0 f_0' = -\mathrm{tr}_{\omega_0}(\omega_0 - T\,\mathrm{Ric}(\omega_0)) =: F_E$$

holds in the weak sense of currents on $E$ for $f_0' \in W^{1,2}(E)$ and $(g_0)^{i\bar{j}}|_E, F_E \in C^\infty(E)$. By applying the regularity theory for weak solutions, we have $f_0' \in W^{m,2}(E)$ for any $m \in \mathbb{N}$, and by the Sobolev imbedding theorem, we have $f_0' \in C^\infty(E)$.

From $\sqrt{-1}\partial\bar{\partial}u_0|_E = 0$ in the weak sense on $E$, the following equation

$$\Delta_0 u_0 = 0$$

holds in the weak sense of currents on $E$ for $u_0 \in W^{1,2}(E)$ and $(g_0)^{i\bar{j}}|_E \in C^\infty(E)$. By applying the regularity theory for weak solutions, we have $u_0 \in W^{m,2}(E)$ for any $m \in \mathbb{N}$, and by the Sobolev imbedding theorem, we have $u_0 \in C^\infty(E)$ and $u_0|_E$ is a constant function on $E$ since $E$ is compact.

Hence, combining these, we have $u_0' = f_0' + u_0 \in C^\infty(E)$ and then together with $u_0' \in C^\infty(M \setminus E)$, we obtain that

$$u_0' \in C^\infty(M).$$

As a consequence, there exist a smooth function $u_0'$ on $M$ and a Gauduchon metric $\hat{\omega}_N$ on $N$ such that

$$\omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}u_0' = \pi^*\hat{\omega}_N,$$

where $\hat{\omega}_N = c\omega_N$.

Therefore, we conclude that we can remove the assumption (†) from the convergence theorem in the Gromov-Hausdorff sense in [70, Theorem 1.3] on a non-Kähler compact complex surface.

# Chapter 4

# Continuity of the Chern-Ricci flow after the singular time on non-Kähler compact complex surfaces

## 4.1 Continuous existence on the space-time region

Let $M$ be a non-Kähler compact complex surface, and let $\omega_0$ be a Gauduchon metric on $M$. The Chern-Ricci flow $\omega(t)$ starting at $\omega_0$ is a flow of Gauduchon metrics

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = -\operatorname{Ric}(\omega(t)), \\ \\ \omega(t)|_{t=0} = \omega_0, \end{cases}$$

for $t \in [0, T)$ where $T = T(\omega_0)$ is a finite singular time with $0 < T \leq \infty$ stated by

$$T = \sup\{t \geq 0 | \exists \psi \in C^\infty(M) \text{ with } \omega_0 - t\operatorname{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi > 0\},$$

where $\operatorname{Ric}(\omega_0)$ is the Chern-Ricci form associated to $\omega_0$. It was shown that a unique maximal solution of the Chern-Ricci flow $\omega(t)$ for $t \in [0, T)$ for a number $T \in (0, \infty]$ determined by $\omega_0$. If the volume of $M$ with respect to $\omega(t)$ tends to zero as $t \to T$, we say that $\omega(t)$ is collapsing at $T$. Otherwise, we say that $\omega(t)$ is non-collapsing at $T$.

Let $N$ be a non-Kähler compact complex surface and $\pi$ be a blow-down map of disjoint irreducible finitely many $(-1)$-curves to some points. For simplicity, we consider the map $\pi$ blows down the only one $(-1)$-curve $E$ to a point $y_0 \in N$. Note that then we have $M \setminus E \cong N \setminus \{y_0\}$ biholomorphic via $\pi|_{M\setminus E}$. We are going to show that there exists a smooth solution of the Chern-Ricci flow $\omega(t)$ on $N$ for $t \in (T, T']$ for some $T' > T$, where $T > 0$ is the singular time of the Chern-Ricci flow $\omega(t)$ on $M$. Then we can prove that the Chern-Ricci flow $\omega(t)$ can be smoothly connected at time $T$ between $[0, T) \times M$ and $(T, T'] \times N$, outside $T \times \{y_0\} \cong T \times E$ via the map $\pi$. We define the space-time region

$$\mathcal{R} := ([0, T) \times M) \cup (T \times (N \setminus \{y_0\})) \cup ((T, T'] \times N).$$

We specify the meaning of that $\omega(t)$ is smooth on the region $\mathcal{R}$ in the following.

**Remark 4.1.1.** Consider a family of metrics $\omega(t,x)$ for $(t,x) \in \mathcal{R}$. For $t \in [0,T)$, $t \in (T,T']$, we require $\omega(t)$ to be smooth at $t$ in the usual sense, in $M, N$ respectively. On the other hand, if $(t,x) = (T,x) \in T \times (N \setminus \{y_0\}) \cong T \times (M \setminus E)$, then we choose a sufficiently small neighborhood $U$ of $x$ in $M \setminus E$ and we consider $\omega$ as a metric on $(T-\delta, T+\delta) \times U$ for some $\delta > 0$ via the map $\pi$. We say $\omega(t)$ is smooth at $(T,x)$ if $\omega(t)$ is smooth at $(T,x)$ in $(T-\delta, T+\delta) \times U$. In the same way, we can define what it means for $\omega(t)$ to satisfy a PDE at an arbitrary point of $\mathcal{R}$.

In this sense, we can continue the Chern-Ricci flow starting at a Gauduchon metric until we contract all finitely many $(-1)$-curves on a given non-Kähler compact complex surface and eventually reach a minimal surface. Additionally, that $(N, \omega(t))$ converge to $(N, d_T)$ in the Gromov-Hausdorff sense can be shown by the same way as in section 6 in [59] with using Lemma 3.4 and Lemma 3.5 in [70].

The result of Theorem 3.1.1 indicates that the requirement of the cohomology classes for the convergence of the Chern-Ricci flow:

$$(\dagger) \quad [\omega_0] + Tc_1^{BC}(K_M) = [\pi^* \hat{\omega}_N]$$

holds under the assumptions in Theorem 3.1.1. Then, we can say that the Chern-Ricci flow performs a canonical surgical contraction in the sense of Definition 1.2.5:

**Theorem 4.1.1.** Let $\omega(t)$ be a smooth solution of the Chern-Ricci flow on $M$ starting at $\omega_0$ for $t \in [0,T)$, $0 < T < \infty$. Assume that $\omega(t)$ is non-collapsing at $T$. Suppose that there exists a blow-down map $\pi : M \to N$ contracting the only one $(-1)$-curve $E$ to the point $y_0 \in N$. Then the Chern-Ricci flow $\omega(t)$ performs a canonical surgical contraction with respect to the data $E$, $N$ and $\pi$.

As considering the definition in [70], in order to say that $g(t)$ performs a canonical surgical contraction in the sense of [70], it additionally requires to show that $(N, d_T)$ is the metric completion of $(N \setminus \{y_0\}, d_{g_T})$, where these notations are the same as in Defenition 1.2.5. It only suffices to prove that $d_{g_T} = d_T|_{N \setminus \{y_0\}}$. In the Kähler case (cf. [60]), this can be shown with using the fact that any Kähler metrics are locally given by Kähler potentials. Hence we expect that it requres new techniques in the non-Kähler case.

## 4.2 Key estimates

We will proceed our argument along the way of Section 5 of [59] and we will state some of its results in the Hermitian case. Remark that our computations are valid for general complex dimension $n$, but we will only focus on surfaces.

From Theorem 3.1.1, we may assume to always have the condition

$$(\dagger) \qquad \omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}u_0' = \pi^* \hat{\omega}_N$$

for a smooth real function $u_0'$ on $M$ and a Gauduchon metric $\hat{\omega}_N$ on $N$. Then the Chern-Ricci flow

$$(CRF) \qquad \frac{\partial}{\partial t}\omega(t) = -\,\mathrm{Ric}(\omega(t)), \quad \omega(t)|_{t=0} = \omega_0$$

on $M$ is written with using $u_0'$ and $\hat{\omega}_N$ in the following way:

$$\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_t,$$

where $\hat{\omega}_t := \frac{1}{T}((T-t)\omega_0 + t\pi^*\hat{\omega}_N)$ and $\varphi_t$ solves the parabolic Monge-Ampère flow:

$$(MAF) \qquad \frac{\partial}{\partial t}\varphi_t = \log\frac{\omega(t)^2}{\Omega}, \quad \varphi_t|_{t=0} = 0,$$

with $\Omega = \omega_0^2 e^{\frac{u_0'}{T}}$. Note that $\varphi_t$ is uniformly bounded from above and below on $M \times [0,T)$.

One can show that the two flows $(CRF)$ and $(MAF)$ are essentially equivalent:

If $\varphi_t$ solves $(MAF)$, then taking $\sqrt{-1}\partial\bar{\partial}$ of $(MAF)$ shows that

$$\begin{aligned}
\sqrt{-1}\partial\bar{\partial}\Big(\frac{\partial}{\partial t}\varphi_t\Big) &= \sqrt{-1}\partial\bar{\partial}\log\frac{\omega(t)^2}{\omega_0^2} - \frac{1}{T}\sqrt{-1}\partial\bar{\partial}u_0' \\
&= -\operatorname{Ric}(\omega(t)) + \operatorname{Ric}(\omega_0) - \frac{1}{T}\pi^*\hat{\omega}_N + \frac{1}{T}(\omega_0 - T\operatorname{Ric}(\omega_0)) \\
&= -\operatorname{Ric}(\omega(t)) - \frac{\partial}{\partial t}\hat{\omega}_t,
\end{aligned}$$

which implies we have $(CRF)$. Conversely, if $\omega(t)$ solves $(CRF)$, then we have

$$\begin{aligned}
\frac{\partial}{\partial t}(\omega(t) - \hat{\omega}_t) &= -\operatorname{Ric}(\omega(t)) - \frac{1}{T}\omega_0 + \frac{1}{T}\pi^*\hat{\omega}_N \\
&= \sqrt{-1}\partial\bar{\partial}\Big(\log\frac{\omega(t)^2}{\omega_0^2} - \frac{1}{T}u_0'\Big) \\
&= \sqrt{-1}\partial\bar{\partial}\log\frac{\omega(t)^2}{\Omega}
\end{aligned}$$

so if we choose $\varphi_t$ to solve $(MAF)$, which is an ODE in $t$ for fixed point on $M$, then we obtain

$$\frac{\partial}{\partial t}(\omega(t) - \hat{\omega}_t - \sqrt{-1}\partial\bar{\partial}\varphi_t) = 0$$

so that indeed $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_t$ and $\varphi_t$ satisfies $(MAF)$.

Since the positive current $\omega(T)$, which is smooth on $M'$, can be written by

$$\omega(T) = \pi^*\hat{\omega}_N + \sqrt{-1}\partial\bar{\partial}\varphi_T \geq 0,$$

where $\varphi_T$ is a bounded function satisfies $\varphi_T|_E \equiv$ constant since we have

$$\sqrt{-1}\partial\bar{\partial}\varphi_T|_E = \omega(T)|_E \geq 0$$

and then we apply the strong maximum principle. Hence, from the properties of the blow-down map $\pi$, there exists a bounded function $\psi_T$ on $N$, smooth on $N \setminus \{y_0\}$, with $\varphi_T = \pi^*\psi_T$. Especially, we have $\psi_T \in \mathrm{PSH}(N \setminus \{y_0\}, \hat{\omega}_N) \cap C^0(N \setminus \{y_0\})$.

We here define a $\partial\bar{\partial}$-closed positive $(1,1)$-current $\omega'$ on $N$ by

$$\omega' := \hat{\omega}_N + \sqrt{-1}\partial\bar{\partial}\psi_T \geq 0,$$

which is smooth and positive on $N \setminus \{y_0\}$ and satisfies $\pi^*\omega' = \omega(T)$. We have

$$0 \leq \frac{\omega'^2}{\hat{\omega}_N^2} \in L^p(N, \hat{\omega}_N^2)$$

for some $p > 1$ sufficiently close to 1 (cf. [59, Lemma 5.4]) and $\omega'^2 > 0$ on $N \setminus \{y_0\}$. We consider the equation

$$\omega'^2 = \hat{f}\hat{\omega}_N^2$$

on $N \setminus \{y_0\}$, where we put $\hat{f} := \frac{\omega'^2}{\hat{\omega}_N^2}$. We normalize $\psi_T$ such that $\sup_{N\setminus\{y_0\}} \psi_T = 0$. Then $\psi_T$ is a unique continuous $\hat{\omega}_N$-psh solution of the equation $\omega'^2 = \hat{f}\hat{\omega}_N^2$ on $N \setminus \{y_0\}$.

We would like to construct a solution of the Chern-Ricci flow on $N$ starting at the metric $\omega'$. We fix a smooth $d$-closed $(1,1)$ form $-\chi \in c_1^{BC}(N)$. Then there exists $T' > T$ sufficiently close to $T$ such that for all $t \in [T, T']$, the following $(1,1)$-form

$$\hat{\omega}_{t,N} := \hat{\omega}_N + (t - T)\chi$$

is Gauduchon. We also fix a smooth volume form $\Omega_N$ on $N$ satisfying

$$\mathrm{Ric}(\Omega_N) = -\sqrt{-1}\partial\bar{\partial}\log\Omega_N = -\chi \in c_1^{BC}(N).$$

For $\varepsilon > 0$ sufficiently small, and $A$ sufficiently large, define a family of volume forms $\Omega_\varepsilon$ on $N$ by

$$\Omega_\varepsilon := (\pi|_{M'}^{-1})^*\Big(\frac{|s|_h^{2A}\omega(T-\varepsilon)^2}{|s|_h^{2A} + \varepsilon}\Big) + \varepsilon\Omega_N$$

on $N \setminus \{y_0\}$, and $\Omega_\varepsilon|_{y_0} = \varepsilon\Omega_N|_{y_0}$ where $s$ is a holomorphic section with $E = (s)$, where $(s)$ is a principal divisor defined by $s$, and $h$ is a smooth Hermitian metric on the holomorphic line bundle $[E]$ associated to the effective divisor $E$ respectively. Note that $\Omega_\varepsilon$ is smooth on $N \setminus \{y_0\}$. By choosing $A$ sufficiently large , the volume form $\Omega_\varepsilon$ lies in $C^l(N)$ for a fixed large constant $l$. And note that $\Omega_\varepsilon$ converges to $\omega'^2$ in $C^\infty$ on any compact subsets of $N \setminus \{y_0\}$ as $\varepsilon \to 0$. Now, for each $\varepsilon > 0$, by the theorem of Tosatti and Weinkove (Theorem 2.5.5), there exist a unique constant $C_\varepsilon \in \mathbb{R}_{>0}$ and a unique function $\psi_{T,\varepsilon} \in C^k(N) \cap C^\infty(N \setminus \{y_0\})$ for some positive integer $k$ with $\sup_{N\setminus\{y_0\}} \psi_{T,\varepsilon} = 0$ such that

$$(\hat{\omega}_N + \sqrt{-1}\partial\bar{\partial}\psi_{T,\varepsilon})^2 = C_\varepsilon \hat{f}_\varepsilon \hat{\omega}_N^2,$$

where we put $\hat{f}_\varepsilon := \frac{\Omega_\varepsilon}{\hat{\omega}_N^2}$. Since we have $0 \leq \frac{\omega'^2}{\hat{\omega}_N^2} \in L^p(N, \hat{\omega}_N^2)$ for some $p > 1$, $\hat{f}_\varepsilon$'s are uniformly bounded in $L^p(N, \hat{\omega}_N^2)$ for some $p > 1$. Notice that we can freely rase $k$ by increasing $l$ and $A$. The constants $C_\varepsilon > 0$ satisfy that $C_\varepsilon \to 1$ as $\varepsilon \to 0$. We define the admissible function $h(x) := CC_\varepsilon^{-1}||\hat{f}_\varepsilon||_{L^p(N,\hat{\omega}_N^2)}^{-1}\exp(ax)$ for some uniform constants $C, a > 0$. Then $(\hat{\omega}_N + \sqrt{-1}\partial\bar{\partial}\psi_{T,\varepsilon})^2$ satisfies $(\clubsuit)_{\hat{\omega}_N}$ from Proposition 2.5.3, and then we may apply Proposition 2.5.4.

**Lemma 4.2.1.** There exists a uniform positive constant $C = C(||f_\varepsilon||_{L^p}, N, \hat{\omega}_N) > 0$ such that

$$\frac{1}{C} \leq C_\varepsilon \leq C.$$

PROOF. Fix $0<\delta<1$. Define $S_\varepsilon := \inf_N \psi_{T,\varepsilon}$ and

$$\delta_0 := \frac{1}{3}\min\{\delta^2, \frac{\delta^3}{16B}, 4(1-\delta)\delta^2, 4(1-\delta)\frac{\delta^3}{16B}\}.$$

Then for $0<s,t<\delta_0$, we have (Remark 2.5.1)

$$\begin{aligned} t^2\mathrm{cap}_{\hat{\omega}_N}(\{\psi_{T,\varepsilon}<S_\varepsilon+s\}) &\leq C\int_{\{\psi_{T,\varepsilon}<S_\varepsilon+s+t\}} C_\varepsilon \hat{f}_\varepsilon \hat{\omega}_N^2 \\ &\leq CC_\varepsilon ||\hat{f}_\varepsilon||_{L^p(N,\hat{\omega}_N^2)} \mathrm{Vol}_{\hat{\omega}_N}(\{\psi_{T,\varepsilon}<S_\varepsilon+s+t\})^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Hence for fixed $0<s=t<\delta_0$, we obtain

$$\begin{aligned} \mathrm{cap}_{\hat{\omega}_N}(\{\psi_{T,\varepsilon}<S_\varepsilon+s\}) &\leq \frac{CC_\varepsilon}{s^n}||\hat{f}_\varepsilon||_{L^p(N,\hat{\omega}_N^2))} \mathrm{Vol}_{\hat{\omega}_N}(\{\psi_{T,\varepsilon}<S_\varepsilon+2s\})^{\frac{1}{q}} \\ &\leq \frac{C'C_\varepsilon}{s^n}\mathrm{Vol}_{\hat{\omega}_N}(N)^{\frac{1}{q}} =: C_\varepsilon C_1 s^{-n} \end{aligned}$$

for some uniform constant $C'>0$, where we used that $||\hat{f}_\varepsilon||_{L^p(N,\hat{\omega}_N^2))}$ is uniformly bounded from above. and then from Proposition 2.5.4,

$$s \leq \kappa(\mathrm{cap}_{\hat{\omega}_N}(\{\psi_{T,\varepsilon}<S_\varepsilon+s\})) \leq \kappa(C_\varepsilon C_1 s^{-n}).$$

Since $\lim_{x\to 0^+}\kappa(x)=0$, $C_\varepsilon$ must be uniformly bounded away from 0.

Since $\hat{f}_\varepsilon \to \hat{f}$ in $L^1(N,\hat{\omega}_N^2)$, we also have $\hat{f}_\varepsilon^{\frac{1}{2}} \to \hat{f}^{\frac{1}{2}}$ in $L^1(N,\hat{\omega}_N^2)$. Since we have $\int_N \hat{f}^{\frac{1}{2}}\hat{\omega}_N^2>0$, for $\varepsilon$ sufficiently small, we obtain

$$\int_N \hat{f}_\varepsilon^{\frac{1}{2}}\hat{\omega}_N^2 > \frac{1}{2}\int_N \hat{f}^{\frac{1}{2}}\hat{\omega}_N^2>0.$$

By the pointwise arithmetic-geometric means inequality implies that

$$(\hat{\omega}_N+\sqrt{-1}\partial\bar{\partial}\psi_{T,\varepsilon})\wedge\hat{\omega}_N \geq \Big(\frac{(\hat{\omega}_N+\sqrt{-1}\partial\bar{\partial}\psi_{T,\varepsilon})^2}{\hat{\omega}_N^2}\Big)^{\frac{1}{2}}\hat{\omega}_N^2 = (C_\varepsilon\hat{f}_\varepsilon)^{\frac{1}{2}}\hat{\omega}_N^2.$$

It follows that for sufficiently small $\varepsilon$,

$$C_\varepsilon^{\frac{1}{2}} \leq \frac{2}{\int_N \hat{f}^{\frac{1}{2}}\hat{\omega}_N^2}\int_N(\hat{\omega}_N+\sqrt{-1}\partial\bar{\partial}\psi_{T,\varepsilon})\wedge\hat{\omega}_N = \frac{2}{\int_N \hat{f}^{\frac{1}{2}}\hat{\omega}_N^2}\int_N \hat{\omega}_N^2,$$

where we used the Stokes theorem and that $\hat{\omega}_N$ is Gauduchon. □

Suppose that there exists a subsequence $C_{\varepsilon_k}\to c\neq 1$ as $k\to\infty$. Consider the equation

$$(\hat{\omega}_N+\sqrt{-1}\partial\bar{\partial}\psi_{T,\varepsilon_k})^2 = C_{\varepsilon_k}\hat{f}_{\varepsilon_k}\hat{\omega}_N^2.$$

Then since the family $\{\psi_{T,\varepsilon_k}\in \mathrm{PSH}(\hat{\omega}_N)\cap C^0(N\setminus\{y_0\}); \sup_{N\setminus\{y_0\}}\psi_{T,\varepsilon_k}=0\}$ is relatively compact in $L^1(N\setminus\{y_0\},\hat{\omega}_N^2)$, after passing a subsequence, still write $\psi_{T,\varepsilon_k}$, since $C_{\varepsilon_k}\hat{f}_{\varepsilon_k}$ are uniformly bounded in $L^p(N,\hat{\omega}_N^2)$ for some $p>1$ sufficiently close to 1, we have that $\{\psi_{T,\varepsilon_k}\}$

is a Cauchy sequence in $C^0(N \setminus \{y_0\})$ (Corollary 2.5.3). This means that $\psi_{T,\varepsilon_k} \to \psi'_T$ for some $\psi'_T \in \mathrm{PSH}(\hat{\omega}_N) \cap C^0(N \setminus \{y_0\})$ in $C^0(N \setminus \{y_0\})$-topology with $\sup_{N\setminus\{y_0\}} \psi'_T = 0$. By the Bedford-Taylor convergence theorem (Theorem 2.5.1), we obtain by taking the limit on $N \setminus \{y_0\}$, since $\psi_T$ is the unique solution of the equation $(\omega')^2 = \hat{f}\hat{\omega}_N^2$,

$$(\omega')^2 = (\hat{\omega}_N + \sqrt{-1}\partial\bar{\partial}\psi'_T)^2 = c\hat{f}\hat{\omega}_N^2 = c(\omega')^2,$$

which is a contradiction. Hence we conclude that $C_\varepsilon \to 1$ as $\varepsilon \to 0$.

For the following two equations

$$(\hat{\omega}_N + \sqrt{-1}\partial\bar{\partial}\psi_T)^2 = \frac{\omega'^2}{\hat{\omega}_N^2}\hat{\omega}_N^2, \quad (\hat{\omega}_N + \sqrt{-1}\partial\bar{\partial}\psi_{T,\varepsilon})^2 = \frac{C_\varepsilon\Omega_\varepsilon}{\hat{\omega}_N^2}\hat{\omega}_N^2,$$

we apply the stability theorem:

**Proposition 4.2.1.** ([50, Theorem A.])Let $(X^n, \omega)$ be a compact $n$-dimensional Hermitian manifold. Let $0 \leq f, g \in L^p(X, \omega^n)$, $p > 1$, be such that $\int_X f\omega^n > 0$, $\int_X g\omega^n > 0$. Consider two continuous $\omega$-psh solutions of the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f\omega^n, \quad (\omega + \sqrt{-1}\partial\bar{\partial}v)^n = g\omega^n$$

with $\sup_X u = \sup_X v = 0$. Assume that $f$ that

$$f \geq c_0 > 0$$

for some uniform positive constant $c_0 > 0$. Fix $0 < \alpha < \frac{1}{n+1}$. Then, there exists a positive constant $C = C(c_0, \alpha, ||f||_{L^p}, ||g||_{L^p}) > 0$ such that

$$||u - v||_{L^\infty} \leq C||f - g||_{L^p}^\alpha.$$

Now we apply Proposition 4.2.1 for $X = N \setminus \{y_0\}$, $u = \psi_{T,\varepsilon}$, $v = \psi_T$, $f = \frac{C_\varepsilon\Omega_\varepsilon}{\hat{\omega}_N^2}$ and $g = \frac{\omega'^2}{\hat{\omega}_N^2}$, since we have $0 \leq \frac{\omega'^2}{\hat{\omega}_N^2}, \frac{C_\varepsilon\Omega_\varepsilon}{\hat{\omega}_N^2} \in L^p(N, \hat{\omega}_N^2)$ for some $p > 1$ and $(\omega')^2 > 0$ on $N \setminus \{y_0\}$, which indicates that we can choose a uniform constant $c_0 > 0$ independent of $\varepsilon$ such that

$$\frac{C_\varepsilon\Omega_\varepsilon}{\hat{\omega}_N^2} \geq c_0 > 0.$$

Then we obtain, for arbitrary fixed $0 < \alpha < \frac{1}{3}$,

$$||\psi_{T,\varepsilon} - \psi_T||_{L^\infty(N\setminus\{y_0\})} \leq C\left|\left|\frac{C_\varepsilon\Omega_\varepsilon}{\hat{\omega}_N^2} - \frac{\omega'^2}{\hat{\omega}_N^2}\right|\right|_{L^p(N\setminus\{y_0\})}^\alpha.$$

Hence $\psi_{T,\varepsilon}$ converges to $\psi_T$ on $N \setminus \{y_0\}$ in $L^\infty$-topology as $\varepsilon \to 0$. It follows that we obtain that as $\varepsilon \to 0$,

$$||\psi_{T,\varepsilon} - \psi_T||_{L^\infty(N)} = \sup_N |\psi_{T,\varepsilon} - \psi_T| = \sup_{N\setminus\{y_0\}} |\psi_{T,\varepsilon} - \psi_T| \to 0.$$

Thus we have that $\psi_T \in \mathrm{PSH}(N, \hat{\omega}_N) \cap C^0(N)$ with $\sup_N \psi_T = 0$.

With using the regularity of the functions $\psi_{T,\varepsilon}$, we can show the following result for solutions $\varphi_\varepsilon = \varphi_\varepsilon(t)$ of the parabolic complex Monge-Ampère equations

$$\frac{\partial}{\partial t}\varphi_\varepsilon = \log \frac{(\hat{\omega}_{t,N} + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon)^2}{\Omega_N}, \quad \text{for } t \in [T, T'], \quad \varphi_\varepsilon|_{t=T} = \psi_{T,\varepsilon},$$

which is equivalent to the Chern-Ricci flow

$$\frac{\partial}{\partial t}\omega_\varepsilon(t) = -\operatorname{Ric}(\omega_\varepsilon(t)), \quad \text{for } t \in [T, T'], \quad \omega_\varepsilon(T) = \omega_{T,\varepsilon}$$

where $\omega_\varepsilon = \omega_\varepsilon(t) := \hat{\omega}_{t,N} + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon$ and $\omega_{T,\varepsilon} := \hat{\omega}_N + \sqrt{-1}\partial\bar{\partial}\psi_{T,\varepsilon}$. Then we obtain the following results as in [59].

**Proposition 4.2.2.** ([59, Proposition 5.1])
There exists a function $\varphi \in C^0([T, T'] \times N) \cap C^\infty((T, T'] \times N)$ such that

(1) $||\varphi_\varepsilon||_{L^\infty} \leq C$ for some uniform constant $C > 0$ for all $\varepsilon > 0$ sufficiently small.

(2) $\varphi_\varepsilon \to \varphi$ in $L^\infty([T, T'] \times N)$.

(3) The convergence $\varphi_\varepsilon \to \varphi$ is $C^\infty$ on compact subsets of $(T, T'] \times N$.

(4) $\varphi$ is the unique solution of

$$\frac{\partial}{\partial t}\varphi = \log \frac{(\hat{\omega}_{t,N} + \sqrt{-1}\partial\bar{\partial}\varphi)^2}{\Omega_N}, \quad \varphi|_{t=T} = \psi_T$$

for $t \in (T, T']$ in the space $C^0([T, T'] \times N) \cap C^\infty((T, T'] \times N)$.

We take advantage of the following result:

**Proposition 4.2.3.** ([54,Theorem 1.1, Corollary 1.2.])
Fix $r$ with $0 < r < 1$. Let $\omega(t)$ solve the Chern-Ricci flow for $t \in [0, T_0]$, $T_0 < \infty$, starting at $\omega_0$; a Hermitian metric on a Hermitian manifold $M$, in a neighborhood of $B_r$, which is the ball of radius $r$ at the origen in $\mathbb{C}^n$, for $t \in [0, T_0]$. Assume $R > 1$ satisfies

$$\frac{1}{R}\omega_0 \leq \omega(t) \leq R\omega_0 \quad \text{on } B_r \times [0, T_0].$$

Then there exist positive constants $C$, $\alpha$, $\beta$ depending only on $\omega_0$ such that

(1) $|\nabla^0\omega|^2_\omega \leq \frac{CR^\alpha}{r^2}$ on $B_{\frac{r}{2}} \times [0, T_0]$, where $\nabla^0$ is the Chern connection of $\omega_0$.

(2) $|Rm|^2_\omega \leq \frac{CR^\beta}{r^4}$ on $B_{\frac{r}{4}} \times [0, T_0]$, for $Rm$ the Chern curvature tensor of $\omega$.

(3) For any $\delta > 0$ with $0 < \delta < T_0$, there exist constants $C_m$, $\alpha_m$ and $\gamma_m$ for $m = 1, 2, 3, \ldots$ depending only on $\omega_0$ and $\delta$ such that

$$|(\nabla^0_{\mathbb{R}})^m\omega|^2_{\omega_0} \leq \frac{C_m R^{\alpha_m}}{r^{\gamma_m}} \quad \text{on } B_{\frac{r}{8}} \times [\delta, T_0],$$

where $\nabla^0_{\mathbb{R}}$ is the Levi-Civita covariant derivative associated to $\omega_0$.

For the metric $\omega_{T,\varepsilon}$, which is smooth away from $y_0$, we have the following estimate:

**Lemma 4.2.2.** For all sufficiently small $\varepsilon > 0$, there exist positive uniform constants $C$, $\alpha$, independent of such $\varepsilon$, such that on $N \setminus \{y_0\}$,

$$\frac{|s|_h^{2\alpha}}{C}\hat{\omega}_N \leq \omega_{T,\varepsilon} \leq \frac{C}{|s|_h^{2\alpha}}\hat{\omega}_N.$$

Fix a large positive integer $K$. Then for each integer $0 \leq k \leq K$ there exist $C_k, \alpha_k > 0$ such that

$$|(\hat{\nabla}^N_{\mathbb{R}})^k \omega_{T,\varepsilon}|^2_{\hat{\omega}_N} \leq \frac{C_k}{|s|_h^{2\alpha_k}},$$

where $\hat{\nabla}^N_{\mathbb{R}}$ denotes the real Levi Civita covariant derivative with respect to the metric $\hat{\omega}_N$.

**Remark 4.2.1.** We identify a small neighborhood of $y_0 \in Y$ with a small ball $B$ centered at the origin of $\mathbb{C}^2$. From the property of the blow-down map $\pi$, we identify via $\pi$ the sets $\pi^{-1}(B \setminus \{0\})$ and $B \setminus \{0\}$, and for the various functions and $(1,1)$-forms on these sets. For instance, we write $|s|_h^{2\alpha}$ as $(\pi|^{-1}_{M\setminus E})^*(|s|_h^{2\alpha})$ for simplicity.

Proof. We fix arbitrary sufficiently small real numbers $\varepsilon_0$ and $\delta$ with $\varepsilon_0 > \delta > 0$ and consider $\varepsilon \in [\delta, \varepsilon_0]$. From the definition of $\Omega_\varepsilon$, $\omega_{T,\varepsilon}^2 = C_\varepsilon \Omega_\varepsilon$, together with

$$(*) \qquad \frac{|s|_h^{2\eta}}{C}\omega_0 \leq \omega(t) \leq \frac{C}{|s|_h^{2\eta}}\omega_0$$

for $t \in [0,T)$ and for some uniform positive constants $C$, $\eta$, where $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_t$, $\hat{\omega}_t = \frac{1}{T}((T-t)\omega_0 + t\pi^*\hat{\omega}_N)$ (cf. [54, Theorem 1.1] and [59 ,Lemma 2.5]), we have, for

$$F_\varepsilon := \log\frac{\omega_{T,\varepsilon}^2}{\hat{\omega}_N^2} = \log\frac{C_\varepsilon\Omega_\varepsilon}{\hat{\omega}_N^2},$$

$$\left|\hat{\Delta}F_\varepsilon\right| = \left|\hat{\Delta}\log\left(\frac{\omega_{T,\varepsilon}^2}{\hat{\omega}_N^2}\right)\right| = \left|-\operatorname{tr}_{\hat{\omega}_N}\operatorname{Ric}(\Omega_\varepsilon) + \operatorname{tr}_{\hat{\omega}_N}\operatorname{Ric}(\hat{\omega}_N)\right| \leq \frac{C}{|s|_h^{2\beta}}$$

for some uniform constants $\beta, C > 0$, where $\hat{\Delta}$ for the Laplacian with respect to $\hat{g}$ (cf. [59,Lemma 5.3]).

By choosing local coordinates $(z_1, z_2)$, then locally we will write $\omega_{T,\varepsilon} = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j$, $\hat{\omega}_N = \sqrt{-1}\hat{g}_{i\bar{j}}dz^i \wedge d\bar{z}^j$. We write $\nabla$, $\hat{\nabla}$ for the Chern connections associated to $g$, $\hat{g}$ respectively. We also write $\Delta$ for the Laplacian of $g$.

Then we can estimate (cf. [72, Proposition 3.1])

$$\begin{aligned}
\Delta\log\operatorname{tr}_{\hat{g}}g &\geq -\frac{2}{(\operatorname{tr}_{\hat{g}}g)^2}\operatorname{Re}\Big(g^{k\bar{l}}\hat{T}^i_{ki}\hat{\nabla}_{\bar{l}}\operatorname{tr}_{\hat{g}}g\Big) - C\operatorname{tr}_g\hat{g} - \frac{1}{\operatorname{tr}_{\hat{g}}g}\operatorname{tr}_{\hat{g}}\operatorname{Ric}(g) \\
&\geq -\frac{2}{(\operatorname{tr}_{\hat{g}}g)^2}\operatorname{Re}\Big(g^{k\bar{l}}\hat{T}^i_{ki}\hat{\nabla}_{\bar{l}}\operatorname{tr}_{\hat{g}}g\Big) - C\operatorname{tr}_g\hat{g} - \frac{1}{\operatorname{tr}_{\hat{g}}g}\frac{C}{|s|_h^{2\beta}}
\end{aligned}$$

for some uniform constants $\beta, C > 0$, where $\hat{T}$ is the torsion tensor of $\hat{g}$ and $\mathrm{Ric}(g)$ is the second Ricci tensor with respect to $g$ and we used that $|\hat{\Delta} F_\varepsilon| \le \frac{C}{|s|_h^{2\beta}}$ for estimating

$$\Big|\mathrm{tr}_{\hat{g}}\, \mathrm{Ric}(g)\Big| \le \frac{C}{|s|_h^{2\beta}}$$

for some constant $C > 0$. Here note that we will write $\mathrm{Ric}_1$, Ric for the first Ricci tensor and the second Ricci tensor with respect to $g$:

$$(\mathrm{Ric}_1)_{k\bar{l}} = g^{i\bar{j}} R_{i\bar{j}k\bar{l}}, \quad \mathrm{Ric}_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}.$$

We define

$$Q := \log \mathrm{tr}_{\hat{g}} g - A\tilde{\psi}_{T,\varepsilon} + \frac{1}{\tilde{\psi}_{T,\varepsilon} + \tilde{C}}$$

for sufficiently large $A, \alpha > 0$, where $\tilde{\psi}_{T,\varepsilon} := \psi_{T,\varepsilon} - \frac{1}{A}\log |s|_h^{2\alpha}$ and $\tilde{C}$ is a constant such that $\tilde{\psi}_{T,\varepsilon} + \tilde{C} \ge 1$. Since $Q \to -\infty$ as $x \to y_0$, we may assume that $Q$ achieves its maximum at a point $x_0 \in N \setminus \{y_0\}$. Note that we may assume that $\mathrm{tr}_{\hat{g}} g \ge 1$ and $|s|_h^{2\beta} \le 1$ at $x_0$.

At the point $x_0$, we have

$$\frac{1}{\mathrm{tr}_{\hat{g}} g} \hat{\nabla}_{\bar{l}} \mathrm{tr}_{\hat{g}} g = \Big(A + \frac{1}{(\tilde{\psi}_{T,\varepsilon} + \tilde{C})^2}\Big) \partial_{\bar{l}} \tilde{\psi}_{T,\varepsilon}.$$

We compute at $x_0$,

$$\begin{aligned} 0 \ge \Delta Q \ &\ge\ -\frac{2}{(\mathrm{tr}_{\hat{g}} g)^2} \mathrm{Re}\Big(g^{k\bar{l}} \hat{T}^i_{ki} \hat{\nabla}_{\bar{l}} \mathrm{tr}_{\hat{g}} g\Big) - C \mathrm{tr}_g \hat{g} - \frac{C}{|s|_h^{2\beta}} \\ &\quad -\Big(A + \frac{1}{(\tilde{\psi}_{T,\varepsilon} + \tilde{C})^2}\Big) \mathrm{tr}_g\big(g - \hat{g} + \frac{\alpha}{A} R_h\big) + \frac{2|\partial \tilde{\psi}_{T,\varepsilon}|_g^2}{(\tilde{\psi}_{T,\varepsilon} + \tilde{C})^3} \\ &\ge\ (-C + Ac_0)\mathrm{tr}_g \hat{g} - \frac{C_A}{|s|_h^{2\beta}} \end{aligned}$$

for some constant $C_A > 0$, where we used that for an arbitrary fixed constant $1 > c_0 > 0$, $\hat{g} - \frac{\alpha}{A} R_h \ge c_0 \hat{g}$ for sufficiently large $A$, $R_h$ is the curvature of the smooth Hermitian metric $h$ given locally by

$$R_h = -\sqrt{-1}\partial\bar{\partial} \log h.$$

Remark that we have $\sqrt{-1}\partial\bar{\partial}\log h = \sqrt{-1}\partial\bar{\partial} \log |s|_h^2$ away from $y_0$. And we also estimated in the following way:

$$(\mho) \quad \Big|\frac{2}{(\mathrm{tr}_{\hat{g}} g)^2} \mathrm{Re}\Big(g^{k\bar{l}} \hat{T}^i_{ki} \hat{\nabla}_{\bar{l}} \mathrm{tr}_{\hat{g}} g\Big)\Big| \ \le\ \Big|\frac{2}{\mathrm{tr}_{\hat{g}} g} \mathrm{Re}\Big(\big(A + \frac{1}{(\tilde{\psi}_{T,\varepsilon} + \tilde{C})^2}\big) g^{k\bar{l}} \hat{T}^i_{ki} \partial_{\bar{l}} \tilde{\psi}_{T,\varepsilon}\Big)\Big|$$
$$\le\ \frac{|\partial \tilde{\psi}_{T,\varepsilon}|_g^2}{(\tilde{\psi}_{T,\varepsilon} + \tilde{C})^3} + CA^2 (\tilde{\psi}_{T,\varepsilon} + \tilde{C})^3 \frac{\mathrm{tr}_g \hat{g}}{(\mathrm{tr}_{\hat{g}} g)^2}.$$

Since we may assume that $(\mathrm{tr}_{\hat{g}} g)^2 \geq A^2(\tilde{\psi}_{T,\varepsilon} + \tilde{C})^3$, we have

$$-\frac{2}{(\mathrm{tr}_{\hat{g}} g)^2}\mathrm{Re}\Big(g^{k\bar{l}}\hat{T}^i_{ki}\hat{\nabla}_{\bar{l}}\mathrm{tr}_{\hat{g}} g\Big) \geq -\frac{|\partial\tilde{\psi}_{T,\varepsilon}|^2_g}{(\tilde{\psi}_{T,\varepsilon} + \tilde{C})^3} - C\mathrm{tr}_g \hat{g}.$$

If necessary, we again choose a much larger constant $A$ and then we have $Ac_0 > C$ in the estimate above. Therefore, we obtain

$$\mathrm{tr}_g \hat{g}(x_0) \leq \frac{C}{|s|_h^{2\beta}}.$$

Hence we have

$$\mathrm{tr}_{\hat{g}} g(x_0) \leq \mathrm{tr}_g \hat{g}(x_0) e^{F_\varepsilon} \leq \frac{C}{|s|_h^{2\beta}}.$$

Since $\psi_{T,\varepsilon}$ is uniformly bounded, we obtain

$$Q \leq Q(x_0) \leq \log(C|s|_h^{2(\alpha-\beta)}) + C \leq C$$

for $\alpha$ sufficiently large so that $\alpha > \beta$ and we obtain the desired estimate.

For the higher order estimates for $\omega_{T,\varepsilon}$, we firstly consider the quantity

$$S_{T,\varepsilon} := |(\nabla H_{T,\varepsilon})H_{T,\varepsilon}^{-1}|_g^2$$

where $(H_{T,\varepsilon})^i_l := \hat{g}^{i\bar{j}} g_{l\bar{j}}$, $\nabla$ is the covariant derivative with respect to $\omega_{T,\varepsilon} = g$ and we here write $\Delta$ for the rough Laplacian of $\omega_{T,\varepsilon}$, $\Delta = \nabla^{\bar{k}}\nabla_k$, where $\nabla^{\bar{k}} = g^{\bar{k}l}\nabla_l$ (cf. [52], [54]).

Note that we compute

$$H^i_{jl} := ((\nabla_j H_{T,\varepsilon})H_{T,\varepsilon}^{-1})^i_l = \Gamma^i_{jl} - \hat{\Gamma}^i_{jl}$$

where $\Gamma^i_{jl}$, $\hat{\Gamma}^i_{jl}$ denote the Christoffel symbols of $\omega_{T,\varepsilon} = g$, $\hat{\omega}_N = \hat{g}$ respectively and then we have

$$S_{T,\varepsilon} = |H|_g^2.$$

By commuting $\nabla$ and $\bar{\nabla}$, we obtain

$$\bar{\Delta} H^i_{jl} - \Delta H^i_{jl} = (\mathrm{Ric}_1)_j{}^r H^i_{rl} + (\mathrm{Ric}_1)_l{}^r H^i_{jr} - (\mathrm{Ric}_1)_r{}^i H^r_{jl}$$

for some constant $C > 0$, where $(\mathrm{Ric}_1)_j{}^r$ is the first Ricci tensor with respect to $\omega_{T,\varepsilon}$.

With using the inequality

$$(**) \qquad \frac{|s|_h^{2\alpha}}{C}\hat{\omega}_N \leq \omega_{T,\varepsilon} \leq \frac{C}{|s|_h^{2\alpha}}\hat{\omega}_N$$

and the following;

$$\Delta H^i_{jl} = \nabla^{\bar{k}}\hat{R}_{j\bar{k}l}{}^i - \nabla^{\bar{k}} R_{j\bar{k}l}{}^i,$$

where $R_{j\bar{k}l}{}^i$, $\hat{R}_{j\bar{k}l}{}^i$ are the Chern curvature tensors of $\omega_{T,\varepsilon}$, $\hat{\omega}_N$ respectively.

Here we notice that the Bianchi identities will not hold necessarily for general Hermitian manifolds: Let $(M, g)$ be a $n$-dimensional compact Hermitian manifold and let $\nabla$ be the Chern connection of $g$ with Christoffel symbols $\Gamma_{ij}^k$ and torsion $T$ given by:

$$\Gamma_{ij}^k = g^{k\bar{l}}\partial_i g_{j\bar{l}}, \quad T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$$

and

$$T_{ik\bar{l}} = T_{ik}^j g_{j\bar{l}} = \hat{T}_{ik}^j \hat{g}_{j\bar{l}} = \hat{T}_{ik\bar{l}}$$

since $g_{i\bar{j}} = \hat{g}_{i\bar{j}} + \partial_i\partial_{\bar{j}}\psi_{T,\varepsilon}$. There are extra torsion terms in the following identities:

$$R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} = -\nabla_{\bar{j}}T_{ik\bar{l}}$$

$$R_{i\bar{j}k\bar{l}} - R_{k\bar{l}i\bar{j}} = -\nabla_i T_{\bar{j}\bar{l}k}$$

$$R_{i\bar{j}k\bar{l}} - R_{k\bar{l}i\bar{j}} = -\nabla_{\bar{j}}T_{ik\bar{l}} - \nabla_k T_{\bar{j}\bar{l}i}$$

$$\nabla_p R_{i\bar{j}k\bar{l}} - \nabla_i R_{p\bar{j}k\bar{l}} = -T_{pi}^r R_{r\bar{j}k\bar{l}}$$

$$\nabla_{\bar{q}} R_{i\bar{j}k\bar{l}} - \nabla \bar{j} R_{i\bar{q}k\bar{l}} = -T_{\bar{q}\bar{j}}^{\bar{s}} R_{i\bar{s}k\bar{l}}.$$

With using the identities above, we then compute

$$\begin{aligned}
\Delta S_{T,\varepsilon} &= g^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\Big(g^{i\bar{a}}g^{j\bar{b}}g_{k\bar{c}}H_{ij}^k\overline{H_{ab}^c}\Big) \\
&= |\bar{\nabla}H|_g^2 + |\nabla H|_g^2 + g^{i\bar{a}}g^{j\bar{b}}g_{k\bar{c}}\Big(\Delta H_{ij}^k \cdot \overline{H_{ab}^c} + H_{ij}^k \cdot \overline{\bar{\Delta} H_{ab}^c}\Big) \\
&= |\bar{\nabla}H|_g^2 + |\nabla H|_g^2 + 2\mathrm{Re}\Big((-\nabla_p R_{i\bar{p}j\bar{l}}g^{k\bar{l}} + \nabla_p \hat{R}_{i\bar{p}j}{}^k)H_k^{ij}\Big) \\
&\quad + g^{i\bar{a}}g^{j\bar{b}}g_{k\bar{c}}H_{ij}^k\Big((\mathrm{Ric}_1)^{\bar{r}}_{\bar{a}}\overline{H_{rb}^c} + (\mathrm{Ric}_1)^{\bar{r}}_{\bar{b}}\overline{H_{ar}^c} - (\mathrm{Ric}_1)^{\bar{c}}_{\bar{r}}\overline{H_{ab}^r}\Big) \\
&= |\bar{\nabla}H|_g^2 + |\nabla H|_g^2 \\
&\quad + 2\mathrm{Re}\Big( - (\hat{\nabla}_i \hat{\mathrm{Ric}}_{j\bar{k}} - H_{ij}^r \hat{\mathrm{Ric}}_{r\bar{k}} - \hat{\nabla}_i\partial_j\partial_{\bar{k}}F_\varepsilon + H_{ij}^r\partial_r\partial_{\bar{k}}F_\varepsilon)H_k^{ij} \\
&\quad + \nabla_i\nabla_{\bar{p}}\hat{T}_{pj\bar{l}}g^{k\bar{l}}H_k^{ij} + \nabla_i\nabla_j\hat{T}_{\bar{p}\bar{l}p}g^{k\bar{l}}H_k^{ij} + \hat{T}_{pi\bar{s}}g^{r\bar{s}}R_{r\bar{p}j\bar{l}}g^{k\bar{l}}H_k^{ij} \\
&\quad + (\hat{\nabla}_p\hat{R}_{i\bar{p}j}{}^k - H_{pi}^r\hat{R}_{r\bar{p}j}{}^k - H_{pj}^r\hat{R}_{i\bar{p}r}{}^k + H_{pr}^k\hat{R}_{i\bar{p}j}{}^r)H_k^{ij}\Big) \\
&\quad + g^{i\bar{a}}g^{j\bar{b}}g_{k\bar{c}}H_{ij}^k \\
&\quad \Big(\mathrm{Ric}_{r\bar{a}}\,\overline{H_{rb}^c} - \hat{\nabla}_{\bar{p}}\hat{T}_{ps\bar{a}}g^{s\bar{r}}\overline{H_{rb}^c} + \overline{H_{pa}^m H_{rb}^c}\hat{T}_{ps\bar{m}}g^{s\bar{r}} - \hat{\nabla}_s\hat{T}_{\bar{p}\bar{a}p}g^{s\bar{r}}\overline{H_{rb}^c} + H_{sp}^m\overline{H_{rb}^c}\hat{T}_{\bar{p}\bar{a}m}g^{s\bar{r}} \\
&\quad + \mathrm{Ric}_{r\bar{b}}\,\overline{H_{ar}^c} - \hat{\nabla}_{\bar{p}}\hat{T}_{ps\bar{b}}g^{s\bar{r}}\overline{H_{ar}^c} + \overline{H_{pb}^m H_{ar}^c}\hat{T}_{ps\bar{m}}g^{s\bar{r}} - \hat{\nabla}_s\hat{T}_{\bar{p}\bar{b}p}g^{s\bar{r}}\overline{H_{ar}^c} + H_{sp}^m\overline{H_{ar}^c}\hat{T}_{\bar{p}\bar{b}m}g^{s\bar{r}} \\
&\quad - (\mathrm{Ric}_{c\bar{r}}\,\overline{H_{ab}^r} - \hat{\nabla}_{\bar{p}}\hat{T}_{ps\bar{r}}g^{s\bar{c}}\overline{H_{ab}^r} + \overline{H_{pr}^m H_{ab}^r}\hat{T}_{ps\bar{m}}g^{s\bar{c}} - \hat{\nabla}_s\hat{T}_{\bar{p}\bar{r}p}g^{s\bar{c}}\overline{H_{ab}^r} + H_{sp}^m\overline{H_{ab}^r}\hat{T}_{\bar{p}\bar{r}m}g^{s\bar{c}})\Big) \\
&\geq \frac{1}{2}|\bar{\nabla}H|_g^2 + \frac{1}{2}|\nabla H|_g^2 - \frac{C}{|s|_h^{2\beta}}(S_{T,\varepsilon}^{\frac{3}{2}} + S_{T,\varepsilon} + S_{T,\varepsilon}^{\frac{1}{2}} + 1)
\end{aligned}$$

since we have $\mathrm{Ric}_{r\bar{a}} = \hat{\mathrm{Ric}}_{r\bar{a}} - \partial_r\partial_{\bar{a}}F_\varepsilon$ and $|\hat{\Delta}F_\varepsilon| \leq \frac{C}{|s|_h^{2\beta}}$, where Ric and $\hat{\mathrm{Ric}}$ are the second

Ricci curvatures with respect to $g$ and $\hat{g}$ respectively and we used that $T_{ij\bar{k}} = \hat{T}_{ij\bar{k}}$,

$$\begin{aligned}
\nabla_p R_{i\bar{p}j\bar{l}} &= \nabla_i R_{p\bar{p}j\bar{l}} - T^r_{pi} R_{r\bar{p}j\bar{l}} \\
&= \nabla_i R_{j\bar{l}p\bar{p}} - \nabla_i \nabla_{\bar{p}} T_{pj\bar{l}} - \nabla_i \nabla_j T_{\bar{p}\bar{l}p} - \hat{T}_{pi\bar{s}} g^{r\bar{s}} R_{r\bar{p}j\bar{l}} \\
&= \nabla_i \hat{\mathrm{Ric}}_{j\bar{l}} - \nabla_i \partial_j \partial_{\bar{l}} F_\varepsilon - \nabla_i \nabla_{\bar{p}} \hat{T}_{pj\bar{l}} - \nabla_i \nabla_j \hat{T}_{\bar{p}\bar{l}p} - \hat{T}_{pi\bar{s}} g^{r\bar{s}} R_{r\bar{p}j\bar{l}} \\
&= \hat{\nabla}_i \hat{\mathrm{Ric}}_{j\bar{l}} - H^r_{ij} \hat{\mathrm{Ric}}_{r\bar{l}} - \hat{\nabla}_i \partial_j \partial_{\bar{l}} F_\varepsilon + H^r_{ij} \partial_r \partial_{\bar{l}} F_\varepsilon \\
&\quad - \nabla_i \nabla_{\bar{p}} \hat{T}_{pj\bar{l}} - \nabla_i \nabla_j \hat{T}_{\bar{p}\bar{l}p} - \hat{T}_{pi\bar{s}} g^{r\bar{s}} R_{r\bar{p}j\bar{l}},
\end{aligned}$$

$$\begin{aligned}
\nabla_i \nabla_{\bar{p}} \hat{T}_{pj\bar{l}} &= \nabla_i (\hat{\nabla}_{\bar{p}} \hat{T}_{pj\bar{l}} - \overline{H^s_{pl}} \hat{T}_{pj\bar{s}}) \\
&= \hat{\nabla}_i \hat{\nabla}_{\bar{p}} \hat{T}_{pj\bar{l}} - H^r_{ip} \hat{\nabla}_{\bar{p}} \hat{T}_{rj\bar{l}} - H^r_{ij} \hat{\nabla}_{\bar{p}} \hat{T}_{pr\bar{l}} - \overline{\nabla_{\bar{i}} H^s_{pl}} \hat{T}_{pj\bar{s}} \\
&\quad - \overline{H^s_{pl}} \hat{\nabla}_i \hat{T}_{pj\bar{s}} + \overline{H^s_{pl}} H^r_{ip} \hat{T}_{rj\bar{s}} + \overline{H^s_{pl}} H^r_{ip} \hat{T}_{pr\bar{s}},
\end{aligned}$$

$$\begin{aligned}
\nabla_i \nabla_j \hat{T}_{\bar{p}\bar{l}p} &= \nabla_i (\hat{\nabla}_j \hat{T}_{\bar{p}\bar{l}p} - H^r_{jp} \hat{T}_{\bar{p}\bar{l}r}) \\
&= \hat{\nabla}_i \hat{\nabla}_j \hat{T}_{\bar{p}\bar{l}p} - H^r_{ij} \hat{\nabla}_r \hat{T}_{\bar{p}\bar{l}p} - H^r_{ip} \hat{\nabla}_j \hat{T}_{\bar{p}\bar{l}r} - \nabla_i H^r_{jp} \hat{T}_{\bar{p}\bar{l}r} \\
&\quad - H^r_{jp} \hat{\nabla}_i \hat{T}_{\bar{p}\bar{l}r} + H^r_{jp} H^s_{ir} \hat{T}_{\bar{p}\bar{l}s},
\end{aligned}$$

$$\left| \overline{\nabla_{\bar{i}} H^s_{pl}} \hat{T}_{pj\bar{s}} g^{k\bar{l}} H^{ij}_k \right| \le C S_{T,\varepsilon} + \frac{1}{4} |\bar{\nabla} H|^2_g,$$

$$\left| \nabla_i H^r_{jp} \hat{T}_{\bar{p}\bar{l}r} g^{k\bar{l}} H^{ij}_k \right| \le C S_{T,\varepsilon} + \frac{1}{2} |\nabla H|^2_g$$

and

$$\left| 2\mathrm{Re}(\hat{T}_{pi\bar{s}} g^{r\bar{s}} R_{r\bar{p}j\bar{l}} g^{k\bar{l}} H^{ij}_k) \right| \le C S_{T,\varepsilon} + \frac{1}{4} |\bar{\nabla} H|^2_g$$

for some constant $C > 0$.

We also compute

$$\begin{aligned}
\Delta \mathrm{tr}_{\hat{g}} g &= -\hat{g}^{k\bar{l}} \mathrm{Ric}_{k\bar{l}} \\
&\quad - g^{i\bar{j}} \hat{g}^{k\bar{l}} \Big( \overline{\hat{\Gamma}^p_{lj}} \nabla_k g_{i\bar{p}} + \hat{R}_{k\bar{l}i\bar{q}} \hat{g}^{p\bar{q}} g_{p\bar{j}} - \hat{\Gamma}^p_{ki} \hat{\nabla}_{\bar{l}} g_{p\bar{j}} - \hat{\Gamma}^p_{ki} \overline{\hat{\Gamma}^q_{lj}} g_{p\bar{q}} \Big) \\
&\quad + g^{i\bar{j}} \hat{g}^{k\bar{l}} g^{p\bar{q}} \Big( \hat{\nabla}_k g_{i\bar{q}} + \hat{\Gamma}^s_{ki} g_{s\bar{q}} \Big) \Big( \hat{\nabla}_{\bar{l}} g_{p\bar{j}} + \overline{\hat{\Gamma}^m_{lj}} g_{p\bar{m}} \Big) \\
&\quad + g^{i\bar{j}} \hat{g}^{k\bar{l}} \Big( (\hat{\nabla}_i \overline{\hat{T}^p_{jl}}) \hat{g}_{k\bar{p}} + (\hat{\nabla}_{\bar{l}} \hat{T}^p_{ik}) \hat{g}_{p\bar{j}} \Big) \\
&\quad - g^{i\bar{j}} \hat{g}^{k\bar{l}} \Big( \hat{\nabla}_i \overline{\hat{T}^p_{jl}} - \hat{R}_{i\bar{l}s\bar{j}} \hat{g}^{s\bar{p}} \Big) g_{k\bar{p}} - g^{i\bar{j}} \hat{g}^{k\bar{l}} \Big( \hat{\nabla}_{\bar{l}} \hat{T}^p_{ik} - \hat{R}_{i\bar{l}k\bar{q}} \hat{g}^{p\bar{q}} \Big) g_{p\bar{j}} \\
&\quad - g^{i\bar{j}} \hat{g}^{k\bar{l}} \Big( \overline{\hat{T}^p_{jl}} \hat{\nabla}_i g_{k\bar{p}} + \hat{T}^p_{ik} \hat{\nabla}_{\bar{l}} g_{p\bar{j}} \Big) \\
&\ge \frac{C_1}{|s|^{2\beta}_h} S_{T,\varepsilon} - \frac{C}{|s|^{2\beta}_h} (S^{\frac{1}{2}}_{T,\varepsilon} + 1)
\end{aligned}$$

for some sufficiently large $\beta > 0$ and for some constant $C_1 > 0$. Note that we have

$$|\nabla \mathrm{tr}_{\hat{g}} g|^2_g \le \frac{C}{|s|^{2\beta}_h} S_{T,\varepsilon}$$

for some sufficiently large $\beta > 0$, and

$$|\nabla S_{T,\varepsilon}|_g^2 \leq 2S_{T,\varepsilon}(|\bar{\nabla}H|_g^2 + |\nabla H|_g^2).$$

Let $B_r$ be a small ball centered at the origin in $\mathbb{C}^2$ with radius $r > 0$. Let $\rho$ be a smooth cut off function with $\mathrm{supp}\rho \subset B_r$ and $\rho \equiv 1$ on $\overline{B_{\frac{r}{2}}}$ such that $|\nabla\rho|_g^2 + |\Delta\rho| \leq \frac{C}{r^2}$. We define $K := \frac{C}{|s|_h^{2\beta}}$ for sufficiently large $\beta > 0$ and for the constant $C > 0$ in the inequality $(**)$ such that

$$\frac{K}{2} \leq K - \mathrm{tr}_{\hat{g}}g \leq K.$$

Additionally, for sufficiently large $\beta > 0$, we may assume that $|s|_h^{2\beta} << 1$ on $B_r$ and then we have

$$|\nabla K|_g \leq \frac{C}{|s|_h^{3\beta}}, \quad |\Delta K|_g \leq \frac{C}{|s|_h^{4\beta}}.$$

For $\alpha_0, \alpha_1 > 0$ sufficiently large with $\alpha_0 = 3\beta < \alpha_1$, we define

$$f := \rho^2|s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{K - \mathrm{tr}_{\hat{g}}g} + A|s|_h^{2\alpha_0}\mathrm{tr}_{\hat{g}}g.$$

Note that we may suppose that we have for any sufficiently large $\alpha_0 > 0$,

$$|\nabla|s|_h^{2\alpha_0}|_g \leq C|s|_h^{\alpha_0}, \quad |\Delta|s|_h^{2\alpha_0}| \leq C|s|_h^{\alpha_0}.$$

We may assume that $f$ achieves its maximum at a point $x_0 \in B_r \setminus \{0\}$. Then, at $x_0$, we compute

$$\begin{aligned}
0 = \bar{\nabla}f \quad &= \quad \bar{\nabla}(\rho^2)|s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{K - \mathrm{tr}_{\hat{g}}g} + \rho^2\bar{\nabla}(|s|_h^{2\alpha_1})\frac{S_{T,\varepsilon}}{K - \mathrm{tr}_{\hat{g}}g} + \rho^2|s|_h^{2\alpha_1}\frac{\bar{\nabla}S_{T,\varepsilon}}{K - \mathrm{tr}_{\hat{g}}g} \\
&\quad +\rho^2|s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{(K - \mathrm{tr}_{\hat{g}}g)^2}(\bar{\nabla}\mathrm{tr}_{\hat{g}}g - \bar{\nabla}K) + A\bar{\nabla}(|s|_h^{2\alpha_0})\mathrm{tr}_{\hat{g}}g + A|s|_h^{2\alpha_0}\bar{\nabla}\mathrm{tr}_{\hat{g}}g.
\end{aligned}$$

And then, with using this computation, we have at $x_0$,

$$\begin{aligned}
0 \geq \Delta f \quad &= \quad \Delta(\rho^2)|s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{K - \mathrm{tr}_{\hat{g}}g} + \rho^2\Delta(|s|_h^{2\alpha_1})\frac{S_{T,\varepsilon}}{K - \mathrm{tr}_{\hat{g}}g} + \rho^2|s|_h^{2\alpha_1}\frac{\Delta S_{T,\varepsilon}}{K - \mathrm{tr}_{\hat{g}}g} \\
&\quad +\rho^2|s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{(K - \mathrm{tr}_{\hat{g}}g)^2}\Big(\Delta\mathrm{tr}_{\hat{g}}g - \Delta K\Big) + A\Delta(|s|_h^{2\alpha_0})\mathrm{tr}_{\hat{g}}g + A|s|_h^{2\alpha_0}\Delta\mathrm{tr}_{\hat{g}}g \\
&\quad +4\mathrm{Re}\Big(\rho\nabla(\rho)\cdot\bar{\nabla}(|s|_h^{2\alpha_1})\frac{S_{T,\varepsilon}}{K - \mathrm{tr}_{\hat{g}}g}\Big) + 4\mathrm{Re}\Big(\rho\nabla(\rho)\cdot\bar{\nabla}S_{T,\varepsilon}\frac{|s|_h^{2\alpha_1}}{K - \mathrm{tr}_{\hat{g}}g}\Big) \\
&\quad +2\rho^2\mathrm{Re}\Big(\nabla(|s|_h^{2\alpha_1})\cdot\bar{\nabla}S_{T,\varepsilon}\frac{1}{K - \mathrm{tr}_{\hat{g}}g}\Big) + 2A\mathrm{Re}\Big(\nabla(|s|_h^{2\alpha_0})\cdot\bar{\nabla}\mathrm{tr}_{\hat{g}}g\Big) \\
&\quad -2\mathrm{Re}\Big(\frac{A\mathrm{tr}_{\hat{g}}g}{K - \mathrm{tr}_{\hat{g}}g}(\nabla\mathrm{tr}_{\hat{g}}g - \nabla K)\cdot\bar{\nabla}(|s|_h^{2\alpha_0})\Big) - 2\frac{A|s|_h^{2\alpha_0}}{K - \mathrm{tr}_{\hat{g}}g}|\nabla\mathrm{tr}_{\hat{g}}g|_g^2 \\
&\quad +2\frac{A|s|_h^{2\alpha_0}}{K - \mathrm{tr}_{\hat{g}}g}\mathrm{Re}\Big(\nabla\mathrm{tr}_{\hat{g}}g\cdot\bar{\nabla}K\Big)
\end{aligned}$$

We estimate the each term above in the following ways:

$$|\Delta(\rho^2)||s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{K-\mathrm{tr}_{\hat{g}}g} \le \frac{C}{r^2}|s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{K},$$

$$\rho^2|\Delta(|s|_h^{2\alpha_1})|\frac{S_{T,\varepsilon}}{K-\mathrm{tr}_{\hat{g}}g} \le C\rho^2|s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{K},$$

$$\begin{aligned}\rho^2|s|_h^{2\alpha_1}\frac{\Delta S_{T,\varepsilon}}{K-\mathrm{tr}_{\hat{g}}g} &\ge \frac{\rho^2|s|_h^{2\alpha_1}}{K-\mathrm{tr}_{\hat{g}}g}\Big(\frac{1}{2}|\bar{\nabla}H|_g^2+\frac{1}{2}|\nabla H|_g^2-\frac{C}{|s|_h^{2\beta}}(S_{T,\varepsilon}^{\frac{3}{2}}+S_{T,\varepsilon}+S_{T,\varepsilon}^{\frac{1}{2}}+1)\Big)\\ &\ge \frac{\rho^2|s|_h^{2\alpha_1}}{K}\Big(|\bar{\nabla}H|_g^2+|\nabla H|_g^2\Big)-\frac{C\rho^2}{K}|s|_h^{2(\alpha_1-\beta)}(S_{T,\varepsilon}^{\frac{3}{2}}+S_{T,\varepsilon}+S_{T,\varepsilon}^{\frac{1}{2}}+1),\end{aligned}$$

$$\begin{aligned}&\Big(\rho^2|s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{(K-\mathrm{tr}_{\hat{g}}g)^2}+A|s|_h^{2\alpha_0}\Big)\Delta\mathrm{tr}_{\hat{g}}g\\ \ge\ &\Big(\rho^2|s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{(K-\mathrm{tr}_{\hat{g}}g)^2}+A|s|_h^{2\alpha_0}\Big)\Big(\frac{C_1}{|s|_h^{2\beta}}S_{T,\varepsilon}-\frac{C}{|s|_h^{2\beta}}(S_{T,\varepsilon}^{\frac{1}{2}}+1)\Big)\\ \ge\ &C'\frac{4\rho^2|s|_h^{2(\alpha_1-\beta)}}{K^2}S_{T,\varepsilon}^2+AC_1|s|_h^{2(\alpha_0-\beta)}S_{T,\varepsilon}\\ &-\frac{C\rho^2|s|_h^{2(\alpha_1-\beta)}}{K^2}(S_{T,\varepsilon}^{\frac{3}{2}}+S_{T,\varepsilon})-CA|s|_h^{2(\alpha_0-\beta)}(S_{T,\varepsilon}^{\frac{1}{2}}+1),\end{aligned}$$

$$A|\Delta(|s|_h^{2\alpha_0})|\mathrm{tr}_{\hat{g}}g \le CA|s|_H^{\alpha_0-2\beta},$$

$$4\Big|\mathrm{Re}\Big(\rho\nabla(\rho)\cdot\bar{\nabla}(|s|_h^{2\alpha_1})\frac{S_{T,\varepsilon}}{K-\mathrm{tr}_{\hat{g}}g}\Big)\Big| \le \frac{C}{rK}|s|_h^{\alpha_1}S_{T,\varepsilon},$$

$$\begin{aligned}4\Big|\mathrm{Re}\Big(\rho\nabla(\rho)\cdot\bar{\nabla}S_{T,\varepsilon}\frac{|s|_h^{2\alpha_1}}{K-\mathrm{tr}_{\hat{g}}g}\Big)\Big| &\le \frac{C\rho}{rK}|s|_h^{2\alpha_1}|\bar{\nabla}S_{T,\varepsilon}|_g\\ &\le \frac{C\rho}{rK}|s|_h^{2\alpha_1}S_{T,\varepsilon}^{\frac{1}{2}}\Big(|\bar{\nabla}H|_g^2+|\nabla H|_g^2\Big)^{\frac{1}{2}}\\ &\le \frac{\rho^2}{2K}|s|_h^{2\alpha_1}\Big(|\bar{\nabla}H|_g^2+|\nabla H|_g^2\Big)+\frac{CS_{T,\varepsilon}}{r^2K}|s|_h^{2\alpha_1},\end{aligned}$$

$$\begin{aligned}2\rho^2\Big|\mathrm{Re}\Big(\nabla(|s|_h^{2\alpha_1})\cdot\bar{\nabla}S_{T,\varepsilon}\frac{1}{K-\mathrm{tr}_{\hat{g}}g}\Big)\Big| &\le \frac{C\rho^2}{K}|\nabla(|s|_h^{2\alpha_1})|\cdot|\bar{\nabla}S_{T,\varepsilon}|_g\\ &\le \frac{C\rho^2}{K}|s|_h^{\alpha_1}S_{T,\varepsilon}^{\frac{1}{2}}\Big(|\bar{\nabla}H|_g^2+|\nabla H|_g^2\Big)^{\frac{1}{2}}\\ &\le \frac{\rho^2}{2K}|s|_h^{2\alpha_1}\Big(|\bar{\nabla}H|_g^2+|\nabla H|_g^2\Big)+\frac{C}{K}S_{T,\varepsilon},\end{aligned}$$

$$2A\Big|\mathrm{Re}\Big(\nabla(|s|_h^{2\alpha_0})\cdot\bar{\nabla}\mathrm{tr}_{\hat{g}}g\Big)\Big| \le CA|s|_h^{\alpha_0}|\bar{\nabla}\mathrm{tr}_{\hat{g}}g|_g \le CA|s|_h^{\alpha_0-\beta}S_{T,\varepsilon}^{\frac{1}{2}},$$

$$2\Big|\mathrm{Re}\Big(\frac{A\mathrm{tr}_{\hat{g}}g}{K-\mathrm{tr}_{\hat{g}}g}(\nabla\mathrm{tr}_{\hat{g}}g-\nabla K)\cdot\bar{\nabla}(|s|_h^{2\alpha_0})\Big)\Big|\leq CA|s|_h^{\alpha_0-\beta}S_{T,\varepsilon}^{\frac{1}{2}}+CA|s|_h^{\alpha_0-3\beta},$$

$$2\frac{A|s|_h^{2\alpha_0}}{K-\mathrm{tr}_{\hat{g}}g}|\nabla\mathrm{tr}_{\hat{g}}g|_g^2\leq\frac{CA}{K}|s|_h^{2(\alpha_0-\beta)}S_{T,\varepsilon}\leq C_2A|s|_h^{2\alpha_0}S_{T,\varepsilon},$$

$$\begin{aligned}2\frac{A|s|_h^{2\alpha_0}}{K-\mathrm{tr}_{\hat{g}}g}\Big|\mathrm{Re}\Big(\nabla\mathrm{tr}_{\hat{g}}g\cdot\bar{\nabla}K\Big)\Big|\leq\frac{4A|s|_h^{2\alpha_0}}{K}|\nabla\mathrm{tr}_{\hat{g}}g|_g\cdot|\bar{\nabla}K|_g &\leq CA|s|^{2\alpha_0+2\beta-\beta-3\beta}S_{T,\varepsilon}^{\frac{1}{2}}\\ &= CA|s|_h^{4\beta}S_{T,\varepsilon}^{\frac{1}{2}},\end{aligned}$$

and finally,

$$\rho^2|s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{(K-\mathrm{tr}_{\hat{g}}g)^2}\Big|\Delta K\Big|\leq\rho^2|s|_h^{2\alpha_1}\frac{S_{T,\varepsilon}}{K^2}\frac{C}{|s|_h^{4\beta}}\leq C\rho^2|s|_h^{2\alpha_1}S_{T,\varepsilon}$$

for some constants $C,C_2>0$, $C$'s are different from each other in these estimates.

Since we may assume that $|s|_h^2<1$, by choosing $\alpha_0=3\beta<\alpha_1$, we obtain at $x_0$,

$$\begin{aligned}0 &\geq -CA+\frac{4C_1\rho^2|s|_h^{2(\alpha_1-\beta)}}{K^2}S_{T,\varepsilon}\Big(S_{T,\varepsilon}-\frac{C}{4C_1}(S_{T,\varepsilon}^{\frac{1}{2}}+1)-\frac{CK}{4C_1}S_{T,\varepsilon}^{\frac{1}{2}}\Big)\\ &\quad+AC_1|s|_h^{4\beta}S_{T,\varepsilon}-C_2A|s|^{2\alpha_0}S_{T,\varepsilon}-C_3AS_{T,\varepsilon}^{\frac{1}{2}}-C_4S_{T,\varepsilon}\end{aligned}$$

for some constants $C,C_3,C_4>0$.

We may assume that $S_{T,\varepsilon}>1$ at $x_0$ and then we may say that there exists a small constant $\kappa>0$ such that

$$\Big(S_{T,\varepsilon}-\frac{C}{4C_1}(S_{T,\varepsilon}^{\frac{1}{2}}+1)-\frac{CK}{4C_1}S_{T,\varepsilon}^{\frac{1}{2}}\Big)>\kappa>0.$$

And also we can say that, for sufficiently large $A,\beta>0$, we have at $x_0$,

$$\begin{aligned}&|s|_h^{4\beta}\Big(AC_1S_{T,\varepsilon}-C_2A|s|_h^{2\beta}S_{T,\varepsilon}-\frac{C_3A}{|s|_h^{4\beta}}S_{T,\varepsilon}^{\frac{1}{2}}-\frac{C_4}{|s|_h^{4\beta}}S_{T,\varepsilon}\Big)\\ >\ & A\kappa'^2\Big((\kappa''-\frac{C_4}{A\kappa'^2})S_{T,\varepsilon}-\frac{C_3}{\kappa'^2}S_{T,\varepsilon}^{\frac{1}{2}}\Big)\\ >\ & A\kappa'^2\Big(\kappa'''S_{T,\varepsilon}-\frac{C_3}{\kappa'^2}S_{T,\varepsilon}^{\frac{1}{2}}\Big)\end{aligned}$$

since $0<\kappa'<|s|_h^{2\beta}(x_0)<<1$ sufficiently small so that $C_1-C_2|s|_h^{2\beta}(x_0)>\kappa''>0$ by choosing a sufficiently large $\beta$ for some small constant $\kappa'',\kappa'''>0$ with $\kappa''-\frac{C_4}{A\kappa'^2}>\kappa'''$ for sufficiently large $A$.. If $\kappa'''S_{T,\varepsilon}-\frac{C_3}{\kappa'^2}S_{T,\varepsilon}^{\frac{1}{2}}\leq 0$ at $x_0$, we obtain

$$S_{T,\varepsilon}^{\frac{1}{2}}\leq\frac{C_3}{\kappa'''\kappa'^2},$$

hence we obtain the upper bound for $S_{T,\varepsilon}$ at $x_0$.

On the other hand, if $\kappa''' S_{T,\varepsilon} - \frac{C_3}{\kappa'^2} S_{T,\varepsilon}^{\frac{1}{2}} > 0$ at $x_0$, we then have

$$|s|_h^{4\beta}(x_0)\Big(AC_1 S_{T,\varepsilon} - C_2 A|s|_h^{2\beta} S_{T,\varepsilon} - \frac{C_3 A}{|s|_h^{4\beta}} S_{T,\varepsilon}^{\frac{1}{2}} - \frac{C_4}{|s|_h^{4\beta}} S_{T,\varepsilon}\Big)(x_0) > 0.$$

Therefore, at $x_0$, in this case we have

$$\frac{\rho^2 |s|^{2\alpha_1}}{K} S_{T,\varepsilon} \leq \frac{C_5 A}{\kappa}$$

for some constant $C_5 > 0$. Putting these together, we have

$$f(x) \leq f(x_0) \leq \Big(2\rho^2(x_0)\max\Big\{\frac{C_5 A}{\kappa}, \frac{|s|_h^{2\alpha_1}}{K}\Big(\frac{C_3}{\kappa'''\kappa'^2}\Big)^2\Big\} + C_6 A|s|_h^{2(\alpha_0-\beta)}(x_0)\Big) \leq A(C_7 + C_6)$$

for some constant $C_6, C_7 > 0$.

Hence, on $\overline{B_{\frac{r}{2}}}$, we obtain

$$S_{T,\varepsilon} \leq \frac{AC_8}{|s|_h^{2\alpha_2}}$$

for some uniform constants $C_8, \alpha_2 := \alpha_1 + \beta > 0$. With using this computation for $S_{T,\varepsilon}$, we can obtain the upper bound for the curvature of $\omega_{T,\varepsilon}$ and then we also have bounds on its all covariant derivatives by an analogue of [59, Proposition 4.2]. Additionally, with using the Sobolev inequality and a bootstrap argument, we obtain the higher order estimates. □

We firstly show an estimate for its volume form and after that, with using the estimate, we can show estimates for $\omega_\varepsilon$ as in Lemma 4.2.2 by applying [59, Lemma 5.4] and [70, Lemma 3.5] respectively.

**Lemma 4.2.3.** ([59, Lemma 5.4])There exist positive constants $\alpha$ and $C$, independent of $\varepsilon$, such that

$$\frac{\omega_\varepsilon^2}{\Omega_N} \leq \frac{C}{|s|_h^{2\alpha}}$$

on $[T, T'] \times (N \setminus \{y_0\})$.

**Lemma 4.2.4.** For all sufficiently small $\varepsilon > 0$, there exist positive uniform constants $C$, $\alpha$, independent of $\varepsilon$, such that on $[T, T'] \times (N \setminus \{y_0\})$,

$$\frac{|s|_h^{2\alpha}}{C}\hat{\omega}_N \leq \omega_\varepsilon \leq \frac{C}{|s|_h^{2\alpha}}\hat{\omega}_N.$$

Fix a large positive integer $L$. Then for each integer $0 \leq k \leq L$ there exist $C_k, \alpha_k > 0$ such that

$$|(\nabla_{\mathbb{R}})^k \omega_\varepsilon|^2_{\omega_{T,\varepsilon}} \leq \frac{C_k}{|s|_h^{2\alpha_k}}$$

for $t \in (T, T']$, where $\nabla_{\mathbb{R}}$ is the Levi-Civita covariant derivative associated to the metric $\omega_{T,\varepsilon} = g$.

PROOF. We write $\omega_\varepsilon = \sqrt{-1}(g_\varepsilon)_{i\bar{j}}dz^i \wedge d\bar{z}^j$, $\hat{\omega}_N = \sqrt{-1}\hat{g}_{i\bar{j}}dz^i \wedge d\bar{z}^j$ and $\hat{\omega}_{t,N} = \sqrt{-1}(\hat{g}_{t,N})_{i\bar{j}}dz^i \wedge d\bar{z}^j$ with local coordinates.

We define the quantity

$$Q'_\varepsilon := \log(|s|_h^{2\alpha}\mathrm{tr}_{\hat{g}}g_\varepsilon) - A\varphi_\varepsilon + \frac{1}{\tilde{\varphi}_\varepsilon + C_0},$$

for sufficiently large $\alpha > 0$, where $\tilde{\varphi}_\varepsilon := \varphi_\varepsilon - \frac{1}{A}\log|s|_h^{2\alpha}$ and choose a constant $C_0$ such that $\tilde{\varphi}_\varepsilon + C_0 \geq 1$, for $A$ a large constant to be determined. Observe that $Q'_\varepsilon$ tends to negative infinity as $x \in N$ tends to $y_0$, for any $t \in [T, T']$. From Lemma 4.2.2, $Q'_\varepsilon|_{t=T}$ is uniformly bounded from above by choosing $\alpha$ sufficiently large.

We apply [72, Proposition 3.1] to $\log\mathrm{tr}_{\hat{g}}g_\varepsilon$, then we have

$$\Big(\frac{\partial}{\partial t} - \Delta_\varepsilon\Big)\log\mathrm{tr}_{\hat{g}}g_\varepsilon \leq \frac{2}{(\mathrm{tr}_{\hat{g}}g_\varepsilon)^2}\mathrm{Re}\Big(g_\varepsilon^{\bar{l}k}\hat{T}^p_{kp}\hat{\nabla}_{\bar{l}}\mathrm{tr}_{\hat{g}}g_\varepsilon\Big) + C\mathrm{tr}_{g_\varepsilon}\hat{g},$$

where $\Delta_\varepsilon$ is the Laplacian with respect to $g_\varepsilon$, $\hat{\nabla}$ is the covariant derivative with respect to $\hat{g}$, $\hat{T}$ is the torsion tensor of $\hat{g}$, and assuming that we compute at a point where we have $\mathrm{tr}_{\hat{g}}g_\varepsilon \geq 1$. Suppose that $Q'_\varepsilon$ achieves its maximum at $x_0 \in N \setminus \{y_0\}$. Then we have at $x_0$,

$$\frac{\hat{\nabla}_{\bar{l}}\mathrm{tr}_{\hat{g}}g_\varepsilon}{\mathrm{tr}_{\hat{g}}g_\varepsilon} = A\partial_{\bar{l}}\tilde{\varphi}_\varepsilon + \frac{\partial_{\bar{l}}\tilde{\varphi}_\varepsilon}{(\tilde{\varphi}_\varepsilon + C_0)^2}$$

and with using this equality, we compute as in the estimate ($\mho$):

$$\Big|\frac{2}{\mathrm{tr}_{\hat{g}}g_\varepsilon}\mathrm{Re}\Big(g_\varepsilon^{\bar{l}k}\hat{T}^p_{kp}\frac{\hat{\nabla}_{\bar{l}}\mathrm{tr}_{\hat{g}}g_\varepsilon}{\mathrm{tr}_{\hat{g}}g_\varepsilon}\Big)\Big| \leq \frac{|\partial\tilde{\varphi}_\varepsilon|^2_{g_\varepsilon}}{(\tilde{\varphi}_\varepsilon + C_0)^3} + CA^2(\tilde{\varphi}_\varepsilon + C_0)^3\frac{\mathrm{tr}_{g_\varepsilon}\hat{g}}{(\mathrm{tr}_{\hat{g}}g_\varepsilon)^2}.$$

We compute

$$\Big(\frac{\partial}{\partial t} - \Delta_\varepsilon\Big)\tilde{\varphi}_\varepsilon = \log\frac{\omega_\varepsilon^2}{\Omega_N} - \mathrm{tr}_{g_\varepsilon}(g_\varepsilon - \hat{g}_{t,N} + \frac{\alpha}{A}R_h),$$

where we used that $\sqrt{-1}\partial\bar{\partial}\log|s|_h^2 = \sqrt{-1}\partial\bar{\partial}\log h$ away from $y_0$.

Since we may assume that at the maximum of $Q'_\varepsilon$ we have $(\mathrm{tr}_{\hat{g}}g_\varepsilon)^2 \geq A^2(\tilde{\varphi}_\varepsilon + C_0)^3$, we have at $x_0$,

$$\begin{aligned}
0 \;&\leq\; \Big(\frac{\partial}{\partial t} - \Delta_\varepsilon\Big)Q'_\varepsilon \\
&\leq\; \frac{|\partial\tilde{\varphi}_\varepsilon|^2_{g_\varepsilon}}{(\tilde{\varphi}_\varepsilon + C_0)^3} + C'\mathrm{tr}_{g_\varepsilon}\hat{g} + \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big)\Big(\log\frac{\Omega_N}{\omega_\varepsilon^2} + 2\Big) \\
&\quad - \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big)\mathrm{tr}_{g_\varepsilon}(\hat{g}_{t,N} - \frac{\alpha}{A}R_h) - \frac{2|\partial\tilde{\varphi}_\varepsilon|^2_{g_\varepsilon}}{(\tilde{\varphi}_\varepsilon + C_0)^3} \\
&\leq\; C'\mathrm{tr}_{g_\varepsilon}\hat{g} + \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big)\log\frac{\hat{\omega}_N^2}{\omega_\varepsilon^2} - \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big)\mathrm{tr}_{g_\varepsilon}(\hat{g}_{t,N} - \frac{\alpha}{A}R_h) \\
&\quad + \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big)\log\frac{\Omega_N}{\hat{\omega}_N^2}\Big) + 2(A+1)
\end{aligned}$$

for some uniform constant $C', C > 0$.

For an arbitrary fixed constant $1 > c_0 > 0$, we have

$$\hat{g}_{t,N} - \frac{\alpha}{A} R_h \geq c_0 \hat{g}_{t,N}$$

for any $t \in [T, T']$ and for sufficiently large $A > 0$ and for all $t \in [T, T']$. If necessary, we again choose a much larger constant $A$, and then we have

$$A c_0 \mathrm{tr}_{g_\varepsilon} \hat{g}_{t,N} \geq (C' + 1) \mathrm{tr}_{g_\varepsilon} \hat{g}.$$

With using these estimates, we obtain at $x_0$,

$$\begin{aligned}
0 \;\leq\;& C' \mathrm{tr}_{g_\varepsilon} \hat{g} + \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big) \log \frac{\hat{\omega}_N^2}{\omega_\varepsilon^2} - A c_0 \mathrm{tr}_{g_\varepsilon} \hat{g}_{t,N} \\
&+ \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big) \log \frac{\Omega_N}{\hat{\omega}_N^2} + 2(A+1) \\
\leq\;& C' \mathrm{tr}_{g_\varepsilon} \hat{g} + \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big) \log \frac{\hat{\omega}_N^2}{\omega_\varepsilon^2} - (C'+1) \mathrm{tr}_{g_\varepsilon} \hat{g} + C
\end{aligned}$$

Then at $x_0$, we have

$$\mathrm{tr}_{g_\varepsilon} \hat{g} + \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big) \log \frac{\omega_\varepsilon^2}{\hat{\omega}_N^2} \leq C$$

for some uniform constant $C > 0$. Now we choose local coordinates around the point $x_0$ such that $\hat{g}_{i\bar{j}}(x_0) = \delta_{ij}$ and $(g_\varepsilon)_{i\bar{j}}(x_0) = \lambda_i \delta_{ij}$ with $\lambda_1, \lambda_2 > 0$. Then we have

$$\sum_{i=1}^{2} \Big( \frac{1}{\lambda_i} + \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big) \log \lambda_i \Big) \leq C.$$

Note that we have

$$A \leq \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big) \leq A + 1.$$

For any $\lambda > 0$, since the function $\lambda \mapsto \frac{1}{\lambda} + (A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}) \log \lambda$ is uniformly bounded from below for sufficiently large $A$, for each $i$ we have

$$\frac{1}{\lambda_i} + \Big(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}\Big) \log \lambda_i \leq C$$

for some uniform constant $C > 0$. And then for each $i$, we obtain $(A + \frac{1}{(\tilde{\varphi}_\varepsilon + C_0)^2}) \log \lambda_i \leq C$, which gives a uniform upper bound $\lambda_i \leq C$ for some uniform constant $C > 0$. Therefore, we have

$$\mathrm{tr}_{\hat{g}} g_\varepsilon(x_0) \leq C$$

and

$$Q'_\varepsilon \leq Q'_\varepsilon(x_0) \leq C$$

since $\varphi_\varepsilon$ is uniformly bounded for all sufficiently small $\varepsilon > 0$ as we see in Proposition 2.3. Then we obtain, on $[T, T'] \times (N \setminus \{y_0\})$,

$$\omega_\varepsilon \leq \frac{C}{|s|_h^{2\alpha}} \hat{\omega}_N.$$

We can also obtain the bound of the Chern curvature tensor with respect to $\omega_\varepsilon$, the bound of its covariant derivatives and the higher order estimates with the application of [45, Theorem 8.11.1&Theorem 8.12.1] by the same way as in [54].

□

Since $\varphi_t = \varphi(t)$ for $t \in [T, T']$ is the limit of $\varphi_\varepsilon$ as $\varepsilon \to 0$, the metric

$$\omega(t) = \hat{\omega}_{t,N} + \sqrt{-1}\partial\bar{\partial}\varphi_t$$

for $t \in [T, T']$ is a solution of the Chern-Ricci flow on $N$:

$$\frac{\partial}{\partial t}\omega(t) = -\operatorname{Ric}(\omega(t)) \quad \text{for } t \in (T, T'], \quad \omega(T) = \omega'.$$

Lemma 4.2.4 gives estimates on $\omega(t)$ for $t \in [T, T']$ on $N \setminus \{y_0\}$ and Proposition 4.2.3 gives us estimates on $\omega(t)$ for $t \in (0, T)$ on $M \setminus E$. We can show that the Chern-Ricci flow can be smoothly connected at time $T$ between $[0, T) \times M$ and $(T, T'] \times N$, outside $T \times \{y_0\} \cong T \times E$ via the map $\pi$.

**Theorem 4.2.1.** The solution $\omega(t)$ is a smooth solution of the Chern-Ricci flow in the space-time region $\mathcal{R}$.

PROOF. From Lemma 4.2.4 and Proposition 4.2.3, $\omega(t)$ satisfies the Chern-Ricci flow and is smooth at time $T$ in the sense of Remark 4.1.1. □

This completes the proof of (4) in Definition 1.2.5.

It remains to show that $(N, \omega(t))$ converges in the Gromov-Hausdorff sense to $(N, d_T)$ as $t \to T^+$. We obtain the following estimate by the same proof as in [59].

**Proposition 4.2.4.** ([59, Proposition 6.1]) There exist $\delta > 0$ and a uniform constant $C > 0$ such that for $t \in [T, T']$,

(1) $\omega(t) \leq \frac{C}{|s|_h^2}\hat{\omega}_N$,

(2) $\omega(t) \leq \frac{C}{|s|_h^{2(1-\delta)}}(\pi|_{M\setminus E}^{-1})^*\omega_0$,

where $\omega_0$ is the initial metric of the Chern-Ricci flow on $M$.

PROOF. We identify a coordinate chart $U$ at $y_0 \in N$ via coordinates $(z_1, z_2)$ with the unit ball $D$ in $\mathbb{C}^2$

$$D = \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 < 1\}.$$

Put $r^2 := |z_1|^2 + |z_2|^2$. Let $f_\varepsilon$ be a family of positive smooth functions on $N$ of the form $f_\varepsilon(z) = \varepsilon + r^2$ on $D$, which converges to a function $f$ which is of the form $f(z) = r^2$ on $D$ and is positive on $M \setminus D$. By the definition of the blow-down map, there is a smooth volume form $\Omega_M$ on $M$ such that $\pi^*\Omega_N = (\pi^* f)\Omega_M$. Note that $\hat{\omega}_{t,N} - \frac{\varepsilon}{T}\hat{\omega}_N$ is positive definite for sufficiently small $\varepsilon > 0$ on $N$ for $t \in [T, T']$. For such sufficiently small $\varepsilon > 0$, we consider the following family of Monge-Ampère flows on $M$:

$$\frac{\partial}{\partial t}\rho_\varepsilon = \log\frac{(\pi^*(\hat{\omega}_{t,N} - \frac{\varepsilon}{T}\hat{\omega}_N) + \frac{\varepsilon}{T}\omega_0 + \sqrt{-1}\partial\bar{\partial}\rho_\varepsilon)^2}{(\pi^* f_\varepsilon)\Omega_M}, \quad \rho_\varepsilon|_{t=T} = \varphi_{T-\varepsilon}.$$

Observe that at $t = T$,

$$\left(\pi^*(\hat{\omega}_{t,N} - \frac{\varepsilon}{T}\hat{\omega}_N) + \frac{\varepsilon}{T}\omega_0\right)\Big|_{t=T} = \left(1 - \frac{\varepsilon}{T}\right)\pi^*\hat{\omega}_N + \frac{\varepsilon}{T}\omega_0$$

is equal to $\hat{\omega}_{T-\varepsilon}$, where $\hat{\omega}_t$ is a family of reference metrics for $t \in [0, T)$, of form

$$\hat{\omega}_t = \frac{1}{T}((T-t)\omega_0 + t\pi^*\hat{\omega}_N) \in [\omega_0] + tc_1^{BC}(K_M)$$

since we have $\pi^*\hat{\omega}_N = \omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}u_0'$ for a smooth function $u_0'$ on $M$.

We can obtain a uniform bound for $|\rho_\varepsilon|$ independent of $\varepsilon$ by considering $\rho_\varepsilon \pm A_\pm(T-t)$ for sufficiently large uniform constants $A_\pm > 0$ as in [61, Lemma 3.2]. For the upper bound of $\rho_\varepsilon$, we apply the maximum principle to

$$\theta_{\varepsilon,+} := \rho_\varepsilon + A_+(T-t)$$

for $A_+ > 0$ a uniform constant to be determined later. Then we have

$$\frac{\partial}{\partial t}\theta_{\varepsilon,+} = \log\frac{(\pi^*(\hat{\omega}_{t,N} - \frac{\varepsilon}{T}\hat{\omega}_N) + \frac{\varepsilon}{T}\omega_0 + \sqrt{-1}\partial\bar{\partial}\theta_{\varepsilon,+})^2}{(\pi^* f_\varepsilon)\Omega_M} - A_+.$$

Since $M \times [T, T']$ is compact, $\theta_{\varepsilon,+}$ attains a maximum at some point $(x_0, t_0) \in M \times [T, T']$. We claim that if $A_+$ is sufficiently large we have $t_0 = T$. Otherwise we have $t_0 > T$ and then by applying Proposition 1.6 in [61], at $(x_0, t_0)$,

$$0 \le \frac{\partial}{\partial t}\theta_{\varepsilon,+} \le \log\frac{(\pi^*(\hat{\omega}_{t_0,N} - \frac{\varepsilon}{T}\hat{\omega}_N) + \frac{\varepsilon}{T}\omega_0)^2}{(\pi^* f_\varepsilon)\Omega_M} - A_+ \le -1,$$

which is a contradiction, where we have chosen the uniform constant $A$ so that

$$A_+ \ge 1 + \sup_{M\times[T,T']}\log\frac{(\pi^*(\hat{\omega}_{t,N} - \frac{\varepsilon}{T}\hat{\omega}_N) + \frac{\varepsilon}{T}\omega_0)^2}{(\pi^* f_\varepsilon)\Omega_M}.$$

Hence we have proved the claim that $t_0 = T$, which gives that

$$\sup_{M\times[T,T']}\theta_{\varepsilon,+} \le \sup_M \theta_{\varepsilon,+}|_{t=T} = \sup_M \varphi_{T-\varepsilon} \le C_+$$

for some uniform constant $C_+ > 0$ and therefore

$$\rho_\varepsilon(x,t) \le A_+(t-T) + C_+ \le A_+T' + C_+$$

for any $(x,t) \in M \times [T, T']$. We apply a similar argument to

$$\theta_{\varepsilon,-} := \rho_\varepsilon - A_-(T-t)$$

for $A_- > 0$ a uniform constant with

$$A_- \ge 1 - \inf_{M\times[T,T']}\log\frac{(\pi^*(\hat{\omega}_{t,N} - \frac{\varepsilon}{T}\hat{\omega}_N) + \frac{\varepsilon}{T}\omega_0)^2}{(\pi^* f_\varepsilon)\Omega_M}.$$

Assume that $\theta_{\varepsilon,-}$ attains a minimum at some point $(x_0, t_0) \in M \times [T, T']$ with $t_0 > T$ and then we have at $(x_0, t_0)$,

$$0 \geq \frac{\partial}{\partial t}\theta_{\varepsilon,-} \geq \log \frac{(\pi^*(\hat{\omega}_{t_0,N} - \frac{\varepsilon}{T}\hat{\omega}_N) + \frac{\varepsilon}{T}\omega_0)^2}{(\pi^* f_\varepsilon)\Omega_M} + A_- \geq 1,$$

which is a contradiction. Hence we obtain $t_0 = T$ and

$$\inf_{M\times[T,T']} \theta_{\varepsilon,-} \geq \inf_M \theta_{\varepsilon,-}|_{t=T} = \inf_M \varphi_{T-\varepsilon} \geq -C_-$$

for some uniform constant $C_- > 0$,

$$\rho_\varepsilon(x,t) \geq -A_- T' - C_-$$

for any $(x,t) \in M \times [T, T']$, which gives the lower bound of $\rho_\varepsilon$.

And also, by modifying the argument in [59, Lemma 2.5] to deal with the extra terms coming from $f_\varepsilon$, we obtain, for

$$\omega_\varepsilon := \pi^*\Big(\hat{\omega}_{t,N} - \frac{\varepsilon}{T}\hat{\omega}_N\Big) + \frac{\varepsilon}{T}\omega_0 + \sqrt{-1}\partial\bar{\partial}\rho_\varepsilon,$$

$$\omega_\varepsilon \leq \frac{C}{|s|_h^2}\pi^*\omega_N, \quad \omega_\varepsilon \leq \frac{C}{|s|_h^{2(1-\delta)}}\omega_0$$

on $M \setminus E \times [T, T']$ and $C^\infty$-estimates for $\omega_\varepsilon$ on compact subsets away from $E$. By letting $\varepsilon \to 0$, and pushing forward to $N$, we obtain a smooth solution $\tilde{\rho}$ of the following parabolic complex Monge-Ampère equation

$$\frac{\partial}{\partial t}\tilde{\rho} = \log \frac{(\hat{\omega}_{t,N} + \sqrt{-1}\partial\bar{\partial}\tilde{\rho})^2}{\Omega_N}, \quad \tilde{\rho}|_{t=T} = \psi_T$$

on $N \setminus \{y_0\} \times [T, T']$ with $\hat{\omega}_{t,N} + \sqrt{-1}\partial\bar{\partial}\tilde{\rho}$ satisfying the estimates (1), (2). On the other hand, $\tilde{\rho}$ is equal to the solution $\varphi$ on $N$ in Proposition 4.2.2. Hence, the estimates (1), (2) holds for $\omega(t)$. □

Then we can obtain an analogue of [59, Lemma 2.6, Lemma 2.7] and then the convergence in the Gromov-Hausdorff sense follows by the argument in [59, Section 3].

**Theorem 4.2.2.** $(N, \omega(t))$ converges in the Gromov-Hausdorff sense to $(N, d_T)$ as $t \to T^+$.

# Chapter 5

# $C^{\alpha}$-convergence of the solution of the Chern-Ricci flow on elliptic surfaces

## 5.1 A non-Kähler properly elliptic surface

The normalized Chern-Ricci flow is given by

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) - \omega(t), \\ \\ \omega(t)|_{t=0} = \omega_0, \end{cases}$$

where $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is a starting Gauduchon metric and the globally defined smooth real $(1,1)$-form locally given by

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det(g)$$

is the Chern-Ricci form of $\omega$.

A non-Kähler properly elliptic surface $M$ is a compact complex surface with its first Betti number $b_1(M) =$ odd and the Kodaira dimension $\mathrm{Kod}(M) = 1$ which admits an elliptic fibration $\pi : M \to S$ to a smooth compact curve $S$. The Kodaira-Enriques classification tells us that properly elliptic surfaces are the only one case for minimal non-Kähler complex surfaces with $\mathrm{Kod} = 1$ (cf. [3, p.244]).

We assume that $M$ is minimal, that is, there is no $(-1)$-curve on $M$. It has been shown that the universal cover of $M$ is $\mathbb{C} \times H$ [38, Theorem 28], where $H$ is the upper half plane in $\mathbb{C}$. Also, it is known that there is a finite unramified covering $p : M' \to M$ which is a minimal properly elliptic surface $\pi' : M' \to S'$ and $\pi'$ is an elliptic fiber bundle over a compact Riemann surface $S'$ of genus at least 2, with fiber an elliptic curve $E$ (cf. [12, Lemmas 1, 2]). So we firstly assume that $\pi : M \to S$ is an elliptic bundle with fiber $E$ with genus $g(S) \geq 2$, with $M$ minimal, non-Kähler and $\mathrm{Kod}(M) = 1$. That $g(S) \geq 2$ implies that the universal cover of $S$ is the upper half plane $H$ in $\mathbb{C}$ and there exists a metric on $S$ with negative constant curvature induced by the Poincaré metric on $H$, then we have $c_1(S) < 0$. And also we have $\mathrm{Kod}(S) = 1$.

It will be more convenient for us to work with $\mathbb{C}^* \times H$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. We define

$$h : \mathbb{C} \times H \to \mathbb{C}^* \times H, \quad h(z, z') = (e^{-\frac{z}{2}}, z'),$$

which is a holomorphic covering map. We will write $(z_1, z_2)$ for the coordinates on $\mathbb{C}^* \times H$ and $z_i = x_i + \sqrt{-1}y_i$, $x_i, y_i \in \mathbb{R}$ for $i = 1, 2$, which means that we have $y_2 > 0$.

It has been shown by Maehara (cf. [46]) that there exists a discrete subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ with $H/\Gamma = S$, together with $\lambda \in \mathbb{C}^*$ with $|\lambda| \neq 1$ and $\mathbb{C}^*/\langle\lambda\rangle = E$ and with a character $\chi : \Gamma \to \mathbb{C}^*$ such that $M$ is biholomorphic to the quotient of $\mathbb{C}^* \times H$ by the $\Gamma \times \mathbb{Z}$-action defined by

$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, n \right) \cdot (z_1, z_2) = \left( (cz_2 + d) \cdot z_1 \cdot \lambda^n \cdot \chi\Big( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big), \frac{az_2 + b}{cz_2 + d} \right)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $n \in \mathbb{Z}$, and then the map $\pi : M \to S$ is induced by the projection $\mathbb{C}^* \times H \to H$ (cf. [6, Proposition 2], [75, Theorem 7.4]). Note that all orientation preserving isometries of the complex upper half plane $H$ coincide with all linear fractional transformations of the form

$$z \mapsto \frac{az + b}{cz + d} \quad \text{with } ad - bc = 1 \text{ for } z \in H,\ a, b, c, d \in \mathbb{R}.$$

We define two forms on $\mathbb{C}^* \times H$ below:

$$\alpha := \frac{\sqrt{-1}}{2y_2^2} dz_2 \wedge d\bar{z}_2, \quad \gamma := \sqrt{-1}\Big(-\frac{2}{z_1} dz_1 + \frac{\sqrt{-1}}{y_2} dz_2\Big) \wedge \Big(-\frac{2}{\bar{z}_1} d\bar{z}_1 - \frac{\sqrt{-1}}{y_2} d\bar{z}_2\Big).$$

The unique Kähler-Einstein metric $\omega_S$ on $S$ with $\mathrm{Ric}(\omega_S) = -\omega_S$ is induced by the form $\alpha$. Since we can check that the forms on $\mathbb{C}^* \times H$; $\frac{\sqrt{-1}}{y_2^2} dz_2 \wedge d\bar{z}_2$ and $-\frac{2}{z_1} dz_1 + \frac{\sqrt{-1}}{y_2} dz_2$ are $\Gamma \times \mathbb{Z}$-invariant, these forms $\alpha$ and $\gamma$ are invariant under the $\Gamma \times \mathbb{Z}$-action. Hence they descend to $M$ and we define a Hermitian metric discovered by Vaisman in [74].:

$$\omega_V = 2\alpha + \gamma,$$

which is a Gauduchon metric, i.e., $\omega_V$ is a $\partial\bar{\partial}$-closed Hermitian metric. Indeed, it satisfies that

$$\bar{\partial}\omega_V = -\frac{\sqrt{-1}}{y_2^2 \bar{z}_1} d\bar{z}_2 \wedge dz_2 \wedge d\bar{z}_1, \quad \partial\bar{\partial}\omega_V = 0.$$

In [74, (2.9)], Vaisman introduced its pullback $h^*\omega_V$ by the holomorphic covering map $h$ on $\mathbb{C} \times H$.

Note that we may work in a single compact fundamental domain for $M$ in $\mathbb{C}^* \times H$ using $z_1, z_2$ as local coordinates and we may assume that $z_1, z_2$ are uniformly bounded and that $y_2$ is uniformly bounded from below away from zero.

Our main result is as follows:

**Theorem 5.1.1.** Let $M$ be a minimal non-Kähler properly elliptic surface and let $\omega(t)$ be the solution of the normalized Chern-Ricci flow starting at a Gauduchon metric of the form

$$\omega_0 = \omega_V + \sqrt{-1}\partial\bar{\partial}\psi > 0.$$

Then the metrics $\omega(t)$ are uniformly bounded in the $C^1$-topology, and as $t \to \infty$,

$$\omega(t) \to \pi^*\omega_S,$$

in the $C^\alpha$-topology, for every $0 < \alpha < 1$, where $\omega_S$ is the orbifold Kähler-Einstein metric on $S$ with $\mathrm{Ric}(\omega_S) = -\omega_S$ away from finitely many orbifold points induced by the form $\frac{\sqrt{-1}}{2y^2} dz \wedge d\bar{z}$ on $\mathbb{C}^* \times H$, $H$ is the upper half palne in $\mathbb{C}$, $z \in H$ is the variable, $y = \mathrm{Im} z$.

## 5.2 Proof of Theorem 5.1.1

We define reference metrics

$$\tilde{\omega} := e^{-t}\omega_V + (1 - e^{-t})\alpha = e^{-t}\gamma + (1 + e^{-t})\alpha,$$

which are Hermitian metrics for any $t \geq 0$. We denote these metrics $\tilde{g}$ and also denote quantities with respect to $\tilde{g}$ with using a tilde such as the torsion tensor, the Chern connection and the Chern curvature tensor.

We define a volume form $\Omega$ by

$$\Omega = 2\alpha \wedge \gamma$$

and we write $\mathrm{Ric}(\Omega)$ for the globally defined real $(1,1)$-form given locally by $-\sqrt{-1}\partial\bar{\partial}\log\Omega$. Then we have

$$\mathrm{Ric}(\Omega) = -\alpha \in c_1^{BC}(M) = -c_1^{BC}(K_M), .$$

which implies that $c_1^{BC}(M) = \pi^* c_1(S)$. Since we have assumed that $g(S) \geq 2$, we have $c_1(S) < 0$. So we have $c_1^{BC}(K_M) \geq 0$, which means that the first Bott-Chern class of the canonical bundle $c_1^{BC}(K_M)$ is nef. Here, we say that $c_1^{BC}(K_M)$ is nef if for any $\varepsilon > 0$, there exists a real smooth function $f_\varepsilon$ on $M$ such that $-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}f_\varepsilon > -\varepsilon\omega_0$, or equivalently for any $\varepsilon > 0$, there exists a smooth Hermitian metric $h_\varepsilon$ on the fibers of the canonical bundle $K_M$ with its curvature form bigger than $-\varepsilon\omega_0$. Hence from [71, Theorem 2.1], the normalized Chern-Ricci flow (equivalently the Chern-Ricci flow) has a smooth solution defined for all $t \geq 0$. For instance, the following time-metric scaling for the solution of the Chern-Ricci flow

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)), \\ \\ \omega(t)|_{t=0} = \omega_0, \end{cases}$$

allows us to transform a solution of the normalized Chern-Ricci flow:

$$\omega(t) = e^s\tilde{\omega}(s), \quad s(t) = \log(t+1),$$

$$\begin{cases} \frac{\partial}{\partial s}\tilde{\omega}(s) = -\mathrm{Ric}(\tilde{\omega}(s)) - \tilde{\omega}(s), \\ \\ \tilde{\omega}(s)|_{s=0} = \omega_0, \end{cases}$$

where $\omega_0 = \omega_V + \sqrt{-1}\partial\bar{\partial}\psi$.

We can observe that the following normalized Chern-Ricci flow is equivalent to the parabolic Monge-Ampère flow for $t \in [0, \infty)$

$$(\dagger) \qquad \frac{\partial}{\partial t}\varphi = \log \frac{e^t(\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^2}{\Omega} - \varphi, \quad \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \quad \varphi(0) = \psi.$$

If $\varphi = \varphi(t)$ solves ($\dagger$), then $\omega(t) = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$ is the solution of the normalized Chern-Ricci flow. On the other hand, given a solution $\omega(t)$ of the normalized Chern-Ricci flow, we can find a solution $\varphi = \varphi(t)$ of the equation ($\dagger$) with $\omega(t) = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$.

Here we let $\varphi = \varphi(t)$ solves the equation above and we will write

$$\omega = \omega(t) = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$$

with $\omega_0 = \omega(0) = \omega_V + \sqrt{-1}\partial\bar{\partial}\psi$.

We have the following lemma (cf. [22, Lemma 2.2], [71, Lemma 3.4]):

**Lemma 5.2.1.** There exists a uniform constant $C > 0$ such that for all $t \geq 0$,

(1) $|\varphi| \leq C(1+t)e^{-t}$

(2) $|\dot{\varphi}| \leq C$

(3) $C^{-1}\tilde{\omega}^2 \leq \omega^2 \leq C\tilde{\omega}^2$.

Proof. We firstly observe that

$$e^t \log \frac{e^t\tilde{\omega}}{\Omega} = e^t \log\Big(\frac{2\alpha\wedge\gamma + e^{-t}(\gamma^2 + 2\gamma\wedge\alpha)}{2\alpha\wedge\gamma}\Big) = e^t\log(1 + O(e^{-t})).$$

Hence we have

$$\Big|e^t \log \frac{e^t\tilde{\omega}^2}{\Omega}\Big| \leq C_1,$$

for uniform positive constant $C_1$. Define

$$W := e^t\varphi - (C_1 + 1)t.$$

We assume that $W$ achieves its maximum on $M \times [0, t_0]$ for some $t_0 > 0$ at $(x_0, t_0) \in M \times [0, t_0]$. Then we have at $(x_0, t_0)$,

$$0 \leq \frac{\partial}{\partial t}W \leq e^t \log \frac{e^t\tilde{\omega}^2}{\Omega} - C_1 - 1 \leq -1,$$

which is a contradiction. It follows that the maximum value of $W$ on $M$ must be bounded from above by its value at time 0. This gives us $\varphi \leq C(1+t)e^{-t}$ for some uniform positive constant $C$. For the lower bound, we similarly consider

$$W' := e^t\varphi + (C_1 + 1)t$$

and get a contradiction. Then by combining these, we obtain $|\varphi| \leq C(1+t)e^{-t}$ for some uniform positive constant $C$.

We now choose a positive constant $C_0$ so that $C_0\tilde{\omega} > \alpha$ for all $t \geq 0$. For the Laplacian $\Delta$ of $g = g(t)$, metrics corresponding to $\omega(t)$, we compute

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)(\dot{\varphi} - (C_0 - 1)\varphi) &= \mathrm{tr}_\omega(\alpha - \tilde{\omega}) + 1 - C_0\dot{\varphi} + (C_0 - 1)\mathrm{tr}_\omega(\omega - \tilde{\omega}) \\ &< 1 - C_0\dot{\varphi} + 2(C_0 - 1).\end{aligned}$$

By the maximum principle, we obtain the upper bound for $\dot{\varphi}$. For the lower bound, we compute

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)(\dot{\varphi} + 2\varphi) &= \mathrm{tr}_\omega(\alpha - \tilde{\omega}) + 1 + \dot{\varphi} - 2\mathrm{tr}_\omega(\omega - \tilde{\omega}) \\ &\geq \mathrm{tr}_\omega(\tilde{\omega}) - 3 + \dot{\varphi} \\ &\geq \frac{1}{C}e^{-\frac{\varphi + \dot{\varphi}}{2}} + \dot{\varphi} - 3,\end{aligned}$$

for a uniform positive constant $C$, where we used the geometric-arithmetic means inequality and that $e^t\tilde{\omega}^2$ and $\Omega$ are uniformly equivalent. It follows that $\dot{\varphi}$ is bounded from below by the maximum principle. $\square$

We can show the desired result by computing directly as in [22] and the following estimates play the most important role in our argument.

**Lemma 5.2.2.** There exists a uniform constant $C > 0$ such that

(1) $|\tilde{T}|_{\tilde{g}} \leq C$.

(2) $|\bar{\partial}\tilde{T}|_{\tilde{g}} + |\tilde{\nabla}\tilde{T}|_{\tilde{g}} + |\tilde{R}|_{\tilde{g}} \leq C$.

(3) $|\tilde{\nabla}\tilde{R}|_{\tilde{g}} + |\overline{\tilde{\nabla}}\tilde{\nabla}\tilde{T}|_{\tilde{g}} + |\tilde{\nabla}\overline{\tilde{\nabla}}\tilde{T}|_{\tilde{g}} \leq C$,

where $\tilde{T}$ is the torsion tensor of $\tilde{g}$, written locally as $\tilde{T}^k_{ij} = \tilde{\Gamma}^k_{ij} - \tilde{\Gamma}^k_{ji}$, $\tilde{T}_{ij\bar{l}} = \tilde{T}^k_{ij}g_{k\bar{l}}$, $\tilde{R}$ is the Chern curvature tensor of $\tilde{g}$, locally written as $\tilde{R}_{i\bar{j}k}{}^{l} = -\partial_{\bar{j}}\tilde{\Gamma}^l_{ik}$ and $\tilde{\nabla}$ is the Chern connection associated to $\tilde{g}$.

Proof. Using the local coordinates $(z_1, z_2)$ as in the previous section, we will write $\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j$, $\tilde{\omega} = \sqrt{-1}\tilde{g}_{i\bar{j}}dz^i \wedge d\bar{z}^j$ and we have

$$\tilde{g}_{1\bar{1}} = \frac{4}{|z_1|^2}e^{-t}, \quad \tilde{g}_{1\bar{2}} = \sqrt{-1}\frac{2}{z_1y_2}e^{-t}, \quad \tilde{g}_{2\bar{1}} = -\sqrt{-1}\frac{2}{\bar{z}_1y_2}e^{-t},$$

$$\tilde{g}_{2\bar{2}} = \frac{1+e^{-t}}{2y_2^2} + \frac{e^{-t}}{y_2^2}, \quad \det\tilde{g} = \frac{2e^{-t}(1+e^{-t})}{|z_1|^2y_2^2},$$

and

$$\tilde{g}^{1\bar{1}} = \frac{e^t(1+3e^{-t})|z_1|^2}{4(1+e^{-t})}, \quad \tilde{g}^{1\bar{2}} = \sqrt{-1}\frac{z_1y_2}{1+e^{-t}}, \quad \tilde{g}^{2\bar{1}} = -\sqrt{-1}\frac{\bar{z}_1y_2}{1+e^{-t}}, \quad \tilde{g}^{2\bar{2}} = \frac{2y_2^2}{1+e^{-t}}.$$

The Christoffel symbols $\tilde{\Gamma}^k_{ij}$ of the Chern connection of $\tilde{g}$ are as follows:

$$\tilde{\Gamma}^2_{11} = \tilde{\Gamma}^1_{12} = \tilde{\Gamma}^2_{12} = 0,$$

$$\tilde\Gamma^1_{21} = -\sqrt{-1}\frac{1}{y_2(1+e^t)}, \quad \tilde\Gamma^2_{21} = -\frac{2}{z_1(1+e^t)},$$
$$\tilde\Gamma^1_{22} = -\frac{z_1(1+3e^{-t})}{4y_2^2(1+e^{-t})}, \quad \tilde\Gamma^1_{11} = -\frac{1+3e^{-t}}{z_1(1+e^{-t})} + \frac{2}{z_1(1+e^t)},$$

and

$$\tilde\Gamma^2_{22} = -\sqrt{-1}\frac{1}{y_2(1+e^t)} + \sqrt{-1}\frac{1+3e^{-t}}{y_2(1+e^{-t})}.$$

Hence the torsion tensor $\tilde T$ of $\tilde g$ can be given by

$$\tilde T^1_{21} = -\sqrt{-1}\frac{1}{y_2(1+e^t)}, \quad \tilde T^2_{12} = \frac{2}{z_1(1+e^t)}.$$

The Chern curvature tensor $\tilde R$ of $\tilde g$ can be computed in the following way:

$$\tilde R_{2\bar 21}{}^1 = \frac{1}{2y_2^2(1+e^t)}, \quad \tilde R_{2\bar 22}{}^2 = -\frac{2+e^t}{2y_2^2(1+e^t)},$$
$$\tilde R_{2\bar 22}{}^1 = -\sqrt{-1}\frac{z_1(1+3e^{-t})}{4y_2^3(1+e^{-t})}$$

and other components of the tensor $\tilde R$

$$\tilde R_{2\bar 12}{}^1, \tilde R_{2\bar 11}{}^1, \tilde R_{1\bar 21}{}^2, \tilde R_{1\bar 11}{}^2, \tilde R_{1\bar 11}{}^1, \tilde R_{2\bar 12}{}^2, \tilde R_{1\bar 21}{}^1, \tilde R_{1\bar 12}{}^2, \tilde R_{1\bar 22}{}^2, \tilde R_{2\bar 11}{}^2, \tilde R_{2\bar 21}{}^2, \tilde R_{1\bar 12}{}^1, \tilde R_{1\bar 22}{}^1$$

are all equal to zero.

We compute

$$\partial_{\bar 1}\tilde T^2_{12} = \partial_{\bar 2}\tilde T^2_{12} = 0, \quad \partial_{\bar 1}\tilde T^1_{21} = 0, \quad \partial_{\bar 2}\tilde T^1_{21} = -\frac{1}{2y_2^2(1+e^t)}$$

and

$$\tilde\nabla_1\tilde T^2_{12} = \partial_1\tilde T^2_{12} - \tilde\Gamma^1_{11}\tilde T^2_{12} = O(e^{-t}), \quad \tilde\nabla_1\tilde T^1_{21} = 0,$$
$$\tilde\nabla_2\tilde T^1_{21} = \partial_2\tilde T^1_{21} - \tilde\Gamma^2_{22}\tilde T^1_{21} + \tilde\Gamma^1_{22}\tilde T^2_{21} = O(e^{-t}), \quad \tilde\nabla_2\tilde T^2_{12} = -\tilde\Gamma^1_{21}\tilde T^2_{12} + \tilde\Gamma^2_{21}\tilde T^1_{12} = O(e^{-2t}).$$

By direct calculation, we have

$$\tilde\nabla_1\tilde\nabla_{\bar 2}\tilde T^1_{21} = 0, \quad \tilde\nabla_2\tilde\nabla_{\bar 2}\tilde T^1_{21} = \partial_2\partial_{\bar 2}\tilde T^1_{21} + (\tilde\Gamma^1_{22} - \tilde\Gamma^2_{22})\partial_{\bar 2}\tilde T^1_{21} = O(e^{-t}),$$

$$\tilde\nabla_{\bar 2}\tilde\nabla_{\bar 1}\tilde T^1_{21} = -\overline{\tilde\Gamma^2_{21}}\partial_{\bar 2}\tilde T^1_{21} = O(e^{-2t}), \quad \tilde\nabla_{\bar 2}\tilde\nabla_{\bar 2}\tilde T^1_{21} = \partial_{\bar 2}\partial_{\bar 2}\tilde T^1_{21} - \overline{\tilde\Gamma^2_{22}}\partial_{\bar 2}\tilde T^1_{21} = O(e^{-t})$$

and

$$\tilde\nabla_2\tilde\nabla_{\bar 2}\tilde T^2_{12} = \tilde\Gamma^2_{21}\partial_{\bar 2}\tilde T^1_{12} = O(e^{-2t}), \quad \tilde\nabla_2\tilde\nabla_{\bar 1}\tilde T^2_{12} = 0, \quad \tilde\nabla_1\tilde\nabla_{\bar 1}\tilde T^2_{12} = 0,$$
$$\tilde\nabla_1\tilde\nabla_{\bar 2}\tilde T^2_{12} = \partial_1\partial_{\bar 2}\tilde T^2_{12} - \tilde\Gamma^1_{11}\partial_{\bar 2}\tilde T^2_{12} + \tilde\Gamma^2_{11}\partial_{\bar 2}\tilde T^1_{12} = 0.$$

For any $i, j = 1, 2$, we have

$$\tilde\nabla_i\tilde\nabla_{\bar 1}\tilde T^1_{21} = 0, \quad \tilde\nabla_{\bar 1}\tilde\nabla_{\bar j}\tilde T^1_{21} = 0$$

and
$$\tilde{\nabla}_{\bar{i}}\tilde{\nabla}_{\bar{j}}\tilde{T}^2_{12} = 0.$$

We can also check that

$$\tilde{\nabla}_2\tilde{R}_{2\bar{2}2}{}^2, \tilde{\nabla}_1\tilde{R}_{2\bar{2}2}{}^1, \tilde{\nabla}_2\tilde{R}_{2\bar{2}2}{}^1$$

are of order $O(1)$,

$$\tilde{\nabla}_2\tilde{R}_{2\bar{2}1}{}^1$$

is of order $O(e^{-t})$, and other components

$$\tilde{\nabla}_1\tilde{R}_{2\bar{1}2}{}^2, \tilde{\nabla}_2\tilde{R}_{2\bar{1}2}{}^2, \tilde{\nabla}_1\tilde{R}_{2\bar{2}2}{}^2, \tilde{\nabla}_1\tilde{R}_{1\bar{1}1}{}^1, \tilde{\nabla}_2\tilde{R}_{1\bar{1}1}{}^1, \tilde{\nabla}_1\tilde{R}_{1\bar{1}1}{}^2, \tilde{\nabla}_2\tilde{R}_{1\bar{1}1}{}^2, \tilde{\nabla}_1\tilde{R}_{1\bar{2}1}{}^2, \tilde{\nabla}_2\tilde{R}_{1\bar{2}1}{}^2,$$
$$\tilde{\nabla}_1\tilde{R}_{2\bar{1}1}{}^1, \tilde{\nabla}_2\tilde{R}_{2\bar{1}1}{}^1, \tilde{\nabla}_1\tilde{R}_{2\bar{2}1}{}^1, \tilde{\nabla}_1\tilde{R}_{2\bar{1}2}{}^1, \tilde{\nabla}_2\tilde{R}_{2\bar{1}2}{}^1$$

are all equal to zero. $\square$

Using the estimates in Lemma 5.2.1, we can obtain the following estimates (cf. [22, Theorem 2.4], [71, Section 5,6,7]):

**Lemma 5.2.3.** For $\varphi = \varphi(t)$ solving (†) on $M$, the estimates below hold.

(1) There exists a uniform constant $C > 0$ such that

$$\frac{1}{C}\tilde{\omega} \leq \omega(t) \leq C\tilde{\omega}.$$

(2) There exists a uniform constant $C > 0$ such that the Chern scalar curvature $\mathrm{Scal}_{g(t)}$ of $g(t)$ satisfies the bound
$$-C \leq \mathrm{Scal}_{g(t)} \leq C.$$

(3) For any $\eta, \sigma$ with $0 < \eta, \sigma < \frac{1}{2}$, there exists a constant $C_{\eta,\sigma} > 0$ such that

$$-C_{\eta,\sigma}e^{-\eta t} \leq \dot{\varphi}(t) \leq C_{\eta,\sigma}e^{-\sigma t}.$$

(4) For any $\varepsilon$ with $0 < \varepsilon < \frac{1}{4}$, there exists a constant $C_\varepsilon > 0$ such that

$$(1 - C_\varepsilon e^{-\varepsilon t})\tilde{\omega} \leq \omega(t) \leq (1 + C_\varepsilon e^{-\varepsilon t})\tilde{\omega}.$$

**Remark 5.2.1.** Even if we choose initial Gauduchon metric $\omega_0$ arbitrary, we can have the same estimates in Lemma 5.2.3 above except for the estimate in (2) by choosing $0 < \sigma < \frac{1}{4}$ and $0 < \varepsilon < \frac{1}{8}$ in (3) and (4) respectively [71, Lemma 6.4 & Theorem 7.1]. Speaking of the estimate for the scalar curvature in (2), when choosing $\omega_0$ arbitrary, although the lower bound can be chosen uniformly, the upper bound depends on $t$ [71, Theorem 6.1].

For proving these estimats above, we firstly require the following lemma.

**Lemma 5.2.4.** For $t \geq 0$, the following evolution inequality holds:

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) \log \mathrm{tr}_{\tilde{\omega}}\omega \leq \frac{2}{(\mathrm{tr}_{\tilde{\omega}}\omega)^2}\mathrm{Re}\Big(\tilde{g}^{i\bar{l}} g^{k\bar{q}} \tilde{T}_{ki\bar{l}} \partial_{\bar{q}} \mathrm{tr}_{\tilde{\omega}}\omega\Big) + C\mathrm{tr}_{\omega}\tilde{\omega}.$$

Proof. Firstly we compute

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{tr}_{\tilde{\omega}}\omega &= \tilde{g}^{k\bar{l}}\partial_k\partial_{\bar{l}} \log\det(g) - \mathrm{tr}_{\tilde{\omega}}\omega - \tilde{g}^{i\bar{l}}\tilde{g}^{k\bar{j}} g_{i\bar{j}} \frac{\partial}{\partial t}\tilde{g}_{k\bar{l}} \\
&= g^{i\bar{j}}\tilde{g}^{k\bar{l}}\tilde{\nabla}_{\bar{l}}\tilde{\nabla}_k g_{i\bar{j}} - g^{p\bar{j}} g^{i\bar{q}} \tilde{g}^{k\bar{l}} \tilde{\nabla}_k g_{i\bar{j}} \tilde{\nabla}_{\bar{l}} g_{p\bar{q}} - \tilde{g}^{k\bar{l}} \tilde{g}^{i\bar{j}} \tilde{R}_{k\bar{l}i\bar{j}} - \mathrm{tr}_{\tilde{\omega}}\omega - \tilde{g}^{i\bar{l}}\tilde{g}^{k\bar{j}} g_{i\bar{j}} \frac{\partial}{\partial t}\tilde{g}_{k\bar{l}}
\end{aligned}$$

From $T_{ij\bar{k}} = \tilde{T}_{ij\bar{k}}$, we have

$$\tilde{\nabla}_i \tilde{\nabla}_{\bar{j}} g_{k\bar{l}} = \tilde{\nabla}_i \tilde{\nabla}_{\bar{l}} g_{k\bar{j}} + (\tilde{\nabla}_i \overline{\tilde{T}^p_{jl}}) \tilde{g}_{k\bar{p}} - (\tilde{\nabla}_i \overline{\tilde{T}^q_{jl}}) g_{k\bar{q}} - \overline{\tilde{T}^q_{jl}} \tilde{\nabla}_i g_{k\bar{q}}.$$

Switching covariant derivatives and arguing as above,

$$\begin{aligned}
\tilde{\nabla}_i \tilde{\nabla}_{\bar{l}} g_{k\bar{j}} &= \tilde{\nabla}_{\bar{l}} \tilde{\nabla}_i g_{k\bar{j}} - \tilde{R}_{i\bar{l}k\bar{q}} \tilde{g}^{p\bar{q}} g_{p\bar{j}} + \tilde{R}_{i\bar{l}p\bar{j}} \tilde{g}^{p\bar{q}} g_{k\bar{q}} \\
&= \tilde{\nabla}_{\bar{l}} \tilde{\nabla}_k g_{i\bar{j}} + (\tilde{\nabla}_{\bar{l}} \tilde{T}^p_{ik}) \tilde{g}_{p\bar{j}} - (\tilde{\nabla}_{\bar{l}} \tilde{T}^p_{ik}) g_{p\bar{j}} - \tilde{T}^p_{ik} \tilde{\nabla}_{\bar{l}} g_{p\bar{j}} - \tilde{R}_{i\bar{l}k\bar{q}} \tilde{g}^{p\bar{q}} g_{p\bar{j}} + \tilde{R}_{i\bar{l}p\bar{j}} \tilde{g}^{p\bar{q}} g_{k\bar{q}}
\end{aligned}$$

It follows we have

$$\begin{aligned}
\Delta \mathrm{tr}_{\tilde{\omega}}\omega &= g^{i\bar{j}} \tilde{g}^{k\bar{l}} \tilde{\nabla}_i \tilde{\nabla}_{\bar{j}} g_{k\bar{l}} \\
&= g^{i\bar{j}} \tilde{g}^{k\bar{l}} \tilde{\nabla}_{\bar{l}} \tilde{\nabla}_k g_{i\bar{j}} + g^{i\bar{j}} \tilde{g}^{k\bar{l}} \Big( (\tilde{\nabla}_i \overline{\tilde{T}^p_{jl}}) \tilde{g}_{k\bar{p}} + (\tilde{\nabla}_{\bar{l}} \tilde{T}^p_{ik}) \tilde{g}_{p\bar{j}} - (\tilde{\nabla}_i \overline{\tilde{T}^q_{jl}} - \tilde{R}_{i\bar{l}p\bar{j}} \tilde{g}^{p\bar{q}}) g_{k\bar{q}} \\
&\quad -(\tilde{\nabla}_{\bar{l}} \tilde{T}^p_{ik} + \tilde{R}_{i\bar{l}k\bar{q}} \tilde{g}^{p\bar{q}}) g_{p\bar{j}} - \overline{\tilde{T}^q_{jl}} \tilde{\nabla}_i g_{k\bar{q}} - \tilde{T}^p_{ik} \tilde{\nabla}_{\bar{l}} g_{p\bar{j}} \Big).
\end{aligned}$$

Putting together, we obtain

$$\begin{aligned}
&\Big(\frac{\partial}{\partial t} - \Delta\Big) \log \mathrm{tr}_{\tilde{\omega}}\omega \\
= \ & \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega} \Big( - g^{p\bar{j}} g^{i\bar{q}} \tilde{g}^{k\bar{l}} \tilde{\nabla}_k g_{i\bar{j}} \tilde{\nabla}_{\bar{l}} g_{p\bar{q}} + \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega} g^{k\bar{l}} \partial_k \mathrm{tr}_{\tilde{\omega}}\omega \partial_{\bar{l}} \mathrm{tr}_{\tilde{\omega}}\omega \\
& -2\mathrm{Re}\Big( g^{i\bar{j}} \tilde{g}^{k\bar{l}} \tilde{T}^p_{ki} \tilde{\nabla}_{\bar{l}} g_{p\bar{j}} \Big) - g^{i\bar{j}} \tilde{g}^{k\bar{l}} \tilde{T}^p_{ik} \overline{\tilde{T}^q_{jl}} g_{p\bar{q}} \\
& + g^{i\bar{j}} \tilde{g}^{k\bar{l}} (\tilde{\nabla}_i \overline{\tilde{T}^q_{jl}} - \tilde{R}_{i\bar{l}p\bar{j}} \tilde{g}^{p\bar{q}}) g_{k\bar{q}} - g^{i\bar{j}} \tilde{\nabla}_i \overline{\tilde{T}^l_{jl}} - g^{i\bar{j}} \tilde{g}^{k\bar{l}} \tilde{g}_{p\bar{j}} \tilde{\nabla}_{\bar{l}} \tilde{T}^p_{ik} \\
& + g^{i\bar{j}} \tilde{g}^{k\bar{l}} \tilde{T}^p_{ik} \overline{\tilde{T}^q_{jl}} \tilde{g}_{p\bar{q}} - \mathrm{tr}_{\tilde{\omega}}\omega - \tilde{g}^{i\bar{l}} \tilde{g}^{k\bar{j}} g_{i\bar{j}} \frac{\partial}{\partial t} \tilde{g}_{k\bar{l}} \Big).
\end{aligned}$$

Note that we have $T_{ij\bar{k}} = \tilde{T}_{ij\bar{k}}$. We observe that $\frac{\partial}{\partial t}\tilde{g} = \alpha - \tilde{g} \geq -\tilde{g}$ and then we have $-\mathrm{tr}_{\tilde{\omega}}\omega - \tilde{g}^{i\bar{l}}\tilde{g}^{k\bar{j}} g_{i\bar{j}} \frac{\partial}{\partial t}\tilde{g}_{k\bar{l}} \leq 0$. We can estimate

$$\begin{aligned}
& \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega} \Big( - g^{p\bar{j}} g^{i\bar{q}} \tilde{g}^{k\bar{l}} \tilde{\nabla}_k g_{i\bar{j}} \tilde{\nabla}_{\bar{l}} g_{p\bar{q}} + \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega} g^{k\bar{l}} \partial_k \mathrm{tr}_{\tilde{\omega}}\omega \partial_{\bar{l}} \mathrm{tr}_{\tilde{\omega}}\omega \\
& -2\mathrm{Re}\Big( g^{i\bar{j}} \tilde{g}^{k\bar{l}} \tilde{T}^p_{ki} \tilde{\nabla}_{\bar{l}} g_{p\bar{j}} \Big) - g^{i\bar{j}} \tilde{g}^{k\bar{l}} \tilde{T}^p_{ik} \overline{\tilde{T}^q_{jl}} g_{p\bar{q}} \Big) \\
\leq \ & \frac{2}{(\mathrm{tr}_{\tilde{\omega}}\omega)^2} \mathrm{Re}\Big( \tilde{g}^{i\bar{l}} g^{k\bar{q}} \tilde{T}_{ki\bar{l}} \partial_{\bar{q}} \mathrm{tr}_{\tilde{\omega}}\omega \Big)
\end{aligned}$$

and

$$
\begin{aligned}
& \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}\Big(g^{i\bar{j}}\tilde{g}^{k\bar{l}}(\tilde{\nabla}_i\overline{\tilde{T}^q_{jl}}-\tilde{R}_{i\bar{l}p\bar{j}}\tilde{g}^{p\bar{q}})g_{k\bar{q}}-g^{i\bar{j}}\tilde{\nabla}_i\overline{\tilde{T}^l_{jl}}-g^{i\bar{j}}\tilde{g}^{k\bar{l}}\tilde{g}_{p\bar{j}}\tilde{\nabla}_{\bar{l}}\tilde{T}^p_{ik} \\
& +g^{i\bar{j}}\tilde{g}^{k\bar{l}}\tilde{T}^p_{ik}\overline{\tilde{T}^q_{jl}}\tilde{g}_{p\bar{q}}-\mathrm{tr}_{\tilde{\omega}}\omega-\tilde{g}^{i\bar{l}}\tilde{g}^{k\bar{j}}g_{i\bar{j}}\frac{\partial}{\partial t}\tilde{g}_{k\bar{l}}\Big) \\
\leq\ & C\mathrm{tr}_{\omega}\tilde{\omega}.
\end{aligned}
$$

□

### 5.2.1 Proof of Lemma 5.2.3

We consider the quantity

$$Q := \log \mathrm{tr}_{\tilde{\omega}}\omega - A\varphi + \frac{1}{\tilde{C}+\varphi},$$

where $\tilde{C}$ is a unform constant chosen so that $\tilde{C}+\varphi \geq 1$, and $A$ ia a large constant too be determined later. We assume that $Q$ achives it maximum at a point $(x_0, t_0)$ with $t_0 > 0$. At the point $(x_0, t_0)$, we have

$$0 = \partial_{\bar{q}}Q = \frac{\partial_{\bar{q}}\mathrm{tr}_{\tilde{\omega}}\omega}{\mathrm{tr}_{\tilde{\omega}}\omega} - \Big(A + \frac{1}{(\tilde{C}+\varphi)^2}\Big)\partial_{\bar{q}}\varphi.$$

Then we have at $(x_0, t_0)$,

$$
\begin{aligned}
& \frac{2}{(\mathrm{tr}_{\tilde{\omega}}\omega)^2}\mathrm{Re}\Big(\tilde{g}^{i\bar{l}}g^{k\bar{q}}\tilde{T}_{ki\bar{l}}\partial_{\bar{q}}\mathrm{tr}_{\tilde{\omega}}\omega\Big) \\
=\ & \frac{2}{\mathrm{tr}_{\tilde{\omega}}\omega}\mathrm{Re}\Big(\tilde{g}^{i\bar{l}}g^{k\bar{q}}\tilde{T}_{ki\bar{l}}\Big(A+\frac{1}{(\tilde{C}+\varphi)^2}\Big)\partial_{\bar{q}}\varphi\Big) \\
\leq\ & \frac{CA^2}{(\mathrm{tr}_{\tilde{\omega}}\omega)^2}(\tilde{C}+\varphi)^3 g^{k\bar{q}}\tilde{g}^{i\bar{l}}\tilde{T}_{ki\bar{l}}\tilde{g}^{m\bar{j}}\overline{\tilde{T}_{qj\bar{m}}} + \frac{|\partial\varphi|^2_g}{(\tilde{C}+\varphi)^3} \\
\leq\ & \frac{CA^2}{\mathrm{tr}_{\tilde{\omega}}\omega} + \frac{|\partial\varphi|^2_g}{(\tilde{C}+\varphi)^3},
\end{aligned}
$$

where we used that $\mathrm{tr}_{\tilde{\omega}}\omega$ and $\mathrm{tr}_{\omega}\tilde{\omega}$ are uniformly equivalent. Since we have

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\Big(-A\varphi + \frac{1}{\tilde{C}+\varphi}\Big) \leq CA - A\mathrm{tr}_{\omega}\tilde{\omega} - \frac{2|\partial\varphi|^2_g}{(\tilde{C}+\varphi)^3},$$

we obtain at the point $(x_0, t_0)$,

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)Q \leq CA^2 + C\mathrm{tr}_{\omega}\tilde{\omega} + CA - A\mathrm{tr}_{\omega}\tilde{\omega},$$

where we are assuming without loss of generality we have $\mathrm{tr}_{\tilde{\omega}}\omega(x_0, t_0) \geq 1$. By choosing $A$ sufficiently large so that $A \geq C + 1$, we obtain at the point,

$$\mathrm{tr}_{\omega}\tilde{\omega} \leq CA^2 + CA.$$

It flows taht $\mathrm{tr}_{\tilde\omega}\omega(x_0,t_0)$ is uniformly bounded from above, and then we obtain the uniform upper bound of $\mathrm{tr}_{\tilde\omega}\omega$.

We put $S_t := \mathrm{Scal}_{g(t)} = -g^{i\bar j}\partial_i\partial_{\bar j}\log\det g$. We compute

$$\begin{aligned}\frac{\partial}{\partial t}S_t &= \Delta S_t + |\,\mathrm{Ric}\,|_g^2 + S_t\\ &\geq \Delta S_t + \frac{1}{2}S_t^2 + S_t,\end{aligned}$$

and then we obtain the lower bound for $S_t$.

For the upper bound of $S_t$, we require the following evolution inequalities.

**Lemma 5.2.5.** There exists a uniform constant such that for $t \geq 0$, we have

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\mathrm{tr}_{\tilde\omega}\omega \leq -\frac{1}{C}|\tilde\nabla g|_g^2 + C,$$

and

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\mathrm{tr}_{\omega}\alpha \leq |\tilde\nabla g|_g^2 - \frac{1}{C}|\nabla \mathrm{tr}_\omega\alpha|_g^2 + C.$$

Combining these, there are uniform positive constants $C_0, C_1$ such that for $t \geq 0$,

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)(\mathrm{tr}_\omega\alpha + C_0\mathrm{tr}_{\tilde\omega}\omega) \leq -|\tilde\nabla g|_g^2 - \frac{1}{C_1}|\nabla \mathrm{tr}_\omega\alpha|_g^2 + C_1.$$

Proof. For $t \geq 0$, we can compute

$$\begin{aligned}&\Big(\frac{\partial}{\partial t} - \Delta\Big)\mathrm{tr}_{\tilde\omega}\omega\\ =\ & -g^{p\bar j}g^{i\bar q}\tilde g^{k\bar l}\tilde\nabla_k g_{i\bar j}\tilde\nabla_{\bar l}g_{p\bar q} - 2\mathrm{Re}\Big(g^{i\bar j}\tilde g^{k\bar l}\tilde T^p_{ki}\tilde\nabla_{\bar l}g_{p\bar j}\Big)\\ & -g^{i\bar j}\tilde g^{k\bar l}\tilde T^p_{ik}\overline{\tilde T^q_{jl}}g_{p\bar q} + g^{i\bar j}\tilde g^{k\bar l}(\tilde\nabla_i\overline{\tilde T^q_{jl}} - \tilde R_{i\bar l p\bar j}\tilde g^{p\bar q})g_{k\bar q}\\ & -g^{i\bar j}\tilde\nabla_i\overline{\tilde T^l_{jl}} - g^{i\bar j}\tilde g^{k\bar l}\tilde g_{p\bar j}\tilde\nabla_{\bar l}\tilde T^p_{ik} + g^{i\bar j}\tilde g^{k\bar l}\tilde T^p_{ik}\overline{\tilde T^q_{jl}}\tilde g_{p\bar q} - \mathrm{tr}_{\tilde\omega}\omega\\ & -\tilde g^{i\bar l}\tilde g^{k\bar j}g_{i\bar j}(\alpha_{k\bar l} - \tilde g_{k\bar l})\\ \leq\ & -\frac{1}{C}|\tilde\nabla g|_g^2 + C.\end{aligned}$$

The second inequality is a parabolic Schwarz Lemma for the map $\pi : M \to S$. Since we have showed that $\omega$ and $\tilde\omega$ are uniformly equivalent, we obtain $\mathrm{tr}_\omega\alpha \leq C$ for some uniform positive constant $C$. Given any point $x \in M$, we choose local coordinates $(z_1, z_2)$ centered at $x$ such that $g$ is the identity at $x$, and a coodinate $w$ on $S$ near $\pi(x) \in S$, which may be assumed to be normal for $g_P$, where $g_P$ is the Poincaré metric

$$\omega_P = \sqrt{-1}g_P dz_2\wedge d\bar z_2 = \frac{\sqrt{-1}}{2y_2^2}dz_2\wedge d\bar z_2$$

on the upper half plane $H$. In these coordinates, we can represent the map $\pi$ as a local holomorphic function $f$. We will write $f_i := \nabla_i f = \partial_i f$, $f_{ij} := \nabla_i\nabla_j f = \partial_j f_i - \Gamma^k_{ji}f_k$ and we have $f_{i\bar{j}} = f_{\bar{j}i} = 0$. The pullback of the metric $\alpha = f^*\omega_P$ is given by $f_i \overline{f_{j}} g_P$. Put $h^{i\bar{j}} := g^{i\bar{l}} g^{k\bar{j}} f_k \overline{f_l} g_P$, which is semipositive definite and satisfies $|h|_g^2 := h^{i\bar{j}} h^{k\bar{l}} g_{i\bar{l}} g_{k\bar{j}} \leq C$. We have at the point $x$,

$$\begin{aligned}
\Delta \mathrm{tr}_\omega \alpha &= g^{i\bar{j}} \partial_i \partial_{\bar{j}} \Big( g^{k\bar{l}} f_k \overline{f_l} g_P \Big) \\
&= g^{i\bar{j}} g^{k\bar{l}} f_{ki} \overline{f_{lj}} g_P + g^{i\bar{j}} h^{p\bar{q}} R_{i\bar{j}p\bar{q}} - g^{i\bar{j}} g^{k\bar{l}} f_i \overline{f_j} f_k \overline{f_l} R_P \\
&\geq g^{i\bar{j}} g^{k\bar{l}} f_{ki} \overline{f_{lj}} g_P + g^{i\bar{j}} h^{p\bar{q}} R_{i\bar{j}p\bar{q}},
\end{aligned}$$

where $R_P < 0$ is the scalar curvature of $g_P$. Note that we have at the point $x$,

$$\partial_i \mathrm{tr}_\omega \alpha = \sum_k f_{ki} \overline{f_k}.$$

Then we can compute at $x$,

$$\begin{aligned}
|\nabla \mathrm{tr}_\omega \alpha|_g^2 &= \sum_{i,k,p} f_{ki} \overline{f_{pi}} f_p \overline{f_k} \\
&\leq \Big( \sum_k |f_k| \Big( \sum_i |f_{ki}|^2 \Big)^{\frac{1}{2}} \Big)^2 \\
&\leq \Big( \sum_l |f_l|^2 \Big) \Big( \sum_{i,k} |f_{ki}|^2 \Big) \\
&= \mathrm{tr}_\omega \alpha\, g^{i\bar{j}} g^{k\bar{l}} f_{ki} \overline{f_{lj}} g_P \\
&\leq C g^{i\bar{j}} g^{k\bar{l}} f_{ki} \overline{f_{lj}} g_P.
\end{aligned}$$

We need the following calculations for completing the proof.

$$\begin{aligned}
R_{i\bar{j}p\bar{q}} &= -g_{r\bar{q}} \partial_{\bar{j}} \Gamma^r_{ip} \\
&= -g_{r\bar{q}} \partial_{\bar{j}} \Gamma^r_{pi} + g_{r\bar{q}} \partial_{\bar{j}} T^r_{pi} \\
&= \overline{R_{j\bar{p}q\bar{i}}} + g_{r\bar{q}} \partial_{\bar{j}} T^r_{pi} \\
&= R_{p\bar{q}i\bar{j}} + g_{i\bar{s}} \partial_p \overline{T^s_{qj}} + g_{r\bar{q}} \partial_{\bar{j}} T^r_{pi}
\end{aligned}$$

Since we have $T_{ij\bar{l}} = \tilde{T}_{ij\bar{l}}$, we differentiate both side

$$g_{k\bar{l}} \partial_p T^k_{ij} + T^k_{ij} \tilde{\nabla}_p g_{k\bar{l}} = \tilde{\nabla}_p T_{ij\bar{l}} = \tilde{\nabla}_p \tilde{T}_{ij\bar{l}} = \tilde{g}_{r\bar{l}} \tilde{\nabla}_p \tilde{T}^r_{ij},$$

which is

$$\partial_p T^s_{ij} = g^{s\bar{l}} \tilde{g}_{r\bar{l}} \tilde{\nabla}_p \tilde{T}^r_{ij} - g^{s\bar{l}} g^{k\bar{b}} \tilde{g}_{r\bar{b}} \tilde{T}^r_{ij} \tilde{\nabla}_p g_{k\bar{l}}.$$

With using this, we compute

$$\partial_p \overline{T^s_{qj}} = \tilde{g}_{r\bar{s}} g^{r\bar{j}} \partial_p \overline{\tilde{T}^s_{qj}} - \tilde{g}_{r\bar{s}} g^{r\bar{b}} g^{k\bar{j}} \overline{\tilde{T}^s_{qj}} \tilde{\nabla}_p g_{k\bar{b}} = g^{r\bar{j}} \tilde{\nabla}_p \overline{\tilde{T}_{qj\bar{r}}} - g^{r\bar{b}} g^{k\bar{j}} \overline{\tilde{T}_{qj\bar{r}}} \tilde{\nabla}_p g_{k\bar{b}},$$

$$\partial_{\bar{j}} T^r_{pi} = g^{r\bar{l}} \partial_{\bar{j}} \tilde{T}^s_{pi} \tilde{g}_{s\bar{l}} - g^{r\bar{l}} T^s_{pi} \tilde{\nabla}_{\bar{j}} g_{s\bar{l}} = g^{r\bar{l}} \tilde{\nabla}_{\bar{j}} \tilde{T}_{pi\bar{l}} - g^{r\bar{l}} g^{s\bar{b}} \tilde{T}_{pi\bar{b}} \tilde{\nabla}_{\bar{j}} g_{s\bar{l}}.$$

Then we obtain

$$\begin{aligned} g^{i\bar{j}}h^{p\bar{q}}(R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}}) &= -h^{p\bar{q}}\partial_p\overline{T^j_{qj}} - g_{r\bar{q}}g^{i\bar{j}}h^{p\bar{q}}\partial_{\bar{j}}T^r_{pi} \\ &= -h^{p\bar{q}}g^{r\bar{j}}\tilde{\nabla}_p\overline{\tilde{T}_{qj\bar{r}}} + h^{p\bar{q}}g^{r\bar{b}}g^{a\bar{j}}\overline{\tilde{T}_{qj\bar{r}}}\tilde{\nabla}_p g_{a\bar{b}} \\ &\quad -h^{p\bar{q}}g^{i\bar{j}}\tilde{\nabla}_{\bar{j}}\tilde{T}_{pi\bar{q}} + h^{p\bar{q}}g^{i\bar{j}}g^{a\bar{r}}\tilde{T}_{pi\bar{r}}\tilde{\nabla}_{\bar{j}}g_{a\bar{q}}. \end{aligned}$$

Finally we have

$$|g^{i\bar{j}}h^{p\bar{q}}(R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}})| \leq |\tilde{\nabla}g|^2_g + C,$$

where we used that the metrics $g$ and $\tilde{g}$ are uniformly equivalent and $|h|_g \leq C$.

□

We consider the quality $u := \dot{\varphi} + \varphi$, we know that $|u| \leq C$ for some uniform positive constant $C$ and can compute $-\Delta u = S_t + \mathrm{tr}_\omega\alpha \geq S_t$. Hence, we require to get an upper bound for $-\Delta u$ in order to obtain the upper bound for $S_t$. We compute

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)u = \mathrm{tr}_\omega\alpha - 1,$$

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\Delta u = S_t^{i\bar{j}}u_{i\bar{j}} + \Delta u + \Delta\mathrm{tr}_\omega\alpha.$$

$$\begin{aligned} \Big(\frac{\partial}{\partial t} - \Delta\Big)\Delta u &= -|\,\mathrm{Ric}\,|^2_g - S_t - g^{i\bar{q}}g^{p\bar{j}}R_{p\bar{q}} - \mathrm{tr}_\omega\alpha + \Delta\mathrm{tr}_\omega\alpha \\ &= -|\nabla\bar{\nabla}u|^2_g - g^{i\bar{q}}g^{p\bar{j}}u_{p\bar{q}}\alpha_{i\bar{j}} + \Delta u + \Delta\mathrm{tr}_\omega\alpha \\ &\geq -\frac{3}{2}|\nabla\bar{\nabla}u|^2_g + \Delta u + \Delta\mathrm{tr}_\omega\alpha - C, \end{aligned}$$

where we used that $R_{i\bar{j}} = -u_{i\bar{j}} - \alpha_{i\bar{j}}$ and $|\alpha|_g \leq C$.

From the second inequality in Lemma 5.2.5, we have

$$\begin{aligned} -\Delta\mathrm{tr}_\omega\alpha &\leq C + |\tilde{\nabla}g|^2_g - \frac{1}{C}|\nabla\mathrm{tr}_\omega\alpha|^2_g + h^{i\bar{j}}\frac{\partial}{\partial t}g_{i\bar{j}} \\ &\leq C + |\tilde{\nabla}g|^2_g - \frac{1}{C}|\nabla\mathrm{tr}_\omega\alpha|^2_g + h^{i\bar{j}}(u_{i\bar{j}} + \alpha_{i\bar{j}}) \\ &\leq C + |\tilde{\nabla}g|^2_g - \frac{1}{C}|\nabla\mathrm{tr}_\omega\alpha|^2_g + \frac{1}{2}|\nabla\bar{\nabla}u|^2_g. \end{aligned}$$

ByCombining these inequalties, we have

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)(-\Delta u) \leq 2|\nabla\bar{\nabla}u|^2_g - \Delta u + C + |\tilde{\nabla}g|^2_g - \frac{1}{C}|\nabla\mathrm{tr}_\omega\alpha|^2_g.$$

We compute

$$
\begin{aligned}
&\Delta|\nabla u|_g^2 \\
=\ & g^{i\bar{j}}g^{k\bar{l}}\Big(\nabla_i\nabla_{\bar{j}}\nabla_k u\nabla_{\bar{l}}u + \nabla_k u\nabla_i\nabla_{\bar{j}}\nabla_{\bar{l}}u + \nabla_i\nabla_k u\nabla_{\bar{j}}\nabla_{\bar{l}}u + \nabla_i\nabla_{\bar{l}}u\nabla_{\bar{j}}\nabla_k u\Big) \\
=\ & |\nabla\bar{\nabla}u|_g^2 + |\nabla\nabla u|_g^2 + g^{i\bar{j}}g^{k\bar{l}}\Big(\nabla_i\nabla_k\nabla_{\bar{l}}u\nabla_{\bar{l}}u + \nabla_k u\nabla_i\nabla_{\bar{l}}\nabla_{\bar{j}}u - \nabla_k u\nabla_i(\overline{T_{jl}^p}\nabla_{\bar{p}}u)\Big) \\
=\ & |\nabla\bar{\nabla}u|_g^2 + |\nabla\nabla u|_g^2 + 2\mathrm{Re}\langle\nabla\Delta u, \nabla u\rangle_g - g^{i\bar{j}}g^{k\bar{l}}T_{ik}^p\nabla_p\nabla_{\bar{j}}u\nabla_{\bar{l}}u \\
& + g^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}R_{i\bar{l}p\bar{j}}\nabla_{\bar{q}}u\nabla_k u - g^{i\bar{j}}g^{k\bar{l}}\overline{T_{jl}^p}\nabla_k u\nabla_i\nabla_{\bar{p}}u - g^{i\bar{j}}g^{k\bar{l}}\partial_i\overline{T_{jl}^p}\nabla_k u\nabla_{\bar{p}}u \\
=\ & |\nabla\bar{\nabla}u|_g^2 + |\nabla\nabla u|_g^2 + 2\mathrm{Re}\langle\nabla\Delta u, \nabla u\rangle_g + g^{k\bar{l}}g^{p\bar{q}}R_{p\bar{l}}\nabla_{\bar{q}}u\nabla_k u \\
& - 2\mathrm{Re}\Big(g^{i\bar{j}}g^{k\bar{l}}T_{ik}^p\nabla_p\nabla_{\bar{j}}u\nabla_{\bar{l}}u\Big) + g^{k\bar{l}}g^{p\bar{q}}\partial_{\bar{l}}T_{pi}^i\nabla_{\bar{q}}u\nabla_k u - g^{i\bar{j}}g^{k\bar{l}}\partial_i\overline{T_{jl}^p}\nabla_k u\nabla_{\bar{p}}u \\
\geq\ & \frac{1}{2}|\nabla\bar{\nabla}u|_g^2 + |\nabla\nabla u|_g^2 + 2\mathrm{Re}\langle\nabla\Delta u, \nabla u\rangle_g + g^{k\bar{l}}g^{p\bar{q}}R_{p\bar{l}}\nabla_{\bar{q}}u\nabla_k u \\
& - C|\nabla u|_g^2 - C|\tilde{\nabla}g|_g|\nabla u|_g^2
\end{aligned}
$$

and

$$
\frac{\partial}{\partial t}|\nabla u|_g^2 = g^{k\bar{l}}g^{p\bar{q}}R_{p\bar{l}}\nabla_k u\nabla_{\bar{q}}u + |\nabla u|_g^2 + 2\mathrm{Re}\langle\nabla\Delta u, \nabla u\rangle_g + 2\mathrm{Re}\langle\nabla\mathrm{tr}_\omega\alpha, \nabla u\rangle_g.
$$

Therefore we obtain

$$
\Big(\frac{\partial}{\partial t} - \Delta\Big)|\nabla u|_g^2 \leq -\frac{1}{2}|\nabla\bar{\nabla}u|_g^2 - |\nabla\nabla u|_g^2 + 2\mathrm{Re}\langle\nabla\mathrm{tr}_\omega\alpha, \nabla u\rangle_g + C|\nabla u|_g^2 + C|\tilde{\nabla}g|_g|\nabla u|_g^2.
$$

Now we fix a sufficiently large constant $A$ so that $|u|+1 \leq A$ and compute the following evolution inequality:

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta\Big)\Big(\frac{|\nabla u|_g^2}{A-u}\Big) =\ & \frac{1}{A-u}\Big(\frac{\partial}{\partial t} - \Delta\Big)|\nabla u|_g^2 + \frac{|\nabla u|_g^2}{(A-u)^2}\Big(\frac{\partial}{\partial t} - \Delta\Big)u \\
& - \frac{2}{(A-u)^2}\mathrm{Re}\langle\nabla|\nabla u|_g^2, \nabla u\rangle_g - \frac{2|\nabla u|_g^4}{(A-u)^3} \\
\leq\ & \frac{1}{A-u}\Big(-\frac{1}{2}|\nabla\bar{\nabla}u|_g^2 - |\nabla\nabla u|_g^2 + 2\mathrm{Re}\langle\nabla\mathrm{tr}_\omega\alpha, \nabla u\rangle_g \\
& + C|\nabla u|_g^2 + C|\tilde{\nabla}g|_g|\nabla u|_g^2\Big) \\
& + \frac{|\nabla u|_g^2}{(A-u)^2}(\mathrm{tr}_\omega\alpha - 1) - \frac{2}{(A-u)^2}\mathrm{Re}\langle\nabla|\nabla u|_g^2, \nabla u\rangle_g - \frac{2|\nabla u|_g^4}{(A-u)^3}
\end{aligned}
$$

For $\varepsilon > 0$ small, we rewite a term in the evolution inequality above.

$$
\begin{aligned}
\frac{2}{(A-u)^2}\mathrm{Re}\langle\nabla|\nabla u|_g^2, \nabla u\rangle_g =\ & \varepsilon\frac{2}{(A-u)^2}\mathrm{Re}\langle\nabla|\nabla u|_g^2, \nabla u\rangle_g \\
& + \frac{2(1-\varepsilon)}{A-u}\mathrm{Re}\Big\langle\nabla\Big(\frac{|\nabla u|_g^2}{A-u}\Big), \nabla u\Big\rangle_g - \frac{2(1-\varepsilon)|\nabla u|_g^4}{(A-u)^3},
\end{aligned}
$$

Then the first term on the right hand side can be estimated in the following way.

$$
\begin{aligned}
-\varepsilon\frac{2}{(A-u)^2}\mathrm{Re}\langle\nabla|\nabla u|_g^2,\nabla u\rangle_g &\le \varepsilon\frac{2\sqrt{2}}{(A-u)^2}|\nabla u|_g^2(|\nabla\bar{\nabla}u|_g^2+|\nabla\nabla u|_g^2)^{\frac{1}{2}}\\
&\le \frac{\varepsilon}{2}\frac{|\nabla u|_g^4}{(A-u)^2}+4\varepsilon\frac{|\nabla\bar{\nabla}u|_g^2+|\nabla\nabla u|_g^2}{(A-u)^2}\\
&\le \frac{\varepsilon}{2}\frac{|\nabla u|_g^4}{(A-u)^2}+\frac{1}{2}\frac{|\nabla\bar{\nabla}u|_g^2+|\nabla\nabla u|_g^2}{(A-u)^2},
\end{aligned}
$$

where provided that $\varepsilon\le\frac{1}{8}$ and we now fix such $\varepsilon$. By estimating

$$
\frac{C|\tilde{\nabla}g|_g|\nabla u|_g^2}{A-u}\le C|\tilde{\nabla}g|_g^2+\frac{\varepsilon}{2}\frac{|\nabla u|_g^4}{(A-u)^2},
$$

and absorbing the term $\frac{|\nabla u|_g^2}{(A-u)^2}(\mathrm{tr}_\omega\alpha-1)$ in the term $\frac{C|\nabla u|_g^2}{A-u}$, we obtain

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)\Big(\frac{|\nabla u|_g^2}{A-u}\Big) &\le \frac{1}{A-u}\Big(2\mathrm{Re}\langle\nabla\mathrm{tr}_\omega\alpha,\nabla u\rangle_g+C|\nabla u|_g^2\Big)+C|\tilde{\nabla}g|_g^2\\
&\quad-\varepsilon\frac{|\nabla u|_g^4}{(A-u)^3}-\frac{2(1-\varepsilon)}{A-u}\mathrm{Re}\Big\langle\nabla\Big(\frac{|\nabla u|_g^2}{A-u}\Big),\nabla u\Big\rangle_g.
\end{aligned}
$$

We define

$$
H:=\frac{|\nabla u|_g^2}{A-u}+C_2(\mathrm{tr}_\omega\alpha+C_0\mathrm{tr}_{\tilde{\omega}}\omega)
$$

for a sufficiently large uniform positive constant $C_2$ to be fixed later. Then we have

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)H &\le \frac{1}{A-u}\Big(2\mathrm{Re}\langle\nabla\mathrm{tr}_\omega\alpha,\nabla u\rangle_g+C|\nabla u|_g^2\Big)+C|\tilde{\nabla}g|_g^2\\
&\quad-\varepsilon\frac{|\nabla u|_g^4}{(A-u)^3}-\frac{2(1-\varepsilon)}{A-u}\mathrm{Re}\Big\langle\nabla\Big(\frac{|\nabla u|_g^2}{A-u}\Big),\nabla u\Big\rangle_g\\
&\quad-\frac{C_2}{2}|\tilde{\nabla}g|_g^2-2|\nabla\mathrm{tr}_\omega\alpha|_g^2+C.
\end{aligned}
$$

Note that we can show that $|\nabla\mathrm{tr}_{\tilde{\omega}}\omega|_{\tilde{g}}^2\le 2|\tilde{\nabla}g|_{\tilde{g}}^2$ by computing with local coordinates around a point such that $\tilde{g}$ is identity at the point, and since $\tilde{g}$ and $g$ are uniformly equivalent, we obtain

$$
|\nabla\mathrm{tr}_{\tilde{\omega}}\omega|_g^2\le C|\tilde{\nabla}g|_g^2.
$$

If needed, we choose much larger $C_2$ so that $-\frac{C_2}{2}|\tilde{\nabla}g|_g^2\le-|\nabla\mathrm{tr}_{\tilde{\omega}}\omega|_g^2$ and we fix such constant $C_2$. By estimating

$$
\frac{C|\nabla u|_g^2}{A-u}\le\frac{\varepsilon}{4}\frac{|\nabla u|_g^4}{(A-u)^3}+C,
$$

$$
\frac{2}{A-u}\mathrm{Re}\langle\nabla\mathrm{tr}_\omega\alpha,\nabla u\rangle_g\le|\nabla\mathrm{tr}_\omega\alpha|_g^2+\frac{\varepsilon}{4}\frac{|\nabla u|_g^4}{(A-u)^3}+C,
$$

and combining these, we have

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)H &\le -\frac{\varepsilon}{2}\frac{|\nabla u|_g^4}{(A-u)^3}-\frac{2(1-\varepsilon)}{A-u}\mathrm{Re}\Big\langle\nabla\Big(\frac{|\nabla u|_g^2}{A-u}\Big),\nabla u\Big\rangle_g\\
&\quad -|\nabla \mathrm{tr}_{\tilde\omega}\omega|_g^2-|\nabla \mathrm{tr}_{\omega}\alpha|_g^2+C\\
&= -\frac{\varepsilon}{2}\frac{|\nabla u|_g^4}{(A-u)^3}-\frac{2(1-\varepsilon)}{A-u}\mathrm{Re}\langle\nabla H,\nabla u\rangle_g\\
&\quad +\frac{2(1-\varepsilon)C_2}{A-u}\mathrm{Re}\langle\nabla \mathrm{tr}_\omega\alpha,\nabla u\rangle_g+\frac{2(1-\varepsilon)C_0C_2}{A-u}\mathrm{Re}\langle\nabla \mathrm{tr}_{\tilde\omega}\omega,\nabla u\rangle_g\\
&\quad -|\nabla \mathrm{tr}_{\tilde\omega}\omega|_g^2-|\nabla \mathrm{tr}_{\omega}\alpha|_g^2+C\\
&\le -\frac{\varepsilon}{4}\frac{|\nabla u|_g^4}{(A-u)^3}-\frac{2(1-\varepsilon)}{A-u}\mathrm{Re}\langle\nabla H,\nabla u\rangle_g+C,
\end{aligned}
$$

where we used the following bounds at the last line:

$$
\frac{2(1-\varepsilon)C_2}{A-u}\mathrm{Re}\langle\nabla \mathrm{tr}_\omega\alpha,\nabla u\rangle_g\le|\nabla \mathrm{tr}_\omega\alpha|_g^2+\frac{\varepsilon}{8}\frac{|\nabla u|_g^4}{(A-u)^3}+C,
$$

$$
\frac{2(1-\varepsilon)C_0C_2}{A-u}\mathrm{Re}\langle\nabla \mathrm{tr}_{\tilde\omega}\omega,\nabla u\rangle_g\le|\nabla \mathrm{tr}_{\tilde\omega}\omega|_g^2+\frac{\varepsilon}{8}\frac{|\nabla u|_g^4}{(A-u)^3}+C.
$$

We may assume that $H$ achieves its maximum at $x_0\in M$, $t_0>0$, and then at the point we have

$$
|\nabla u|_g^4(x_0,t_0)\le C
$$

for some uniform constant $C>0$. Therefore, we conclude that $H\le C$ uniformly bounded and that $|\nabla u|_g^2\le C$ everywhere for some uniform positive constant $C$. It follows that we have

$$
\Big(\frac{\partial}{\partial t}-\Delta\Big)|\nabla u|_g^2\le-\frac{1}{2}|\nabla\bar\nabla u|_g^2-|\nabla\nabla u|_g^2+|\nabla \mathrm{tr}_\omega\alpha|_g^2+|\tilde\nabla g|_g^2+C,
$$

and then for sufficiently large constant $C_3>0$, we obtain

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t}-\Delta\Big)(-\Delta u+6|\nabla u|_g^2+C_3(\mathrm{tr}_\omega\alpha+C_0\mathrm{tr}_{\tilde\omega}\omega)) &\le -|\nabla\bar\nabla u|_g^2-\Delta u+C\\
&\le -\frac{1}{2}(-\Delta u)^2+(-\Delta u)+C,
\end{aligned}
$$

where we used the Cauchy-Schwarz inequalty at the last line. We may assume that $-\Delta u+6|\nabla u|_g^2+C_3(\mathrm{tr}_\omega\alpha+C_0\mathrm{tr}_{\tilde\omega}\omega)$ achieves its maximum at $x_0\in M$, $t_0>0$ and then we have $\frac{1}{2}(-\Delta u)^2(x_0,t_0)\le-\Delta u(x_0,t_0)+C$. It follows that we have that $-\Delta u(x_0,t_0)\le C$ and

$$
-\Delta u+6|\nabla u|_g^2+C_3(\mathrm{tr}_\omega\alpha+C_0\mathrm{tr}_{\tilde\omega}\omega)\le C
$$

for some uniform constant $C>0$ everywhere. Therefore, we conclude that we have the uniform upper bound $-\Delta u$, which implies that we obtain the uniform upper bound also for $S_t$.

Third, we observe that $\dot{\varphi}$ decays exponentially fast as $t \to \infty$. Since we have

$$\frac{\partial}{\partial t}\dot{\varphi} = -S_t - 1 - \dot{\varphi},$$

and $|\dot{\varphi}|$, $|S_t|$ are uniformly bounded, we obtain

$$\Big|\frac{\partial}{\partial t}\dot{\varphi}\Big| \leq C_0$$

for some uniform constant $C_0 > 0$. Suppose that we do not have the bound $\dot{\varphi} \leq Ce^{-\sigma t}$ for any constant $C > 0$. Then there exists a sequence $(x_k, t_k) \in M \times [0, \infty)$ with $t_k \to \infty$ as $k \to \infty$ such that

$$\dot{\varphi}(x_k, t_k) \geq ke^{-\sigma t_k}.$$

Define

$$\gamma_k := \frac{k}{2C_0}e^{-\sigma t_k}.$$

We work at the point $x_k$. Then by $\frac{\partial}{\partial t}\dot{\varphi} \geq -C_0$, we obtain for any $a \in [0, \gamma_k]$,

$$\dot{\varphi}(t_k + a) - \dot{\varphi}(t_k) = \int_{t_k}^{t_k+a} \frac{\partial}{\partial t}\dot{\varphi}dt \geq -C_0\gamma_k,$$

which implies that we have for any $t \in [t_k, t_k + \gamma_k]$,

$$\dot{\varphi}(t) \geq \frac{k}{2}e^{-\sigma t_k}.$$

Thus, we have

$$\frac{k^2}{4C_0}e^{-2\sigma t_k} = \gamma_k \frac{k}{2}e^{-\sigma t_k} \leq \int_{t_k}^{t_k+\gamma_k} \dot{\varphi}dt = \varphi(t_k + \gamma_k) - \varphi(t_k) \leq C(1 + t_k)e^{-t_k},$$

which leads a contradiction for $\sigma < \frac{1}{2}$ when $k \to \infty$. The lower bound $\dot{\varphi} \geq -Ce^{-\eta t}$ for any $0 < \eta < \frac{1}{2}$ and some uniform constant $C > 0$ can be obtaind similarly.

Forth, we show that $\omega(t)$ and $\tilde{\omega}$ approach each other exponentially fast as $t \to \infty$. We start with the evolution of $\mathrm{tr}_\omega\tilde{\omega}$. Firstly, we compute

$$\frac{\partial}{\partial t}\mathrm{tr}_\omega\tilde{\omega} = \mathrm{tr}_\omega\tilde{\omega} + \mathrm{tr}_\omega(\alpha - \tilde{\omega}) + g^{i\bar{j}}h^{p\bar{q}}R_{p\bar{q}i\bar{j}}.$$

And we have, since we can compute

$$\begin{aligned}
\tilde{\nabla}_i\tilde{\nabla}_{\bar{j}}g_{p\bar{q}} &= \tilde{\nabla}_i\Big(\partial_{\bar{j}}g_{p\bar{q}} - \overline{\tilde{\Gamma}^s_{jq}}g_{p\bar{s}}\Big) \\
&= \partial_i\partial_{\bar{j}}g_{p\bar{q}} - \tilde{\Gamma}^r_{ip}\partial_{\bar{j}}g_{r\bar{q}} - g_{p\bar{s}}\partial_i\overline{\tilde{\Gamma}^s_{jq}} - \overline{\tilde{\Gamma}^s_{jq}}\partial_i g_{p\bar{s}} + \tilde{\Gamma}^r_{ip}\overline{\tilde{\Gamma}^s_{jq}}g_{r\bar{s}} \\
&= \tilde{R}_{i\bar{j}r\bar{q}}\tilde{g}^{r\bar{s}}g_{p\bar{s}} - R_{i\bar{j}p\bar{q}} + g^{r\bar{s}}\tilde{\nabla}_i g_{p\bar{s}}\tilde{\nabla}_{\bar{j}}g_{r\bar{q}},
\end{aligned}$$

$$
\begin{aligned}
\Delta \mathrm{tr}_\omega \tilde{\omega} &= g^{i\bar{j}} \tilde{\nabla}_i \tilde{\nabla}_{\bar{j}} (g^{k\bar{l}} \tilde{g}_{k\bar{l}}) \\
&= -g^{i\bar{j}} \tilde{\nabla}_i (g^{k\bar{q}} g^{p\bar{l}} (\tilde{\nabla}_{\bar{j}} g_{p\bar{q}}) \tilde{g}_{k\bar{l}}) \\
&= g^{i\bar{j}} g^{k\bar{s}} g^{r\bar{q}} g^{p\bar{l}} \tilde{g}_{k\bar{l}} \tilde{\nabla}_i g_{r\bar{s}} \tilde{\nabla}_{\bar{j}} g_{p\bar{q}} + g^{i\bar{j}} g^{p\bar{s}} g^{r\bar{l}} g^{k\bar{q}} \tilde{g}_{k\bar{l}} \tilde{\nabla}_i g_{r\bar{s}} \tilde{\nabla}_{\bar{j}} g_{p\bar{q}} - g^{i\bar{j}} g^{k\bar{q}} g^{p\bar{l}} \tilde{g}_{k\bar{l}} \tilde{\nabla}_i \tilde{\nabla}_{\bar{j}} g_{p\bar{q}} \\
&= g^{i\bar{j}} g^{k\bar{s}} g^{r\bar{q}} g^{p\bar{l}} \tilde{g}_{k\bar{l}} \tilde{\nabla}_i g_{r\bar{s}} \tilde{\nabla}_{\bar{j}} g_{p\bar{q}} + g^{i\bar{j}} g^{k\bar{q}} g^{p\bar{l}} \tilde{g}_{k\bar{l}} R_{i\bar{j}p\bar{q}} - g^{i\bar{j}} g^{p\bar{q}} \tilde{R}_{i\bar{j}p\bar{q}}.
\end{aligned}
$$

By putting together, we obtain

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta\Big) \mathrm{tr}_\omega \tilde{\omega} &\leq C + g^{i\bar{j}} h^{p\bar{q}} (R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}}) - g^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} \tilde{\nabla}_i g_{k\bar{q}} \tilde{\nabla}_{\bar{j}} g_{p\bar{l}} \\
&\leq C - \frac{1}{2C_0} |\tilde{\nabla} g|_g^2,
\end{aligned}
$$

where we used that $g$ and $h$ are uniformly equivalent and the bounds

$$
-g^{i\bar{j}} g^{k\bar{l}} h^{p\bar{q}} \tilde{\nabla}_i g_{k\bar{q}} \tilde{\nabla}_{\bar{j}} g_{p\bar{l}} \leq -\frac{1}{C_0} |\tilde{\nabla} g|_g^2,
$$

$$
|g^{i\bar{j}} h^{p\bar{q}} (R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}})| \leq C + \frac{1}{2C_0} |\tilde{\nabla} g|_g^2.
$$

For arbitrary given $0 < \varepsilon < \frac{1}{2}$, we choose $\frac{1}{2} > \eta > \varepsilon$ such that $\varepsilon + \eta < 1$ and $2\varepsilon < \delta < \varepsilon + \eta$. Define

$$
J_1 := e^{\varepsilon t} (\mathrm{tr}_\omega \tilde{\omega} - 2) - e^{\delta t} \varphi
$$

and compute the evolution of $J_1$,

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta\Big) J_1 &\leq C e^{\varepsilon t} + \varepsilon e^{\varepsilon t} (\mathrm{tr}_\omega \tilde{\omega} - 2) - \delta e^{\delta t} \varphi - e^{\delta t} \dot{\varphi} - e^{\delta t} (\mathrm{tr}_\omega \tilde{\omega} - 2) \\
&\leq C e^{\varepsilon t} + C e^{(\delta - \eta) t} - e^{\delta t} (\mathrm{tr}_\omega \tilde{\omega} - 2),
\end{aligned}
$$

where we used that $\dot{\varphi} \geq -C e^{-\eta t}$ and $\mathrm{tr}_\omega \tilde{\omega} \leq C$. Since $\delta - \eta < \varepsilon$, at a maximum point of $J_1$,

$$
e^{\delta t} (\mathrm{tr}_\omega \tilde{\omega} - 2) \leq C e^{\varepsilon t}
$$

and hence

$$
e^{\varepsilon t} (\mathrm{tr}_\omega \tilde{\omega} - 2) \leq C e^{(2\varepsilon - \delta) t} \leq C
$$

since $2\varepsilon < \delta$. Thus $J_1$ has the uniform upper bound everywhere. It follows that for any $0 < \varepsilon < \frac{1}{2}$, there exists a uniform positive constant $C$ such that for $t \geq 0$,

$$
\mathrm{tr}_\omega \tilde{\omega} - 2 \leq C e^{-\varepsilon t}.
$$

Recall that for $t \geq 0$,

$$
\Big(\frac{\partial}{\partial t} - \Delta\Big) \mathrm{tr}_{\tilde{\omega}} \omega \leq C.
$$

Define

$$
J_2 := e^{\varepsilon t} (\mathrm{tr}_{\tilde{\omega}} \omega - 2) - e^{\delta t} \varphi
$$

and compute the evolution of $J_2$,

$$
\Big(\frac{\partial}{\partial t} - \Delta\Big) J_2 \leq C e^{\varepsilon t} - e^{\delta t} (\mathrm{tr}_{\tilde{\omega}} \omega - 2),
$$

where we used $|\dot\varphi| \le Ce^{-\eta t}$ for $0 < \varepsilon < \eta < \frac{1}{2}$, and $\delta - \eta < \varepsilon < \frac{1}{2}$. We have

$$\mathrm{tr}_\omega \tilde\omega = \frac{\tilde\omega^2}{\omega^2}\mathrm{tr}_{\tilde\omega}\omega = \mathrm{tr}_{\tilde\omega}\omega + \Big(\frac{\tilde\omega^2}{\omega^2} - 1\Big)\mathrm{tr}_{\tilde\omega}\omega.$$

Since we have $\dot\varphi = \log \frac{\omega^2}{\tilde\omega^2} + O(e^{-\eta t})$, we obtain

$$\Big|\frac{\tilde\omega^2}{\omega^2} - 1\Big| = |e^{O(e^{-\eta t})} - 1| \le Ce^{-\eta t}.$$

Since $\mathrm{tr}_{\tilde\omega}\omega$ is uniformly bounded, for $t \ge 0$, we have

$$\Big(\frac{\partial}{\partial t} - \Delta\Big) J_2 \le Ce^{\varepsilon t} - e^{\delta t}(\mathrm{tr}_{\tilde\omega}\omega - 2).$$

Then at a maximum point of $J_2$, we have, since $\delta > 2\varepsilon$,

$$e^{\varepsilon t}(\mathrm{tr}_{\tilde\omega}\omega - 2) \le Ce^{(2\varepsilon - \delta)t} \le C.$$

It follows that for any $0 < \varepsilon < \frac{1}{2}$, there exists a uniform positive constant $C$ such that for $t \ge 0$,

$$\mathrm{tr}_{\tilde\omega}\omega - 2 \le Ce^{-\varepsilon t}.$$

Finally, by applying the following lemma, we conclude the forth inequality in Lemma 5.2.3.

**Lemma 5.2.6.** Let $\varepsilon > 0$ be small. Suppose that $\mathrm{tr}_\omega\tilde\omega - 2 \le \varepsilon$ and $\mathrm{tr}_{\tilde\omega}\omega - 2 \le \varepsilon$. Then we have

$$(1 - 2\sqrt{\varepsilon})\tilde\omega \le \omega \le (1 + 2\sqrt{\varepsilon})\tilde\omega.$$

Proof. Choose local coordinates around a point at which $\tilde g$ is the identity and $g$ is diagonal with eigenvalues $\lambda_1, \lambda_2 > 0$. From our assumption, we have

$$\lambda_1 \le 2 + \varepsilon - \lambda_2, \quad \frac{1}{\lambda_2} \le \frac{(2+\varepsilon)\lambda_1 - 1}{\lambda_1},$$

which imply $(2+\varepsilon)\lambda_1 - 1 > 0$,

$$-\lambda_2 \le -\frac{\lambda_1}{(2+\varepsilon)\lambda_1 - 1}$$

and

$$\lambda_1 \le 2 + \varepsilon - \frac{\lambda_1}{(2+\varepsilon)\lambda_1 - 1}.$$

Then, we obtain

$$\lambda_1^2 - (2+\varepsilon)\lambda_1 + 1 \le 0$$

and by completing the square,

$$(\lambda_1 - (1 + \frac{1}{2}))^2 \le \varepsilon + \frac{\varepsilon^2}{4}.$$

Assuming that $\varepsilon > 0$ is smaller that some universal constant, by symmetry, we obtain for $i = 1, 2$,

$$1 - 2\sqrt{\varepsilon} \le \lambda_i \le 1 + 2\sqrt{\varepsilon}.$$

□

### 5.2.2 A third order estimate

Denote by $\Psi_{ij}^k := \Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k$, the difference of the Christoffel symbols of $g$ and $\tilde{g}$, which satisfies $\mathcal{S} := |\tilde{\nabla} g|_g^2 = |\Psi|_g^2$. The quantity $|\tilde{\nabla} g|_g^2$ is equivalent to $|\tilde{\nabla} g|_{\tilde{g}}^2$ from the result (1) in Lemma 5.2.3. Note that we will write locally $\alpha = \sqrt{-1}\alpha_{i\bar{j}} dz^i \wedge d\bar{z}^j$.

Then we compute the evolution of $\mathcal{S}$ (cf. [52], [54]):

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta\Big)\mathcal{S} \;\; = \;\; & \mathcal{S} - |\overline{\nabla}\Psi|_g^2 - |\nabla\Psi|_g^2 \\
& + g^{i\bar{j}} g^{r\bar{s}} g^{a\bar{b}} \Big(\nabla_r \overline{T_{bj\bar{a}}} + \nabla_{\bar{b}} T_{ar\bar{j}}\Big) \Psi_{ip}^k \overline{\Psi_{sq}^l} g^{p\bar{q}} g_{k\bar{l}} \\
& + g^{i\bar{j}} g^{r\bar{s}} g^{a\bar{b}} \Big(\nabla_r \overline{T_{bj\bar{a}}} + \nabla_{\bar{b}} T_{ar\bar{j}}\Big) \Psi_{pi}^k \overline{\Psi_{qs}^l} g^{p\bar{q}} g_{k\bar{l}} \\
& - g^{i\bar{j}} g^{r\bar{s}} g^{a\bar{b}} \Big(\nabla_k \overline{T_{bs\bar{a}}} + \nabla_{\bar{b}} T_{ak\bar{s}}\Big) \Psi_{ip}^k \overline{\Psi_{jq}^m} g^{p\bar{q}} g_{r\bar{m}} \\
& - 2\mathrm{Re}\Big[[g^{r\bar{s}}(\nabla_i \nabla_p \overline{T_{sl\bar{r}}} + \nabla_i \nabla_{\bar{s}} T_{rp\bar{l}} \\
& - T_{ir}^a R_{a\bar{s}p\bar{l}} + g_{k\bar{l}} \nabla_r \tilde{R}_{i\bar{s}p}{}^{k}) + g_{k\bar{l}} \tilde{g}^{k\bar{s}} \tilde{\nabla}_i \alpha_{p\bar{s}}] \overline{\Psi_{jq}^l} g^{i\bar{j}} g^{p\bar{q}}\Big],
\end{aligned}$$

where $\nabla$, $\Delta$ are the Chern connection and the Laplacian with respect to $g$, and in this computation, we used especially that $\frac{\partial}{\partial t}\tilde{g}_{k\bar{l}} = -\tilde{g}_{k\bar{l}} + \alpha_{k\bar{l}}$ and $\frac{\partial}{\partial t}\tilde{\Gamma}_{ip}^k = \tilde{g}^{k\bar{\delta}} \tilde{\nabla}_i \alpha_{p\bar{\delta}}$.

With using $\tilde{T}_{ij\bar{k}} = \tilde{T}_{ij}^k \tilde{g}_{k\bar{l}} = T_{ij}^k g_{k\bar{l}} = T_{ij\bar{k}}$, we can compute as follows:

$$\nabla_{\bar{b}} T_{ar\bar{j}} = \tilde{\nabla}_{\bar{b}} \tilde{T}_{ar\bar{j}} - \overline{\Psi_{bj}^s} \tilde{T}_{ar\bar{s}},$$

$$\begin{aligned}
\nabla_i \nabla_{\bar{s}} T_{rp}^k \;\; = \;\; & g^{\bar{l}k}\Big(\tilde{\nabla}_i \tilde{\nabla}_{\bar{s}} \tilde{T}_{rp\bar{l}} - \Psi_{ir}^a \tilde{\nabla}_{\bar{s}} \tilde{T}_{ap\bar{l}} - \Psi_{ip}^a \tilde{\nabla}_{\bar{s}} \tilde{T}_{ra\bar{l}} \\
& - (\tilde{\nabla}_i \overline{\Psi_{sl}^q}) \tilde{T}_{rp\bar{q}} - \overline{\Psi_{sl}^q} (\tilde{\nabla}_i \tilde{T}_{rp\bar{q}} - \Psi_{ir}^a \tilde{T}_{ap\bar{q}} - \Psi_{ip}^a \tilde{T}_{ra\bar{q}})\Big)
\end{aligned}$$

$$g^{r\bar{s}} T_{ir}^a R_{a\bar{s}p\bar{l}} = g^{r\bar{s}} g^{a\bar{b}} \tilde{T}_{ir\bar{b}} \Big(\tilde{R}_{a\bar{s}p}{}^{\delta} g_{\delta\bar{l}} - \nabla_{\bar{s}} \Psi_{ap}^{\delta} g_{\delta\bar{l}}\Big).$$

And we also can easily compute

$$\nabla_r \tilde{R}_{i\bar{s}p}{}^{k} = \tilde{\nabla}_r \tilde{R}_{i\bar{s}p}{}^{k} - \Psi_{ri}^a \tilde{R}_{a\bar{s}p}{}^{k} - \Psi_{rp}^a \tilde{R}_{i\bar{s}a}{}^{k} + \Psi_{ra}^k \tilde{R}_{i\bar{s}p}{}^{a},$$

and

$$|\tilde{\nabla}\alpha|_{\tilde{g}} \leq C$$

since the only nonzero component of $\alpha$ is $\alpha_{2\bar{2}} = \frac{1}{2y_2^2}$ and we can compute in the following:

$$\tilde{\nabla}_1 \alpha_{1\bar{2}} = -\tilde{\Gamma}_{11}^2 \alpha_{2\bar{2}} = 0, \quad \tilde{\nabla}_1 \alpha_{2\bar{2}} = -\tilde{\Gamma}_{12}^2 \alpha_{2\bar{2}} = 0,$$

$$\tilde{\nabla}_2 \alpha_{1\bar{2}} = -\tilde{\Gamma}_{21}^2 \alpha_{2\bar{2}} = \frac{1}{z_1 y_2^2 (1 + e^t)} = O(e^{-t}), \quad \tilde{\nabla}_2 \alpha_{2\bar{2}} = \partial_2 \alpha_{2\bar{2}} - \tilde{\Gamma}_{22}^2 \alpha_{2\bar{2}} = O(1).$$

Therefore, with using the estimate in Lemma 5.2.2 and (1) in Lemma 5.2.3, we obtain

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\mathcal{S} \leq C(\mathcal{S}^{\frac{3}{2}} + 1) - \frac{1}{2}(|\overline{\nabla}\Psi|_g^2 + |\nabla\Psi|_g^2).$$

We also have the evolution of $\mathrm{tr}_{\tilde{g}} g$ (cf. [54]):

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)\mathrm{tr}_{\tilde{g}} g &= -g^{\bar{j}p}g^{\bar{q}i}\tilde{g}^{\bar{l}k}\tilde{\nabla}_k g_{i\bar{j}}\tilde{\nabla}_{\bar{l}} g_{p\bar{q}} - 2\mathrm{Re}\Big(g^{\bar{j}i}\tilde{g}^{\bar{l}k}\tilde{T}^p_{ki}\tilde{\nabla}_{\bar{l}} g_{p\bar{j}}\Big)\\ &\quad -g^{\bar{j}i}\Big(\tilde{\nabla}_i\overline{\tilde{T}^p_{jp}}+\tilde{g}^{\bar{l}k}\tilde{\nabla}_{\bar{l}}\tilde{T}_{ik\bar{j}}\Big)+g^{\bar{j}i}\tilde{g}^{\bar{l}k}\Big(\tilde{\nabla}_i\overline{\tilde{T}^q_{jl}}-\tilde{R}_{i\bar{l}p}{}^{s}\tilde{g}_{s\bar{j}}\tilde{g}^{\bar{q}p}\Big)g_{k\bar{q}}\\ &\quad +g^{\bar{j}i}\tilde{g}^{\bar{l}k}\tilde{T}^p_{ik}\overline{\tilde{T}^q_{jl}}(\tilde{g}-g)_{p\bar{q}}-\tilde{g}^{k\bar{2}}\tilde{g}^{2\bar{l}}g_{k\bar{l}}\alpha_{2\bar{2}}.\end{aligned}$$

We use the fact that $g$ and $\tilde{g}$ are uniformly equivalent in Lemma 5.2.3 (1). We compute that

$$\tilde{g}^{k\bar{2}}\tilde{g}^{2\bar{l}}g_{k\bar{l}}\alpha_{2\bar{2}} \le C\tilde{g}^{k\bar{2}}\tilde{g}^{2\bar{l}}\tilde{g}_{k\bar{l}}\alpha_{2\bar{2}} = C\tilde{g}^{2\bar{2}}\alpha_{2\bar{2}} = \frac{C}{1+e^{-t}} \le C$$

for some constant $C>0$ independent of $t$, then we again use the result in Lemma 5.2.2 and can obtain

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\mathrm{tr}_{\tilde{g}} g \le -\frac{1}{C_0}\mathcal{S}+C(\mathcal{S}^{\frac{1}{2}}+1),$$

for a uniform constant $C_0, C>0$. Then we apply the way in [54, Section 3] and we have the uniform estimate $\mathcal{S}\le C$: Since our estimates are local, we work in a small open ball $B_r$ of radius $r>0$ centered at the origin in $\mathbb{C}^n$. Choose a smooth cutoff function $\rho$ supported in $B_r$ and which is identically 1 on $\overline{B_{\frac{r}{2}}}$. We may assume that $|\nabla\rho|^2$, $|\Delta\rho|$ are bounded by $\frac{C}{r^2}$. Let $K$ be a large uniform constant, at least sufficiently large so that

$$\frac{K}{2}\le K-\mathrm{tr}_{\hat{g}} g \le K.$$

Let $A$ be another sufficiently large constant to be determined later. Then we define

$$\Phi := \rho^2\frac{\mathcal{S}}{K-\mathrm{tr}_{\hat{g}} g}+A\mathrm{tr}_{\hat{g}} g.$$

Suppose that $f$ achieves its maximum on $\overline{B_r}\times[0,T]$ at a point $(x_0,t_0)$. We assume for the moment that $t_0>0$ and that $x_0$ does not lie on the boundary of $\overline{B_r}$. We may assume without loss of generality that $\mathcal{S}>1$ at $(x_0,t_0)$. At $(x_0,t_0)$, we have

$$0=\bar{\nabla}\Phi = 2\rho\bar{\nabla}\frac{\mathcal{S}}{K-\mathrm{tr}_{\hat{g}} g}+\rho^2\frac{\bar{\nabla}\mathcal{S}}{K-\mathrm{tr}_{\hat{g}} g}+\rho^2\frac{\mathcal{S}\bar{\nabla}\mathrm{tr}_{\hat{g}} g}{(K-\mathrm{tr}_{\hat{g}} g)^2}+A\bar{\nabla}\mathrm{tr}_{\hat{g}} g.$$

Then we have at $(x_0,t_0)$,

$$\begin{aligned}0\le\Big(\frac{\partial}{\partial t}-\Delta\Big)\Phi &= A\Big(\frac{\partial}{\partial t}-\Delta\Big)\mathrm{tr}_{\hat{g}} g+(-\Delta(\rho^2))\frac{\mathcal{S}}{K-\mathrm{tr}_{\hat{g}} g}+\rho^2\frac{\mathcal{S}}{(K-\mathrm{tr}_{\hat{g}} g)^2}\Big(\frac{\partial}{\partial t}-\Delta\Big)\mathrm{tr}_{\hat{g}} g\\ &\quad +\rho^2\frac{1}{K-\mathrm{tr}_{\hat{g}} g}\Big(\frac{\partial}{\partial t}-\Delta\Big)\mathcal{S}-4\mathrm{Re}\Big(\rho\frac{1}{K-\mathrm{tr}_{\hat{g}} g}\nabla\rho\cdot\bar{\nabla}\mathcal{S}\Big)+\frac{2A|\nabla\mathrm{tr}_{\hat{g}} g|^2}{K-\mathrm{tr}_{\hat{g}} g}\\ &\le \Big(-\frac{A}{2C_0}\mathcal{S}+CA\Big)+\frac{C\mathcal{S}}{r^2K}+\Big(-\frac{\rho^2}{2K^2C_0}\mathcal{S}^2+\frac{C\rho^2}{K^2}\mathcal{S}\Big)\\ &\quad +\Big(-\frac{\rho^2}{2K}(|\bar{\nabla}\Psi|^2+|\nabla\Psi|^2)+\frac{\rho^2}{4K^2C_0}\mathcal{S}^2+C\rho^2\mathcal{S}\Big)\\ &\quad +\Big(\frac{\rho^2}{4K}(|\bar{\nabla}\Psi|^2+|\nabla\Psi|^2)+\frac{C}{Kr^2}\mathcal{S}\Big)+\frac{CA}{K}\mathcal{S}\\ &\le -\frac{A}{2C_0}\mathcal{S}+CA+\frac{C'}{r^2}\mathcal{S}+\frac{CA}{K}\mathcal{S}.\end{aligned}$$

Choose $K \geq 4C_0C$ so that at $(x_0, t_0)$,

$$0 \leq -\frac{A}{4C_0}\mathcal{S} + CA + \frac{C'}{r^2}\mathcal{S}.$$

Then choose $A = \frac{8C'C_0}{r^2}$ so that at $(x_0, t_0)$,

$$\frac{C'}{r^2}\mathcal{S} \leq CA.$$

It follows that $\Phi$ is uniformly bounded from above by $\frac{C}{r^2}$. Thus, we conclude that $\mathcal{S}$ on $\overline{B_{\frac{r}{2}}}$ is uniformly bounded from above by $\frac{C}{r^2}$. It remains to deal with the cases when $t_0 = 0$ or $x_0$ lies on the boundary of $\overline{B_r}$. In either case, we obtain

$$\Phi(x_0, t_0) \leq A\mathrm{tr}_{\hat{g}}g(x_0, t_0) \leq \frac{C}{r^2}$$

and the same bound holds.

Note that we write locally $\omega_V = \sqrt{-1}(g_V)_{i\bar{j}}dz^i \wedge d\bar{z}^j$. As we confirmed in Lemma 2.1, since all components of the Christoffel symbols of $\tilde{g}$ are uniformly bounded as $t$ approaches infinity, we have that

$$|\tilde{\Gamma} - \Gamma_V|_{g_V} \leq C$$

for some uniform constant $C > 0$, where $\Gamma_V$ are the Christoffel symbols of $g_V$. Together with the fact that $\tilde{g} \leq Cg_V$ for some uniform constant $C > 0$, we finally obtain

$$|\nabla_V g|_{g_V} \leq |\tilde{\nabla} g|_{g_V} + C \leq C|\tilde{\nabla} g|_{\tilde{g}} + C \leq C$$

for some uniform constant $C > 0$.

Then it only suffices to apply the same way in the proof of [71, Corollary 1.2] and the result holds also on a minimal non-Kähler properly elliptic surface: Considering the general case when $\pi : M \to S$ is not a fiber bundle, it is known that $\pi$ is a quasi-bundle [12, Lemma 1] i.e., $\pi$ has no singular fibers but it might have multiple fibers. Recall that there exists a finite unramified covering $p : M' \to M$ with a covering transformation group $\Gamma(p) := \mathrm{Aut}(p)$, where $\mathrm{Aut}(p)$ is the set of automorphisms of $p$, i.e., any $\tau \in \mathrm{Aut}(p)$ is biholomorphic $\tau : M' \cong M'$, satisfies $p \circ \tau = p$ and is called a covering transformation. Here $M'$ is a minimal properly elliptic surface, $\pi' : M' \to S'$ is an elliptic fiber bundle over a compact Riemann surface $S'$ of genus at least 2 (since $\Gamma(p)$ acts also $S'$, $\pi'$ is $\Gamma(p)$-equivalent) and $M$ is a non-Kähler minimal properly elliptic surface which admits an elliptic fibration $\pi : M \to S$ to a smooth compact curve $S$. The curve $S'$ is a finite cover of $S$ ramified at the images of the multiple fibers of $\pi$ (precisely equal to the image of the quotient map $q : S' \to S$ of the set of finitely many fixed points under the $\Gamma(p)$-action), with quotient $S = S'/\Gamma(p)$, $\pi : M \to S$ is equal to the $\Gamma(p)$-quotient of $\pi' : M' \to S'$ and so that the map $q$ satisfies $q \circ \pi' = \pi \circ p$.

Note that when $\pi : M \to S$ is not a fiber bundle, $\pi$ has no singular fibers, but it might have multiple fibers. Let $D \subset M$ be the set of all multiple fibers of $\pi$, so that $\pi(D)$ consists of finitely many orbifold points, which is precisely equal to the set of branch points, also equal to the image of the map $q$ of fixed points under the $\Gamma(p)$-action on $S'$.

Then from [6, Proposition 2], [46] and [75, Theorem 7.4], we have that $M$ is a quotient of $\mathbb{C}^* \times H$ by a discrete subgroup $\Gamma'$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{C}^*$, which acts by

$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, t \right) \cdot (z_1, z_2) = \left( (cz_2 + d) \cdot z_1 \cdot t, \frac{az_2 + b}{cz_2 + d} \right)$$

for $(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, t) \in \Gamma'$, and the map $\pi : M \to S$ is induced by the projection $\mathbb{C}^* \times H \to H$. The case we were considering in the previous section can be obtained by mapping

$$\mathrm{SL}(2,\mathbb{R}) \times \mathbb{Z} \ni (A, n) \mapsto (A, \lambda^n \chi(A)) \in \mathrm{SL}(2,\mathbb{R}) \times \mathbb{C}^*,$$

where $\lambda \in \mathbb{C}^*$ with $|\lambda| \neq 1$ and $\mathbb{C}^*/\langle\lambda\rangle = E$ and with a character $\chi : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{C}^*$. If we consider the projection $\Gamma''$ of $\Gamma'$ to $\mathrm{SL}(2,\mathbb{R})$, the $\Gamma''$-action on $H$ is generally not free. Note that $\Gamma''$ acts properly discontinuously on $H$. Hence the quotient $S = H/\Gamma''$ is an orbifold, especially it is called a good orbifold (cf. [75, p.139]), i.e., which is a global finite quotient of a manifold.

Since the two forms $\alpha$ and $\gamma$ on $\mathbb{C}^* \times H$ are still invariant under the $\Gamma'$-action, they descend to $M$. We can then define $\omega_0$ as in the case of the fiber bundle. The form $\alpha = \frac{\sqrt{-1}}{2y_2^2} dz_2 \wedge d\bar{z}_2$ on $\mathbb{C}^* \times H$ induces the unique Kähler-Einstein metric $\omega_{S'}$ on $S'$ with $\mathrm{Ric}(\omega_{S'}) = -\omega_{S'}$ and also induces the orbifold Kähler-Einstein metric $\omega_S$ on $S$ with $\mathrm{Ric}(\omega_S) = -\omega_S$ away from finitely many orbifold points and we have $q^*\omega_S = \omega_{S'}$ since $\alpha$ is $\Gamma$-invariant and also $\Gamma''$-invariant. Since we see that $\pi^*\omega_S$ and $\pi'^*\omega_{S'}$ are induced by $\alpha$, we have that $\pi'^*\omega_{S'}$ is a smooth real $(1,1)$-form on $M'$ and that $p^*\pi^*\omega_S = \pi'^*\omega_{S'}$ since $\alpha$ is $\Gamma \times \mathbb{Z}$-invariant and also $\Gamma'$-invariant.

Given any initial metric $\omega_0$ in the $\partial\bar{\partial}$-class of the Vaisman metric on $M$, we denote $\omega'_0 = p^*\omega_0$, which is a $\Gamma(p)$-invariant Gauduchon metric in the $\partial\bar{\partial}$-class of the Vaisman metric on $M'$. Then, let $\omega(t)$, $\omega'(t)$ be solutions of the normalized Chern-Ricci flow on each surface $M'$ and $M$ starting at $\omega_0$, $\omega'_0$ respectively. Note that $p^*\omega(t)$ is equal to $\omega'(t)$, which is also $\Gamma(p)$-invariant, and $\Gamma(p)$ acts by isometries of $p^*\omega(t)$.

For a sufficiently small open set $U \subset M$ so that $p^{-1}(U)$ is a disjoint union of finitly many copies $U_j$ of $U$. Then $p : U_j \to U$ is a biholomorphism for each $j$ and the $\Gamma(p)$-action on $p^{-1}(U)$ permutes the $U_j$'s. Hence for each $j$, the map $p : U_j \to U$ gives an isometry between $(U_j, \omega'(t)|_{U_j})$ and $(U, \omega(t)|_U)$ and also between $(U_j, (\pi'^*\omega_{S'})|_{U_j})$ and $(U, (\pi^*\omega_S)|_U)$ since we have that $U_j \overset{p}{\cong} U$ is biholomorphic, $\omega'(t) = p^*\omega(t)$ and $\pi'^*\omega_{S'} = p^*\pi^*\omega_S$.

We now apply the argument we discussed above to the elliptic bundle $\pi' : M' \to S'$. Since we have

$$||\omega'(t)|_{U_j} - (\pi'^*\omega_{S'})|_{U_j}||_{C^\alpha(U_j, g'_0)} \to 0$$

as $t \to \infty$, it follows that we have, as $t \to \infty$,

$$||\omega(t)|_U - (\pi^*\omega_S)|_U||_{C^\alpha(U, g_0)} \to 0$$

for any $\alpha \in (0,1)$ as $t \to \infty$, where we write locally $\omega'_0 = \sqrt{-1}(g'_0)_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Hence, we conclude that the solution of the normalized Chern-Ricci flow $\omega(t)$ on a non-Kähler minimal properly elliptic surface $M$ starting at a Gauduchon metric $\omega_0$ converges to $\pi^*\omega_S$ in $C^\alpha$-topology for any $\alpha \in (0,1)$ as $t \to \infty$.

# Chapter 6

# Conclusion and research plan

In the setting of Theorem 3.1.1, we removed the condition (†) and completed the argument for proving the correspondence between canonical surgical contraction (Definition 1.2.5) and the blow-down of $(-1)$-curves $E_1, \ldots, E_k$ on $M$ to points $y_1, \ldots, y_k \in N$. Additionally, in order to say that the Chern-Ricci flow performs the canonical surgical contraction in the sense of [70], we would like to prove that $(N, d_T)$ is the metric completion of $(N', d_{g_T})$, where $d_T$ is the distance function in Definition 1.2.6 and $N' = N \backslash \{y_1, \ldots, y_k\}$. We showed that the normalized Chern-Ricci flow on non-Kähler minimal properly elliptic surface converges in $C^\alpha$-topology by choosing the initial metric from the $\partial\bar{\partial}$-class of the Vaisman metric. We would like to obtain much better convergence results on elliptic surfaces, Hopf surfaces and Inoue surfaces. Moreover, we are interested in extending these results to the higher-dimensional manifolds. In fact, there are higher-dimensional analogues of Inoue surfaces, constructed by Oeljeklaus-Toma [51], and it is natural to conjecture that similar behavior occurs. Similarly, there are non-Kähler higher dimensional torus bundles with $c_1 = 0$ but with $c_1^{BC} \neq 0$ over compact Riemann surfaces of genus at least 2 [66], and one would expect that at least some of the results of [71] on elliptic bundles should generalize to these higher-dimensional torus bundles. These researchs are useful to improve the applicability of the Chern-Ricci flow. For instance, surfaces of class $VII_0$ with the second Betti number $b_2 = 0$ are classified completely and these are Hopf surfaces or Inoue surfaces. In the case of $b_2 = 1$, Teleman proved the global spherical conjecture and these surfaces are classified into Kato surfaces. On the other hand, class $VII_0$ surfaces with $b_2 > 1$ are still unclassified. It is known as Kato's conjecture proven by Dloussky, Oeljeklaus and Toma [20] that if surfaces of class $VII_0$ with $b_2 > 0$ have $b_2$-rational curves, then they admit global spherical shells, which implies that they are classified into Kato surfaces as well. We hope that eventually the Chern-Ricci flow will be applied to solving these classification problems of minimal complex surfaces.

# Bibliography


[1] Adem, A., Leida, J., Ruan, Y. *Orbifolds and Stringy Topology*, Cambridge Tracts In Mathematics 171, Cambridge University Press, 2007.

[2] Andrews, B., Hopper, C. *The Ricci Flow in Riemannian Geometry : A Complete Proof of the Differentiable 1/4-Pinching Sphere Theorem*, Lecture Notes in Mathematics 2011, 2011.

[3] Barth, W.P., Hulek, K., Peters, C.A.M., Van de Ven, A. *Compact complex surfaces*, Springer-Verlag, Berlin, 2004.

[4] Bedford, E., Taylor, B.A. *The Dirichlet problem for a complex Monge-Ampère operator*, Invent. math. **37** (1976), 1-44.

[5] Bedford, E., Taylor, B.A. *A new capacity for plurisubharmonic functions*, Acta. Math. **149** (1982), 1-40.

[6] Belgun, F.A. *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000) no. 1, 1-40.

[7] Birkar, C., Cascini, P., Hacon, C. and McKernan, J. *Existence of minimal models for varieties of log general type*, preprint, arXiv:math/0610203.

[8] Blocki, Z. *The complex Monge-Ampère operator in pluripotential theory*, unfinished lecture notes based on graduate course at Jagiellonian University, 1997, last modified: November 2002.

[9] Blocki, Z., Kolodziej, S. *On regularization of plurisubharmonic functions on manifolds*, Proc. Amer. Math. Soc. **135** (2007), no. 7, 2089-2093.

[10] Bogomolov, F. A. *Classification of surfaces of class $VII_0$ with $b_2 = 0$*, Math. USSR Izv. **40** (1976), 255-269 (1977).

[11] Boucksom, S. *On the volume of a line bundle*, Internat. J. Math. **13** (2002), no. 10, 1043-1063.

[12] Brînzănescu, V. *Néron-Severi group for nonalgebraic elliptic surfaces. II. Non-Kählerian case*, Manuscripta Math. **84** (1994), no. 3-4, 415-420.

[13] Buchdahl, N. *On compact Kähler surfaces*, Ann. Inst. Fourier **49**, no 1 (1999), 287-302

[14] Buchdahl, N. *A Nakai-Moishezon criterion for non-Kähler surfaces*, Ann. Inst. Fourier (Grenoble) **50**, no 5, 1533-1538.

[15] Cao, H.-D. *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. **81** (1985), no. 2, 359-372.

[16] Davis, M. *Lectures on orbifolds and reflection groups*, Transformation Groups and Moduli Spaces of Curves (eds, L. Ji, S-T Yau) International Press, 2010, pp. 63–93.

[17] Demailly, J.-P. *Complex analytic and differential geometry*, self published e-book.

[18] Dinew, S. *An inequality for mixed Monge-Ampère measures*, Math. Zeit. **262** (2009), 1-15.

[19] Dinew, S., Kolodziej, S. *Pluripotential estimates on compact Hermitian manifolds*, Advances in Geometric Analysis, 69-86, Advanced Lectures in Math. **21**, International Press, 2012.

[20] Dloussky, G., Oeljeklaus, K. and Toma, M. *Class $VII_0$ surfaces with $b_2$ curves*, Tohoku Math. J. **55** (2003), 283-309.

[21] Essidieux, P., Guedj, V. and Zeriahi, A. *Singular Kähler-Einstein metrics*, J. Amer. Math. Soc. **22** (2009), no. 3, 607-639.

[22] Fang, S., Tosatti, V., Weinkove, B. and Zheng, T. *Inoue surfaces and the Chern-Ricci flow*, preprint, arXiv: 1501.07578v1.

[23] Fong, T.-H. F., Zhang, Z. *The collapsing rate of the Kähler-Ricci flow with regular infinite time singularity*, J. Reine Angew. Math. **2015**(2015), Issue 703, 95-113.

[24] Gauduchon, P. *Le théorème de l'excentricité nulle*, C. R. Acad. Sci. Paris Sér. A-B **285** (1977), no. 5, A387-390.

[25] Gilbarg, D., Trudinger, N. *Elliptic Partial Differential Equations of Second order*, Springer, Paperback edition, 2001, MR 0473443.

[26] Gill, M. *Convergence of the parabolic complex Monge-Ampère equation on compact Hermitian manifolds*, Communications in Analysis and Geometry. **19** (2011), no. 2, 277-304.

[27] Gill, M. *Collapsing of products along the Kähler-Ricci flow*, Trans. Amer. Math. Soc. **19** 366 (2014), 3907-3924.

[28] Griffiths, P., Harris, J. *Principles of Algebraic Geometry*, Pure and Applied Mathematics. Wiley-Interscience, New York, 1978.

[29] Guan, B. Li, Q. *Complex Monge-Ampère equations and totally real submanifolds*, Adv. Math. **225** (2010), 1185-1223.

[30] Guo, B., Song, J., Weinkove, B. *Geometric convergence of the Kähler-Ricci flow on complex surfaces of general type*, preprint, arXiv: 1505.00705v1.

[31] Hamilton, R. *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255-306.

[32] Inoue, M. *On surfaces of class $VII_0$*, Invent. Math. **24** (1974), 269-310.

[33] Inoue, M. *New surfaces with no meromorphic functions*, Proc. Int. Congress of Math. Vancouver (1974).

[34] Kato, M. *Compact complex manifolds containing "global" spherical shells*, Proceeding of the Int. Symp. Alg. Geometry (Kyoto Univ., Kyoto, 1977), 45-84, Kinokuniya Book Store, Tokyo,1978.

[35] Kawamura, M. *Convergence in the Gromov-Hausdorff sense and the Chern-Ricci flow on complex surfaces*, preprint, submitted to Osaka J. Math.

[36] Kawamura, M. *Continuity of the Chern-Ricci flow after the singular time on non-Kähler compact complex surfaces*, preprint.

[37] Kawamura, M. *On the $C^\alpha$-convergence of the solution of the Chern-Ricci flow on elliptic surfaces*, to appear in Tokyo J. Math.

[38] Kodaira, K. *On the structure of compact complex analytic surfaces, II* , Amer. J. Math. **88** (1966), no. 3, 682-721.

[39] Kolodziej, S. *Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge Ampère operator*, Ann. Polon. Math. **65** (1996), 11-21.

[40] Kolodziej, S. *The complex Monge-Ampère equation*, Acta. Math. **180** (1998), 69-117.

[41] Kolodziej, S. *Equicontinuity of families of plurisubharmonic functions with bounds on their Monge-Ampère masses*, Math. Z. **240** (2002), 835-847.

[42] Kolodziej, S. *The complex Monge-Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. J. **52** (2003), no. 3, 667-686.

[43] Kolodziej, S. *Stability of solutions to the Monge-Ampère on compact Kähler manifolds*, Indiana Univ. Math. J. **52** (2003), no. 3, 667-686.

[44] Kolodziej, S. *The complex Monge-Ampère equation and pluripotential theory*, Memoirs Amer. Math. Soc. **178** (2005), pp. 64.

[45] Krylov, N. V. *Lectures on elliptic and parabolic equations in Hölder spaces*, Graduate Studies in Mathematics, 12. American Mathematics society, Providence, RI, 1996.

[46] Maehara, K. *On elliptic surfaces whose first Betti numbers are odd*, in *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)*, 565-574, Kinokuniya Book Store, Tokyo, 1978.

[47] Moichezon, B.G. *On n-dimensional compact varieties with n algebraically independent meromorphic functions, I, II and III*, Izv. Akad. Nauk SSSR Ser. Mat., **30** (1966): 133174 345386 621656 English translation. AMS Translation Ser. 2, 63 51-177.

[48] Nguyen, N.C, Kolodziej, S. *Weak solutions to the complex Monge-Ampère equation on Hermitian manifolds*, preprint, arXiv: 1312.5491v1.

[49] Nguyen, N.C. *The complex Monge-Ampère type equation on compact Hermitian manifolds and applications*, preprint, arXiv: 1501.00891v1.

[50] Nguyen, N.C, Kolodziej, S. *Stability and regularity of solutions of the Monge-Ampère equation on Hermitian manifolds*, preprint, arXiv: 1501.05749v2.

[51] Oeljeklaus, K., Toma, M. *Non-Kähler compact complex manifolds associated to number fields*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 161-171.

[52] Phong, D. H., Sesum, N. and Sturm, J. *Multiplier ideal sheaves and the Kähler-Ricci flow*, Comm. Anal. Geom. **15** (2007), no. 3, 613-632.

[53] Popovici, D. *Sufficient bigness criterion for differences of two nef classes*, Math. Ann. DOI 10.1007/s00208-015-1230-z.

[54] Sherman, M., Weinkove, B. *Local Calabi and curvature estimates for the Chern-Ricci flow*, New York J. Math. **19** (2013), 565-582.

[55] Song, J., Tian, G. *The Kähler-Ricci flow through singularities*, preprint, arXiv: 0909.4898.

[56] Song, J., Tian, G. *Kähler-Ricci flow on surfaces of positive Kodaira dimension*, Invent. Math. **170** (2007), no. 3, 609-653.

[57] Song, J., Tian, G. *Canonical measures and Kähler-Ricci flow*, J. Amer. Math. Soc. **25** (2012), no. 2, 303353.

[58] Song, J., Weinkove, B. *The Kähler-Ricci flow on surfaces of positive Kodaira dimention*, Invent. Math. **170** (2007), no. 3, 609-653.

[59] Song, J., Weinkove, B. *Contracting exceptional divisors by The Kähler-Ricci flow*, Duke Math. J. **162** (2013), no. 2, 367-415.

[60] Song, J., Weinkove, B. *Contracting exceptional divisors by The Kähler-Ricci flow II*, Proc. Lond. Math. Soc. (3) 108 (2014), no. 6, 1529-1561.

[61] Song, J., Weinkove, B. *Lecture notes on the Kähler-Ricci flow*, preprint, arXiv: 1212.3653.

[62] Teleman, A. *Projectively flat surfaces and Bogomolov's theorem on class $VII_0$-surfaces*, Int. J.Math. **5** (1994), 253-264.

[63] Teleman, A. *Donaldson theory on non-Kählerian surfaces and class $VII$ surfaces with* $b_2 = 1$, Invent. Math. **162** (2005), no. 3, 493-521.

[64] Teleman, A. *The pseude-effective cone of a non-Kählerian surface and applications*, Math. Ann. **335** (2006), no. 4, 965-989.

[65] Tian, G., Zhang, Z. *On the Kähler-Ricci flow on projective manifolds of general type*, Chinese Ann. Math. Ser. B **27** (2006), no. 2, 179-192.

[66] Tosatti, V. *Non-Kähler Calabi-Yau manifolds*, Contemp. Math. **644** (2015), 261-277.

[67] Tosatti, V. Weinkove, B. *The Calabi-Yau equation, symplectic forms and almost complex structures*, in Geometry and Analysis, Vol. I, 475-493, Advanced Lectures in Math. **17**, International Press, 2010.

[68] Tosatti, V. Weinkove, B. *The complex Monge-Ampère equation on compact Hermitian manifolds*, J. Amer. Math. Soc. 23 (2010), no.4, 1187-1195.

[69] Tosatti, V. Weinkove, B. *Plurisubharmonic functions and nef classes on compact manifolds*, Proc. Amer. Math. Soc. 140 (2012), 4003-4010.

[70] Tosatti, V. Weinkove, B. *The Chern-Ricci flow on complex surfaces*, Compos. Math. **149** (2013), no. 12, 2101-2138.

[71] Tosatti, V. Weinkove, B., Yang, X. *Collapsing of the Chern-Ricci flow on elliptic surfaces*, Math. Ann. **362** (2015), no. 3-4, 1223-1271.

[72] Tosatti, V. Weinkove, B. *On the evolution of a Hermitian metric by its Chern-Ricci form*, J. Differential Geom. 99 (2015), 125-163.

[73] Tosatti, V. Weinkove, B., Yau, S.-T. *Taming symplectic forms and the Calabi-Yau equation*, Proc. London Math. Soc. **97** (2) (2008), 401-424.

[74] Vaisman, I. *Non-Kähler metrics on geometric complex surfaces*, Rend. Sem. Mat. Univ. Politec. Torino **45** (1987), no. 3, 117-123.

[75] Wall, C.T.C. *Geometric structure on compact complex analytic surfaces*, Topology **25** (1986), no. 2, 119-153.

[76] Yau, S.-T. *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. **100** (1978), no. 1, 197-203.

[77] Yau, S.-T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, Comm. Pure Appl. Math. **31** (1978), 339-411.

[78] Zhang, Z. *On Degenerated Monge-Ampère Equations over Closed Kähler Manifold.* Int. Math. Res. Not. 2006, Art. ID 63640, 18 pp.

[79] Zhang, Z. *Scalar curvature bound for Kähler-Ricci flows over minimal manifolds of general type*, Int. Math. Res. Not. 2009; doi: 1093/imrn/rnp073.