Massive gravity:

14879323
ShinnIchiro Shimizu
Department of Physics, Tokyo Metropolitan University

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Abstract

Massive gravity is the theory of graviton with mass. This idea has been researched for long time but all attempts had failed because of some ghost problems. In 2012, for the fist time, consistent theory of massive gravity was constructed as dGRT ghost-free massive gravity. And now it is expected to solve some cosmological problems. I introduce the theoretical aspects of massive gravity and its cosmological applications, as a review based on Refs [1] ∽ [9] .

TO JASMINE

Contents

Notation

Natural units $c = 1$, $\hbar = 1$

Minkowski metric $\eta_{\mu\nu}$is defined as $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

M_{pl} : Planck mass$\backsimeq 2.4 \times 10^{18}\ GeV$
$g_{\mu\nu}$: space-time metric in 4 dimensions
Christoffel symbols are

$$\Gamma^{\mu}_{\nu\gamma}(g) \equiv \frac{1}{2} g^{\mu\lambda}(\partial_{\lambda} g_{\nu\gamma} - \partial_{\gamma} g_{\lambda\nu} - \partial_{\nu} g_{\gamma\lambda}) \tag{1}$$

Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$ is

$$R^{\mu}{}_{\nu\rho\sigma} \equiv \partial_{\sigma}\Gamma^{\mu}_{\nu\rho} - \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} + \Gamma^{\lambda}_{\nu\rho}\Gamma^{\mu}_{\lambda\sigma} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\mu}_{\lambda\rho} \tag{2}$$

$R_{\mu\nu}$ = Ricci tensor
is

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} \tag{3}$$

Ricci scalar R
is

$$R = g^{\mu\nu} R_{\mu\nu} \tag{4}$$

Einstein contraction rule is

$$A^{\mu} B_{\mu} \equiv \sum_{\mu} A^{\mu} B_{\mu} \tag{5}$$

Chapter 1

Introduction

1.1 Brief Summary of General Relativity

1.1.1 Principles of General Relativity

General Relativity(GR) is the theory of dynamical space-time. It's based on two principles.
1. Any observers observe same physical laws.(General Covariance)
2. In small space-time region, one can set up local inertial frame in which one can cancel out Gravitational field.(Equivalence principle)

In more detail, 1. Any physical laws are invariant under general coordinate transformation. The general coordinate transformation from x coordinate system to $x^{'}$coordinate system in 4 dimension space time is

$$x^\mu \to x'^\mu(x) \qquad\qquad \mu = 0,1,2,3 \tag{1.1}$$

2. intends inertial mass corresponding to gravitational mass.

$$M_{inertial} = M_{gravitational} \tag{1.2}$$

In other words, one can cancel out gravitational field by moving along with the direction of gravity as free fall. But it is possible only in the small region because the general gravitational field is not homogeneous.
The action of GR is (with matter and a cosmological constant)

$$\mathcal{S}[g^{\mu\nu}] = \frac{M_{pl}^2}{2}\int dx^4\,\sqrt{-g}\{R - 2\Lambda + \mathcal{L}_M\} \tag{1.3}$$

M_{pl} is Planck mass.
Lagrangian $\mathcal{L}_M$ is matter which depends on the physical situation. The variable of (1.3) is $g^{\mu\nu}(x)$ defined by inverse matrix of length of world line of space-time

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{1.4}$$

$g_{\mu\nu}$ represents the metric of curved space time. $\sqrt{-g}$ is the determinant of $g_{\mu\nu}$.
$R[g_{\mu\nu}]$ is Ricci scalar.

$$R \equiv\; g^{\mu\nu}R_{\mu\nu} \tag{1.5}$$

$\{g^{\mu\nu}\}$ is 4×4 inverse matrix of $\{g_{\mu\nu}\}$,

$$g^{\mu\lambda} g_{\lambda\nu} \equiv \delta^{\mu}_{\nu} \tag{1.6}$$

∂_μ means $\partial/\partial x^\mu$. (2) is derived from parallel transportation of arbitrary vector along a small closed region. (1) is from$\nabla_\lambda g_{\mu\nu} = 0$. When one takes variation of (1.3) respect to $g^{\mu\nu}$, one gets Einstein equation

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.7}$$

Λ is a cosmological constant. $T_{\mu\nu}$ is energy momentum tensor defined as

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}_{matter}}{\delta g^{\mu\nu}} \tag{1.8}$$

1.1.2 Einstein-Hilbert action and its Linearization

One can expand the Einstein–HIlbert action to 2nd order of perturbation field $h_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1.9}$$

one consider $h_{\mu\nu}$ small relative to $\eta_{\mu\nu}$ $|h_{\mu\nu}| \ll 1$. Cosmological constant Λ is small enough to ignore, so one starts from the free Einstein Hilbert action

$$\mathcal{S}[g^{\mu\nu}] = \frac{M_{pl}^2}{2} \int dx^4 [\sqrt{-g} R]_{linearized} \tag{1.10}$$

One expands it to 2nd order of $h_{\mu\nu}$ by using

$$\sqrt{det(\eta_{\mu\nu} + h_{\mu\nu})} = 1 + \frac{1}{2} h + O({h_{\mu\nu}}^3) \tag{1.11}$$

h is the trace of $h_{\mu\nu}$. And the Christoffel symbols are

$$\Gamma^{\rho}_{\mu\nu}(\eta_{\mu\nu} + h_{\mu\nu}) = -\eta^{\rho\alpha}(\partial_\nu h_{\mu\alpha} + \partial_\mu h_{\alpha\nu} - \partial_\alpha h_{\mu\nu}) + O(h^2_{\mu\nu}) \tag{1.12}$$

So one gets

$$R = -\partial^\mu h^{\alpha\beta} + 2\partial_\mu h_{\alpha\beta} \partial^\alpha h^{\mu\beta} - (2\partial_\beta h \partial_\mu h^{\mu\beta} - \partial_\beta h \partial^\beta h) + O(h_{\mu\nu^2}) \tag{1.13}$$

From (1.11) and (1.13), the action of 2nd order of $h_{\mu\nu}$ is

$$\mathcal{S}_{linear} = \frac{M_{pl}^2}{2} \int d^4x \frac{1}{2}(-\partial^\lambda h^{\mu\nu} \partial_\lambda h_{\mu\nu} + \partial_\mu h_{\nu\lambda} \partial^\nu h^{\mu\lambda} - \partial_\mu h \partial_\nu h^{\nu\mu} + \partial_\mu h \partial^\mu h) \tag{1.14}$$

In this equation, the upper index such as $h^{\mu\nu}$ is defined by $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}$. The equation of motion is derived from variation of (1.14)

$$\Box h_{\mu\nu} - \partial_\lambda \partial_\nu h^\lambda{}_\mu - \partial_\lambda \partial_\mu h^\lambda{}_\nu + \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h = 0 \tag{1.15}$$

So one gets the linearlized equation of motion. If there are matter sources, one adds an energy momentum tensor $T_{\mu\nu}$

$$\Box h_{\mu\nu} - \partial_\lambda \partial_\nu h^\lambda{}_\mu - \partial_\lambda \partial_\mu h^\lambda{}_\nu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h = -kT_{\mu\nu} \tag{1.16}$$

In this equation, $k = 8\pi G$, G is Newton constant.
Gauge invariance :
(1.16) is invariant under the transformation

$$h'_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{1.17}$$

This transformation is called gauge transformation and such invariance is called gauge invariance. This transformation is relative to the small coordinate transformation

$$x'^\mu = x^\mu + \xi^\mu \tag{1.18}$$

because $h_{\mu\nu}$ transforms under the small coordinate transformation as

$$h'_{\mu\nu}(x_\mu + \xi_\mu) = h_{\mu\nu}(x_\mu) + \frac{\partial \xi^\alpha}{\partial x^\mu}\eta_{\alpha\nu} + \frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\mu\beta} \tag{1.19}$$

So, explicitly, one gets the gauge transformation

$$h'_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{1.20}$$

Back to (1.16). If one sets gauge as

$$\partial_\mu h^{\mu\nu} = 0 \qquad h = 0 \tag{1.21}$$

then, (1.15) yields

$$\Box h_{\mu\nu} = 0 \tag{1.22}$$

This is the free wave equation. So gravitational field $h_{\mu\nu}$ propagates as waves with the speed of light. This is the so called Gravitational wave which one refer in section 4.

1.1.3 Solution with spherical symmetry

The existence of matter makes the space-time curved. In that space-time, particles -even the light- move along geodesics, for example, in our Solar system, the light came from vary large distant star is bended by the Sun. This phenomenon is called lensing effect. In this section, one consider lensing effect in GR and later compare it to Massive Gravity theory.

The geodesic Line:
First, one get a equation of motion of a particle in gravity field. If there is no fields except gravity. The particles move the shortest path in the space-time. The path is called the geodesic line. The length of world line L is

$$dl^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{1.23}$$

λ is the parameter of the geodesic line. dl is line element. So one can write down

$$L(x^\mu, \frac{dx^\mu}{d\lambda}) = \int |g_{\mu\nu}dx^\mu dx^\nu|^{\frac{1}{2}} \tag{1.24}$$

$$= \int |g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}|^{\frac{1}{2}} d\lambda \tag{1.25}$$

This is the functional of x^μ and $\frac{dx^\mu}{d\lambda}$. one take variation and get

$$\frac{d}{d\lambda}(U^\mu) + \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta = 0 \tag{1.26}$$

This is the geodesic equation, where $U^\alpha = \frac{dx^\alpha}{d\lambda}$.

Correspondence to Newton potential:
In non-relativistic limit, The geodesic equation (1.26) should correspond to Newton's equation of motion

$$\frac{d^2x_i}{dt^2} = -\frac{\partial\phi}{\partial x^i} \tag{1.27}$$

ϕ is the potential of gravity of Newton.
(1) The oneak gravity:

$$g_{\mu\nu} = \eta_{mu\nu} + h_{\mu\nu} \tag{1.28}$$

$$|h_{\mu\nu}| \ll 1 \tag{1.29}$$

(2) The stationarity:
The gravity field does not depend on time x^0, and

$$g_{0i} = 0 \quad i = 1, 2, 3 \tag{1.30}$$

(3) The speed of motion is slooner than that of light

$$\frac{dx^i}{d\tau} \ll 1 \tag{1.31}$$

Using these assumptions, one can write the (1.26) as

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{00} = 0 \tag{1.32}$$

So comparing to (1.27), one finds

$$-\Gamma^\mu_{00} = -\frac{\partial\phi}{\partial x_\mu} \tag{1.33}$$

By using (1) and (1.30) , one can deduce

$$h_{00} = -2\phi \tag{1.34}$$

So one gets ($c = 1$)

$$g_{00} = -1 - 2\phi \tag{1.35}$$

Spherical symmetry solution:
Let us consider as source the stress energy tensor of mass M point particle at rest at the origin

$$T_{\mu\nu} = M\delta^0_\mu \delta^0_\nu \delta^3(\mathbf{x}) \tag{1.36}$$

like the Sun at rest at the origin. δ^ν_μ is Kronecker delta and $\delta^3(\mathbf{x})$ is Dirac delta function. In this situation, one need the spherical symmetric metric such as

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.37}$$

$\nu(r)$ and $\lambda(r)$ are functions of r. r is the radius of polar coordinate system. θ and ϕ are angle of the system. The corresponding metric $g_{\mu\nu}$ is

$$g_{00} = -e^{\nu(r)} \quad g_{11} = e^{\lambda(r)} \quad g_{22} = r^2 \quad g_{33} = r^2 sin^2\theta \tag{1.38}$$

Inverse of the metric is

$$g^{00} = -e^{-\nu(r)} \quad g^{11} = e^{-\lambda(r)} \quad g^{22} = \frac{1}{r^2} \quad g^{33} = \frac{1}{r^2 sin^2\theta} \tag{1.39}$$

The Ricci tensor of the space time is derived from this $g_{\mu\nu}$ as

$$R_{00} = e^{\nu-\lambda}(\frac{1}{2}\nu'' - \frac{1}{4}\nu'\lambda' + \frac{\nu'}{r}) \tag{1.40}$$

$$R_{11} = -\frac{1}{2}\nu'' + \frac{1}{4}\nu'\lambda' - (\frac{1}{4}\nu')^2 + \frac{\lambda'}{r} \tag{1.41}$$

$$R_{22} = -e^{-\lambda} + \frac{1}{2}re^{-\lambda}(\lambda' - \nu') + 1 \tag{1.42}$$

$$R_{33} = s^2_\theta(-e^{-\lambda} + \frac{1}{2}re^{-\lambda}(\lambda' - \nu') + 1) \tag{1.43}$$

other components are 0. Also one need Ricci scalar R

$$R = e^{-\lambda}(-\nu'' - \frac{1}{2}\nu'^2 + \frac{2}{r}(\lambda' - \nu') + \frac{2}{r^2} + \frac{1}{2}\nu'\lambda') + \frac{2}{r^2} \tag{1.44}$$

and put these expression into the Einstein equations

$$e^{\nu-\lambda}(\frac{\lambda'}{r} - \frac{1}{r^2}) + \frac{e^\nu}{r^2} = 8\pi G T_{00} \tag{1.45}$$

$$\frac{\nu'}{r} + \frac{1}{r^2}(1 - e^\lambda) = 8\pi G T_{11} \tag{1.46}$$

$$\frac{e^{-\lambda}}{2}r^2\{-\frac{(\lambda' - \nu')}{r} + \nu'' + \frac{1}{2}\nu'^2 - \frac{\lambda'\nu'}{2}\} = 8\pi G T_{22} \tag{1.47}$$

$$r^2 sin^2\theta \ \frac{e^{-\lambda}}{2}\{-\frac{(\lambda' - \nu')}{r} + \nu'' + \frac{1}{2}\nu'^2 - \frac{\lambda'\nu'}{2}\} = 8\pi G T_{33} \tag{1.48}$$

λ' is derivative of λ respect to r. Now one put $R_{\mu\nu}$ into (1.36), and get

$$e^{-\lambda} = 1 + \frac{D_1}{r} \tag{1.49}$$

$$e^\nu = D_2(1 + \frac{D_2}{r}) \tag{1.50}$$

D_1 and D_2 are the constants. D_2 is set to 1 by redefining of time t. D_1 is determined by the boundary condition such the field becomes Newton like one at large distance from the origin. From (1.35)

$$D_2 = -2GM \tag{1.51}$$

So one gets the spherical symmetric solution

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{1}{1 - \frac{2GM}{r}}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.52}$$

This is called the Swartzschild solution. And

$$r_g = 2GM \tag{1.53}$$

is called Swartzschild radius.

1.1.4 Gravitational Waves and Polarizations

Let us discuss the propagation of gravity filed (1.16) in more detail, and start from (1.22). Let us investigate the degrees of freedom of Gravitational Waves. (1.22) is free wave function so that one can put plane wave as the solution

$$h_{\mu\nu} = a_{\mu\nu}e^{ik_l x^l} \tag{1.54}$$

$a_{\mu\nu}$ is 4×4 constant matrix. k is the wave length and $k^2 = 0$ from (1.22). For a convenience, let us choose the direction of wave to x^3 . So one can choose $k_\mu = (-k, 0, 0, k)$. From the gauge constraints one gets

$$a_{\nu 0}k + a_{3\nu}k = 0 \qquad (\partial_\mu h^{\mu\nu} = 0) \tag{1.55}$$

$$-a_{00} + a_{11} + a_{22} + a_{33} = 0 \qquad (h = 0) \tag{1.56}$$

From these equations one gets

$$\{a_{\mu\nu}\} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & -a_{00} \\ \vdots & a_{11} & a_{12} & -a_{01} \\ \vdots & \cdots & -a_{11} & -a_{02} \\ \vdots & \cdots & \cdots & a_{00} \end{pmatrix} \tag{1.57}$$

($h_{\mu\nu} = h_{\nu\mu}$.) From (1.57), it looks that $a_{\mu\nu}$ has 5 degrees of freedom. But some possibility remained that one can eliminate such degrees by choosing a coordinate transformation.

One sets gauge

$$\xi_\mu = b_\mu exp\{ik_l x^l\} \tag{1.58}$$

and get new $h'_{\mu\nu}$ and $a'_{\mu\nu}$ as

$$h'_{\mu\nu} = a'_{\mu\nu}e^{ik_l x^l} \tag{1.59}$$

$$a'_{\mu\nu} = a_{\mu\nu} + ik_\mu b_\nu + ik_\nu b_\mu \tag{1.60}$$

And investigating the 5 components

$$a'_{00} = a_{00} - 2ikb_0 \tag{1.61}$$
$$a'_{01} = a_{01} + ik_0b_1 + ik_1b_0 \tag{1.62}$$
$$= a_{01} - ikb_1 \tag{1.63}$$
$$a'_{02} = a_{02} - ikb_2 \tag{1.64}$$
$$a'_{11} = a_{11} \tag{1.65}$$
$$a'_{12} = a_{12} \tag{1.66}$$
$$\tag{1.67}$$

one can cancel out the $a'_{00}, a'_{01}, a'_{02}$ if one chooses

$$b_0 = \frac{a_{00}}{2ik} \qquad b_1 = \frac{a_{01}}{ik} \qquad b_2 = \frac{a_{02}}{ik} \tag{1.68}$$

So the true number of degrees of freedom of GW is 2. And note that a'_{11} and a'_{12} are vertical to the direction of the propagating of the 2 waves. So GW is the transverse wave.
To make sure, investigate other components

$$a'_{03} = a_{03} - ikb_3 + ikb_0 \tag{1.69}$$
$$= a_{03} + ik\frac{a_{00}}{2ik} - ikb_3 = 0 \tag{1.70}$$
$$a'_{13} = a_{13} + ikb_1 \tag{1.71}$$
$$= -a_{01} + ik\frac{a_{01}}{ik} = 0 \tag{1.72}$$
$$a'_{23} = a_{23} + ikb_2 \tag{1.73}$$
$$= -a_{02} + ik\frac{a_{02}}{ik} = 0 \tag{1.74}$$
$$a'_{33} = a33 + ikb_3 + ikb_3 \tag{1.75}$$
$$= a_{00} + 2ik(-\frac{a_{00}}{2ik}) = 0 \tag{1.76}$$

where

$$b_3 = -\frac{a_{00}}{2ik} \tag{1.77}$$

So, the non zero components are only a'_{11} and a'_{12}.

1.1.5 Dark Energy and Cosmological Constant

In this section let us assume the metric to depend on time. In other words, one assumes general homogeneous isotropic universe metric. And let us see how the self accelerating universes arise.
The solution to the isotropic metric reads

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.78}$$

Fundamental Observer:

To assume the homogeneous isotropic solution, let us require the existence of fundamental observer who can see the universe as homogeneous and isotropic. And one sets the scale of t as fundamental observers one. then

$$ds^2 = -d(x^0)^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1.79}$$

And what one wants is

$$ds^2 = -d(x^0)^2 + a(t)[e^{\lambda(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)] \tag{1.80}$$

where $a(t)$ is time dependent 3 dimension scale factor.

The homogeneous isotropic solution:
To calculate the curvature, one can use the result of (1.44),

$${}^{(3)}R = e^{-\lambda}(-\frac{2}{r}\lambda' + \frac{2}{r^2}) + \frac{2}{r^2} \tag{1.81}$$

${}^{(3)}R$ is the 3 dimensional part of the R. Then, since the space-time is homogeneous, ${}^{(3)}R$ should be constant so one can integrate and get solution

$${}^{(3)}R = \frac{2}{r^2}\frac{d}{dr}(1 - e^{-\lambda}) \tag{1.82}$$

$$\frac{R}{6}r^3 = r(1 - e^{\lambda}) + D \tag{1.83}$$

$$\frac{R}{6}r^2 = (1 - e^{\lambda}) + \frac{D}{r} \tag{1.84}$$

D is some constant. One sets the boundary condition that e^{λ} should be 0 for $r \to 0$. So D is 0. After that,

$$e^{-\lambda} = 1 - Kr^2 \tag{1.85}$$

$$K = \frac{{}^{(3)}R}{6} \tag{1.86}$$

So one gets the solution as

$$ds^2 = -d(x^0)^2 + a(t)[\frac{1}{1 - Kr^2}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)] \tag{1.87}$$

This is so called FLRW solution.
If one takes another gauge as $r \to |K|^{-1/2}\, r$, one gets

$$ds^2 = -d(x^0)^2 + R^2(t)[\frac{1}{1 - kr^2}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)] \tag{1.88}$$

where

$$k = \begin{cases} -1 & (K < 0) \\ 0 & (K = 0) \\ 1 & (K > 0) \end{cases} \tag{1.89}$$

$$R(t) = \begin{cases} |K|^{-1/2}a(t) & (K \neq 0) \\ a(t) & (K = 0) \end{cases} \tag{1.90}$$

So if curvature K is not 0, r is dimensionless and k takes discrete numbers $-1, 0, 1$. To get this equation, no Einstein equation was used. So (1.87), (1.88) are right even in the situation where Einstein equation is modified.

Freedman equation and acceleration of expanding universe :
Now the metric of the universe scale depends on time by $a(t)$. Next target is the way to know the value of the components of the metric. Let us plug (1.88) into the Einstein equation, to gets

$$R^0_0 = \frac{3\ddot{a}}{a} \tag{1.91}$$

$$R^i_j = [\frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2 + \frac{2}{a^2}K]\delta^i_j \tag{1.92}$$

Other components are equal to 0. $\dot{a}$ is the time derivative of a and the double dots one is the 2nd derivative. So one gets Einstein tensor as

$$G^0_0 = -3[(\frac{\dot{a}}{a})^2 + \frac{K}{a^2}] \tag{1.93}$$

$$G^i_j = -[2(\frac{\ddot{a}}{a})(\frac{\dot{a}}{a})^2 + \frac{K}{a^2}]\delta^i_j \tag{1.94}$$

and other components are 0. Energy momentum tensor $T_{\mu\nu}$ need to satisfy the Einstein equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.95}$$

So $T_{\mu\nu}$ should be the form as,

$$\{T^\mu{}_\nu\} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{1.96}$$

(the space components are corresponding to each other.) This $T_{\mu\nu}$ is the so called perfect fluid. ρ is energy density and p is the pressure of the space-time. ρ and p only depend on the time. Now one gets the Einstein equations of the homogeneous isotropic space-time.

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \tag{1.97}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1.98}$$

This is the Freedman equation.

Dark Energy;
From the observations, one know our universe is expanding and it's speed is increasing. To express this situation, one should assume the universe is filled with the exotic matter which create negative pressure. If one imposes the role to cosmological constant (as unordinary matter), the Einstein equations become

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3} \tag{1.99}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{1.100}$$

So one gets

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{K}{a^2} \tag{1.101}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}[(\rho + \rho_\Lambda + 3(p + p_\Lambda)] \tag{1.102}$$

where

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \tag{1.103}$$

The observed cosmological constant Λ_{obs} is

$$\Lambda_{obs} < 10^{-120} M_{pl}^2 \tag{1.104}$$

while $\Lambda_{quantum}$ calculated as energy of vacuum from QFT is about

$$\Lambda_{quantum} \cong M_{pl}^2 \tag{1.105}$$

then

$$\frac{\Lambda_{obs}}{\Lambda_{quantum}} \approx 10^{-120} \tag{1.106}$$

So there is the huge gap betoneen observations and theoretical predictions of cosmological constant. It is called the cosmological constant problem. One needs some new theory beyond the GR.

Chapter 2

How to build the Consistent Massive Gravity

As one have seen, weak gravity field propagate as waves. Things getting better, (1.22) looks like spin-2 massless particle as the perturbation theory of GR. In this situation, one can also find the possibility of massive spin-2 particle. But there are several difficulties to realize it.

2.1 Fierz-Pauli action

In,1939, Fiertz and Pauli built first massive gravity [1]. Not to describe massive graviton, but massive spin-2 particle. Let us discuss the possibility of a mass term. As introduced in the previous section, the graviton is described as the propagation of quantized fields $h_{\mu\nu}$. It is spin-2 massless particle and should be described in the field theory.

2.1.1 Fierz-Pauli mass term

As the temporary field theory, one imposes the constrains on the action as below:
1. The action may contain 2nd order of ∂_μ at most but to keep the equation of motion with at most 2nd derivatives of $h_{\mu\nu}$.
2 The action should invariant under global Lorentz transformation, to covariant the equation of motion under global Lorentz transforms.
The possible mass terms without derivatives are $h_{\mu\nu}h^{\mu\nu}$ and h^2. And the possible mass part is

$$\mathcal{L}_{mass} = -\frac{1}{4}(h_{\mu\nu}h^{\mu\nu} + Ah^2) \tag{2.1}$$

A is the constant. Notice that it has no diffeomorphism invariance. Let us investigate whether the A is constrained one or not. To do that, one decompose the $h_{\mu\nu}$ as

$$h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)} \tag{2.2}$$

where $\partial_{(\mu}\xi_{\nu)} \equiv \frac{1}{2}(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$ Then the mass part becomes

$$\mathcal{L}_{mass} = \frac{1}{8}m^2\{(h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)})^2 + A(h + 2\partial^\alpha \chi_\alpha)^2\} \tag{2.3}$$

This terms have the gauge invariance under

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} \tag{2.4}$$

$$\chi_\mu \to \chi_\mu - \frac{1}{2}\xi_\mu \tag{2.5}$$

The kinetic terms of χ_μ are

$$\mathcal{L}_{kinetic\chi} = 4\{\partial_{(\mu}\chi_{\nu)}\}^2 + 4A\{\partial_\alpha \chi^\alpha\}^2 \tag{2.6}$$

$$= (\partial_{(\mu}\chi_{\nu)})^2 + 4A(\partial_\alpha \chi^\alpha)^2 \tag{2.7}$$

one can decompose it by scalar field π as

$$\chi_\mu \to \chi_\mu + \partial_\mu \pi \tag{2.8}$$

then, one gets

$$\mathcal{L}_{kinetic\ \pi} = \partial_\mu\partial_\nu\pi\partial^\mu\partial^\nu\pi + 2\partial_\mu\partial_\nu\pi\partial^\nu\partial^\mu\pi + \partial_\nu\partial_\mu\pi\partial^\nu\partial^\mu\pi + 4A\partial_\alpha\partial^\alpha\pi\partial_\lambda\partial^\lambda\pi \tag{2.9}$$

and one can rewrite this by considering surface term

$$\mathcal{L}_{kinetic\pi} = 4(1+A)\partial_\alpha\partial^\alpha\pi\partial_\lambda\partial^\lambda\pi = 4(1+A)(\Box\pi)^2 \tag{2.10}$$

It is known that these terms such as higher (more than 2nd) derivatives have some instability (called Ostrogradsky instability). More explicitly, by rewriting with $\tilde{\pi} = 2\Box\pi$

$$(\Box\pi)^2 = \tilde{\pi}\Box\pi - \frac{1}{4}\tilde{\pi}^2 \tag{2.11}$$

So finally, one chooses $\pi = \phi_1 + \phi_2$ and $\tilde{\pi} = \phi_1 - \phi_2$. One gets

$$\mathcal{L}_{kinetic\pi} = 4(1+A)\{\phi_1\Box\phi_1 - \phi_2\Box\phi_2 - \frac{1}{4}(\phi_1 - \phi_2)^2\} \tag{2.12}$$

As you can see, these kinetic terms have opposite sign, while connect to each other by $\frac{1}{4}(\phi_1 - \phi_2)^2$. So there exist the negative energy particle. Once you allow to exist the negative kinetic terms, negative kinetic particle can get infinite energy from other positive kinetic particles. So the system will be break down. Only what one can do to avoid the instability is to set $A = -1$. So the possible mass term is allowed only when

$$\mathcal{L}_{mass} = -\frac{1}{8}m^2(h_{\mu\nu}{}^2 - h^2) \tag{2.13}$$

this is called Fierz-Pauli mass term. And it's action

$$\mathcal{S} = \frac{M_{pl}^2}{2}\int dx^4 \mathcal{L}_{FP} \tag{2.14}$$

called Fierz-Pauli action(FP action), where

$$\mathcal{L}_{FP} = -\frac{1}{4}h^{\mu\nu}\hat{\epsilon}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{8}m^2(h^2_{\mu\nu} - h^2) \tag{2.15}$$

$$\hat{\epsilon}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = -\frac{1}{2}[\Box h_{\mu\nu} - 2\partial_{(\mu}\partial_\alpha h^\alpha_{\nu)} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta})]$$

2.1.2 vDVZ discontinuity problem

In previous section, the linearlized massive graviton was introduced. If it's mass is small, one expects to see the correspondence of the limit of massive gravity to GR. Of course, If one takes the limit in the action, one sees the correspondence. In this section, I review the massless limit of MG and compare it to GR for physical observables, such as quantum transition amplitudes (one follow [2] here).

Massive spin-2:
Let us start from the equation of motion of (2.15)

$$\hat{\epsilon}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + \frac{1}{2}m^2(h_{\mu\nu} - h\eta_{\mu\nu}) = \frac{1}{M_{pl}}T_{\mu\nu} \tag{2.16}$$

the constrains are derived from the trace and the divergence of (2.16)

$$h = -\frac{1}{3m^2M_{pl}}(T + \frac{2}{m^2}\partial_\alpha\partial_\beta T^{\alpha\beta}) \tag{2.17}$$

$$\partial_\mu h^\mu{}_\nu = \frac{1}{m^2M_{pl}}(\partial_\mu T^\mu{}_\nu + \frac{1}{3}\partial_\nu T + \frac{2}{3m^2}\partial_\nu\partial_\alpha\partial_\beta T^{\alpha\beta}) \tag{2.18}$$

put back into (2.16)

$$(\Box - m^2)h_{\mu\nu} = \frac{1}{M_{pl}}[\tilde{\eta}_{\mu(\alpha}\tilde{\eta}_{\nu\beta)} - \frac{1}{3}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta}]T^{\alpha\beta} \tag{2.19}$$

where

$$\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu \tag{2.20}$$

Then the propagator of $h_{\mu\nu}$ is

$$G^{massive}_{\mu\nu\alpha\beta}(x,x') = \frac{f^{massive}_{\mu\nu\alpha\beta}}{\Box - m^2} \tag{2.21}$$

where

$$f^{massive}_{\mu\nu\alpha\beta} = \tilde{\eta}_{\mu(\alpha}\tilde{\eta}_{\nu\beta)} - \frac{1}{3}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta} \tag{2.22}$$

Finally one gets the amplitude transition between two sources as

$$\mathcal{A}^{massive}_{TT'} = \int d^4x\, h_{\mu\nu}T'^{\mu\nu} = \int d^4x\, T'^{\mu\nu}\frac{f^{massive}_{\mu\nu\alpha\beta}}{\Box - m^2}T^{\alpha\beta} \tag{2.23}$$

one wants to take the massless limit of this. In the massless limit, $\partial_\mu T^{\mu\nu} = 0$. one gets

$$\mathcal{A}_{TT'}^{m\to 0} = \int d^4x \;\; T'^{\mu\nu}\frac{1}{\Box}(T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu}) \tag{2.24}$$

Massless spin-2:
The equation of motion is

$$\hat{\epsilon}_{\mu\nu}^{\alpha\beta}h_{\alpha\beta} = \frac{1}{M_{pl}}T_{\mu\nu} \tag{2.25}$$

In massless case, $\partial_\mu T^{\mu\nu} = 0$ is automatically (because of $\partial^\mu \hat{\epsilon}_{\mu\nu}^{\alpha\beta}h_{\alpha\beta} = 0$). In the de Donder gauge as

$$\partial_\mu h^\mu{}_\nu = \frac{1}{2}p_\nu \tag{2.26}$$

where $p_\mu p^\mu = -m^2$ then, (2.25) become,

$$(\Box - m^2)h_{\mu\nu} = -\frac{2}{M_{pl}}(T_\mu - \frac{1}{2}T\eta_{\mu\nu}) \tag{2.27}$$

So the propagator is

$$G_{\mu\nu\alpha\beta}^{massless} = \frac{f_{\mu\nu\alpha\beta}^{massless}}{\Box} \tag{2.28}$$

where

$$f_{\mu\nu\alpha\beta}^{massless} = \tilde{\eta}_{\mu(\alpha}\tilde{\eta}_{\nu\beta)} - \frac{1}{2}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta} \tag{2.29}$$

Finally one gets the $\mathcal{A}$ as

$$\mathcal{A}_{TT'}^{GR} = \int d^4x \;\; T'^{\mu\nu}\frac{1}{\Box}(T_{\mu\nu} - \frac{1}{2}T\eta_{\mu\nu}) \tag{2.30}$$

So it is different to (2.24) of massless limit of MG. This discontinuity is called vDVZ discontinuity. And it is the general problem of FP massive gravity that different physical predictions from GR in the massive gravity limit $m \to 0$.

Another example of the discontinuity:
Now the vDVZ discontinuity in spherical symmetric metric is studied. Let us from the general spherical symmetric metric

$$ds^2 = -A(r)(dt)^2 + f(r)(dr)^2 + k(r)d\Omega^2 \tag{2.31}$$

In GR, it is transformed the $k(r) \to r^2$ and the metric is rewritten to

$$ds^2 = -e^{\nu(r)}(dt)^2 + e^{\lambda(r)}(dr^)2 + r^2 d\Omega^2 \tag{2.32}$$

without the loss of generality. Because GR has diffeomorphism invariance. In the FP action, notice that no more that invariance, so (2.31) and (2.32) yield different equation of motions.
Let us rewrite the metric to convenience to

$$ds^2 = -e^{\nu(r)}(dt)^2 + e^{\lambda(r)\frac{d}{dr}(re^{\mu(r)/2})}(dr)^2 + r^2e^{\mu(r)}\Omega^2 \tag{2.33}$$

So linearized $h_{\mu\nu}$ is

$$h_{00} = \nu \quad h_{rr} = -\lambda - (r\mu)' \quad h_{\theta\theta} = -r^2\mu \quad h_{\phi\phi} = -r^2\mu sin^2\theta \tag{2.34}$$

putting into FP equation (2.16) and yield 3 equations as

$$\frac{1}{r}\lambda' + \frac{1}{r^2}\lambda = -\frac{m^2}{2}(\lambda + 3\mu + r\mu') \tag{2.35}$$

$$-\frac{1}{r}\nu' + \frac{1}{r^2}\lambda = -m^2(\mu + \frac{\nu}{2}) \tag{2.36}$$

$$m^2(\frac{\nu'}{2} - \frac{\lambda}{r}) = 0 \tag{2.37}$$

If one sets $m = 0$, one gets the GR results as

$$\lambda = -\nu = \frac{r_g}{r} \equiv \frac{2m}{M_{pl}r} \tag{2.38}$$

and one has the property

$$\nu + \mu = 0 \tag{2.39}$$

There is no surprise. On the contrary, If one solves (2.37) one gets

$$\nu = -\frac{2C}{r}e^{-mr} \tag{2.40}$$

$$\lambda = \frac{C}{r}(1 + mr)e^{-mr} \tag{2.41}$$

$$\mu = C\frac{1 + mr + (mr)^2}{m^2r^3}e^{-mr} \tag{2.42}$$

but the massless limit of the equations are

$$\nu = -\frac{2C}{r} \quad \lambda = \frac{C}{r} \quad \mu = \frac{C}{r(mr)^2} \tag{2.43}$$

and

$$\nu + \lambda \neq 0 \tag{2.44}$$

so one sees that $m = 0$ and $m \to 0$ are totally different. Moreover, FP massive gravity has no smooth limit of $m \to 0$ in this case ($\mu \to \infty$).

2.2 Non linear Fierz Pauli action

No smooth limit means no consistency. To avoid the vDVZ discontinuity, one expands the linear FP massive gravity to Non-linear one by introducing the reference metric $f_{\mu\nu}$ built of four scalar fields ϕ^A.

$$\mathcal{S} = \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g}\{R - m^2\mathcal{U}\} + \mathcal{S}_{matter} \tag{2.45}$$

where

$$\mathcal{U} = H^\mu_\nu H^\nu_\mu - [H]^2 \tag{2.46}$$

$$H^\mu_\nu = g^{\mu\alpha} f_{\alpha\nu} - \delta^\mu_{nu} \tag{2.47}$$

and

$$f_{\mu\nu} = \eta_{AB}\partial_\mu\phi^A\partial_\nu\phi^B \tag{2.48}$$

$f_{\mu\nu}$ is consist of the stuckelberg scalar fileds ϕ^A. Notice that $f_{\mu\nu}$ keeps the (2.45) diffeomrphism invariant. From here, all upper and looner indices are transfered by $g_{\mu\nu}$. In the gauge $\phi^A = \delta^\mu_\nu x^\mu$, one gets $f_{\mu\nu} = \delta_{\mu\nu}$, and gets back the FP theory after linearization.

2.2.1 Vainshtein mechanism

Let us study whether the Non-linear FP action avoid the vDVZ discontinuity. To see it, let us reintroduce the spherical metric

$$ds^2 = -e^{\nu(r)}(dt)^2 + e^{\lambda(r)}\frac{d}{dr}(re^{\mu(r)/2})(dr)^2 + r^2 e^{\mu(r)}\Omega^2 \tag{2.49}$$

After one calculates the equation of motion, one compares to linear one. So expanding $\nu(r)$, $\lambda(r)$ and $\mu(r)$ with a parameter ϵ

$$\nu(r) = \nu_0(r) + \epsilon\nu_1(r) + \epsilon^2\nu_2(r) + \dots \tag{2.50}$$

$$\lambda(r) = \lambda_0(r) + \epsilon\lambda_1(r) + \epsilon^2\lambda_2(r) + \dots \tag{2.51}$$

$$\mu(r) = \mu_0(r) + \epsilon\mu_1(r) + \epsilon^2\mu_2(r) + \dots \tag{2.52}$$

put it into the equation

$$G_{\mu\nu} = m^2 T_{\mu\nu} + kT^{matter}_{\mu\nu} \tag{2.53}$$

$$T_{\mu\nu} = 2\frac{\partial\mathcal{U}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{U} \tag{2.54}$$

where $G_{\mu\nu}$ is the variation of (2.45) respect to $g_{\mu\nu}$, $T_{\mu\nu}$ is the contribution from mass terms, and $T^{matter}_{\mu\nu}$ is from source. If one sets $T^{matter}_{\mu\nu}$ as a point source, one can set $T^{matter}_{\mu\nu} = 0$ except $x = 0$. Then one gets

$$\nu = -\frac{2r_g}{r}(1 + c_1\frac{r_g}{m^4r^5} + \dots) \tag{2.55}$$

$$\lambda = \frac{r_g}{r}(1 + c_2\frac{r_g}{m^4r^5} + \dots) \tag{2.56}$$

$$\mu = \frac{r_g}{m^2r^3}(1 + c_3\frac{r_g}{m^4r^5} + \dots) \tag{2.57}$$

Notice that parameter is set to correspond the first term of the Linear FP results (2.43) . To roughly estimate the scale of second term, one can choose $m \sim (10^{25}cm)^{-1}$ and at the edge of the Solar system

$$\frac{r_g}{m^4 r^5} \sim 10^{32} \tag{2.58}$$

one cannot ignore such large terms. This is the cause of vDVZ discontinuity. So one cannot take the linear approximation in large r. How large ? The second term small enough where

$$r \gg r_V = (\frac{r_g}{m^4})^{\frac{1}{5}} \sim 100Kpc \tag{2.59}$$

these results imply that the linearization is broken for $r \gg r_V$. This r_V is called Vainshtein radius. It is argued by Vainshtein [4]. If r is small enough to $r \ll r_V$, one can take linear approximation. To do that, one should choose the parameter as mass parameter m, then expands as the corrections of mass order

$$\nu = -\frac{r_g}{r}(1 + a_1(mr)^2\sqrt{\frac{r}{r_g}} + \dots) \tag{2.60}$$

$$\lambda = \frac{r_g}{r}(1 + a_2(mr)^2\sqrt{\frac{r}{r_g}} + \dots) \tag{2.61}$$

$$\mu = \sqrt{\frac{ar_g}{r}}(1 + a_3(mr)^2\sqrt{\frac{r}{r_g}} + \dots) \tag{2.62}$$

$a_{1\sim 3}$ are the constants adjusting the dimensions. Here one takes massless limit $m \to 0$ and the solution is corresponding to (2.38). And notice, when one takes the linear approximation for first order, one loses the extra degrees of freedom from the mass because mass is strongly coupled with non-linearity. These results imply that one cannot detect the mass effect by linearization . It agrees with observations in the Solar system. i.e. inside r_V of gravitational body of mass M having the gravitational radius $r_g = 2GM$, the gravitational field in massive gravity (with mass m) is the same as in GR with $m = 0$. But outside of r_V, the extra degrees of freedom of massive gravity significantly change the gravitational shield VS GR.

2.2.2 Boulware Deser ghost problem again

Now one solves the mechanism of smooth massless limit by non-linearization. Next one should investigate the existence of ghost in the non-linear theory. It was investigated by Boulware and Deser in 1972[6]. The mass term of non-linear FP action is

$$\mathcal{U} = \frac{1}{8}(H^\mu_\nu H^\nu_\mu - (H^\alpha_\alpha)^2) \tag{2.63}$$

$$H^\mu_\nu = g^{\mu\alpha} f_{\alpha\nu} - \delta^\mu_\nu, \qquad f_{\mu\nu} = \eta_{AB}\partial_\mu\phi^A\partial_\nu\phi^B \tag{2.64}$$

and the equation of motion

$$G_{\mu\nu} = m^2 T_{\mu\nu} + kT^{matter}_{\mu\nu} \tag{2.65}$$

where

$$T_{\mu\nu} = 2\frac{\partial \mathcal{U}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{U} \qquad \nabla^{\mu}T_{\mu\nu} = 0 \tag{2.66}$$

To investigate the existence of ghost, one decomposes the $H_{\mu\nu}$ with St*ü*kelberg fields. First one divides $\phi^a = x^a - \frac{1}{m_{pl}}\chi^a$ to cancel out the $\eta_{\mu\nu}$ which come from the decomposition of $g_{\mu\nu}$. Then $f_{\mu\nu}$ is

$$\begin{aligned} f_{\mu\nu} &= \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab} \\ &= \partial_\mu (x^a - \frac{1}{M_{pl}}\chi^a)\partial_\nu (x^b - \frac{1}{M_{pl}}\chi^b)\eta_{ab} \\ &= \eta_{\mu\nu} - \frac{2}{M_{pl}}\partial_{(\mu}\chi_{\nu)} + \frac{1}{M_{pl}^2}\partial_\mu \chi^a \partial_\nu \chi^b \eta_{ab} \end{aligned} \tag{2.67}$$

where x^a is the coordinate variables. one decomposes the χ_μ again,

$$\chi_\mu = A_\mu + \partial_\mu \pi \tag{2.68}$$

So

$$f_{\mu\nu} = \eta_{\mu\nu} - \frac{2}{M_{pl}}\partial_{(\mu}A_{\nu)} - \frac{2}{M_{pl}m^2}\Pi_{\mu\nu} + \frac{1}{M_{pl}^2}\partial_\mu A^\alpha \partial_\nu A_\alpha + \frac{2}{M_{pl}^2 m^4}\Pi^2_{\mu\nu} \tag{2.69}$$

where $\Pi_{\mu\nu} = \partial_\mu \pi \partial_\nu \pi$. Here one can rewrite the $H_{\mu\nu}$ as

$$\begin{aligned} H_{\mu\nu} &= M_{pl}(f_{\mu\nu} - (\eta_{\mu\nu} + \frac{1}{M_{pl}}h_{\mu\nu})) \\ &= M_{pl}(-\frac{2}{M_{pl}}\partial_{(\mu}A_{\nu)} - \frac{2}{M_{pl}m^2}\Pi_{\mu\nu} + \frac{1}{M_{pl}^2}\partial_\mu A^\alpha \partial_\nu A_\alpha + \frac{2}{M_{pl}^2 m^4}\Pi^2_{\mu\nu} - \frac{1}{M_{pl}}h_{\mu\nu}) \\ &= -2\partial_{(\mu}A_{\nu)} - \frac{2}{m^2}\Pi_{\mu\nu} + \frac{1}{M_{pl}}\partial_\mu A^\alpha \partial_\nu A_\alpha + \frac{2}{M_{pl}m^4}\Pi^2_{\mu\nu} - h_{\mu\nu} \end{aligned} \tag{2.70}$$

Notice that $h_{\mu\nu}$ is the helicity -2 part of the graviton, A_μ is the helicity-1 part and π is the helicity-0. The graviton is decomposed to the helicity modes$(2+2+1)$. The kinetic term of π is most important part for existence of the ghost

$$\begin{aligned} \mathcal{L}_{FPmass} &= H^\mu{}_\nu H^\nu{}_\mu - H^\alpha{}_\alpha \\ \mathcal{L}_{FP,\pi} &= \frac{4}{m^2}([\Pi^2] - [\Pi]^2) - \frac{4}{M_{pl}m^4}([\Pi^3] - [\Pi][\Pi^2]) - \frac{1}{M_{pl}^2 m^6}([\Pi^4] - [\Pi^2]^2) \end{aligned} \tag{2.71}$$

The first term is the one of linear FP action (2.10). As one has seen, this quadratic term is a total derivative and be canceled out as FP structure. But notice that quartic and cubic terms are not. So these higher derivative terms yield negative kinetic term as (2.12). Such ghost term called Boulware Deser ghost (BD ghosts). Of course, the possibility to avoid the BD ghost is still remained by rewriting the mass term. The non-linear FP mass term is second order of $H_{\mu\nu}$. One can generalize it by considering higher order mass terms

$$\mathcal{L}_{mass} = -\sqrt{(-g)}\frac{1}{4}m^2\mathcal{U}(g_{\mu\nu}, f_{\mu\nu}) \tag{2.72}$$

where

$$\mathcal{U}(g,f) = \mathcal{U}_2(g,f) + \mathcal{U}_3(g,f) + \mathcal{U}_4(g,f) + \mathcal{U}_5(g,f) + \dots \tag{2.73}$$

$$\mathcal{U}_2(g,f) = [f^2] - [f]^2 \tag{2.74}$$
$$\mathcal{U}_3(g,f) = C_1[f^3] + C_2[f^2][f] + C_3[f]^3 \tag{2.75}$$
$$\mathcal{U}_4(g,f) = D_1[f^4] + D_2[f^3][f] + D_3[f^2]^2 + D_4[f^2][f]^2 + D_5[f]^4 \tag{2.76}$$
$$\mathcal{U}_5(g,f) = F_1[f^5] + F_2[f^4][f] + F_3[f^3][f]^2 + F_4[f^3][f^2] + F_5[f^2]^2[f] \tag{2.77}$$
$$+ F_6[f^2][f]^3 + F_7[f]^5 \tag{2.78}$$
$$\vdots$$

Where C, D and F are constants. One can analyze the BD ghost for each combination of constants. But they have not been able to avoid the BD ghost problem for 40 years.

2.3 dGRT Ghost-free Massive Gravity

In 2010, for the first time, de Rham, Gabadaze and Tolley found the consistent Massive gravity (dRGT model) [7] without BD ghosts.

$$\mathcal{L}_{EH}[g_{\mu\nu}] = \frac{M_{pl}^2}{2}\int d^4x\,\sqrt{-g}R \tag{2.79}$$
$$\mathcal{L}_{potential}[g_{\mu\nu}, f_{\mu\nu}] = M_{pl}^2 m_g^2 \int d^4x\,\sqrt{-g}(\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4) \tag{2.80}$$

$$\mathcal{L}_{dRGT}[g_{\mu\nu}, f_{\mu\nu}] = \frac{M_{pl}^2}{2}\int d^4x\,\sqrt{-g}(R + \mathcal{L}_{potential}[g_{\mu\nu}, f_{\mu\nu}]\,) \tag{2.81}$$
$$\mathcal{L}_{potential}[g_{\mu\nu}, f_{\mu\nu}] = M_{pl}^2 m_g^2 \int d^4x\,\sqrt{-g}(\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4) \tag{2.82}$$

by choosing the extra terms as

$$\mathcal{L}_2 = \frac{1}{2}([\mathcal{K}]^2 - [\mathcal{K}^2]) \tag{2.83}$$
$$\mathcal{L}_3 = \frac{1}{6}([\mathcal{K}]^3 - 3[\mathcal{K}][K^2] + 2[\mathcal{K}^3])$$
$$\mathcal{L}_4 = \frac{1}{24}([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4])$$

where

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - (\sqrt{g^{-1}f}\;)^\mu_\nu \tag{2.84}$$

This square root is defined as

$$(\sqrt{g^{-1}f}\;)^\mu_\rho(\sqrt{g^{-1}f}\;)^\rho_\nu = f^\mu_\nu \quad = g^{\mu\rho}f_{\rho\nu} \tag{2.85}$$

As one can see, dRGT model has 2 arbitrary constants α_3 and α_4. One can confirm the ghost free property by the decomposition $\phi^a = x^a - \frac{1}{m_{pl}}\chi^a$ and (2.68). Then

$$m^2 M_{pl}^2 \mathcal{U}\sqrt{-g} = (\partial_\mu \phi)^2 + \frac{1}{(\Lambda_5)^5}(()^2 + \dots) + \frac{1}{(\Lambda_3)^3}(h()^2 + \dots) + \dots \tag{2.86}$$

where

$$\Lambda_5 = (M_{pl}m^4)^{\frac{1}{5}} \sim \frac{1}{10^{11}km} \tag{2.87}$$

$$\Lambda_3 = (M_{pl}m^2)^{\frac{1}{3}} \sim \frac{1}{10^{11}km} \tag{2.88}$$

(one set $A_\mu = 0$). Therefore, one can adjust the higher derivatives to total derivatives that vanish on integration. one gets the consistent ghost free massive gravity.

2.4 Bi-Gravity

The dGRT action, has the form

$$\mathcal{S}_{MG} = \mathcal{S}_{EH}[g_{\mu\nu}] + \mathcal{S}_{mass}[g_{\mu\nu}, f_{\mu\nu}] + \mathcal{S}_{matter}[g_{\mu\nu}\Phi] \tag{2.89}$$

So, the action consist of 1.kinetic term of $g_{\mu\nu}$. 2.mass term = interaction terms betoneen $g_{\mu\nu}$ and $f_{\mu\nu}$ and interaction to matter. The metric $f_{\mu\nu}$ is non dynamical one. One can generalize if by making the reference metric $f_{\mu\nu}$ also dynamical, by adding kinetic terms of $f_{\mu\nu}$.

$$\mathcal{S}_{MG} = \mathcal{S}_{EH}[g_{\mu\nu}] + \mathcal{S}_{EH}[f_{\mu\nu}] + \mathcal{S}_{mass}[g_{\mu\nu}, f_{\mu\nu}] + \mathcal{S}_{matter}[g_{\mu\nu}\Phi] \tag{2.90}$$

This is so called bi-metric gravity (or bi-gravity) theory. Here, $f_{\mu\nu}$ does not interact with the matter and may be expected to act as the dark energy. dRGT model is special case of this bi-metric gravity theory.

Chapter 3

Cosmological Aspects of Massive Gravity

3.1 FLRW self-accelerating universe and dark energy in massive gravity

In this section, one review the FLRW solution of dRGT massive gravity [8]. Whether one could build isotopic and homogeneous solution or not is good test for massive gravity. Let us start from the dGRT action (2.82)

In massive gravity, one needs two metric information to get a solution.

$$ds_1^2 = g_{\mu\nu}dx^{\mu\nu} = -N(t)^2dt^2 + a(t)^2\Omega_{ij}dx^idx^i \tag{3.1}$$

and

$$ds_2^2 = f_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b dx^\mu dx^\nu \tag{3.2}$$

one redefines $f_{\mu\nu}$ by new fields φ^a and divide it to time and space components as

$$f_{\mu\nu} = -n^2(\varphi^0)\partial_\mu\varphi^0\partial_\nu\varphi^0 + \alpha^2(\varphi^0)\Omega_{ij}(\varphi^k)\partial_\mu\varphi^i\partial_\nu\varphi^j \tag{3.3}$$

$i, j = 1, 2, 3$ and

$$\Omega_{ij}(\varphi^k) = \delta_{ij} + \frac{K\delta_{il}\delta_{jm}\varphi^l\varphi^m}{1 - K\delta_{l,}\varphi^l\varphi^m} \tag{3.4}$$

$K(< 0)$ is the curvature constant. This is achieved by

$$\phi^0 = f(\varphi^0)\sqrt{1 - K\delta_{ij}\varphi^i\varphi^j}, \qquad \varphi^i = \sqrt{-K}f(\varphi^0)\varphi^i \tag{3.5}$$

and

$$n(\varphi^0) =\mid \dot{f}(\varphi^0) \mid, \qquad \alpha(\varphi^0) = \sqrt{-K} \mid f(\varphi^0) \mid \tag{3.6}$$

When one adopts the unitary gauge

$$\varphi^0 = t, \qquad \varphi^i = x^i \tag{3.7}$$

one gets

$$f_{\mu\nu}dx^\mu dx^\nu = -(\dot{f}(t))^2 dt^2 + \mid K \mid (f(t))^2 \Omega_{ij}(x^k) dx^i dx j \tag{3.8}$$

that the Minkowski line element in the open chart.

In this space-time, Einstein-Hilbert term becomes

$$\mathcal{L}_{EH} = \frac{3K}{a^2} - \frac{3\dot{a}}{a^2 N^2} \tag{3.9}$$

and contribution of mass terms are

$$\mathcal{L}_2 = 3(1 - \frac{\sqrt{-K} \mid f \mid}{a})(2 - \frac{\dot{f}}{N} - \frac{\sqrt{-k} \mid f \mid}{N}) \tag{3.10}$$

$$\mathcal{L}_3 = (1 - \frac{\sqrt{-K} \mid f \mid}{a})^2 (4 - 3\frac{\dot{f}}{N} - \frac{\sqrt{-k} \mid f \mid}{N}) \tag{3.11}$$

$$\mathcal{L}_4 = (1 - \frac{\sqrt{-K} \mid f \mid}{a})^3 (1 - \frac{\dot{f}}{N}) \tag{3.12}$$

So, varying with respect to f yields

$$[H - sgn(\frac{\dot{f}}{f})\frac{\sqrt{-K}}{a}][3 + 3\alpha_3 + \alpha_4 - \frac{2\sqrt{-K} \mid f \mid}{a}(1 + 2\alpha_3 + \alpha_4) - \frac{K \mid f \mid^2}{a^2}(\alpha_3 + \alpha_4)] = 0 \tag{3.13}$$

H is the Hubble expansion rate of physical metric $g_{\mu\nu}$

$$H \equiv \frac{\dot{a}}{Na} \tag{3.14}$$

(3.13) has three solutions, the trivial one $\dot{a} = sgn(\dot{f}/f)\sqrt{-K}N$ corresponds to the Minkowski space time. So what one needs remaining two solutions

$$\alpha(t) = X_\pm a(t), \qquad X_\pm = \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{\alpha_3 + \alpha_4} (> 0) \tag{3.15}$$

To derive Friedman equation, varying the action respect to $N(t)$ and $a(t)$ yields

$$3H^2 + \frac{3K}{a^2} = \frac{1}{M_{pl}^2}\rho_m + \rho_g \tag{3.16}$$

$$-\frac{2\dot{N}}{N} + \frac{2K}{a^2} = \frac{1}{M_{pl}^2}(\rho_m + p_m) + (\rho_g + p_g) \tag{3.17}$$

ρ_m and p_m are energy density and pressure contribution from the matter action $\mathcal{L}_{matter}$. And

$$\rho_g = -m_g^2(1 - \frac{\sqrt{-K} \mid f \mid}{a})[3(2 - \frac{\sqrt{-K} \mid f \mid}{a})]$$

$$+ \alpha_3(1 - \frac{\sqrt{-K} \mid f \mid}{a})(4 - \frac{\sqrt{-K} \mid f \mid}{a}) + \alpha_4(1 - \frac{\sqrt{-K} \mid f \mid}{a})^2] \tag{3.18}$$

$$\rho_g + p_g = -m_g^2(\frac{\dot{f}}{N} - \frac{\sqrt{-K} \mid f \mid}{a})[(3 - \frac{2\sqrt{-K} \mid f \mid}{a}) \tag{3.19}$$

$$+ \alpha_3(3 - \frac{\sqrt{-K} \mid f \mid}{a})(1 - \frac{\sqrt{-K} \mid f \mid}{a}) + \alpha_4(1 - \frac{\sqrt{-K} \mid f \mid}{a})^2] \tag{3.20}$$

ρ_g and p_g are contribution from the graviton mass. From the (3.13), $\rho_g + p_g = 0$ So finally one gets Friedmann equation for massive gravity

$$3(\frac{\dot{a}}{a})^2 + \frac{3K}{a^2} = \frac{1}{M_{pl}^2}\rho_m + c_\pm m_g^2 \tag{3.21}$$

$$-\frac{2\dot{N}}{N} + \frac{2K}{a^2} = \frac{1}{M_{pl}^2}(\rho_m + p_m) \tag{3.22}$$

where $c_\pm$ is

$$c_\pm = -\frac{1}{(\alpha_3 + \alpha_4)^2}[(1 + \alpha_3)(2 + \alpha_3 + 2\alpha_3^2 - 3\alpha_4) \pm 2(1 + \alpha_3 + \alpha_3^2 - \alpha_4)^{3/2}] \tag{3.23}$$

(3.21) correspond to Friedmann equation of Einstein gravity. And note that mass contribution term $c_\pm m_g^2$ roles cosmological constant as of GR (1.101) and (1.102) when $c_\pm > 0$. So dRGT massive gravity allows a self-accelerating universe without negative pressure source and a cosmological constant ($\Lambda = 0$).

Chapter 4

Gravitational Waves in massive gravity

By the observed cosmological constant (refDEo), one can roughly estimate the mass of graviton. (If the source of dark energy is effect of the mass of graviton). After easy calculation (use $m_g^2 c^4 < \Lambda$). one considers the mass value as $m_g < 10^{-33}[eV]$. Even comparing to the electron neutrino, their mass is $m_{\nu_e} < 2.5$ [eV]. So it is too small to detect the mass effect of massive gravity by today' experiments.
If graviton has no mass, gravitational wave propagate as speed of light. By contrast, if it has mass, the speed of propagation is

$$v_g^2 = 1 - \frac{m_g^2}{E^2} \tag{4.1}$$

using this relation, one could bound graviton mass. The representative example is observation of supernova [9]. one can limit value of mass $m_g < 10^{-23}$ [eV]. for 100$[Hz]$ gravitational wave. In detail [9], there is a relationship

$$1 - \frac{v_g}{c} = 5 \times 10^{-17} (\frac{200Mpc}{D})(\frac{\Delta t}{1s}) \tag{4.2}$$

D is a distance from source. And Δt is time distance which represents arriving time difference between photon and gravitational wave. one can rewrite the restriction by gravitational wave length of λ_g as

$$\lambda_g > 3 \times 10^{12} km (\frac{D}{200Mpc} \frac{100Hz}{f})^{1/2} (\frac{1}{f \Delta t})^{1/2} \tag{4.3}$$

As one mentioned in Chapter1, gravitational waves propagate 2 degrees of freedom in GR. In Massive Gravity, there are 5 polarizations, and it is the general structure of massive particle include dRGT ghost-free Massive Gravity. It consist of the helicity-2 mode (tensor mode), helicity-1 mode(vector), and helicity-0 mode (scalar). It is easily seen by calculating the FP action using st*ü*kelberg fileds.

The helicity-2 mode are produced in the same way as GR and indistinguishable when travel distances smaller than the graviton Comptom wavelength

($\lambda_g = \frac{h}{mc}$). The helicity-1 mode (vector mode) are expected not to be produced from ordinary source. Because the vector modes don't couple with matter. The helicity-0 mode (scalar mode) travel but be suppressed by the Vainshtein mechanism. To summarize, Massive Gravity has additional 3 polarizations, but additional scalar mode is suppressed compared to the helicity-2 mode, and additional vector mode does not couple to known matters.

Chapter 5

Conclusion and Outlook

I reviewed the possible mass term to add to GR. Finally, I reviewed dRGT ghost-free massive gravity. The scalar graviton is strongly coupled within Vainshtein radius $r_V = (r_g/m^4)^{1/5}$. one cannot take linear approximation for $r \gg r_V$. But for $r \ll r_V$, one can restore GR. It agrees with today's observations when Vainshtein radius is much larger than the edge of Solar system.
dRGT model also allows us to build FLRW homogeneous isotropic solution. Therefore, it can be applied to a clarification of Dark Energy. The value of mass is very small $m < 10^{-33}$ [eV] to detect. But out of the Solar system, dRGT model has different predictions from GR. The massive gravitational waves have extra degrees of freedom such as massive scalar graviton and massive vector graviton, which are difficult to detect.

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