Point-condensation phenomena and saturation effect for the Gierer-Meinhardt system

ギーラー・マインハルト系における点凝集現象と飽和効果 (英文)

Kotaro Morimoto
森本 光太郎
# Contents

1 Background and main results ........................................... 3  
1.1 Reaction-diffusion system .......................................... 3  
1.1.1 Scalar reaction-diffusion equation .............................. 4  
1.1.2 Diffusion-induced instability .................................... 5  
1.2 Gierer-Meinhardt system ............................................ 6  
1.2.1 Shadow system of the Gierer-Meinhardt system without saturation ......................................................... 6  
1.2.2 Main result: instability result for the shadow system ...... 8  
1.2.3 Solutions of the full system without saturation .............. 9  
1.2.4 Saturation effect for the Gierer-Meinhardt system ........... 10  
1.2.5 Main results: weak saturation case ............................. 12  
1.2.6 Other topics for the Gierer-Meinhardt system and biological pattern formation .............................................. 13  
1.3 Schnakenberg model ................................................. 14  
1.3.1 Main result: Schnakenberg model with saturation .......... 15  
1.4 Chemotaxis model .................................................. 17  
1.4.1 Main results: chemotaxis model with saturation .......... 20  
1.5 About this thesis .................................................. 21  
1.6 Figures ................................................................... 25  

2 Analysis for equations in whole space .............................. 27  
2.1 Uniqueness and nondegeneracy ..................................... 27  
2.2 Properties on parameter ............................................. 33  

3 Semilinear Neumann problems with parameter ..................... 40  
3.1 Introduction and main results ..................................... 40  
3.2 Basic analysis ........................................................ 45  
3.3 Basic estimates ....................................................... 56  
3.4 Proof of Theorem 3.1 ............................................... 61  
3.5 Proof of Theorem 3.2 ............................................... 68  
3.6 Remark: nonlocal problems ......................................... 72  
3.7 Appendix ............................................................. 73
Chapter 1

Background and main results

In this thesis, we will study some systems of nonlinear partial differential equations arising as models of a biological pattern formation or a chemical reaction. We are mainly concerned with the Gierer-Meinhardt system which is a model of an activator and an inhibitor in the field of biological pattern formation. The most fundamental problem in morphogenesis is how an inhomogeneous pattern is constructed. A. Gierer and H. Meinhardt (1972) explained the phenomenon by using two biochemical substances an activator and an inhibitor, and they proposed a model equation of the activator and the inhibitor in the form of a reaction-diffusion system. Today, the model equation is called the Gierer-Meinhardt system. In this thesis, we will mainly study the steady-state problem of the Gierer-Meinhardt system. The analytical methods which will be used to construct solutions to the Gierer-Meinhardt system also work for other models so-called the Schnakenberg model and the chemotaxis model. We will investigate the “point-condensation phenomena” and the “saturation effect” for each model. Let us state the background of our study and main results of this thesis (Theorems A-G). We first introduce a reaction-diffusion system.

1.1 Reaction-diffusion system

In general, the n-component reaction-diffusion system is described as follows:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 + f_1(u_1, \cdots, u_n) \text{ in } \Omega, \ t > 0, \\
\frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + f_2(u_1, \cdots, u_n) \text{ in } \Omega, \ t > 0, \\
&\vdots \\
\frac{\partial u_n}{\partial t} &= d_n \Delta u_n + f_n(u_1, \cdots, u_n) \text{ in } \Omega, \ t > 0,
\end{align*}
\]

(1.1)
where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$. $\Delta$ is the Laplace operator in $\mathbb{R}^N$, namely,

$$\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}.$$ 

Each $u_i(x,t)$, $i = 1, \cdots, n$, denotes the concentration or density of a single species in isotropic diffusitive medium at time $t$ and position $x$. $d_i > 0$, $i = 1, \cdots, n$, denotes the diffusion constant of $u_i$. The boundary and initial conditions are

$$\frac{\partial u_i}{\partial \nu} = 0 \text{ on } \partial \Omega,$$

$$u_i(x,0) = a_i(x) \text{ in } \Omega,$$

for $i = 1, \cdots, n$, where $\nu$ is the outer normal vector on the boundary. The condition (1.3) is called a homogeneous Neumann boundary condition or zero flux boundary condition. The each term $f_i(u)$ is called a reaction term of $u_i$. In the fields of chemistry, ecology and biology, a lot of models described in the form (1.1) have been proposed.

### 1.1.1 Scalar reaction-diffusion equation

When $n = 1$ for (1.1), it is particularly called a scalar reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = d\Delta u + f(u) \text{ in } \Omega, \ t > 0,$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,$$

$$u(x,0) = a(x) \text{ in } \Omega.$$ 

Formally, the equation (1.4) consists of the following well-known equations:

$$\frac{\partial u}{\partial t} = d\Delta u,$$ 

$$\frac{\partial u}{\partial \nu} = f(u).$$

(1.6) is a simple diffusion equation (or heat equation). The diffusion equation (1.6) describe the diffusion-phenomena of diffusive substances. If we consider the problem (1.6) under the conditions (1.5), then it is known that

$$\lim_{t \to \infty} u(x,t) = \frac{1}{|\Omega|} \int_{\Omega} a(x) dx.$$ 

That is, by the effect of diffusion, $u(x,t)$ tends to the average of its initial data (see e.g. [99]). The fact is very natural. On the other hand, if we consider $u$ to
be independent of \(x\), namely, \(u(x, t) = u(t)\), then the reaction of \(u\) is described by the kinetic equation (1.7). We note that, if the algebraic equation
\[
f(u) = 0, \quad u \in \mathbb{R},
\]
has a solution \(u_c \in \mathbb{R}\), then it is also a constant solution of (1.4). By the analogy from the heat equation, we can similarly expect that any solution of (1.4) tends to a uniform state, namely, some constant solution \(u_c\). This expectation is true when \(d\) is large enough (see [12]). Moreover, the following facts are known for (1.4)-(1.5):

(i) The \(\omega\)-limit set consists of equilibrium solutions only [32].

(ii) If the domain \(\Omega\) is convex, any nonconstant solutions are unstable, even if they exist [10, 58].

(iii) There are suitable nonconvex domain \(\Omega\) and \(f(u)\) such that there exists stable nonconstant equilibrium solutions [58].

Here, an equilibrium solution (or a stationary solution, a steady-state solution) is a solution which is independent of time \(t\), that is, \(u(x, t) = u(x)\), and it satisfies
\[
0 = d\Delta u + f(u) \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega.
\] (1.9)

The results (i)-(iii) indicate that any bounded solution generically approaches one of the stable constant solutions provided the domain \(\Omega\) is convex. This result is also natural because the diffusion enhances spatial homogeneity. Therefore, it was long believed that a phenomenon which is described by reaction-diffusion systems were not interesting from pattern formation viewpoints. However, the situation turned by Turing’s suggestion.

1.1.2 Diffusion-induced instability

In 1952, A. Turing [102] insisted that diffusion enhances spatial inhomogeneities. His insistence was surprising then because many mathematicians believed that diffusion enhances spatial homogeneities. The key of his insistence is that, for two diffusivity substances, let their diffusion-rate be different widely, then the homogeneous steady-state may become unstable. He indeed demonstrated it by using some linear model. Today, it is called diffusion-induced instability or Turing instability. Although his model is linear, his idea is naturally extended to the nonlinear model as two-component reaction-diffusion systems:

\[
\begin{cases}
\frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + f_1(u_1, u_2) \text{ in } \Omega, \quad t > 0, \\
\frac{\partial u_2}{\partial t} = d_2 \Delta u_1 + f_2(u_1, u_2) \text{ in } \Omega, \quad t > 0,
\end{cases}
\] (1.10)

with zero flux boundary condition. By many mathematicians, for suitable reaction terms \(f_1\) and \(f_2\), the diffusion-induced instability was substantiated in mathematical and numerical way. One of the popular model which realizes the diffusion-induced instability is the Gierer-Meinhardt system.
1.2 Gierer-Meinhardt system

One of the models which realize the Turing instability is the activator-inhibitor model which was proposed by A. Gierer and H. Meinhardt [22]. An activator and an inhibitor are biochemical substances and are supposed to be satisfied the following conditions:

(i) Activator grows in an autocatalytic way.

(ii) Activator produces also inhibitor.

(iii) Inhibitor inhibits activator’s growth.

Under these conditions, Gierer and Meinhardt proposed the following reaction-diffusion system which is called the Gierer-Meinhardt system:

\[
\begin{align*}
\frac{\partial A}{\partial t} &= \epsilon^2 \Delta A - A + \frac{A^p}{(1 + A^q)} + \sigma_0, \quad A > 0 \text{ in } \Omega \times (0, \infty), \\
\tau \frac{\partial H}{\partial t} &= D \Delta H - H + \frac{A^r}{H^s}, \quad H > 0 \text{ in } \Omega \times (0, \infty), \\
\frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} &= 0 \text{ on } \partial \Omega \times (0, \infty), \\
A(x, 0) &= A_0(x), \quad H(x, 0) = H_0(x) \text{ in } \Omega,
\end{align*}
\]

(1.11)

where \(\epsilon > 0, \tau > 0, \kappa \geq 0\). \(\Omega\) is a bounded domain in \(\mathbb{R}^N (N \geq 1)\) with smooth boundary \(\partial \Omega\). \(A = A(x, t)\) and \(H = H(x, t)\) represent the concentrations of the activator and the inhibitor at \(x \in \Omega\) and \(t \in (0, \infty)\), respectively. \(A_0\) and \(H_0\) are their initial data. \(\epsilon\) and \(D\) stand for the diffusion constants of the activator and the inhibitor, respectively. The constant \(\kappa\) stands for the degree of saturation effect. The term \(\sigma_0 = \sigma_0(x)\) is a source term which means a source rate of the activator. The exponents are assumed to be satisfied \(p > 1, q, r > 0, s \geq 0, 0 < (p-1)/q < r/(s+1)\). This assumption ensures the existence of a (unique) constant solution to (1.11). For this system, it is known that the constant solution becomes unstable when \(\epsilon^2 << D\) (diffusion-induced instability) even if it is stable when \(\epsilon^2\) and \(D\) are not so different.

We will see that the Gierer-Meinhardt system has various kinds of stationary solutions and that the stability depends on the parameters. In this thesis, we will address the saturation effect, namely, the parameter \(\kappa\). We first introduce some interesting investigations for the Gierer-Meinhardt system of no saturation case \(\kappa = 0\).

1.2.1 Shadow system of the Gierer-Meinhardt system without saturation

We first introduce a limit system of the Gierer-Meinhardt system which was proposed by Nishiura [80]. By taking the limit \(D \to \infty\) formally after dividing the second equation of (1.11) by \(D\), we have \(\Delta H = 0\) and \(\frac{\partial H}{\partial \nu} = 0\) on \(\partial \Omega\). Then we notice that \(H(x, t)\) is spatial homogeneous, namely, \(H(x, t) = \xi(t)\), and we
obtain a system for $A(x, t)$ and $\xi(t)$ which is called the shadow system of the Gierer-Meinhardt system:

\[
\begin{align*}
\frac{\partial A}{\partial t} &= \varepsilon^2 \Delta A - A + \frac{A^p}{\xi^{(1+p+A)}} + \sigma_0, \quad A > 0 \text{ in } \Omega, \quad t > 0, \\
\tau \frac{\partial \xi}{\partial t} &= -\xi + \frac{1}{|\Omega|^s} \int_\Omega A^r dx, \quad \xi > 0, \quad t > 0, \\
\frac{\partial A}{\partial \nu} &= 0 \text{ on } \partial \Omega, \quad t > 0.
\end{align*}
\]

Let $\kappa = 0$ and $\sigma_0 = 0$. Let us consider stationary solutions of (1.12). If we put

\[ A(x) = \xi^{q/(p-1)} u(x), \]

and substitute this into the steady-state shadow system of (1.12), then we have the following equations for $(u, \xi)$:

\[
\begin{align*}
\varepsilon^2 \Delta u - u + u^p &= 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial \Omega, \\
\xi^q &= \int_\Omega u^r(x) dx,
\end{align*}
\]

where $\gamma := qr/(p-1) - (s+1) > 0$. Therefore, we notice that, if we get a solution to (1.14), then defining $\xi$ by (1.15) we obtain a stationary solution to the shadow system (1.12) in the case $\kappa = 0$ and $\sigma_0 = 0$.

Let us consider the scalar reaction-diffusion equation corresponding to (1.16):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \varepsilon^2 \Delta u - u + u^p \text{ in } \Omega, \quad t > 0, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial \Omega, \quad t > 0, \\
u(x, 0) &= u_0(x) \text{ in } \Omega.
\end{align*}
\]

If we consider the diffusionless equation of (1.16), the ordinary differential equation is given by

\[
\begin{align*}
\frac{du(t)}{dt} &= -u(t) + u^p(t), \quad t > 0, \\
u(0) &= u_0.
\end{align*}
\]

Note that this ODE has two equilibrium solutions $u = 0, 1$, and $u = 0$ is stable, while $u = 1$ is unstable. When the domain $\Omega$ is convex, one easily knows that a stable stationary solution to (1.16) is $u = 0$ only. Moreover, it is known that (1.16) loses all its stationary solutions except the trivial ones $u = 0, 1$ when $\varepsilon$ is large (see [53]).

Let us consider the case where $\varepsilon$ is small. In the case $N = 1$, a lot of works have been done by I. Takagi [101]. In the case $N \geq 2$, the situation becomes more interesting. By W.-M. Ni et al. [53, 75, 76], a nonconstant stationary solution to (1.14) was constructed for sufficiently small $\varepsilon$ by the variational approach due to the Mountain Pass Theorem, namely, so-called a least-energy solution to (1.14), under the restriction:

\[ 1 < p < \infty \text{ if } N = 1, 2, \quad 1 < p < \frac{N+2}{N-2} \text{ if } N \geq 3. \]
Moreover, they showed that the least-energy solution is concentrated at only one point on the boundary. More precisely, for $\varepsilon$ sufficiently small, the least-energy solution attains its maximum only on one point $P_{\varepsilon} \in \partial \Omega$. Moreover, $H(P_{\varepsilon}) \to \max_{P \in \partial \Omega} H(P)$ as $\varepsilon \to 0$ holds, where $H(P)$ is a mean curvature function with respect to the inner normal of the boundary $\partial \Omega$. If $\Omega$ is convex, then $H(P) \geq 0$. By using the least-energy solution to (1.14), we obtain a stationary solution, so-called a least-energy pattern, to the shadow system (1.12) in the case $\kappa = 0, \sigma_0 = 0$. It is known that, for sufficiently small $\tau$, the least-energy pattern is weakly stable if $r = p + 1$ holds, and it is stable if the domain $\Omega$ is an annulus (see [78]). Such a spiky-shaped solution is called a peak solution or a spike-layer solution. Although the least-energy solution to (1.16) was a single peak solution, after their work, multi-peak solutions were constructed by many authors. For boundary peaks see [29, 107, 17, 26, 50, 110], for interior peaks see [27, 108, 41], for mixed boundary and interior peaks see [28], and the references therein. By their works, multi-peak solutions to the shadow system (1.12) in the case $\kappa = 0$ and $\sigma_0 = 0$ were established. However, it seems that multi-peak solutions to the shadow system are unstable. Indeed, Y. Nishiura [81] and W.-M. Ni, P. Poláčik and E. Yanagida [73] proved that, for the one-dimensional shadow systems with $\Omega = (0, 1)$, only monotone stationary solutions could be stable. In the case $N = 2$, Y. Miyamoto [63] showed that, for two-dimensional shadow system, if $\Omega$ is a disc in $\mathbb{R}^2$, then a stationary solution $(A, \xi)$ which has more than two peaks at the boundary is unstable. Moreover, it was proven by F. Li, K. Nakashima and W.-M. Ni [52] that, if $\Omega$ is a convex domain in $\mathbb{R}^N$ with smooth boundary, then a non-constant stationary solution $(A, \xi)$ is unstable for all large $\tau$. In addition, they showed that, if $N \geq 2$ and $\Omega$ is a ball or an annulus, then a non-constant radially symmetric stationary solution $(A, \xi)$ is unstable. Recently, K. Ikeda, S. Ei and E. Yanagida proved that multi-peak solutions to the shadow system are always unstable for general domains. Namely, stationary peak solutions which has $k$-peaks ($k \geq 2$) on $\mathbb{R}^N$ are always unstable. In W.-M. Ni’s survey paper [71], one can find some results on the qualitative properties for some elliptic problems such as (1.16).

1.2.2 Main result: instability result for the shadow system

One of main results in this thesis is on the instability for some solutions to the shadow system (1.12) on cylindric domains. Let $\Omega$ be a cylindric domain in $\mathbb{R}^N$ such that $\Omega = G \times (-T, T)$ where $G$ is a smooth bounded domain in $\mathbb{R}^{N-1}$ and $T \in (0, \infty)$. Then we have the following result:

**Theorem A.** Suppose that $p = r - 1$ and $\kappa = \sigma_0 = 0$ in (1.12). Let $(A(x), \xi)$ be a stationary solution to (1.12). If $A(x)$ is symmetric with respect to the $x_N$-direction, i.e., $A(x', x_N) = A(x', -x_N)$ for $(x', x_N) \in \Omega$, and $\frac{\partial A}{\partial x_N} \neq 0$, then $(A(x), \xi)$ is unstable for all $\tau > 0$.

Actually, we can show the result above for more general shadow system. For more precise statement, see Chapter 9.
1.2.3 Solutions of the full system without saturation

As we observed, a lot of stationary solutions to the shadow system have been constructed. However, to construct stationary solutions to the full system (1.11) is more difficult than the shadow system. We mention in the case $\kappa = 0$.

(Solutions near the shadow system with $\kappa = 0$) In some cases, solutions to the full system can be constructed near the one to the shadow system for sufficiently small $\varepsilon$ and sufficiently large $D$. In the case $N = 1$, I. Takagi [101] showed the existence of a boundary-peak solution. However, we note that, if single boundary-peak solution is obtained, then we also obtain multi-peak solutions by reflection in the case $N = 1$. In the case $N = 2, 3$, M. A. del Pino et al. [18] showed the existence of a single boundary-peak solution when $(p, q, r, s) = (2, 1, 2, 0)$. In the case $N \geq 2$ and $r = p + 1$, Y. Miyamoto [62] showed the existence of single boundary-peak solution, and it is stable provided $\tau > 0$ is small enough. In the case where $N \geq 2$ and $\Omega$ is axially symmetric, boundary multi-peak solutions were constructed by W.-M. Ni and I. Takagi [77]. From the bifurcation theory viewpoint, if $N = 1$, more precise information is known. The bifurcating solution emanating from a uniform state was studied in [100, 101, 24] (see also [59]). In particular, in [101], it was shown that, for sufficiently large $D$, whenever bifurcation from the constant solution occurs, there exists a continuum of nonconstant solutions which connects the peak solutions with the bifurcating solutions.

(Strong coupling case with $\kappa = 0$) The case where $D > 0$ is finite and not so large is called a strong coupling case. In that case, J. Wei and M. Winter [111] showed the existence of interior multi-peak solutions provided $\varepsilon$ is small enough, and their stability were studied in [113, 112]. In the case $N = 1$, symmetry and asymmetry multi-peak solutions were constructed and their stability were studied in [115]. In the case where $D$ is not so large, then an interesting phenomena emerges. Let $N = 1$. By using matched asymptotic analysis, the following stability result was given by formal argument in [35].

(Stability of symmetric $k$-peak solutions (formal result)) For some $\tau_0 > 0$, there exists a sequence of positive numbers

$$D_1 > D_2 > \ldots > D_k > \ldots$$

such that, for $\varepsilon << 1$, the symmetric $k$-interior peak stationary solution is stable provided $D < D_k$, while it is unstable provided $D > D_k$.

We emphasize that the shadow system which is limit system of the Gierer-Meinhardt system of $D \to \infty$ has not any stable multi-peak stationary solutions, however, when $D$ becomes finite and smaller, multi-peak stationary solutions get back their stability (see Figure 1.5 in Section 1.6). When $D = D_k$, it was suggested that asymmetric $k$-peak stationary solutions appear by formal approach [105]. Although they are formal approach, a rigorous approach was done in [115]. The existence of $k$-peak stationary solutions (symmetric or asymmetric) was established by Liapunov-Schmidt reduction. The stability of symmetric $k$-peak stationary solutions was given rigorously as follows.
(Stability of symmetric $k$-peak solutions (rigorous result))

Let $k \in \mathbb{N}$. Assume that $\varepsilon \ll 1$ and $0 < \tau < \tau_0$ for some small $\tau_0$ and that

$$r = 2, \ 1 < p < 5 \ \text{or} \ r = p + 1, \ 1 < p < \infty.$$

Then there exists $D_k > 0$ (which can be written explicitly) such that the symmetric $k$-interior peak solution is linearly stable provided $D < D_k$, while it is linearly unstable for all $\tau > 0$ provided $D > D_k$.

Recently, J. Wei’s survey paper [109] was published. One can find many interesting results for the Gierer-Meinhardt system.

1.2.4 Saturation effect for the Gierer-Meinhardt system

Hitherto we considered the case $\kappa = 0$. Let us consider the case $\kappa > 0$. Then the nonlinearity changes into the one like a bistable. We remark that the Gierer-Meinhardt (1.11) system possesses exactly one positive constant solution. Let us rewrite the system (1.11) simply as follows:

$$
\begin{align*}
\frac{\partial A}{\partial t} &= \varepsilon^2 \Delta A + f(A, H), \ A > 0 \ \text{in} \ \Omega \times (0, \infty), \\
\tau \frac{\partial H}{\partial t} &= D \Delta H + g(A, H), \ H > 0 \ \text{in} \ \Omega \times (0, \infty), \\
\frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} &= 0 \ \text{on} \ \partial \Omega \times (0, \infty).
\end{align*}
$$

Here, $f(A, H) = -A + A^p / (H^q(1 + \kappa A^p)) + \sigma_0$ and $g(A, H) = -H + A^r / H^s$. Let $\kappa > 0$ and $\sigma_0 \geq 0$ be small constants. Then we notice that $0 = f(A, H)$ possesses three roots denoted by $A = h_-(H), h_0(H), h_+(H)$ ($h_-(H) < h_0(H) < h_+(H)$) for each $H \in I$ (see Figure 1.1), where $I$ is a suitable open interval.

![Graph of $f(A, H)$ with $\kappa = 0.2$, $\sigma_0 = 0.1$, $H = 0.95$.](image.png)

For each $H \in I$, let us consider a scalar reaction-diffusion equation:

$$
\frac{\partial A}{\partial t} = \varepsilon^2 \Delta A + f(A, H) \ \text{in} \ \Omega, \ t > 0,
$$

(1.20)
under the homogeneous Neumann boundary condition. If $\Omega$ is a convex domain, then one easily knows that stable equilibrium solutions are $A = h_-(H)$ and $A = h_+(H)$ only, while the rest $A = h_0(H)$ is unstable. Therefore, we call (1.20) a scalar bistable reaction-diffusion equation. Set

\[ J(H) = \int_{h_-(H)}^{h_+(H)} f(s, H) ds, \quad H \in I. \]

Then we can see that there exists a unique $H^* \in I$ such that $J(H^*) = 0$, and $J'(H^*) \neq 0$. In the case $H = H^*$, two stable solutions $h_-(H)$ and $h_+(H)$ are equi-stable. Therefore, one can expect an internal layer equilibrium solution to (1.20). Therefore, one can also expect an internal layer solution to (1.19). Indeed, when $N = 1$, for sufficiently large $D > 0$ and sufficiently small $\varepsilon$, M. Mimura et al. [61] showed that there exists internal multi-layer stationary solutions to (1.19) by using the singular perturbation method for more general nonlinear terms (see Figure 1.2). The stability was shown by Y. Nishiura and H. Fujii [82]. See also [88]. When also $N \geq 2$ and $\Omega$ is a ball in $\mathbb{R}^N$, the internal layer stationary solution exists (see [16] and [89]). From the bifurcation theory viewpoint, in [80], it is known that in the case $N = 1$, for sufficiently large $D$, the bifurcating branch emanating from a constant solution to exist until it is connected to the internal layer solutions.

![Figure 1.2: Internal layer. $\varepsilon^2 = 5 \times 10^{-6}$, $D = 1.0$, $\kappa = 0.5$, $\tau = 0.1$. $(p, q, r, s) = (2, 1, 2, 0)$](image)

For large amplitude stationary solutions, it seems that $\kappa \geq 0$ must be small. M. del Pino [15] showed the a priori estimate for a stationary solutions to (1.19) with $(p, q, r, s) = (2, 1, 2, 0)$, $\kappa, \sigma_0 \in \mathbb{R}_+$, such that

\[ \sigma_0 \leq A \leq \frac{1}{\kappa \sigma_0^2} + \sigma_0, \quad \sigma_0^2 \leq H \leq \left( \frac{1}{\kappa \sigma_0^2} + \sigma_0 \right)^2. \]  

In general, if a stationary peak solution $(A(x), H(x))$ to the Gierer-Meinhardt system exists, then the maximum value of $A(x)$ on $\Omega$ becomes large in the order of $\varepsilon^{-N}$ as $\varepsilon \to 0$. Hence, peak solutions has a large amplitude for sufficiently small $\varepsilon$. However, from the estimate (1.21), we see that such a large amplitude stationary solutions does not exist provided $\kappa \sigma_0^2$ is large.
1.2.5 Main results: weak saturation case

As we observed, stable peak solutions appear when $\kappa = 0$, and stable internal layer solutions appear when $\kappa > 0$. Now, does a peak stationary solution with large amplitude exist even if $\kappa$ is positive? One of the answers is given under the weak saturation effect. For the shadow system of the Gierer-Meinhardt system with $(p, q, r, s) = (2, 1, 2, 0)$, $\kappa > 0$, $\sigma_0 = 0$,

\[
\begin{aligned}
\frac{\partial A}{\partial t} &= \varepsilon^2 \Delta A - A + \frac{A^2}{\varepsilon^{2(N+\kappa)-2}} \text{ in } \Omega \times (0, \infty), \\
\tau \frac{\partial \xi}{\partial t} &= -\xi + \frac{1}{|\Omega|} \int_{\partial \Omega} A^2 \, dx \text{ in } (0, \infty), \\
\frac{\partial A}{\partial \nu} &= 0 \text{ on } \partial \Omega \times (0, \infty),
\end{aligned}
\] 

(1.22)

J. Wei and M. Winter subjected the following condition:

(A.I) (weak saturation condition) $\kappa > 0$ depends on $\varepsilon$ and there exists a limit $\lim_{\varepsilon \to 0} \varepsilon^{-2N} \kappa = \kappa_0 \in [0, \infty)$.

Under this condition, they showed the existence of the boundary single-peak stationary solution for sufficiently small $\varepsilon$. When $N = 1$, the single-peak solution is stable provided $\tau$ is small. When $N = 2, 3$, it is stable provided $\kappa_0$ and $\tau$ are small and the peak point is a nondegenerate local maximum point of the mean curvature function of the boundary. Although their result was the one for the shadow system, they give one of sufficient conditions to exist a peak solution in the case $\kappa > 0$.

Inspired from their work, we obtain a result on the existence of multi-peak stationary solutions to the full system (1.19) with $\kappa > 0$ and $\sigma_0 = 0$ for sufficiently small $\varepsilon$ and large $D$ when the domain $\Omega$ is axially symmetric. Moreover, we obtain a similar result for $\sigma_0 \neq 0$ in the case $(p, q, r, s) = (2, 1, 2, 0)$.

**Theorem B.** Let $\sigma_0 = 0$. We assume (1.18). Suppose the weak saturation condition (A.I), and let $\kappa_0$ be small enough. Let $\Omega$ be an axially-symmetric domain in $\mathbb{R}^N$ with respect to the $x_N$-axis. Let $P_1, \cdots, P_{2n}$ be the intersections of $\partial \Omega$ and the $x_N$-axis. We choose points $P_{j_1}, \cdots, P_{j_m}$ from $P_1, \cdots, P_{2n}$ arbitrarily. Then, for sufficiently small $\varepsilon$ and large $D$, there exist a boundary multi-peak stationary solution to (1.19), which concentrates at $P_{j_1}, \cdots, P_{j_m}$.

**Theorem C.** Let $\sigma_0(x)$ be a nonnegative axially symmetric function on $\overline{\Omega}$ of class $C^\alpha(\Omega)$, $\alpha \in (0, 1)$. Let $(p, q, r, s) = (2, 1, 2, 0)$ and $2 \leq N \leq 5$. Suppose the same assumptions on $\kappa$ and $\Omega$, and let $P_{j_1}, \cdots, P_{j_m}$ be points, as in Theorem B. Then, for sufficiently small $\varepsilon$ and large $D$, there exist a boundary multi-peak stationary solution to (1.19), which concentrates at $P_{j_1}, \cdots, P_{j_m}$.

The results above are on the stationary solutions to (1.19) near the shadow system. Next, we consider the strong coupling case. In the case, we can show that multi-peak stationary solutions exist under the same weak saturation condition when $N = 1$, and we obtain the following theorem. Here, we remark that the construction of solutions to the Gierer-Meinhardt system is more difficult in the strong coupling case than in the shadow system case.
Theorem D. Let $N = 1$, $\Omega = (-1, 1)$ and $(p, q, r, s) = (2, 1, 2, 0)$. We assume that $\sigma_0$ is a nonnegative constant, and assume (A.1) for sufficiently small $\kappa_0$. Let $D > 0$ be given arbitrarily. Then, for sufficiently small $\varepsilon > 0$, there exists a peak stationary solution to (1.19) which concentrates at the origin.

We note that, under the same situation as that in Theorem D, the existence of multi-peak solutions are ensured by reflections. See Figure 1.6 in the last section of this chapter, which is a numerical simulation of the one-dimensional Gierer-Meinhardt system. We see that multi-peak patterns appear if $D$ is not so large.

1.2.6 Other topics for the Gierer-Meinhardt system and biological pattern formation

Other types of solutions are also studied. When $\Omega$ is a ball in $\mathbb{R}^N$ ($N \geq 2$), it is known that (1.11) ($\kappa = 0$, $\sigma_0 = 0$) has a stationary solution which concentrate on a ($N - 1$)-dimensional sphere for finite $D$ and sufficiently small $\varepsilon$ [79]. The Hopf bifurcations and Oscillatory instability of peak solutions for one-dimensional Gierer-meinhardt system were studied in [106]. Some interesting numerical simulations can be found in [86].

There is a characteristic solution, so-called a stripe pattern. A stripe pattern means that activator’s concentration localizes along the mid-line of the rectangular domain. K. Ikeda [34] rigorously proved that a stripe pattern is unstable when the saturation effect is neglected. On the other hand, a lot of interesting numerical simulations were done by T. Kolokolnikov et al. [44]. They showed some patterns for the Gierer-Meinhardt system in a rectangular domain, stripes, wriggled stripes, self-replication of spots and so on. And they studied the stability and instability of the stripe pattern. In the process of collapse of a stripe pattern, several instabilities has been found, so-called, zigzag instability, breakup instability. However, it seems that a stripe pattern becomes stable provided the saturation effect is enough. In fact, they showed a numerical simulation whereby a single stripe splits into two, with the two stripes undergoing a further splitting at later times, however, the stripe-shape does not collapse.

A Turing model has been suggested to explain the development of pigmentation patterns on certain species of growing angle-fish such as Pomacanthus semicirculans where colored stripes are observed which change their number, size and orientation (see [45]). After this model was refined, adding effects such as cell growth and movement, also stripes of various thickness could be explained (see [85]). For reaction-diffusion systems on growing domains, which is a good model for the growth of organisms, see [13, 14, 55, 56].

From mathematical viewpoints, the uniqueness and the global existence of solutions to the initial-boundary value problem (1.11) are also interesting problems. For this direction, see [60, 57, 87, 96, 49, 37].

When $\kappa = 0$, the behavior of solutions to the following diffusionless system
was studied completely in [74]:

$$\begin{align*}
\frac{dA}{dt} &= -A + \frac{Ap}{Hq}, \quad t > 0, \\
\tau \frac{dH}{dt} &= -H + \frac{Ar}{Hs}, \quad t > 0, \\
A(0) &= A_0, \quad H(0) = H_0.
\end{align*}$$

(1.23)

Moreover, it was shown that (1.23) has a finite-time blow-up solution under some suitable condition.

### 1.3 Schnakenberg model

The Schnakenberg model [92] is a model equation of some chemical reaction on an activator and a substrate. The system is written as follows:

$$\begin{align*}
\frac{\partial a}{\partial t} &= \varepsilon^2 \Delta a - a + ha^2, \quad a > 0 \text{ in } \Omega \times (0, \infty), \\
\tau \frac{\partial h}{\partial t} &= D \Delta h - ha^2 + \rho, \quad h > 0 \text{ in } \Omega \times (0, \infty), \\
\frac{\partial a}{\partial \nu} = \frac{\partial h}{\partial \nu} &= 0 \text{ on } \partial \Omega \times (0, \infty), \\
a(x, 0) &= a_0(x), \quad h(x, 0) = h_0(x) \text{ in } \Omega,
\end{align*}$$

(1.24)

where $\varepsilon, D > 0$, $\tau > 0$, $k \geq 0$. $a = a(x,t)$ and $h = h(x,t)$ represent the concentrations of the activator and the substrate at $x \in \Omega$ and $t \in (0, \infty)$, respectively. $a_0$ and $h_0$ are their initial data. $\rho = \rho(x)$ represents the feed-rate of the substrate at $x \in \Omega$. Let $A$ and $H$ be diffusive substances. Let $P$ be a product. We consider the following chemical reaction processes:

$$2A + H \rightarrow 3A,$$

$$A \rightarrow P.$$

Note that the product $P$ is independent of the reaction between $A$ and $H$.

The Schnakenberg model describe these reaction processes. We note that the Gierer-Meinhardt system is a model of the activator and the inhibitor, and the inhibitor inhibits the reaction of the activator. Conversely, for the Schnakenberg model, the activator ($A$) is provided by substrate ($H$), and the substrate decays. Therefore, the feed of the substrate from outside is needed, and hence the feed rate $\rho$ must be positive.

It is known that the Schnakenberg model also realizes the Turing instability. Namely, the spatial homogeneous state becomes unstable when $\varepsilon^2 << D$.

Similarly to the Gierer-Meinhard system, it is known that (1.24) admits multi-peak stationary solutions. The existence and the stability of interior multi-peak solutions have been studied in [36, 104] in the case $N = 1$ (see Figure 1.3). In the case $N = 2$, interior multi-peak stationary solutions were constructed in [116]. Moreover, the Schnakenberg model has been widely studied by analytical and numerical methods. We refer to [4] and the references therein. The Schnakenberg model is a simple model of a chemical reaction. However, many
patterns observed experimentally can be computed, such as multi-spot forming hexagonal arrays, stripes and wiggled stripes (see [19]).

As in the case of the Gierer-Meinhardt system, we can consider the shadow system (which is a limit system of $D \to \infty$) also for the Schnakenberg model, and the same analysis can be done. The steady-state shadow system can be written as follows:

$$
\begin{align*}
0 &= \varepsilon^2 \Delta a - a + \xi a^2 \text{ in } \Omega, \\
0 &= \int_{\Omega} (-\xi a^2 + \rho) \, dx, \\
\frac{\partial a}{\partial \nu} &= 0 \text{ on } \partial \Omega.
\end{align*}
$$

By putting $a(x) = \xi^{-1} u(x)$, the first equation of (1.25) becomes

$$
0 = \varepsilon^2 \Delta u - u + u^2 \text{ in } \Omega.
$$

Hence, under the homogeneous Neumann boundary condition, we can show the multi-peak stationary solutions to (1.25) by the same consideration as in the case of the Gierer-Meinhardt system.

![Figure 1.3: Peak solution. $\varepsilon^2 = 0.0001$, $D = 0.01$, $\rho = 0.1$, $\tau = 0.01$.](image)

### 1.3.1 Main result: Schnakenberg model with saturation

Now, we note that the Schnakenberg model can be regarded as a reaction-diffusion system of the resource-consumer type. A resource-consumer reaction-diffusion system is described as follows:

$$
\begin{align*}
\frac{\partial u}{\partial t} &= d_u \Delta u + \omega f(u, v) - g_1(u) \text{ in } \Omega \times (0, \infty), \\
\frac{\partial v}{\partial t} &= d_v \Delta v - f(u, v) + g_2(v) \text{ in } \Omega \times (0, \infty),
\end{align*}
$$

under suitable boundary condition, where $d_u, d_v > 0$. $u$ represents a concentration of the resource, and $v$ represents a density of the consumer. The term $f(u, v)$ means an interaction between the consumption and the production. The constant $\omega$ means the conversion-rate from consumption to production. $g_1(u)$ is a dissipation term, and $g_2(v)$ is a source term. From this viewpoint, it is not
unnatural to consider the saturation effect to the Schnakenberg model which is written as follows:
\[
\begin{align*}
\frac{\partial a}{\partial t} &= \varepsilon^2 \Delta a - a + \frac{a^2}{1 + ka^2}, \quad a > 0 \text{ in } \Omega \times (0, \infty), \\
\tau \frac{\partial h}{\partial t} &= D \Delta h - h + \frac{a^2}{1 + ka^2} + \rho, \quad h > 0 \text{ in } \Omega \times (0, \infty), \\
\frac{\partial a}{\partial \nu} = \frac{\partial h}{\partial \nu} &= 0 \text{ on } \partial \Omega \times (0, \infty), \\
a(x, 0) &= a_0(x), \quad h(x, 0) = h_0(x) \text{ in } \Omega,
\end{align*}
\]
(1.28)

where $k > 0$. Because, the Schnakenberg model is originally a model of chemical reactions, such a model (1.28) with saturating growth has not be considered. However, the author thinks it is interesting to consider the problem with saturation as in the case of the Gierer-Meinhardt system. Actually, we can find that stationary internal layer solutions exists (see Figure 1.4). Indeed, by some transformation, the results in [61, 16, 89], which showed the existence of the internal layer solutions, can be applied (for the detail, see Section 5.3).

![Figure 1.4: Internal layer. $\varepsilon^2 = 10^{-6}$, $D = 0.05$, $\rho = 0.1$, $\kappa = 1.0$, $\tau = 0.01$.](image)

Similarly to the Gierer-Meinhardt system, under the weak saturation condition, we can construct multi-peak stationary solutions to (1.28) with $k > 0$ for sufficiently small $\varepsilon$ and large $D$ in the case where $\Omega$ is an axially symmetric domain. The weak saturation condition for (1.28) is given as follows:

**A.II** $k$ and $\rho$ depend on $\varepsilon$ and there exists a limit
\[
\lim_{\varepsilon \to 0} \varepsilon^{-2N} k \left( \int_\Omega \rho(x) \, dx \right)^2 = k_0 \in [0, \infty).
\]
In this case, the balance of $k$ and $\rho$ is important. We take $P_{j_1}, \ldots, P_{j_m}$ as in Theorem B. Then we can construct a multi-peak stationary solution to (1.28), and we obtain the following theorem:

**Theorem E.** Let $2 \leq N \leq 5$, $\Omega$ be an axially symmetric domain, and $\rho$ be an axially symmetric function of class $C^\alpha(\Omega)$, $\alpha \in (0, 1)$, such that $\max_{\Omega} \rho(x) > 0$. Suppose the condition (A.II) for sufficiently small $k_0$. Then, for sufficiently small $\varepsilon$ and large $D$, there exists a multi-peak stationary solution to (1.28) which concentrates at $P_{j_1}, \ldots, P_{j_m}$. 

16
1.4 Chemotaxis model

Chemotaxis. In the movement of biological individuals, microorganisms or cells sometimes move by responding to some chemical substance. This property is called a chemotaxis, and such a chemical substance is called a chemotactic substance. For example, Dictyostelium discoideum (D. discoideum) is a kind of microorganisms which show chemotaxis. D. discoideum is a single-celled organism like an amoeba. D. discoideum moves at random and propagates when the feed of bacterium is enough. However, if the feed is not enough, then they secrete chemotactic substances each other, and move by recognizing the gradient of density, and form cellular aggregates.

Keller and Segel [39] proposed a mathematical model of chemotaxis. A simple form of the model can be described as follows:

\[
\begin{align*}
\frac{\partial P}{\partial t} &= d_1 \Delta P - r \chi(W) \cdot (P \nabla \chi(W)) \quad \text{in } \Omega \times (0,\infty), \\
\frac{\partial W}{\partial t} &= d_2 \Delta + F(P, W) \quad \text{in } \Omega \times (0,\infty), \\
\frac{\partial P}{\partial \nu} &= \frac{\partial W}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0,\infty), \\
P(x, 0) &= P_0(x), \ W(x, 0) = W_0, \ x \in \Omega,
\end{align*}
\]

(1.29)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary. \( P = P(x, t) \) is the population density of individuals and \( W = W(x, t) \) is the concentration of chemotactic substance. The constants \( d_1, d_2 > 0 \) are the diffusion constants of \( P \) and \( W \), respectively. \( \chi(W) \) is called a sensitivity function of chemotaxis, and \( r \chi(W) \) is the velocity of the direct movement of \( P \) due to chemotaxis. The function \( \chi(W) \) is generally assumed to be satisfied \( \chi(W) \geq 0 \) for \( W > 0 \). For example,

\[ \chi(W) = p \log(W), \quad pW, \quad \frac{pW}{1+W}, \]

where \( p > 0 \) is a constant. And \( -\nabla \cdot (P \nabla \chi(W)) \) stands for the movement of individuals. \( F(P, W) \) is the reaction term leading to production and degradation of \( W \). The simplest form is \( F(P, W) = -b_1 W + b_2 P \) with positive constants \( b_1 \) and \( b_2 \).

Steady-state solutions. Under the concept of chemotaxis-induced instability, R. Schaaf [91] showed that there exist stable nonconstant equilibrium solutions, which indicate chemotactic aggregation of individuals. C.-S. Lin, W.-M. Ni and I. Takagi [53] treated the case \( \chi(W) = p \log(W) \) which had not been treated in [91], and they showed the existence of a large amplitude nonconstant stationary solution by using the Mountain Pass Theorem. More precisely, they showed the following:

(Result in [53]) Let \( F(P, W) = -b_1 W + b_2 P \) and \( \chi(W) = p \log(W), \ p > 0 \). Suppose that, \( p > d_1 \) if \( N = 1, 2, \ 1 < p/d_1 < (N + 2)/(N - 2) \) if \( N \geq 3 \). Then there exists a large amplitude nonconstant stationary solution provided \( d_2/b_1 \) is small enough.

After that, W.-M. Ni and I. Takagi [75] studied the shape of the least-energy
solution (which was given by the Mountain Pass Theorem), and showed the following:

(Result in [75]) If $d_2/b_1$ is small enough, then the least-energy solution attains its maximum at exactly one point on the boundary $\partial \Omega$ (the point tends to the maximum point of the mean curvature function of the boundary), and the solution tends to 0 at the interior of the domain $\Omega$ as $d_2/b_1 \to 0$.

Here, we recall that the works [53, 75] contributed also in the field of the Gierer-Meinhardt system as a least-energy pattern. In the case $N \geq 3$ and $p/d_1 = (N+2)/(N-2)$, see [1, 9]. They assumed that $\Omega$ is an open ball, and studied the existence of radially symmetric stationary solutions. For general domain, the similar results as that in the case $p/d_1 < (N+2)/(N-2)$ were given in [72] with respect to the least-energy solution.

**From the viewpoint of equation of evolution.** In this viewpoint, the blow-up phenomenon is widely studied. Here, we say that $u(x,t)$ blows up in finite time if there exists $T \in (0, \infty)$ such that

$$\limsup_{t \to T} \max_{x \in \Omega} u(x,t) = \infty,$$

and $T$ is called a blow-up time. Additionally, we say that $x_0 \in \overline{\Omega}$ is a blow-up point if there exists $\{(x_n, t_n)\}_{n=1}^{\infty} \subset \overline{\Omega} \times (0, T)$ such that $x_n \to x_0$, $t_n \to T$, $u(x_n, t_n) \to \infty$ as $n \to \infty$. Let $\chi(W) = pW$ and $F(P, W) = -b_1 W + b_2 P$. V. Nanjundiah [70] conjectured that there exists a blow-up solution in finite time in the case $N = 2$. After that, S. Childress and J. K. Percus [11] conjectured the following:

1. In the case $N = 1$, blow-up solutions does not exists.

2. In the case $N = 2$, there exists a certain $c > 0$ such that,
   - if $\int_{\Omega} P_0(x)dx < c$, then the blow-up does not arise,
   - if $\int_{\Omega} P_0(x)dx > c$, then the blow-up in finite time may arise, and the blow-up solution tends to some delta function.

3. In the case $N \geq 3$, independently of $\int_{\Omega} P_0(x)dx$, the blow-up in finite time may arise.

It is known that the conjecture above is correct when $N = 1$. In the case $N = 2$, it is known that there exists a finite time blow-up solution provided $\Omega$ is a disk, and the blow-up point is the origin, and the blow-up solution is radially symmetric (see [30, 31]). Inversely, when $\Omega$ is a disk in $\mathbb{R}^2$, if there exists a finite time blow-up solution, then the blow-up point must be the origin only (see [68]).

**Remark 1.1.** Strictly speaking, the results in [30, 31, 68] above are the ones on the dimensionless system. The dimensionless system is given as follows:

In (1.29) with $\chi(W) = pW$ ($p > 0$) and $F(P, W) = -b_1 W + b_2 P$ ($b_1, b_2 > 0$),
transforming $d_1 t \mapsto t$, and putting
\[
a = \frac{p}{d_1}, \quad \tau = \frac{d_1}{d_2}, \quad \gamma = \frac{b_1}{d_2}, \quad \alpha = \frac{b_2}{d_2},
\]
we have the following dimensionless system:
\[
\begin{aligned}
\frac{\partial P}{\partial t} &= \Delta P - a \nabla \cdot (P \nabla W) \text{ in } \Omega \times (0, \infty), \\
\frac{\partial W}{\partial t} &= \Delta W - \gamma W + \alpha P \text{ in } \Omega \times (0, \infty), \\
\frac{\partial P}{\partial \nu} = \frac{\partial W}{\partial \nu} &= 0 \text{ on } \partial \Omega \times (0, \infty).
\end{aligned}
\] (1.30)

This system can be treated more easily than original one.

Let $N = 2$. The unique existence of nonnegative local solution to (1.30) for the initial data $(P_0, W_0)$ was given in [117]. For nonnegative classical solution $(P(x,t), W(x,t))$ to (1.30), the following properties are known (see [117, 69]):

1. Let $T_{max}$ be a maximum existence time for $(P, W)$. If $P_0 \geq 0, W_0 \geq 0$ and $P_0 \neq 0$, then
   \[P(x,t), \ W(x,t) > 0, \ x \in \Omega, \ 0 < t < T_{max}.\]

2. If $T_{max} < \infty$, then
   \[\limsup_{t \to T_{max}} \int_{\Omega} P(x,t) = \infty, \quad \limsup_{t \to T_{max}} \int_{\Omega} W(x,t) = \infty.\]

3. If $\Omega$ is a disk in $\mathbb{R}^N$, then $P$ and $W$ are radially symmetric provided $P_0$ and $W_0$ are radially symmetric.

The following results on the global existence of a solution were known (see [7, 21, 69]). Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$. For nonnegative initial data $(P_0, W_0)$, the following hold:

1. If $\int_{\Omega} P_0(x) dx < 4\pi/(a\alpha)$, then the nonnegative global solution to (1.30) exists boundedly.

2. In particular, let $\Omega$ be a disk and $P_0, W_0$ be radially symmetric. If $\int_{\Omega} P_0(x) dx < 8\pi/(a\alpha)$, then the nonnegative global solution exists boundedly.

As we observed above, the chemotaxis model gave many interesting problems (existence of stationary spiky patterns, finite time blow-up solutions) to mathematicians. For other related results and surveys, see [67, 93, 8, 94, 103, 54, 83, 95, 98, 97]. The works on the chemotaxis model which stem from Keller and Segel are organized in [33].
1.4.1 Main results: chemotaxis model with saturation

Recently, the chemotaxis model with saturating growth has been studied, namely, such a model where the reaction term $F(W, P)$ in (1.29) includes a saturation effect. In [83], the following case was treated

$$F(P, W) = PW - \mu W + \gamma W, \gamma > 0,$$

where $\mu > 0, \gamma > 0$. In [98], the case

$$F(P, W) = PW - \frac{\mu W}{1 + \gamma W}, \gamma > 0,$$

was treated, and the existence and the stability of the boundary single peak stationary solution were studied. Inspired from their works, we treat two cases as $F(P, W)$:

$$F_1(P, W) = -W + \frac{PW^q}{\alpha + \gamma W^q}, \quad (\text{case A})$$

$$F_2(P, W) = -W + \frac{P}{1 + kP}, \quad (\text{case B})$$

where $q > 0, \alpha, \gamma, k \geq 0$, and we assume $\chi(W) = p \log(W)$ and $1 < p < \infty$ if $N = 1, 2; 1 < p < \frac{N + 2}{N - 2}$ if $N \geq 3$.

Let $d_1 = 1, d_2 = \varepsilon^2$ in (1.29). Then the system can be rewritten as follows:

$$\begin{align*}
\frac{\partial P}{\partial t} &= \nabla \cdot (P \nabla \left( \log \frac{P}{\chi(W)} \right)), \quad (x, t) \in \Omega \times (0, \infty), \\
\frac{\partial W}{\partial t} &= \varepsilon^2 \Delta W + F(P, W), \quad (x, t) \in \Omega \times (0, \infty), \\
\frac{\partial P}{\partial \nu} = \frac{\partial W}{\partial \nu} &= 0, \quad (x, t) \in \partial\Omega \times (0, \infty),
\end{align*}$$

(1.31)

with $F(P, W)$ being $F_1$ or $F_2$. We assume that $\Omega$ is axially symmetric bounded domain in $\mathbb{R}^N$ with smooth boundary. Let $P_{j_1}, \ldots, P_{j_m}$ be points as in Theorems B. With respect to the saturation parameters, we assume the following:

(A.III) In the case A, $\alpha$ and $\gamma$ depend on $\varepsilon$ and there exists a limit

$$\lim_{\varepsilon \to 0} \varepsilon^N (\alpha^N \gamma^{N-1})^{1/q} = \alpha_0 \in [0, \infty).$$

(A.IV) In the case B, $k$ depends on $\varepsilon$ and there exists a limit

$$\lim_{\varepsilon \to 0} \varepsilon^{-N} k = k_0 \in [0, \infty).$$

Under these situations, we obtain the following results:

**Theorem F. (case A)** Suppose (A.III). For some $\alpha_1 \in (0, \infty]$, if $0 \leq \alpha_0 < \alpha_1$, then, for sufficiently small $\varepsilon$, there exists a multi-peak stationary solution to (1.31) which concentrates at $P_{j_1}, \ldots, P_{j_m}$. 20
It is a delicate problem whether the value $\alpha_1$ above can be taken to be infinite or finite. For the detail, see Chapter 4.

**Theorem G. (case B)** Suppose (A.IV). For each $k_0 \in [0, \infty)$, if $\varepsilon$ is sufficiently small, then there exists multi-peak stationary solution to (1.31) which concentrates at $P_{j_1}, \ldots, P_{j_m}$.

### 1.5 About this thesis

In later chapters, we will study several partial differential systems stated before, the Gierer-Meinhardt system, the Schnakenberg model, the chemotaxis model. For these models, we are mainly concerned with steady-state patterns, and we will study the existence of stationary peak solutions under the saturation effect. The existence of stationary peak solutions represents the point-condensation phenomena. We will study the relations between the point-condensation phenomena and the saturation effect.

**Motivation for our study**

In the first place, we will study the following equation in whole space:

\[
\begin{align*}
\Delta w - w + f_\delta(w) &= 0, \quad w > 0 \text{ in } \mathbb{R}^N, \\
\max_{\mathbb{R}^N} w &= w(0), \quad w(z) \to 0 \text{ as } |z| \to \infty,
\end{align*}
\]

where $f_\delta(w)$ is a nonlinear term of $w$ with a parameter $\delta$. For example, $f_\delta(w) = w^p/(1 + \delta w^p)$, $p > 1$. In the case where $f_\delta(w)$ does not possess a parameter, for example $f_\delta(w) = w^p$, the equation (1.32) has been studied widely. However, we will need to treat the problem (1.32) including a parameter. Let us show why we must treat such a problem, and why the solution to (1.32) will be needed for our analysis. We explain it using the Gierer-Meinhardt system in the case $\sigma_0 = 0$, as an example. The steady-state Gierer-Meinhardt system is written as follows:

\[
\begin{align*}
0 &= \varepsilon^2 \Delta A - A + \frac{A^p}{\Omega(1 + \kappa A^p)}, \quad A > 0 \text{ in } \Omega, \\
0 &= D\Delta H - H + \frac{H^s}{\Omega}, \quad H > 0 \text{ in } \Omega, \\
\frac{\partial A}{\partial \nu} &= \frac{\partial H}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{align*}
\]

Let $P_1, \ldots, P_m$ be m certain points on $\overline{\Omega}$. Let us assume that one wants to construct a multi-peak solution to (1.33) such that $A(x)$ concentrates at the points $P_1, \ldots, P_m$. However, it is hard to construct a solution $(A, H)$ to (1.33) directly. We first construct a solution to the steady-state shadow system:

\[
\begin{align*}
0 &= \varepsilon^2 \Delta A - A + \frac{A^p}{\xi^p(1 + \kappa A^p)}, \quad A > 0 \text{ in } \Omega, \\
0 &= -\xi + \frac{1}{\mu^2} \int_{\Omega} A^\nu dx, \quad \xi > 0, \\
\frac{\partial A}{\partial \nu} &= 0 \text{ on } \partial \Omega.
\end{align*}
\]
For convenience, we set
\[\gamma := \frac{qr - (s + 1)(p - 1)}{pq}, \quad \gamma' := \frac{qr}{p - 1} - (s + 1).\]
Note that \(\gamma, \gamma' > 0\). If we put
\[A(x) = \xi^{q/(p-1)}u(x),\]
and substitute this into the shadow system (1.34), then we have the following equation for \((u, \xi)\):
\[
\begin{cases}
\varepsilon^2 \Delta u - u + \frac{u^p}{1 + \kappa \xi^{pq/(p-1)}u^p} = 0 \text{ in } \Omega, \\
\xi' = \frac{\varepsilon^q}{\int_{\Omega} \xi^{q/(p-1)}u^p dx}, \\
\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{cases}
\] (1.36)
Note that \(\xi\) includes the integral of \(u^r\). Hence, we remark that this system can be regarded as a nonlocal scalar equation. Now, we put \(\delta = \kappa \xi^{q/(p-1)}\), then problem (1.36) is reduced to the problem: find the pair of \(u\) and \(\delta\) satisfying
\[
\begin{cases}
\varepsilon^2 \Delta u - u + \frac{u^p}{1 + \delta u^p} = 0 \text{ in } \Omega, \\
\delta^\gamma \int_{\Omega} u^r(x) dx = \kappa^\gamma |\Omega|, \\
\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{cases}
\] (1.37)
If we can find a pair of \((u, \delta)\) satisfying (1.37), then the solution to (1.36) is also obtained. For the purpose, we first consider the single Neumann problem with parameters \(\varepsilon\) and \(\delta\):
\[
\begin{cases}
\varepsilon^2 \Delta u - u + f_\delta(u) = 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,
\end{cases}
\] (1.38)
\[f_\delta(u) = \frac{u^p}{1 + \delta u^p}.\] (1.39)
Then, our construction of a peak solution to (1.33) is organized as follows:
Step 1. We construct a solution to (1.38) peaked at \(P_1, \ldots, P_m\), denoted by \(u_\delta(x; \varepsilon)\), for sufficiently small \(\varepsilon > 0\).
Step 2. We find \(\delta = \delta_\varepsilon\) satisfying the second equation of (1.37) with \(u = u_\delta(\cdot, \varepsilon)\).
Step 3. By putting
\[A_\varepsilon(x) = \xi^{q/(p-1)}u_\delta(x; \varepsilon), \quad \xi_\varepsilon = \left(\frac{|\Omega|}{\int_{\Omega} u_\delta^r(x; \varepsilon) dx}\right)^{1/\gamma'},\]
we obtain a one-parameter family of solution \((A_\varepsilon, \xi_\varepsilon)\) to the shadow system (1.34).
Step 4. We construct a solution to (1.33) near \((A_{\varepsilon}, \xi_{\varepsilon})\) for sufficiently large \(D\) by the implicit function theorem. Therefore, to begin with we construct a peak solution to (1.38). For the purpose, we note that the equation (1.32) appears in the formal limit \(\varepsilon \to 0\) of the problem (1.38) after the rescaling \(x = \varepsilon y\) around the peak points \(P_1, \ldots, P_m\). Hence, the solution to (1.32) gives a good approximation of a peak solution to (1.38), around the peak points if they exist. We will construct a suitable approximate function \(U_{\varepsilon,\delta}\) by using the solution to the limiting problem (1.32), and we will construct a solution \(u_\delta(x; \varepsilon)\) to (1.38) near \(U_{\varepsilon,\delta}\) by using the contraction mapping principle. In general, the location of the peak points \(P_1, \ldots, P_m\) strongly depends on the shape of the domain \(\Omega\). In this thesis, we assume that the domain is axially symmetric with respect to the \(x_N\)-axis. We will construct a peak solution which has peaks on the intersections of the boundary \(\partial \Omega\) and the \(x_N\)-axis.

**Composition of this thesis.**

Our main aim is to construct peak solutions under the weak saturation effect for the Gierer-Meinhardt system. However, the method can be applied for the Schnakenberg model and the chemotaxis model. We will study the point-condensation phenomena and the saturation effect for these models. Thus, this thesis is composed as follows:

Chapter 2. We study the solution to (1.32) for more general nonlinear term \(f_\delta\).

Chapter 3. Construction of peak solutions to (1.38) on axially symmetric domains.

Chapter 4. Construction of stationary peak solutions to the chemotaxis model.

Chapter 5. Construction of stationary peak solutions to the Schnakenberg model.

Chapter 6. Construction of stationary peak solutions to the Gierer-Meinhardt system in the case \(\sigma_0 = 0\).

Chapter 7. Construction of stationary peak solutions to the Gierer-Meinhardt system in the case \(\sigma_0 \neq 0\).

The method stated above, namely, to construct a solution near the shadow system, is one of the methods to construct a solution to the Gierer-Meinhardt system. However, \(D\) must be sufficiently large for small \(\varepsilon\) in the method. In the case where \(D\) is not large (the case is called a strong coupling case), the construction is more difficult. However, for the one-dimensional case, we can construct a stationary peak solution to the Gierer-Meinhardt system. We emphasize that the method for the strong coupling case is different widely from the ones for the shadow system case.

Chapter 8. Construction of stationary peak solutions for the one-dimensional Gierer-Meinhardt system in the strong coupling case.

In the final chapter, we will treat a stability problem for general shadow system on cylindric domain. We will study the instability of some symmetric stationary solutions to the shadow system.
Chapter 9. Instability of some symmetric solutions to the general shadow system.

Most of this thesis except Chapter 9 are results given in the papers [46, 47, 64, 65, 66]. The results in each chapter correspond as follows:
Chapter 2,3... [46, 47], Chapter 4... [47], Chapter 5... [65], Chapter 6... [46, 65]
Chapter 7... [64], Chapter 8... [66].

Remarks on notations

For a domain $\Omega \subset \mathbb{R}^N$, we use standard Lebesgue spaces and Sobolev spaces $L^p(\Omega)$, $W^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,2}(\Omega)$ and so on, with the usual norms. We define a radially symmetric function spaces as follows: for $1 \leq p \leq \infty$ and $m \in \mathbb{N},$

$$L^p_r(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : u(x) = u(|x|)\},$$
$$W^{m,p}_r(\mathbb{R}^N) := W^{m,p}(\mathbb{R}^N) \cap L^p_r(\mathbb{R}^N),$$
$$H^m_r(\mathbb{R}^N) := W^{m,2}_r(\mathbb{R}^N).$$

Throughout this thesis, unless otherwise stated, we use the symbols $C$, $C'$, $C''$, $c$, $c'$, $c''$ as positive constants, but they need not have the same value in each situation.

We sometimes use the same symbols even if they have different meanings, for example, $w_\delta$, $u_\delta(x; \varepsilon)$ and so on. Then, although we will give remarks in each situation, the reader should not be confused.

We use the following notations:

$$\mathbb{R}^N_+ := \{x = (x',x_N) \in \mathbb{R}^N : x' = (x_1, \cdots, x_{N-1}) \in \mathbb{R}^{N-1}, x_N > 0\},$$
$$B_r(a) := \{x \in \mathbb{R}^N : |x - a| < r\}, \quad B_r := B_r(0),$$
$$B^+_r := \{x \in B_r : x_N > 0\}, \quad B^-_r := \{x \in B_r : x_N < 0\}. $$
1.6 Figures

Figure 1.5: Dynamics of multi-peak. \((p, q, r, s) = (2, 1, 2, 0), \varepsilon^2 = 0.001, D = 0.006, \tau = 0.001, \kappa = 0, \Omega = (0, 1)\). The figures show the numerical simulation for the one-dimensional Gierer-Meinhardt system. We notice that a multi-peak solution could be formed as a stable pattern.
Figure 1.6: Dynamics of multi-peak. \((p, q, r, s) = (2, 1, 2, 0), \varepsilon^2 = 0.001, D = 0.006, \tau = 0.001, \kappa = 0.2, \Omega = (0, 1)\).
Chapter 2

Analysis for equations in whole space

2.1 Uniqueness and nondegeneracy

In this chapter, we consider the existence, the uniqueness and the nondegeneracy of solutions to the following problem:

\[
\begin{align*}
\Delta w + g(w) &= 0, \quad w > 0 \text{ in } \mathbb{R}^N, \\
w(0) &= \max_{\mathbb{R}^N} w, \quad w(z) \to 0 \text{ as } |z| \to \infty,
\end{align*}
\]

where \( g(w) \) is a nonlinear term of \( w \). We always assume \( g \in C^1([0, \infty)) \), and define \( G(v) := vg'(v)/g(v) \). We first present useful well-known conditions to assure the uniqueness and the nondegeneracy of a solution to this problem.

We consider the two types of nonlinearities for \( g \), named type A and type B.

**Definition 2.1.** We say \( g \) is **type A** if \( g \) satisfies the following conditions:

\( (g_1) \) \( g(0) = 0, \quad g'(0) < 0, \) and there exists \( a > 0 \) such that \( g(a) = 0, \quad g(v) < 0 \) for \( v \in (0, a) \), and \( g'(a) > 0 \).

\( (g_2) \) There exists \( \theta > a \) such that \( \int_0^\theta g(t)dt = 0 \) and \( g(v) > 0 \) for \( v \in (a, \theta) \).

\( (g_3a) \) \( g(v) > 0 \) for \( v > a \).

\( (g_4a) \) The function \( G(v) \) is nonincreasing in \( [\theta, \infty) \) and converges to a finite limit \( L \geq 1 \) as \( v \to \infty \).

\( (g_5a) \) \( G(v) \geq G(\theta) \) for \( v \in (a, \theta) \), and \( G(v) \leq L \) for \( v \in (0, a) \).

\( (g_6a) \) \( \lim_{v \to \infty} g(v)/v^l = 0 \) for some \( l \in [0, \infty) \) in case \( N = 1, 2 \), \( l \in [0, \frac{N+2}{N-2}) \) in case \( N \geq 3 \).
Definition 2.2. We say $g$ is type B if $g$ satisfies the following conditions in addition to (g1) and (g2):

**(g3b)** There exists $b > \theta$ such that $g(b) = 0$, $g(v) > 0$ for $v \in (a, b)$, $g(v) < 0$ for $v \in (b, \infty)$.

**(g4b)** Let $\rho \in [a, b]$ be the smallest number such that $(v - \rho)g'(v) \leq g(v)$ for $v \in (\rho, b)$. Then either (i) or (ii) holds:

(i) $\theta \geq \rho$,
(ii) $\theta < \rho$ and that $G(v)$ is nonincreasing in $(\theta, \rho)$, $G(v) \geq G(\theta)$ for $v \in (\theta, \rho)$, $G(v) \leq G(\rho)$ for $v \in (0, a) \cup (\rho, b)$.

For these nonlinearities, the following proposition hold for a solution to (2.1). For type A, see [3, 48]. For type B, see [2, 3, 5, 6, 84].

**Proposition 2.1.** If $g$ is type A or B, then (2.1) has a unique solution $w$, and it satisfies the following:

(i) $w \in C^2(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$.

(ii) $w$ is radially symmetric, i.e., $w(x) = w(|x|)$, and $w'(r) < 0$ for $r = |x| > 0$.

(iii) $w$ decays exponentially together with its derivatives up to the order of 2, that is, for any $|\alpha| \leq 2$,

$$|D^\alpha w(x)| \leq Ce^{-c|x|}, \quad x \in \mathbb{R}^N,$$

for some $C, c > 0$.

(iv) The linearized operator $L = \Delta + g'(w)$ on $L^2(\mathbb{R}^N)$ with domain $\text{Dom}(L) = H^2(\mathbb{R}^N)$ satisfies

$$\text{Ker}(L) = \text{span}\{\frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_N}\},$$

and if we restrict its domain to $H^2_r(\mathbb{R}^N)$, then $L$ is bijective from $H^2_r(\mathbb{R}^N)$ onto $L^2_r(\mathbb{R}^N)$.

Let us consider the following problem including parameter $\delta$.

$$\begin{cases}
\Delta w - w + \frac{w^p}{\delta + w} = 0, & w > 0 \text{ in } \mathbb{R}^N, \\
w(0) = \max_{\mathbb{R}^N} w, & w(z) \to 0 \text{ as } |z| \to \infty.
\end{cases} \quad (2.2)$$

**Proposition 2.2.** Suppose that $s - t > 1$, $t > 1$, $\delta \geq 0$. In addition, if $N \geq 3$, we assume that $s - t < (N + 2)/(N - 2)$. Then (2.2) has a unique radial solution $w_\delta$, and it has the properties (i)-(iv) in Proposition 2.1.

When $\delta = 0$, the unique existence and the nondegeneracy of the solution to (2.2) is well-known (see e.g. [43, 42]). So, we consider the case $\delta > 0$. Put
\[ w(x) = \delta^{t/1}(\delta^{(s-t-1)/2t}x) = \delta^{t/1}v(y), \]
and substitute this into (2.2), then we have the equation for \( v \):

\[
\begin{cases}
  \Delta v - \zeta v + \frac{v^s}{1 + v^t} = 0, & v > 0 \text{ in } \mathbb{R}^N, \\
  v(0) = \max_{\mathbb{R}^N} v, & v(y) \to 0 \text{ as } |y| \to \infty,
\end{cases}
\]

(2.3)

where \( \zeta := \delta^{-(s-t-1)/t} > 0 \). We put

\[ \hat{g}(v) = -\zeta v + \frac{v^s}{1 + v^t}, \quad \hat{G}(v) := \frac{\hat{g}'(v)v}{\hat{g}(v)}. \]

To show Proposition 2.2, it suffices to have the following lemma.

**Lemma 2.1.** Suppose the same conditions on \( s \) and \( t \) as in Proposition 2.2. For each \( \zeta > 0 \), \( \hat{g} \) is type A. Hence, the nonlinear term \( -v + \frac{v^s}{1 + v^t} \) is also type A for all \( \delta \in (0, \infty) \).

**Proof.** The conditions (g1), (g2), (g3a) and (g6a) obviously hold true. Hence we show only (g4a) and (g5a). First, it is easy to see that

\[ L = \lim_{v \to \infty} \hat{G}(v) = s - t > 1. \]

Secondly, we note that \( a > 0 \) is a unique positive root of \( \hat{g}(v) = 0 \), and thus

\[ -\zeta + \frac{v^{s-1}}{1 + v^t} < 0 \text{ for } v \in (0, a), \quad -\zeta + \frac{v^{s-1}}{1 + v^t} > 0 \text{ for } v \in (a, \infty) \]

(2.4)

hold. By a direct calculation, we have

\[ \hat{G}(v) = 1 + \frac{\hat{f}(v)}{\hat{f}(\zeta)} (s - 1 - \frac{tv^t}{1 + v^t}), \quad \hat{f}(v) := \frac{v^{s-1}}{1 + v^t}, \]

(2.5)

Hence its derivative is written as follows:

\[
\frac{d\hat{G}}{dv}(v) = \frac{d\hat{f}}{dv} \frac{d}{df} \left( \frac{\hat{f}}{\hat{f} - \zeta} \right) \left( s - 1 - \frac{tv^t}{1 + v^t} \right) + \frac{\hat{f}}{\hat{f} - \zeta} \left( \frac{-f^2v^{t-1}}{1 + v^t} + \frac{f^2v^{2t-1}}{(1 + v^t)^2} \right).
\]

Here, for \( v > a \), the following inequalities hold by noting (2.4) and \( s > t + 1, t > 0 \):

\[
\frac{d\hat{f}}{dv} = \frac{v^{t-2}}{(1 + v^t)^2} \{(s - 1) + (s - t - 1)v^t\} > 0,
\]

\[
\frac{d}{df} \left( \frac{\hat{f}}{\hat{f} - \zeta} \right) = \frac{-\zeta}{(\hat{f} - \zeta)^2} < 0,
\]

\[
s - 1 - \frac{tv^t}{1 + v^t} = \frac{(s - 1) + (s - t - 1)v^t}{1 + v^t} > 0,
\]

\[
\frac{\hat{f}}{\hat{f} - \zeta} > 0,
\]

\[
-\frac{f^2v^{t-1}}{1 + v^t} + \frac{f^2v^{2t-1}}{(1 + v^t)^2} = -\frac{f^2v^{t-1}}{(1 + v^t)^2} < 0.
\]

29
Therefore \( \frac{dG}{dv}(v) < 0 \) for \( v > a \). On the other hand, for \( v \in (0, a) \), from (2.5) and (2.4), we can see that

\[
\hat{G}(v) = 1 + \frac{\hat{f}(v)}{\hat{f}(v) - \zeta} \cdot \frac{1}{1 + v^t} \{(s - 1) + (s - t - 1)v^t\}
\]

\[
1 \leq L = \lim_{v \to \infty} \hat{G}(v).
\]

These observations imply (g4a) and (g5a).

Next, we consider the following problem:

\[
\begin{cases}
\Delta w - w + \frac{w^p}{1 + \delta w^p} = 0, & w > 0 \text{ in } \mathbb{R}^N, \\
w(0) = \max_{\mathbb{R}^N} w, & w(z) \to 0 \text{ as } |z| \to \infty.
\end{cases}
\] (2.6)

**Proposition 2.3.** Assume that, \( 1 < p < (N + 2)/(N - 2) \) if \( \delta = 0 \) and \( N \geq 3 \), \( 1 < p < \infty \) if \( \delta > 0 \) or \( N = 1, 2 \). Then there exists \( \delta_* > 0 \) such that, if \( 0 < \delta < \delta_* \), (2.6) has a unique radial solution \( w_\delta \), and it has the properties (i)-(iv) in Proposition 2.1.

**Remark 2.1.** When \( p = 2 \), the same statement has been proven in [114]. Proposition 2.3 generalizes the result.

In the case \( \delta = 0 \), it is well-known that Proposition 2.3 holds true. Let us show that, even if \( \delta > 0 \), the uniqueness and the nondegeneracy hold true for \( \delta \in (0, \delta_*) \) where \( \delta_* \) is a positive number defined below. For \( \delta > 0 \), put

\[
w(x) = \delta^{-1/p} \nu(\delta(1 - p)/2p, x),
\]

and substitute this into (2.6), then we have

\[
\begin{cases}
\Delta v - \nu^p \frac{v^p}{1 + v^p} = 0, & v > 0 \text{ in } \mathbb{R}^N, \\
v(0) = \max_{\mathbb{R}^N} v, & v(z) \to 0 \text{ as } |z| \to \infty.
\end{cases}
\] (2.7)

where \( \eta := 1 - 1/p \). We put \( \tilde{g}(v) := -\nu^p + \frac{v^p}{1 + v^p} \). Note that, if \( \delta \) is small enough, \( \tilde{g}(v) = 0 \) has two positive roots \( a(\delta) < b(\delta) \), and satisfies (g1), and there exists \( \theta \) of (g2). However, \( g(v) \) satisfies neither (g1) nor (g2) when \( \delta \) is large. Let us define \( \delta^* > 0 \) so that

\[\delta_* := \sup\{\delta > 0 : \tilde{g}(v) \text{ satisfies (g1) and (g2)}\} \]

In this case, \( \delta_* \) is characterized as such a number that

\[\int_0^{b(\delta_*)} \tilde{u}_{\delta_*(t)} dt = 0 \] (2.8)

holds, here we write \( \tilde{g}(v) = \tilde{g}_\delta(v) \), and \( b = b(\delta) \) is a number in (g3b). Then, to show Proposition 2.3 it suffices to have the following lemma.
Lemma 2.2. Suppose the same conditions on \( p \) as in Proposition 2.3. Let 
\[ 0 < \delta < \delta^* \]. Then \( \tilde{g} \) is type B. Hence, the nonlinear term \(-v + \frac{\nu}{1 + \nu}g\) is also type B for \( \delta \in (0, \delta^*) \).

Proof. (g1), (g2) and (g3b) are obviously satisfied. Hence we check only (g4b).

For the purpose, we show the following properties which imply (g4b):

(i) \( \tilde{G}(v) = v\tilde{g}'(v)/\tilde{g}(v) \) is monotone decreasing in \( (a(\delta), b(\delta)) \).

(ii) \( \tilde{G}(v) \leq 1 \) in \( (0, a(\delta)) \).

(iii) \( 1 \leq \tilde{G}(\rho) \).

We show (i). Put \( \tilde{f}(v) := v^{p-1}/(1 + v^p) \). Then \( \tilde{g}(v) = v(-\delta^p + \tilde{f}(v)) \), and it holds that

\[-\delta^p + \tilde{f}(v) < 0 \text{ in } (0, a(\delta)), \]
\[-\delta^p + \tilde{f}(v) > 0 \text{ in } (a(\delta), b(\delta)), \]
\[-\delta^p + \tilde{f}(v) < 0 \text{ in } (b(\delta), \infty). \] (2.9)

By a direct calculation, we have

\[ \tilde{G}(v) = 1 + \frac{\tilde{f}(v)}{-\delta^p + \tilde{f}(v)} \left( \frac{p - 1 - v^p}{1 + v^p} \right). \] (2.10)

Hence,

\[ \frac{d\tilde{G}}{dv}(v) = \frac{d\tilde{f}}{dv} \frac{d}{d\tilde{f}} \left( \frac{\tilde{f}}{-\delta^p + \tilde{f}} \right) \left( \frac{p - 1 - v^p}{1 + v^p} \right) + \frac{\tilde{f}(v)}{-\delta^p + \tilde{f}(v)} \left( -p^2 v^{p-1} \right), \] (2.11)

and we can calculate as follows:

\[ \frac{d\tilde{f}}{dv} \frac{d}{d\tilde{f}} \left( \frac{\tilde{f}}{-\delta^p + \tilde{f}} \right) = \frac{-\delta^p v^{p-2}(p - 1 - v^p)}{(1 + v^p)^2(-\delta^p + \tilde{f})^2}. \]

Thus, we have

\[ \frac{d\tilde{G}}{dv}(v) = -\left( \frac{\delta^p v^{p-2}(p - 1 - v^p)^2}{(1 + v^p)^3(-\delta^p + \tilde{f}(v))^2} + \frac{\tilde{f}(v)}{-\delta^p + \tilde{f}(v)} \cdot \frac{p^2 v^{p-1}}{(1 + v^p)^2} \right). \]

Therefore, by (2.9), we can see that \( \frac{d\tilde{G}}{dv}(v) < 0 \) in \( (a(\delta), b(\delta)) \), and (i) is verified.

We show (ii). Because \( \tilde{g}(v) < 0 \) in \( (0, a(\delta)) \), (ii) is equivalent to \( \tilde{g}'(v) \geq \tilde{g}(v)/v \) in \( (0, a(\delta)) \), and it is equivalent to \( p - 1 \geq v^p \) in \( (0, a(\delta)) \) by a direct calculation. Thus, it is sufficient to show that

\[ a(\delta)^p \leq p - 1, \quad 0 < \delta < \delta^*. \] (2.12)

Let us define \( h_\delta(v) := \tilde{g}(v)/v \), namely, \( h_\delta(v) = -\delta^p + \tilde{f}(v) \). Let \( t_0(\delta) > 0 \) be a point such that \( h_\delta'(t_0(\delta)) = 0 \). Then

\[ t_0(\delta) = \frac{p - 1}{p - \delta^p}. \] (2.13)
Let us define a number $\delta_{**} > 0$ by

$$
\delta_{**} := \sup \{ \delta > 0 : g(v) = 0 \text{ has two positive roots} \}.
$$

Then $\delta < \delta_{**}$, $a(\delta_{**}) = b(\delta_{**}) = t_0(\delta_{**})$, and $a(\delta) \nearrow t_0(\delta_{**})$ as $\delta \nearrow \delta_{**}$. On the other hand, by noting that

$$
h_{\delta_{**}}(t_0(\delta_{**})) = 0, \quad \text{and} \quad t_0(\delta_{**}) = \frac{p-1}{p} \frac{1}{(\delta_{**})^q}
$$

from (2.13), we can solve as

$$
\delta_{**} = \frac{p-1}{p^{p/(p-1)}}.
$$

Thus, by (2.13) and (2.14), we have

$$
t_0(\delta_{**}) = \frac{p-1}{p} (\delta_{**})^q = \frac{p-1}{p} \left( \frac{p-1}{p^{p/(p-1)}} \right)^q = (p-1)^{1/p}.
$$

Hence, we have

$$
a(\delta)^p < t_0(\delta_{**})^p = p-1, \quad 0 < \delta < \delta_{**}.
$$

Thus (2.12) is verified.

We show (iii). Let $v_1 > 0$ be a point such that $\tilde{g}''(v_1) = 0$. Then we have

$$
v_1 = \frac{p-1}{p+1}.
$$

We may decide $\rho$ by $\tilde{g}(v_1) = \tilde{g}'(v_1)(v_1 - \rho)$. By (2.15), there hold that

$$
\tilde{g}(v_1) = -\delta^q + \frac{p-1}{2p},
$$

$$
\tilde{g}'(v_1) = -\delta^q + \frac{(p-1)(p+1)}{4v_1^p}.
$$

Hence, we have

$$
\rho = \frac{v_1 \tilde{g}'(v_1) - \tilde{g}(v_1)}{\tilde{g}'(v_1)} = \left[ \frac{p+1}{v_1(p-1)} - \frac{4p\delta^q}{(p-1)^2} \right]^{-1}.
$$

Now, by (2.9) and (2.10), it is enough for (iii) to show that $p - 1 - \rho^p \geq 0$, namely, $\rho \leq (p-1)^{1/p}$. However, it is equivalent to the following:

$$
1 \leq (p-1)^{1/p} \left[ \frac{p+1}{v_1(p-1)} - \frac{4p\delta^q}{(p-1)^2} \right] = \frac{(p+1)^{1+1/p}}{p-1} - 4p(p-1)^{1/p-2}\delta^q
$$

by (2.15),

$$
1 + 4p(p-1)^{1/p-2}\delta^q \leq \frac{(p+1)^{1+1/p}}{p-1}.
$$

(2.19)
Because \( \delta < \delta_* < \delta_{**} = (p - 1)/(p^{p/(p-1)}) \) from (2.14), we have

\[
1 + 4p(p - 1)^{1/p-2} \delta^n < 1 + \frac{4}{p-1}.
\]

Therefore, it is sufficient for (2.19) to show that

\[
1 + \frac{4}{p-1} \leq \frac{(p+1)^{1+1/p}}{p-1}, \quad p > 1.
\]  \(\text{(2.20)}\)

Now, (2.20) is equivalent to that

\[
0 \leq (p + 1)(p+1)/p - (p + 1) - 2, \quad p > 1.
\]

Put \( y = p + 1 \). Then \( y + 2 \leq y^p/(y-1) \), \( y > 2 \). It suffices to show that

\[
0 \leq f(y) := y \log y - (y - 1) \log(y + 2), \quad y > 2.
\]  \(\text{(2.21)}\)

Note that \( f(2) = 0 \), hence it is sufficient to show that \( f'(y) > 0, \ y > 2 \). By a direct calculation, we have

\[
f'(y) = \log \frac{y}{y+2} + \frac{3}{y+2}, \quad f''(y) = \frac{4 - y(y+2)^2}{y(y+2)^2}.
\]

Therefore, \( f'(y) \) attains a local maximum at \( y = 4 \) and is monotone decreasing in \( y > 4 \). By noting that \( f'(2) = \log(e^{3/4}/2) > 0 \) and \( f'(y) \to 0 \) as \( y \to \infty \), we can see that \( f'(y) > 0, \ y > 2 \). Thus, we complete the proof.

\[
\Box
\]

### 2.2 Properties on parameter

We consider the following problem including a parameter \( \delta \):

\[
\begin{align*}
\Delta w - w + f_\delta(w) &= 0, \ w > 0 \text{ in } \mathbb{R}^N, \\
 w(0) &= \max_{\mathbb{R}^N} w, \ w(z) \to 0 \text{ as } |z| \to \infty. 
\end{align*}
\]  \(\text{(2.22)}\)

The term \( f_\delta(w) \) is a nonlinear term with parameter \( \delta \). We require the following conditions:

\begin{itemize}
  \item[(f1)] \( f_\delta(t) \) is defined on \( (\delta, t) \in [0, \delta_0) \times [0, \infty) \) for some \( \delta_0 \in (0, \infty) \), and the derivative \( f'_\delta(t) \equiv \frac{\partial f_\delta}{\partial t}(t) \) exists for all \( t \in [0, \infty) \) and is continuous on \( (\delta, t) \in [0, \delta_0) \times [0, \infty) \).
  \item[(f2)] For \( \delta \in (0, \delta_0) \) fixed arbitrarily, there exist \( p_1 > 1, \ p_2 > 0 \) and \( C > 0 \), there hold that
    \[
    |f_\delta(t)| \leq Ct^{p_1}, \quad |f'_\delta(t)| \leq Ct^{p_2}, \quad (\delta, t) \in [0, \delta] \times [0, \infty).
    \]
  \item[(f3)] The term \( g_\delta(t) := -t + f_\delta(t) \) is type A or B for each \( \delta \in [0, \delta_0) \).
\end{itemize}
By (f1) and (f2), if we define \( \tilde{f}_\delta(t) = f_3(t) = f'_3(t) \equiv 0 \) for \( t \leq 0 \), then \( \frac{f_\delta(t)}{t} \) and \( f'_\delta(t) \) are continuous on \( (\delta, t) \in [0, \delta_0) \times (-\infty, \infty) \). Thus, we regard henceforth
\[
\frac{f_\delta(t)}{t} = f_\delta(t) = f'_\delta(t) \equiv 0 \text{ for } t \leq 0.
\]

By (f3) and Proposition 2.1, the problem (2.22) possess a unique radial solution. We denote the solution by \( w_\delta \). The solution \( w_\delta \) satisfies (i)-(iv) of Proposition 2.1.

We will use the following elementary properties.

**Lemma 2.3.** Assume (f1). Let \( \Omega \) be an open set in \( \mathbb{R}^N \). Fix \( \bar{u} > 0 \) and \( \delta \in (0, \delta_0) \), and put
\[
\mathcal{A} = \{ u \in L^\infty(\Omega) : \| u \|_{L^\infty(\Omega)} \leq \bar{u} \}.
\]

Then, the following estimate holds uniformly in \( u \in \mathcal{A} \) and \( \delta, \delta' \in [0, \bar{\delta}] \):
\[
\| f_\delta(u) - f_{\delta'}(u) \|_{L^\infty(\Omega)}, \| f'_\delta(u) - f'_{\delta'}(u) \|_{L^\infty(\Omega)} \leq \omega(\delta, \delta'),
\]
where \( \omega(\delta, \delta') \) is a certain quantity depends only on \( \bar{u}, \bar{\delta}, \delta, \delta' \) such that \( \omega(\delta, \delta') \to 0 \) as \( \delta' \to \delta \). In addition, for \( L > 0 \), there exists a certain quantity \( \omega(L) \) depends only on \( \bar{u}, \bar{\delta}, L \) such that
\[
\| f_\delta(u) - f_\delta(v) \|_{L^\infty(\Omega)}, \| f'_\delta(u) - f'_\delta(v) \|_{L^\infty(\Omega)} \leq \omega(L),
\]
hold for any \( u, v \in \mathcal{A} \) such that \( \| u - v \|_{L^\infty(\Omega)} \leq L \) and \( \delta \in [0, \bar{\delta}] \), where \( \omega(L) \to 0 \) as \( L \to 0 \).

**Proof.** Note that \( f_\delta(t) \) and \( f'_\delta(t) \) are particularly uniformly continuous on \( [0, \bar{\delta}] \times [-\bar{u}, \bar{u}] \). Hence, we can see that, for any \( \varepsilon > 0 \), there exists \( L > 0 \) such that,
\[
\| f_\delta(u) - f_{\delta'}(u) \|_{L^\infty(\Omega)} \leq \sup \{ |f_\delta(y) - f_{\delta'}(y)| : y \in [-\bar{u}, \bar{u}], \delta, \delta' \in [0, \bar{\delta}], |\delta - \delta'| < L \} < \varepsilon,
\]
and for any \( u, v \in \mathcal{A} \) satisfying \( \| u - v \|_{L^\infty(\Omega)} < L \),
\[
\| f_\delta(u) - f_\delta(v) \|_{L^\infty(\Omega)} \leq \sup \{ |f_\delta(y') - f_\delta(y)| : y, y' \in [-\bar{u}, \bar{u}], |y - y'| < L, \delta \in [0, \bar{\delta}] \} < \varepsilon.
\]
Thus, the first estimates of (2.23) and (2.24) hold. We can prove similarly the estimates for the derivative \( f'_\delta(u) \).

For applications, we had better treat \( L_\delta = \Delta - 1 + f_\delta'(w_\delta) \) as an operator on \( L^t(\mathbb{R}^N) \), \( 1 < t < \infty \). We have the following lemma.

**Lemma 2.4.** Suppose (f1)-(f3). For each \( \delta \in [0, \delta_0) \), if we regard \( L_\delta \) as an operator on \( L^t(\mathbb{R}^N) \), \( 1 < t < \infty \), with domain \( \text{Dom}(L_\delta) = W^{2,t}_\nu(\mathbb{R}^N) \), then \( L_\delta \) has a bounded inverse \( L^{-1}_\delta \). Moreover, \( L^{-1}_\delta \) is bounded uniformly in \( \delta \in [0, \bar{\delta}] \) for fixed \( \bar{\delta} \in (0, \delta_0) \).

34
Proof. Let \( P \) be an operator defined by \( P = -\Delta + 1 : W^{2, t}(\mathbb{R}^N) \to L^t(\mathbb{R}^N) \). Then it is known that \( P \) has a bounded inverse \( P^{-1} \). We first show that, if \( L_\delta \phi = 0, \phi \in W^{2, t}(\mathbb{R}^N) \), then \( \phi = 0 \). It is known that the estimate \( \sup_{x \in \mathbb{R}^n} |\phi(x)| \leq C\|\phi\|_{L^t(\mathbb{R}^N)} \) hold for some \( C > 0 \) provided \( L_\delta \phi = 0, \phi \in W^{2, t}(\mathbb{R}^N) \). Hence, \( \phi \) satisfies \( \Delta \phi - \phi + f'_\delta(w_\delta) \phi = 0 \) in \( \mathbb{R}^N \), and the term \( |f'_\delta(w_\delta) \phi| \) decays exponentially at infinity because of (P2) and Proposition 2.1(iii).

Under these situation, we can see that \( \phi = P^{-1}[f'_\delta(w_\delta) \phi] \in W^{2, t}(\mathbb{R}^N) \), we have \( \phi = 0 \) by Proposition 2.1(iv).

Next, we show that \( L_\delta \) is invertible. For the purpose, we put

\[ K[u] := P^{-1}[f'_\delta(w_\delta)u], \quad u \in L^t(\mathbb{R}^N), \]

and we show \( K \) is a compact operator on \( L^t(\mathbb{R}^N) \). We follow the argument used in the proof of Lemma 5.1 in [76]. Let \( u_n \in L^t(\mathbb{R}^N) \), \( n = 1, 2, \cdots \), and \( \|u_n\|_{L^t(\mathbb{R}^N)} \leq 1 \). Then we can estimate as follows:

\[
\|Ku_n\|_{W^{2, 1}(\mathbb{R}^N)} = \|P^{-1}[f'_\delta(w_\delta)u_n]\|_{W^{2, 1}(\mathbb{R}^N)} \leq C\|f'_\delta(w_\delta)u_n\|_{L^t(\mathbb{R}^N)} \leq C\|f'_\delta(w_\delta)\|_{L^\infty(\mathbb{R}^N)}\|u_n\|_{L^t(\mathbb{R}^N)} = C',
\]

for some constant \( C' > 0 \) independent of \( n \). By Rellich’s theorem and the diagonal process, one can extract a subsequence \( \{Ku_n\} \) which converges in \( L^t_{loc}(\mathbb{R}^N) \). For simplicity, we denote the subsequence by \( \{Ku_n\} \). Moreover, we claim that \( \{Ku_n\} \) is a Cauchy sequence in \( L^t(\mathbb{R}^N) \). To see this, for any \( \eta > 0 \), let \( R_0 > 0 \) be a number such that \( |f'_\delta(w_\delta(y))| < \eta \) for \( |y| > R_0 \). Let \( G \) be the Green function for \( -\Delta + 1 \) on \( \mathbb{R}^N \). By using the property of Green function (see the proof of Lemma 5.1 in [76]), we can estimate as follows: for \( |x| > 2R_0 \),

\[
|Ku_n(x) - Ku_m(x)| = \int_{\mathbb{R}^N} G(x, y)f'_\delta(w_\delta(y))(u_m(y) - u_n(y))dy \leq C\int_{|y| < R_0} e^{-|x-y|} |u_m(y) - u_n(y)|dy + \int_{|y| > R_0} G(x, y)|f'_\delta(w_\delta(y))|u_m(y) - u_n(y))dy \leq C e^{-|x|}e^{R_0 N/t'} + \eta \int_{|y| > R_0} G(x, y)|u_m(y) - u_n(y))dy.
\]

Here we used the estimate \( e^{-|x-y|} = e^{-|x|}e^{|x|-|y|} \leq e^{-|x|}e^{R_0} \) and Hölder’s inequality for \( 1/t + 1/t' = 1 \). Therefore, if we choose \( R_0 \) large enough, then we
have

\[
\|K u_m - K u_n\|_{L^t(|x|>2 R_0)} \\
\leq C e^{R_0 R_0^{-N/t'}} \left( \int_{|x|>2 R_0} e^{-|x|^{1/t'}} dx \right)^{1/t'} \\
+ \eta \left( \int_{|x|>2 R_0} \left( \int_{|y|>2 R_0} G(x,y) |u_m(y) - u_n(y)| dy \right)^{1/t} dx \right)^{1/t'} \\
\leq C e^{-R_0/2 R_0^{N/t'}} \left( \int_{|x|>2 R_0} e^{-|x|^{1/4}} dx \right)^{1/t'} \\
+ \eta \left( \int_{|x|>2 R_0} \left( \int_{|y|>2 R_0} G(x,y) |u_m(y) - u_n(y)| dy \right)^{1/t} dx \right)^{1/t'} \\
\leq C' \eta,
\]

for some constant \( C' > 0 \) independent of \( m, n \). Moreover, since \( K u_n \) converges in \( L^t_{\text{loc}}(\mathbb{R}^N) \), there exists \( M \in \mathbb{N} \) such that

\[
\|K u_m - K u_n\|_{L^t(|x|<2 R_0)} \leq \eta \quad (m, n \geq M).
\]

and hence,

\[
\|K u_m - K u_n\|_{L^t(\mathbb{R}^N)} \leq \eta(1 + C')
\]

for \( m, n \geq M \). Thus, \( \{K u_n\} \) is a Cauchy sequence in \( L^t(\mathbb{R}^N) \), and it converges in \( L^t(\mathbb{R}^N) \). Therefore, \( K \) is compact. Let \( f \in L^t(\mathbb{R}^N) \). We notice that \( L_\delta u = f \), \( u \in W^{2,t}(\mathbb{R}^N) \), is equivalent to

\[
u - Ku = -P^{-1} f. \tag{2.25}
\]

By the Riesz-Schauder theory for compact linear operators, \( \text{Ker}(I - K) \cap L^t(\mathbb{R}^N) \) is of finite dimension and (2.25) is solvable if and only if \( \langle P^{-1} f, \psi \rangle = 0 \) holds for all \( \psi \in \text{Ker}(I - K^*) \), where \( K^* \) is the adjoint operator of \( K \) and it is a compact operator on \( L^t(\mathbb{R}^N) \), \( 1/t + 1/t' = 1 \). However, we notice that \( \text{Ker}(I - K) \cap L^t(\mathbb{R}^N) = \{0\} \) from \( \text{Ker}(L_\delta) \cap L^t(\mathbb{R}^N) = 0 \). Hence we have \( \text{Ker}(I - K^*) \cap L^t(\mathbb{R}^N) = \{0\} \). Thus (2.25) is solvable. Furthermore, 1 is not in the spectrum of \( K|_{L^t(\mathbb{R}^N)} \), and hence \( (I - K)^{-1} : L^t(\mathbb{R}^N) \rightarrow L^t(\mathbb{R}^N) \) is bounded. From \( (I - K)u = -P^{-1} f \), we have

\[
\|u\|_{W^{2,t}(\mathbb{R}^N)} \leq \|K u\|_{W^{2,t}(\mathbb{R}^N)} + \|P^{-1} f\|_{W^{2,t}(\mathbb{R}^N)} \\
\leq C(\|u\|_{L^t(\mathbb{R}^N)} + \|f\|_{L^t(\mathbb{R}^N)}) \\
= C(\|(I - K)^{-1} P^{-1} f\|_{L^t(\mathbb{R}^N)} + \|f\|_{L^2(\mathbb{R}^N)}) \\
\leq C' \|f\|_{L^t(\mathbb{R}^N)}.
\]

This implies the invertibility of \( L_\delta \) for each \( \delta \in [0, \delta_0] \).

Let us show the uniform boundedness of \( L^{-1}_\delta \) in \( \delta \in [0, \delta] \). Let the contrary be true. Then there exist \( \delta_n \in [0, \delta] \) and \( v_n \in L^t(\mathbb{R}^N) \) such that

\[
\|L^{-1}_\delta v_n\|_{W^{2,t}(\mathbb{R}^N)} > n \|v_n\|_{L^t(\mathbb{R}^N)}, \quad n = 1, 2, \ldots. \tag{2.26}
\]
By the compactness of the interval $[0, \delta]$, $\delta_n$ has an accumulating point $\delta \in [0, \delta]$. On the other hand, for the accumulating point $\delta \in [0, \delta]$, we know that $L_\delta^{-1}$ is bounded, namely,
\[
\|L_\delta^{-1}v\|_{W^{2,1}(\mathbb{R}^N)} \leq C_\delta \|v\|_{L^1(\mathbb{R}^N)}, \quad v \in L^1_c(\mathbb{R}^N),
\]
for some $C_\delta > 0$. Let $\delta' \in [0, \delta]$, $v \in L^1_c(\mathbb{R}^N)$. For $u \in W^{2,1}(\mathbb{R}^N)$, the following equations are equivalent:
\[
\begin{align*}
L_{\delta'}u &= v \\
L_{\delta}u - (L_{\delta}u - L_{\delta'}u) &= v \\
(I - K_{\delta,\delta'}u) &:= u - L_{\delta}^{-1}(L_{\delta}u - L_{\delta'}u) = L_{\delta}^{-1}v. & (2.27)
\end{align*}
\]
We can estimate so that
\[
\begin{align*}
\|K_{\delta,\delta'}u\|_{L^1(\mathbb{R}^N)} &\leq \|K_{\delta,\delta'}u\|_{W^{2,1}(\mathbb{R}^N)} \\
&\leq C_\delta \|L_{\delta}u - L_{\delta'}u\|_{L^1(\mathbb{R}^N)} \\
&= C_\delta \|f_\delta'(w_\delta - f_\delta'(w_{\delta'}))u\|_{L^1(\mathbb{R}^N)} \\
&\leq C_\delta \|f_\delta'(w_\delta - f_\delta'(w_{\delta'}))\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^1(\mathbb{R}^N)}.
\end{align*}
\]
By Lemma 2.4, there exists an $\varepsilon$-neighborhood $U_{\delta}(\varepsilon) = \{\delta' : |\delta - \delta'| < \varepsilon\}$ of $\delta$, it holds that
\[
\|K_{\delta,\delta'}u\|_{L^1(\mathbb{R}^N)} \leq \frac{1}{2}\|u\|_{L^1(\mathbb{R}^N)}, \quad u \in L^1(\mathbb{R}^N), \quad \delta' \in U_{\delta}(\varepsilon) \cap [0, \delta].
\]
Moreover, the following estimate holds from (2.27):
\[
\begin{align*}
\|u\|_{W^{2,1}(\mathbb{R}^N)} &\leq \|K_{\delta,\delta'}u\|_{W^{2,1}(\mathbb{R}^N)} + \|L_{\delta}^{-1}v\|_{W^{2,1}(\mathbb{R}^N)} \\
&\leq \frac{1}{2}\|u\|_{L^1(\mathbb{R}^N)} + C_\delta \|v\|_{L^1(\mathbb{R}^N)}, \quad \delta' \in U_{\delta}(\varepsilon) \cap [0, \delta].
\end{align*}
\]
Therefore,
\[
\|L_{\delta}^{-1}v\|_{W^{2,1}(\mathbb{R}^N)} = \|u\|_{W^{2,1}(\mathbb{R}^N)} \leq 2C_\delta \|v\|_{L^1(\mathbb{R}^N)}
\]
holds provided $L_{\delta'}u = v$ and $\delta' \in U_{\delta}(\varepsilon) \cap [0, \delta]$. This implies that
\[
\|L_{\delta}^{-1}v\|_{W^{2,1}(\mathbb{R}^N)} \leq 2C_\delta \|v\|_{L^1(\mathbb{R}^N)}, \quad v \in L^1(\mathbb{R}^N), \quad \delta' \in U_{\delta}(\varepsilon) \cap [0, \delta].
\]
This estimate contradict (2.26). Thus we complete the proof. \hfill \Box

Next, we state a properties of $w_\delta$ with respect to the continuity and differentiability on $\delta$.

**Lemma 2.5.** Let $N/2 < t < \infty$. For the solution $w_\delta$ to (2.22), as a $W^{2,t}(\mathbb{R}^N)$-valued function of $\delta$, the following properties hold.

(i) $w_\delta \in C([0, \delta_0], W^{2,t}(\mathbb{R}^N))$. 

37
(ii) If $f_\delta(t)$ is differentiable with respect to $\delta \in (0, \delta_0)$, and the derivative is continuous in $(\delta, t) \in (0, \delta_0) \times [0, \infty)$, then $w_\delta \in C^1((0, \delta_0), W^{2,t}(\mathbb{R}^N))$.

Proof. Put $F(\eta, w) := \Delta w - w + f_\delta(w)$ for $\eta := \delta^2$. Then $F(\eta, w) \in C((\delta_0^2, \delta_0^2) \times \mathbb{R}^{N^2}, L^1(\mathbb{R}^N))$. We note that $w \in W^{2,t}(\mathbb{R}^N)$ is a continuous and bounded function by Sobolev’s embedding theorem. By (11) and Lemma 2.3, we notice that $F(\eta, w)$ is partially differentiable in $w$, and $f_w(\eta, w) = \Delta - 1 + f'_\delta(w)$, and it is continuous as a mapping from $(-\delta_0^2, \delta_0^2) \times W^{2,t}(\mathbb{R}^N)$ into $B(W^{2,t}_r(\mathbb{R}^N), L^1_r(\mathbb{R}^N))$. Now, let $w_\eta$ be a unique solution to (2.22) with $\eta = \delta^2$. Then $F(\eta, w_\eta) = 0$, and the mapping $F_w(\eta, w_\eta) : W^{2,t}(\mathbb{R}^N) \to L^1_r(\mathbb{R}^N)$ is bijective by Lemma 2.4. By the uniqueness of the solution to (2.22) and the implicit function theorem, we obtain

$$w_\eta \in C((-\delta_0^2, \delta_0^2), W^{2,t}(\mathbb{R}^N)).$$

Thus (i) follows. Next, under the assumption in (ii), we notice that $F(\eta, w)$ is partially differentiable in $\eta$, and the derivative $F_\eta(\eta, w)$ is continuous as a mapping from $(0, \delta_0^2) \times W^{2,t}(\mathbb{R}^N)$ into $B((0, \delta_0^2), L^1_r(\mathbb{R}^N))$. As a conclusion of the uniqueness and the implicit function theorem similarly, we have

$$w_\eta \in C^1((0, \delta_0^2), W^{2,t}(\mathbb{R}^N)).$$

Therefore, (ii) follows. \[\square\]

Remark 2.2. In particular, $w_\delta$ is continuous in $\delta \in [0, \delta_0)$ with respect to the norm $C^1(\mathbb{R}^N)$ by Sobolev’s embedding theorem.

Let us establish uniform exponentially decay estimate of $w_\delta$ with respect to $\delta$.

Lemma 2.6. Let $\delta \in (0, \delta_0)$ be fixed arbitrarily. Then $w_\delta$ decays exponentially together with its derivatives up to the order of 2 uniformly in $\delta \in [0, \delta]$, namely, for some constants $C, c > 0$ independent of $\delta$, it holds that

$$|D^\alpha w_\delta(x)| \leq Ce^{-c|x|}, \ x \in \mathbb{R}^N, \delta \in [0, \delta],$$

(2.28)

for any $|\alpha| \leq 2$.

Proof. Let $N/2 < t < \infty$. By Lemma 2.5, it holds that $\|w_\delta\|_{W^{2,2}(\mathbb{R}^N)}, \|w_\delta\|_{C^1(\mathbb{R}^N)} \leq C, \delta \in [0, \delta]$, for some $C > 0$ independent of $\delta$. Noting $w_\delta(r) < 0$, $r = |x| > 0$, for any $R > 0$, we have

$$w_\delta(R^t R^N |B_1|) \leq \int_{B_R} w_\delta(x)dx \leq \int_{\mathbb{R}^N} w_\delta(x)dx \leq C,$$

where $|B_1|$ is the volume of the unit ball in $\mathbb{R}^N$ and $B_R = \{ x \in \mathbb{R}^N : |x| < R \}$. Hence

$$w_\delta(R) \leq \left( \frac{C}{R^N |B_1|} \right)^{1/t}, \ \delta \in [0, \delta].$$
By this and (f2), when \( R \) is large enough, it holds that

\[
0 = \Delta w_\delta - w_\delta + f_\delta(w_\delta) \leq \Delta w_\delta - w_\delta (1 - \frac{f_\delta(w_\delta)}{w_\delta}) \leq \Delta w_\delta - \frac{1}{2} w_\delta \text{ in } \mathbb{R}^N \setminus B_R, \ \delta \in [0, \delta].
\]

Now we put \( \varphi(x) = \varphi(|x|) := e^{-|x|/\sqrt{2}} \). \( \varphi \) satisfies

\[
\Delta \varphi - \frac{1}{2} \varphi = -\frac{N - 1}{|x|} \sqrt{2} < 0 \text{ in } \mathbb{R}^N \setminus B_R.
\]

We may assume \( w_\delta(R) < C' \varphi(R) \) for \( \delta \in [0, \delta] \) by taking \( C' > 0 \) large enough. Hence, we have

\[
w_\delta(x) \leq C' \varphi(x), \ \mathbb{R}^N \setminus B_R, \ \delta \in [0, \delta],
\]

by the maximum principle. Because \( w_\delta \) is bounded in \( B_R \) uniformly in \( \delta \in [0, \delta] \), (2.28) holds in the case \( |\alpha| = 0 \).

Because \( w_\delta \) is radially symmetric, it holds that

\[
(r^{N-1} w_\delta')' = r^{N-1} (w_\delta - f_\delta(w_\delta)).
\]

Integrating (2.30) from \( r \geq R \) to \( a > r \), and by (2.28) in the case \( |\alpha| = 0 \), we can estimate as follows:

\[
|r^{N-1} w_\delta'(r) - a^{N-1} w_\delta'(a)| = \int_r^a r^{N-1} (w_\delta - f_\delta(w_\delta)) dr
\leq C \int_r^a e^{-cr} dr
= C \left\{ -\frac{1}{c} e^{-ca} + \frac{1}{c} e^{-cr} \right\}, \ \delta \in [0, \delta].
\]

By taking the limit \( a \to \infty \), we have

\[
|r^{N-1} w_\delta'(r)| \leq \frac{C}{c} e^{-cr}, \ r > R, \ \delta \in [0, \delta].
\]

Hence, we have

\[
|w_\delta'(r)| \leq \frac{C}{R^{N-1}} e^{-cr}, \ r > R.
\]

Therefore, (2.28) holds in the case \( |\alpha| = 1 \) because \( w_\delta' \) is bounded in \( B_R \) uniformly in \( \delta \in [0, \delta] \).

By (2.28) in the case \( |\alpha| = 0, 1 \), we have

\[
|w_\delta''(r)| = \left| -\frac{N - 1}{r} w_\delta'(r) + w_\delta'(r) + f_\delta(w_\delta(r)) \right| \leq \left( \frac{N - 1}{R} + 2 \right) C e^{-cr}, \ r > R.
\]

Moreover, by Schauder’s estimate in \( B_R \), we have

\[
\|w_\delta\|_{C^{2+\alpha}(\overline{B_{3R}})} \leq C' \{ \|w_\delta\|_{L^\infty(B_{3R})} + \|w_\delta - f_\delta(w_\delta)\|_{C^{\alpha}(\overline{B_{3R}})} \}.
\]

The right hand side is bounded uniformly in \( \delta \in [0, \delta] \). Hence \( w_\delta'' \) is bounded in \( B_R \) uniformly in \( \delta \in [0, \delta] \). Hence, (2.28) holds also in the case \( |\alpha| = 2 \).  \( \square \)
Chapter 3

Semilinear Neumann problems with parameter

3.1 Introduction and main results

Based on the analysis in the previous chapter, we consider the following semilinear Neumann problem:

\[
\begin{aligned}
\varepsilon^2 \Delta u - u + f_\delta(u) + \sigma = 0, & \quad u > 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \varepsilon > 0, \) \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N, \) \( N \geq 1, \) \( \nu = \nu(x) \) is the outer unit normal at \( x \in \partial \Omega. \) \( \sigma = \sigma(x) \) is a nonnegative function of \( x. \) The main purpose in this chapter is to construct multi-peak solution to (3.1), which is continuous in \( \delta, \) for \( \varepsilon \) sufficiently small.

We use the notation \((x_1, \cdots, x_{N-1}, x_N) = (x', x_N), x' \in \mathbb{R}^{N-1}, \) henceforth.

We suppose the following condition:

(A0) \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \) and is axially symmetric with respect to the \( x_N \)-axis if \( N \geq 2. \)

(A1) \( \sigma \in C^\alpha(\Omega), \) \( \alpha \in (0,1), \) and \( \sigma \) is axially symmetric, namely, \( \sigma(x) = \sigma(|x'|, x_N). \)

Let us state our construction of solutions. In this chapter, we always assume (f1)-(f3) in the previous chapter for the nonlinear term \( f_\delta. \) Then the problem (2.22) has a unique radial solution \( w_\delta, \) and it is nondegenerate. Using the function \( w_\delta, \) we will make an approximate function \( U_{\varepsilon, \delta}. \) By using the symmetry of the domain \( \Omega \) and the nondegeneracy of \( w_\delta, \) we can show that the linearized operator \( L_{\varepsilon, \delta} = \Delta - 1 + f_\delta'(U_{\varepsilon, \delta}) \) is invertible on a suitable function space. Then, we can find a solution to (3.1) in the form \( u(x) = U_{\varepsilon, \delta}(x) + \varepsilon \phi \) for some \( \phi \) due to the contraction mapping principle.
Remark 3.1. When $N = 1$, the $x_N$-axially symmetric domain can not be defined. However, we can apply the method in this chapter to the one-dimensional case, and we can construct a solution to (3.1) peaked at the boundary.

Remark 3.2. For a domain $\Omega$ which has other types of symmetry, we can construct multi-peak solutions to (3.1) in a similar way. For example, the following domains $\Omega_1$, $\Omega_2$:

(i) $\Omega_1$ is symmetric with respect to each hyperplane $\{x_1 = 0\}, \ldots, \{x_{N-1} = 0\}$.

(ii) Let $(r, \theta_1, \ldots, \theta_{N-1})$ be a polar coordinates in $\mathbb{R}^N$. $\Omega_2$ is invariant for some rotation with respect to the $\theta_{N-1}$-direction and is symmetric with respect to each hyperplane $\{x_1 = 0\}, \ldots, \{x_{N-1} = 0\}$.

For $\Omega_1$, we can construct a solution to (3.1) peaked at the intersections of $\partial \Omega_1$ and the $x_N$-axis. For $\Omega_2$, we can construct a solution to (3.1) peaked at some points on the boundary, the peak-points are located at a position which is invariant for rotations with respect to the $\theta_{N-1}$-direction. For the details, see [47] and [65].

Diffeomorphism

Let $P_1, \ldots, P_{2n}$ be the points at which $\partial \Omega$ and the $x_N$-axis intersect. We number these points so that $p_1^N < \cdots < p_{2n}^N$, where $p_j^N$ is the $x_N$-coordinate of $P_j$. For each point $P_j$, $j = 1, \ldots, 2n$, we introduce a diffeomorphism which straightens a boundary portion around $P_j$. From the smoothness of $\partial \Omega$ and the assumption (A0), there is a smooth function $\psi_j \in C^\infty([-\tau, \tau])$ depending only on $|x'|$, $x' \in \mathbb{R}^{N-1}$, for some $\tau > 0$, such that $\psi_j(0) = \psi_j'(0) = 0$, and, denoting a neighborhood of $P_j$ by $U_j$, there hold that

\[
\partial \Omega \cap U_j = \{(x', x_N) : x_N = \psi_j(|x'|) + p_j^N, |x'| < \tau\}, \quad (3.2)
\]

\[
\Omega \cap U_1 = \{(x', x_N) : x_N > \psi_1(|x'|) + p_1^N, |x'| < \tau\} \text{ if } k = \text{odd}, \quad (3.3)
\]

\[
\Omega \cap U_j = \{(x', x_N) : x_N < \psi_j(|x'|) + p_j^N, |x'| < \tau\} \text{ if } k = \text{even}. \quad (3.4)
\]

Put

\[
\Phi_k(y; P_j) = \begin{cases} 
  y_k - y_N \psi_j(|y'|) \frac{\partial \psi_j}{\partial |y'|}, & k = 1, \ldots, N-1, \\
  y_N + \psi_j(|y'|) + p_j^N, & k = N.
\end{cases} \quad (3.5)
\]

Then

\[
x = \Phi(y; P_j) := (\Phi_1(y; P_j), \ldots, \Phi_N(y; P_j))
\]

is a diffeomorphism from an open set containing the closed ball $B_{3r_0} = \{|y| \leq 3r_0\}$, with $r_0 > 0$ sufficiently small, onto a neighborhood of $P_j$, since $\psi_j'(0) = 0$ implies $D\Phi(0; P_j) = I$ (the identity map). Clearly, we may assume $r_0$ is uniform in $j$. Note that $x = \Phi(y; P_j)$ maps the hyperplane $\{y_N = 0\}$ near $y = 0$ into $\partial \Omega$ near $P_j$, and the normal to $\{y_N = 0\}$ is mapped to a normal to the boundary portion $x_N = \psi(|x'|) + p_j^N$. We shall write $\Psi(\cdot; P_j)$ for $\Phi^{-1}(\cdot; P_j)$. 

41
Approximate function

We choose \( m \) points \( P_j, \ldots, P_m \) from \( P_1, \ldots, P_n \) arbitrarily. Let us relabel these points as \( P_1, \ldots, P_m \) for the simplicity of notation. Let \( \chi \in C_0^\infty(\mathbb{R}^n) \) be a function such that \( 0 \leq \chi \leq 1, \chi(t) = 1 \) for \( |t| \leq 1, \chi(t) = 0 \) for \( |t| \geq 2 \). Suppose (1)-(f3) in the previous chapter for \( f_\delta \). Then (2.22) has a unique radial solution \( w_\delta \). We define an approximate function \( U_{\varepsilon, \delta} \) by

\[
U_{\varepsilon, \delta}(x) = \sum_{k=1}^{m} \chi\left(\frac{1}{r_0} |\Psi(x; P_k)| \right) w_\delta\left(\frac{1}{\varepsilon} \Psi(x; P_k) \right).
\]

(3.6)

Obviously, \( U_{\varepsilon, \delta} \in C^\infty(\Omega) \) and satisfies the homogeneous Neumann boundary condition on \( \partial \Omega \). \( U_{\varepsilon, \delta} \) attains the maximum at points \( P_1, \ldots, P_m \), and \( U_{\varepsilon, \delta}(x) = 0 \) for \( x \notin \bigcup_{k=1}^{m} \Psi(B_{2r_0}; P_k), B_{2r_0} = \{|x| < 2r_0\} \).

Main results

Now, let us state main theorems in this chapter. We first consider in the case \( \sigma = 0 \).

**Theorem 3.1.** Suppose (A0) and (f1)-(f3). Let \( \sigma = 0 \). We fix \( \bar{\delta} \in (0, \delta_0) \) arbitrarily. Then there exists \( \varepsilon_1 > 0 \) such that, if \( \varepsilon \in (0, \varepsilon_1) \), then, for each \( \delta \in [0, \bar{\delta}] \), (3.1) admits an axially symmetric positive solution \( u_\delta(x; \varepsilon) \) with the following form:

\[
u_\delta(x; \varepsilon) = U_{\varepsilon, \delta}(x) + \varepsilon \phi_{\varepsilon, \delta}(x), \quad x \in \Omega,
\]

(3.7)

where \( \phi_{\varepsilon, \delta} \) is a function such that

\[
\| \phi_{\varepsilon, \delta} \|_{L^\infty(\Omega)} \leq C
\]

(3.8)

holds for some constant \( C > 0 \) independent of \( \varepsilon \) and \( \delta \in [0, \bar{\delta}] \). Moreover, \( u_\delta(x; \varepsilon) \) is continuous in \( \delta \in [0, \bar{\delta}] \) with respect to the \( C^0(\overline{\Omega}) \)-norm. Furthermore, \( u_\delta(x; \varepsilon) \) satisfies the following properties:

\[
0 < u_\delta(x; \varepsilon) \leq C \exp\left\{ -\frac{c}{\varepsilon} \text{dist}(x, \mathcal{P}) \right\}, \quad x \in \Omega, \quad \delta \in [0, \bar{\delta}],
\]

(3.9)

where \( \mathcal{P} := \{P_1, \ldots, P_m\} \), the constants \( C, c > 0 \) are independent of \( \varepsilon, \delta \) and \( x \). For each \( r > 0 \),

\[
\int_\Omega w_\delta(x; \varepsilon)dx = \varepsilon^N m \int_{\mathbb{R}^N} w_\delta(y)dy + o(\varepsilon^N)
\]

(3.10)

as \( \varepsilon \to 0 \) uniformly in \( \delta \in [0, \bar{\delta}] \).

Next, we consider in the case \( \sigma \neq 0 \).
Theorem 3.2. Suppose \((A0), (A1) \text{ and } (f1)-(f3)\). We fix \(\delta \in [0, \delta_0)\) arbitrarily.
There exist constants \(\varepsilon_1 > 0\) and \(\sigma_1 > 0\) such that, if \(\varepsilon \in (0, \varepsilon_1), \delta \in [0, \delta]\)
and \(\|\sigma\|_{L^\infty(\Omega)} \leq \sigma_1\), then \((3.1)\) admits an axially symmetric solution \(u_\delta(x; \varepsilon, \sigma)\)
which satisfies
\[
\|u_\delta(x; \varepsilon, \sigma) - u_\delta(x; \varepsilon)\|_{L^\infty(\Omega)} \leq C\|\sigma\|_{L^\infty(\Omega)}
\]  \hspace{1cm} (3.11)
for some constant \(C > 0\) independent of \(\varepsilon, \sigma\) and \(\delta\), where \(u_\delta(x; \varepsilon)\) is a function
given in Theorem 3.1. Moreover, \(u_\delta(x; \varepsilon, \sigma)\) is continuous in \((\delta, \sigma) \in [0, \delta] \times C^0(\bar{\Omega})\)
with respect to \(C^0(\bar{\Omega})\)-norm, namely,
\[
\|u_\delta(\cdot; \varepsilon, \sigma) - u_{\delta'}(\cdot; \varepsilon, \sigma')\|_{C^0(\bar{\Omega})} \to 0,
\]  \hspace{1cm} (3.12)
as \(|\delta - \delta'| \to 0\) and \(\|\sigma - \sigma'\|_{C^0(\bar{\Omega})} \to 0\).

Remark 3.3 (Positivity of solutions). We note that a nontrivial classical solution \(u \in C^2(\Omega) \cap C^1(\bar{\Omega})\) to \((3.1)\) satisfies \(u(x) > 0\), \(x \in \bar{\Omega}\), automatically.
Indeed, let some nontrivial solution \(u\) satisfy \(u(x_0) = \min_{x \in \bar{\Omega}} u(x) < 0\), \(x_0 \in \bar{\Omega}\).
In the case \(x_0 \in \Omega\), because \(x_0\) is a minimum point of \(u\),
\[
\varepsilon^2 \Delta u(x_0) = \left(1 - \frac{f_\delta(u(x_0))}{u(x_0)}\right) u(x_0) - \sigma(x_0) \geq 0
\]
holds. Since \(u(x_0) < 0\) and \(\sigma(x_0) \geq 0\), we see that \(1 - \frac{f_\delta(u(x_0))}{u(x_0)} \leq 0\). Note that
\[
1 - \frac{f_\delta(t)}{t} > 0, \quad t \leq t_0,
\]
holds for some \(t_0 > 0\) by \((f2)\). Hence, \(u(x_0) > 0\) must hold. This is a contradiction. In the case \(x_0 \in \partial\Omega\), \(u\) attains its minimum on \(\bar{\Omega}\) only on \(\partial\Omega\) by the first observation. Denoting the neighborhood of \(x_0\) by \(U_{x_0}\), we have
\[
\varepsilon^2 \Delta u = \left(1 - \frac{f_\delta(u)}{u}\right) u - \sigma < 0, \quad \text{in } \Omega \cap U_{x_0},
\]
\[
u(x_0) < u(x), \quad x \in \Omega \cap U_{x_0},
\]
\[
u(x_0) < 0.
\]
Hence, by Hopf’s lemma, \(\frac{\partial u}{\partial n}(x_0) < 0\). This contradicts to the Neumann boundary condition. Thus, \(u(x) \geq 0, x \in \Omega\), holds. Moreover, since \(\sigma \geq 0\), \(u\) satisfies
\[
\varepsilon^2 \Delta u + \left(-1 + \frac{f_\delta(u)}{u}\right) u \leq 0, \quad u \geq 0 \text{ in } \Omega
\]
Then it is known that either \(u > 0\) in \(\Omega\) or \(u \equiv 0\) holds by the maximum principle. However, since \(u\) is not trivial, \(u > 0\) in \(\Omega\) holds. Furthermore, since \(u\) satisfies the homogeneous Neumann boundary condition, we have \(u(x) > 0\), \(x \in \bar{\Omega}\).
Outline of our construction

We define some function spaces as follows:

**Definition 3.1.** Let $1 \leq t \leq \infty$.

$$X^t := \{ u \in L^t(\Omega) : u(x) = u(|x'|, x_N) \}, \quad X^{2,t} := W^{2,t}(\Omega) \cap X^t,$$

$$W^{2,t}_\nu := \{ u \in W^{2,t}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \}, \quad X^{2,t}_\nu := W^{2,t}_\nu \cap X^t,$$

$$X^0 := \{ u \in C^0(\bar{\Omega}) : u(x) = u(|x'|, x_N) \}.$$

We show the outline of our construction. At first, we suppose $\sigma = 0$. We observe that, if we substitute $u(\sigma) = U_{\sigma, \delta}(x) + \varepsilon \phi(x)$ into (3.1):

$$
\begin{aligned}
\varepsilon^2 \Delta u - u + f_{\sigma}(u) &= 0, \quad u > 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial \Omega,
\end{aligned}
$$

then we have

$$
\begin{aligned}
L_{\sigma, \delta} \phi + g_{\sigma, \delta} + M_{\sigma, \delta}[\phi] &= 0 \text{ in } \Omega, \\
\frac{\partial \phi}{\partial \nu} &= 0 \text{ on } \partial \Omega,
\end{aligned}
$$

(3.13)

where

$$
\begin{aligned}
g_{\sigma, \delta} := \frac{1}{\varepsilon} \left( \varepsilon^2 \Delta U_{\sigma, \delta} - U_{\sigma, \delta} + f_{\delta}(U_{\sigma, \delta}) \right), \\
M_{\sigma, \delta}[\phi] := \frac{1}{\varepsilon} \left( f_{\delta}(U_{\sigma, \delta} + \varepsilon \phi) - f_{\delta}(U_{\sigma, \delta}) - \varepsilon f'_{\delta}(U_{\sigma, \delta}) \phi \right).
\end{aligned}
$$

(3.14)

(3.15)

Thus, to obtain a solution to (3.1) we seek $\phi$ satisfying (3.13) in a suitable function space. Here, we will see that $L_{\sigma, \delta}$ with domain $X^{2,t}_\nu$, $t \in (1, \infty)$, has a bounded inverse $K_{\sigma, \delta} : X^t \to X^{2,t}_\nu$ provided $\varepsilon$ is sufficiently small (see Lemma 3.1). Thus, it suffices to find $\phi \in X^0$ satisfying

$$
\phi = -(K_{\sigma, \delta}[g_{\sigma, \delta}] + K_{\sigma, \delta}[M_{\sigma, \delta}[\phi]]) := T_{\sigma, \delta}[\phi].
$$

(3.16)

Indeed, let $\phi_{\sigma, \delta} \in X^0$ satisfy (3.16). Then $\phi_{\sigma, \delta} \in X^{2,t}_\nu$ since $g_{\sigma, \delta} + M_{\sigma, \delta}[\phi_{\sigma, \delta}] \in X^t$, $t \in (1, \infty)$, and $K_{\sigma, \delta}$ is a mapping form $X^t$ onto $X^{2,t}_\nu$. Hence this $\phi_{\sigma, \delta}$ satisfies (3.13) as a strong solution. By taking $t$ so that $t > N$, then $\phi_{\sigma, \delta} \in C^{1,\theta}(\bar{\Omega})$, $\theta \in (0, 1)$, by Sobolev’s embedding theorem. Moreover, we can see that $\phi_{\sigma, \delta} \in C^{2,\theta}(\bar{\Omega})$ by Schauder’s theory. Thus, $u_{\delta}(x; \varepsilon) = U_{\sigma, \delta} + \varepsilon \phi_{\sigma, \delta}$ becomes a classical solution to (3.1). In the case $\sigma \neq 0$, we can show the existence of a solution to (3.1) near $u_{\delta}(x; \varepsilon)$ by a perturbation argument. To prove theorems, we will find $\phi_{\sigma, \delta} \in X^0$ as a fixed point of $T_{\sigma, \delta}$. For the purpose, we will show that, the mapping $T_{\sigma, \delta}$ becomes a contraction mapping and is continuous in $\delta \in [0, \delta]$ on suitable space provided $\varepsilon$ is small enough. By this, we can assure the existence of a unique fixed point $\phi_{\sigma, \delta}$ of $T_{\sigma, \delta}$ and that $\phi_{\sigma, \delta}$ is continuous in $\delta \in [0, \delta]$ by the contraction mapping principle with parameter (see e.g. [118, Proposition 1.2]).
3.2 Basic analysis

In this section, we study the invertibility of some linearized operator and lead some basic estimates. We first consider the following operator \( L_{\varepsilon, \delta} \) on \( L^t(\Omega) \) with domain \( W^{2,t}_{\nu} \), \( 1 < t < \infty \):

\[
L_{\varepsilon, \delta} := \varepsilon^2 - 1 + f'_\delta(U_{\varepsilon, \delta}).
\]  

(3.17)

When \( \varepsilon \) is sufficiently small, we will show that \( L_{\varepsilon, \delta} \) has a bounded inverse if the domain is restricted to \( X^{2,t}_{\nu} \).

As in the Theorem 3.1, let \( \delta \in (0, \delta_0) \) be a constant fixed arbitrarily. To establish some uniform estimates with respect to \( \delta \), we always suppose \( \delta \in [0, \delta] \) henceforth.

Now, we first state properties of \( L_{\varepsilon, \delta} \) in a series of lemmas.

**Lemma 3.1.** There exists \( \varepsilon_2 > 0 \) such that, for each \( \varepsilon \in (0, \varepsilon_2) \) and \( \delta \in [0, \delta] \), the operator \( L_{\varepsilon, \delta} \) has a bounded inverse \( K_{\varepsilon, \delta} : X^t \to X^{2,t}_{\nu} \) if we restrict the domain to \( \text{Dom}(L_{\varepsilon, \delta}) = X^{2,t}_{\nu} \), \( 1 < t < \infty \).

**Remark 3.4.** Although the operator \( L_{\varepsilon, \delta} \), \( \text{Dom}(L_{\varepsilon, \delta}) = X^{2,t}_{\nu} \), has a bounded inverse \( K_{\varepsilon, \delta} \) for \( \delta \in [0, \delta] \) and \( \varepsilon \in (0, \varepsilon_2) \), it is unclear that \( K_{\varepsilon, \delta} \) is bounded uniformly in \( \varepsilon \) and \( \delta \). However, we will need some uniform estimates with respect to \( \varepsilon \) and \( \delta \). However, we can lead a uniform estimate if we regard \( K_{\varepsilon, \delta} \) as a mapping from \( X^t \) into \( X^t \). See Lemma 3.2.

We notice that, for given \( g \in X^\infty \), the equation \( L_{\varepsilon, \delta}v = g \) has a unique solution \( v \in X^{2,t}_{\nu} \) for each \( t \in (1, \infty) \). However, it is easy to see that \( v \) is independent of the choice of \( t \). In particular, if we choose \( t \in (N/2, \infty) \), then \( v \in C^0(\Omega) \) by Sobolev's embedding theorem for some \( \theta \in (0, 1) \), and hence \( v \in X^\infty \). Therefore, we can regard \( K_{\varepsilon, \delta} \) as a mapping from \( X^\infty \) into \( X^\infty \).

Now, we state uniform estimates with respect to \( \delta \).

**Lemma 3.2.** Let \( t \in (1, \infty] \). Then there exists a constant \( C_t > 0 \), the following estimate holds:

\[
\|K_{\varepsilon, \delta}g\|_{L^t(\Omega)} \leq C_t\|g\|_{L^t(\Omega)}, \quad g \in X^t, \quad \delta \in [0, \delta],
\]

(3.18)

for all \( \varepsilon \) sufficiently small.

**Proofs of Lemmas 3.1, 3.2.** The proofs of Lemmas 3.1, 3.1 are complicated and intricate since \( L_{\varepsilon, \delta} \) has two parameters \( \varepsilon \) and \( \delta \). Let us primarily state an outline. We will prove along the following flowchart:

**Step 1.** For each \( \delta \in [0, \delta] \), the mapping \( L_{\varepsilon, \delta} : X^{2,t}_{\nu} \to X^t \) is one to one provided \( \varepsilon \) is sufficiently small.

**Step 2.** \( L_{\varepsilon, \delta} : X^{2,t}_{\nu} \to X^t \) has a bounded inverse for each \( \delta \in [0, \delta] \) and \( \varepsilon \) sufficiently small.

**Step 3.** For each \( \delta \in [0, \delta] \), the estimate (3.18) holds in the case \( t = \infty \).
Step 4. For each $\delta \in [0, \tilde{\delta}]$, the estimate (3.18) holds in the case $t \in (1, \infty)$.

Step 5. The constant $\epsilon_2$ in Lemma 3.1 can be taken uniformly in $\delta \in [0, \tilde{\delta}]$.

Step 6. The constant $C_\epsilon$ in (3.18) can be taken uniformly in $\delta \in [0, \tilde{\delta}]$.

Now, we begin the proofs according to these steps.

Step 1. We show that, for each $\delta \in [0, \tilde{\delta}]$, $L_{\epsilon, \delta} : X^2_{\epsilon, t} \to X^t$ is one to one if $\epsilon$ is sufficiently small. Let the contrary be true. Then there exists a positive monotone decreasing sequence $\{\epsilon_i\}$ such that $\text{Ker}(L_{\epsilon, \delta}) \cap X^t \neq \{0\}$, $i = 1, 2, \ldots$, namely, there exists $\{\varphi_i\} \subset X^2_{\epsilon, t}$ such that $\varphi_i \neq 0$ and

$$
\begin{aligned}
L_{\epsilon, \delta} \varphi_i &= 0 \text{ in } \Omega, \\
\frac{\partial \varphi_i}{\partial \nu} &= 0 \text{ on } \partial \Omega.
\end{aligned}
$$

(3.19)

By the elliptic regularity theory, $\varphi_i$ becomes the classical solution to (3.19). Hence we may assume $\sup_{\Omega} |\varphi_i| = 1$ by rewriting $\varphi_i / \sup_{\Omega} |\varphi_i|$ as $\varphi_i$. Let $Q_i \in \overline{\Omega}$ be a point such that $|\varphi_i(Q_i)| = 1$. Then we claim that

$$
f_\delta(U_{\epsilon, \delta}(Q_i)) \geq 1.
$$

(3.20)

Now, we show (3.20) only in the case $\varphi(Q_i) = 1$ since we can show similarly in the case $\varphi(Q_i) = -1$. Let $f_\delta(U_{\epsilon, \delta}(Q_i)) < 1$ and $Q_i \in \Omega$. Then we have

$$
\epsilon_i^2 \Delta \varphi_i(Q_i) = (1 - f_\delta(U_{\epsilon, \delta}(Q_i))) \varphi_i(Q_i) = 1 - f_\delta(U_{\epsilon, \delta}(Q_i)) > 0.
$$

On the other hand, $\epsilon_i^2 \Delta \varphi_i(Q_i) \leq 0$ holds since $\varphi_i$ attain its local maximum at $Q_i \in \Omega$. This is a contradiction. Let $f_\delta(U_{\epsilon, \delta}(Q_i)) < 1$ and $Q_i \in \partial \Omega$. Then

$$
\epsilon_i^2 \Delta \varphi_i = 1 - f_\delta(U_{\epsilon, \delta}) > 0 \text{ near } Q_i,
$$

and it must hold that $\varphi(Q_i) > \varphi(x)$, $x \in \Omega$, by the first observation. By Hopf’s boundary lemma, we conclude $\frac{\partial \varphi_i}{\partial \nu}(Q_i) > 0$. This contradicts the Neumann boundary condition. Therefore, (3.20) holds.

Noting the definition of $U_{\epsilon, \delta}$ and the assumption (f2) on $f_\delta$, there exists $R \geq 0$ independent of $i$ such that

$$
Q_i \in ( \bigcup_{k=1}^m \Phi(B_{\epsilon_i R}; P_k) ) \cap \overline{\Omega}, \quad B_{\epsilon_i R} = \{|y| < \epsilon_i R\}, \quad i = 1, 2, \ldots.
$$

Without loss of generality, we may assume that

$$
Q_i \in \Phi(B_{\epsilon_i R}; P_1) \cap \overline{\Omega}, \quad i = 1, 2, \ldots,
$$

(3.21)

and

$$
\Omega \cap U_1 = \{(x', x_N) \colon x_N > \psi_1(|x'|) + p_1^N, \ |x'| < \tau\},
$$

46
$U_1$ is some neighborhood of $P_i = (p_i^1, \cdots, p_i^N) \in \partial \Omega$. Put

$$Q_i := \frac{1}{\varepsilon_i} \Psi(Q_i; P_i).$$

Obviously, $Q_i \in B_R$. Therefore, it holds that $Q_i \to Q$ ($i \to \infty$) for some $Q \in \overline{B_R}$ by taking a subsequence if necessary. Put

$$\varphi_i(z) := \varphi_i(\Phi(\varepsilon_i z; P_i)), \quad z \in \overline{B_{3r_0/\varepsilon_i}},$$

and extend to the function on $\overline{B_{3r_0/\varepsilon_i}}$ by reflection, namely,

$$\varphi_i(z) := \varphi_i(\Phi(\varepsilon_i (z', z_N); P_i)), \quad z = (z', z_N) \in \overline{B_{3r_0/\varepsilon_i}} \setminus \{z_N = 0\}. \quad (3.23)$$

Substituting $\varphi_i(x) = \varphi_i^*(\varepsilon_i^{-1} \Psi(x; P_i))$ into

$$\varepsilon_i^2 \Delta \varphi_i(x) - \varphi_i(x) + f_i^0(Uc, \delta(x)) \varphi_i(x) = 0,$$

we obtain a equation for $\varphi_i^*$:

$$\sum_{m,n=1}^N a_{mn}(z) D_{mn} \varphi_i^*(z) + \varepsilon_i \sum_{m=1}^N b_m(z) D_m \varphi_i^*(z) - \varphi_i^*(z)$$

$$+ f_i^0(\chi(\varepsilon_i/\varepsilon_0) w_3(z)) \varphi_i^*(z) = 0, \quad z \in \overline{B_{3r_0/\varepsilon_i}},$$

where $D_{mn} = \frac{\partial^2}{\partial z_m \partial z_n}, \ D_m = \frac{\partial}{\partial z_m},$ and

$$a_{mn}(z) = \sum_{l=1}^N \frac{\partial \Psi_m}{\partial x_l}(\Phi(\varepsilon_i z; P_i); P_1) \frac{\partial \Psi_n}{\partial x_l}(\Phi(\varepsilon_i z; P_i); P_1),$$

$$b_m(z) = \Delta \Psi_m(\Phi(\varepsilon_i z; P_i); P_1).$$

We put

$$a_{mn}^*(z) = \begin{cases} a_{mn}(z), & z \in \overline{B_{3r_0/\varepsilon_i}}, \\ (-1)^{\delta_{mN} + \delta_{nN}} a_{mn}(z', -z_N), & z \in \overline{B_{3r_0/\varepsilon_i}} \setminus \{z_N = 0\}, \end{cases} \quad (3.24)$$

$$b_m^*(z) = \begin{cases} b_m(z), & z \in \overline{B_{3r_0/\varepsilon_i}}, \\ (-1)^{\delta_{mN}} b_m(z', -z_N), & z \in \overline{B_{3r_0/\varepsilon_i}} \setminus \{z_N = 0\}, \end{cases} \quad (3.25)$$

where $\delta_{mn}$ is Kronecker’s delta. Then we can see that $\varphi_i^*$ satisfies the following equation:

$$\sum_{m,n=1}^N a_{mn}^*(z) D_{mn} \varphi_i^*(z) + \varepsilon_i \sum_{m=1}^N b_m^*(z) D_m \varphi_i^*(z) - \varphi_i^*(z)$$

$$+ f_i^0(\chi(\varepsilon_i/\varepsilon_0) w_3(z)) \varphi_i^*(z) = 0, \quad z \in \overline{B_{3r_0/\varepsilon_i}}. \quad (3.26)$$
It is easy to see that \( a_{mn}^*, m, n = 1, 2, \cdots, N, \) and \( b_{m}^*, m = 1, 2, \cdots, N - 1, \) are Lipschitz continuous on \( \overline{B}_{3\epsilon_i}/\epsilon_i \) with a Lipschitz constant independent of \( i. \) Moreover, there exists a constant \( M > 0 \) independent of \( i \) such that

\[
|a_{mn}^*(z) - a_{mn}^*(\tilde{z})| \leq M|z - \tilde{z}|, \quad z, \tilde{z} \in \overline{B}_{3\epsilon_i}/\epsilon_i, \quad m, n = 1, \cdots, N. \tag{3.27}
\]

Now, by the \( L^p \)-estimate, we can see that, for each \( j \in \mathbb{N} \) and \( p \in (1, \infty), \) there exists a constant \( C_{j,p} > 0 \) such that

\[
\|\varphi_i^*\|_{W^{2,p}(B_{2j})} \leq C_{j,p} \tag{3.28}
\]

holds for all \( i \) large enough. We will give the proof of (3.28) in Appendix. By taking \( N < p, \) we have

\[
\|\varphi_i^*\|_{C^{1,\theta}(\overline{B}_{2j})} \leq C'_{j,p}, \tag{3.29}
\]

for some \( \theta \in (0, 1) \) by Sobolev’s embedding theorem. Although \( b_{m}^* \) is not continuous at \( z_N = 0, \) note that \( \varepsilon_i b_{m}^* \frac{\partial \varphi_i^*}{\partial z_m} \) is Lipschitz continuous. Regarding the term \( \varepsilon_i b_{m}^* \frac{\partial \varphi_i^*}{\partial z_m} \) as an inhomogeneous term, we can lead the following estimate by Schauder’s interior estimate:

\[
\|\varphi_i^*\|_{C^{2,\theta}(\overline{B}_{2j})} \leq C''_{j,p} \tag{3.30}
\]

for all \( i \) large enough. Hence, by Ascoli-Arzela’s theorem, \( \{\varphi^*_i\} \) has a subsequence which converges in \( C^{2,\theta'}(\overline{B}_{j}), \theta' \in (0, \theta). \) By the diagonal process, one obtains a subsequence which converges to some \( \varphi^* \in C^{2,\theta'}(\mathbb{R}^N) \) in the topology of \( C^{2,\theta'}_{loc}(\mathbb{R}^N). \) We denote the subsequence \( \varphi^*_i \) again for simplicity. Then \( \varphi^* \) satisfies the following equation:

\[
\Delta \varphi^* - \varphi^* + f^*_3(w_3)\varphi^* = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{3.31}
\]

Indeed, \( a_{mn}^*, b_{m}^*, \) and \( f^*_3(\chi(\frac{z}{\epsilon_i}|z|)w_3(z)) \) converges to \( a_{mn}(0), b_{m}(0) \) and \( f^*_3(w_3(z)) \) uniformly in an arbitrary compact set in \( \mathbb{R}^N \) as \( i \to \infty, \) respectively. Note that

\[
a_{mn}(0) = \sum_{l=1}^{N} \partial \Psi_m \partial x_l(0; P_1) \frac{\partial \Psi_n}{\partial x_l}(0; P_1). \tag{3.32}
\]

The right hand side of (3.32) is equal to the \((m, n)\)-element of the matrix

\[
(D\Psi(0; P_1))(D\Psi(0; P_1))^t,
\]

and \( D\Psi(0; P_1) = [D\Psi(0; P_1)]^{-1} = I \) (identity map). Thus (3.31) holds. Noting \( |\varphi^*(z)| \leq 1 \) and the assumption (f2), we see that \( |f^*_3(w_3)| \) decays exponentially at infinity. Therefore, by Lemma 5.1 in [76], we can see that \( \varphi^* \) also decays exponentially at infinity. In particular, \( \varphi^* \in L^2(\mathbb{R}^N). \) By Proposition 2.1(iv) and (3.31), there exist \( a_l \in \mathbb{R}, \quad l = 1, \cdots, N, \) such that

\[
\varphi^* = \sum_{l=1}^{N} a_l \frac{\partial w_3}{\partial z_l}. \tag{3.33}
\]

48
Because $\varphi^*$ is axially symmetric, i.e., $\varphi^*(z) = \varphi(|z'|, z_N)$, it follows that

$$\frac{\partial \varphi^*}{\partial z_j}(0) = 0, \quad j = 1, \cdots, N - 1.$$ (3.34)

Moreover, because $\varphi^*$ is an even function with respect to the hyperplane $\{z_N = 0\}$, it follows that

$$\frac{\partial \varphi^*}{\partial z_N}(0) = 0.$$ (3.35)

On the other hand, noting $\frac{\partial w^*}{\partial z_i}(z) = u'_i(r) \frac{z_i}{r}$ for $r = |z| > 0$, we have

$$\nabla \frac{\partial w^*}{\partial z_i}(z) = \left( \frac{z_1 z_i}{r^2} \{w''_i(r) - \frac{u'_i(r)}{r}\}, \cdots, \frac{z_i^2}{r^2} w''_i(r) - \frac{z^2 i}{r^3} u'_i(r), \cdots \right).$$

Note that

$$\left| \frac{z_1 z_i}{r^2} \{w''_i(r) - \frac{u'_i(r)}{r}\} \right| \leq |w''_i(r) - \frac{1}{r}(w'_i(r) - u'_i(0))| \to 0$$

and

$$\frac{1}{r} w'_i(r) = \frac{1}{r}(w'_i(r) - u'_i(0)) \to w''_i(0)$$

as $r \to 0$ since $w''_i(0) = 0$. Therefore, it follows that

$$\nabla \frac{\partial w^*}{\partial z_i}(0) = w''_i(0)e_i,$$ (3.36)

where $e_i$ is the unit vector of $z_i$-direction. By (3.33)-(3.36), we have

$$(0, \cdots, 0) = \nabla \varphi^*(0) = \sum_{l=1}^{N} a_l w''_i(0)e_i.$$ (3.37)

Then $a_l = 0$, $l = 1, \cdots, N$, must hold since $w''_i(0) < 0$. Thus we conclude $\varphi^* = 0$ from (3.33). However, note that

$$|\varphi^*(Q) - 1| = |\varphi(Q) - \varphi^*(Q)| \leq |\varphi^*(Q) - \varphi^*(Q_i^*)| + |\varphi^*(Q_i^*) - \varphi^*(Q_i^*)| \to 0$$

as $i \to \infty$. Hence $\varphi^*(Q) = 1$ holds. Thus we obtain a contradiction. We complete step 1.

**Remark 3.5.** By step 1, we conclude that there exists $\varepsilon_2(\delta) > 0$ for each $\delta \in [0, 1]$ such that, if $\varepsilon \in (0, \varepsilon_2(\delta))$, then $\text{Ker}(L_{c, \delta}) \cap X^t = \{0\}$. We remark that $\varepsilon \varphi(\delta)$ is independent of the choice of $t \in (1, \infty)$. Indeed, suppose that, although $\text{Ker}(L_{c, \delta}) \cap X^t = \{0\}$, $\varepsilon \in (0, \varepsilon_2(\delta))$, for some $t \in (1, \infty)$, there exists $\varphi \neq 0$ and $\varepsilon' \in (0, \varepsilon(\delta))$ such that $L_{c', \delta} \varphi = 0$, $\varphi \in X_{c', \delta}^t$, for some $s \in (1, \infty)$, $s \neq t$. However, by the elliptic regularity theory, $\varphi$ turns out to be of class $C^2 \cap (\overline{\Omega})$, $\alpha \in (0, 1)$. In particular, $\varphi \in X_{\alpha}^{2, t}$. Thus it is a contradiction.
Step 2. We show that $L_{\varepsilon, \delta}$ with domain $X^2_t$ is invertible. We can take a constant $M > 0$ so that $1 - f'(U_{\varepsilon, \delta}) - M \geq c > 0$ in $\Omega$ for some constant $c > 0$. Put $\tilde{L}_{\varepsilon, \delta} = \varepsilon^2 \Delta - 1 + f'(U_{\varepsilon, \delta}) - M$. Then $\tilde{L}_{\varepsilon, \delta} : X^2_t \to X^t$ becomes an isomorphism and there exists $C > 0$ such that

$$\|u\|_{W^2, t(\Omega)} \leq C\|\tilde{L}_{\varepsilon, \delta}u\|_{L^t(\Omega)}, \quad u \in X^2_t. \quad (3.38)$$

Let $\tilde{L}_{\varepsilon, \delta}^{-1}$ be the inverse. Because the embedding $X^2_t \hookrightarrow X^t$ is compact, (3.38) implies $\tilde{L}_{\varepsilon, \delta}^{-1}$ is a compact operator on $X^t$. Note that, for $u \in X^2_t$, if

$$\tilde{L}_{\varepsilon, \delta}^{-1}u - \frac{1}{M} u = 0, \quad u \in X^2_t \quad (3.39)$$

holds, then $u \in X^2_t$, and (3.39) is equivalent to

$$L_{\varepsilon, \delta}u = 0. \quad (3.40)$$

We know that $u$ satisfying (3.40) must be $u = 0$ for sufficiently small $\varepsilon$ by step 1. Hence $-1/M$ belongs to the resolvent set of $\tilde{L}_{\varepsilon, \delta}$, and $\tilde{L}_{\varepsilon, \delta}^{-1} + 1/M : X^t \to X^t$ is invertible. Now, for given $v \in X^t$, we show that there exists unique $u \in X^2_t$ such that $L_{\varepsilon, \delta}u = v$. For the purpose, we note that the following equations are equivalent:

$$L_{\varepsilon, \delta}u = v,$$

$$\tilde{L}_{\varepsilon, \delta}u = -Mu + v,$$

$$u = -M\tilde{L}_{\varepsilon, \delta}^{-1}u + \tilde{L}_{\varepsilon, \delta}^{-1}v, \quad (3.41)$$

$$\left(\tilde{L}_{\varepsilon, \delta}^{-1} + \frac{1}{M}\right)u = \frac{1}{M}\tilde{L}_{\varepsilon, \delta}^{-1}v. \quad (3.42)$$

We see that there exists $u \in X^t$ satisfying (3.42) for any $v \in X^t$. This $u$ turns out to be $u \in X^2_t$ since $\tilde{L}_{\varepsilon, \delta}^{-1}$ is a mapping from $X^t$ onto $X^2_t$. Therefore, $L_{\varepsilon, \delta} : X^2_t \to X^t$ has an inverse denoted by $K_{\varepsilon, \delta}$. The boundedness of $K_{\varepsilon, \delta}$ is assured from (3.41) and (3.42) as follows:

$$\|K_{\varepsilon, \delta}v\|_{W^2, t(\Omega)} = \|u\|_{W^2, t(\Omega)} \leq M\|\tilde{L}_{\varepsilon, \delta}^{-1}u\|_{W^2, t(\Omega)} + \|\tilde{L}_{\varepsilon, \delta}^{-1}v\|_{W^2, t(\Omega)} \leq CM\|u\|_{L^t(\Omega)} + C\|v\|_{L^t(\Omega)}.$$

and

$$\|u\|_{L^t(\Omega)} \leq \left\|\left(\tilde{L}_{\varepsilon, \delta}^{-1} + \frac{1}{M}\right)^{-1}\left(\frac{1}{M}\tilde{L}_{\varepsilon, \delta}^{-1}v\right)\right\|_{L^t(\Omega)} \leq C\|v\|_{L^t(\Omega)}.$$

These estimates imply the boundedness of $K_{\varepsilon, \delta}$.

Step 3. We show that, for each $\delta \in [0, \bar{\delta}]$, the estimate (3.18) holds in the case $t = \infty$. This proof is carried out in a way similar to that in step 1. Suppose
that there exists a positive monotone decreasing sequence \( \{ \varepsilon_i \} \) which converges to 0 and \( \{ g_i \} \subset X^\infty \) such that

\[
\| g_i \|_{L^\infty(\Omega)} = 1, \quad \| K_{\varepsilon_i, \delta} g_i \|_{L^\infty(\Omega)} \geq i, \quad i = 1, 2, \ldots.
\]

Put

\[
u_i := \frac{1}{\| K_{\varepsilon_i, \delta} g_i \|_{L^\infty(\Omega)}} K_{\varepsilon_i, \delta} g_i, \quad h_i := \frac{1}{\| K_{\varepsilon_i, \delta} \|_{L^\infty(\Omega)}} g_i.
\]

Then \( u_i \) satisfies the following equation as a strong solution:

\[
\varepsilon_i^2 \Delta u_i - u_i + f_\delta'(U_{\varepsilon_i, \delta}) u_i = h_i \text{ in } \Omega,
\]

and there hold that

\[
\| u_i \|_{L^\infty(\Omega)} = 1, \quad \| h_i \|_{L^\infty(\Omega)} \leq \frac{1}{i}, \quad i = 1, 2, \ldots.
\]

As we observed before Lemma 3.2, \( u_i \) is continuous on \( \overline{\Omega} \). Let \( Q_i \in \overline{\Omega} \) be the maximum point of \( |u_i| \), i.e., \( |u_i(Q_i)| = 1 \). By the continuity, there exists \( r_i > 0 \) such that

\[
|u_i(x)| \geq \frac{1}{2}, \quad x \in B_{r_i}(Q_i) \cap \Omega, \quad B_{r_i}(Q_i) = \{ |x - Q_i| < r_i \}.
\]

We claim that

\[
f_\delta'(U_{\varepsilon_i, \delta}(Q_i)) > 1 + o(1) \quad (3.44)
\]

as \( i \to \infty \). Let us verify this claim only in the case \( u_i(Q_i) = 1 \). In the case \( u_i(Q_i) = -1 \), we can also verify in the same way. For any \( \tau > 0 \), we may assume \( |h_i(x)| < \tau \) a.e. \( x \in \Omega \) for large \( i \). Then \( u_i \) satisfies the following equation as a strong solution:

\[
\varepsilon_i^2 \Delta u_i + \left( -1 + f_\delta'(U_{\varepsilon_i, \delta}) + \frac{\tau}{u_i} \right) u_i = h_i + \tau > 0, \quad a.e. x \in B_{r_i}(Q_i),
\]

for large \( i \). In the case \( Q_i \in \Omega \), we can see that

\[
f_\delta'(U_{\varepsilon_i, \delta}(Q_i)) > 1 - \frac{\tau}{u_i(Q_i)} > 1 - 2\tau
\]

holds for large \( i \) by the maximum principle for a strong solution. In the case where \( u_i \) attain its maximum only on \( \partial \Omega \), let

\[
-1 + f_\delta'(U_{\varepsilon_i, \delta}(Q_i)) + \frac{\tau}{u_i(Q_i)} < 0.
\]

Then, by the continuity, it holds that

\[
\varepsilon_i^2 \Delta u_i > \left( 1 - f_\delta'(U_{\varepsilon_i, \delta}) - \frac{\tau}{u_i} \right) u_i > 0 \text{ near } Q_i.
\]
Hence, \( \frac{\partial u}{\partial r} (Q_i) > 0 \) holds by Hopf's lemma. This contradicts the Neumann boundary condition. Thus our claim is verified. Then we can see that there exists \( R > 0 \) independent of \( i \) such that

\[
Q_i \in \left( \bigcup_{k=1}^{m} \Phi(B_{\varepsilon,R}; P_k) \right) \cap \bar{\Omega}
\]

for all \( i \) sufficiently large. Without loss of generality, we may assume that

\[
Q_i \in \Phi(B_{\varepsilon,R}; P_i) \cap \bar{\Omega}, \quad i = 1, 2, \ldots ,
\]

and

\[
\Omega \cap U_i = \{(x', x_N) : x_N > \psi_i(|x'|) + p_i N, \ |x'| < \tau \}.
\]

\( U_1 \) is some neighborhood of \( P_i = (p_1^i, \ldots , p_N^i) \in \partial \Omega \). Put

\[
u_i(z) := u_i(\Phi(\varepsilon z; P_i)), \quad h_i(z) := h_i(\Phi(\varepsilon z; P_i)), \quad z \in \bar{B_{3\rho_i/\varepsilon}}.
\]

We extend \( \nu_i \) and \( h_i \) to functions on \( \bar{B_{3\rho_i/\varepsilon}} \) by reflections. As in step 1, the following equation holds:

\[
\sum_{m,n=1}^{N} a_{mn}^* (z) D_{mn} \nu_i^* (z) + \varepsilon \sum_{m=1}^{N} b_i^* (z) D_m \nu_i^* (z) - \nu_i^* (z) + f_i'(\chi(\varepsilon z)) \nu_i^* (z) = h_i^*, \quad z \in \bar{B_{3\rho_i/\varepsilon}}.
\]

Note that \( h_i^* \) converges to 0 as \( i \to \infty \) uniformly on \( B_{3\rho_i/\varepsilon} \). By the same argument as was used in step 1 except Schaunader's interior estimate, we can pick up a subsequence of \( \{u_i^*\} \) which converges some \( u^* \in C^1,\theta' (\mathbb{R}^N) \) in the topology of \( C^1_{\text{loc}} (\mathbb{R}^N) \), \( \theta' \in (0, 1) \). Then \( u^* \) satisfies the following equation in the distribution sense:

\[
\Delta u^* - u^* + f_i'(w_i)u^* = 0 \text{ in } \mathbb{R}^N.
\]

By the elliptic regularity theory, \( u^* \) becomes a classical solution. The rest of this proof is carried out by the same argument as in step 1. We can conclude \( u^* = 0 \) and lead a contradiction.

**Step 4.** We show that, for each \( \delta \in [0, \bar{\delta}] \), the estimate (3.18) holds in the case \( t \in (1, \infty) \). We have already known in the case \( t = \infty \). Let us first show in the case \( t = 1 \). For the purpose, it suffices to show that

\[
\|K_{\varepsilon,\delta} g \|_{L^1(\Omega)} \leq C \|g \|_{L^1(\Omega)}, \quad g \in X^1 \cap C^\infty_0 (\Omega)
\]

for some \( C > 0 \). Here, we note that \( K_{\varepsilon,\delta} \) has not been defined on \( X^1 \). However, if (3.47) holds, then we can extend \( K_{\varepsilon,\delta} \) to an operator on \( X^1 \). More precisely, for \( g \in X^1 \), we take a sequence \( g_n \in X^1 \cap C^\infty_0 (\Omega) \) which converges to \( g \) in \( L^1(\Omega) \). Then, by the estimate (3.47), \( K_{\varepsilon,\delta} g_n \) becomes a Cauchy sequence in
respectively. We normalize \( \phi_f \) adjoint compact operator on e.g. [23, Theorem 9.8]), we see that (3.18) holds for

\[ \|u\|_{L^1(\Omega)} = \sup \{ |(f, u)| : \|f\|_{(L^1(\Omega))'} = 1, f \in (L^1(\Omega))' \}, u \in L^1(\Omega), \]

where \((L^1(\Omega))'\) denotes the dual space of \(L^1(\Omega)\), \((\cdot, \cdot)'\) stands for a duality pairing. Because the dual space of \(L^1(\Omega)\) is \(L^\infty(\Omega)\), we see that

\[ \|u\|_{L^1(\Omega)} = \sup \{ \left| \int_\Omega f(x)u(x)dx \right| : f \in L^\infty(\Omega), \|f\|_{L^\infty(\Omega)} = 1 \}, u \in L^1(\Omega). \]

By the result for \( t = \infty \) in step 3, the following estimate holds:

\[ \left| \int_\Omega f(x)K_{\varepsilon, \delta}[g](x)dx \right| = \left| \int_\Omega K_{\varepsilon, \delta}[f](x)g(x)dx \right| \leq C_{\infty}\|g\|_{L^1(\Omega)}, \]

for \( g \in X^1 \cap C_0^\infty(\Omega) \) and \( f \in L^\infty(\Omega) \), \( \|f\|_{L^\infty(\Omega)} = 1 \). Therefore, (3.47) holds, and hence (3.18) holds for \( t = \infty \).

Next, we show in the case \( t = 2 \). It is easy to see that \( K_{\varepsilon, \delta} \) is a self-adjoint compact operator on \( X^2 \), and it does not have 0 as an eigenvalue. Let \( \{\mu_{\varepsilon}^n\}_{n=1}^\infty \) and \( \{\phi_{\varepsilon}^n\}_{n=1}^\infty \) be the eigenvalues and the corresponding eigenfunctions, respectively. We normalize \( \phi_{\varepsilon}^n \) in advance, i.e., \( \|\phi_{\varepsilon}^n\|_{L^2(\Omega)} = 1 \). Then \( \{\phi_{\varepsilon}^n\} \) becomes a complete orthonormal system of \( X^2 \). We claim that, for each \( \delta \) in \([0, \delta]\), there exists a constant \( M > 0 \) such that

\[ |\mu_{\varepsilon}^n| \leq M < \infty, \ n = 1, 2, \ldots, \]  \hspace{2cm} (3.48)

for \( \varepsilon \) sufficiently small. Let (3.48) be false. Then there exists \( \{n_i\}_{i=1}^\infty \) and \( \{\varepsilon_i\}_{i=1}^\infty \) such that \( |\mu_{\varepsilon_i}^{n_i}| > i, \ i = 1, 2, \ldots, \) and \( \varepsilon_i \to 0 \) as \( i \to \infty \). Because \( K_{\varepsilon_i, \delta}\phi_{\varepsilon_i}^{n_i} = \mu_{\varepsilon_i}^{n_i}\phi_{\varepsilon_i}^{n_i} \) holds, we have

\[ \hat{L}_{\varepsilon_i, \delta}\phi_{\varepsilon_i}^{n_i} := \varepsilon_i^2\Delta\phi_{\varepsilon_i}^{n_i} - \left( 1 + \frac{1}{\mu_{\varepsilon_i}^{n_i}} \right)\phi_{\varepsilon_i}^{n_i} + f_u(U_{\varepsilon_i, \delta})\phi_{\varepsilon_i}^{n_i} = 0 \text{ in } \Omega. \]

Noting \( (\mu_{\varepsilon_i}^{n_i})^{-1} \to 0 \) as \( i \to \infty \), we can show that

\[ \text{Ker}(\hat{L}_{\varepsilon_i, \delta} \cap X^2 = \{0\} \]

for \( \varepsilon_i \) sufficiently small, by the same argument as in step 1. This contradicts \( \phi_{\varepsilon_i}^{n_i} \neq 0 \). Thus (3.48) is verified. Hence, we have

\[ \|K_{\varepsilon, \delta}g\|_{L^2(\Omega)} = |\mu_{\varepsilon}^n| \left\| \sum_{n=1}^\infty (g, \phi_{\varepsilon}^n)_{L^2(\Omega)} \phi_{\varepsilon}^n \right\|_{L^2(\Omega)} \leq M\|g\|_{L^2(\Omega)}, \ g \in X^2. \]

Therefore (3.18) holds for \( t = 2 \).

By Riesz’s convexity theorem or Marcinkiewicz’s interpolation theorem (see e.g. [23, Theorem 9.8]), we see that (3.18) holds for \( t \in (1, 2) \).
Note that the dual space of $L^t(\Omega)$, $t \in [1, \infty)$, is identified with $L^{t'}(\Omega)$, $1/t + 1/t' = 1$. Therefore, if (3.18) holds for $t \in (1, 2)$, then (3.18) also holds for $t' \in (2, \infty)$. Indeed, for $v \in X^{t'}$, let

$$F_v(u) := \int_{\Omega} K_{\varepsilon, \delta}[u](x)v(x)dx, \ u \in X^t.$$  

Then $F_v$ is a bounded linear functional on $X^t$. Hence, there exists a unique $w \in L^{t'}(\Omega)$ such that

$$F_v(u) = \int_{\Omega} w(x)u(x)dx, \ u \in X^t,$$

$$\|F_v\|_{(X^{t'})'} = \|w\|_{L^{t'}(\Omega)}.$$  

Note that

$$\int_{\Omega} L_{\varepsilon, \delta}[u'](x)v'(x)dx = \int_{\Omega} u'(x)L_{\varepsilon, \delta}[v'](x)dx, \ u' \in W^{2,t}_0, \ v' \in W^{2,t}_0,'$$

by Green’s formula. Hence, we have

$$F_v(u) = \int_{\Omega} K_{\varepsilon, \delta}[u](x)v(x)dx = \int_{\Omega} K_{\varepsilon, \delta}[u](x)L_{\varepsilon, \delta}[v](x)dx$$

$$= \int_{\Omega} L_{\varepsilon, \delta}[K_{\varepsilon, \delta}[u]](x)K_{\varepsilon, \delta}[v](x)dx$$

$$= \int_{\Omega} u(x)K_{\varepsilon, \delta}[v](x)dx, \ u \in X^t.$$  

Therefore, $w = K_{\varepsilon, \delta}v$ holds from (3.49) and (3.50). By Hölder’s inequality, we see that

$$|F_v(u)| \leq \|K_{\varepsilon, \delta}u\|_{L^t(\Omega)}\|v\|_{L^{t'}(\Omega)} \leq C_t\|u\|_{L^t(\Omega)}\|v\|_{L^{t'}(\Omega)}, \ u \in X^t,$$

and hence $\|F_v\|_{(X^{t'})'} \leq C_t\|v\|_{L^{t'}(\Omega)}$. Thus we have

$$\|K_{\varepsilon, \delta}v\|_{L^{t'}(\Omega)} = \|w\|_{L^{t'}(\Omega)} = \|F_v\|_{(X^{t'})'} \leq C_t\|v\|_{L^{t'}(\Omega)}, \ v \in X^{t'}.$$  

By This duality argument, we obtain (3.18) for $t \in (2, \infty)$ from the result for $t \in (1, 2)$. Thus we complete step 4.

**Step 5.** We show that the constant $\varepsilon_2$ in Lemma 3.1 can be taken uniformly in $\delta \in [0, \delta]$. Let the contrary be holds. Then, there exists monotone decreasing sequence $\{\varepsilon_n\}$ and $\delta_n \in [0, \delta]$, such that

$$Ker(L_{\varepsilon_n, \delta_n}) \cap X^t = \{0\}, \ n = 1, 2, \ldots.$$  

Therefore, there exists $\phi_n \neq 0$ such that $\phi_n \in X^{2,t}_0$ and $L_{\varepsilon_n, \delta_n}\phi_n = 0$. Note that $\delta_n \in [0, \delta]$ has an accumulating point in $[0, \delta]$. Let $\delta \in [0, \delta]$ be the accumulating
We show that the constant $\varepsilon_2(\delta) > 0$ such that

$$Ker(L_{\varepsilon, \delta}) \cap X^t \neq \{0\}, \ \varepsilon \in (0, \varepsilon_2(\delta)),$$

and the bounded inverse $K_{\varepsilon, \delta}$ of $L_{\varepsilon, \delta}$ exists for all $\varepsilon \in (0, \varepsilon_2(\delta))$. In particular, $K_{\varepsilon, \delta}$ exists for all $n$ large enough. Thus we can rewrite as follows:

$$L_{\varepsilon_n, \delta_n} \phi_n = 0$$

$$L_{\varepsilon_n, \delta_n} \phi_n - (L_{\varepsilon_n, \delta_n} \phi_n - L_{\varepsilon_n, \delta_n} \phi_n) = 0$$

$$\phi_n - K_{\varepsilon, \delta}[L_{\varepsilon_n, \delta_n} \phi_n - L_{\varepsilon_n, \delta_n} \phi_n] = 0$$

$$(I - Q_n) \phi_n := \phi_n - K_{\varepsilon, \delta}[(f'_n(U_{\varepsilon_n, \delta}) - f'_n(U_{\varepsilon_n, \delta_n})) \phi_n] = 0. \quad (3.52)$$

By step 4, there exists $C_t(\delta) > 0$, we can estimate as follows:

$$\|Q_n \phi\|_{L^t(\Omega)} \leq C_t \|[(f'_n(U_{\varepsilon_n, \delta}) - f'_n(U_{\varepsilon_n, \delta_n})) \phi]\|_{L^t(\Omega)}$$

$$\leq \|f'_n(U_{\varepsilon_n, \delta}) - f'_n(U_{\varepsilon_n, \delta_n})\|_{L^\infty(\Omega)} \|\phi\|_{L^t(\Omega)}, \ \phi \in X^t.$$

Note that, by the definition of $U_{\varepsilon, \delta}$, the following estimate holds:

$$|U_{\varepsilon_n, \delta}(x) - U_{\varepsilon_n, \delta_n}(x)| \leq \sum_{k=1}^m \chi\left(\frac{1}{r_0}|\Psi(x; P_k)|\right) w_{\delta_n}(\frac{1}{\varepsilon_n} \Psi(x; P_k)) - w_{\delta_n}(\frac{1}{\varepsilon_n} \Psi(x; P_k))$$

$$\leq m \|w_{\delta_n} - w_{\delta_n}\|_{L^\infty(\mathbb{R}^N)}. \quad (3.53)$$

By this estimate and Lemma 2.3, we see that

$$\|f'_n(U_{\varepsilon_n, \delta}) - f'_n(U_{\varepsilon_n, \delta_n})\|_{L^\infty(\Omega)} = o(1)$$

as $n \to \infty$. Therefore, we notice that $\|Q\|_{X^t \to X^t} \leq 1/2$ for all $n$ large enough. By the Neumann series theory, a bounded inverse $(I - Q_n)^{-1} : X^t \to X^t$ exists. Thus $\phi_n = 0$ hold for all $n$ large enough from (3.52). This is a contradiction. \qed

**Step 6.** We show that the constant $C_t$ in (3.18) can be taken uniformly in $\delta \in [0, \delta]$. We have already known the following:

(i) For each $\delta \in [0, \delta]$, the estimate (3.18) holds for each $t \in (1, \infty]$.

(ii) There exists $\varepsilon_2 > 0$ independent of $t$ and $\delta$, $L_{\varepsilon, \delta} : X^{2, t}_{\varepsilon, \delta} \to X^t$, $t \in (1, \infty)$,

has a bounded inverse for all $\varepsilon \in (0, \varepsilon_2)$ and $\delta \in [0, \delta]$.

Under these situation, we can prove in the same way as that in the proof of the uniform boundedness in Lemma 2.4. Thus, we omit the details. \qed

Thus, we complete the proofs of Lemmas 3.1 and 3.2.
3.3 Basic estimates

In this section, we deduce some estimates.

**Lemma 3.3.** For \( g_{\varepsilon, \delta} \) defined by (3.14), \( g_{\varepsilon, \delta} \) is continuous in \( \delta \in [0, \delta] \) with respect to the \( C^0(\Omega) \)-norm for each \( \varepsilon \).

**Proof.** Let \( \delta, \delta' \in [0, \delta] \). Then we have

\[
|g_{\varepsilon, \delta}(x) - g_{\varepsilon, \delta'}(x)| \leq \varepsilon^{-1}\{\varepsilon^2 |\Delta U_{\varepsilon, \delta}(x) - \Delta U_{\varepsilon, \delta'}(x)| + |U_{\varepsilon, \delta}(x) - U_{\varepsilon, \delta'}(x)|
+ |f_\delta(U_{\varepsilon, \delta}(x)) - f_{\delta'}(U_{\varepsilon, \delta'}(x))|\}.
\]

We shall estimate each term. We can first see that

\[
\|U_{\varepsilon, \delta} - U_{\varepsilon, \delta'}\|_{L^\infty(\Omega)} \leq m\|w_\delta - w_{\delta'}\|_{L^\infty(\mathbb{R}^N)},
\]

and the right hand side tends to 0 as \( \delta' \to \delta \) by the continuity of \( w_\delta \). Secondly, we note that \( \|U_{\varepsilon, \delta}\|_{L^\infty(\Omega)} \) is bounded uniformly in \( \delta \in [0, \delta] \). We can estimate by Lemma 2.3 as follows:

\[
\|f_\delta(U_{\varepsilon, \delta}) - f_{\delta'}(U_{\varepsilon, \delta'})\|_{L^\infty(\Omega)} \leq \|f_\delta(U_{\varepsilon, \delta}(x)) - f_{\delta'}(U_{\varepsilon, \delta}(x))\|_{L^\infty(\Omega)}
+ \|f_{\delta'}(U_{\varepsilon, \delta}(x)) - f_{\delta'}(U_{\varepsilon, \delta'}(x))\|_{L^\infty(\Omega)}
\leq \omega(\delta, \delta') + o(1),
\]

as \( \delta' \to \delta \), where \( \omega(\delta, \delta') \) is some quantity depending only on the \( L^\infty \)-bounds of \( U_{\varepsilon, \delta} \) and \( \delta, \delta' \) and \( \delta \) such that \( \omega(\delta, \delta') \to 0 \) as \( \delta' \to \delta \). Thirdly, we observe that

\[
\|\Delta U_{\varepsilon, \delta}(x) - \Delta U_{\varepsilon, \delta'}(x)\| \leq \sum_{k=1}^m \left\{ \left| \sum_{k=1}^m \left[ \Delta_x \left( \frac{1}{r_0} |y_k| \right) w_\delta \left( \frac{1}{\varepsilon} y_k \right) - w_{\delta'} \left( \frac{1}{\varepsilon} y_k \right) \right] \right| + \left| \sum_{k=1}^m \left[ \Delta_x \left( \frac{1}{r_0} |y_k| \right) w_\delta \left( \frac{1}{\varepsilon} y_k \right) - w_{\delta'} \left( \frac{1}{\varepsilon} y_k \right) \right] \right| \}
\]

where \( y_k := \Psi(x; P_k) \). The first term and the second term of the right hand side tend to 0 as \( \delta' \to \delta \) in \( L^\infty(\mathbb{R}^N) \) by the continuity of \( w_\delta \) with respect to \( \delta \). After the differentiation of the composition function, the third term can be estimated as follows:

\[
\left| \Delta_x w_\delta \left( \frac{1}{\varepsilon} y_k \right) - \Delta_x w_{\delta'} \left( \frac{1}{\varepsilon} y_k \right) \right| \leq C \left\{ \left| \sum_{m=1}^N \frac{\partial w_\delta}{\partial z_m} - \frac{\partial w_{\delta'}}{\partial z_m} \right| + \sum_{m,n=1}^N \left| \frac{\partial^2 w_\delta}{\partial z_m \partial z_n} - \frac{\partial^2 w_{\delta'}}{\partial z_m \partial z_n} \right| \}
\]

The first term of the right hand side tends to 0 as \( \delta' \to \delta \). For any \( R > 0 \), the following estimate holds by the Schauder estimate:

\[
\|w_\delta - w_{\delta'}\|_{C^{2,\alpha}(B_{2R})} \leq C' \left\{ \|w_\delta w_{\delta'}\|_{L^\infty(B_{3R})} + \|(w_\delta - f_\delta(w_\delta)) - (w_{\delta'} - f_{\delta'}(w_{\delta'}))\|_{C^\alpha(B_{3R})} \right\}.
\]

56
The first term tends to 0 as \( \delta' \to \delta \). Moreover, it is easy to see that the second term also tends to 0 as \( \delta' \to \delta \) by using Lemma 2.3 and the continuity of \( w_\delta \) in \( \delta \) after the estimation:

\[
\|(w_\delta - f_\delta(w_\delta)) - (w_{\delta'} - f_{\delta'}(w_{\delta'}))\|_{C^0(B_{2R})} \\
\leq C_R(\|w_\delta - w_{\delta'}\|_{C^1(B_{2R})} + \|f_\delta - f_{\delta'}(w_{\delta'})\|_{C^1(B_{2R})}).
\]

For \( r = |z| \geq R \), we can estimate as follows:

\[
\sum_{m,n=1}^{N} \left| \frac{\partial^2 w_\delta}{\partial z_m \partial z_n}(z) - \frac{\partial^2 w_{\delta'}}{\partial z_m \partial z_n}(z) \right| \\
\leq C\{(w_\delta''(r) - w_{\delta'}''(r)) + \frac{1}{r}|w_\delta'(r) - w_{\delta'}'(r)| + |w_\delta'(r) - w_{\delta'}'(r)|\} \\
\leq C'(1 + \frac{1}{R})\|(w_\delta' - w_{\delta'}) L^\infty(r \geq R) + \|w_\delta - w_{\delta'}\|_{L^\infty(r \geq R)} \\
+ \|f_\delta - f_{\delta'}(w_{\delta'})\|_{L^\infty(r \geq R)}\}.
\]

Here, we used \( w_\delta''(r) = -\frac{N-1}{r}w_\delta'(r) + w_\delta'(r) - f_\delta(w_\delta(r)) \). All the terms of the right hand side tends to 0 as \( \delta' \to \delta \) by the continuity of \( w_\delta \) in \( \delta \).

By these estimates, we have

\[
\sup_{x \in \Omega} |g_{\epsilon,\delta}(x) - g_{\epsilon,\delta'}(x)| \to 0 \text{ as } \delta' \to \delta,
\]

and we complete the proof. \( \square \)

**Lemma 3.4.** There exists a constant \( C_1 > 0 \) such that

\[
\sup_{x \in \Omega} |g_{\epsilon,\delta}| \leq C_1, \ \delta \in [0, \tilde{\delta}],
\]

for all \( \epsilon \) sufficiently small.

**Proof.** Because

\[
g_{\epsilon,\delta}(x) = 0, \ x \in \Omega \setminus \bigcup_{k=1}^{m} \Phi(B_{2R_0}; P_k),
\]

if we put

\[
g_{\epsilon,\delta,k}^*: = g_{\epsilon,\delta}(\Phi(\epsilon z; P_k)),
\]

it holds that

\[
\sup_{x \in \Omega} |g_{\epsilon,\delta}(x)| = \max_{1 \leq k \leq m |z| < 2R_0/\epsilon} \sup_{1 \leq k \leq m |z| < 2R_0/\epsilon} |g_{\epsilon,\delta,k}^*(z)|.
\]

Therefore, it suffices to show that

\[
\sup_{|z| < 2R_0/\epsilon} |g_{\epsilon,\delta,k}(z)| \leq C(\sup_{|z|} e^{-|z|} + \frac{1}{\epsilon} e^{-2\epsilon/\epsilon}), \ \delta \in [0, \tilde{\delta}],
\]

(3.56)
holds for some constant $C, c > 0$ independent of $\varepsilon$. For $z \in B_{2r_0/\varepsilon}$, it follows that
\[
\varepsilon g^*_{\varepsilon,\delta,k}(z) = \varepsilon g_{\varepsilon,\delta}(\Phi(\varepsilon z; P_{k}))
\]
\[
= \varepsilon^2 \Delta U_{\varepsilon,\delta}(\Phi(\varepsilon z; P_{k})) - U_{\varepsilon,\delta}(\Phi(\varepsilon z; P_{k})) + f_\delta(U_{\varepsilon,\delta}(\Phi(\varepsilon z; P_{k})))
\]
\[
= \sum_{i,j=1}^{N} a^*_{ij}(z)D_{ij}(\chi(\varepsilon \frac{\rho}{r_0}|z|)w_\delta(z)) + \varepsilon \sum_{j=1}^{N} b^*_{j}(z)D_{j}(\chi(\varepsilon \frac{\rho}{r_0}|z|)w_\delta(z))
\]
\[
- \chi(\varepsilon \frac{\rho}{r_0}|z|)w_\delta(z) + f_\delta(\chi(\varepsilon \frac{\rho}{r_0}|z|)w_\delta(z)),
\]
where the coefficients $a_{ij}^*$ and $b_j^*$ are given in the same form as that in (3.24) and (3.25). It is known that (see [76, Lemma 4.1])
\[
a_{ij}^*(z) = \delta_{ij} + 2\varepsilon \delta_{ij}^\prime(0)|z_N| + \alpha_{ij}(z),
\]
\[
b_j^*(z) = -(N-1)\delta_{ijN}^\prime(0)sgn(z_N) + \beta_j(z),
\]
where $\delta_{ij}$ is Kronecker’s delta, $\delta_{ij}^\prime := (1 - \delta_{ij})\chi(1 - \delta_{ij})\delta_{ij}$, $\alpha_{ij}$ and $\beta_j$ satisfy
\[
|\alpha_{ij}(z)| \leq C\varepsilon^2 |z|^2, \quad z \in B_{2r_0/\varepsilon},
\]
\[
|\beta_j(z)| \leq C|z|, \quad z \in B_{2r_0/\varepsilon}.
\]
Therefore, it follows that
\[
\varepsilon g^*_{\varepsilon,\delta,k}(z) = \sum_{i,j=1}^{N} \{ \delta_{ij} + 2\varepsilon \delta_{ij}^\prime(0)|z_N| + \alpha(z) \}
\]
\[
\times \{ D_{ij}(\chi(\varepsilon \frac{\rho}{r_0}|z|))w_\delta(z) + 2D_i(\chi(\varepsilon \frac{\rho}{r_0}|z|))D_jw_\delta(z) + \chi(\varepsilon \frac{\rho}{r_0}|z|)D_{ij}w_\delta \}
\]
\[
+ \varepsilon \sum_{j=1}^{N} (-(N-1)\delta_{ijN}^\prime(0)sgn(z_N) + \beta_j(z))D_{j}(\chi(\varepsilon \frac{\rho}{r_0}|z|)w_\delta(z))
\]
\[
- \chi(\varepsilon \frac{\rho}{r_0}|z|)w_\delta(z) + f_\delta(\chi(\varepsilon \frac{\rho}{r_0}|z|)w_\delta(z)).
\]
Here, note that
\[
\sum_{i,j=1}^{N} \delta_{ij}\chi(\varepsilon \frac{\rho}{r_0}|z|)D_{ij}w_\delta(z) = \chi(\varepsilon \frac{\rho}{r_0}|z|)\Delta w_\delta(z),
\]
\[
-\chi(\varepsilon \frac{\rho}{r_0}|z|)w_\delta(z) = -\chi(\varepsilon \frac{\rho}{r_0}(\Delta w_\delta(z) + f_\delta(w_\delta(z))).
\]
Hence, $g^*_{\varepsilon, \delta, k}(z)$ is written as follows:

$$g^*_{\varepsilon, \delta, k}(z) = \frac{1}{\varepsilon} \left\{ f_3(\chi(\frac{\varepsilon}{r_0}|z|)w_3(z)) - \chi(\frac{\varepsilon}{r_0}|z|)f_3(w_3(z)) \right\}$$

$$+ \chi(\frac{\varepsilon}{r_0}|z|) \sum_{i,j=1}^{N} \{ 2\delta_{ij} \psi''_{k}(0)|z_N| + \frac{1}{\varepsilon} \alpha_{ij}(z) \} D_{ij}w_{3}(z)$$

$$+ \sum_{j=1}^{N} \left\{ -(N-1)\delta_{jN} \psi''_{k}(0)sgn(z_N) + \beta_j(z) \right\} D_j(\chi(\frac{\varepsilon}{r_0}|z|)w_{3}(z))$$

$$+ \frac{1}{\varepsilon} \sum_{i,j=1}^{N} \{ \delta_{ij} + 2\varepsilon \delta_{ij} \psi''_{k}(0)|z_N| + \alpha_{ij}(z) \}$$

$$\times \{ w_3(z)D_{ij}(\chi(\frac{\varepsilon}{r_0}|z|)) + 2D_jw_{3}(z)D_j(\chi(\frac{\varepsilon}{r_0}|z|)) \}.$$
and

$$|M_{\varepsilon,\delta}[\phi_1](x) - M_{\varepsilon,\delta}[\phi_2](x)|$$

$$= \frac{1}{\varepsilon} |f_\delta(U_{\varepsilon,\delta} + \varepsilon \phi_1) - f_\delta(U_{\varepsilon,\delta} + \varepsilon \phi_2) - \varepsilon f'_\delta(U_{\varepsilon,\delta})(\phi_1 - \phi_2)|$$

$$= |\int_0^1 \{f'_\delta(U_{\varepsilon,\delta} + \varepsilon \phi_2 + \varepsilon(\phi_1 - \phi_2)t) - f'_\delta(U_{\varepsilon,\delta})\} dt||\phi_1 - \phi_2|.$$

Noting that

$$\| (U_{\varepsilon,\delta} + \varepsilon \phi t) - U_{\varepsilon,\delta} \|_{L^\infty(\Omega)} \leq \varepsilon L,$$

$$\| (U_{\varepsilon,\delta} + \varepsilon(\phi_1 - \phi_2)t) - U_{\varepsilon,\delta} \|_{L^\infty(\Omega)} \leq 2\varepsilon L,$$

we can estimate by Lemma 2.3 so that

$$\| f'_\delta(U_{\varepsilon,\delta} + \varepsilon \phi t) - f'_\delta(U_{\varepsilon,\delta}) \|_{L^\infty(\Omega)} \leq \omega_1(\varepsilon),$$

$$\| f'_\delta(U_{\varepsilon,\delta} + \varepsilon(\phi_1 - \phi_2)t) - f'_\delta(U_{\varepsilon,\delta}) \|_{L^\infty(\Omega)} \leq \omega_1(\varepsilon),$$

for some quantity $\omega_1(\varepsilon)$ satisfying the lemma’s assertion.

**Lemma 3.6.** For $\delta, \delta' \in [0, \bar{\delta}]$ and $\varepsilon \in (0, \varepsilon_2)$, it holds that

$$\| (K_{\varepsilon,\delta} - K_{\varepsilon,\delta'})\phi \|_{L^\infty(\Omega)} \leq \omega_2(\delta, \delta')\|\phi\|_{L^\infty(\Omega)}, \quad \phi \in X^\infty,$$

for some quantity $\omega_2(\delta, \delta')$ independent of $\varepsilon$ such that $\omega_2(\delta, \delta') \to 0$ as $\delta' \to \delta$.

**Proof.** Noting

$$\phi = L_{\varepsilon,\delta'}K_{\varepsilon,\delta'}[\phi] = \{L_{\varepsilon,\delta} + (f'_\delta(U_{\varepsilon,\delta'}) - f'_\delta(U_{\varepsilon,\delta}))\}K_{\varepsilon,\delta'}[\phi],$$

we have

$$K_{\varepsilon,\delta}[\phi] = K_{\varepsilon,\delta}[\{L_{\varepsilon,\delta} + (f'_\delta(U_{\varepsilon,\delta'}) - f'_\delta(U_{\varepsilon,\delta}))\}K_{\varepsilon,\delta'}[\phi]]$$

$$= K_{\varepsilon,\delta'}[\phi] + K_{\varepsilon,\delta}[\{f'_\delta(U_{\varepsilon,\delta'}) - f'_\delta(U_{\varepsilon,\delta})\}K_{\varepsilon,\delta'}[\phi]].$$

Therefore, we have the following estimate by Lemmas 2.3 and 3.2:

$$\| K_{\varepsilon,\delta'}[\phi] - K_{\varepsilon,\delta}[\phi] \|_{L^\infty(\Omega)} = \| K_{\varepsilon,\delta}[\{f'_\delta(U_{\varepsilon,\delta'}) - f'_\delta(U_{\varepsilon,\delta})\}K_{\varepsilon,\delta'}[\phi]] \|_{L^\infty(\Omega)}$$

$$\leq C_\infty \| f'_\delta(U_{\varepsilon,\delta'}) - f'_\delta(U_{\varepsilon,\delta}) \|_{L^\infty(\Omega)}\| K_{\varepsilon,\delta'}[\phi] \|_{L^\infty(\Omega)}$$

$$\leq C_\infty \| f'_\delta(U_{\varepsilon,\delta'}) - f'_\delta(U_{\varepsilon,\delta}) \|_{L^\infty(\Omega)}\| \phi \|_{L^\infty(\Omega)}$$

$$\leq \omega_2(\delta, \delta')\|\phi\|_{L^\infty(\Omega)},$$

for some quantity $\omega_2(\delta, \delta')$ satisfying the lemma’s assertion. □
3.4 Proof of Theorem 3.1

We prove Theorem 3.1 in a series of lemmas.

We prepare the following well-known contradiction mapping principle with parameter.

Lemma 3.7. [118, Proposition 1.2] Let the following conditions be satisfied.

1. Let \( X \) and \( \Lambda \) be a complete metric space and a metric space, respectively.
2. For each \( \lambda \in \Lambda \), let \( f_\lambda \) be a continuous mapping from \( X \) into \( X \).
3. There exists a constant \( k \in [0,1) \) independent of \( \lambda \in \Lambda \) such that
   \[ \text{dist}(f_\lambda(x), f_\lambda(y)) \leq k \cdot \text{dist}(x, y) \] for all \( x, y \in X \).
4. For any \( \lambda_0 \in \Lambda \), and for all \( x \in X \), \( \lim_{\lambda \to \lambda_0} f_\lambda(x) = f_{\lambda_0}(x) \).

Then for each \( \lambda \in \Lambda \), there exists a unique fixed point \( x = x(\lambda) \in X \). Moreover, this \( x(\lambda) \) is continuous with respect to \( \lambda \in \Lambda \).

Let us find a fixed point of \( T_{\varepsilon, \delta} \) defined by (3.16). We put
   \[ B := \{ \phi \in X^0 : \| \phi \|_{L^\infty(\Omega)} \leq 2C_\infty C_1 \}, \]
where \( C_\infty \) is a constant given in Lemma 3.2 for \( t = \infty \), \( C_1 \) is a constant given in Lemma 3.4.

Lemma 3.8. There exists \( \varepsilon_1 > 0 \), if \( \varepsilon \in (0, \varepsilon_1) \) and \( \delta \in [0, \bar{\delta}] \), then there exists a unique fixed point \( \phi_{\varepsilon, \delta} \in B \) of \( T_{\varepsilon, \delta} \). Moreover, \( \phi_{\varepsilon, \delta} \) is continuous in \( \delta \in [0, \bar{\delta}] \) with respect to \( C^0(\Omega) \)-norm, namely, for \( \delta, \delta' \in [0, \bar{\delta}] \), we have
   \[ \lim_{\delta \to \delta'} \sup_{x \in \Omega} |\phi_{\varepsilon, \delta}(x) - \phi_{\varepsilon, \delta'}(x)| = 0. \]

Proof. We show that \( T_{\varepsilon, \delta} \) is a contraction mapping on \( B \) and is continuous in \( \delta \).

Existence. We first show the unique existence of the fixed point. Let \( \varepsilon \in (0, \varepsilon_2) \), \( \delta \in [0, \bar{\delta}] \). \( \varepsilon_2 \) was given in Lemma 3.1. Then we can estimate as follows:
\[
\| T_{\varepsilon, \delta}[\phi] \|_{L^\infty(\Omega)} \leq \| K_{\varepsilon, \delta} g_{\varepsilon, \delta} \|_{L^\infty(\Omega)} + \| K_{\varepsilon, \delta} M_{\varepsilon, \delta}[\phi] \|_{L^\infty(\Omega)} \\
\leq C_\infty (\| g_{\varepsilon, \delta} \|_{L^\infty(\Omega)} + \| M_{\varepsilon, \delta}[\phi] \|_{L^\infty(\Omega)} ) \\
\leq C_\infty (C_1 + 2\omega_1(\varepsilon)C_\infty C_1),
\]
where \( \omega_1(\varepsilon) \) is given in Lemma 3.5 with \( L = 2C_\infty C_1 \). Here we note that the constants \( C_\infty \), \( C_1 \) are independent of \( \varepsilon \) and \( \delta \in [0, \bar{\delta}] \), and hence \( \omega_1(\varepsilon) \) tends to \( 0 \) as \( \varepsilon \to 0 \) uniformly in \( \delta \in [0, \bar{\delta}] \). Therefore, if we take \( \varepsilon_0' \) small so that
   \[ 0 < \omega_1(\varepsilon_0) \leq 1/(2C_\infty) \] for \( \varepsilon \in (0, \varepsilon_0') \), then \( \| T_{\varepsilon, \delta}[\phi] \|_{L^\infty(\Omega)} \leq 2C_1 C_\infty \) holds for
any \( \phi \in \mathcal{B} \). Hence we have \( T_{\epsilon, \delta} [\phi] \in \mathcal{B} \) for \( \phi \in \mathcal{B} \). Moreover, it follows from Lemma 3.5 that

\[
\| T_{\epsilon, \delta} [\phi_1] - T_{\epsilon, \delta} [\phi_2] \|_{L^\infty (\Omega)} \leq C_\infty \| M_{\epsilon, \delta} [\phi_1] - M_{\epsilon, \delta} [\phi_2] \|_{L^\infty (\Omega)} \\
\leq 2 C_\infty^2 C_1 \omega_1 (\epsilon) \| \phi_1 - \phi_2 \|_{L^\infty (\Omega)}
\]

for any \( \phi_1, \phi_2 \in \mathcal{B} \). Therefore, if we take \( \epsilon'' \) small so that \( \omega_1 (\epsilon) \leq 1/(4C_\infty^2 C_1) \) for \( \epsilon \in (0, \epsilon'') \), then

\[
\| T_{\epsilon, \delta} [\phi_1] - T_{\epsilon, \delta} [\phi_2] \|_{L^\infty (\Omega)} \leq \frac{1}{2} \| \phi_1 - \phi_2 \|_{L^\infty (\Omega)} \tag{3.58}
\]

holds for any \( \phi_1, \phi_2 \in \mathcal{B} \) and \( \delta \in [0, \delta] \). Thus, if we take \( \epsilon_0 := \min \{ \epsilon', \epsilon'' \} \), then \( T_{\epsilon, \delta} : \mathcal{B} \to \mathcal{B} \) is a contraction mapping for \( 0 < \epsilon \leq \epsilon_0 \) and \( \delta \in [0, \delta] \). Therefore, by Lemma 3.7, \( T_{\epsilon, \delta} \) has a unique fixed point in \( \mathcal{B} \), which is denoted by \( \phi_{\epsilon, \delta} \).

**Continuity.** We show that the fixed point \( \phi_{\epsilon, \delta} \) is continuous in \( \delta \in [0, \delta] \). It suffices to show that the mapping \( T_{\epsilon, \delta} (\phi) \) is continuous with respect to \( (\delta, \phi) \in \mathcal{B} \times [0, \delta] \) in the space \( \mathcal{B} \) with \( L^\infty (\Omega) \)-norm. However, by the uniform estimate (3.58) with respect to \( \delta \in [0, \delta] \), we can see that \( T_{\epsilon, \delta} [\phi] \) is continuous in \( \phi \in \mathcal{B} \) uniformly with respect to \( \delta \in [0, \delta] \). Thus, it suffices to show that \( T_{\epsilon, \delta} [\phi] \) is continuous in \( \delta \in [0, \delta] \) for any \( \phi \in \mathcal{B} \). For any \( \delta, \delta' \in [0, \delta] \) and any \( \phi \in \mathcal{B} \), we can estimate as follows:

\[
\| T_{\epsilon, \delta} [\phi] - T_{\epsilon, \delta'} [\phi] \|_{L^\infty (\Omega)} \\
\leq \| K_{\epsilon, \delta} g_{\epsilon, \delta} - K_{\epsilon, \delta'} g_{\epsilon, \delta'} \|_{L^\infty (\Omega)} + \| K_{\epsilon, \delta} M_{\epsilon, \delta} [\phi] - K_{\epsilon, \delta'} M_{\epsilon, \delta'} [\phi] \|_{L^\infty (\Omega)} \\
\leq \| K_{\epsilon, \delta} g_{\epsilon, \delta} - K_{\epsilon, \delta'} g_{\epsilon, \delta'} \|_{L^\infty (\Omega)} + \| K_{\epsilon, \delta} M_{\epsilon, \delta} [\phi] - K_{\epsilon, \delta'} M_{\epsilon, \delta'} [\phi] \|_{L^\infty (\Omega)} \\
+ \| K_{\epsilon, \delta} M_{\epsilon, \delta} [\phi] - K_{\epsilon, \delta'} M_{\epsilon, \delta'} [\phi] \|_{L^\infty (\Omega)} \\
\leq \| K_{\epsilon, \delta} g_{\epsilon, \delta} - K_{\epsilon, \delta'} g_{\epsilon, \delta'} \|_{L^\infty (\Omega)} + \| K_{\epsilon, \delta} M_{\epsilon, \delta} [\phi] - K_{\epsilon, \delta'} M_{\epsilon, \delta'} [\phi] \|_{L^\infty (\Omega)} \\
+ \| K_{\epsilon, \delta} M_{\epsilon, \delta} [\phi] - K_{\epsilon, \delta'} M_{\epsilon, \delta'} [\phi] \|_{L^\infty (\Omega)} \\
+ C_\infty \| g_{\epsilon, \delta} - g_{\epsilon, \delta'} \|_{L^\infty (\Omega)} \\
+ C_\infty \| M_{\epsilon, \delta} [\phi] - M_{\epsilon, \delta'} [\phi] \|_{L^\infty (\Omega)}.
\]

The first term and the third term of the right hand side tend to 0 as \( \delta' \to \delta \) by Lemma 3.5. The second term also tends to 0 as \( \delta' \to \delta \) by Lemma 3.3. The fourth term is estimated as follows:

\[
\| M_{\epsilon, \delta} [\phi] - M_{\epsilon, \delta'} [\phi] \|_{L^\infty (\Omega)} \\
\leq \frac{1}{\epsilon} \{ \| f_0(U_{\epsilon, \delta} + \epsilon \phi) - f_0(U_{\epsilon, \delta'} + \epsilon \phi) \|_{L^\infty (\Omega)} \\
+ \| f_0(U_{\epsilon, \delta} - \epsilon \phi) - f_0(U_{\epsilon, \delta'}) \|_{L^\infty (\Omega)} \}
\]

By Lemma 2.3, all the terms of this right hand side tend to 0 as \( \delta' \to \delta \). Thus, we conclude that \( \| \phi_{\epsilon, \delta} - \phi_{\epsilon, \delta'} \|_{L^\infty (\Omega)} \to 0 \) as \( \delta' \to \delta \). \( \square \)

As we observed before, by Lemma 3.8, we obtain an axially symmetric solution to (3.1) for \( \sigma = 0 \) by putting

\[
us(x; \epsilon) = U_{\epsilon, \delta}(x) + \epsilon \phi_{\epsilon, \delta}.
\tag{3.59}
\]
From the continuity of $u_\delta$ in $\delta$, we immediately see that $U_{\varepsilon,\delta}$ is continuous in $\delta \in [0, \delta]$ with respect to $C^0(\Omega)$-norm. Hence, $u_\delta(x; \varepsilon)$ is continuous in $\delta \in [0, \delta]$ with respect to $C^0(\Omega)$-norm. Because $\phi \in \mathcal{B}$, the estimate (3.8) is obtained with $C = 2C_\infty C_1$.

Let us prove the estimate (3.9). To show it, we prepare a lemma which was given in [20].

Lemma 3.9. [20, Lemma 4.2] Let $d \in (0, \infty)$ be fixed, and let $u$ be a solution of class $C^2(G) \cap C(\overline{G})$ to the following elliptic equation:

$$
\varepsilon^2 \left( \sum_{i,j=1}^N a_{ij} D_{ij} u + \sum_{i=1}^N b_i D_i u \right) - cu = 0 \text{ in } G,
$$

where $\varepsilon > 0$ and $G$ is a bounded domain in $\mathbb{R}^N$. If $a_{ij}$ and $b_i$ are bounded in $G$, and $c \geq \mu^*$ in $G$ for some $\mu^* > 0$, and $\text{diam}(G) \leq d$, then it holds that

$$
|u(x)| \leq 2 \left( \sup_{x \in G} |u(x)| \right) e^{-\mu \text{dist}(x, \partial G)/\varepsilon}, \ x \in G,
$$

the constant $\mu > 0$ is depending only on $\mu^*$, $d$, and the $L^\infty$-bounds of $a_{ij}$ and $b_i$.

Lemma 3.10. For $u_\delta(x; \varepsilon)$ given by (3.59), the estimate (3.9) holds.

Proof. We first note that $u_\delta(x; \varepsilon)$ satisfies the following:

$$
\varepsilon^2 \Delta u_\delta(x; \varepsilon) - c(x) u_\delta(x; \varepsilon) = 0 \text{ in } \Omega, \quad (3.60)
$$

$$
c(x) := 1 - \frac{f_\delta(u_\delta(x; \varepsilon))}{u_\delta(x; \varepsilon)}.
$$

Recall that $u_\delta(x; \varepsilon)$ takes the form:

$$
u_\delta(x; \varepsilon) = \sum_{k=1}^m \chi_{\left[ \frac{1}{\varepsilon}, \frac{1}{\varepsilon} \right]}[\Psi(x; P_k)] u_\delta \left( \frac{1}{\varepsilon} \Psi(x; P_k) \right) + \varepsilon \phi_{\varepsilon, \delta}(x),
$$

$u_\delta(y)$ decays exponentially at infinity, and $\phi_{\varepsilon, \delta}$ is bounded. Hence, by the assumption $(\Omega 2)$, it is easy to see that, if we take $R > 0$ large enough, then it holds that

$$
c(x) \geq \frac{1}{2}, \ x \in \Omega_\delta^c := \Omega \setminus \bigcup_{k=1}^m \Phi(B_{\varepsilon R}; P_k), \ \delta \in [0, \delta] \quad (3.61)
$$

for sufficiently small $\varepsilon$. Put $\Omega_\delta^c := \Omega \setminus \Omega_\delta$. We may assume that $\Phi(B_{\varepsilon R}; P_k) \subset \Phi(B_{\varepsilon R}; P_k), \ k = 1, 2, \cdots, m$, since we consider $\varepsilon$ to be so small.

We first prove in the case $x \in \Omega_\delta^c$. For $x \in \Phi(B_{\varepsilon R}; P_k) \cap \Omega, \ k = 1, 2, \cdots, m$, we can estimate by Lemma 2.6 as follows:

$$
u_\delta(x; \varepsilon) \leq C \exp \left\{ - \frac{c}{\varepsilon} |\Psi(x; P_k)| \right\} + C' \varepsilon
$$

$$
\leq C \exp \left\{ - \frac{c}{\varepsilon} \text{dist}(x, P) \right\} + C' \varepsilon
$$

$$
\leq (C + C' \varepsilon e^{C' R}) e^{-c' \text{dist}(x, P)/\varepsilon}, \ x \in \Phi(B_{\varepsilon R}; P_k) \cap \Omega,
$$

where $C' > 0$.
where \( P = \{P_1, \ldots, P_m\} \), the constants \( C, C', c, c' > 0 \) are independent of \( \delta \) and \( \epsilon \). Hence (3.9) follows in the case \( x \in \Omega'_\epsilon \).

Next, we prove in the case \( x \in \Omega'_\epsilon \). For \( P_k \), we assume the case (3.3). We can do similarly also in the case (3.4). Put

\[
v_\delta(y; \epsilon) := u_\delta(\Phi(y; P_k)), \quad \tilde{c}(y) = c(\Phi(y; P_k)), \quad y = \Psi(x; P_k), \quad x \in \Phi(B^+_r; P_k),
\]

and we extend \( v_\delta(y; \epsilon) \) and \( \tilde{c} \) to functions on \( B_{3\delta_0} \) by reflection. Note that the extended function \( v_\delta(y; \epsilon) \) is of class \( C^2 \) on \( B_{3\delta_0} \). By an argument analogous to that used in step 1 in the proof of Lemma 3.1, we see that \( v_\delta(x; \epsilon) \) satisfies the following equation:

\[
\varepsilon^2 \left( \sum_{m,n=1}^N \tilde{a}_{mn} D_{mn} v_\delta(y; \epsilon) + \sum_{m=1}^N \tilde{b}_m D_m v_\delta(y; \epsilon) \right) - \tilde{c} v_\delta(y; \epsilon) = 0 \text{ in } B_{3\delta_0} \setminus B_{\varepsilon R}.
\]

The coefficients \( \tilde{a}_{mn} \) and \( \tilde{b}_m \) are bounded independently of \( \delta \) and \( \varepsilon \), and \( \tilde{c}(y) \geq 1/2, y \in B_{3\delta_0} \). Hence, by using Lemma 3.9 for \( G = B_{3\delta_0} \setminus B_{\varepsilon R} \), we have

\[
v_\delta(y; \epsilon) \leq 2 \left( \sup_{y \in B_{3\delta_0} \setminus B_{\epsilon R}} \{v_\delta(x; \epsilon)\} \right) \exp \left\{ -\frac{\mu}{\epsilon} \text{dist}(y, \partial (B_{3\delta_0} \setminus B_{\varepsilon R})) \right\}, \quad y \in B_{3\delta_0} \setminus B_{\epsilon R},
\]

for some \( \mu > 0 \) independent of \( \varepsilon \) and \( \delta \). Noting that \( v_\delta(y; \epsilon) \) is bounded independently of \( \varepsilon \) and \( \delta \), and it particularly holds that

\[
\text{dist}(y, \partial (B_{3\delta_0} \setminus B_{\varepsilon R})) = \text{dist}(y, \partial B_{\varepsilon R}) = |y| - \epsilon R, \quad y \in B_{3\delta_0/2} \setminus B_{\varepsilon R},
\]

we see from (3.64) that

\[
v_\delta(y; \epsilon) \leq C \exp \left\{ -\frac{\epsilon}{\varepsilon} |y| \right\}, \quad y \in B_{3\delta_0} \setminus B_{\varepsilon R}.
\]

(3.65)

for some \( C, c > 0 \) independent of \( \varepsilon \) and \( \delta \). Clearly, these constants \( C, c > 0 \) can be taken uniformly in \( k = 1, \ldots, m \). Thus we have

\[
u_\delta(x; P_k) \leq C \exp \left\{ -\frac{\epsilon}{\varepsilon} \text{dist}(x, P) \right\}, \quad x \in \Omega'_\epsilon \cap \left( \bigcup_{k=1}^m \Phi(B_{3\delta_0/2}; P_k) \right).
\]

(3.66)

It remains to show in the case \( x \in \Omega' := \Omega \setminus \bigcup_{k=1}^m \Phi(B_{3\delta_0/2}; P_k) \). We claim that

\[
u_\delta(x; \epsilon) \leq C e^{-c/\epsilon}, \quad x \in \Omega',
\]

(3.67)

holds for some constants \( C, c > 0 \) independent of \( \varepsilon \) and \( \delta \). Noting (3.60) and (3.61), by applying Lemma 3.9 for \( G = \Omega'_\epsilon \), we obtain the following estimate:

\[
u_\delta(x; \epsilon) \leq C \exp \left\{ -\frac{\mu}{\varepsilon} \text{dist}(x, \partial \Omega'_{\epsilon}) \right\}, \quad x \in \Omega'_{\epsilon},
\]

(3.68)

for some \( C, \mu > 0 \) independent of \( \varepsilon \) and \( \delta \). For sufficiently small \( \kappa > 0 \), we put

\[D = \{x \in \Omega' : \text{dist}(x, \partial \Omega'_{\epsilon}) \geq \kappa\}, \quad D' = \{x \in \Omega' : \text{dist}(x, \partial \Omega'_{\epsilon}) < \kappa\}
\]

\[\Gamma_1 = \partial D' \cap \partial \left( \bigcup_{k=1}^m \Phi(B_{3\delta_0/2}; P_k) \right), \quad \Gamma_2 = \partial D' \cap \partial D.
\]

64
From (3.68), we have
\[ u_\delta(x; \varepsilon) \leq C e^{-\mu \varepsilon / \varepsilon}, \quad x \in D. \]

By (3.60) and (3.61), we note that
\[ \varepsilon^2 \Delta u_\delta(x; \varepsilon) = c(x) u_\delta(x; \varepsilon) \geq \frac{1}{2} u_\delta(x; \varepsilon) > 0, \quad x \in D', \]
\[ \frac{\partial u_\delta(x; \varepsilon)}{\partial \nu} = 0, \quad x \in \partial \Omega \cap \partial D'. \]

By the maximum principle and Hopf’s lemma, \( u_\delta(x; \varepsilon) \) attain its local maximum neither in \( D' \) nor on \( \partial \Omega \cap \partial D' \). Hence we see that
\[ u_\delta(x; \varepsilon) \leq \sup_{x \in \Gamma_1 \cap \Gamma_2} u_\delta(x; \varepsilon), \quad x \in D'. \]

Using (3.66) for \( x \in \Gamma_1 \) and (3.68) for \( x \in \Gamma_2 \), we have
\[ \sup_{x \in \Gamma_1 \cap \Gamma_2} u_\delta(x; \varepsilon) \leq C e^{-c / \varepsilon}. \]

Thus we have (3.67). Noting that
\[ \gamma_1 \leq \text{dist}(x, P) \leq \gamma_2, \quad x \in \Omega' \]
holds for some \( \gamma_1, \gamma_2 > 0 \), we see by (3.67) that
\[ u_\delta(x; \varepsilon) \leq C \exp \left\{ - \frac{c}{\gamma_1 \varepsilon} \text{dist}(x, P) \right\}, \quad x \in \Omega'. \quad (3.69) \]

Thus, by (3.62), (3.66) and (3.69), we have a conclusion. \( \square \)

Next, to show (3.10) we first show the following lemma.

**Lemma 3.11.** The following statements (i) and (ii) hold:

(i) For each \( r > 0 \) and \( \eta \in (0, 1) \), there exists a constant \( C > 0 \) independent of \( \varepsilon \) and \( \delta \in [0, \bar{\delta}] \) such that
\[ \int_\Omega |\phi_{\varepsilon, \delta}(x)|^r \, dx \leq C \varepsilon^{N \eta}. \]

(ii) For each \( r > 0 \), it holds that
\[ \int_\Omega U_{\varepsilon, \delta}^r(x) \, dx = \varepsilon^{N m / 2} \int_{\mathbb{R}^n} w_\delta(y) \, dy + O(\varepsilon^{N+1}), \]
as \( \varepsilon \to 0 \) uniformly in \( \delta \in [0, \bar{\delta}] \).
Proof. (i) We note that the function $U_{\varepsilon,\delta}$ can be estimate as follows:

$$U_{\varepsilon,\delta}(x) \leq C\exp\left(-\frac{c}{\varepsilon} \text{dist}(x,P)\right),\quad P = \{P_1,\ldots,P_m\},$$  \hspace{1cm} (3.70)

for some constants $C, c > 0$ independent of $\varepsilon$ and $\delta \in [0,\bar{\delta}]$. Indeed, $U_{\varepsilon,\delta}(x) = 0$ for $x \not\in \bigcup_{k=1}^m \Phi(B_{2r_0}; P_k)$, and we can see that, for $x \in \Phi(B_{2r_0}; P_k)$,

$$0 \leq U_{\varepsilon,\delta}(x) \leq w_\delta\left(\frac{1}{\varepsilon} \Phi(x;P_k)\right) \leq C \exp\left(\frac{c}{\varepsilon} \text{dist}(x,P)\right),$$

holds for each $k = 1,\ldots,m$, for some constants $C, c > 0$ by Lemma 2.6. Thus (3.70) holds.

Combining (3.70) and (3.9), $\phi_{\varepsilon,\delta}(x) = \frac{1}{\varepsilon}(u_\delta(x;\varepsilon) + U_{\varepsilon,\delta}(x))$ satisfies

$$|\phi_{\varepsilon,\delta}(x)| \leq C\exp\left(-\frac{c}{\varepsilon} \text{dist}(x,P)\right),$$  \hspace{1cm} (3.71)

for some constants $C, c > 0$ independent of $\varepsilon$ and $\delta \in [0,\bar{\delta}]$. Noting this regard, we divide $\Omega$ as $\Omega = I_\varepsilon \cup I'_\varepsilon$ where $I_\varepsilon := \bigcup_{k=1}^m B_{\varepsilon}\Phi(P_k) \cap \Omega$, $I'_\varepsilon := \Omega \setminus I_\varepsilon$, and divide the integral on $\Omega$ as follows:

$$\int_{\Omega} |\phi_{\varepsilon,\delta}(x)|^r dx = \int_{I_\varepsilon} |\phi_{\varepsilon,\delta}(x)|^r dx + \int_{I'_\varepsilon} |\phi_{\varepsilon,\delta}(x)|^r dx. \hspace{1cm} (3.72)$$

The first term is estimated as follows:

$$\int_{I_\varepsilon} |\phi_{\varepsilon,\delta}(x)|^r dx \leq Cm\varepsilon^{\eta},$$  \hspace{1cm} (3.73)

because $|\phi_{\varepsilon,\delta}|$ is bounded uniformly in $\delta \in [0,\bar{\delta}]$ and $\varepsilon$ sufficiently small. The second term is estimated as follows:

$$\int_{I'_\varepsilon} |\phi_{\varepsilon,\delta}(x)|^r dx \leq C'\varepsilon^{-c'/\varepsilon^{1-\eta}} \leq C''\varepsilon^{\eta},$$  \hspace{1cm} (3.74)

for some constants $C', c', C'' > 0$ independent of $\delta \in [0,\bar{\delta}]$ and $\varepsilon$ sufficiently small by (3.71). Thus we complete the proof.

(ii) For each $k = 1,\ldots,m$, we put

$$D^k_\varepsilon := \Phi(B_{r_0};P_k) \cap \Omega, \quad i = 1,2,$$

$$I^k_{\varepsilon,\delta} := \varepsilon^{-N} \int_{D^k_\varepsilon} \lambda^r\left(\frac{1}{r_0} |\Phi(x;P_k)|\right) \frac{1}{\varepsilon} \Phi(x;P_k) dx.$$ 

By the definition of $U_{\varepsilon,\delta}(x)$, we see that

$$\varepsilon^{-N} \int_{\Omega} U^r_{\varepsilon,\delta}(x) dx = \sum_{k=1}^m I^k_{\varepsilon,\delta}.$$
Here we notice that
\[
\epsilon^{-N} \int_{D^k_1} w_\delta^r(\frac{1}{\epsilon} \Psi(x; P_k)) dx \leq I^k_{\epsilon, \delta} \leq \epsilon^{-N} \int_{D^k_2} w_\delta^r(\frac{1}{\epsilon} \Psi(x; P_k)) dx. \tag{3.75}
\]

In Lemma A.1 of [75], it was shown that
\[
det(D\Phi)(y; P_k) = 1 - \alpha y_N + O(|y|^2), \quad \alpha := \Delta \psi_k(0).
\]

Therefore, for the left hand side of (3.75), putting \( y = \frac{1}{\epsilon} \Psi(x; P_k) \),
\[
(l.h.s.) = \int_{B^+_r(y)} w_\delta^r(y)(1 - \alpha \epsilon y_N + O(\epsilon^2 |y|^2)) dy
\]
\[
= \frac{1}{2} \int_{B_{r^*}} w_\delta^r(y) dy + O(\epsilon) \]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} w_\delta^r(y) dy - \frac{1}{2} \int_{\mathbb{R}^N \setminus B_{r^*}} w_\delta^r(y) dy + O(\epsilon)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} w_\delta^r(y) dy + O(\epsilon)
\]
holds as \( \epsilon \to 0 \) uniformly in \( \delta \) by Lemma 2.6. We can see that the right hand side of (3.75) also has the same behavior as \( \epsilon \to 0 \). Thus we have
\[
\epsilon^{-N} \int_{\Omega} U^r_{\epsilon, \delta}(x) dx = \sum_{k=1}^{m} I^k_{\epsilon, \delta} = \frac{m}{2} \int_{\mathbb{R}^N} w_\delta^r(y) dy + O(\epsilon)
\]
as \( \epsilon \to 0 \) uniformly in \( \delta \), and we complete the proof.

\[\square\]

**Lemma 3.12.** For \( u_\delta(x; \epsilon) \) given by (3.59), (3.10) holds.

**Proof.** We first claim that
\[
| \int_{\Omega} (u_\delta^r(x; \epsilon) - U^r_{\epsilon, \delta}(x)) dx | = o(\epsilon^N) \tag{3.76}
\]
holds as \( \epsilon \to 0 \) uniformly in \( \delta \in [0, \bar{\delta}] \).

In the case \( 0 < r \leq 1 \),
\[
| \int_{\Omega} (u_\delta^r(x; \epsilon) - U^r_{\epsilon, \delta}(x)) dx | \leq \int_{\Omega} |u_\delta(x; \epsilon) - U_{\epsilon, \delta}(x)|^r dx
\]
\[
= \int_{\Omega} |\epsilon \phi_\delta(x)|^r dx
\]
\[
\leq C \epsilon^r \epsilon^{N\eta},
\]
holds for some \( C > 0 \) independent of \( \delta \in [\delta_0, \delta_1] \) by Lemma 3.11(i). Hence, by taking \( \eta \in (0, 1) \) near 1 enough in advance, we have (3.76).

In the case \( 1 < r \). We use the inequality: for positive \( u \) and \( v \),
\[
|u^r - v^r| \leq r(u^{r-1} + v^{r-1})|u - v|.
\]
Because $u_3(x; \varepsilon)$ and $U_{\varepsilon, \delta}$ are bounded in $\Omega$ uniformly in $\delta \in [0, \bar{\delta}]$ and $\varepsilon$ sufficiently small, we have the following estimate:

\[
| \int_{\Omega} (u_3^\prime(x; \varepsilon) - U_{\varepsilon, \delta}^\prime(x)) dx | \leq C \varepsilon \| \phi_{\varepsilon, \delta} \|_{L^1(\Omega)} \leq C \varepsilon \varepsilon^N \eta = o(\varepsilon^N),
\]

as $\varepsilon \to 0$ for some constants $C, C' > 0$ independent of $\delta$ by Lemma 3.11(i) and taking $\eta \in (0, 1)$ near 1 enough in advance.

Thus, we can see by (3.76) and Lemma 3.11(ii) that

\[
\int_{\Omega} u_3^\prime(x; \varepsilon) dx = \int_{\Omega} U_{\varepsilon, \delta}^\prime(x) dx + o(\varepsilon^N) = \varepsilon^N m_1 \int_{\mathbb{R}^N} w_k^\prime(y) dy + o(\varepsilon^N),
\]

holds as $\varepsilon \to 0$ uniformly in $\delta \in [0, \bar{\delta}]$. Thus we complete the proof. 

By these lemmas, we complete the proof of Theorem 3.1.

### 3.5 Proof of Theorem 3.2

The solution to (3.1) will be given as a perturbation of the solution $u_3(x; \varepsilon)$ given by Theorem 3.1. For the purpose, we first show the following lemma.

**Lemma 3.13.** Let $u_3(x; \varepsilon)$ be a solution to (3.1) for $\sigma = 0$, given by Theorem 3.1. Then there exists $\varepsilon_3 > 0$, the following linearized operator

\[
L_{u_3(\varepsilon, \delta)} := \varepsilon^2 \Delta - 1 + f_3^\prime(u_3(x; \varepsilon))
\]

on $L^1(\Omega)$ with $\text{Dom}(L_{u_3(\varepsilon, \delta)}) = X^{2, t}_0$, $1 < t < \infty$, has a bounded inverse $K_{u_3(\varepsilon, \delta)} : X^t \to X^{2, t}$ for $\varepsilon \in (0, \varepsilon_3)$ and $\delta \in [0, \bar{\delta}]$. Moreover, for each $t \in (1, \infty]$, there exists a constant $C_1^* > 0$ independent of $\varepsilon$ and $\delta$, it holds that

\[
\| K_{u_3(\varepsilon, \delta)} g \|_{L^1(\Omega)} \leq C_1^* \| g \|_{L^1(\Omega)}, \quad g \in X^t, \quad \delta \in [0, \bar{\delta}].
\]

**Remark 3.6.** If the inverse $K_{u_3(\varepsilon, \delta)} : X^t \to X^{2, t}$ of $L_{u_3(\varepsilon, \delta)}$ exists for $t \in (1, \infty)$, then we can regard $K_{u_3(\varepsilon, \delta)}$ as a mapping from $X^\infty$ into $X^\infty$ by Sobolev’s embedding theorem. Therefore, the estimate (3.78) is meaningful even if $t = \infty$.

**Proof.** Let $g \in X^t$ be given. For $v \in X^{2, t} \nu$, the following equations are equivalent:

\[
L_{u_3(\varepsilon, \delta)} v = g, \quad L_{\varepsilon, \delta} + (L_{u_3(\varepsilon, \delta)} - L_{\varepsilon, \delta}) v = g, \quad v + K_{\varepsilon, \delta} f_3^\prime(u_3(x; \varepsilon)) - f_3^\prime(U_{\varepsilon, \delta}(x)) v = K_{\varepsilon, \delta} g.
\]

Recall that $u_3(x; \varepsilon)$ and $U_{\varepsilon, \delta}(x)$ are bounded uniformly in $\delta \in [0, \bar{\delta}]$ and $\varepsilon$ sufficiently small, and

\[
\| u_3(\cdot; \varepsilon) - U_{\varepsilon, \delta} \|_{L^\infty(\Omega)} \leq C \varepsilon, \quad \delta \in [0, \bar{\delta}].
\]
By Lemma 2.3, it holds that
\[ \| f_\delta'(u_\delta(\cdot; \varepsilon)) - f_\delta'(U_{\varepsilon, \delta}) \|_{L^\infty(\Omega)} \leq \omega(\varepsilon), \]
the quantity \( \omega(\varepsilon) \) converges to 0 as \( \varepsilon \to 0 \) uniformly in \( \delta \in [0, \delta] \). By Lemma 3.2, we can estimate as follows:
\[ \| K_{\varepsilon, \delta}([f_\delta'(u_\delta(\cdot; \varepsilon)) - f_\delta'(U_{\varepsilon, \delta}))v] \|_{L^1(\Omega)} \leq C_\varepsilon \omega(\varepsilon) \| v \|_{L^1(\Omega)}. \]
Therefore, for given \( g \in X^t \), there exists unique \( K_{u(\varepsilon, \delta)}g := v \in X^t \) satisfying (3.81) provided \( \varepsilon \) is small enough by the Neumann series theory. We know that \( K_{\varepsilon, \delta} \) is a mapping from \( X^t \) onto \( X^t_{\varepsilon, \delta} \), hence we have \( v \in X^t_{\varepsilon, \delta} \) from (3.81). Hence we have \( K_{u(\varepsilon, \delta)}g = v \). In particular, for \( t \in (1, \infty) \), if \( \varepsilon \) is small enough so that \( C_\varepsilon \leq 1/2 \) then we have
\[ \| v \|_{L^1(\Omega)} \leq \| K_{\varepsilon, \delta}([f_\delta'(u_\delta(\cdot; \varepsilon)) - f_\delta'(U_{\varepsilon, \delta}))v] \|_{L^1(\Omega)} + \| K_{\varepsilon, \delta}g \|_{L^1(\Omega)} \]
\[ \leq \frac{1}{2} \| v \|_{L^1(\Omega)} + C_\varepsilon \| g \|_{L^1(\Omega)}. \]
Thus we have \( \| v \|_{W^{2,1}(\Omega)} \leq 2C_\varepsilon \| g \|_{L^1(\Omega)} \). We complete the proof.

**Proof of Theorem 3.2.** **Existence.** For \( u_\delta(x; \varepsilon) \) given in Theorem 3.1, put
\[ u(x) = u_\delta(x; \varepsilon) + \phi(x), \]
and substitute this into (3.1), then we have
\[ 0 = L_{u(\varepsilon, \delta)} \phi + M_{\delta}(\phi) + \sigma, \tag{3.82} \]
where
\[ M_{\delta}(\phi) := f_\delta(u_\delta + \phi) - f_\delta(u_\delta) - f_\delta'(u_\delta)\phi(x). \]
Here, we write \( u_\delta = u_\delta(\cdot; \varepsilon) \) for simplicity. Multiplying both side of (3.82) by \( K_{u(\varepsilon, \delta)} \), we have
\[ \phi = -(K_{u(\varepsilon, \delta)}[M_{\delta}(\phi)] + K_{u(\varepsilon, \delta)}[\sigma]) =: T_{\varepsilon, \sigma}(\phi). \tag{3.83} \]
Let us write \( \| \cdot \| = \| \cdot \|_{L^\infty(\Omega)} \) simply. We put \( \sigma_* := \| \sigma_0 \|_{\infty} \), and define
\[ \bar{\mathcal{B}} = \{ \phi \in X^0 : \| \phi \|_{\infty} \leq 2C_* \sigma_* \}, \]
where \( C_* \) is a constant given in Lemma 3.13 for \( t = \infty \). Let us find a fixed point of the mapping \( T_{\varepsilon, \sigma} \) in \( \bar{\mathcal{B}} \). First, note that \( M_{\delta}(\phi) \) is written as follows:
\[ M_{\delta}(\phi) = \int_0^1 \left\{ f_\delta'(u_\delta + \phi t) - f_\delta'(u_\delta) \right\} dt \cdot \phi. \tag{3.84} \]
Note that
\[ \| (u_\delta + \phi t) - u_\delta \|_{\infty} \leq \| \phi \|_{\infty} \leq 2C_* \sigma_*, \phi \in \bar{\mathcal{B}}. \]
By Lemma 2.3, it follows that
\[ \| M_\delta (\phi) \|_\infty \leq \omega (\sigma_*), \quad \phi \in \bar{B}, \]
where \( \omega (\sigma_*) \) is certain value such that \( \omega (\sigma_*) \to 0 \) as \( \sigma_* \to 0 \) uniformly in \( \delta \in [0, \delta] \). Here, we recall that there is a certain \( \varepsilon_1 > 0 \), the estimate (3.8) with \( t = \infty \) holds for \( \varepsilon \in (0, \varepsilon_1) \) and \( \delta \in [0, \delta] \). We assume \( \varepsilon \in (0, \varepsilon_1) \) henceforth. Then we can estimate as follows:
\[
\| T_{\delta, \sigma} (\phi) \|_\infty = \| K_{u(\varepsilon, \delta)} [M_\delta (\phi) + \sigma] \|_\infty \\
\leq C^* (|M_\delta (\phi)|_\infty + \sigma) \\
\leq C^* (\omega (\sigma_*) |\phi|_\infty + \sigma) \\
\leq C^* (2C^* \omega (\sigma_*))_\sigma + \sigma_*. 
\]
Therefore, if \( \omega (\sigma_*) \leq 1/(2C^*_\infty) \), then it holds that \( T_{\delta, \sigma} (\phi) \in \bar{B} \) for \( \phi \in \bar{B} \). On the other hand, we can estimate similarly to (3.85) as follows:
\[
| M_\delta (\phi_1) - M_\delta (\phi_2) | = | f_\delta (u_\delta + \phi_2) - f_\delta (u_\delta + \phi_1) + f_\delta (u_\delta) (\phi_1 - \phi_2) | \\
= \left| \int_0^1 f_\delta (u_\delta + \phi_2 + (\phi_1 - \phi_2) t) - f_\delta (u_\delta) dt \right| (\phi_1 - \phi_2) \\
\leq \omega (\sigma_*) |\phi_1 - \phi_2|. 
\]
By this, we have
\[
\| T_{\delta, \sigma} (\phi_1) - T_{\delta, \sigma} (\phi_2) \|_\infty = \| K_{u(\varepsilon, \delta)} [M_\delta (\phi_1) - M_\delta (\phi_2)] \|_\infty \\
\leq C^* (\omega (\sigma_*) |\phi_1 - \phi_2|_\infty. 
\]
Therefore, if \( 0 \leq \omega (\sigma_*) \leq 1/(2C^*_\infty) \), then
\[
\| T_{\delta, \sigma} (\phi_1) - T_{\delta, \sigma} (\phi_2) \|_\infty \leq \frac{1}{2} |\phi_1 - \phi_2|_\infty, \quad \delta \in [0, \delta], 
\]
holds for any \( \phi_1, \phi_2 \in \bar{B} \). Thus, if we take \( \sigma_1 > 0 \) small enough, there exists a unique fixed point \( \phi_{\delta, \sigma} \) of \( T_{\delta, \sigma} \) in \( \bar{B} \) for every \( \sigma \) satisfying \( |\sigma|_\infty \leq \sigma_1 \) and for \( \delta \in [0, \delta] \). Now, noting that \( K_{u(\varepsilon, \delta)} \) is a mapping from \( X^t \) into \( X^{2j}_\nu \), and (3.83), we obtain \( \phi_{\delta, \sigma} \in X^{2j}_\nu \). By choosing \( t > N \), \( \phi_{\delta, \sigma} \in C^{1, \beta} (\bar{\Omega}) \) holds for some \( \theta \in (0, 1) \). Furthermore, applying Schauder’s theory, \( \phi_{\delta, \sigma} \in C^{2, \theta} (\bar{\Omega}) \) follows. Thus, \( u_\delta (x; \varepsilon, \sigma) := u_\delta (x) + \phi_{\delta, \sigma} \) becomes a solution to (3.1).

**Continuity.** Because of the contraction mapping theorem with parameter (Lemma 3.7), to prove the continuity of \( \phi_{\delta, \sigma} \) with respect to parameters \( \delta \) and \( \sigma \), it suffices to show that, for any \( \phi \in \bar{B}, \delta \in [0, \delta] \) and \( \sigma \in C^0 (\bar{\Omega}) \) with \( |\sigma|_\infty \leq \sigma_1 \),
\[
\| T_{\delta, \sigma} (\phi) - T_{\delta', \sigma'} (\phi) \|_\infty \to 0 
\]
holds as \( \delta' \to \delta \) and \( \sigma' \to \sigma \) in \( C^0 (\bar{\Omega}) \). To show this, we first note that
\[
\| K_{u(\varepsilon, \delta)} (\phi) - K_{u(\varepsilon, \delta')} (\phi) \|_\infty \leq \omega (\delta, \delta') |\phi|_\infty, 
\]
70
where \( \omega(\delta, \delta') > 0 \) is a certain value independent of \( \sigma \) such that \( \omega(\delta, \delta') \to 0 \) as \( \delta' \to \delta \). We omit the proof of (3.88) since this claim is verified easily by the same argument as in the proof of Lemma 3.6. Secondly, we note that, by the definition of \( T_{\delta, \sigma} \) and Lemma 3.13,

\[
\|T_{\delta, \sigma}(\phi) - T_{\delta', \sigma'}(\phi)\| \leq C^*_{\infty} \|\sigma - \sigma'\|_{\infty} \tag{3.89}
\]

holds. Thirdly, we claim that

\[
\|T_{\delta, \sigma}(\phi) - T_{\delta', \sigma'}(\phi)\|_{\infty} \leq C^*_{\infty} \left( \omega(\delta, \delta') + \omega(\delta', \sigma') \|\phi\|_{\infty} \right) \tag{3.90}
\]

holds for some value \( \omega(\delta, \delta') \) which tends to 0 as \( \delta' \to \delta \) uniformly in \( \sigma \) such that \( \|\sigma\|_{\infty} \leq \sigma_1 \). Indeed,

\[
\|T_{\delta, \sigma}(\phi) - T_{\delta', \sigma'}(\phi)\|_{\infty} \leq \|K_{u(\varepsilon, \delta)}[M_\delta(\phi)] - K_{u(\varepsilon, \delta')}[M_{\delta'}(\phi)]\|_{\infty} \leq C^*_{\infty} \left( \|M_\delta(\phi) - M_{\delta'}(\phi)\|_{\infty} + \omega(\delta, \delta') \|M_{\delta'}(\phi)\|_{\infty} \right). \tag{3.91}
\]

The first term of the right hand side is estimated as follows:

\[
\|M_\delta(\phi) - M_{\delta'}(\phi)\|_{\infty} \leq |f_\delta'(u_\delta - f_{\delta'}(u_{\delta'})| \|\phi\| + |f_\delta(u_\delta) - f_{\delta'}(u_{\delta'})| + |f_\delta(u_\delta + \phi) - f_{\delta'}(u_{\delta'} + \phi)|
\]

By using Lemma 2.3, we can see that all the terms of the right hand side tends to 0 as \( \delta' \to \delta \) uniformly \( \sigma \) such that \( \|\sigma\|_{\infty} \leq \sigma_1 \). The second term of the right hand side in (3.91) is estimated from above by \( \omega(\delta, \delta') \|\sigma\|_{\infty} \|\phi\|_{\infty} \). Hence (3.90) is verified.

Thus (3.87) is verified by (3.89) and (3.90).

Finally, we show the invertibility of the following linearized operator:

\[
L_{u(\varepsilon, \delta, \sigma)} := \varepsilon^2 \Delta - 1 + f_\delta'(u_\delta(\cdot; \varepsilon, \sigma)). \tag{3.92}
\]

As before, we restrict the domain to \( X_0^{2, t}, t \in (1, \infty) \). Then we can see that \( L_{u(\varepsilon, \delta, \sigma)} \) has a bounded inverse provided \( \|\sigma\|_{L^\infty(\Omega)} \) is small enough. More precisely, we have the following lemma.

**Lemma 3.14.** There exists \( \bar{\sigma}_1 > 0 \) such that, if \( \varepsilon \in (0, \bar{\varepsilon}_1) \), \( \|\sigma\|_{L^\infty(\Omega)} < \bar{\sigma}_1 \) and \( \delta \in [0, \bar{\delta}] \), then the operator \( L_{u(\varepsilon, \delta, \sigma)} \) with domain \( \text{Dom}(L_{u(\varepsilon, \delta, \sigma)}) = X_0^{2, t}, t \in (1, \infty) \), has a bounded inverse \( K_{u(\varepsilon, \delta, \sigma)} : X^1 \to X_0^{2, t} \). Moreover, for each \( t \in (1, \infty) \), the following estimate holds:

\[
\|K_{u(\varepsilon, \delta, \sigma)} g\|_{L^1(\Omega)} \leq C^*_{1} \|g\|_{L^1(\Omega)}, \quad g \in X^1, \tag{3.93}
\]

for some constant \( C^*_{1} > 0 \) independent of \( \varepsilon, \delta \) and \( \sigma \).
Remark 3.7. $K_{u(\varepsilon, \delta, \sigma)}$ can be regarded as a mapping from $X^\infty$ into $X^\infty$ similarly to $K_{u(\varepsilon, \delta)}$ and $K_{\varepsilon, \delta}$.

Proof. Recall that, if $\varepsilon \in (0, \varepsilon_1)$, $\|\sigma\|_{L^\infty(\Omega)} \leq \sigma_1$ and $\delta \in [0, \delta]$, then

$$\|u_{\delta}(x; \varepsilon, \sigma) - u_{\delta}(x; \varepsilon)\|_{L^\infty(\Omega)} \leq 2C^*_\infty \|\sigma\|_{L^\infty(\Omega)},$$

Thus we can see that

$$\|f_{\delta}^\prime(u_{\delta}(\cdot; \varepsilon, \sigma)) - f_{\delta}^\prime(u_{\delta}(\cdot; \varepsilon))\|_{L^\infty(\Omega)} \leq \omega(\|\sigma\|_{L^\infty(\Omega)}),$$

holds for some quantity $\omega(\|\sigma\|_{L^\infty(\Omega)})$ which converges to 0 as $\|\sigma\|_{L^\infty(\Omega)} \to 0$ uniformly in $\delta \in [0, \delta]$ and $\varepsilon \in (0, \varepsilon_1)$. Noting this regard, by using the invertibility of $L_{u(\varepsilon, \delta)}$ given in Lemma 3.13, we can prove by the same argument as in the proof of Lemma 3.13. Thus we omit the details. \hfill \Box

3.6 Remark: nonlocal problems

In this section, let us present how the result in Theorem 3.1 will be applied in later chapters. Let some nonlocal equation be given as follows:

$$\begin{align*}
0 &= \varepsilon^2 \Delta u - u + F(u, g(u)), \quad u > 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial \Omega,
\end{align*}$$

(3.94)

where $\varepsilon > 0$, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ satisfying (A0), $F(u, v)$ is a given nonlinear term, $g(u)$ is a given nonlocal function with respect to $u$. Now, let us introduce a new parameter $\delta$, and put

$$\delta = g(u).$$

(3.95)

Then the problem (3.94) is reduced to the problem:

$$\begin{align*}
0 &= \varepsilon^2 \Delta u - u + F(u, \delta), \quad u > 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial \Omega,
\end{align*}$$

(3.96)

$$\delta = g(u).$$

(3.97)

The equation (3.96) is the type of (3.1). Thus, if the nonlinear term $F(u, \delta)$ satisfies (Ω)-(Ω3), then we can apply Theorem 3.1, and we obtain a solution $u_{\delta}(x; \varepsilon)$ to (3.96), which is continuous in $\delta$. For this $u_{\delta}(x; \varepsilon)$, if we can find $\delta = \delta_{\varepsilon}$ such that (3.97) holds, namely,

$$\delta_{\varepsilon} = g(u_{\delta_{\varepsilon}}(x; \varepsilon)),$$

(3.98)

then $u_{\delta_{\varepsilon}}(x; \varepsilon)$ becomes a solution to the original equation (3.94).

Let us call the new parameter $\delta$ a matching parameter for convenience. The equation (3.96) gives a matching condition for which the parameterized solution $u_{\delta}(x; \varepsilon)$ becomes a solution to the original equation (3.94). This method is sometimes powerful for some nonlocal Neumann problems. By using this method, we will solve some type of the chemotaxis models, the Gierer-Meinhardt system and the Schnakenberg model in later chapters.
3.7 Appendix

We prove (3.28). For each $j = 1, 2, \ldots$, we may assume $B_{3j} \subset B_{r_0/\varepsilon}$, considering $\varepsilon_i$ is small enough. Put

$$\Phi_l := \sup_{\rho \in (0,1)} (1 - \rho)^l (3j)^l \| D^l \varphi^*_i \|_{L^p(B_{3\rho})}, \quad l = 0, 1, 2. \quad (3.99)$$

then we have the following lemma.

**Lemma 3.15.** The following estimates hold:

$$\Phi_2 \leq C \{(3j)^2 \| \varphi^*_i - f'_3(\varepsilon^i r_0)w_3(z)\|_{L^p(B_{3\rho})} + \Phi_1 + \Phi_0\}, \quad (3.100)$$

$$\Phi_1 \leq t\Phi_2 + \frac{1}{t}C(N)\Phi_0 \text{ for any } t > 0, \quad (3.101)$$

where $C > 0$ is a constant depending only on $N, p$ and the bounds of the coefficients $a_{mn}^*, b_{mn}^*$, and the constant $M$ in (3.27), and it is independent of $\varepsilon_i$ and $j$.

Let this lemma holds true. Note that

$$\varphi^*_i(z) - f'_3(\varepsilon^i r_0)w_3(z)\varphi^*_i(z)$$

is bounded in $B_{r_0/\varepsilon}$ uniformly in $i$. Hence, (3.100) and (3.101) with $t = 1/(2C)$ yield

$$\Phi_2 \leq C \{(3j)^2 M_j + M'_j\}, \quad (3.102)$$

for some constants $M_j, M'_j > 0$ independent of $i$. Taking $\rho = 2/3$ in $\Phi_2$, we have

$$j^2 \| D^2 \varphi^*_i \|_{L^p(B_{2j})} \leq \Phi_2. \quad (3.103)$$

Combining (3.102) and (3.103), we have

$$\| D^2 \varphi^*_i \|_{L^p(B_{2j})} \leq C(9M_j + M'_j), \quad (3.104)$$

for all $i$ sufficiently large. On the other hand, taking $t = 1/j$ in (3.101) and (3.102), we have

$$\Phi_1 \leq \frac{1}{j} \Phi_2 + jC(N)\Phi_0 \leq C(9jM_j + \frac{M'_j}{j} + C(N)j)\Phi_0. \quad (3.105)$$

By taking $\rho = 2/3$ in $\Phi_1$, we have

$$j \| D\varphi^*_i \|_{L^p(B_{2j})} \leq \Phi_1. \quad (3.106)$$

Combining (3.105) and (3.106), we have

$$\| D\varphi^*_i \|_{L^p(B_{2j})} \leq C(9M_j + M'_j) + C(N)M'_j, \quad (3.107)$$

for all $i$ sufficiently large. By (3.104) and (3.107), we obtain the desired estimate (3.28).
Proof of Lemma 3.15. Here, we denote $3j$ by $R$ for simplicity. We use the following known fact (see [23, Corollary 9.10]).

**Lemma 3.16.** Let $U$ be a domain in $\mathbb{R}^N$, $u \in W_0^{2,p}(U)$, $1 < p < \infty$. Then
\[
\|D^2 u\|_{L^p(U)} \leq C \|\Delta u\|_{L^p(U)}
\]
holds, where $C = C(N,p)$.

From this lemma and the relation $\sum_{m,n=1}^N a_{mn}^*(0) D_{mn} u = \Delta u$, we can estimate as follows: for all $v \in W_0^{2,p}(B_R)$,
\[
\|D^2 v\|_{L^p(B_R)} \leq C \|\Delta v\|_{L^p(B_R)}
\]
\[
= C \| \sum_{m,n=1}^N a_{mn}^*(0) D_{mn} v \|_{L^p(B_R)}
\]
\[
\leq C \{ \| \sum_{m,n=1}^N (a_{mn}^*(0) - a_{mn}^*) D_{mn} v \|_{L^p(B_R)} + \| \sum_{m,n=1}^N a_{mn}^* D_{mn} v \|_{L^p(B_R)} \}
\]
\[
\leq CMr_0 \|D^2 v\|_{L^p(B_R)} + C \| \sum_{m,n=1}^N a_{mn}^* D_{mn} v \|_{L^p(B_R)},
\]
because
\[
|a_{mn}^*(0) - a_{mn}^*(z)| \leq M \varepsilon_i |z| \leq M \varepsilon_i R \leq Mr_0, \ z \in B_R
\]
holds by (3j =) $R < r_0/\varepsilon_i$. Therefore, by taking $r_0$ sufficiently small so that $CMr_0 \leq 1/2$ in advance, we have
\[
\|D^2 v\|_{L^p(B_R)} \leq 2C \| \sum_{m,n=1}^N a_{mn}^* D_{mn} v \|_{L^p(B_R)} \text{ for all } v \in W_0^{2,p}(B_R). \tag{3.108}
\]

Now, for $\rho \in (0,1)$, we define a cut-off function. For $\rho' := \frac{1+\rho}{2}$, let $\eta \in C_0^\infty(B_R)$ be a function such that $0 \leq \eta \leq 1$, $\eta(z) = 1 \ (|z| \leq \rho R)$, $\eta(z) = 0 \ (|z| \geq \rho' R)$, $|\eta'| \leq \frac{4}{(1-\rho) R}$, $|\eta''| \leq \frac{16}{(1-\rho)^2 R^2}$. Then, obviously $\eta \varphi_i^* \in W_0^{2,p}(B_R)$. By noting that $a_{mn}^*, b_{mn}^* \leq C'$ (independent of $\varepsilon_i$) holds on $B_{r_0/\varepsilon_i}$, we can estimate
as follows:
\[
\|D^2 \varphi_1^*\|_{L^p(B^p, \mathbb{R})} \leq \|D^2 (\eta \varphi_1^*)\|_{L^p(B^p)} \\
\leq 2C \| \sum_{m,n=1}^{N} a_{mn}^* D_{mn} (\eta \varphi_1^*)\|_{L^p(B^p)} \\
\leq 2C \| \|\eta \sum_{m,n=1}^{N} a_{mn} D_{mn} \varphi_1^*\|_{L^p(B^p)} + 2 \| \sum_{m,n=1}^{N} a_{mn}^* D_{mn} \eta D_{u} \varphi_1^*\|_{L^p(B^p, \mathbb{R})} \\
+ \| \varphi_1^* \sum_{m,n=1}^{N} a_{mn} D_{mn} \eta\|_{L^p(B^p)} \}
\]
\[
\leq C'' \| \|\varphi_1^* - f_1^*(\chi (\frac{\xi}{r_0} | \cdot |) w_\delta (\cdot))\|_{L^p(B^p)} + \| \varphi_1^* \|_{L^p(B^p, \mathbb{R})} \\
+ \frac{1}{1-\rho^2} \| D \varphi_1^*\|_{L^p(B^p, \mathbb{R})} + \frac{1}{1-\rho^2 R^2} \| \varphi_1^*\|_{L^p(B^p)} \}
\]
for some \( C'' > 0 \) independent \( \epsilon_i \) and \( R \). Hence, we have
\[
(1-\rho^2) R^2 \| D^2 \varphi_1^*\|_{L^p(B^p)} \leq C'' \left( R^2 \| \varphi_1^* - f_1^*(\chi (\frac{\xi}{r_0} | \cdot |) w_\delta (\cdot))\|_{L^p(B^p)} + 4(1+\rho^2) R \| D \varphi_1^*\|_{L^p(B^p, \mathbb{R})} + \| \varphi_1^*\|_{L^p(B^p)} \right). 
\]
Here we used \( (1-\rho^2)/(1-\rho^2) \leq 4(1+\rho^2) \). Therefore,
\[
\Phi_2 \leq C \{ R^2 \| \varphi_1^* - f_1^*(\chi (\frac{\xi}{r_0} | \cdot |) w_\delta (\cdot))\|_{L^p(B^p)} + \Phi_1 + \Phi_0 \}
\]
holds, and we complete the proof of (3.100).

Second, we prove (3.101). By its invariance under coordinate stretching, it suffices to prove (3.101) for the case \( R = 1 \). We use the following known fact (see [23, Theorem 7.28]).

**Lemma 3.17.** Let \( B_1 \) be a unit ball centered at the origin. Then there exists a constant \( C = C(N) > 0 \) such that
\[
\|D u\|_{L^p(B_1)} \leq t \|D^2 u\|_{L^p(B_1)} + \frac{C}{t} \|u\|_{L^p(B_1)} 
\]
holds for any \( u \in W_0^{2,p}(B_1) \) and any \( t \) > 0.

Note that for \( v \in W_0^{2,p}(B_\rho), (\rho \in (0,1)) \), if we put \( u(x) = v(\rho x) \) then \( u \in W_0^{2,p}(B_1) \). Therefore, by substituting \( u(x) = v(\rho x) \) into (3.109), we have
\[
\rho \|D u\|_{L^p(B_\rho)} \leq \rho^2 t \|D^2 u\|_{L^p(B_\rho)} + \frac{C}{t} \|u\|_{L^p(B_\rho)}. 
\]
(3.110)
Therefore, for any \( t' > 0 \), if we put \( t = \frac{1-\rho}{\rho} t' \) in (3.110), we have
\[
\|D v\|_{L^p(B_\rho)} \leq (1-\rho) t' \|D^2 v\|_{L^p(B_\rho)} + \frac{C}{t'} \|v\|_{L^p(B_\rho)}, 
\]
(3.111)
for any \( v \in W^{2,p}_0(B_\rho) \). By the definition of \( \Phi \) and (3.111), for any \( \gamma > 0 \), there is \( \rho \in (0, 1) \) such that

\[
\Phi \leq (1 - \rho) \| D\varphi \|_{L^p(B_\rho)} + \gamma \\
\leq t'(1 - \rho)^2 \| D^2 \varphi \|_{L^p(B_\rho)} + \frac{C}{\nu} \| \varphi \|_{L^p(B_\rho)} + \gamma \\
\leq t'\Phi_2 + \frac{C}{\nu} \Phi_0 + \gamma.
\]

Hence, by taking the limit \( \gamma \to 0 \), we have (3.101). \( \square \)
Chapter 4

Chemotaxis model

4.1 Introduction and main results

In this chapter, we will present an application of Theorem 3.1 for chemotaxis models. We are interested in qualitative properties of solutions to the steady-state problem of the following generalized chemotaxis system:

\[
\begin{aligned}
\frac{\partial P}{\partial t} &= r \cdot \left( P \nabla \left( \frac{P}{\Phi(W)} \right) \right), \quad (x, t) \in \Omega \times (0, \infty), \\
\frac{\partial W}{\partial t} &= \varepsilon^2 \Delta W + F(P, W), \quad (x, t) \in \Omega \times (0, \infty), \\
\frac{\partial P}{\partial \nu} &= \frac{\partial W}{\partial \nu} = 0, \quad (x, t) \in \partial \Omega \times (0, \infty),
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \). Here \( P(x, t) \) is a population density, \( W(x, t) \) is a density of chemotaxis substance which influences the movement of \( P \), and \( \log \Phi(W) \) is the chemotactical sensitivity function. We note that \( P \) and \( W \) must be nonnegative. This system appears as a chemotaxis model in biology and structure of solutions depends on \( \Phi(W) \) and \( F(P, W) \). Interesting phenomena including aggregation, blow-up, and collapse of solutions have been studied by many mathematicians. We are interested in a chemotaxis model with saturation growth, and study conditions for the existence of boundary multi-peak solutions.

We note that, by integrating the equation for \( P \) over \( \Omega \) and using the divergence theorem, \( L^1 \)-norm of \( P \) is preserved in \( t \), namely,

\[
\int_{\Omega} P(x, t) dx = \int_{\Omega} P(x, 0) dx = \lambda.
\]

Then, the steady-state problem for (4.1) becomes

\[
\begin{aligned}
\nabla \cdot \left( P \nabla \left( \frac{P}{\Phi(W)} \right) \right) &= 0 \text{ in } \Omega, \\
\varepsilon^2 \Delta W + F(P, W) &= 0 \text{ in } \Omega, \\
\frac{\partial P}{\partial \nu} = \frac{\partial W}{\partial \nu} &= 0 \text{ on } \partial \Omega, \\
\int_{\Omega} P(x) dx &= \lambda,
\end{aligned}
\]

77
To simplify the computations, we assume $\lambda = 1$. Throughout this chapter, we take $\Phi(W) = W^p$, $p > 1$, which corresponds to a logarithmic chemotactical sensitivity function. Moreover, we note that

$$ P(x) = \frac{1}{\int \Omega} W^p(x) \quad (4.4) $$

must be satisfied from the first equation in (4.1) and the condition $\int \Omega P = 1$.

We shall consider the two cases as $F(P, W)$:

$$ F_1(P, W) := -W + \frac{PW^q}{\alpha + \gamma W^q}, \quad \text{(case A)} $$

$$ F_2(P, W) := -W + \frac{P}{1 + kP}, \quad \text{(case B)} $$

where $q > 0$, $\alpha, \gamma, k \geq 0$. In this chapter, we always assume that

$$ 1 < p < \infty \text{ if } N = 1, 2, \quad 1 < p < \frac{N + 2}{N - 2} \text{ if } N \geq 3, \quad q > 0. \quad (4.5) $$

In the case $F(P, W) = F_1(P, W)$ with $\alpha = 1$ and $q = 1$, the existence and stability of a single boundary spike solution has been studied by B. D. Sleeman, M. J. Ward and J. Wei [98]. Inspired with their works, the purpose of this chapter is to construct boundary multi-peak solutions to (4.3) in case A and case B. For the existence of spike patterns to (4.3), the shape of the domain $\Omega$ and the saturation parameters $\alpha, \gamma, k$ play an important role.

**Main results**

For a domain $\Omega$, we assume (A0) in Chapter 3. Let $P_1, \ldots, P_m$ be $m$ points taken from the intersections of $\partial \Omega$ and the $x_N$-axis, and let $\Phi(\cdot; P_k)$ and $\Psi(\cdot; P_k)$ are diffeomorphisms given in Chapter 3. In case A, we further assume that

$$ \text{(A2)} \lim_{\varepsilon \to 0} \varepsilon^N (\alpha \gamma q^{-1})^{1/q} = \alpha_0 \text{ for some } \alpha_0 \in [0, \infty). $$

In case B, we further assume that

$$ \text{(A3)} \lim_{\varepsilon \to 0} \varepsilon^{-N} k = k_0 \text{ for some } k_0 \in [0, \infty). $$

Let us state main results in this chapter.

**Theorem 4.1** (case A). Assume $F(P, W) = F_1(P, W)$, (A0) and (A2). Then there exists a constant $\alpha_1 \in (0, \infty)$ such that, if $0 \leq \alpha_0 < \alpha_1$, then, for sufficiently small $\varepsilon$, there exists an axially symmetric solution $(\hat{P}_\varepsilon, \hat{W}_\varepsilon)$ to (4.3) of the following form:

$$ (\hat{P}_\varepsilon(x), \hat{W}_\varepsilon(x)) = \left( \frac{1}{\int \Omega} \hat{u}_\varepsilon^p(x), \frac{1}{\gamma} \frac{1}{\int \Omega} \hat{u}_\varepsilon^q(x) \right), \quad (4.6) $$
and the function \( \hat{u}_\varepsilon \) is a solution to the following problem:

\[
\begin{cases}
\varepsilon^2 \Delta \hat{u} - \hat{u} + \frac{\hat{u}^{p+q}}{\alpha \gamma^{-1}(\int_\Omega \hat{w}^p(x)dx)^q + \hat{w}^q} = 0, & \hat{u} > 0, \text{ in } \Omega, \\
\frac{\partial \hat{u}}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(4.7)

such that, by taking a subsequence \( \varepsilon_i \to 0 \) as \( i \to \infty \) if necessary,

\[
\hat{u}_{\varepsilon_i}(x) = \hat{w}_{\varepsilon_i} \left( \frac{1}{\varepsilon_i} \Psi(x; P_k) \right) + o(1), \quad x \in \Psi(B_{\varepsilon_i}; P_k) \cap \Omega, \quad k = 1, \ldots, m,
\]

(4.8)

as \( \varepsilon_i \to 0 \), and

\[
\hat{u}_x(x) \leq C \exp \left\{ -\frac{c}{\varepsilon} \text{dist}(x, \mathcal{P}) \right\}, \quad x \in \Omega,
\]

(4.9)

where \( \mathcal{P} := \{ P_1, \ldots, P_m \} \), and the constants \( C, c > 0 \) are independent of \( \varepsilon \) and \( x \). Here \( \hat{w}_\varepsilon \) is the unique solution to the following problem:

\[
\begin{cases}
\Delta w - w + \frac{w^{p+q}}{\delta + w^q} = 0, & w > 0 \text{ in } \mathbb{R}^N, \\
w(0) = \max_{\mathbb{R}^N} w, & w(z) \to 0 \text{ as } |z| \to \infty,
\end{cases}
\]

(4.11)

and \( \delta_1 \geq 0 \) is a number satisfying

\[
\frac{\alpha_1^{1/q}}{\frac{m}{2} \int_{\mathbb{R}^N} \hat{w}^p_{\delta_1}} = \alpha_0.
\]

(4.12)

Remark 4.1. If \( \alpha_0 \) is small enough, then (4.8) and (4.9) in Theorem 4.1 hold true without taking a subsequence \( \{ \varepsilon_i \} \). Moreover, if \( p + q < \infty \) (\( N = 1, 2 \)), \( p + q < \frac{N+2}{N} \) (\( N \geq 3 \)), the constant \( \alpha_1 \) that appeared in the statement of Theorem 4.1 can be taken such that

\[
\left( \frac{m}{2} \int_{\mathbb{R}^N} v_0^p(x)dx \right)^{-1} \leq \alpha_1 \leq \infty \quad \text{if } q = 1, \quad \alpha_1 = \infty \quad \text{if } q > 1, \quad \alpha_1 < \infty \quad \text{if } q < 1,
\]

(4.13)

where \( v_0 \) is the unique solution to

\[
\begin{cases}
\Delta v - v + v^{p+q} = 0, & v > 0 \text{ in } \mathbb{R}^N, \\
v(0) = \max_{\mathbb{R}^N} v, & v(z) \to 0 \text{ as } |z| \to \infty.
\end{cases}
\]

(4.14)

We shall explain the details after the proof of Theorem 4.1 (see Remark 4.7).

Theorem 4.2 (case B). Assume \( F(P, W) = F_2(P, W) \), (A0) and (A3). Then, for each \( k_0 \in [0, \infty) \), if \( \varepsilon \) is sufficiently small, there exists an axially symmetric solution \( (\tilde{P}_\varepsilon, \tilde{W}_\varepsilon) \) to (4.3) of the following form:

\[
(\tilde{P}_\varepsilon(x), \tilde{W}_\varepsilon(x)) = \left( \frac{1}{\int_\Omega \hat{w}_\varepsilon^p(x)}, \frac{1}{\int_\Omega \hat{w}_\varepsilon^q} \hat{u}_\varepsilon(x) \right),
\]

(4.15)

79
and the function $\tilde{u}_\varepsilon$ is a solution to the following problem:

$$
\begin{cases}
\varepsilon^2 \Delta \tilde{u} - \tilde{u} + \frac{\tilde{u}^p}{1 + k(\int_{\Omega} \tilde{u}^p(x)dx)^{1-p}} = 0, \quad \tilde{u} > 0 \text{ in } \Omega, \\
\frac{\partial \tilde{u}}{\partial \nu} = 0 \text{ on } \partial \Omega,
\end{cases}
$$

(4.16)

such that, by taking a subsequence $\varepsilon_i \to 0$ as $i \to \infty$ if necessary,

$$
\tilde{u}_{\varepsilon_i}(x) = \tilde{w}_{\varepsilon_i} \left( \frac{1}{\varepsilon_i} \Psi(x; P_k) \right) + o(1), \quad x \in \Phi(B_k; P_k) \cap \Omega, \quad k = 1, \ldots, m, \quad (4.17)
$$

$$
\int_{\Omega} \tilde{u}_{\varepsilon_i}^p(x)dx = \varepsilon_i^N \frac{m}{2} \int_{\mathbb{R}^N} \tilde{w}_{\varepsilon_i}^p(x)dx + o(\varepsilon_i^N) \quad (4.18)
$$

as $\varepsilon_i \to 0$, and

$$
\tilde{u}_\varepsilon(x) \leq C \exp \left\{ -\frac{c}{\varepsilon} \text{dist}(x, P) \right\}, \quad x \in \Omega, \quad (4.19)
$$

where $P := \{P_1, \ldots, P_m\}$, and the constants $C, c > 0$ are independent of $\varepsilon$ and $x$. $\tilde{w}_{\varepsilon_i}$ is the unique solution to the following problem:

$$
\begin{cases}
\Delta w - w + \frac{w^p}{1 + \delta w^p} = 0, \quad w > 0 \text{ in } \mathbb{R}^N, \\
w(0) = \max_{\mathbb{R}^N} w, \quad w(z) \to 0 \text{ as } |z| \to \infty,
\end{cases}
$$

(4.20)

and $\delta_2 \geq 0$ is a number satisfying

$$
\delta_2 \int_{\mathbb{R}^N} \tilde{w}_{\varepsilon_i}^p(x)dx = k_0. \quad (4.21)
$$

**Remark 4.2.** If $k_0$ is small enough or in the case $N = 1$ and $p = 2$, then (4.17) and (4.18) in Theorem 4.2 hold true without taking a subsequence $\{\varepsilon_i\}$. We shall explain this in Remark 4.9.

**Remark 4.3.** If $k_0$ is large enough, then we can relax our assumption on $p$ in (4.5) into $1 < p < \infty$ for any dimension. We shall explain the reason in Remark 4.10 after the proof of Theorem 4.2.

**Remark 4.4.** The case $\alpha = 0$ in Theorem 4.1 and the case $k = 0$ in Theorem 4.2 correspond to the Keller-Segel model. In these cases, the statements coincide with the one in [77].

**Remark 4.5.** The uniqueness and the nondegeneracy of solutions to (4.11) and (4.20) were given in Chapter 2.

**Remark 4.6.** It would be interesting to study in the case the saturation effect is strong, namely, $\lim_{\varepsilon \to 0} \varepsilon^N (\alpha \gamma^{q-1})^{1/q} = +\infty$ or $\lim_{\varepsilon \to 0} \varepsilon^{-N} k = +\infty$. In those cases, stripe patterns or patterns with inner transition layers may occur (see e.g. [40, 16, 89]).
Outline of our construction

Let us explain a basic strategy to construct solutions to (4.3). We use the method introduced in Section 3.5. The problem (4.3) is reduced into some nonlocal problem for \( W \) by substituting \( P(x) \) of (4.4) into (4.3).

We first consider case A, let us define the function \( ˆ{u} \) by

\[
W(x) = \frac{1}{\gamma \int_{\Omega} ˆ{u} dx} ˆ{u}(x),
\]

(4.22)

and substituting (4.4) and (4.22) into (4.3), we have the equation (4.7) for \( ˆ{u} \):

\[
\begin{aligned}
\varepsilon^2 \Delta ˆ{u} - ˆ{u} + \frac{ ˆ{u}^{p+q}}{\delta + \hat{u}^q} &= 0, \; ˆ{u} > 0, \; \text{in} \; \Omega, \\
\frac{\partial ˆ{u}}{\partial \nu} &= 0 \; \text{on} \; \partial \Omega,
\end{aligned}
\]

((4.7))

To seek a solution to (4.7), we a matching parameter \( \delta \), then we reduce the problem into the following problem: find \(( ˆ{u}, \delta)\) such that

\[
\begin{aligned}
\varepsilon^2 \Delta ˆ{u} - ˆ{u} + \frac{ ˆ{u}^{p+q}}{\delta + \hat{u}^q} &= 0, \; ˆ{u} > 0, \; \text{in} \; \Omega, \\
\delta &= \alpha \gamma^{q-1} \left( \int_{\Omega} ˆ{u}^p dx \right)^q + \hat{u}^q, \\
\frac{\partial ˆ{u}}{\partial \nu} &= 0 \; \text{on} \; \partial \Omega.
\end{aligned}
\]

(4.23)

We notice that the first equation of (4.23) is just the type of (3.1) since the nonlinearity \(-t + \frac{\hat{u}^{p+q}}{\delta + \hat{u}^q}\) satisfies the conditions (f1)-(f3) in Section 2.2 (see Lemma 2.1). Therefore, we can apply Theorem 3.1. Hence, for \( \delta > 0 \), under the conditions (A0) and (A2), we know that there exists multi-peak solution denoted by \( ˆ{u}_\delta(x; \varepsilon) \) of the first equation in (4.23) for small \( \varepsilon \), which is continuous in \( \delta \). By using the continuity of \( ˆ{u}_\delta(x; \varepsilon) \) in \( \delta \) and asymptotic behavior of the integral \( \int_{\Omega} ˆ{u}^p(x; \varepsilon) dx \) as \( \varepsilon \to 0 \), we show that there exists \( \delta = \delta_\varepsilon \) such that the second equation holds for each \( \varepsilon \) under the condition (A2), namely,

\[
\delta_\varepsilon = \alpha \gamma^{q-1} \left( \int_{\Omega} ˆ{u}_\delta^p(x; \varepsilon) dx \right)^q.
\]

Then, we obtain a solution \( ˆ{u}_\varepsilon \) to (4.23) for each small \( \varepsilon \) by putting \( ˆ{u}_\varepsilon(x) = ˆ{u}_\delta(x; \varepsilon) \), and we obtain a solution to (4.3).

In case B, let us define the function \( ˜{u} \) by

\[
W(x) = \frac{1}{\int_{\Omega} ˜{u}^p dx} ˜{u}(x),
\]

(4.24)

Substituting (4.4) and (4.24) into (4.3), we have (4.16). As in the case A, introducing a matching parameter \( \delta \), we can reduce the problem to the following
problem: to find \((\tilde{u}, \delta)\) such that
\[
\begin{cases}
\varepsilon^2 \Delta \tilde{u} - \tilde{u} + \frac{\tilde{u}^p}{1 + \delta \tilde{u}^p} = 0, & \tilde{u} > 0 \text{ in } \Omega, \\
\delta = \frac{1}{\int_{\Omega} \tilde{u}^p(x) dx}, \\
\frac{\partial \tilde{u}}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{cases}
\] (4.25)

Also in this case, note that the nonlinearity \(-t + \frac{t^p}{1 + \delta t^p}\) satisfies the conditions (f1)-(f3) (see Lemma 2.2). Hence, we can apply Theorem 3.1. Then, by the same strategy as that in case A, we can construct a multiple spike solution \(\tilde{u}_\varepsilon\) to (4.25).

### 4.2 Proof of Theorem 4.1

As we observed, once we obtain a solution to (4.7), then we obtain a solution \((P, W)\) of (4.3) by the relations (4.4) and (4.22).

\[
\begin{cases}
\varepsilon^2 \Delta \hat{u} - \hat{u} + \frac{\hat{u}^p + \hat{u}^q}{\alpha \gamma^{q-1} \left( \int_{\Omega} \hat{u}^p dx \right)^{q} + \hat{u}^q} = 0, & \hat{u} > 0 \text{ in } \Omega, \\
\frac{\partial \hat{u}}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{cases}
\] (4.7)

To seek a solution to (4.7), by introducing a matching parameter \(\delta\), the problem (4.7) is reduced to the problem: find \((\delta, \hat{u})\) such that
\[
\begin{cases}
\varepsilon^2 \Delta \hat{u} - \hat{u} + \hat{f}_\delta(\hat{u}) = 0, & \hat{u} > 0 \text{ in } \Omega, \\
\hat{f}_\delta(\hat{u}) = \frac{\hat{u}^{p+q}}{\delta^{1/q}}, \\
\int_{\Omega} \frac{\hat{u}^p dx}{\delta^{1/q}} = \left( \alpha \gamma^{q-1} \right)^{1/q}, \\
\frac{\partial \hat{u}}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{cases}
\] (4.26)

Note that \(\hat{f}_\delta\) satisfies (f1)-(f3) with \(\delta_0 = \infty\). By Theorem 3.1, we obtain a solution denoted by \(\hat{u}_\delta(x; \varepsilon)\) of the first equation in (4.26) for each \(\delta \in [0, \infty)\), and \(\hat{u}_\delta(x; \varepsilon)\) is continuous in \(\delta\). Hence, let us seek \(\delta_\varepsilon \in [0, \infty)\) for each \(\varepsilon\) sufficiently small such that matching equation is satisfied, namely,
\[
\int_{\Omega} \frac{\delta_\varepsilon^{1/q}}{\int_{\Omega} \hat{u}_\delta^p(x; \varepsilon) dx} dx = \left( \alpha \gamma^{q-1} \right)^{1/q}.
\] (4.27)

To seek \(\delta_\varepsilon\) satisfying (4.27), it is important to study the properties \(\hat{\beta}(\delta)\) defined by
\[
\hat{\beta}(\delta) := \frac{\delta^{1/q}}{2 \int_{\mathbb{R}^N} \hat{w}_\delta(\delta) dx},
\] (4.28)
where \(\hat{w}_\delta\) denote the unique solution to (4.11).
Lemma 4.1. If $N \geq 3$, we assume $p + q < (N+2)/(N-2)$. Let $v_0$ be a solution to (4.14). Then the following statements hold.

(a) If $q < 1$, then $\hat{\beta}(\delta) \to 0$ as $\delta \to \infty$.

(b) If $q = 1$, then $\hat{\beta}(\delta) \to \left(\frac{m}{2} \int_{\mathbb{R}^N} v_0^q(x)dx\right)^{-1}$ as $\delta \to \infty$.

(c) If $q > 1$, then $\hat{\beta}(\delta) \to \infty$ as $\delta \to \infty$.

Proof. Let $\hat{w}_\delta(x) = \delta^s v(x)$, where $s$ is a number chosen later. By substituting this into (4.11), we have an equation for $v$:

$$
\Delta v - v - \frac{v^{p+q}}{\delta^{1-s(p+q-1)} + \delta^{-s(p-1)}v^q} = 0, \quad v > 0 \text{ in } \mathbb{R}^N, \quad v(0) = \max_{\mathbb{R}^N} v, \quad v(x) \to 0 \text{ as } |x| \to \infty. \quad (4.29)
$$

Let $s = 1/(p + q - 1) > 0$ and $k := \delta^{-s(p-1)} > 0$, we have

$$
\Delta v - v - \frac{v^{p+q}}{1 + kv^q} = 0, \quad v > 0 \text{ in } \mathbb{R}^N, \quad v(0) = \max_{\mathbb{R}^N} v, \quad v(x) \to 0 \text{ as } |x| \to \infty. \quad (4.30)
$$

Thus, it turns out that $v$ is a unique solution to (4.30). Let us write the solution $v_k$ for each $k > 0$. Note that, by Lemma 2.2, the equation (4.30) has a unique solution denoted by $v_k$ even if $k = 0$. We see that $v_k$ is continuous in $k \geq 0$ with respect to the norm $W^{2,r}(\mathbb{R}^N)$, $N/2 < r < \infty$, and $v_k(x)$ decays exponentially as $|x| \to \infty$ for each $k \geq 0$.

On the other hand, $\hat{\beta}(\delta)$ is written as follows:

$$
\hat{\beta}(\delta) = \frac{1}{\frac{m}{2} \int_{\mathbb{R}^N} v_k^q(x)dx} \delta^{1/q - p/(p+q-1)}. \quad (4.31)
$$

The exponent of $\delta$ in (4.31) is, negative if $q < 1$, 0 if $q = 1$, positive if $q > 1$, respectively. From the observation above, by noting that $k \to 0$ as $\delta \to \infty$, we have a conclusion.

Lemma 4.2. For $\hat{\beta}(\delta)$, it holds that

$$
\left. \frac{d}{d\delta} \hat{\beta}(\delta) \right|_{\delta = 0} > 0.
$$

Proof. Note that $\hat{w}_\delta(t)$ is differentiable with respect to $\delta \in (0, \infty)$, and the derivative is continuous in $(\delta, t) \in (0, \infty) \times [0, \infty)$. Hence, we can see by Lemma 2.5 that

$$
\hat{w}_\delta \in C^1((0, \infty), W^{2,r}(\mathbb{R}^N)), \quad N/2 < r < \infty.
$$
Let us write \( \hat{w}_0 '^\delta(x) \) for \( \delta \in (0, \infty) \), and \( f' = \int_{\mathbb{R}^N} \) simply only here.

By a direct calculation, we have

\[
\frac{d}{d\delta} \left( \int \hat{w}_0 '^\delta \right)^q = \frac{1}{(\int \hat{w}_0 '^\delta)^q} + \frac{\delta pq f(\hat{w}_0 '^\delta - 1)}{(\int \hat{w}_0 '^\delta)^{2q}}.
\] (4.32)

We note that

\[
L_\delta \hat{w}_0 '^\delta = \Delta \hat{w}_0 '^\delta + \int \hat{w}_0 '^\delta \hat{f}_0 '^\delta = \frac{\hat{w}_0 '^\delta + \hat{w}_0 '^\delta}{(\hat{w}_0 '^\delta)^2},
\] (4.33)

for \( \delta > 0 \). We note that the inverse \( L_\delta^{-1} \) is bounded uniformly in \( \delta \in [0, \bar{\delta}] \) for fixed \( \bar{\delta} \in (0, \infty) \) by Lemma 2.4. Hence, we can see that the quantity \( |f(\hat{w}_0 '^\delta - 1)| \) remains in a finite value as \( \delta \to 0 \). Thus we have

\[
\frac{d}{d\delta} \left( \int \hat{w}_0 '^\delta \right)^q = \frac{1}{(\int \hat{w}_0 '^\delta)^q} > 0.
\] (4.34)

This implies the assertion of this lemma.

We are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** Put

\[
\alpha_1 := \sup_{\delta > 0} \hat{\beta}(\delta).
\] (4.35)

Then by Lemmas 4.1, 4.2, we see that this \( \alpha_1 \) is positive and satisfies (4.13). As we observed, once if we find \( \delta_1 \in [0, \alpha_1] \) such that

\[
\frac{\delta_1^{1/q}}{\int \hat{u}_0 '^\delta_1} = (\alpha_{\gamma q^{-1}})^{1/q},
\] (4.27)

then by putting \( \hat{u}_0 '^\delta(x) = \hat{u}_0 '^\delta_1(x; \varepsilon) \), we can obtain a solution \((P, W)\) to (4.3) from (4.4) and (4.7). Now, we claim that

**Claim.** For any \( \alpha_0 \in [0, \alpha_1) \), if \( \lim_{\varepsilon \to 0} \varepsilon^N (\alpha_{\gamma q^{-1}})^{1/q} = \alpha_0 \), then there exists \( \delta_1 \in [0, \delta] \) satisfying (4.27) for each \( \varepsilon > 0 \) sufficiently small, where \( \delta_1 \in (0, \infty) \) is a certain number independent of \( \varepsilon \).

Recall by Theorem 3.1 that

\[
\int \hat{u}_0 '^\delta(x; \varepsilon) dx = \varepsilon^N \frac{M}{2} \int \hat{w}_0 '^\delta(x) dx + o(\varepsilon^N),
\] (4.36)

and hence

\[
\hat{\beta}_1 (\delta) := \frac{\delta_1^{1/q}}{\int \hat{u}_0 '^\delta} = \varepsilon^{-N} (\hat{\beta}(\delta) + \delta^{1/q} o(1)),
\] (4.37)
as $\varepsilon \to 0$ uniformly in $\delta \in [0, \delta]$, where $\hat{\beta}(\delta)$ is defined by (4.28), $\bar{\delta} \in (0, \infty)$ is a constant fixed arbitrarily.

From the observation above, if we fix $\bar{\delta} \in (0, \infty)$ so that $\hat{\beta}(\delta) > \alpha_0$ holds in advance, then we can see that

$$\hat{\beta}_c(0) = 0 \leq (\alpha \gamma^{-1})^{1/q}$$

and

$$\hat{\beta}_c(\delta_1) \geq (\alpha \gamma^{-1})^{1/q},$$

when $\varepsilon > 0$ is small enough. Hence, by the intermediate value theorem, there exists $\delta_\varepsilon \in [0, \bar{\delta}]$ such that $\hat{\beta}_c(\delta_\varepsilon) = (\alpha \gamma^{-1})^{1/q}$ for each $\varepsilon > 0$ sufficiently small. Thus the claim is verified, and we obtain a solution $(\hat{P}_\varepsilon, \hat{W}_\varepsilon)$ of the form (4.6) by putting $\hat{u}_\varepsilon(x) = \hat{u}_{\delta_\varepsilon}(x; \varepsilon)$.

Now, let us show the properties (4.8)-(4.10) in Theorem 4.1. (4.10) is a direct consequence of (3.9) in Theorem 3.1 since $\delta_\varepsilon$ is in the finite interval $[0, \bar{\delta}]$.

Because $\delta_\varepsilon \in [0, \bar{\delta}]$, there exists $\{\varepsilon_i\}_{i=1}^\infty$ which converges to 0 such that $\hat{\beta}_c(\delta_{\varepsilon_i}) = (\alpha \gamma^{-1})^{1/q}$ and $\delta_{\varepsilon_i} \to \delta_1$ for some $\delta_1 \in [0, \bar{\delta}]$. Let us write $\delta_i = \delta_{\varepsilon_i}$ simply. Because

$$\lim_{i \to \infty} \varepsilon_i \hat{\beta}_c(\delta_i) = \hat{\beta}(\delta_1)$$

and $\lim_{i \to \infty} \varepsilon_i (\alpha \gamma^{-1})^{1/q} = \alpha_0$, this $\delta_1$ must satisfy (4.12). Note that

$$\hat{u}_\varepsilon(x) = U_{\delta_i, \varepsilon_i}(x) + \varepsilon_i \phi_{\delta_i, \delta_i}$$

$$= \hat{u}_{\delta_i} \left( \frac{1}{\varepsilon_i} \Psi(x; \hat{P}_i) \right) + O(\varepsilon_i), \; x \in \Phi(B_{r_i}; \hat{P}_i) \cap \Omega, \; k = 1, \ldots, m,$$

by the definition (3.6) of $U_{\varepsilon, \delta}$ and the boundedness of $\phi_{\varepsilon, \delta}$. Noting the continuity of $\hat{w}_{\delta_i}$ with respect to $\delta$, we obtain (4.8). Next, by the continuity of $\hat{w}_{\delta_i}$ with respect to $\delta$ again, and (4.36), we obtain (4.9). Thus we complete the proof. \qed

**Remark 4.7.** The assertions in Remark 4.1 are verified as follows. (4.13) follows by Lemma 4.1 and the definition of $\alpha_1$ in this proof. Next, if we define

$$\delta_{*} := \sup \{ \delta > 0 : \frac{d\hat{\beta}}{d\delta}(\delta') > 0, \; \delta' \in (0, \delta) \},$$

then $\delta_{*} > 0$ by Lemma 4.2, and $\hat{\beta}(\delta)$ is strictly monotone increasing in $\delta \in [0, \delta_{*}]$. Hence, if $\alpha_0$ is small enough, then we can take $\delta \in (0, \delta_{*})$ so that $\hat{\beta}(\delta) > \alpha_0$ and $\delta \in (0, \delta_{*})$, and hence $\delta_\varepsilon \in [0, \bar{\delta}]$ exists uniquely. By the uniqueness, it is easy to see that $\delta_\varepsilon \to \delta_1$ as $\varepsilon \to 0$, and that (4.8) and (4.9) hold true without taking a subsequence.

### 4.3 Proof of Theorem 4.2

As we observed, once we obtain a solution to (4.16), then we obtain a solution $(P, W)$ of (4.3) by the relations (4.4) and (4.24):

$$\begin{cases}
\varepsilon^2 \Delta \bar{u} - \bar{u} + \frac{\bar{u}^p}{1 + k \left( \int_{\Omega} \bar{u}^p(x)dx \right) ^{-1}} \bar{u}^p = 0, \; \bar{u} > 0, \; \text{in } \Omega, \\
\frac{\partial \bar{u}}{\partial \nu} = 0 \text{ on } \partial \Omega,
\end{cases}$$

(4.16)
Proof. We consider the positive root of

\( (V) \)

Lemma 4.3. By integrating after multiplying (4.41) by \( x \) for each \( x \), where \( \tilde{w} \)

and the problem is reduced into the problem:

\[
\begin{cases}
\varepsilon^2 \Delta \tilde{u} - \tilde{u} + \tilde{f}_\delta(\tilde{u}) = 0, & \text{in } \Omega, \\
\frac{\partial \tilde{u}}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\] (4.25)

where \( \tilde{f}_\delta(\tilde{u}) = \frac{\tilde{u}^p}{1 + \varepsilon^p} \). Note that \( \tilde{f}_\delta \) satisfies (1)-(3) with \( \delta_0 = \delta_* \). By Theorem 3.1, we obtain a solution denoted by \( \tilde{u}_\delta(x; \varepsilon) \) of the first equation in (4.25) for each \( \delta \in (0, \delta_*) \), and \( \tilde{u}_\delta(x; \varepsilon) \) is continuous in \( \delta \). Hence, let us seek \( \delta_\varepsilon \in (0, \delta_*) \) for each \( \varepsilon \) sufficiently small such that

\[ \delta_\varepsilon = \frac{k}{\int_{\Omega} \tilde{w}_\delta^p(x; \varepsilon) dx}. \] (4.39)

To seek \( \delta_\varepsilon \) satisfying (4.39), it is important to study the properties \( \tilde{\beta}(\delta) \) defined by

\[ \tilde{\beta}(\delta) := \delta \int_{\mathbb{R}^N} \tilde{w}_\delta^p(x) dx, \] (4.40)

where \( \tilde{w}_\delta \) is the unique solution to (2.6).

**Lemma 4.3.** \( \tilde{w}_\delta \to b_* \) in \( C^2_{\text{loc}}(\mathbb{R}^N) \) holds as \( \delta \to \delta_* \), where \( b_* \) is a second positive root of \(-t + \tilde{f}_\delta(t) = 0 \).

**Proof.** By the transformation \( w_\delta(x) = \delta^{-1/p} v_\delta(\delta^{(1-p)/2} x), v_\delta \) satisfies (2.7). Let \( a(\delta) \) and \( b(\delta) \) be the first and the second positive roots of \( \tilde{g}_\delta(v) = 0 \), where \( \tilde{g}_\delta(v) = -v'' + \frac{v^{p-1}}{1 + v^p} \). Note that \( b_* = (\delta_*)^{-1/p} b(\delta_*) \). Hence it suffices to show that \( v_\delta \to b(\delta_*) \) in \( C^2_{\text{loc}}(\mathbb{R}^N) \) as \( \delta \to \delta_* \).

Now, note that \( \tilde{g}_\delta(v)/v = (-\delta v'' + \frac{v^{p-1}}{1 + v^p}) \) is decreasing with respect to \( \delta \) for each \( v > 0 \), and hence \( b(\delta) \) is decreasing with respect to \( \delta \). Moreover, it is easy to see that \( v_\delta(x) < b(\delta) \) by the maximum principle. Thus, if we fix \( \delta_2 \in (0, \delta_*) \), then we have the following estimate:

\[ v_\delta(x) < b(\delta) < b(\delta_2) \]

for \( x \in \mathbb{R}^N \) and \( \delta \in (\delta_2, \delta_*) \). Hence, \( v_\delta(x) \) is bounded uniformly in \( \delta \in (\delta_2, \delta_*) \). Therefore, for any compact subset \( K \subseteq \mathbb{R}^N \), \( \|v_\delta\|_{C^{2+\omega}(K)} \) is bounded uniformly in \( \delta \in (\delta_2, \delta_*) \) by the elliptic estimate. Thus, there exists \( V(x) \) such that \( v_\delta(x) \to V(x) \) in \( C^2_{\text{loc}}(\mathbb{R}^N) \) as \( \delta \to \delta_* \), and \( V(x) \) satisfies \( \Delta V - \tilde{g}_\delta(V) = 0 \).

Since \( v_\delta(x) = v_\delta(r), r = |x|, v_\delta(r) \) satisfies

\[ v_\delta''(r) + \frac{N-1}{r} v_\delta'(r) + \tilde{g}_\delta(v_\delta(r)) = 0. \] (4.41)

By integrate after multiplying (4.41) by \( v_\delta'(r) \), we have

\[ \frac{v_\delta'(r)^2}{2} + \int_0^{v_\delta(r)} \tilde{g}_\delta(t) dt \leq \int_0^{v_\delta(0)} \tilde{g}_\delta(t) dt. \] (4.42)
By taking \( r \to \infty \) in (4.42), we have \( \int_0^{\nu_k(t)} 	ilde{g}_{\delta}(t) dt \geq 0 \). Let \( \theta = \theta(\delta) \) be a number in (g2) for \( g(v) = \tilde{g}_{\delta}(v) \). Then we can see that \( \nu_k(0) \geq \theta(\delta) \). As we observed in (2.8), \( \theta(\delta) \to b(\delta_*) \) holds as \( \delta \to \delta_* \). Thus we have \( V(0) = b(\delta_*) \).

Because \( V(r) \leq V(0) = b(\delta_*) \), \( V'(0) = 0 \) holds. Hence, by the uniqueness of a solution to ODE, we have \( V(x) \equiv b(\delta_*) \).

**Lemma 4.4.** \( \tilde{\beta}(\delta) \to \infty \) holds as \( \delta \to \delta_* \).

**Proof.** This is a direct consequence of Lemma 4.3.

**Lemma 4.5.** For \( \tilde{\beta}(\delta) \) defined by (4.40), it holds that
\[
\frac{d}{d\delta} \tilde{\beta}(\delta) \bigg|_{\delta=0} > 0.
\]

**Proof.** Note that
\[
\frac{d}{d\delta} \tilde{\beta}(\delta) = \int_{\mathbb{R}^N} \tilde{w}_\delta^p(x)dx + p\delta \int_{\mathbb{R}^N} \tilde{w}_\delta^{p-1}(x)\tilde{w}_\delta'(x)dx,
\]
where \( \tilde{w}_\delta'(x) = \frac{\partial \tilde{w}_\delta}{\partial \delta}(x) \). By the same argument as that in the proof of Lemma 4.2, we have
\[
\frac{d}{d\delta} \tilde{\beta}(\delta) \to \int_{\mathbb{R}^N} \tilde{w}_\delta^p(x)dx,
\]
as \( \delta \to 0 \). Thus we complete the proof.

**Remark 4.8.** In the case \( p = 2 \) and \( N = 1 \), it is known that
\[
\frac{d}{d\delta} \int_{\mathbb{R}} \tilde{w}_\delta^2(x)dx > 0
\]
for any \( \delta \in (0, \delta_*) \). This fact was shown in Lemma 2.6 of [114]. So, in this case, \( \tilde{\beta}(\delta) \) is monotone increasing on \( \delta \in (0, \delta_*) \). We do not know whether \( \tilde{\beta}(\delta) \) is monotone increasing on \( \delta \in (0, \delta_*) \) for general \( p > 1 \) and \( N \geq 1 \).

**Proof of Theorem 4.2.** We claim that

**Claim.** For any \( k_0 \in [0, \infty) \), if \( \lim_{\varepsilon \to 0} \varepsilon^{-N} k = k_0 \), then there exists \( \delta_\varepsilon \in [0, \bar{\delta}] \) such that (4.39) holds for \( \varepsilon > 0 \) sufficiently small, where \( \bar{\delta} \) is a certain number in \( (0, \delta_*) \) independent of \( \varepsilon \).

To verify this claim, we put
\[
\tilde{\beta}_\varepsilon(\delta) := \delta \int_{\Omega} \tilde{u}_\delta^p(x; \varepsilon)dx, \tag{4.43}
\]
and we seek \( \delta_\varepsilon \) such that \( \tilde{\beta}_\varepsilon(\delta) = k \) for sufficiently small \( \varepsilon \). By the same argument as that in the proof of Theorem 4.1, we have
\[
\int_{\Omega} \tilde{u}_\delta^p(x; \varepsilon)dx = \varepsilon^{N/2} \int_{\mathbb{R}^N} \tilde{w}_\delta^p(x)dx + o(\varepsilon^N), \tag{4.44}
\]
87
and hence
\[ \tilde{\beta}_\varepsilon(\delta) = \varepsilon^N \left( \frac{m}{2} \tilde{\beta}(\delta) + \delta o(1) \right) \]  
holds as \( \varepsilon \to 0 \) uniformly in \( \delta \in [0, \bar{\delta}] \) for fixed \( \bar{\delta} \in [0, \delta_*] \). Now, by Lemma 4.4, we can take \( \delta \in (0, \delta^*) \) so that \( \frac{m}{2} \tilde{\beta}(\delta) > k_0 \). Then we can see that
\[ \tilde{\beta}_\varepsilon(0) = 0 \leq k \text{ and } \tilde{\beta}_\varepsilon(\bar{\delta}) \geq k, \]
for sufficiently small \( \varepsilon \). Therefore, by the intermediate value theorem, there exists \( \delta_\varepsilon \in [0, \bar{\delta}] \) such that \( \tilde{\beta}_\varepsilon(\delta_\varepsilon) = k \) for sufficiently small \( \varepsilon \). Thus the claim is verified.

Thus, if we put \( \bar{u}_\varepsilon(x) := \bar{u}_{\delta_\varepsilon}(x; \varepsilon) \) and define \( \bar{P}_\varepsilon(x) \) and \( \bar{W}_\varepsilon(x) \) by (4.4) and (4.24), then \( \bar{P}_\varepsilon, \bar{W}_\varepsilon \) becomes a solution to (4.3).

Because \( \delta_\varepsilon \in [0, \bar{\delta}] \), we can pick up a subsequence \( \{ \varepsilon_i \}_{i=1}^\infty \) which converges to 0 as \( i \to \infty \) such that \( \beta_{\delta_\varepsilon}(\delta_\varepsilon) = k \) and \( \delta_{\varepsilon_i} \to \delta_2 \) for some \( \delta_2 \in [0, \bar{\delta}] \). (4.17)-(4.19) and (4.21) can be verified as in the proof of Theorem 4.1. Thus we complete the proof.

**Remark 4.9.** The assertion in Remark 4.2 is verified as follows. If we put
\[ \delta_{**} := \sup \left\{ \delta \in (0, \delta_*): \frac{d}{d\delta} \tilde{\beta}(\delta') > 0 \text{ for } \delta' \in (0, \delta) \right\}, \]  
then \( \tilde{\beta}(\delta) \) is monotone increasing in \( (0, \delta_{**}) \). Note that \( \delta_{**} > 0 \) by Lemma 4.5. In the case \( N = 1 \), we notice that \( \delta_{**} = \delta_* \) by Remark 4.8. Hence, if \( k_0 \) is small enough, then \( \delta_2 \) satisfying
\[ \delta_2 \int_{\mathbb{R}^N} \bar{w}_{\delta_2}^p(x)dx = k_0 \]
is decided uniquely in \( [0, \delta_{**}) \). Moreover, then \( \tilde{\delta}_\varepsilon \) can be also taken in \( [0, \delta_{**}) \) for sufficiently small. Hence, by the uniqueness of \( \tilde{\delta}_\varepsilon \) and \( \delta_2 \), we do not need to take a subsequence \( \{ \varepsilon_i \} \).

**Remark 4.10.** Let us explain Remark 4.3. If \( N \geq 3 \) and \( (N+2)/(N-2) \leq p \), then \( f_3(t) \) do not satisfy (f3) because the uniqueness and the nondegeneracy of a solution to (4.20) do not hold for \( \delta = 0 \) in such a situation. However, if we restrict the range of \( \delta \) to \( [\tilde{\delta}, \delta_*] \) for small \( \tilde{\delta} > 0 \), our analysis in Chapter 3 also works for \( \delta \in [\tilde{\delta}, \bar{\delta}] \) for any fixed \( \tilde{\delta} < \delta_* \), and we obtain \( \bar{u}_\delta(x; \varepsilon) \) for \( \delta \in [\tilde{\delta}, \bar{\delta}] \). Put
\[ k := \inf_{\delta \in (0, \delta_*)} \left\{ \frac{m}{2} \tilde{\beta}(\delta) \right\}. \]  
Recall that
\[ (\tilde{\beta}_\varepsilon(\delta) = \delta \int_{\Omega} \bar{u}_{\delta}^p(x; \varepsilon)dx = \varepsilon^N \frac{m}{2} \tilde{\beta}(\delta) + \delta o(1)), \]  
as \( \varepsilon \to 0 \) uniformly in \( \delta \in [\tilde{\delta}, \bar{\delta}] \). Thus, if \( k_0 \) satisfies \( k_0 > k \), then we can take the constants \( \tilde{\delta}, \delta \in (0, \delta_* \). so that \( \tilde{\delta} < \delta < \frac{m}{2} \tilde{\beta}(\delta) < k_0 \) and \( \frac{m}{2} \tilde{\beta}(\delta) > k_0 \) in advance. Hence,
\[ \tilde{\beta}_\varepsilon(\delta) \leq k \text{ and } \tilde{\beta}_\varepsilon(\bar{\delta}) \geq k, \]
holds for all \( \varepsilon \) sufficiently small. By the continuity of \( \tilde{\beta}_\varepsilon(\delta) \) with respect to \( \delta \in [\delta, \overline{\delta}] \), we can find \( \delta_\varepsilon \in [\delta, \overline{\delta}] \) such that \( \tilde{\beta}_\varepsilon(\delta_\varepsilon) = k \) for each \( \varepsilon \) small enough. Therefore, the same statements as in Theorem 4.2 can be proven under the assumptions \( k_0 > k \) and \( 1 < p < \infty \) for any dimension. We know \( k = 0 \) when \( p \) satisfies (4.5). However, for the case \( N \geq 3 \) and \( \frac{N+2}{N-2} \leq p \), we do not know the precise information on \( k \), since the asymptotic behavior of \( \int_{\mathbb{R}^N} \tilde{w}_\delta^p \, dx \) as \( \delta \to 0 \) is not clear.
Chapter 5

Schnakenberg model

5.1 Introduction and main results

We consider the following activator-substrate system with saturation:

\[
\begin{cases}
\frac{\partial a}{\partial t} = \varepsilon^2 \Delta a - a + \frac{a^2}{1 + ka^2}, & a > 0 \text{ in } \Omega \times (0, \infty), \\
\tau \frac{\partial h}{\partial t} = D \Delta h - h \frac{a^2}{1 + ka^2} + \rho, & h > 0 \text{ in } \Omega \times (0, \infty), \\
\frac{\partial a}{\partial \nu} = \frac{\partial h}{\partial \nu} = 0 \text{ on } \partial \Omega \times (0, \infty),
\end{cases}
\]  

(5.1)

where \( \varepsilon, D > 0, \tau > 0, k \geq 0, \) \( a = a(x,t) \) and \( h = h(x,t) \) represent the concentrations of the activator and the substrate at \( x \in \Omega \) and \( t \in (0, \infty) \), respectively. \( \rho = \rho(x) \) represent the feed-rate of the substrate at \( x \in \Omega \). When \( k = 0 \), this system is called the Schnakenberg model [92]. When \( k > 0 \), (5.1) is the Schnakenberg model with saturation effect. In this activator-substrate system, it is well-known that the feed-rate plays an important role for pattern formation. In fact, if the feed-rate is not enough, then the substrate can not support the production of the activator. Thus we assume \( \rho \neq 0 \) to get a pattern formation.

In this chapter, we show that (5.1) admits boundary multi-peak solutions for sufficiently small \( \varepsilon \) and sufficiently large \( D \) by applying Theorem 3.1, and we consider the effect of the saturation and the feed-rate on the pattern formation.

We assume (A0). The points \( P_1, \ldots, P_m \) and the diffeomorphisms \( \Phi(\cdot; P_k) \) and \( \Psi(\cdot; P_k) \) are the same as in Chapter 3. In addition, we assume the following:

(A4) \( 1 \leq N \leq 5 \).

(A5) \( \rho \) is of class \( C^\alpha(\overline{\Omega}) \) for some \( \alpha \in (0,1) \) and is \( x_N \)-axially symmetric, and \( \rho \neq 0, \rho(x) \geq 0 \) for \( x \in \overline{\Omega} \).

(A6) \( k \) and \( \rho \) depends on \( \varepsilon \) and satisfy

\[
\lim_{\varepsilon \to 0} \varepsilon^{-2N} k \left( \int_\Omega \rho(x) dx \right)^2 = k_0
\]

(5.2)

for some \( k_0 \in [0, \infty) \).
Now, we introduce the shadow system of the Schnakenberg model. The steady-state problem of (5.1) is the following:

\[
\begin{align*}
0 &= \varepsilon^2 \Delta a - a \frac{a^2}{1 + ka^2}, \quad a > 0 \text{ in } \Omega, \\
0 &= D \Delta h - h \frac{a^2}{1 + ka^2} + \rho, \quad h > 0 \text{ in } \Omega, \\
\frac{\partial a}{\partial \nu} &= \frac{\partial h}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{align*}
\] (5.3)

Dividing the second equation by $D$ and taking the limit $D \to \infty$ formally, we have $\Delta h = 0$ and $\frac{\partial h}{\partial \nu} = 0$ on $\partial \Omega$. Thus, we can regard $h = \xi$ (constant). Then we have the shadow system of the Schnakenberg model:

\[
\begin{align*}
0 &= \varepsilon^2 \Delta a - a \frac{a^2}{1 + ka^2}, \quad a > 0 \text{ in } \Omega, \\
0 &= \int_{\Omega} \left( -\xi \frac{a^2}{1 + ka^2} + \rho \right) dx, \quad \xi > 0 \text{ in } \Omega, \\
\frac{\partial a}{\partial \nu} &= 0 \text{ on } \partial \Omega.
\end{align*}
\] (5.4)

We first construct a solution to the shadow system (5.4) for sufficiently small $\varepsilon$. After that we construct a solution to (5.3) near the solution to (5.4) for sufficiently large $D$ by the implicit function theorem.

Let us state main results on the Schnakenberg model with saturation.

**Theorem 5.1.** Assume (A0), and (A4)-(A6). For each $k_0$ in (A6), if $\varepsilon$ is sufficiently small, then (5.4) admits a solution $(a_\varepsilon(x), \xi_\varepsilon)$ which is axially symmetric, and the following properties hold:

\[
a_\varepsilon(x) \leq \frac{c}{\xi_\varepsilon} \exp \left\{ -\frac{c}{\varepsilon} \text{dist}(x, P) \right\}, \quad x \in \Omega, \\
c' \varepsilon^N \left( \int_{\Omega} \rho(x) dx \right)^{-1} \leq \xi_\varepsilon \leq C' \varepsilon^N \left( \int_{\Omega} \rho(x) dx \right)^{-1},
\] (5.5)

where $P = \{P_1, \cdots, P_m\}$, $C, c, C', c'$ are positive constants independent of $\varepsilon$, $x$, and $\rho$. By taking a subsequence $\{\varepsilon_i\}$, if necessary, which converges to 0 as $i \to \infty$,

\[
a_{\varepsilon_i}(x) = \frac{1}{\xi_{\varepsilon_i}} \left\{ w_{\delta_i} \left( \frac{1}{\xi_{\varepsilon_i}} \Psi(x; P_k) \right) + o(1) \right\}, \quad x \in \Phi(B_{\delta_0}; P_k) \cap \Omega, \quad k = 1, \cdots, m,
\] (5.6)

as $i \to \infty$.

\[
\xi_{\varepsilon_i} = \varepsilon_i^N \left( \frac{m}{2} \int_{\mathbb{R}^N} w_{\delta_i}(y) dy + o(1) \right) \left( \int_{\Omega} \rho(x) dx \right)^{-1},
\] (5.7)

as $i \to \infty$. Here, $o(1)$ in (5.6) is uniform with respect to $x$, and $w$ is a unique solution to the following problem:

\[
\begin{align*}
\Delta w - w + f_\delta(w) &= 0, \quad w > 0 \text{ in } \mathbb{R}^N, \\
\max_{\mathbb{R}^N} w &= w(0), \quad w(z) \to 0 \text{ as } |z| \to \infty,
\end{align*}
\] (5.8)

\[
f_\delta(w) := \frac{w^2}{1 + \delta w^2}.
\]
\( \delta_3 \) is a certain number satisfying

\[
\delta_3 \left( \frac{m}{2} \int_{\mathbb{R}^N} w_{\delta_3}(y) \, dy \right)^2 = k_0.
\] (5.9)

**Remark 5.1.** If \( k_0 \) is small enough or in the case \( N = 1 \) and \( p = 2 \), then (5.6) and (5.7) hold without taking a subsequence \( \{ \varepsilon_i \} \) similarly to Remark 4.2.

**Theorem 5.2.** We suppose that \( k_0 \) is sufficiently small. For the solution \((a_\varepsilon(x), \xi_\varepsilon)\) to (5.4) given in Theorems 5.1, there exists \( D_\varepsilon > 0 \) such that, for \( D > D_\varepsilon \), the Schnakenberg model (5.3) admits an \( x_N \)-axially symmetric solution \((a_\varepsilon(x; D), h_{\varepsilon}(x; D))\) which satisfies

\[
\sup_{x \in \Omega} |a_\varepsilon(x; D) - a_\varepsilon(x)|, \quad \sup_{x \in \Omega} |h_\varepsilon(x; D) - \xi_\varepsilon| \to 0, \quad (D \to \infty).
\] (5.10)

Now, let us consider the effect of the saturation and feed-rate. When there is no saturation effect, namely, \( k = 0 \), the assumption (A6) is always satisfied. In this case, we can see by (5.5)-(5.7) that, the larger the value of \( \int_{\Omega} \rho(x) \, dx \) is, the more the maximum value of \( a_\varepsilon \) grows. On the other hand, when \( k > 0 \) is fixed, then \( \rho \) must satisfy \( \int_{\Omega} \rho(x) \, dx = O(\varepsilon^{N}) \) as \( \varepsilon \to 0 \). In this case, the maximum value of \( a_\varepsilon \) does not become large. More generally, we put \( k = \varepsilon^\alpha k_\ast \) and \( \rho = \varepsilon^\beta \rho_\ast \), where \( k_\ast > 0 \) and \( \rho_\ast \) are independent of \( \varepsilon \). The exponents \( \alpha \) and \( \beta \) may be negative. Then the condition for which (A6) is satisfied becomes \( \alpha \geq 2(N - \beta) \). We note that \( \varepsilon^{N} / \int_{\Omega} \rho(x) \, dx = \varepsilon^{N - \beta} / \int_{\Omega} \rho_\ast(x) \, dx \). Because \( N - \beta < 0 \) when \( \alpha < 0 \), we can see that the peaks of the activator \( a_\varepsilon \) vanish as \( \varepsilon \to 0 \) if \( \alpha < 0 \). Inversely, if \( N - \beta > 0 \), then \( \alpha > 0 \). This means that one of the conditions for which spiky pattern appears is \( \alpha > 0 \), namely, the saturation effect is weak.

### 5.2 Proof of Theorems

**Proof of Theorem 5.1.** Integrating the first equation of (5.4) over \( \Omega \), we see that

\[
\int_\Omega a \, dx = \xi \int_\Omega \frac{a^2}{1 + ka^2} \, dx.
\] (5.11)

Therefore, the second equation of (5.4) becomes

\[
0 = \int_\Omega (a(x) + \rho(x)) \, dx.
\] (5.12)

Put \( a = u/\xi \) and \( \delta = k/\xi^2 \), then (5.4) becomes the following problem by introducing a matching parameter \( \delta \): find the pair \((u, \delta)\) satisfying

\[
\begin{cases}
\varepsilon^2 \Delta u - u + \frac{u^2}{1 + \delta u^2} = 0 \text{ in } \Omega,
\delta \left( \int_{\Omega} u(x) \, dx \right)^2 = k \left( \int_{\Omega} \rho(x) \, dx \right)^2, \\
\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,
\end{cases}
\] (5.13)

92
with
\[ \xi = \int_{\Omega} u(x) dx \int_{\Omega} \rho(x) dx. \] (5.14)

The nonlinear term \( \frac{u^2}{1 + \delta u^2} \) satisfies the conditions (f1)-(f3) by Lemma 2.2 under the assumption (A4). Therefore we can apply Theorem 3.1. Thus, for sufficiently small \( \varepsilon \), we obtain a solution \( u_\delta(x; \varepsilon) \), which is continuous in \( \delta \), to the first equation in (5.13) for each \( \delta \in [0, \delta_*] \).

Next, by making use of the asymptotic behavior (3.10) of \( \int_{\Omega} u_\delta(x; \varepsilon) dx \) and the assumption (A6), we can find \( \delta = \delta_\varepsilon \) satisfying the second equation of (5.13) as we did in the proof of Theorem 4.2, namely,
\[ \delta_\varepsilon \left( \int_{\Omega} u_\delta(x; \varepsilon) dx \right)^2 = k \left( \int_{\Omega} \rho(x) dx \right)^2. \] (5.15)

Thus, we obtain a solution to (5.4) by putting
\[ a_\varepsilon(x) = \frac{u_\delta(x; \varepsilon)}{\xi_\varepsilon}, \quad \xi_\varepsilon \frac{\int_{\Omega} u_\delta(x; \varepsilon) dx}{\int_{\Omega} \rho(x) dx}. \] (5.16)

The properties (5.5)-(5.9) are easily to verified by using the properties (3.7)-(3.10). \( \square \)

Proof of Theorem 5.2. Let \( (a_\varepsilon, \xi_\varepsilon) \) be a solution to (5.4) given in Theorem 5.1. We decompose the space \( X^t \) into \( X^1 = \mathbb{R} \oplus X_1 \), where
\[ X_1 := \{ u \in X^t : \int_{\Omega} u(x) dx = 0 \}, \]
and the space of constant functions is identified with \( \mathbb{R} \). Let \( P : X^t \to X_1 \) be the projection associated with this decomposition:
\[ Pu = u - \frac{1}{|\Omega|} \int_{\Omega} u(x) dx. \] (5.17)

Put \( Z = X_0^{2,t} \cap X_1 \). We set \( h(x) \) as follows according to the decomposition:
\[ h(x) = \xi + \psi(x), \quad \xi \in \mathbb{R}, \ \psi \in X_1. \] (5.18)

We put
\[ d := \frac{1}{D}. \] (5.19)

To find a solution to the full system (5.1), we shall find a solution \( (a, \psi, \xi) \)
satisfying the following:

\[0 = \varepsilon^2 \Delta a - a + (\xi + \psi) \frac{a^2}{1 + ka^2} \text{ in } \Omega, \quad (5.20)\]

\[0 = \frac{1}{|\Omega|} \int_{\Omega} \left\{ (\xi + \psi) \frac{a^2}{1 + ka^2} + \rho \right\} dx, \quad (5.21)\]

\[0 = \Delta \psi - d \left\{ P \left( (\xi + \psi) \frac{a^2}{1 + ka^2} + \rho \right) \right\} \text{ in } \Omega, \quad (5.22)\]

\[\frac{\partial \psi}{\partial \nu} = \frac{\partial a}{\partial \nu} = 0 \text{ on } \partial \Omega. \quad (5.23)\]

We define operators \( F_1, F_2, F_3 \) as follows:

\[F_1(d, a, \xi, \psi) = \varepsilon^2 \Delta a - a + (\xi + \psi) \frac{a^2}{1 + ka^2}, \quad (5.24)\]

\[F_2(d, a, \xi, \psi) = \int_{\Omega} \left\{ (\xi + \psi) \frac{a^2}{1 + ka^2} + \rho \right\} dx, \quad (5.25)\]

\[F_3(d, a, \xi, \psi) = \Delta \psi - d \left\{ P \left( (\xi + \psi) \frac{a^2}{1 + ka^2} + \rho \right) \right\}. \quad (5.26)\]

Then, for \( t > N \), \( F = (F_1, F_2, F_3) \) is a \( C^1 \)-mapping from an open set

\[\mathbb{R} \times \{ a \in X_{\nu}^{2,t} : a > 0 \} \times (c, +\infty) \times \{ \psi \in Z : \| \psi \|_{L^\infty(\Omega)} < c \}\]

into \( X^t \times \mathbb{R} \times X_1 \), where \( c \) is a small positive constant. We note that

\[F(0, a_\varepsilon, \xi_\varepsilon, 0) = 0.\]

Therefore, if the partial derivative of \( F \) with respect to \((a, \xi, \psi)\) at \((0, a_\varepsilon, \xi_\varepsilon, 0)\):

\[D_{(a, \xi, \psi)}F|_{(0, a_\varepsilon, \xi_\varepsilon, 0)} = \begin{pmatrix} \varepsilon^2 - 1 + \xi_\varepsilon \frac{2a_\varepsilon}{1 + ka_\varepsilon^2} & \frac{a^2}{1 + ka^2} & \frac{a^2}{1 + ka^2} \\ \int_{\Omega} \xi_\varepsilon \frac{2a_\varepsilon}{1 + ka_\varepsilon^2} \ dx & \int_{\Omega} \frac{a^2}{1 + ka^2} \ dx & \int_{\Omega} \frac{a^2}{1 + ka^2} \ dx \end{pmatrix}\]

has a bounded inverse, then by the implicit function theorem, we obtain a one parameter family of solutions \((a(d), \xi(d), \psi(d)) \in X_{\nu}^{2,t} \times \mathbb{R}_+ \times Z\), which are of class \( C^1 \) with respect to \( d \), such that

\[F(d, A(d), \xi(d), \psi(d)) = 0\]

for \(|d|\) sufficiently small, and

\[(a(0), \xi(0), \psi(0)) = (a_\varepsilon, \xi_\varepsilon, 0).\]

Thus, by applying Taylor’s theorem, we have a conclusion.
Note that $\Delta$ under the homogeneous Neumann boundary condition is an isomorphism from $Z$ onto $X_1$. Thus, from the observation above, we only need to confirm that the operator $L_\infty := \begin{pmatrix} \varepsilon^2 \Delta - 1 + \frac{2\xi \varepsilon^2}{(1 + ka_\varepsilon)^2} & \frac{a_\varepsilon^2}{1 + ka_\varepsilon^2} \\ 2\xi \int_\Omega (1 + ka_\varepsilon^2)^2 \, dx & \int_\Omega \frac{a_\varepsilon^2}{1 + ka_\varepsilon^2} \, dx \end{pmatrix} : X'_N \times \mathbb{R} \to X^t \times \mathbb{R}$ ($N < t < \infty$), is invertible.

We put $u_\varepsilon(x) := u_{\delta_\varepsilon}(x; \varepsilon)$, $L_\varepsilon := \varepsilon^2 \Delta - 1 + f_{\delta_\varepsilon}(u_\varepsilon)$, where $f_\varepsilon(u) = \frac{u^2}{1 + \delta u^2}$. Note that $L_\varepsilon$ has a bounded inverse $K_\varepsilon$ by Lemma 3.13.

We show that $\text{Ker}(L_\infty) = f(0, 0)$. Let $(\phi, \eta) \in \text{Ker}(L_\infty)$, namely,

$$
\begin{cases}
\varepsilon^2 \Delta \phi - \phi + \frac{2\xi \varepsilon^2 \phi}{(1 + ka_\varepsilon^2)^2} + \frac{\eta a_\varepsilon^2}{1 + ka_\varepsilon^2} = 0, \\
2\xi \int_\Omega (1 + ka_\varepsilon^2)^2 \, dx + \eta \int_\Omega \frac{a_\varepsilon^2}{1 + ka_\varepsilon^2} \, dx = 0.
\end{cases}
$$

(5.27)

Recall that $a_\varepsilon = u_\varepsilon/\xi_\varepsilon$, $\delta_\varepsilon = k/\xi_\varepsilon^2$, and put $\hat{\eta} = \eta/\xi_\varepsilon^2$, then we have

$$
\begin{cases}
\varepsilon^2 \Delta \phi - \phi + \frac{2u_\varepsilon \phi}{(1 + \delta u_\varepsilon^2)^2} + \frac{\hat{\eta} u_\varepsilon^2}{1 + \delta u_\varepsilon^2} = 0, \\
2 \int_\Omega \frac{u_\varepsilon \phi}{(1 + \delta u_\varepsilon^2)^2} \, dx + \hat{\eta} \int_\Omega \frac{u_\varepsilon^2}{1 + \delta u_\varepsilon^2} \, dx = 0.
\end{cases}
$$

(5.28)

The first equation in (5.28) is written as $L_\varepsilon \phi = -\hat{\eta} f_{\delta_\varepsilon}(u_\varepsilon)$. Hence,

$$
\phi = -\hat{\eta} K_\varepsilon [f_{\delta_\varepsilon}(u_\varepsilon)].
$$

(5.29)

Now, we can see that

$$
K_\varepsilon [f_{\delta_\varepsilon}(u_\varepsilon)] = u_\varepsilon + K_\varepsilon [gf_{\delta_\varepsilon}(u_\varepsilon)], \quad g := \frac{2\delta u_\varepsilon^2}{1 + \delta u_\varepsilon^2}.
$$

(5.30)

Substituting this into (5.29), we have

$$
\phi = -\hat{\eta}(u_\varepsilon + K_\varepsilon [gf_{\delta_\varepsilon}(u_\varepsilon)]).
$$

(5.31)

Then we have the following equation by substituting this into the second equation in (5.28):

$$
\hat{\eta} \left\{ \int_\Omega \frac{-u_\varepsilon^2}{(1 + \delta u_\varepsilon^2)^2} \, dx + \int_\Omega \frac{\delta u_\varepsilon^4}{(1 + \delta u_\varepsilon^2)^2} \, dx - \int_\Omega \frac{u_\varepsilon K_\varepsilon [gf_{\delta_\varepsilon}(u_\varepsilon)]}{(1 + \delta u_\varepsilon^2)^2} \, dx \right\} = 0.
$$

(5.32)
Noting that \( u_\varepsilon \) is bounded uniformly in \( \varepsilon \) sufficiently small, we can estimate by Lemma 3.13 as follows:

\[
\int_\Omega \frac{u_\varepsilon K_\varepsilon [g f_\varepsilon (u_\varepsilon)]}{(1 + \delta_\varepsilon u_\varepsilon^2)^2} dx = 2\delta_\varepsilon \int_\Omega \frac{u_\varepsilon}{(1 + \delta_\varepsilon u_\varepsilon^2)^2} K_\varepsilon \left[ \frac{u_\varepsilon^4}{(1 + \delta_\varepsilon u_\varepsilon^2)^2} \right] dx \\
\leq C \delta_\varepsilon \left\| \frac{u_\varepsilon}{(1 + \delta_\varepsilon u_\varepsilon^2)^2} \right\|_{L^2(\Omega)} \left\| \frac{u_\varepsilon^4}{(1 + \delta_\varepsilon u_\varepsilon^2)^2} \right\|_{L^2(\Omega)} \\
\leq C' \delta_\varepsilon \left\| \frac{u_\varepsilon}{(1 + \delta_\varepsilon u_\varepsilon^2)^2} \right\|_{L^2(\Omega)},
\]

for some positive constants \( C, C' \) independent of \( \varepsilon \). Hence, the following estimate holds:

\[
\int_\Omega \frac{-u_\varepsilon^2}{(1 + \delta_\varepsilon u_\varepsilon^2)^2} dx + \int_\Omega \frac{\delta_\varepsilon u_\varepsilon^4}{(1 + \delta_\varepsilon u_\varepsilon^2)^2} dx - \int_\Omega \frac{u_\varepsilon K_\varepsilon [g f_\varepsilon (u_\varepsilon)]}{(1 + \delta_\varepsilon u_\varepsilon^2)^2} dx \\
\leq (-1 + C'' \delta_\varepsilon) \int_\Omega \frac{u_\varepsilon^2}{(1 + \delta_\varepsilon u_\varepsilon^2)^2} dx,
\]

for some \( C'' > 0 \) independent of \( \varepsilon \). We note that, if \( k_0 \) is sufficiently small, then \( \delta_\varepsilon \) is also small when \( \varepsilon \) is small, since \( \delta_\varepsilon \to \delta_3 \) as \( \varepsilon \to 0 \) and \( \delta_3 \) is characterized by (5.9). Therefore, if \( k_0 \) is small enough, then \(-1 + C'' \delta_\varepsilon < 0 \) holds for \( \varepsilon \) sufficiently small. Thus we have \( \eta = 0 \), and hence \( \eta = 0 \). Moreover, \( \phi = 0 \) follows from (5.28). Therefore, \( \varepsilon = 1 \) is one to one. Then the invertibility is easily verified. Hence we omit the details. \( \square \)

### 5.3 Internal layer solution

As we observed in Subsection 1.3.1, the Schnakenberg model with saturation possesses an internal layer solution provided \( k > 0 \) is fixed independently of \( \varepsilon \).

Let us confirm the existence of a stationary internal transition layer solution to

\[
\begin{align*}
0 &= \varepsilon^2 \Delta a = a + \frac{h a^2}{1 + ka^2}, \quad a > 0 \text{ in } \Omega \times (0, \infty), \\
\tau \frac{\partial h}{\partial t} &= D \Delta h = \frac{h a^2}{1 + ka^2} + \rho, \quad h > 0 \text{ in } \Omega \times (0, \infty), \\
\frac{\partial a}{\partial \nu} = \frac{\partial h}{\partial \nu} &= 0 \text{ on } \partial \Omega \times (0, \infty), \\
a(x, 0) &= a_0(x), \quad h(x, 0) = h_0(x) \text{ in } \Omega.
\end{align*}
\]

(5.33)

Here, \( k \) and \( \rho \) are positive constants. We use the results of M. del Pino [16]. Del Pino considered the general steady-state problem of two-components reaction-diffusion system:

\[
\begin{align*}
0 &= \varepsilon^2 \Delta u = f(u, v) \text{ in } B, \\
0 &= D \Delta v + g(u, v) \text{ in } B, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0 \text{ on } \partial B,
\end{align*}
\]

(5.34)

where \( B \) is an open unit ball in \( \mathbb{R}^N (N \geq 1) \), namely, \( B = \{ |x| < 1 \} \). The following assumptions are subjected (similarly assumptions were used in [61, 89]):

---

96
(a) \( f \) and \( g \) are functions defined on some open subset of \( \mathbb{R}^2 \), \( f \) of class \( C^2 \), \( g \) of class \( C^1 \).

(b) There exists a bounded open interval \( I \) such that, for all \( v \in \bar{I} \), the function 
\[ u \mapsto f(u,v) \] 
possesses exactly three zeros \( h_{-}(v) < h_0(v) < h_{+}(v) \), two of them are nondegenerate and stable, namely,
\[ f_u(h_{\pm}(v),v) < 0, \quad v \in \bar{I}. \]

(c) For \( v \in I \), put
\[ J(v) = \int_{h_{-}(v)}^{h_{+}(v)} f(s,v)ds. \]
Then there exists a unique \( v^* \in I \) such that \( J(v^*) = 0 \) and \( J'(v^*) \neq 0 \).

(d) For \( v \in \bar{I} \), set
\[ G_{\pm}(v) = g(h_{\pm}(v),v). \]
Then \( G_{-}(v) < 0 < G_{+}(v) \) holds for all \( v \in \bar{I} \).

Del Pino showed that (5.34) possesses a radially symmetric internal layer solution provided the conditions (a)-(d) hold. More precisely, put
\[ G(v) := \begin{cases} 
G_{-}(v), & v < v^*, \\
G_{+}(v), & v \geq v^*.
\end{cases} \]
For the problem
\[ \begin{cases} 
\Delta v = \frac{1}{B} G(v) & \text{in } B, \\
\frac{\partial v}{\partial \nu} = 0 & \text{on } B, 
\end{cases} \tag{5.35} \]
del Pino showed that

**Proposition 5.1.** There exists \( D_0 > 0 \) such that, for \( D_0 < D \), (5.35) admits a radially symmetric weak solution \( v_0(x) \in C^1(\bar{B}) \) whose range lies on \( I \), and it satisfies \( v_0'(r) < 0 \) for \( r = |x| \in (0,1) \) and \( v_0(\lambda) = v^* \) for some unique \( \lambda \in (0,1) \).

And he showed the following:

**Proposition 5.2.** Fix \( D > D_0 \). Then there exists an \( \varepsilon_0 > 0 \) and a family of radially symmetric solutions \( \{u_\varepsilon,v_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)} \) to (5.34) such that
(1) \( v_\varepsilon \to v_0 \) in \( C^{1,\alpha}(\overline{B}) \) as \( \varepsilon \to 0 \),
(2) \( u_\varepsilon \to h^*(v_0) \) as \( \varepsilon \to 0 \) uniformly on any compact subset in \( \overline{B} \setminus \{|x| = \lambda\} \),
where \( h^*(v) \) is a function defined by
\[ h^*(v) := \begin{cases} 
h_{-}(v), & v < v^*, \\
h_{+}(v), & v \geq v^*.
\end{cases} \]
Now, let us show the existence of an internal layer stationary solution to (5.33). Put \( u = a \) and \( v = -h \), and substitute them into steady-state problem of (5.33), then we have the following system:

\[
\begin{aligned}
0 &= \varepsilon^2 \Delta u + f(u,v) \text{ in } B, \\
0 &= D \Delta v + g(u,v) \text{ in } B, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial B.
\end{aligned}
\] (5.36)

where \( f(u,v) = -u - \frac{vu^2}{1 + kuv} \), \( g(u,v) = -\frac{vu^2}{1 + kuv} - \rho \). Let us confirm that the terms \( f(u,v) \) and \( g(u,v) \) satisfy the conditions (a)-(d) for suitable \( k \) and \( \rho \).

(a) The condition (a) is obviously satisfied.

(b) We first notice that \( f(0,v) = 0 \) always holds. Moreover, if \( v < -2\sqrt{k} \), then \( u \mapsto f(u,v) \) possesses exactly three zeros at

\[
h_-(v) = 0, \quad h_0(v) = \frac{-v - \sqrt{v^2 - 4k}}{2k}, \quad h_+(u) = \frac{-v + \sqrt{v^2 - 4k}}{2k}.
\]

Then, the condition (b) is satisfied for \( v < -2\sqrt{k} \).

(c) We can compute as follows: for \( v < -2\sqrt{k} \) and

\[
J(v) = \int_{h_-(v)}^{h_+(v)} f(s,v)ds,
\]

\[
J'(v) = h_+'(v)f(h_+(v),v) + \int_0^{h_+(v)} \frac{d}{ds}f(s,v)ds = -\int_0^{h_+(v)} \frac{s^2}{1 + ks^2}ds < 0.
\]

Therefore, \( J(v) \) is strictly monotone decreasing for \( v < -2\sqrt{k} \). Noting that \( h_0(v) = h_+(v) = 1/\sqrt{k} \) when \( v = -2\sqrt{k} \) and \( f(u,v) < 0 \) for \( u \in (0, h_0(v)) \), we notice that

\[
J(-2\sqrt{k}) < 0, \quad \lim_{v \to -\infty} J(v) = +\infty.
\]

Hence, there exists a unique \( v^* < -2\sqrt{k} \) such that \( J(v^*) = 0 \) and \( J'(v^*) \neq 0 \).

(d) We put \( I = (-a,-b) \in (-\infty, -2\sqrt{k}) \) for arbitrarily fixed \( a, b \in (2\sqrt{k}, \infty) \) so that \(-a < v^* < -b \). For \( G_+(v) = g(h_+(v),v) \), \( G_-(v) = g(0,v) = -\rho < 0 \) always holds. Because \( f(h_+(v),v) = 0 \), we have

\[
-\frac{vh_+(v)^2}{1 + kh_+(v)^2} = h_+(v).
\]

Hence,

\[
G_+(v) = -\frac{vh_+(v)^2}{1 + kh_+(v)^2} - \rho = h_+(v) - \rho.
\]

We note that \( h_+(v) \) is independent of \( \rho \). Let \( h_+ := \inf_{v \in I} h_+(v) > 0 \). If we take \( \rho > 0 \) so that \( h_+ > \rho \), then \( G_+(v) > 0 \) follows for any \( v \in I \).

Thus, we can apply del Pino’s propositions, and we obtain an internal layer solution to (5.36). We note that \( v_\varepsilon(x) < 0, \ x \in B \), for sufficiently small \( \varepsilon \). Put \( (a_\varepsilon(x), h_\varepsilon(x)) = (u_\varepsilon(x), v_\varepsilon(x)) \). Then we obtain a positive internal layer solution to (5.33).
Chapter 6

Gierer-Meinhardt system

6.1 Introduction and main results

We consider the following activator-inhibitor system which was proposed by A. Gierer and H. Meinhardt:

\[
\begin{align*}
\frac{\partial A}{\partial t} &= \varepsilon^2 \Delta A - A + \frac{A^p}{\varepsilon(1+\kappa A^q)} + \sigma_0, \ A > 0 \text{ in } \Omega \times (0, \infty), \\
\tau \frac{\partial H}{\partial t} &= D \Delta H - H + \frac{A^r}{\varepsilon^s}, \ H > 0 \text{ in } \Omega \times (0, \infty), \\
\frac{\partial A}{\partial \nu} &= \frac{\partial H}{\partial \nu} = 0 \text{ on } \partial \Omega \times (0, \infty),
\end{align*}
\]  

(6.1)

where \( \varepsilon > 0, \ \tau > 0, \ \kappa \geq 0. \ A = A(x, t) \) and \( H = H(x, t) \) represent the concentrations of the activator and the inhibitor at \( x \in \Omega \) and \( t \in (0, \infty) \), respectively. \( \varepsilon \) and \( D \) are stand for the diffusion constants of the activator and the inhibitor, respectively. The constant \( \kappa \) stands for the degree of saturation effect. The system (6.1) expresses some models in biological pattern formations. The term \( \sigma_0 = \sigma_0(x) \) is a source term. We refer the readers to [22] for a biological background of this model. We always assume that \( p > 1, \ q, r > 0, \ s \geq 0, \) and

\[
0 < \frac{p-1}{q} < \frac{r}{s+1},
\]

(6.2)

\[
1 < p < \infty \text{ if } N = 1, 2; \ 1 < p < \frac{N+2}{N-2} \text{ if } N \geq 3.
\]

(6.3)

To construct solutions to (6.1), we shall first consider the following system which is called the shadow system of the Gierer-Meinhardt system:

\[
\begin{align*}
\frac{\partial A}{\partial t} &= \varepsilon^2 \Delta A - A + \frac{A^p}{\varepsilon(1+\kappa A^q)} + \sigma_0, \ A > 0 \text{ in } \Omega \times (0, \infty), \\
\frac{\partial \xi}{\partial t} &= \frac{1}{|\Omega|} \int_\Omega (-\xi + \frac{A^r}{\varepsilon^s}), \ \xi > 0 \text{ in } (0, \infty), \\
\frac{\partial A}{\partial \nu} &= 0 \text{ on } \partial \Omega \times (0, \infty).
\end{align*}
\]

(6.4)

The unknowns are \( A = A(x, t) \) and \( \xi = \xi(t). \) The shadow system appears when we take the limit \( D \to \infty \) formally for the Gierer-Meinhardt system.
Assume (A0), and let $P_1, \ldots, P_m$ be $m$ points and $\Phi(\cdot; P_k)$ and $\Psi(\cdot; P_k)$ be diffeomorphisms given in Chapter 3. Now, let us present the existence of multi-peak stationary solution peaked at the points $P_1, \ldots, P_m$ to (6.4) and (6.1). We define constants $\gamma$ and $\gamma'$ as follows:

$$\gamma := \frac{qr - (s + 1)(p - 1)}{pq}, \quad \gamma' := \frac{qr}{p - 1} - (s + 1). \quad (6.5)$$

We note that $\gamma, \gamma' > 0$ by (6.2). For this system, we suppose the following condition:

(A7) $\kappa \geq 0$ depends on $\varepsilon$ and satisfies

$$\lim_{\varepsilon \to 0} \varepsilon^{-N} \kappa \gamma = \kappa_0$$

for some $\kappa_0 \in [0, \infty)$.

The assumption (A7) is a weak saturation condition for the Gierer-Meinhardt system. For the existence of spiky solutions, it seems that $k$ must be small according to $\varepsilon \ll 1$. When $\kappa > 0$ is fixed, due to the bistable nonlinearity, solutions with transition layers may exist (see [15], [16], [38], [44], [89]).

We consider the case $\sigma_0 = 0$. The case $\sigma_0 \neq 0$ will be treated in the next chapter. Let us state main results on the multi-peak stationary solutions to the Gierer-Meinhardt system.

**Theorem 6.1.** Assume (A0) and (A7). For each $\kappa_0 \in [0, \infty)$ in (A7), if $\varepsilon$ is sufficiently small, then (6.4) admits an $x_N$-symmetric solution $(A_\varepsilon(x), \xi_\varepsilon)$, and the following properties hold:

$$A_\varepsilon(x) \leq C \xi_\varepsilon^{q/(p-1)} \exp \left\{ - \frac{\varepsilon}{\xi} \text{dist}(x, P) \right\}, \quad x \in \Omega,$$

$$\left( c' \varepsilon^{-N} \right)^{1/\gamma'} \leq \xi_\varepsilon \leq \left( C' \varepsilon^{-N} \right)^{1/\gamma'},$$

where $P = \{P_1, \ldots, P_m\}$, $C, C', c'$ are positive constants independent of $\varepsilon$ and $x$. By taking a subsequence $\{\varepsilon_i\}$ if necessary, which converges to 0 as $i \to \infty$,

$$A_\varepsilon(x) = \xi_\varepsilon^{q/(p-1)} \left\{ w_{\delta_i} \left( \frac{1}{\xi_{\varepsilon_i}} \Psi(x; P_k) \right) + o(1) \right\}, \quad x \in \Phi(B_{\varepsilon_i}; P_k) \cap \Omega, \quad k = 1, \ldots, m,$$

as $i \to \infty$.

$$\xi_{\varepsilon_i} = |\Omega|^{1/\gamma'} \left( \frac{\varepsilon_i^N}{2} \int_{\mathbb{R}^N} w_{\delta_i}(y) dy + o(1) \right)^{-1/\gamma'},$$

as $i \to \infty$. Here, $o(1)$ in (6.7) is uniform with respect to $x$, and $w_{\delta}$ is a unique solution to the following problem:

$$\begin{cases} \Delta w - w + f_\delta(w) = 0, \quad x > 0 \text{ in } \Omega, \\ \max_{\mathbb{R}^N} w = w(0), \quad w(z) \to 0 \text{ as } |z| \to \infty, \end{cases} \quad (6.9)$$
\[ f_\delta(w) := \frac{w^p}{1 + \delta w^p}, \]
and \( \delta_4 \geq 0 \) is a certain number satisfying
\[
\delta_4 \left( \frac{m}{2|\Omega|} \int_{\mathbb{R}^N} w_\delta^p(y) dy \right) = \kappa_0.
\] (6.10)

**Remark 6.1.** If \( \kappa_0 \) is sufficiently small or in the case \( N = 1 \) and \( r = 2 \), then (6.7) and (6.8) hold without taking a subsequence \( \{ \varepsilon_i \} \) by a reason similarly to Remark 4.2.

**Theorem 6.2.** We suppose that \( \kappa_0 \) is sufficiently small. For the solution \((A_\varepsilon(x), \xi_\varepsilon)\) to (6.4) given in Theorem 6.1, there exists \( D_\varepsilon > 0 \) such that, for \( D > D_\varepsilon \), the Gierer-Meinhardt system (6.1) admits an \( x_N \)-axially symmetric solution \((A_\varepsilon(x; D), H_\varepsilon(x; D))\) which satisfies
\[
\sup_{x \in \Omega} |A_\varepsilon(x; D) - A_\varepsilon(x)|, \quad \sup_{x \in \Omega} |H_\varepsilon(x; D) - \xi_\varepsilon| \to 0, \quad (D \to \infty).
\] (6.11)

**Outline of our construction**

We state an outline of our construction of solutions. We put
\[
A(x) = \xi^{p/(p-1)}u(x),
\] (6.12)
and substitute this into the steady-state problem of the shadow system (6.4). Then we have the following equation on \((u, \xi)\):
\[
\begin{align*}
\varepsilon^2 \Delta u - u + \frac{u^p}{1 + \kappa \xi^{pq/(p-1)}w^p} &= 0 \text{ in } \Omega, \\
\kappa \gamma' &= \frac{\int_\Omega u w(x) dx}{|\Omega|}, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial \Omega.
\end{align*}
\] (6.13)

Now, we introduce a matching parameter \( \delta = \kappa \xi^{pq/(p-1)} \), then the first equation of (6.13) is reduced to the problem:
\[
\begin{align*}
\varepsilon^2 \Delta u - u + f_\delta(u) &= 0 \text{ in } \Omega, \\
\delta \gamma' \int_\Omega u w(x) dx &= \kappa |\Omega|, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial \Omega,
\end{align*}
\] (6.14)
where \( f_\delta(u) := \frac{u^p}{1 + \delta u^p} \). If we can find a pair \((u, \delta)\) satisfying (6.14), then the stationary solution \((A, \xi)\) to (6.4) is also obtained by (6.12) and the second equation of (6.13). For the purpose, we first consider the single equation:
\[
\begin{align*}
\varepsilon^2 \Delta u - u + f_\delta(u) &= 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial \Omega.
\end{align*}
\] (6.15)

By Lemma 2.2, the nonlinear term \( f_\delta(u) \) satisfies (\text{f1})-(\text{f3}). Hence, if we fix \( \delta \in (0, \delta_*), \) we can apply Theorem 3.1, and we obtain a solution denoted by
Next, we will seek \( \delta = \delta_\varepsilon \) satisfying the matching condition (6.14) for each \( \varepsilon \) sufficiently small, namely,

\[
\delta_\varepsilon^\gamma \int_\Omega u_{\delta_\varepsilon}^\varepsilon(x;\varepsilon)dx = \kappa^\gamma|\Omega|.
\]  

Finally, we shall construct a stationary solution to (6.1), near the solution to the shadow system, by using the implicit function theorem.

### 6.2 Proof of Theorems

**Proof of Theorem 6.1.** As we observed above, for \( u_\delta(x;\varepsilon) \) given by Theorem 3.1, if we can find \( \delta_\varepsilon \) such that

\[
\delta_\varepsilon^\gamma \int_\Omega u_{\delta_\varepsilon}^\varepsilon(x;\varepsilon)dx = \kappa^\gamma|\Omega|,
\]  

then we obtain a solution to the shadow system (6.4). Put

\[
\beta(\delta) := \delta^\gamma \int_{\mathbb{R}^N} w_\varepsilon^\delta(y)dy, \quad \beta_\varepsilon(\delta) := \varepsilon^{-N}\delta^\gamma \int_\Omega u_{\delta_\varepsilon}^\varepsilon(x;\varepsilon)dx.
\]  

Then, \( \beta(\delta) \) is continuous in \( \delta \in [0, \delta_*] \), and \( \beta(0) = 0, \beta(\delta) \to \infty \) as \( \delta \to \delta_* \) by Lemma 4.3. We take a number \( \kappa_1 \) so that \( \kappa_1 > \kappa_0 \). Then we can see that there exists \( \overline{\delta} \in [0, \delta_*] \) such that

\[
\beta(\overline{\delta}) = \delta^\gamma \int_{\mathbb{R}^N} w_\varepsilon^\delta(y)dy = \kappa_1|\Omega|(6.18)
\]  

holds. We take \( \delta \) such this in advance. We recall that \( \beta_\varepsilon(\delta) \) is continuous in \( \delta \in [0, \delta_*] \). Now, by (3.10), it holds that

\[
\beta_\varepsilon(\delta) = \varepsilon^{-N}\delta^\gamma \int_\Omega u_{\delta_\varepsilon}^\varepsilon(x;\varepsilon)dx = \delta^\gamma \left( \frac{m}{2} \int_{\mathbb{R}^N} w_\varepsilon^\delta(y)dy + o(1) \right),
\]  

as \( \varepsilon \to 0 \) uniformly in \( \delta \in [0, \delta_*] \). Note that \( \beta_\varepsilon(0) = 0 \) and \( \beta_\varepsilon(\delta) = \kappa_1|\Omega| + o(1) \). Then, by the assumption (A7) and the intermediate value theorem, we can see that there exists \( \delta_\varepsilon \in [0, \delta_*] \) for sufficiently small \( \varepsilon \) such that \( \beta_\varepsilon(\delta_\varepsilon) = \varepsilon^{-N}\kappa^\gamma|\Omega| \) which implies (6.16). Thus, by putting

\[
\xi_\varepsilon = \left( \frac{|\Omega|}{\int_\Omega u_{\delta_\varepsilon}^\varepsilon(x;\varepsilon)dx} \right)^{1/\gamma'}, \quad A_\varepsilon(x) = \xi_\varepsilon^{\gamma/(p-1)} u_{\delta_\varepsilon}(x;\varepsilon),
\]  

we obtain a solution to (6.4).

By the definition of \( A_\varepsilon \) and (3.9), the first inequality in (6.6) holds. By the definition of \( \xi_\varepsilon \) and (3.10), the second inequality in (6.6) can be verified.

Note that, \( \delta \) is independent of \( \varepsilon \), and \( \delta_\varepsilon \in [0, \delta_*] \). Thus, there exists a subsequence \( \{\varepsilon_i\} \) such that \( \delta_{\varepsilon_i} \) converges to some \( \delta_4 \in [0, \delta_*] \) and \( \varepsilon_i \to 0 \) as \( i \to \infty \).
By (6.19), \( \beta_\epsilon(\delta_{\epsilon_i}) = \epsilon^{-N}\kappa^i|\Omega| \), and the assumption (A7), it is easy to see that \( \delta_4 \) satisfies
\[
\delta_4^\gamma \left( \frac{m}{2|\Omega|} \right) \int_{\mathbb{R}^N} w_{\delta_4}(y)dy = \kappa_0. \tag{6.20}
\]
Here, we write \( \delta_i := \delta_{\epsilon_i} \) simply. Let us show (6.7). We can see that
\[
A_{\epsilon_i}(x) = \xi_{\epsilon_i} + A_{\epsilon_i}(x; \epsilon_i, \delta_{\epsilon_i}(x)) = \xi_{\epsilon_i} + (\Omega_{\epsilon_i}(x; P_k) + \epsilon_{\delta_{\epsilon_i}, \delta_{\epsilon_i}}(x)), \quad x \in \Phi(B_{\rho_0}; P_k) \cap \Omega,
\]
holds as \( i \to \infty \) for each \( k = 1, \ldots, m \) by the continuity of \( w_\delta \) with respect to \( \delta \) and the boundedness of \( \phi_{\epsilon, \delta} \). Thus (6.7) holds.

Next, by (3.10), the definition of \( \xi_{\epsilon_i} \), and the continuity of \( w_\delta \) with respect to \( \delta \), (6.8) follows. Thus we complete the proof.

**Proof of Theorem 6.2.** Let \( (A_\epsilon, \xi_\epsilon) \) be a solution to (6.4) given in Theorem 6.1. As in the proof of Theorem 5.1, we decompose the space \( X^t \) into \( X^t = \mathbb{R} \oplus X_1 \). Let \( P : X^t \to X_1 \) be the projection associated with this decomposition. We put \( Z = X_2 \rightleftharpoons X_1 \). We set \( H(x) \) as follows according to the decomposition above:
\[
H(x) = \xi + \psi(x), \quad \xi \in \mathbb{R}, \quad \psi \in X_1. \tag{6.21}
\]
And we put
\[
d = 1/D.
\]
To find a stationary solution to the full system (6.1), we define operators \( F_1, F_2, F_3 \) by
\[
F_1(d, A, \xi, \psi) = \epsilon^2 \Delta A - A + \frac{A^p}{(\xi + \psi)^q(1 + \kappa A^p)},
\]
\[
F_2(d, A, \xi, \psi) = \int_\Omega \left\{ -\xi + \frac{A^r}{(\xi + \psi)^s} \right\} dx,
\]
\[
F_3(d, A, \xi, \psi) = \Delta \psi + d \left\{ -\psi + P \left( \frac{A^r}{(\xi + \psi)^s} \right) \right\}.
\]
Then, for \( t > N \),
\[
F = (F_1, F_2, F_3)
\]
is a \( C^1 \) mapping from an open set
\[
\mathbb{R} \times \{ A \in X_2^{2,t} : A > 0 \text{ on } \Omega \} \times (c, +\infty) \times \{ \psi \in Z : \| \psi \|_{L^\infty(\Omega)} < c \}
\]
into \( X^t \times \mathbb{R} \times X_1 \), where \( c \) is a small positive number.

Let \( (A_\epsilon, \xi_\epsilon) \) be a solution to (6.4) given in Theorem 6.1. Then
\[
F(0, A_\epsilon, \xi_\epsilon, 0) = 0.
\]

103
Therefore, if the partial derivative of \( F \) with respect to \((A, \xi, \psi)\) at \((0, A_\varepsilon, \xi_\varepsilon, 0)\):

\[
D_{(A, \xi, \psi)}F|_{(0, A_\varepsilon, \xi_\varepsilon, 0)} = \begin{pmatrix}
\varepsilon^2 \Delta - 1 + \frac{pA^{-1}}{\xi^2(1 + \kappa A^2)^2} & -\frac{qA^p}{\xi^{q+1}}(1 + \kappa A^2) & -\frac{qA^p}{\xi^{q+1}}(1 + \kappa A^2) \\
\frac{r}{\xi^s} \int_{\Omega} A^{-1} \cdot \Delta d \xi - |\Omega| - s \int_{\Omega} A^{-1} \Delta d \xi & -s \int_{\Omega} A^{-1} \Delta d \xi & -\frac{s}{\xi^{q+1}} \int_{\Omega} A^{-1} \Delta d \xi \\
\end{pmatrix}
\]

is boundedly invertible, then by the implicit function theorem, we have a one parameter family of solutions \((A(d), \xi(d), \psi(d)) \in X_{a,t} \times \mathbb{R}_+ \times Z\), which are of class \( C^1 \) with respect to \( d \), such that

\[
F(d, A(d), \xi(d), \psi(d)) = 0
\]

for \( |d| \) sufficiently small, and

\[
(A(0), \xi(0), \psi(0)) = (A_\varepsilon, \xi_\varepsilon, 0).
\]

Thus, by applying Taylor’s theorem, we have a conclusion.

Let us show that (6.22) has a bounded inverse. Note that \( \Delta \) under the homogeneous Neumann boundary conditions is an isomorphism from \( Z \) onto \( X_1 \). Thus (6.22) is invertible if and only if the linearized shadow operator

\[
L_\infty := \begin{pmatrix}
\varepsilon^2 \Delta - 1 + \frac{pA^{-1}}{\xi^2(1 + \kappa A^2)^2} & -\frac{qA^p}{\xi^{q+1}}(1 + \kappa A^2) & -\frac{qA^p}{\xi^{q+1}}(1 + \kappa A^2) \\
\frac{r}{\xi^s} \int_{\Omega} A^{-1} \cdot \Delta d \xi - |\Omega| - s \int_{\Omega} A^{-1} \Delta d \xi & -s \int_{\Omega} A^{-1} \Delta d \xi & -\frac{s}{\xi^{q+1}} \int_{\Omega} A^{-1} \Delta d \xi \\
\end{pmatrix} : X_{a,t} \times \mathbb{R} \to X' \times \mathbb{R},
\]

has a bounded inverse. Here, we used the relation \( \int_{\Omega} A^r d \xi = |\Omega|/\xi^{q+1} \). To show this, we first show that \( \text{Ker}(L_\infty) = \{(0, 0)\} \). Let \((\phi, \eta) \in \text{Ker}(L_\infty)\), namely,

\[
\varepsilon^2 \Delta \phi - \phi + \frac{pA^{-1}}{\xi^2(1 + \kappa A^2)^2} \phi - \frac{qA^p \eta}{\xi^{q+1}}(1 + \kappa A^2) = 0,
\]

\[
\frac{r}{\xi^s} \int_{\Omega} A^{-1} \phi d \xi - (1 + s)|\Omega| \eta = 0.
\]

Now we put \( \phi = \xi^2/(p-1) \phi \) and \( \eta = \xi_\varepsilon \eta \). Recall that

\[
A_\varepsilon = \xi^2/(p-1) u_{\delta_\varepsilon} \phi, \quad \xi_\varepsilon = \left(\frac{|\Omega|}{\int_{\Omega} u_{\delta_\varepsilon} \phi d \xi}\right)^{1/\gamma}
\]

and \( \delta_\varepsilon = k \xi^{pq/(p-1)} \). We put \( u_{\varepsilon}(x) := u_{\delta_\varepsilon} \phi, \xi_\varepsilon := \varepsilon^2 \Delta - 1 + f_{\delta_\varepsilon} \phi \). Note that, by Lemma 3.13, \( L_\varepsilon \) has a bounded inverse \( K_\varepsilon \). Then we have

\[
\varepsilon^2 \Delta \phi - \phi + \frac{pu_{\varepsilon}^{-1}}{1 + \delta_{\varepsilon} u_{\varepsilon}^2} \phi - \frac{q u_{\varepsilon} \eta}{1 + \delta_{\varepsilon} u_{\varepsilon}^2} = 0,
\]

\[
\frac{r}{\int_{\Omega} u_{\varepsilon}^{-1} \phi d \xi} \int_{\Omega} u_{\varepsilon}^{-1} \phi d \xi - (1 + s) \eta = 0.
\]
(6.25) is written as $\mathcal{L}_\varepsilon \hat{\phi} = q\hat{\eta} f_{\delta_4}(u_\varepsilon)$. Therefore

$$\hat{\phi} = q\hat{\eta} \mathcal{K}_\varepsilon [f_{\delta_4}(u_\varepsilon)]$$

(6.27)

On the other hand, since $\varepsilon^2 \Delta u_\varepsilon - u_\varepsilon + f_{\delta_4}(u_\varepsilon) = 0$, it holds that

$$\mathcal{L}_\varepsilon u_\varepsilon = \varepsilon^2 \Delta u_\varepsilon - u_\varepsilon + f'_{\delta_4}(u_\varepsilon) u_\varepsilon$$

$$= -f_{\delta_4}(u_\varepsilon) + f'_{\delta_4}(u_\varepsilon) u_\varepsilon$$

$$= -u_\varepsilon^p + \frac{\mu u_\varepsilon^p}{1 + \delta_\varepsilon u_\varepsilon^p} + \frac{(1 + \delta_\varepsilon u_\varepsilon^p)^{q-1}}{1 + \delta_\varepsilon u_\varepsilon^p}$$

$$= (p-1-g) f_{\delta_4}(u_\varepsilon),$$

where $g := \frac{p\delta_\varepsilon u_\varepsilon^p}{1 + \delta_\varepsilon u_\varepsilon^p}$. Therefore,

$$\mathcal{K}_\varepsilon [f_{\delta_4}(u_\varepsilon)] = \frac{1}{p-1} u_\varepsilon + \mathcal{K}_\varepsilon [g f_{\delta_4}(u_\varepsilon)].$$

(6.28)

We substitute this into (6.27), and we have

$$\hat{\phi} = q\hat{\eta} \left( \frac{1}{p-1} u_\varepsilon + \mathcal{K}_\varepsilon [g f_{\delta_4}(u_\varepsilon)] \right).$$

Hence, by substituting this into (6.26), we obtain

$$\hat{\eta} \left\{ \gamma' + \frac{qr}{\Omega} u_\varepsilon^p \int_\Omega \mathcal{K}_\varepsilon [g f_{\delta_4}(u_\varepsilon)] u_\varepsilon^{p-1} dx \right\} = 0,$$

(6.29)

where $\gamma' = qr/(p-1) - (s+1)$. Now, we can estimate as follows:

$$\left| \int_\Omega \frac{qr}{\Omega} u_\varepsilon^p \int_\Omega \mathcal{K}_\varepsilon [g f_{\delta_4}(u_\varepsilon)] u_\varepsilon^{p-1} dx \right| \leq \frac{qr}{\Omega} u_\varepsilon^{p-1} \mathcal{K}_\varepsilon [g f_{\delta_4}(u_\varepsilon)] \| \mathcal{L}_\varepsilon \|_{L^\infty(\Omega)} \int_\Omega u_\varepsilon^p dx$$

$$\leq qr \| \mathcal{K}_\varepsilon [g f_{\delta_4}(u_\varepsilon)] \|_{L^\infty(\Omega)} \leq qr C \| \mathcal{L}_\varepsilon \|_{L^\infty(\Omega)} \| g f_{\delta_4}(u_\varepsilon) \|_{L^\infty(\Omega)}$$

$$= qr C \| \mathcal{K}_\varepsilon [g f_{\delta_4}(u_\varepsilon)] \|_{L^\infty(\Omega)} \| \mathcal{K}_\varepsilon [g f_{\delta_4}(u_\varepsilon)] \|_{L^\infty(\Omega)}$$

$$\leq pqr \delta_\varepsilon C \| u_\varepsilon \|_{L^\infty(\Omega)}^{2p-1}$$

$$\leq C' \delta_\varepsilon,$$

for some constant $C' > 0$ independent of $\varepsilon$ sufficiently small. Here, we used (3.78) and the fact $u_\varepsilon$ is bounded uniformly in $\varepsilon$ sufficiently small. Now, recall that if $\kappa_0 \geq 0$ is sufficiently small then $\delta_\varepsilon \to \delta_\varepsilon (\varepsilon \to 0)$ and the $\delta_\varepsilon$ must be small, since the $\delta_\varepsilon$ is characterized by

$$\delta_\varepsilon \left( \frac{m}{2\Omega} \int_{\mathbb{R}^N} w_\delta^*(y) dy \right) = \kappa_0.$$ 

(6.30)

Therefore, if $\kappa_0 \geq 0$ and $\varepsilon > 0$ are sufficiently small, then it follows that

$$\left| \int_\Omega \frac{qr}{\Omega} u_\varepsilon^p \int_\Omega \mathcal{K}_\varepsilon [g f_{\delta_4}(u_\varepsilon)] u_\varepsilon^{p-1} dx \right| \leq \gamma'.$$
Thus, from (6.29), we conclude that \( \hat{\eta} = 0 \). Then it is easy to see that \( \hat{\phi} = 0 \) from (6.25) or (6.26). Thus \( L_\infty \) is one to one.

Secondly, for given \((\hat{\phi}, \hat{\eta}) \in X^t \times \mathbb{R}\), we seek \((\phi, \eta) \in X^2 \times X^2 \) such that

\[
L_\infty (\phi, \eta) = (\hat{\phi}, \hat{\eta}).
\]

(6.31)

We put \( \phi = \xi^{q/(p-1)} \hat{\phi} \) and \( \eta = \xi \hat{\eta} \), and we note the relations (6.24). Then (6.31) is written as follows:

\[
\varepsilon^2 \Delta \hat{\phi} - \hat{\phi} + \frac{pu^{p-1}_\varepsilon \hat{\phi}}{(1 + \delta \varepsilon u^p_\varepsilon)^2} - \frac{q u^q_\varepsilon \hat{\eta}}{1 + \delta \varepsilon u^q_\varepsilon} = \xi^{(p-1)/q} \hat{\phi},
\]

(6.32)

\[
\frac{r \int_{\Omega} u^{r-1}_\varepsilon \hat{\phi} dx}{\int_{\Omega} u^r_\varepsilon dx} - (1 + s) \hat{\eta} = \frac{\xi_\varepsilon}{|\Omega|} \hat{\eta}.
\]

(6.33)

Since (6.32) is written as follows:

\[
L_\varepsilon \hat{\phi} = \xi^{(p-1)/q} \hat{\phi} + q \hat{\eta} f_{\delta \varepsilon}(u_\varepsilon),
\]

we have

\[
\hat{\phi} = \xi^{(p-1)/q} K_c [\hat{\phi}] + q \hat{\eta} K_c [f_{\delta \varepsilon}(u_\varepsilon)].
\]

(6.34)

Substitute this into (6.33), and use (6.28), then we have

\[
\xi^{-1} \left\{ \gamma' + \frac{qr}{\int_{\Omega} u^r_\varepsilon dx} \int_{\Omega} K_c [g f_{\delta \varepsilon}(u_\varepsilon)] u^{r-1}_\varepsilon \right\} = \frac{\xi_{\varepsilon}}{|\Omega|} \hat{\eta} - \frac{\xi^{(p-1)/q}}{\int_{\Omega} u^r_\varepsilon dx} \int_{\Omega} u^{r-1}_\varepsilon K_c [\hat{\phi}] dx.
\]

Because

\[
\gamma' + \frac{qr}{\int_{\Omega} u^r_\varepsilon dx} \int_{\Omega} K_c [g f_{\delta \varepsilon}(u_\varepsilon)] u^{r-1}_\varepsilon dx \neq 0
\]

(6.35)

for sufficiently small \( \varepsilon \) from the argument above, we obtain \( \hat{\eta} \) by

\[
\hat{\eta} = \left( \frac{qr}{\int_{\Omega} u^r_\varepsilon dx} \int_{\Omega} K_c [g f_{\delta \varepsilon}(u_\varepsilon)] u^{r-1}_\varepsilon dx \right)^{-1} \left( \frac{\xi_{\varepsilon}}{|\Omega|} \hat{\eta} - \frac{\xi^{(p-1)/q}}{\int_{\Omega} u^r_\varepsilon dx} \int_{\Omega} u^{r-1}_\varepsilon K_c [\hat{\phi}] dx \right).
\]

(6.36)

and we obtain \( \eta \) by \( \eta = \xi \hat{\eta} \). By substituting this \( \hat{\eta} \) into (6.34), we obtain \( \hat{\phi} \), and we obtain \( \phi \) by \( \phi = \xi^{q/(p-1)} \hat{\phi} \). Moreover, it is easy to see that

\[
\|\phi\|_{W^2(t \Omega)} \leq C (\|\hat{\phi}\|_{L^1(\Omega)} + |\hat{\eta}|)
\]

\[
|\eta| \leq C' (\|\hat{\phi}\|_{L^1(\Omega)} + |\hat{\eta}|)
\]

holds for some \( C, C' > 0 \) independent of \( \hat{\phi}, \hat{\eta} \). Thus we complete the proof. 

\[
106
\]
Chapter 7

Effect of the source term for the Gierer-Meinhardt system

7.1 Main results

In this chapter, we consider the case $\kappa \geq 0$ and $\sigma_0 \neq 0$ for (6.1). For simplicity, we consider the case $(p, q, r, s) = (2, 1, 2, 0)$, namely, the following steady-state Gierer-Meinhardt system:

$$
\begin{align*}
0 &= \varepsilon^2 \Delta A - A + \frac{A^2}{\eta(1 + \kappa A^2)} + \sigma_0, \quad A > 0 \text{ in } \Omega, \\
0 &= D\delta H - H + A^2, \quad H > 0 \text{ in } \Omega, \\
\frac{\partial A}{\partial \nu} &= \frac{\partial H}{\partial \nu} = 0 \text{ on } \partial \Omega. 
\end{align*}
\tag{7.1}
$$

Corresponding shadow system is as follows:

$$
\begin{align*}
0 &= \varepsilon^2 \Delta A - A + \frac{A^2}{\eta(1 + \kappa A^2)} + \sigma_0, \quad A > 0 \text{ in } \Omega, \\
\xi &= \frac{1}{\Omega} \int_{\Omega} A^2 \, dx, \\
\frac{\partial A}{\partial \nu} &= 0 \text{ on } \partial \Omega. 
\end{align*}
\tag{7.2}
$$

We assume (A0) and (A7). Corresponding to the condition (6.3), we restrict the dimension $N$ to

$$
1 \leq N \leq 5. 
\tag{7.3}
$$

For the source term $\sigma_0$, we suppose the following assumption.

(A8) $\sigma_0$ is a nonnegative function of class $C^\alpha(\overline{\Omega})$, $\alpha \in (0, 1)$, and is $x_N$-axially symmetric.

Let $P_1, \ldots, P_m$ and $P(\cdot; P_k), \Psi(\cdot; P_k)$ are as before. Needless to say, the solution for the case $\sigma_0 \neq 0$ may differ from the one of the case $\sigma_0 = 0$. In this chapter, we
show the existence of solutions to (7.1) and (7.2), and considering the asymptotic behavior of the solution as \( \varepsilon \to 0 \), we consider the effect of the source term.

We state a result on the existence of multi-peak solution to the shadow system (7.2).

**Theorem 7.1.** Assume (A0), (A7) and (A8). Then for \( \varepsilon \) sufficiently small, the shadow system (6.4) admits an \( x_N \)-axially symmetric solution \((A_\varepsilon, \xi_\varepsilon)\), the solution satisfies

\[
A_\varepsilon(x) \to \sigma_0(x) \quad \text{as} \quad \varepsilon \to 0
\]

for each point \( x \in \Omega \). Moreover, by taking a subsequence \( \{\varepsilon_i\} \) which converges to 0 as \( i \to \infty \) if necessary,

\[
A_{\varepsilon_i}(x) = \xi_{\varepsilon_i} \left\{ w_{\delta_5} \left( \frac{1}{\varepsilon_i} \Psi(x; P_k) \right) + o(1) \right\}, \quad x \in \Phi(B_{r_0}; P_k) \cap \Omega, \quad k = 1, \ldots, m,
\]

\[
\xi_{\varepsilon_i} = |\Omega| \left( \frac{m}{2} \int_{\mathbb{R}^N} w_{\delta_5}^2(y)dy + o(1) \right)^{-1},
\]

\[
\int_\Omega A_{\varepsilon_i}(x)dx = |\Omega| \left( \frac{m}{2} \int_{\mathbb{R}^N} w_{\delta_5}(y) \left( \int_{\mathbb{R}^N} w_{\delta_5}^2(y)dy \right)^{-1} + \int_\Omega \sigma_0(x)dx + o(1) \right),
\]

as \( i \to \infty \), the term \( o(1) \) in (7.5) is uniform in \( x \), \( \delta_5 \) is a certain number satisfying

\[
\delta_5 \left( \frac{m}{2|\Omega|} \int_{\mathbb{R}^N} w_{\delta_5}^2(y)dy \right)^2 = \kappa_0^2,
\]

and \( w_5 \) is a unique solution to the following problem:

\[
\begin{cases}
\Delta w - w + f_\delta(w) = 0, \quad w > 0 \quad \text{in} \quad \mathbb{R}^N, \\
\max_{\mathbb{R}^N} w = w(0), \quad \text{as} \quad |z| \to \infty,
\end{cases}
\]

\[
f_\delta(w) = \frac{w^2}{1 + \delta w^2},
\]

Moreover, if \( \kappa_0 \) is sufficiently small, then we need not to take the subsequence \( \{\varepsilon_i\} \) above.

We have the following global estimate.

**Theorem 7.2.** Assume \( \sigma_0 \in C^2(\overline{\Omega}) \) and \( \frac{\partial \sigma_0}{\partial \nu} = 0 \) on \( \partial \Omega \). Let \((A_\varepsilon, \xi_\varepsilon)\) be the solution to (7.2) given in Theorem 7.1. Then the following estimate holds:

\[
\sigma_0(x) - \varepsilon^2 \|\Delta \sigma_0\|_{L^\infty(\Omega)} < A_\varepsilon(x), \quad x \in \Omega,
\]

\[
A_\varepsilon(x) < C_\varepsilon e^{-c \text{dist}(x, P)/\varepsilon} + \sigma_0(x) + \varepsilon^2 \|\Delta \sigma_0\|_{L^\infty(\Omega)} + \frac{\sigma_0^2(x)}{\xi_\varepsilon} + o(\varepsilon^N), \quad x \in \Omega,
\]

as \( \varepsilon \to 0 \), where \( P = \{P_1, \ldots, P_m\} \), the term \( o(\varepsilon^N) \) is uniform in \( x \), the constants \( C, c > 0 \) are independent of \( \varepsilon \) and \( x \).
Theorem 7.3. Suppose that $\kappa_0$ is sufficiently small. Let $(A_\varepsilon, \xi_\varepsilon)$ be the solution to (7.2) given in Theorem 7.1. Then there exists $D_\varepsilon > 0$ such that, if $D > D_\varepsilon$, then (7.1) admits an $x_N$-axially symmetric solution $(A_\varepsilon(x; D), H_\varepsilon(x; D))$ satisfying
\[
\sup_{x \in \Omega} |A_\varepsilon(x) - A_\varepsilon(x; D)| \to 0, \quad \sup_{x \in \Omega} |\xi_\varepsilon - H_\varepsilon(x; D)| \to 0,
\]
as $D \to \infty$.

Remark 7.1. Suzuki and Takagi [96] studied the effect of the source term on the asymptotic behavior of the Gierer-Meinhardt system with out saturation. Our global estimate and its proof are different from the one in [96].

7.2 Construction of a solution to the shadow system

In this section, we construct a solution to the shadow system (7.2) and prove Theorem 7.1. As we did in Chapter 6, we put $A = \xi v$ and substitute it into (7.2) and introduce a matching parameter $\delta$. Then we have the following problem equivalent to (7.2):
\[
\begin{dcases}
\varepsilon^2 \Delta v - v + f_\delta(v) + \frac{\sigma_0}{\varepsilon} \xi = 0, \quad v > 0 \text{ in } \Omega, \\
\frac{\partial v}{\partial \nu} = 0 \text{ on } \partial \Omega,
\end{dcases}
\] (7.12)
with
\[
\xi = \frac{|\Omega|}{\int_\Omega v^2 \, dx},
\] (7.13)
\[
\delta \left( \int_\Omega v^2 \, dx \right)^2 = \kappa |\Omega|^2.
\] (7.14)
The matching condition is (7.14). Comparing with the case $\sigma_0 = 0$, the problem is complex since $\xi$ remains in the equation (7.12). Hence, we must consider the equation (7.12) together with (7.13). Now, we substitute $\xi$ in (7.13) into (7.12). Then we have the following nonlocal problem with parameter $\delta$:
\[
\begin{dcases}
\varepsilon^2 \Delta v - v + f_\delta(v) + \frac{\sigma_0}{\varepsilon} \int_\Omega v^2 \, dx = 0, \quad v > 0 \text{ in } \Omega, \\
\frac{\partial v}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{dcases}
\] (7.15)

We know that the problem (7.9) has a unique solution for $\delta \in [0, \delta_*)$ (see Lemma 2.2). In the case $\sigma_0 = 0$, we have already known that the equation (7.15) has a solution $u_{\delta}(x; \varepsilon)$ for each $\delta \in [0, \delta_*)$ provided $\varepsilon$ is small enough by Theorem 3.1. Let us write $u_{\varepsilon, \delta}(x) = u_{\delta}(x; \varepsilon)$. We will find a solution $v$ to (7.15) by perturbation argument. As before, we consider $\delta \in [0, \bar{\delta}]$ for $\bar{\delta} \in (0, \delta_*)$ fixed arbitrarily. We first show the following proposition.
Proposition 7.1. Let $\sigma_2 \in \mathbb{R}_+$ be fixed arbitrarily. There exists $\varepsilon_4$ depending on $\delta$ and $\|\sigma_0\|_{L^\infty(\Omega)}$, the problem (7.15) admits an $x_N$-axially symmetric solution $v_\delta(x; \varepsilon)$ provided $\varepsilon \in (0, \varepsilon_4)$, $\delta \in [0, \delta]$ and $\|\sigma_0\|_{L^\infty(\Omega)} \leq \sigma_2$, such that

$$v_\delta(x; \varepsilon) = u_{\varepsilon, \delta}(x) + \varepsilon N \phi_{\varepsilon, \delta},$$

(7.16)

where $u_{\varepsilon, \delta}(x)$ is a solution to (7.15) in the case $\sigma_0 = 0$ given in Theorem 3.1, and $\phi_{\varepsilon, \delta}$ satisfies

$$\|\phi_{\varepsilon, \delta}\|_{L^\infty(\Omega)} \leq C\sigma_2$$

(7.17)

for some constant $C > 0$ independent of $\varepsilon$, $\sigma_0$ and $\delta \in [0, \delta]$. Moreover, $v_\delta(x; \varepsilon)$ is continuous in $\delta \in [0, \delta]$ with respect to $C^0(\bar{\Omega})$-norm.

We shall prove Proposition 7.1 in a series of lemmas. Now we put $v(x) = u_{\varepsilon, \delta}(x) + \varepsilon N \phi(x)$ and substitute it into (7.15). If $\phi \in X_{\nu}^{2,t}$, then the following equations are equivalent:

$$\varepsilon^2 \Delta (u_{\varepsilon, \delta} + \varepsilon N \phi) - (u_{\varepsilon, \delta} + \varepsilon N \phi) + f_\delta(u_{\varepsilon, \delta} + \varepsilon N \phi) + \frac{\sigma_0}{\|\Omega\|} \int_\Omega (u_{\varepsilon, \delta} + \varepsilon N \phi)^2 dx = 0,$$

$$\varepsilon^2 N L_{u(\varepsilon, \delta)} \phi + M_{1, \delta}^1(\phi) + M_{2, \delta}^2(\phi) = 0,$$

(7.18)

where

$$M_{1, \delta}^1(\phi) := f_\delta(u_{\varepsilon, \delta} + \varepsilon N \phi) - f_\delta(u_{\varepsilon, \delta}) - \varepsilon N f_\delta^1(u_{\varepsilon, \delta}) \phi,$$

$$M_{2, \delta}^2(\phi) := \frac{\sigma_0}{\|\Omega\|} \int_\Omega (u_{\varepsilon, \delta} + \varepsilon N \phi)^2 dx.$$  

(7.19)

(7.20)

Here, we used $\varepsilon^2 \Delta u_{\varepsilon, \delta} - u_{\varepsilon, \delta} + f_\delta(u_{\varepsilon, \delta}) = 0$, and

$$L_{u(\varepsilon, \delta)} = \varepsilon^2 \Delta - 1 + f_\delta^1(u_{\varepsilon, \delta}) : X_{\nu}^{2,t} \to X^t.$$  

Recall that $L_{u(\varepsilon, \delta)}$ has a bounded inverse $K_{u(\varepsilon, \delta)} : X^t \to X_{\nu}^{2,t}$, $t \in (1, \infty)$, by Lemma 3.13. Hence (7.18) is equivalent to

$$\phi = -\frac{1}{\varepsilon N} K_{u(\varepsilon, \delta)} [M_{1, \delta}^1(\phi) + M_{2, \delta}^2(\phi)] =: T_{\varepsilon, \delta}(\phi).$$

(7.21)

Thus, we only need to find a fixed point denoted by $\bar{\phi}_{\varepsilon, \delta}$ of $T_{\varepsilon, \delta}$ defined above. Note that if $\phi_{\varepsilon, \delta} \in X^t$ is a fixed point of $T_{\varepsilon, \delta}$, then $\phi_{\varepsilon, \delta} \in X_{\nu}^{2,t}$ because of (7.21) and $K_{u(\varepsilon, \delta)}$ is a mapping form $X^t$ onto $X_{\nu}^{2,t}$. Hence, by Schauder’s theory, $v_\delta(x; \varepsilon) := u_{\varepsilon, \delta} + \varepsilon N \phi_{\varepsilon, \delta}$ becomes a classical solution to (7.15).

Lemma 7.1. For $M_{1, \delta}^1$ defined by (7.19), there exists a constant $C_2 > 0$, the following estimates hold:

$$\|M_{1, \delta}^1(\phi)\|_{L^\infty(\Omega)} \leq \varepsilon^{2N} C_2 \|\phi\|_{L^\infty(\Omega)}^2,$$

(7.22)

$$\|M_{1, \delta}^1(\phi_1) - M_{1, \delta}^1(\phi_2)\|_{L^\infty(\Omega)} \leq \varepsilon^{2N} C_2 (\|\phi_1\|_{L^\infty(\Omega)} + \|\phi_2\|_{L^\infty(\Omega)}) \|\phi_1 - \phi_2\|_{L^\infty(\Omega)}.$$  

(7.23)

for any $\phi, \phi_1, \phi_2 \in C^0(\bar{\Omega})$ and for all $\varepsilon$ sufficiently small and $\delta \in [0, \delta]$. 

110
Remark 7.2. Although we have regarded \( f_\delta(t) \equiv 0 \) for \( t \geq 0 \) in the previous chapters, we might as well regard \( f_\delta(t) = \frac{t^2}{1 + t^2 \delta} \) for all \( t \in (-\infty, \infty) \) in this case.

Proof. We note that
\[
    f''_\delta(t) = \frac{2 - 6\delta t^2}{(1 + \delta t^2)^3}.
\]

We see that \(|f''_\delta(t)| \leq C\) holds for some uniform constant \( C > 0 \). Hence, by the mean value theorem, we can estimate as follows:

\[
|M^1_{\epsilon, \delta}(\phi)| = |f_\delta(u_{\epsilon, \delta} + \epsilon^N \phi) - f_\delta(u_{\epsilon, \delta}) - \epsilon^N f'_\delta(u_{\epsilon, \delta})\phi| \\
= \epsilon^N \left| \int_0^1 \{f'_\delta(u_{\epsilon, \delta} + \epsilon^N \phi t) - f'_\delta(u_{\epsilon, \delta})\} dt \right| |\phi| \\
\leq \epsilon^{2N} C\|\phi\|_{L^\infty(\Omega)}^2 ;
\]

and

\[
|M^1_{\epsilon, \delta}(\phi_1) - M^1_{\epsilon, \delta}(\phi_2)| \\
= |f_\delta(u_{\epsilon, \delta} + \epsilon^N \phi_1) - f_\delta(u_{\epsilon, \delta} + \epsilon^N \phi_2) - \epsilon^N f'_\delta(u_{\epsilon, \delta})(\phi_1 - \phi_2)| \\
= \epsilon^N \left| \int_0^1 \{f'_\delta(u_{\epsilon, \delta} + \epsilon^N \phi_2 + \epsilon^N (\phi_1 - \phi_2) t) - f'_\delta(u_{\epsilon, \delta})\} dt \right| |\phi_1 - \phi_2| \\
\leq \epsilon^{2N} C(\|\phi_1\|_{L^\infty(\Omega)} + \|\phi_2\|_{L^\infty(\Omega)})\|\phi_1 - \phi_2\|_{L^\infty(\Omega)} .
\]

Thus, we complete the proof. \( \Box \)

Lemma 7.2. For \( M^2_{\epsilon, \delta} \) defined by (7.20), there exists a constant \( C_\delta > 0 \) depending only on \( |\Omega| \) and \( \delta \), the following estimates hold:

\[
\|M^2_{\epsilon, \delta}(\phi)\|_{L^\infty(\Omega)} \leq \epsilon^{N} \|\sigma_0\|_{L^\infty(\Omega)}(C_\delta + \epsilon^N C_\delta\|\phi\|_{L^\infty(\Omega)} + \epsilon^N \|\phi\|_{L^\infty(\Omega)}^2), \quad (7.24)
\]

\[
\|M^2_{\epsilon, \delta}(\phi_1) - M^2_{\epsilon, \delta}(\phi_2)\|_{L^\infty(\Omega)} \leq \epsilon^{2N} \|\sigma_0\|_{L^\infty(\Omega)}(C_\delta\|\phi_1 - \phi_2\|_{L^\infty(\Omega)} \\
+ (\|\phi_1\|_{L^\infty(\Omega)} + \|\phi_2\|_{L^\infty(\Omega)})\|\phi_1 - \phi_2\|_{L^\infty(\Omega)}), \quad (7.25)
\]

for any \( \phi, \phi_1, \phi_2 \in C^0(\Omega) \) and for all \( \epsilon \) sufficiently small and \( \delta \in [0, \delta] \).

Proof. In this proof, we write \( \|\cdot\|_{L^\infty} \) instead of \( \|\cdot\|_{L^\infty(\Omega)} \). We first note that, for each \( r > 0 \), there exists a constant \( C_{r, \delta} > 0 \) depending only on \( r \) and \( \delta \), the following estimate holds by (3.10):

\[
\int_\Omega u_{\epsilon, \delta}^r(x) dx \leq \epsilon^{N} C_{r, \delta}, \quad \delta \in [0, \delta].
\]

for \( \epsilon \) sufficiently small. By this estimate, we have

\[
\|M^2_{\epsilon, \delta}(\phi)\|_{L^\infty} \leq \frac{\|\sigma\|_{L^\infty}}{|\Omega|} \int_\Omega \left\{ u_{\epsilon, \delta}^2 + 2\epsilon^N u_{\epsilon, \delta}\phi + \epsilon^{2N} \phi^2 \right\} dx \\
\leq \frac{\|\sigma\|_{L^\infty}}{|\Omega|} \left( \epsilon^{N} C_{2, \delta} + 2\epsilon^{2N} C_{1, \delta}\|\phi\|_{L^\infty} + \epsilon^{2N} |\Omega|\|\phi\|_{L^\infty}^2 \right) ,
\]

111
There exists a mapping \( T_{\varepsilon, \delta} \) defined by (7.21) is a contraction mapping on \( \hat{B} \) for any \( \delta \in [0, \tilde{\delta}] \) provided \( \varepsilon \in (0, \varepsilon_4) \) and \( \|\sigma_0\|_{L^\infty(\Omega)} \leq \sigma_2 \). Moreover, the mapping \( T_{\varepsilon, \delta} \) is continuous in \( \varepsilon, \delta \), namely, for \( \delta, \delta' \in [0, \tilde{\delta}] \),

\[
\|T_{\varepsilon, \delta}(\phi) - T_{\varepsilon, \delta'}(\phi)\|_{L^\infty(\Omega)} = 0
\]
as \( \delta' \to \delta \) for all \( \phi \in \hat{B} \).

Proof. Let \( \phi \in \hat{B} \), \( \|\sigma_0\|_{L^\infty(\Omega)} \leq \sigma_2 \). We can estimate by Lemmas 3.13, 7.1, 7.2 as follows:

\[
\|T_{\varepsilon, \delta}(\phi)\|_{L^\infty(\Omega)} \leq \frac{1}{\varepsilon N} C^*_\infty(\|M^1_{\varepsilon, \delta}(\phi)\|_{L^\infty(\Omega)} + \|M^2_{\varepsilon, \delta}(\phi)\|_{L^\infty(\Omega)}) \\
\leq \sigma_2 C^*_\infty \delta + \varepsilon N P(\sigma_2, C^*_\infty, C_\delta, C_2),
\]

where \( P(\sigma_2, C^*_\infty, C_\delta, C_2) \) is some polynomial of at most degree three. Hence, if \( \varepsilon \) is small enough, then

\[
\|T_{\varepsilon, \delta}(\phi)\|_{L^\infty(\Omega)} \leq 2\sigma_2 C^*_\infty C_\delta, \ \delta \in [0, \tilde{\delta}],
\]

holds for any \( \phi \in \hat{B} \), and hence \( T_{\varepsilon, \delta} \) is a mapping from \( \hat{B} \) into itself.

Let \( \phi_1, \phi_2 \in \hat{B} \), \( \|\sigma_0\|_{L^\infty(\Omega)} \leq \sigma_2 \). Then we can estimate similarly as follows:

\[
\|T_{\varepsilon, \delta}(\phi_1) - T_{\varepsilon, \delta}(\phi_2)\|_{L^\infty(\Omega)} \leq \frac{1}{\varepsilon N} C^*_\infty(\|M^1_{\varepsilon, \delta}(\phi_1) - M^1_{\varepsilon, \delta}(\phi_2)\|_{L^\infty(\Omega)}) \\
+ \|M^2_{\varepsilon, \delta}(\phi_1) - M^2_{\varepsilon, \delta}(\phi_2)\|_{L^\infty(\Omega)}) \\
\leq \varepsilon N P(\sigma_2, C^*_\infty, C_\delta, C_2)\|\phi_1 - \phi_2\|_{L^\infty(\Omega)}.
\]

Therefore, \( T_{\varepsilon, \delta} \) becomes a contraction mapping provided \( \varepsilon \) is small enough.

Recall that \( u_{\varepsilon, \delta} \) is continuous in \( \delta \in [0, \tilde{\delta}] \) with respect to \( C^0(\Omega) \)-norm. Hence, it is easy to see that the mapping \( T_{\varepsilon, \delta} \) is also continuous in \( \delta \in [0, \tilde{\delta}] \).

Thus, we complete the proof. \( \square \)
Proof of Proposition 7.1. By Lemma 7.3, \( T_{\varepsilon, \delta} \) has a unique fixed point \( \delta_{\varepsilon, \delta} \in \hat{B} \) provided \( \varepsilon \in (0, \varepsilon_4) \), \( \|\sigma_0\|_{L^\infty(\Omega)} < \sigma_2 \) and \( \delta \in [0, \delta] \). Moreover, we notice that \( \delta_{\varepsilon, \delta} \) is continuous in \( \delta \in [0, \delta] \) with respect to \( C^0(\Omega) \)-norm by the contraction mapping principle (Lemma 3.7). By these observations, Proposition 7.1 is verified.

By Proposition 7.1, we obtain a solution \( v_\delta(x; \varepsilon) = u_{\varepsilon, \delta}(x) + \varepsilon^N \hat{\phi}_{\varepsilon, \delta}(x) \) to (7.15). Next, we seek \( \delta = \delta_\varepsilon \) satisfying the matching condition (7.14).

**Lemma 7.4.** For each \( \varepsilon \) sufficiently small, there exists \( \delta_\varepsilon \in [0, \delta_*] \) satisfying

\[
\delta_\varepsilon \left( \int_\Omega v_\delta^2(x; \varepsilon) \, dx \right)^2 = \kappa |\Omega|^2.
\] (7.27)

Moreover, on taking a subsequence \( \{\varepsilon_i\}_{i=1}^\infty \) which converges to 0 as \( i \to \infty \), \( \delta_{\varepsilon_i} \to \delta_\varepsilon \) holds, where \( \delta_\varepsilon \in [0, \delta_\varepsilon] \) is a number satisfying

\[
\delta_\varepsilon \left( \frac{m}{2|\Omega|} \int_{\mathbb{R}^N} w_\varepsilon^2(y) \, dy \right)^2 = \kappa_0^2.
\] (7.28)

Furthermore, if \( \kappa_0 \) is sufficiently small or in the case \( N = 1 \), then \( \delta_{\varepsilon} \to \delta_5 \) holds as \( \varepsilon \to 0 \) without taking a subsequence.

**Proof.** This proof is carried out in the same way as that in the proof of Theorem 6.1. Put

\[
\beta(\delta) := \delta^{1/2} \frac{m}{2} \int_{\mathbb{R}^N} w_\varepsilon^2(y) \, dy, \quad \beta_\varepsilon(\delta) := \varepsilon^{-N} \delta^{1/2} \int_\Omega v_\delta^2(x; \varepsilon) \, dx.
\] (7.29)

Then, \( \beta(\delta) \) is continuous in \( \delta \in [0, \delta_*] \), and \( \beta(0) = 0, \beta(\delta) \to \infty \) as \( \delta \to \delta_* \) by Lemma 4.3. We take a number \( \kappa_1 \) so that \( \kappa_1 > \kappa_0 \). Then we can see that there exists \( \delta \in [0, \delta_*] \) such that

\[
\beta(\delta) = \delta^{1/2} \frac{m}{2} \int_{\mathbb{R}^N} w_\varepsilon^2(y) \, dy = \kappa_1 |\Omega|.
\] (7.30)

holds. We take \( \delta \) such this in advance. We recall that \( \beta_\varepsilon(\delta) \) is continuous in \( \delta \in [0, \delta] \). By using (3.10), and noting \( v_\delta(x; \varepsilon) = u_{\varepsilon, \delta}(x) + \varepsilon^N \hat{\phi}_{\varepsilon, \delta}(x) \), it holds that

\[
\beta_\varepsilon(\delta) = \varepsilon^{-N} \delta^{1/2} \int_\Omega v_\delta^2(x; \varepsilon) \, dx = \delta^{1/2} \left( \frac{m}{2} \int_{\mathbb{R}^N} w_\varepsilon^2(y) \, dy + o(1) \right),
\] (7.31)

as \( \varepsilon \to 0 \) uniformly in \( \delta \in [0, \delta] \). Note that \( \beta_\varepsilon(0) = 0 \) and \( \beta_\varepsilon(\delta) = \kappa_1 |\Omega| + o(1) \).

Then, by the assumption (A7) and the intermediate value theorem, we can see that there exists \( \delta_\varepsilon \in [0, \delta] \) for sufficiently small \( \varepsilon \) such that \( \beta_\varepsilon(\delta_\varepsilon) = \varepsilon^{-N} \kappa_0^{1/2} |\Omega| \) which implies (7.27). Note that, \( \delta \) is independent of \( \varepsilon \), and \( \delta_\varepsilon \in [0, \delta] \). Thus, there exists a subsequence \( \{\varepsilon_i\} \) such that \( \delta_{\varepsilon_i} \) converges to some \( \delta_\varepsilon \in [0, \delta] \) and \( \varepsilon_i \to 0 \) as \( i \to \infty \). By (7.31), \( \beta_\varepsilon(\delta_{\varepsilon_i}) = \varepsilon_i^{-N} \kappa_0^{1/2} |\Omega| \), and the assumption (A7), it is easy to see that \( \delta_{\varepsilon_i} \) satisfies (7.28). By the same reason stated in Remark 4.2, we notice that the subsequence \( \{\varepsilon_i\} \) is not needed if \( \kappa_0 \) is small enough or in the case \( N = 1 \).
Proof of Theorem 7.1. From the argument above, we have found the pair \((v(x), \delta) = (u_\delta(x; \varepsilon), \delta_x)\) satisfying (7.14) and (7.15) for \(\varepsilon\) sufficiently small. Hence, by putting

\[
A_\varepsilon(x) = \xi_\varepsilon u_\delta(x; \varepsilon), \quad \xi_\varepsilon = \frac{\|\Omega\|}{\Omega v_\delta(x; \varepsilon) dx},
\]

we obtain a solution to the shadow system (7.2). Now, let us show (7.4)-(7.7).

Proof of (7.5) and (7.6). Let us write \(v_\varepsilon(x) = v_\delta(x; \varepsilon)\) and \(u_\varepsilon(x) = u_{\varepsilon, \delta_x}(x)\) in this proof. Recall that \(v_\varepsilon\) and \(u_\varepsilon\) take the forms:

\[
v_\varepsilon(x) = u_\varepsilon(x) + \varepsilon^N \phi_\varepsilon, \quad \|\phi_\varepsilon\|_{L^\infty(\Omega)} \leq C, \tag{7.33}
\]

\[
u_\varepsilon(x) = U_{\varepsilon, \delta_x}(x) + \varepsilon \phi_\varepsilon(x), \quad \|\phi_\varepsilon\|_{L^\infty(\Omega)} \leq C, \tag{7.34}
\]

for some \(C > 0\) independent of \(\varepsilon\) sufficiently small. Now, we take a subsequence \(\{\varepsilon_i\}\) as in Proposition 7.1 if necessary. We notice that\(\gamma_\varepsilon(\sigma_i)\) as in Proposition 7.1 if necessary. We notice that

\[
\gamma_\varepsilon(\sigma_i) = \sigma + \sigma^2 + 2\sigma^3 + O(\sigma^4), \quad \text{as } \sigma \to 0,
\]

by the implicit function theorem. Then we can see that \(\gamma_\varepsilon(\sigma_0/\xi_\varepsilon)\) and \(\gamma_\delta_x(\sigma_0/\xi_\varepsilon)\) becomes a subsolution and a supersolution to the following equation, respectively:

\[
\begin{aligned}
\varepsilon^2 \Delta u - v + f_{\delta_x}(v) + \frac{\sigma_0}{\xi_\varepsilon} &= 0 \text{ in } B_\varepsilon(x_0), \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial B_\varepsilon(x_0).
\end{aligned}
\]

Indeed,

\[
\varepsilon^2 \Delta (\gamma_\delta(\sigma_0/\xi_\varepsilon)) - \gamma_\delta(\sigma_0/\xi_\varepsilon) + f_{\delta_x}(\gamma_\delta(\sigma_0/\xi_\varepsilon)) + \frac{\sigma_0}{\xi_\varepsilon} = \frac{1}{\xi_\varepsilon}(\sigma_0 - \sigma_0) \geq 0 \text{ in } B_\varepsilon(x_0),
\]

\[
\varepsilon^2 \Delta (\gamma_\delta_x(\sigma_0/\xi_\varepsilon)) - \gamma_\delta_x(\sigma_0/\xi_\varepsilon) + f_{\delta_x}(\gamma_\delta_x(\sigma_0/\xi_\varepsilon)) + \frac{\sigma_0}{\xi_\varepsilon} = \frac{1}{\xi_\varepsilon}(\sigma_0 - \sigma_0) \leq 0 \text{ in } B_\varepsilon(x_0).
\]

114
Hence, by the theory of sub-supersolution (see e.g. [90]) there exists a solution 
\( \tilde{v}_\varepsilon \) to (7.37) such that
\[
\gamma_{\delta_v}(\frac{\sigma_0}{\xi_\varepsilon}) < \tilde{v}_\varepsilon(x) < \gamma_{\delta_v}(\frac{\overline{\sigma}_0}{\xi_\varepsilon}), \quad x \in B_r(x_0).
\]

Here, we note that
\[
\gamma_{\delta_v}(\frac{\sigma_0}{\xi_\varepsilon}), \gamma_{\delta_v}(\frac{\overline{\sigma}_0}{\xi_\varepsilon}) = O(\varepsilon^N)
\]
since \( \xi_\varepsilon = O(\varepsilon^N) \) and (7.36). Hence \( \tilde{v}_\varepsilon(x) = O(\varepsilon) \) uniformly in \( x \in B_r(x_0) \).

Now we put
\[
w_\varepsilon(x) := v_\varepsilon(x) - \tilde{v}_\varepsilon(x) = u_\varepsilon(x) + \varepsilon^N \phi_\varepsilon - \tilde{v}_\varepsilon(x).
\]

Noting that \( u_\varepsilon \) decays exponentially as \( \varepsilon \to 0 \) at any interior point of \( \Omega \) (see (3.9)) and \( \phi_\varepsilon \) is bounded uniformly in \( \varepsilon \) sufficiently small, we see that \( w_\varepsilon(x) = O(\varepsilon^N) \) uniformly in \( x \in B_r(x_0) \). We notice that \( w_\varepsilon \) satisfies the following equation:
\[
\varepsilon^2 \Delta w_\varepsilon - cw_\varepsilon = 0 \text{ in } B_r(x_0),
\]
\[
c := 1 - \frac{f_\delta(v_\varepsilon) - f_\delta(\tilde{v}_\varepsilon)}{v_\varepsilon - \tilde{v}_\varepsilon}.
\]

We may assume that \( c(x) \geq 1/2 \) for \( x \in B_r(x_0) \) when \( \varepsilon \) is small enough. Then, by applying Lemma 3.9, we have
\[
|w_\varepsilon(x)| = |v_\varepsilon(x) - \tilde{v}_\varepsilon(x)| \leq Ce^{-\varepsilon/c}, \quad x \in B_{r/2}(x_0),
\]
for some constant \( C, c > 0 \) independent of \( \varepsilon \). Thus we have
\[
\gamma_{\delta_v}(\frac{\sigma_0}{\xi_\varepsilon}) - Ce^{-\varepsilon/c} \leq v_\varepsilon(x) \leq \gamma_{\delta_v}(\frac{\overline{\sigma}_0}{\xi_\varepsilon}) + Ce^{-\varepsilon/c}, \quad x \in B_{r/2}(x_0),
\]
and hence
\[
\xi_\varepsilon(\gamma_{\delta_v}(\frac{\sigma_0}{\xi_\varepsilon}) - Ce^{-\varepsilon/c}) \leq A_\xi(x) \leq \xi_\varepsilon(\gamma_{\delta_v}(\frac{\overline{\sigma}_0}{\xi_\varepsilon}) + Ce^{-\varepsilon/c}), \quad x \in B_{r/2}(x_0),
\]

By using (7.36), we have
\[
\sigma_0(x_0) - \tau = \sigma_0 \leq \liminf_{\varepsilon \to 0} A_\xi(x) \leq \limsup_{\varepsilon \to 0} A_\xi(x) \leq \overline{\sigma}_0 = \sigma_0(x_0) + \tau.
\]

Because \( \tau > 0 \) is arbitrary, \( \lim_{\varepsilon \to 0} A_\xi(x_0) = \sigma_0(x_0) \) holds. Thus \( \lim_{\varepsilon \to 0} A_\xi(x) = \sigma_0(x) \) holds for each \( x \in \Omega \).

**Proof of (7.7).** We investigate the behavior of \( \phi_\varepsilon(x) \) in (7.33) for each point \( x \in \Omega \) as \( \varepsilon \to 0 \). Because \( A_\xi(x) = \xi_\varepsilon v_\varepsilon(x) \to \sigma_0(x) \) as \( \varepsilon \to 0 \), it holds that
\[
\frac{1}{\xi_\varepsilon}(\sigma_0(x) - o(1)) \leq v_\varepsilon(x) = u_\varepsilon(x) + \varepsilon^N \phi_\varepsilon(x) \leq \frac{1}{\xi_\varepsilon}(\sigma_0(x) + o(1))
\]
as $\varepsilon_i \to 0$ for each $x \in \Omega$. Hence,

$$\frac{\varepsilon_i^{-N}}{\xi_{\varepsilon_i}}(\sigma_0(x) - (1)) \leq \frac{\varepsilon_i^{-N}}{\xi_{\varepsilon_i}} u_{\varepsilon_i}(x) + \tilde{\phi}_{\varepsilon_i} \leq \frac{\varepsilon_i^{-N}}{\xi_{\varepsilon_i}}(\sigma_0(x) + o(1)).$$

Then, by (3.9) and (7.6), we have

$$\frac{\sigma_0(x)}{|\Omega|} \left( \frac{m}{2} \int_{R^N} w_{\tilde{\delta}_\varepsilon}(y)dy \right) - o(1) \leq \frac{\sigma_0(x)}{|\Omega|} \left( \frac{m}{2} \int_{R^N} w_{\tilde{\delta}_\varepsilon}(y)dy \right) + o(1).$$

Therefore, it follows that

$$\tilde{\phi}_{\varepsilon_i}(x) = \frac{\sigma_0(x)}{|\Omega|} \left( \frac{m}{2} \int_{R^N} w_{\tilde{\delta}_\varepsilon}(y)dy \right)$$

for each $x \in \Omega$ as $\varepsilon_i \to 0$. Then we can see that

$$\int_{\Omega} \tilde{\phi}_{\varepsilon_i}(x)dx = m \int_{\Omega} \sigma_0(x)dx \int_{R^N} w_{\tilde{\delta}_\varepsilon}(y)dy \quad (7.38)$$

as $\varepsilon_i \to 0$ by Lebesgue’s convergence theorem. Hence, it follows that

$$\int_{\Omega} u_{\varepsilon_i}(x)dx = \int_{\Omega} \left( u_{\varepsilon_i}(x) + \varepsilon_i^N \tilde{\phi}_{\varepsilon_i}(x) \right)dx$$

$$= \int_{\Omega} \left( U_{\varepsilon_i, \varepsilon_i}(x) + \varepsilon_i \phi_{\varepsilon_i}(x) + \varepsilon_i^N \tilde{\phi}_{\varepsilon_i}(x) \right)dx$$

$$= \varepsilon_i^N \frac{m}{2} \left( \int_{R^N} w_{\delta}(y)dy + \frac{1}{|\Omega|} \int_{\Omega} \sigma_0(x)dx \int_{R^N} w_{\tilde{\delta}_\varepsilon}(y)dy \right) + o(\varepsilon_i^N),$$

(7.39)

because we know that

$$\int_{\Omega} U_{\varepsilon_i, \varepsilon_i}(x)dx = \varepsilon_i^N \frac{m}{2} \int_{R^N} w_{\delta}(y)dy + o(\varepsilon_i^N), \quad \varepsilon_i \int_{\Omega} \phi_{\varepsilon_i}(x)dx = o(\varepsilon_i^N),$$

$$\varepsilon_i^N \int_{\Omega} \tilde{\phi}_{\varepsilon_i}(x)dx = \varepsilon_i^N \frac{m}{2|\Omega|} \int_{\Omega} \sigma_0(x)dx \int_{R^N} w_{\tilde{\delta}_\varepsilon}(y)dy + o(\varepsilon_i^N),$$

as $\varepsilon_i \to 0$ by Lemma 3.11 and (7.38). By (7.6) and (7.39), and since $A_{\varepsilon_i}(x)\in \varepsilon_i u_{\varepsilon_i}(x)$, we conclude that

$$\int_{\Omega} A_{\varepsilon_i}(x)dx = \frac{\varepsilon_i^N |\Omega|}{2} \left( \int_{R^N} w_{\delta}(y)dy + \frac{1}{|\Omega|} \int_{\Omega} \sigma_0(x)dx \int_{R^N} w_{\tilde{\delta}_\varepsilon}(y)dy \right) + o(\varepsilon_i^N),$$

$$= |\Omega| \left( \int_{R^N} w_{\delta}(y)dy \right) \left( \int_{R^N} w_{\tilde{\delta}_\varepsilon}(y)dy \right)^{-1} + \int_{\Omega} \sigma_0(x)dx + o(1).$$

Thus, we complete the proof.
7.3 Global estimate

In this section, we consider the asymptotic behavior of the solution \((A_\varepsilon, \xi_\varepsilon)\) to the shadow system under the condition that \(\sigma_0\) is nonnegative function of class \(C^2(\Omega)\) such that \(\frac{\partial \sigma_0}{\partial \nu} = 0\) on \(\partial \Omega\). We recall that the solution to the shadow system given in Theorem 7.1 takes the form \((A_\varepsilon(x), \xi_\varepsilon) = (\xi_\varepsilon v_{\delta_\varepsilon}(x; \varepsilon), \xi_\varepsilon)\), where \((v_{\delta_\varepsilon}(x; \varepsilon), \xi_\varepsilon)\) is a solution to (7.12)-(7.14). We write \(v_\varepsilon(x) := v_{\delta_\varepsilon}(x; \varepsilon)\) simply. We first establish a lower estimates of a non-constant solution to the following equation:

\[
\begin{align*}
\varepsilon^2 \Delta u - u + f_{\delta_\varepsilon}(u) + \sigma_\varepsilon &= 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

(7.40)

where

\[
\sigma_\varepsilon(x) := \frac{\sigma_0(x)}{\xi_\varepsilon}.
\]

Note that \(\|\sigma_\varepsilon\|_{L^\infty(\Omega)}\) is very small if \(\varepsilon\) is sufficiently small, and hence \(0 = g_{\delta_\varepsilon}(t; \sigma_\varepsilon) = -t + f_{\delta_\varepsilon}(t) + \sigma_\varepsilon(x)\) has two or three nonnegative roots for any \(x \in \Omega\). We denote the smallest root by \(\gamma_{\delta_\varepsilon}(\sigma_\varepsilon(x))\) as in (7.36).

**Lemma 7.5.** Let \(u\) be a non-constant solution to (7.40). Then we have, for \(\varepsilon > 0\) sufficiently small,

(a) \(\gamma_{\delta_\varepsilon}(\min_{\overline{\Omega}} \sigma_\varepsilon) < u(x)\) in \(\Omega\),

(b) \(\sigma_\varepsilon(x) - \varepsilon^2 \|\Delta \sigma_\varepsilon\|_{L^\infty(\Omega)} < u(x)\) in \(\Omega\).

**Proof.** Put \(m_\varepsilon := \min_{\overline{\Omega}} \sigma_\varepsilon\). Assume that there exists a point \(x_0 \in \overline{\Omega}\) such that \(\min_{\overline{\Omega}} u = u(x_0) < \gamma_{\delta_\varepsilon}(m_\varepsilon)\). We first note that \(g_{\delta_\varepsilon}(t; \sigma_\varepsilon) > 0\) holds for any \(t \in (-\infty, \gamma_{\delta_\varepsilon}(m_\varepsilon))\).

**Case i** If \(x_0 \in \Omega\), then \(\Delta u(x_0) \geq 0\) holds. However, \(\varepsilon^2 \Delta u(x_0) = -\left(-u(x_0) + f_{\delta_\varepsilon}(u(x_0)) + m_\varepsilon - \sigma_\varepsilon(x_0)\right) < 0\) holds from the shape of the graph of \(g_{\delta_\varepsilon}(t; \sigma_\varepsilon(x_0))\), and thus we get a contradiction.

**Case ii** If \(x_0 \in \partial \Omega\), then it must hold that \(u(x_0) < u(x)\) \((x \in \Omega)\) by (case i). We take an open ball \(B \subset \Omega\) with small radius such that \(x_0 \in \partial B\). Then \(v := \gamma_{\delta_\varepsilon}(m_\varepsilon) - u\) satisfies the following:

\[
\varepsilon^2 \Delta v(x) + c(x)v(x) = \sigma(x) - m_\varepsilon \geq 0, \quad x \in B,
\]

where \(c(x) = -(u + f_{\delta_\varepsilon}(u) + m_\varepsilon)/\left(u - \gamma_{\delta_\varepsilon}(m_\varepsilon)\right)\), and \(c(x) < 0\) \((x \in B)\). By Hopf’s lemma, we have \(\frac{\partial v}{\partial \nu}(x_0) = -\frac{\partial v}{\partial \nu}(x_0) > 0\). This contradicts the Neumann boundary condition.

Therefore, \(\gamma_{\delta_\varepsilon}(m_\varepsilon) \leq u\) holds. However, we can conclude that \(\gamma_{\delta_\varepsilon}(m_\varepsilon) < u\) by using the usual maximum principle. This implies (a).

Let us show (b). We first note that \(m_\varepsilon \leq \gamma_{\delta_\varepsilon}(m_\varepsilon)\), and hence we have \(0 \leq m_\varepsilon < u\) holds in \(\Omega\) by (a). Assume there exists a point \(x_0 \in \overline{\Omega}\) such that \(\min_{\overline{\Omega}} (u - \sigma_\varepsilon) = (u - \sigma_\varepsilon)(x_0) \leq -\varepsilon^2 \|\Delta \sigma_\varepsilon\|_{L^\infty(\Omega)}\).

**Case i’** If \(x_0 \in \Omega\), then \(\Delta (u - \sigma_\varepsilon)(x_0) \geq 0\). However, it holds that

\[
\varepsilon^2 \Delta (u - \sigma_\varepsilon)(x_0) = u(x_0) - f_{\delta_\varepsilon}(u(x_0)) - \sigma_\varepsilon(x_0) - \varepsilon^2 \Delta \sigma_\varepsilon(x_0) \leq -f_{\delta_\varepsilon}(u(x_0)) < 0.
\]

(7.41)
Thus we get a contradiction. 

(Case ii') If $x_0 \in \partial \Omega$, then $(u - \sigma_\varepsilon)(x_0) < (u - \sigma_\varepsilon)(x)$ $(x \in \Omega)$ holds by (case i'). We take an open ball $B \subset \Omega$ with small radius such that $x_0 \in \partial B$. Then we can see that $u - \sigma_\varepsilon$ is superharmonic in $B$ by (7.41). Thus by applying Hopf’s lemma, we can conclude that $\frac{\partial}{\partial \nu}(u - \sigma_\varepsilon)(x_0) < 0$. Hence $\frac{\partial}{\partial \nu}(x_0) < \frac{\partial \sigma_\varepsilon}{\partial \nu}(x_0) = 0$. This contradicts the Neumann boundary condition. Thus we complete the proof. □

**Remark 7.3.** In Lemma 7.5(a), we do not need the condition $\frac{\partial \sigma_\varepsilon}{\partial \nu} = 0$ on $\partial \Omega$. In Lemma 7.5(b), it suffices to assume the condition: $\frac{\partial \sigma_\varepsilon}{\partial \nu} < 0$ on $\partial \Omega$, namely, $\frac{\partial \sigma_\varepsilon}{\partial \nu} < 0$ on $\partial \Omega$.

To obtain an upper bound of the solution $\tilde{u}_\varepsilon$, first we show the existence of a solution to (7.40) near $\sigma_\varepsilon$ by using the sub-supersolution method.

**Lemma 7.6.** For sufficiently small $\varepsilon > 0$, there exists a solution $h_\varepsilon$ to (7.40) such that

$$
\frac{1}{\xi_\varepsilon}(\sigma_\varepsilon - \varepsilon^2 \|\Delta \sigma_0\|_{L^\infty(\Omega)}) \leq h_\varepsilon \leq \frac{1}{\xi_\varepsilon}(\sigma_\varepsilon + \varepsilon^2 \|\Delta \sigma_0\|_{L^\infty(\Omega)} + \frac{\sigma_0^2}{\xi_\varepsilon} + o(\varepsilon^N)) \quad \text{in} \quad \Omega
$$

as $\varepsilon \to 0$, where the term $o(\varepsilon^N)$ is uniform in $\Omega$.

**Proof.** We put

$$
\tilde{h}_\varepsilon := \frac{1}{\xi_\varepsilon}(\sigma_\varepsilon + \varepsilon^2 \|\Delta \sigma_0\|_{L^\infty(\Omega)} + \frac{\sigma_0^2}{\xi_\varepsilon} + C_1 \varepsilon^{N+p}),
$$

where $p$ is a fixed number such that $0 < p < \min\{N, 2\}$, and $C_1 > 0$ is a fixed constant. Because $C\varepsilon^{-N} \leq \xi_\varepsilon^{-1} \leq C'\varepsilon^{-N}$ holds for some $C, C' > 0$ independent of $\varepsilon$, it follows that, if $\varepsilon$ is sufficiently small,

$$
\varepsilon^2 \Delta \tilde{h}_\varepsilon - f_\varepsilon(\tilde{h}_\varepsilon) + \frac{\sigma_0}{\xi_\varepsilon} = \frac{\varepsilon^2}{\xi_\varepsilon} \Delta \sigma_0 + \frac{\varepsilon^2}{\xi_\varepsilon^2} \Delta(\sigma_0^2) - \frac{1}{\xi_\varepsilon}(\varepsilon^2 \|\Delta \sigma_0\|_{L^\infty(\Omega)} + \frac{\sigma_0^2}{\xi_\varepsilon} + C_1 \varepsilon^{N+p})
$$

$$
+ \frac{1}{\xi_\varepsilon^2}(\sigma_\varepsilon + \varepsilon^2 \|\Delta \sigma_0\|_{L^\infty(\Omega)} + \frac{\sigma_0^2}{\xi_\varepsilon} + C_1 \varepsilon^{N+p})^2
$$

$$
\leq \frac{\varepsilon^2}{\xi_\varepsilon^2} \Delta(\sigma_0^2) - \frac{1}{\xi_\varepsilon}(\sigma_0^2 + C_1 \varepsilon^{N+p}) + \frac{1}{\xi_\varepsilon^2}(\sigma_0 + \varepsilon^2 \|\Delta \sigma_0\|_{L^\infty(\Omega)} + \frac{\sigma_0^2}{\xi_\varepsilon} + C_1 \varepsilon^{N+p})^2
$$

$$
\leq C'' \varepsilon^{2N+p} + O(\varepsilon^{2N+2}) \leq 0
$$

for some $C'' > 0$ independent of $\varepsilon$. Hence $\tilde{h}_\varepsilon$ is a supersolution to (7.40) if $\varepsilon$ is small enough. Next we put

$$
h_\varepsilon := \frac{1}{\xi_\varepsilon}(\sigma_\varepsilon - \varepsilon^2 \|\Delta \sigma_0\|_{L^\infty(\Omega)}).$$

118
Then it follows that
\[ \varepsilon^2 \Delta h_\varepsilon - h_\varepsilon + f_\delta(h_\varepsilon) + \frac{\sigma_0}{\varepsilon} = \frac{\varepsilon^2}{\varepsilon} \Delta \sigma_0 + \frac{\varepsilon^2}{\varepsilon} \| \Delta \sigma_0 \|_{L^\infty(\Omega)} + f_\delta(h_\varepsilon) \geq 0. \]

Hence \(\overline{h}_\varepsilon\) is a subsolution to (7.40). Note that \(h_\varepsilon\) and \(\overline{h}_\varepsilon\) satisfy the Neumann boundary condition, obviously. Thus, there exists a solution \(h_\varepsilon\) to (7.40) such that \(h_\varepsilon \leq h_\varepsilon \leq \overline{h}_\varepsilon\) in \(\Omega\), and we complete the proof. \(\Box\)

Proof of Theorem 7.2. Since \(v_\varepsilon\) satisfies (7.40), by Lemma 7.5(b), we have
\[ \sigma_0(x) - \varepsilon^2 \| \Delta \sigma_0 \|_{L^\infty(\Omega)} < \varepsilon v_\varepsilon(x) = A_\varepsilon(x), \quad x \in \Omega. \]

Hence, the first inequality of (7.11) holds true.

Let us show the second inequality of (7.11). For sufficiently small \(\varepsilon > 0\), let \(h_\varepsilon\) be a solution to (7.40) given in Lemma 7.6. Put \(w_\varepsilon := v_\varepsilon - h_\varepsilon\). Then \(w_\varepsilon\) satisfies the following equation:
\[ \begin{cases} 
\varepsilon^2 \Delta w_\varepsilon - cw_\varepsilon = 0 & \text{in } \Omega, \\
\frac{\partial w_\varepsilon}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases} \quad (7.42) \]
\[ c := 1 + \frac{\bar{u}_\varepsilon + h_\varepsilon}{(1 + \delta_\varepsilon h_\varepsilon^2)(1 + \delta_\varepsilon \bar{u}_\varepsilon^2)}. \]

We note that, by our construction, if we put \(v_\varepsilon = u_\varepsilon + \varepsilon^N \bar{\phi}\) for \(u_\varepsilon(x) := u_{\delta_\varepsilon}(x; \varepsilon)\) simply, \(\bar{\phi}\) is bounded uniformly in \(\varepsilon\), and \(u_\varepsilon\) satisfies (3.9), and \(h_\varepsilon(x) = O(\varepsilon^N)\) as \(\varepsilon \to 0\) uniformly in \(x \in \Omega\). Therefore, it is easy to see that, if we take \(R > 0\) large enough, then it holds that
\[ c(x) \geq 1/2, \quad x \in \Omega^{(o)}_\varepsilon := \Omega \setminus \bigcup_{k=1}^m \Phi(B_{\varepsilon R}; P_k). \]

We divide \(\Omega\) into \(\Omega = \Omega^{(o)}_\varepsilon \cap \Omega^{(i)}_\varepsilon\), where \(\Omega^{(i)}_\varepsilon := \Omega \setminus \Omega^{(o)}_\varepsilon\). We first prove that
\[ |w_\varepsilon(x)| \leq Ce^{-c \cdot \text{dist}(x, P)}/\varepsilon, \quad x \in \Omega, \quad (7.43) \]
holds for some constants \(C, c > 0\) independent of \(\varepsilon\), where \(P = \{P_1, \cdots, P_m\}\).

For \(x \in \Omega^{(i)}_\varepsilon\), we can see that
\[ |w_\varepsilon(x)| = |u_\varepsilon(x) + \varepsilon^N \bar{\phi} - h_\varepsilon(x)| \leq Ce^{-c \cdot \text{dist}(x, P)}/\varepsilon + C' \varepsilon^N \]
\[ \leq (C + C' \varepsilon^N c' R) e^{-c \cdot \text{dist}(x, P)}/\varepsilon \]
for some constants \(C, c, C', c' > 0\) independent of \(\varepsilon\). Thus (7.43) is verified for \(x \in \Omega^{(i)}_\varepsilon\).

Because we consider \(\varepsilon > 0\) to be so small, we can assume that \(\Phi(B_{\varepsilon R}; P_k) \subset \Phi(B_{\varepsilon_0}; P_k), k = 1, \cdots, m\). For each \(k = 1, \cdots, m\), put
\[ w_\varepsilon^*(y) = w_\varepsilon(\Phi(y; P_k)), \quad c_\varepsilon(y) = c(\Phi(y; P_k)), \quad y \in B_{3\varepsilon_0}, \]

119
and we extend \( w^\ast_x \) and \( c^\ast \) to the function defined on \( B_{3r_0} \) by reflection. Note that this extended function is of class \( C^2 \). Then we can see that \( w^\ast_x \) satisfies the following elliptic equation:

\[
\varepsilon^2 \left( \sum_{i,j=1}^N a_{ij}(y)D_{ij}w^\ast_x(y) + \sum_{i=1}^N b_i(y)D_iw^\ast_x(y) \right) - c^\ast(y)w^\ast_x(y) = 0, \quad y \in B_{3\kappa},
\]

(7.44)

for some bounded coefficients \( a_{ij} \) and \( b_i \) independent of \( \varepsilon \). When \( y \in B_{r_0} \setminus B_{\varepsilon R} \), \( c^\ast(y) \geq 1/2 \) holds. Hence, by applying Lemma 3.9 for \( G = B_{3r_0} \setminus B_{\varepsilon R} \), we have

\[
|w^\ast_x(y)| \leq C e^{-c\cdot\text{dist}(y, \partial(B_{3\kappa}\setminus B_{\varepsilon R}))/\varepsilon}, \quad y \in B_{3\kappa} \setminus B_{\varepsilon R}
\]

for some \( C, c > 0 \) independent of \( \varepsilon \) since \( w^\ast_x \) is bounded uniformly in \( \varepsilon \) sufficiently small. In particular, for \( y \in B_{r_0} \setminus B_{\varepsilon R} \),

\[
|w^\ast_x(y)| \leq C e^{cR}e^{-|y|/\varepsilon}
\]

holds since \( \text{dist}(y, \partial(B_{3r_0} \setminus B_{\varepsilon R})) = |y| - \varepsilon R \). Therefore, it follows that

\[
|w_x(x)| \leq C e^{cR}e^{-c\cdot\text{dist}(x, P)/\varepsilon}, \quad x \in \Omega^{(o)} \cap (\bigcup_{k=1}^m \Phi(B_{r_0}; P_k)),
\]

for some \( c' > 0 \) independent of \( \varepsilon \). Thus (7.43) follows for \( x \in \Omega^{(o)} \setminus (\bigcup_{k=1}^m \Phi(B_{r_0}; P_k)) \).

Finally, let us show (7.43) for \( \Omega^{(o)} \), for the purpose, it suffices to show that

\[
|w_x(x)| \leq C e^{-c\cdot\text{dist}(x, P)/\varepsilon}, \quad x \in \Omega^{(o)} \cap (\bigcup_{k=1}^m \Phi(B_{r_0}; P_k)) \quad (7.45)
\]

for some constants \( C, c > 0 \) independent of \( \varepsilon \) because \( c > c' \cdot \text{dist}(x, P)/r_0 \) holds for some \( c' > 0 \) independent of \( \varepsilon \) there. By applying Lemma 3.9 for \( G = \Omega^{(o)} \), we have

\[
|w_x(x)| \leq C e^{-c\cdot\text{dist}(x, \partial\Omega^{(o)})/\varepsilon}, \quad x \in \Omega^{(o)},
\]

for some \( C, c > 0 \) independent of \( \varepsilon \). In particular,

\[
|w_x(x)| \leq C e^{-c_1\cdot\varepsilon}, \quad x \in \Omega_{r_1} \cap (\Omega \setminus \bigcup_{k=1}^m \Phi(B_{r_0}; P_k)),
\]

follows, where \( \Omega_{r_1} := \{ x \in \Omega : \text{dist}(x, \partial\Omega) > r_1 \} \), and \( r_1 > 0 \) is a sufficiently small fixed constant. Thus (7.45) follows for \( x \in \Omega_{r_1} \cap (\Omega \setminus \bigcup_{k=1}^m \Phi(B_{r_0}; P_k)) \).

Next, we show (7.45) for \( x \in (\Omega \setminus \bigcup_{k=1}^m \Phi(B_{r_0}; P_k)) \setminus \Omega_{r_1} \). By taking fine number of points \( \{ P_k' \}_{k=1}^m \subset \partial\Omega \), we can cover \( \Omega \setminus \bigcup_{k=1}^m \Phi(B_{r_0}; P_k) \setminus \Omega_{r_1} \) with finite number of balls as follows:

\[
(\Omega \setminus \bigcup_{k=1}^m \Phi(B_{r_0}; P_k)) \setminus \Omega_{r_1} \subset \bigcup_{k=1}^m B_{2r_1}(P_k').
\]

For each ball \( B_{2r_1}(P_k') \), \( k = 1, \ldots, m' \), by taking a new coordinate \( (x_1, x_2, \ldots, x_N) \) so that, \( P_k' \) is the origin, and the positive direction of \( x_N \) is the direction of the inner normal vector of \( \partial\Omega \) at \( P_k' \). Define \( \Phi_j(y; P_k') \), \( j = 1, \ldots, N \), in the same way as in (3.5), and put \( \Phi(y; P_k') = (\Phi_1(y; P_k'), \ldots, \Phi_N(y; P_k')) \). Then \( \Phi(y; P_k') \)
is a diffeomorphism from an open set onto a neighborhood of $P'_k$ (origin). Put $\Psi(\cdot; P'_k) := \Phi^{-1}(\cdot; P'_k)$. We can assume that $\Psi(x; P'_k)$ is defined on a closed ball $B_{3r_1}$ by taking $P'_k$ suitably in advance. Put $w^*_\varepsilon(y) = v_\varepsilon(\Phi(y; P'_k))$, $\tilde{c}(y) = c(\Phi(y; P'_k))$. We extend $w^*_\varepsilon(y)$ and $\tilde{c}$ to being defined on $\Psi(B_{3r_1}; P'_k)$ by reflection. Then $w^*_\varepsilon$ satisfies the following elliptic equation:

$$
\varepsilon^2 \left( \sum_{i,j=1}^N \tilde{a}_{ij} D_{ij} w^*_{\varepsilon} + \sum_{i=1}^N \tilde{b}_i D_i w^*_{\varepsilon} \right) - \tilde{c} w^*_{\varepsilon} = 0, \quad y \in \Psi(B_{3r_1}; P'_k),
$$

for some bounded coefficients $\tilde{a}_{ij}$ and $\tilde{b}_i$ independent of $\varepsilon$. Since $\tilde{c} \geq 1/2$ holds there, we can apply Lemma 3.9 for $G = \Psi(B_{3r_1}; P'_k)$, and we have

$$
|w^*_{\varepsilon}(y)| \leq C e^{-c \text{dist}(y, \partial \Psi(B_{3r_1}; P'_k))/\varepsilon}, \quad y \in \Psi(B_{3r_1}; P'_k),
$$

for some constants $C, c > 0$ independent of $\varepsilon$. Note that we can take the constants $C$ and $c$ uniformly in $k = 1, \ldots, m'$. In particular,

$$
|w^*_{\varepsilon}(y)| \leq C e^{-c'/\varepsilon}, \quad y \in \Psi(B_{2r_1}; P'_k),
$$

holds for some $c' > 0$ independent of $\varepsilon$ and $k$. Therefore, it follows that

$$
|w_{\varepsilon}(x)| \leq C e^{-c'/\varepsilon}, \quad x \in B_{2r_1}(P'_k) \cap \Omega,
$$

and we can conclude that $|w_{\varepsilon}(x)| \leq C e^{-c'/\varepsilon}$ holds for $x \in (\Omega \setminus \cup_{k=1}^m \Phi(B_n; P_k)) \setminus \Omega_{r_1}$. Thus, (7.45) is verified, and (7.43) is also verified.

By (7.43) and Lemma 7.6, we have the following estimate:

$$
v_{\varepsilon} = w_{\varepsilon}(x) + h_{\varepsilon}(x)
\leq C e^{-c \text{dist}(x, P)/\varepsilon} + \frac{1}{\varepsilon} \left( \sigma_0(x) + \varepsilon^2 \| \Delta \sigma_0 \|_{L^\infty(\Omega)} + \frac{\sigma_0^2}{\varepsilon} + o(\varepsilon^N) \right), \quad x \in \Omega.
$$

Because $A_{\varepsilon}(x) = \xi_x v_{\varepsilon}$, we have the second inequality of (7.11). Thus we complete the proof.

### 7.4 Construction of a solution to the full system

As in the previous section, we write $v_{\varepsilon}(x) = v_{\delta, \varepsilon}(x)$ and $u_{\varepsilon}(x) = u_{\delta, \varepsilon}(x)$. In this section, we prove Theorem 7.3 and construct a solution to the full system (7.1). For the purpose, the most important thing is the invertibility of the linearized operator

$$
L_{v_{\varepsilon}} := \varepsilon^2 - 1 + f'_x(v_{\varepsilon}).
$$

Recall that $v_{\varepsilon}$ is a solution with the form

$$
v_{\varepsilon}(x) = u_{\varepsilon}(x) + \varepsilon^N \tilde{\phi}_{\varepsilon}
$$

(7.47)
and solves
\[
\begin{aligned}
\varepsilon^2 \Delta v - v + f_{\delta_\varepsilon}(v) + \frac{2\varepsilon}{\xi} = 0, \quad v > 0 \text{ in } \Omega, \\
\frac{\partial v}{\partial \nu} = 0 \text{ on } \partial \Omega,
\end{aligned}
\]  
(7.48)

with
\[
\xi = \frac{|\Omega|}{\int_\Omega \varepsilon^2 dx}.
\]

On the other hand, by Theorem 3.2, we obtain a solution to (7.48), denoted by \(u_{\delta_\varepsilon}(x; \varepsilon, \frac{2\varepsilon}{\xi})\), for sufficiently small \(\varepsilon\). However, noting the uniqueness of \(\tilde{\phi}_\varepsilon\), we notice that the two solutions are the same, \(u_{\delta_\varepsilon}(x; \varepsilon, \frac{2\varepsilon}{\xi}) \equiv v_\varepsilon(x)\). Hence, by Lemma 3.14, we notice that \(L_{v_\varepsilon}\) with \(\text{Dom}(L_{v_\varepsilon}) = X^{2, t}_{\Omega}\), has a bounded inverse \(K_{v_\varepsilon}\) provided \(\varepsilon\) is small enough.

**Proof of Theorem 4.2.** Let \((A_\varepsilon, \xi_\varepsilon)\) be a solution to the shadow system (7.2) given in Theorem 7.1. By the same consideration as in the proof of Theorem 6.2, we only need to show that
\[
L_\infty := \left(\varepsilon^2 \Delta - 1 + \frac{2A_\varepsilon}{\xi_\varepsilon(1 + \kappa A_\varepsilon^2)^2} \right) : X^{2, t}_{\Omega} \times \mathbb{R} \to X^{t} \times \mathbb{R}
\]  
(7.49)

has a bounded inverse. To show this, first we show that \(\text{Ker}(L_\infty) = \{0, 0\}\). Let \((\phi, \eta) \in \text{Ker}(L_\infty)\), namely,
\[
\varepsilon^2 \Delta \phi - \phi + \frac{2A_\varepsilon \phi}{\xi_\varepsilon(1 + \kappa A_\varepsilon^2)^2} - \frac{A_\varepsilon^2 \eta}{\xi_\varepsilon^2(1 + \kappa A_\varepsilon^2)} = 0,
\]
\[
2 \oint_{\Omega} A_\varepsilon \phi dx = |\Omega| \eta = 0.
\]
Recall that
\[
A_\varepsilon = \xi_\varepsilon v_\varepsilon, \quad \xi_\varepsilon = |\Omega| \left(\int_\Omega v_\varepsilon^2(x) dx\right)^{-1}
\]
and \(\delta_\varepsilon = \kappa \xi_\varepsilon^2\). Then we have
\[
\varepsilon^2 \Delta \phi - \phi + \frac{2v_\varepsilon \phi}{(1 + \delta_\varepsilon v_\varepsilon^2)^2} - \frac{v_\varepsilon^2 \eta}{1 + \delta_\varepsilon v_\varepsilon^2} = 0, \quad \text{(7.50)}
\]
\[
\frac{2 \oint_{\Omega} v_\varepsilon \phi dx}{\oint_{\Omega} v_\varepsilon^2 dx} - \eta = 0. \quad \text{(7.51)}
\]
(7.50) is written as \(L_{v_\varepsilon} \phi = \eta f_{\delta_\varepsilon}(v_\varepsilon)\). Therefore
\[
\phi = \eta K_{v_\varepsilon}(f_{\delta_\varepsilon}(v_\varepsilon)) \quad \text{(7.52)}
\]
holds. On the other hand, since
\[
\varepsilon^2 \Delta v_\varepsilon - v_\varepsilon + f_{\delta_\varepsilon}(v_\varepsilon) + \sigma_0 / \xi_\varepsilon = 0 \text{ in } \Omega,
\]
it holds that

\[ L_{\varepsilon}v_\epsilon = \varepsilon^2 \Delta v_\epsilon - v_\epsilon + f_\theta'(v_\epsilon)v_\epsilon \]
\[ = -f_\theta(v_\epsilon) + f_\theta'(v_\epsilon)v_\epsilon - \sigma_0/\xi \]
\[ = -\frac{v_\epsilon^2}{1 + \delta_\varepsilon v_\epsilon^2} + \frac{2v_\epsilon^2}{(1 + \delta_\varepsilon v_\epsilon^2)^2} - \sigma_0/\xi \]
\[ = (1 - g)f_\theta(v_\epsilon) - \sigma_0/\xi, \]

where \( g := 2\delta_\varepsilon v_\epsilon^2/(1 + \delta_\varepsilon v_\epsilon^2) \). Therefore,

\[ K_{v_\epsilon}(f_\theta(v_\epsilon)) = v_\epsilon + K_{v_\epsilon}(g f_\theta(v_\epsilon)) + K_{v_\epsilon}(\sigma_0/\xi). \]

We substitute this into (7.52), and we have

\[ \phi = \eta(v_\epsilon + K_{v_\epsilon}(g f_\theta(v_\epsilon)) + K_{v_\epsilon}(\sigma_0/\xi)). \]

Hence, by substituting this into (7.51), we obtain

\[ \eta \left( 1 + \frac{2 \int_{\Omega} v_\epsilon K_{v_\epsilon}(g f_\theta(v_\epsilon))dx}{\int_{\Omega} v_\epsilon^2 dx} + \frac{2 \int_{\Omega} v_\epsilon K_{v_\epsilon}(\sigma_0/\xi)dx}{\int_{\Omega} v_\epsilon^2 dx} \right) = 0. \] (7.53)

Here, the following estimates hold:

\[ \left| \frac{2 \int_{\Omega} v_\epsilon K_{v_\epsilon}(g f_\theta(v_\epsilon))dx}{\int_{\Omega} v_\epsilon^2 dx} \right| \leq \frac{2\|v_\epsilon\|_{L^2(\Omega)} \|K_{v_\epsilon}(g f_\theta(v_\epsilon))\|_{L^2(\Omega)}}{\|v_\epsilon\|_{L^2(\Omega)}} \]
\[ \leq \frac{2C\|g f_\theta(v_\epsilon)\|_{L^2(\Omega)}}{\|v_\epsilon\|_{L^2(\Omega)}} \]
\[ \leq 2CC'\delta_\varepsilon, \]

\[ \left| \frac{2 \int_{\Omega} v_\epsilon K_{v_\epsilon}(\sigma_0/\xi)dx}{\int_{\Omega} v_\epsilon^2 dx} \right| \leq 2C\|\sigma_0\|_{L^\infty(\Omega)} \int_{\Omega} |v_\epsilon|dx \leq 2C\|\sigma_0\|_{L^\infty(\Omega)} C', \]

for some constants \( C, C' > 0 \) independent of \( \varepsilon > 0 \), by Lemma 3.14. Here, recall that if \( \kappa_0 \geq 0 \) is sufficiently small then \( \delta_\varepsilon \to \delta_5 (\varepsilon \to 0) \) and the \( \delta_5 \) must be small, since the \( \delta_5 \) was characterized by

\[ \kappa_0^2 = \delta_5 \left( \frac{m}{2|\Omega|} \int_{\mathbb{R}^N} w_\delta^2(y)dy \right)^2. \]

Therefore, if \( \kappa_0 \geq 0 \) is sufficiently small, then it follows that, for \( \varepsilon \) sufficiently small,

\[ \left| \frac{2 \int_{\Omega} v_\epsilon K_{v_\epsilon}(g f_\theta(v_\epsilon))dx}{\int_{\Omega} v_\epsilon^2 dx} + \frac{2 \int_{\Omega} v_\epsilon K_{v_\epsilon}(\sigma_0/\xi)dx}{\int_{\Omega} v_\epsilon^2 dx} \right| < 1. \]

Thus, from (7.53), we conclude that \( \eta = 0 \). Then it is easy to see that \( \phi = 0 \) from (7.50) or (7.51). Thus \( L_\infty \) is one-to-one.
Secondly, for given \((\tilde{\phi}, \tilde{\eta}) \in X^t \times \mathbb{R}\), we seek \((\phi, \eta) \in X^t \times \mathbb{R}\) such that
\[
L_{\infty} \begin{pmatrix} \phi \\ \eta \end{pmatrix} = \begin{pmatrix} \tilde{\phi} \\ \tilde{\eta} \end{pmatrix}. \tag{7.54}
\]
Here (7.54) is written as follows:
\[
L_{\infty} \begin{pmatrix} \phi \\ \eta \end{pmatrix} = \left( \frac{\varepsilon^2 \Delta \phi - \phi + 2\nu_0 \nu_0 \phi}{2 \int_{\Omega} A_v \phi dx - |\Omega| \eta} - \frac{\eta v_2^2}{1 + \delta_2 v_2^2} \right) = \begin{pmatrix} \tilde{\phi} \\ \tilde{\eta} \end{pmatrix}. \tag{7.55}
\]
And the first equation of (7.55) is written as follows:
\[
L_v \phi = \tilde{\phi} + \eta v_2^2 \left( 1 + \delta_2 v_2^2 \right),
\]
therefore
\[
\phi = K_v \left[ \tilde{\phi} \right] + \eta K_v \left[ \frac{v_2^2}{1 + \delta_2 v_2^2} \right]. \tag{7.56}
\]
We substitute this into the second equation of (7.55), and we have
\[
\eta \left( 2 \int_{\Omega} A_v K_v \left[ \frac{v_2^2}{1 + \delta_2 v_2^2} \right] dx - |\Omega| \right) = \tilde{\eta} - 2 \int_{\Omega} A_v K_v \left[ \tilde{\phi} \right] dx,
\]
Hence
\[
\eta = \left( 2 \int_{\Omega} A_v K_v \left[ \frac{v_2^2}{1 + \delta_2 v_2^2} \right] dx - |\Omega| \right)^{-1} \left( \tilde{\eta} - 2 \int_{\Omega} A_v K_v \left[ \tilde{\phi} \right] dx \right). \tag{7.57}
\]
Here, note that
\[
2 \int_{\Omega} A_v K_v \left[ \frac{v_2^2}{1 + \delta_2 v_2^2} \right] dx - |\Omega| \neq 0
\]
by the first argument above. By substituting this \(\eta\) into (7.56), we have
\[
\phi = K_v \left[ \tilde{\phi} \right] + \left( 2 \int_{\Omega} A_v K_v \left[ \frac{v_2^2}{1 + \delta_2 v_2^2} \right] dx - |\Omega| \right)^{-1} \left( \tilde{\eta} - 2 \int_{\Omega} A_v K_v \left[ \tilde{\phi} \right] dx \right) K_v \left[ \frac{v_2^2}{1 + \delta_2 v_2^2} \right]. \tag{7.58}
\]
And we obtain
\[
\|\phi\|_{W^{s,1}(\Omega)} \leq C (|\tilde{\phi}\|_{L^1(\Omega)} + |\tilde{\eta}|),
\]
\[
|\eta| \leq C' (|\tilde{\phi}\|_{L^1(\Omega)} + |\tilde{\eta}|)
\]
for some \(C, C' > 0\) independent of \(\tilde{\phi}, \tilde{\eta}\) from (7.57) and (7.58). Thus we have a conclusion. \(\square\)
Chapter 8

One-dimensional Gierer-Meinhardt system: the strong coupling case

8.1 Introduction and main results

In this chapter, we consider the following steady-state problem of the one-dimensional Gierer-Meinhardt system:

\[
\begin{align*}
0 &= \epsilon^2 A'' - A + \frac{A^2}{M[H + A^2]} + \sigma_0, \quad A > 0, \quad x \in (-1, 1), \\
0 &= DH'' - H + A^2, \quad H > 0, \quad x \in (-1, 1), \\
A'(\pm 1) &= H'(\pm 1) = 0.
\end{align*}
\] (8.1)

The method used in the previous chapters, namely, to find a stationary solution to the Gierer-Meinhardt system near the stationary solution to the shadow system by the implicit function theorem, is one of the methods to construct a solution to the Gierer-Meinhardt system. In general, the number \(D\) must be large enough in the method. However, the following question arises, “for \(D > 0\) given arbitrarily, does the Gierer-Meinhardt system possess a peak solution under the weak saturation condition?” The purpose of this chapter is to construct a 1-peak solution concentrating at \(x = 0\) to the one-dimensional Gierer-Meinhardt system (8.1) for any fixed finite \(D\) (which is called the strong coupling case) under the weak saturation condition.

We need some preliminaries to state our main results. Let \(w_\delta\) be the unique solution to the following problem:

\[
\begin{align*}
\begin{cases}
  w'' - w + f_\delta(w) = 0, & w > 0 \text{ in } \mathbb{R}, \\
  w(0) = \max_{y \in \mathbb{R}} w(y), & w(y) \to 0 \text{ as } |y| \to \infty,
\end{cases}
\end{align*}
\] (8.2)

\[
f_\delta(w) := \frac{w^2}{1 + \delta w^2}.
\] (8.3)
We already known that (see Lemma 2.2), there exists a constant \( \delta^* > 0 \), the problem (8.2) has a unique solution \( w_\delta \) for each \( \delta \in [0, \delta^*] \), and \( w_\delta \) is radially symmetric, namely, \( w_\delta(y) = w_\delta(-y), \ y \in \mathbb{R} \).

For fixed \( D > 0 \), let \( G_D(x, z) \) be Green’s function to

\[
\begin{align*}
DG_{xx}(x, z) - G(x, z) &= -\delta_z(x) \text{ in } (-1, 1), \\
G(x, \pm 1, z) &= 0.
\end{align*}
\]

(8.4)

\( G_D(x, z) \) can be written explicitly

\[
G_D(x, z) = \begin{cases} 
\frac{\theta}{\sinh(2\theta)} \cosh[\theta(1 + x)] \cosh[\theta(1 - z)], & -1 < x < z, \\
\frac{\theta}{\sinh(2\theta)} \cosh[\theta(1 - x)] \cosh[\theta(1 + z)], & z < x < 1,
\end{cases}
\]

(8.5)

where \( \theta := D^{-1/2} \). We put

\[
\alpha_D := \frac{1}{G_D(0, 0)}.
\]

(8.6)

Moreover, the non-smooth part of \( G_D(x, z) \) is given by

\[
K_D(|x - z|) = \frac{1}{2} e^{-1/\sqrt{D} |x-z|}.
\]

(8.7)

Let \( H_D(x, z) \) be the regular part of \( G_D(x, z) \),

\[
G_D(x, z) = K_D(|x - z|) - H_D(x, z).
\]

\( H_D(x, z) \) is \( C^\infty \) in both \( x \) and \( z \).

Next, we prepare a cut-off function. Let \( \chi \in C^\infty_0(\mathbb{R}) \) be a function such that,

\[
0 \leq \chi \leq 1, \ \chi(x) = 0 \text{ for } |x| < 1, \ \chi(x) = 1 \text{ for } |x| > 2.
\]

Let \( r_0 \) be a fixed constant such that \( 0 < r_0 < 1/2 \), for example, \( r_0 = 1/10 \). We will use a cut-off function in the form \( \chi(\frac{x}{r_0}) \). Note that \( \chi(\frac{x}{r_0}) = 0 \) for \( |x| > 2r_0 \).

We suppose the following assumption on the constant \( \kappa \) in (8.1).

\( (A9) \) \( \kappa \geq 0 \) depends on \( \varepsilon \), and there exists a limit

\[
\lim_{\varepsilon \to 0} \kappa \varepsilon^{-2} = \kappa_0
\]

(8.8)

for some \( \kappa_0 \in [0, \infty) \).

This assumption is a weak saturation condition and is equivalent to \( (A7) \) of the case \( (p, q, r, s) = (2, 1, 2, 0) \) and \( N = 1 \) (note that \( \gamma = 1/2 \)).

Let us state our main results. We first state a result in the case \( \sigma_0 = 0 \).

**Theorem 8.1.** Let \( \sigma_0 = 0 \). Fix \( D > 0 \) arbitrarily. Suppose \( (A9) \), and let the value \( \kappa_0 \alpha_D^2 \) be sufficiently small. Then, for sufficiently small \( \varepsilon > 0 \), (8.1) admits a 1-peak radially symmetric solution \((A_\varepsilon(x), H_\varepsilon(x))\) such that \( A_\varepsilon(x) \) concentrates at \( x = 0 \). More precisely, there exists \( \delta_\varepsilon \in [0, \delta^*] \) for each \( \varepsilon \) sufficiently
small such that $\delta_\varepsilon \to \delta_0$ as $\varepsilon \to 0$ for some $\delta_0 \in [0, \delta_\ast)$ which is decided by $\kappa_0$ and $D$ and satisfies

$$\delta_0 \left( \int_\mathbb{R} w_{\delta_0}^2(y) dy \right)^2 = \kappa_0 \alpha_D^2, \quad (8.9)$$

and $A_\varepsilon$ takes the form:

$$A_\varepsilon(x) = \frac{1}{\varepsilon \int_\mathbb{R} w_{\delta_\varepsilon}^2} \left\{ \alpha_D w_{\delta_\varepsilon} \left( \frac{x}{\varepsilon} \right) \chi \left( \frac{x}{r_0} \right) + \varepsilon \phi_\varepsilon \left( \frac{x}{\varepsilon} \right) \right\}, \quad x \in (-1, 1), \quad (8.10)$$

where $\alpha_D$ is defined by (8.6), $w_\delta$ is the unique solution to (8.2), and $\phi_\varepsilon(y)$ is a radially symmetric function on $\Omega_\varepsilon := (-1, 1)$ such that

$$\|\phi_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leq C \quad (8.11)$$

holds for some constant $C > 0$ independent of $\varepsilon$. $H_\varepsilon$ has the following property:

$$H_\varepsilon(0) = \frac{1}{\varepsilon \int_\mathbb{R} w_{\delta_\varepsilon}^2} (\alpha_D + O(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0. \quad (8.12)$$

Next, we state a result in the case $\sigma_0 > 0$.

**Theorem 8.2.** Let $\sigma_0 > 0$. We assume the same assumption on $\kappa$ as in Theorem 8.1. Then, (8.1) admits a radially symmetric solution provided $\varepsilon$ is sufficiently small. More precisely, if we fix $\sigma > 0$ and $\gamma \in \left(0, 1/2\right)$, there exists $\delta_\varepsilon > 0$ such that, for all $\varepsilon \in (0, \delta_\varepsilon)$ and $\sigma_0 \in (0, \overline{\sigma})$, (8.1) admits a radially symmetric solution $(A_{\varepsilon, \sigma}(x), H_{\varepsilon, \sigma}(x))$, and $A_{\varepsilon, \sigma}$ takes the form:

$$A_{\varepsilon, \sigma}(x) = \frac{1}{\varepsilon \int_\mathbb{R} w_{\delta_\varepsilon}^2} \left\{ \alpha_D w_{\delta_\varepsilon} \left( \frac{x}{\varepsilon} \right) \chi \left( \frac{x}{r_0} \right) + \varepsilon \phi_{\varepsilon, \sigma} \left( \frac{x}{\varepsilon} \right) \right\} + \sigma_0, \quad x \in (-1, 1), \quad (8.13)$$

where $\delta_\varepsilon$ and $\phi_\varepsilon$ are given in Theorem 8.1, and $\phi_{\varepsilon, \sigma}(y)$ is a radially symmetric function on $\Omega_\varepsilon$ such that

$$\|\phi_{\varepsilon, \sigma}\|_{H^2(\Omega_\varepsilon)} \leq \overline{\sigma} \quad (8.14)$$

holds, and $H_{\varepsilon, \sigma}$ satisfies

$$H_{\varepsilon, \sigma}(0) = \frac{1}{\varepsilon \int_\mathbb{R} w_{\delta_\varepsilon}^2} (\alpha_D + O(\varepsilon) + O(\varepsilon^\gamma (\sigma + \overline{\sigma}^2))) \quad (8.15)$$

as $\varepsilon \to 0$, where $O(\varepsilon)$ is independent of $\sigma_0$.

**Remark 8.1.** The setting of the domain $(-1, 1)$ is not essential. For given $k \in \mathbb{N}$, if we construct a $1$-peak solution to (8.1) on smaller domain in advance, then we can obtain a $k$-peak symmetric solution to (8.1) by reflections.

**Remark 8.2.** The assumption “$\kappa_0 \alpha_D^2$ is sufficiently small” in Theorem 8.1 is due to some technical reason. See Remark 8.3 stated later.
8.2 Preliminaries

In this section, we prepare some lemmas to prove Theorem 8.1 and state the outline of our construction. We first define some function spaces as follows:

\[ L^2_\delta(\mathbb{R}) := \{ u \in L^2(\mathbb{R}) : u(x) = u(-x), \ x \in \mathbb{R} \}, \tag{8.16} \]
\[ H^2_\delta(\mathbb{R}) := H^2(\mathbb{R}) \cap L^2_\delta(\mathbb{R}), \tag{8.17} \]
and for a domain \((-a, a), \ a \in (0, \infty),\)

\[ L^2_\delta(-a, a) := \{ u \in L^2(-a, a) : u(x) = u(-x), \ x \in (-a, a) \}, \tag{8.18} \]
\[ H^2_\delta(-a, a) := H^2(-a, a) \cap L^2_\delta(-a, a), \tag{8.19} \]
\[ H^2_{\epsilon, \nu}(-a, a) := \{ u \in H^2_\delta(-a, a) : u'(\pm a) = 0 \}. \tag{8.20} \]

Because we will frequently use rescaling, we introduce the following notations.

**Definition 8.1.** Put \( \Omega_\varepsilon := \left( -\frac{1}{\varepsilon}, \frac{1}{\varepsilon} \right). \)

For a function \( u : (-1, 1) \to \mathbb{R}, \) let \( \Pi(y) := u(\varepsilon y), \ y \in \Omega_\varepsilon. \)

Inversely, for a function \( v : \Omega_\varepsilon \to \mathbb{R}, \) let \( v(x) := v(\frac{x}{\varepsilon}), \ x \in (-1, 1). \)

For the unique solution \( w_\delta \) to (8.2), let us restate the properties of \( w_\delta \) given in Chapter 2 for convenience.

**Lemma 8.1.** For each \( \delta \in [0, \delta_*), \) the unique radially symmetric solution \( w_\delta \) has the following properties:

(i) \( w_\delta \in C^\infty(\mathbb{R}), \)

(ii) Let

\[ L_\delta := \frac{d^2}{dx^2} - 1 + f_\delta'(w_\delta) : H^2(\mathbb{R}) \to L^2(\mathbb{R}), \]

where \( f_\delta'(w_\delta) = 2w_\delta/(1 + \delta w_\delta^2). \) Then, \( \text{Ker}(L_\delta) = \text{span}\{w_\delta^*\}. \)

(iii) If we restrict the domain to \( \text{Dom}(L_\delta) = H^2_\delta(\mathbb{R}), \) then \( L_\delta \) has a bounded inverse \( L_\delta^{-1} : L^2_\delta(\mathbb{R}) \to H^2_\delta(\mathbb{R}). \) Moreover, if \( \delta \in (0, \delta_*) \) is fixed, \( L_\delta^{-1} \) is bounded uniformly in \( \delta \in [0, \delta]. \)

(iv) If we fix \( \bar{\delta} \in (0, \delta_*), \) then there exist constants \( C, c > 0 \) such that

\[ w_\delta(y), \ \left| \frac{d^n w_\delta}{dy^n}(y) \right| \leq Ce^{-c|y|}, \ y \in \mathbb{R}, \ n = 1, 2, \tag{8.21} \]

holds for any \( \delta \in [0, \bar{\delta}]. \)

(v) \( w_\delta \) is continuous in \( \delta \in [0, \delta_*) \) with respect to the \( C^1(\mathbb{R}) \)-norm.

(vi) \( w_\delta \) is of class \( C^1((0, \delta_*), C^1(\mathbb{R})). \)

(vii) \( w_\delta \to b_* \) in \( C^2_{\text{loc}}(\mathbb{R}) \) holds as \( \delta \to \delta_*, \) where \( b_* > 0 \) is the second positive root of \(-t + f_{\delta_*}(t) = 0, \ t \in \mathbb{R}. \)
(vii) For any \( \delta \in (0, \delta_*), \) it holds that
\[
\frac{d}{d\delta} \left( \int_{-\infty}^{\infty} w_\delta^2(y) dy \right) > 0.
\tag{8.22}
\]

Let us denote the derivatives of \( w_\delta \) in \( x \) and in \( \delta \) by \( w'_\delta(x) \) and \( \frac{dw_\delta}{d\delta} \), respectively. Next, we state some useful formulae.

**Lemma 8.2.** The following identities hold:
\[
L_\delta w_\delta = f_\delta'(w_\delta) w_\delta - f_\delta(w_\delta),
\tag{8.23}
\]
\[
L_\delta \frac{dw_\delta}{d\delta} = f_\delta''(w_\delta),
\tag{8.24}
\]
\[
L_\delta \left( w_\delta + 2\delta \frac{dw_\delta}{d\delta} + \frac{1}{2} y \cdot w'_\delta \right) = w_\delta,
\tag{8.25}
\]
\[
L_\delta \left( w_\delta + 2\delta \frac{dw_\delta}{d\delta} \right) = f_\delta(w_\delta).
\tag{8.26}
\]

**Proof.** These facts were proven in Lemma 2.3 of [114].

**Lemma 8.3.** For fixed \( \delta \in (0, \delta_*), \) there exists constant \( C > 0 \) such that
\[
\left\| \frac{dw_\delta}{d\delta} \right\|_{H^2(\mathbb{R})} \leq C
\tag{8.27}
\]
holds for any \( \delta \in (0, \delta_*). \)

**Proof.** It is easy to see that \( L_\delta^{-1} \) is bounded uniformly in \( \delta \in [0, \delta_*] \). By using (8.24) and Lemma 8.1(iv), we can estimate by some constants \( C, C' > 0 \) independent of \( \delta \in [0, \delta_*] \) as follows:
\[
\left\| \frac{dw_\delta}{d\delta} \right\|_{H^2(\mathbb{R})} = \| L_\delta^{-1} f_\delta'(w_\delta) \|_{L^2(\mathbb{R})} \leq C \| f_\delta'(w_\delta) \|_{L^2(\mathbb{R})} \leq C'.
\tag{8.28}
\]

Hence we complete the proof.

**Lemma 8.4.** (i) For each \( \delta \in [0, \delta_*], \) if \( \phi \in H^2_r(\mathbb{R}) \) satisfies the following:
\[
\phi^{\prime\prime} - \phi + f_\delta'(w_\delta) \phi - \gamma \frac{\int_{\mathbb{R}} w_\delta^2 \phi}{\int_{\mathbb{R}} w_\delta^2} f_\delta(w_\delta) = 0 \text{ in } \mathbb{R},
\tag{8.29}
\]
\[
\gamma \neq \frac{\int_{\mathbb{R}} w_\delta^2}{\int_{\mathbb{R}} w_\delta^2 + 2\delta \int_{\mathbb{R}} w_\delta \frac{dw_\delta}{d\delta}},
\tag{8.30}
\]
then \( \phi = 0. \)

(ii) There exists \( \delta_1 \in (0, \delta_*) \) such that, for \( \delta \in [0, \delta_1], \) if \( \phi \in H^2_r(\mathbb{R}) \) satisfies the following:
\[
\phi^{\prime\prime} - \phi + f_\delta'(w_\delta) \phi - \gamma \frac{\int_{\mathbb{R}} f_\delta(w_\delta) \phi}{\int_{\mathbb{R}} w_\delta^2} w_\delta = 0 \text{ in } \mathbb{R},
\tag{8.31}
\]
\[
\gamma \neq \frac{\int_{\mathbb{R}} w_\delta^2}{\int_{\mathbb{R}} L_\delta^{-1}(w_\delta) f_\delta(w_\delta)}.
\tag{8.32}
\]
then \( \phi = 0. \)
Before the proof, we state some remarks. Lemma 8.1(viii) implies that
\[ \int_{\mathbb{R}} w_\delta \frac{d\alpha}{\delta} > 0 \] for any \( \delta \in (0, \delta_*) \). Hence, we first notice that
\[ 0 < \frac{\int_{\mathbb{R}} w_\delta^2}{\int_{\mathbb{R}} w_\delta^2 + 2\delta \int_{\mathbb{R}} w_\delta \frac{d\alpha}{\delta}} \leq 1, \ \delta \in [0, \delta_*]. \] (8.33)

Secondly, we consider the value of \( \int_{\mathbb{R}} L_\delta^{-1}(w_\delta)f_\delta(w_\delta) \). By using (8.25) and integration by parts, we have
\[ \lim_{\delta \to 0} \int_{\mathbb{R}} L_\delta^{-1}(w_\delta)f_\delta(w_\delta) = \lim_{\delta \to 0} \int_{\mathbb{R}} (w_\delta^3(y) + \frac{1}{2} y \cdot w_\delta'(y))f_\delta(w_\delta(y))dy = \int_{\mathbb{R}} (w_\delta^3(y) - \frac{1}{6} w_\delta^3(y))dy = \frac{5}{6} \int_{\mathbb{R}} w_\delta^3(y)dy > 0. \] (8.34)

Here, we note that \( \delta \int_{\mathbb{R}} \frac{d\alpha}{\delta} f_\delta(w_\delta)dy \to 0 \) as \( \delta \to 0 \) by Lemma 8.3. Moreover, we see that
\[ w_\delta'' - w_\delta + w_\delta^2 = 0 \]
\[ \int_{\mathbb{R}} w_\delta''w_\delta - \int_{\mathbb{R}} w_\delta^2 + \int_{\mathbb{R}} w_\delta^3 = 0 \] (8.35)
\[ \int_{\mathbb{R}} (w_\delta''')^2 + \int_{\mathbb{R}} w_\delta^2 = \int_{\mathbb{R}} w_\delta^3. \]

Therefore, \( \int_{\mathbb{R}} w_\delta^3 > \int_{\mathbb{R}} w_\delta^2 \). Thus we have
\[ \frac{\int_{\mathbb{R}} w_\delta^2}{\int_{\mathbb{R}} L_\delta^{-1}(w_\delta)f_\delta(w_\delta)} \bigg|_{\delta=0} = \frac{\int_{\mathbb{R}} w_\delta^3}{\int_{\mathbb{R}} w_\delta^2} < \frac{6}{5}, \] (8.36)

**Proof.** (i) By using (8.26), the equation (8.29) can be written as follows:
\[ L_\delta \phi = \gamma \frac{\int_{\mathbb{R}} w_\delta \phi}{\int_{\mathbb{R}} w_\delta^2} f_\delta(w_\delta) \]
\[ \phi = \gamma \frac{\int_{\mathbb{R}} w_\delta \phi}{\int_{\mathbb{R}} w_\delta^2} L_\delta^{-1}(f_\delta(w_\delta)) \]
\[ \int_{\mathbb{R}} w_\delta \phi = \gamma \frac{\int_{\mathbb{R}} w_\delta \phi}{\int_{\mathbb{R}} w_\delta^2} \left( \int_{\mathbb{R}} w_\delta^2 + 2\delta \int_{\mathbb{R}} w_\delta \frac{d\alpha}{\delta} \right). \]

Hence, \( \int_{\mathbb{R}} w_\delta \phi = 0 \) must be hold by (8.30). Thus we have \( L_\delta \phi = 0, \phi \in H^2_\infty(\mathbb{R}) \), and hence \( \phi = 0 \) by Lemma 8.1(iii).
(ii) Define $\delta_1$ by

$$
\delta_1 := \sup \{ \delta \in (0, \delta_*) : \int_{\mathbb{R}} L_{\gamma'}^{-1}(w_{\gamma'}) f'_{\gamma'}(w_{\gamma'}) > 0 \text{ for } \delta' \in (0, \delta) \}. \quad (8.37)
$$

This $\delta_1$ is well-defined by (8.34). Then we can prove by the same argument as in the proof of (i).

Now, we define an operator $L_{\delta}$ on $L^2(\mathbb{R})$ with $\text{Dom}(L_{\delta}) = H^2(\mathbb{R})$ by

$$
L_{\delta} \phi = \phi'' - \phi + f'_1(w_{\delta}) \phi - 2 \int_{\mathbb{R}} \frac{w_{\delta}}{w_{\delta}'} f_1(w_{\delta}) \phi, \ \phi \in H^2(\mathbb{R}). \quad (8.38)
$$

Its conjugate operator is given by

$$
L_{\delta}^* \psi = \psi'' - \psi + f'_1(w_{\delta}) \psi - 2 \int_{\mathbb{R}} f_{\gamma}(w_{\delta}) \psi \frac{w_{\delta}}{w_{\delta}'} - w_{\delta}, \ \psi \in H^2(\mathbb{R}). \quad (8.39)
$$

Let us define $\delta_2$ by

$$
\delta_2 = \sup \{ \delta \in (0, \delta_1) : \frac{\int_{\mathbb{R}} w_{\delta}^2}{\int_{\mathbb{R}} L_{\gamma'}^{-1}(w_{\gamma'}) f_{\gamma'}(w_{\gamma'})} < 2 \text{ for } \delta' \in (0, \delta) \}, \quad (8.40)
$$

where $\delta_1$ is defined by (8.37). This $\delta_2$ is well-defined by (8.36).

**Lemma 8.5.** For the operators $L_{\delta}$ and $L_{\delta}^*$, and $\delta_2$ defined above, there hold that

(i) $\text{Ker}(L_{\delta}) \cap H^2(\mathbb{R}) = \{ 0 \}$ for any $\delta \in [0, \delta_*)$,

(ii) $\text{Ker}(L_{\delta})^* \cap H^2(\mathbb{R}) = \{ 0 \}$ for any $\delta \in [0, \delta_2)$.

**Proof.** This Lemma is a consequence of Lemma 8.4.

**Remark 8.3.** We do not know whether $\text{Ker}(L_{\delta})^* \cap H^2(\mathbb{R})$ is trivial or not for $\delta$ near $\delta_*$. If $\text{Ker}(L_{\delta}^*) \cap H^2(\mathbb{R}) = \{ 0 \}$ holds for all $\delta \in [0, \delta_*)$, then we can remove the assumption "$\kappa_0 \alpha_D^2$ is sufficiently small" in Theorem 8.1. However, it seems to be a difficult problem.

### 8.3 Outline of our construction

We state an outline of our construction. We see by Lemma 8.1(v),(vii) that there exists unique $\delta_\varepsilon \in [0, \delta_*)$ such that

$$
\delta_\varepsilon \left( \int_{\mathbb{R}} w_{\delta_\varepsilon}^2 \right)^2 = \kappa \varepsilon^{-2} \alpha_D^2 \quad (8.41)
$$

holds for each $\varepsilon > 0$. By the assumption (A9), in the limit $\varepsilon \to 0$, there hold that

$$
\delta_\varepsilon \to \delta_0, \quad \delta_0 \left( \int_{\mathbb{R}} w_{\delta_0}^2 \right)^2 = \kappa_0 \alpha_D^2 \quad (8.42)
$$

131
as \( \varepsilon \to 0 \), for some \( \delta_0 \in (0, \delta_*). \) We assume henceforth that \( \kappa_0 a_D^2 \geq 0 \) is small enough so that \( \delta_0 \in (0, \delta_2) \), where \( \delta_2 \) is given by (8.40). Then we note that there exists \( \delta \in (0, \delta_2) \) such that \( \delta \varepsilon \in [0, \delta] \) holds for all \( \varepsilon > 0 \) sufficiently small. Hence, we may assume that \( c < \int_{\mathbb{R}} w^2_{\varepsilon}(y) dy < C \) holds for all \( \varepsilon \) sufficiently small, the constants \( c, C > 0 \) are independent of \( \varepsilon \).

Put
\[
c_\varepsilon := \frac{1}{\varepsilon \int_{\mathbb{R}} w_{\varepsilon}^2}.
\] (8.43)

We consider the following problem for \( a \) and \( h \):
\[
\begin{aligned}
\varepsilon^2 a'' + a + \frac{a^2}{h(1+\delta \alpha_D a^2)} + \sigma_\varepsilon &= 0, \quad a > 0, \quad x \in (-1, 1), \\
Dh'' - h + c_\varepsilon a^2 &= 0, \quad h > 0, \quad x \in (-1, 1), \\
a'(\pm 1) = h'(\pm 1) &= 0,
\end{aligned}
\] (8.44)

where
\[
\sigma_\varepsilon := \frac{\sigma_0}{c_\varepsilon}.
\] (8.45)

If we obtain a solution to (8.44), then we obtain a solution to (8.1) by putting \( A(x) = c_\varepsilon a(x) \) and \( H(x) = c_\varepsilon h(x) \). For \( U \in H^2_{r, \nu}(\Omega_\varepsilon) \), let \( T[U] \) be a unique solution to the following problem for \( v \):
\[
\begin{aligned}
Dv'' - v + c_\varepsilon U^2 &= 0, \quad x \in (-1, 1), \\
v'(\pm 1) &= 0.
\end{aligned}
\] (8.46)

Here, the under-bar and over-bar notation is due to Definition 8.1. Moreover, we put
\[
S[U](y) := U''(y) - U(y) + \frac{U^2(y)}{T[U](y)(1 + \delta \alpha_D^2 U^2(y))}, \quad y \in \Omega_\varepsilon.
\] (8.47)

If we can find \( U \in H^2_{r, \nu}(\Omega_\varepsilon) \) such that \( S[U] + \sigma_\varepsilon = 0, \ U > 0 \) in \( \Omega_\varepsilon \), then we obtain a solution to (8.44) by putting \( a(x) = U(x) \) and \( h(x) = T[U](x) \).

Here, we note that \( T[U] \) is written by using Green’s function as follows:
\[
T[U](x) = c_\varepsilon \int_{-1}^{1} G_D(x, z) U^2(z) dz, \quad x \in (-1, 1),
\] (8.48)

for \( U \in L^2(\Omega_\varepsilon) \). In particular, \( T[U] \) is radially symmetric provided \( U \) is radially symmetric. Now, let us define an approximate function \( w_\varepsilon \) as follows:
\[
w_\varepsilon(x) := \alpha_D w_{\delta_0}(\frac{x}{\varepsilon}) \chi(\frac{x}{r_0}),
\] (8.49)

where \( \alpha_D = G_D(0, 0)^{-1}, \ w_{\delta_0} \) is the unique solution to (8.2) for \( \delta = \delta_* \), \( \chi \) is the cut-off function defined in the previous section. We will first consider the case \( \sigma_0 = 0 \) and prove Theorem 8.1 in Chapter 5. For the purpose, we will seek \( U \in H^2_{r, \nu}(\Omega_\varepsilon) \) such that \( S[U] = 0, \ U > 0 \) in \( \Omega_\varepsilon \) in the form \( U(y) = \bar{w}_\varepsilon(y) + \varepsilon \phi(y) \) for some \( \phi \in H^2_{r, \nu}(\Omega_\varepsilon) \). Next, we will consider the case \( \sigma_0 \neq 0 \) in Chapter 6. Note that \( (\sigma_\varepsilon = ) \sigma_0/c_\varepsilon \leq C \varepsilon \) holds for some constant \( C > 0 \) independent of \( \varepsilon \) sufficiently small. Therefore, we can prove Theorem 8.2 by a perturbation argument.
8.4 Basic estimates

In this section, we show some basic estimates.

**Lemma 8.6.** There exists $c_1 > 0$ such that $T[w_\varepsilon](x) \geq c_1$, $x \in (-1, 1)$, for all $\varepsilon$ sufficiently small.

**Proof.**

$$T[w_\varepsilon](x) = c_\varepsilon \int_{-1}^{1} G_D(x, z) w_\varepsilon^2(z)\,dz$$

$$= \alpha^2_D c_\varepsilon \int_{-1}^{1} G_D(x, z) w_\varepsilon^2(\frac{z}{\varepsilon}) \chi(\frac{z}{r_0})\,dz$$

$$= \alpha^2_D c_\varepsilon \int_{-1/\varepsilon}^{1/\varepsilon} G_D(x, \varepsilon z) w_\varepsilon^2(z) \chi(\frac{z}{r_0})\,dz$$

$$\geq \alpha^2_D \int_{\mathbb{R}} w_\delta^2 \sinh(2\theta) \int_{-1/\varepsilon}^{1/\varepsilon} w_\delta^2(z) \chi(\frac{z}{r_0})\,dz$$

$$= \alpha^2_D \int_{\mathbb{R}} w_\delta^2 \sinh(2\theta) \int_{-\infty}^{\infty} w_\delta^2(z)\,dz + o(1),$$

as $\varepsilon \to 0$, where $o(1)$ is uniform in $x \in (-1, 1)$. This estimate complete the proof. □

Next, we show the following elementary inequality.

**Lemma 8.7.** For the non-smooth part $K_D(|x - z|)$ of $G_D(x, z)$, the following estimate holds:

$$|K_D(|x|) - K_D(|y|)| \leq \frac{1}{2\sqrt{D}} \left\{ \frac{1}{\sqrt{D}} \left( ||x| - |y|| \right) + \frac{1}{2} \left( \frac{1}{\sqrt{D}} \right)^2 (|x|^2 + |y|^2) \right\}. \quad (8.50)$$

**Proof.** This lemma is easily verified by (8.7) and the following elementary inequality:

$$1 - |x| \leq e^{-|x|} \leq 1 - |x| + \frac{1}{2} |x|^2.$$  

Thus, we omit the details. □

**Lemma 8.8.** For $w_\varepsilon$ defined by (8.49), it holds that

$$T[w_\varepsilon](0) = \alpha_D + O(\varepsilon), \quad (8.51)$$

as $\varepsilon \to 0$.

**Proof.** Note that

$$T[w_\varepsilon](0) = c_\varepsilon \int_{-1}^{1} G_D(0, z) w_\varepsilon^2(z)\,dz = \frac{\alpha^2_D}{\int_{\mathbb{R}} w_\delta^2} \int_{-1/\varepsilon}^{1/\varepsilon} G_D(0, \varepsilon z) w_\delta^2(z) \chi(\frac{z}{r_0})\,dz,$$
and the following inequalities holds:

\[ \int_{|z|<\frac{r_0}{\epsilon}} G_D(0, \epsilon z)w_{\delta_0}^2(z)dz \leq \int_{-1/\epsilon}^{1/\epsilon} G_D(0, \epsilon z)w_{\delta_0}^2(z)\chi^2_{r_0}(z)dz \leq \int_{|z|<\frac{r_0}{\epsilon}} G_D(0, \epsilon z)w_{\delta_0}^2(z)dz. \]  

(8.52)

The left hand side of (8.52) is written as follows:

\[ (l.h.s) = G_D(0, 0) \int_{|z|<\frac{r_0}{\epsilon}} w_{\delta_0}^2(z)dz + \int_{|z|<\frac{r_0}{\epsilon}} \{G_D(0, \epsilon z) - G_D(0, 0)\}w_{\delta_0}^2(z)dz \equiv I + II. \]

Moreover, noting \( \alpha^{-1}_D = G_D(0, 0) \), we can estimate by Lemma 8.1(iv) so that

\[ I = \alpha^{-1}_D \left\{ \int_{\mathbb{R}} w_{\delta_0}^2(z)dz - \int_{|z|>\frac{r_0}{\epsilon}} w_{\delta_0}^2(z)dz \right\} = \alpha^{-1}_D \int_{\mathbb{R}} w_{\delta_0}^2(z)dz + e.s.t., \quad (8.53) \]

where “e.s.t.” means “exponentially small term”. Next, we can estimate by Lemma 8.7 and the mean value theorem as follows:

\[ |II| \leq \int_{|z|<\frac{r_0}{\epsilon}} |K_D(\epsilon|z|)| - K_D(0)|w_{\delta_0}^2(z)dz + \int_{|z|<\frac{r_0}{\epsilon}} |H_D(0, \epsilon z) - H_D(0, 0)|w_{\delta_0}^2(z)dz \]

\[ \leq C \int_{|z|<\frac{r_0}{\epsilon}} \epsilon|z|w_{\delta_0}^2(z)dz \]

\[ \leq C\epsilon. \]

Here, we note that, \( \epsilon^2|z|^2 < \epsilon|z|r_0 \) for \( |z| < r_0/\epsilon \), \( \int_{\mathbb{R}} |z|w_{\delta_0}^2(z)dz \) is bounded uniformly in \( \epsilon \) sufficiently small since we may assume \( \delta_0 \in [0, \bar{\delta}] \) and we can apply Lemma 8.1(iv). Hence

\[ (l.h.s of (8.52)) = \alpha^{-1}_D \int_{\mathbb{R}} w_{\delta_0} + O(\epsilon). \]  

(8.54)

We can see that the right hand side of (8.52) have the same behavior as (8.54). Thus we have

\[ T|w_\varepsilon|(0) = \frac{\alpha^2_D}{\int_{\mathbb{R}} w_{\delta_0}^2} \left( \alpha^{-1}_D \int_{\mathbb{R}} w_{\delta_0}^2(z)dz + O(\epsilon) \right) = \alpha_D + O(\epsilon). \]

Thus we complete the proof. \( \square \)

Lemma 8.9. For some constant \( C > 0 \), the following estimate holds:

\[ |T|w_\varepsilon|(\varepsilon y) - T|w_\varepsilon|(0)| \leq C(\varepsilon|y| + \varepsilon), \quad y \in \Omega_\varepsilon, \]  

(8.55)

for all \( \epsilon \) sufficiently small.
Proof.

\[ T[w_ε](εy) - T[w_ε](0) = c_ε \int_{-1}^{1} \{ G_D(εy, z) - G_D(0, z) \} w_ε^2(z)dz \]

\[ = \frac{\alpha_D^2}{J_{\mathbb{R}} w_ε^{2a}} \int_{-1/ε}^{1/ε} \{ G_D(εy, εz) - G_D(0, εz) \} w_ε^2(z)χ^2(\frac{ε}{r_0} z)dz \]

\[ = \frac{\alpha_D^2}{J_{\mathbb{R}} w_ε^{2a}} \left[ \int_{-1/ε}^{1/ε} \{ KD(ε|y - z|) - KD(ε|z|) \} w_ε^2(z)χ^2(\frac{ε}{r_0} z)dz \right. \]

\[ - \left. \int_{-1/ε}^{1/ε} \{ HD(εy, εz) - HD(0, εz) \} w_ε^2(z)χ^2(\frac{ε}{r_0} z)dz \right]. \]

Now, by Lemma 8.7, and noting \( ε|z| \leq 1 \) for \( |z| \leq 1/ε \), the following estimate holds:

\[ |KD(ε|y - z|) - KD(ε|z|)| \leq C(ε||y| - |y - z|| + ε²(|y - z|^2 + |z|^2)) \]

\[ \leq C'ε(|y| + |z|), \ y, z \in Ω_ε, \]

for some \( C, C' > 0 \) independent of \( ε, y \) and \( z \). Moreover, we can estimate by the Maclaurin expansion as follows:

\[ |HD(εy, εz) - HD(0, εz)| \leq C''ε(|y| + |z|), \ y, z \in Ω_ε, \]

for some \( C'' > 0 \) independent of \( ε, y \) and \( z \). Thus we have

\[ |T[w_ε](εy) - T[w_ε](0)| \leq \varepsilon \frac{\alpha_D^2}{J_{\mathbb{R}} w_ε^{2a}} (C' + C'') \int_{-1/ε}^{1/ε} (|y| + |z|)w_ε^2(z)dz \]

\[ \leq C''''(ε|y| + ε), \ y \in Ω_ε, \]

for some \( C''' > 0 \) independent of \( ε \) and \( y \). Thus we complete the proof. \( \square \)

**Lemma 8.10.** There exists \( C_1 > 0 \) such that

\[ ||S[\delta_ε]||_{L^2(Ω_ε)} \leq C_1ε, \quad (8.56) \]

for all \( ε \) sufficiently small.

Proof. It is easy to see that \( Δ\overline{w_ε} - \overline{w_ε} = -f_δ(w_ε)\alpha_D + e.s.t. \) in \( L^2(Ω_ε) \) as \( ε \to 0 \). Hence,

\[ S[\overline{w_ε}](y) = -f_δ(w_ε)\alpha_D + \frac{1}{T[\overline{w_ε}](y)} \overline{w_ε^2}(y) + e.s.t. \]

\[ = -f_δ(w_ε)\alpha_D + \frac{1}{T[\overline{w_ε}](y)} \alpha_D w_ε^2(y)χ^2(\frac{ε}{r_0} y) + e.s.t. \]

\[ = -f_δ(w_ε)\alpha_D + \frac{\alpha_D^2}{T[\overline{w_ε}]} f_δ(w_ε) + e.s.t. \] in \( L^2(Ω_ε) \).
By Lemma 8.8, we have
\[
- f_\varepsilon(w_\varepsilon) \alpha D + \frac{\alpha_D}{T[w_\varepsilon]} f_\varepsilon(w_\varepsilon) \\
= f_\varepsilon(w_\varepsilon) \alpha D \left\{ -1 + \frac{\alpha_D}{T[w_\varepsilon](0)} + \frac{\alpha_D}{T[w_\varepsilon](y)} - \frac{\alpha_D}{T[w_\varepsilon](0)} \right\} \\
= f_\varepsilon(w_\varepsilon) \alpha D \left\{ O(\varepsilon) + \frac{\alpha_D}{T[w_\varepsilon](\varepsilon y)T[w_\varepsilon](0)} (T[w_\varepsilon](0) - T[w_\varepsilon](\varepsilon y)) \right\}.
\]
Moreover, by Lemma 8.9, the following estimate holds:
\[
\| f_\varepsilon(w_\varepsilon) (T[w_\varepsilon](0) - T[w_\varepsilon](\varepsilon y)) \|_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \frac{w_\varepsilon^2}{(1 + \varepsilon w_\varepsilon^2)^2} (T[w_\varepsilon](0) - T[w_\varepsilon](\varepsilon y))^2 dy \\
\leq C \int_{\Omega_\varepsilon} w_\varepsilon(y) (|y| + \varepsilon)^2 dy \\
\leq C' \varepsilon^2,
\]
for some constants $C, C' > 0$ independent of $\varepsilon$ sufficiently small. From these estimates and by Lemma 8.6, we have a conclusion.

Next, we give the derivatives of $T$ and $S$. The proofs of Lemmas 8.11, 8.12 below are uninteresting calculation. So we give their proofs in Appendix.

**Lemma 8.11.** If we regard $T$ as a mapping from $L^2(-1, 1)$ into $L^\infty(-1, 1)$, then $T$ is Fréchet differentiable on $L^2(-1, 1)$, and its derivative at $u \in L^2(-1, 1)$ is given by
\[
T'[u] \phi = 2 \varepsilon_c \int_{-1}^{1} G_D(x, z) u(z) \phi(z) dz, \phi \in L^2(-1, 1).
\]
Moreover, for some constant $C > 0$ independent of $\varepsilon$ sufficiently small, the following estimates hold:
\[
\| T[u + h] - T[u] - T'[u] h \|_{L^\infty(\Omega_\varepsilon)} \leq C \| h \|_{L^2(\Omega_\varepsilon)},
\]
\[
\| T'[u] h \|_{L^\infty(\Omega_\varepsilon)} \leq C \| u \|_{L^2(\Omega_\varepsilon)} \| h \|_{L^2(\Omega_\varepsilon)},
\]
for any $u, h \in L^2(\Omega_\varepsilon)$.

For $\tau > 0$, we define a ball in $H^2(\Omega_\varepsilon)$ as follows:
\[
B_\tau(\nabla_\varepsilon) := \{ u \in H^2(\Omega_\varepsilon) : \| \nabla_\varepsilon - u \|_{H^2(\Omega_\varepsilon)} < \tau \}.
\]

Let us fix $\tau > 0$ so that
\[
T[u](x) \geq \frac{1}{2} c_1, \ x \in (-1, 1),
\]
holds for all $u \in B_\tau(\nabla_\varepsilon)$ and $\varepsilon$ sufficiently small, where $c_1$ is a constant given in Lemma 8.6.
Lemma 8.12. For all ε sufficiently small, \( S : H^2(\Omega) \to L^2(\Omega) \) is Fréchet differentiable on \( B_\varepsilon(w) \), and its derivative at \( u \in B_\varepsilon(w) \) is given by

\[
S'[u]\phi = \phi'' - \phi + \frac{2u\phi}{T[u](1 + \delta_\varepsilon \alpha_D^2 w^2)^2} - \frac{u^2(T'[u]\phi)}{T[u](1 + \delta_\varepsilon \alpha_D^2 w^2)^2}, \quad \phi \in H^2(\Omega).
\]

Moreover, the following estimates hold: for \( u \in B_\varepsilon(w) \), \( \phi \in H^2(\Omega) \) and \( h \in H^2(\Omega) \), \( \|h\|_{H^2(\Omega)} < 1 \),

\[
\|S[u + h] - S[u] - S'[u]h\|_{L^2(\Omega)} \leq C(\|h\|_{L^2(\Omega)}^2 + \|h\|_{L^\infty(\Omega)} \|h\|_{L^2(\Omega)}),
\]

(8.63)

\[
\|S'[u + h]\phi - S'[u]\phi\|_{L^2(\Omega)} \leq C(\|h\|_{L^2(\Omega)} + \|h\|_{L^\infty(\Omega)}) \|\phi\|_{L^2(\Omega)},
\]

(8.64)

where \( C > 0 \) is independent of \( u, \phi, h \) and \( \varepsilon \) sufficiently small.

Remark 8.4. For the estimate (8.64), we note that the term \( \phi'' \) vanishes in \( S'[u + h]\phi - S'[u]\phi \). Actually, (8.64) also holds for \( \phi \in L^2(\Omega) \).

8.5 Construction of a solution for \( \sigma_0 = 0 \)

In this section, we construct the 1-peak solution to (8.1) in the case \( \sigma_0 = 0 \) and prove Theorem 8.1. Therefore, we always assume \( \sigma_0 = 0 \) throughout this section. Our construction is based on the argument due to the contraction mapping principle.

Now we define an operator \( \tilde{L}_\varepsilon \) on \( L^2(\Omega) \) with \( \text{Dom}(\tilde{L}_\varepsilon) = H^2_{\varepsilon,\nu}(\Omega) \)

\[
\tilde{L}_\varepsilon \phi := S'[\tilde{w}_\varepsilon]\phi = \phi'' - \phi + \frac{2\tilde{w}_\varepsilon\phi}{T[\tilde{w}_\varepsilon](1 + \delta_\varepsilon \alpha_D^2 \tilde{w}_\varepsilon^2)^2} - \frac{\tilde{w}_\varepsilon^2(T'[\tilde{w}_\varepsilon]\phi)}{T[\tilde{w}_\varepsilon](1 + \delta_\varepsilon \alpha_D^2 \tilde{w}_\varepsilon^2)^2}.
\]

Then its conjugate operator \( \tilde{L}_\varepsilon^* \) is given by \( \text{Dom}(\tilde{L}_\varepsilon^*) = H^2_{\varepsilon,\nu}(\Omega) \) and

\[
\tilde{L}_\varepsilon^* \psi = \psi'' - \psi + \frac{2\tilde{w}_\varepsilon \psi}{T[\tilde{w}_\varepsilon](1 + \delta_\varepsilon \alpha_D^2 \tilde{w}_\varepsilon^2)^2} - \frac{\tilde{w}_\varepsilon \psi}{\left(\frac{T[\tilde{w}_\varepsilon]}{T[\tilde{w}_\varepsilon]^2(1 + \delta_\varepsilon \alpha_D^2 \tilde{w}_\varepsilon^2)^2}\right)}.
\]

(8.65)

(8.66)

The most important thing for our construction is the invertibility of \( \tilde{L}_\varepsilon \). We will notice that the limits of \( \tilde{L}_\varepsilon \) and \( \tilde{L}_\varepsilon^* \) as \( \varepsilon \to 0 \) are \( \tilde{L}_{\varepsilon_0} \) and \( \tilde{L}_{\varepsilon_0}^* \) in some sense.

Proposition 8.1. There exist \( \varepsilon_0 > 0 \) and \( \lambda > 0 \) such that, for \( \varepsilon \in (0, \varepsilon_0) \), the following inequality holds:

\[
\|\tilde{L}_\varepsilon \phi\|_{L^2(\Omega)} \geq \lambda \|\phi\|_{H^2(\Omega)}, \quad \phi \in H^2_{\varepsilon,\nu}(\Omega).
\]

(8.67)

In particular, if \( \delta_0 \) given in (8.42) is small so that \( \delta_0 \in [0, \delta_2] \), then

\[
\text{Ran}(\tilde{L}_\varepsilon) = L^2_{\varepsilon,\nu}(\Omega), \quad \text{Ran}(\tilde{L}_\varepsilon) = L^2_{\varepsilon,\nu}(\Omega).
\]

(8.68)

holds for \( \varepsilon \in (0, \varepsilon_0) \), and hence, \( \tilde{L}_\varepsilon : H^2_{\varepsilon,\nu}(\Omega) \to L^2_{\varepsilon,\nu}(\Omega) \) has a bounded inverse \( \tilde{L}^{-1}_\varepsilon \).
Before the proof, we make sure of the following extension and embedding lemmas on $\Omega_\varepsilon$ and a priori elliptic estimate. Although they are elementary and well-known facts, we need to state their $\varepsilon$-dependence clearly because our domain $\Omega_\varepsilon$ depends on $\varepsilon$. So, we give their proofs in Appendix for the completeness.

**Lemma 8.13** (Extension lemma). For fixed $\varepsilon > 0$, there exists an extension operator $E$ from $H^2(\Omega_\varepsilon)$ into $H^2(\mathbb{R})$, and there exists $C > 0$ depending only on $\varepsilon$ such that, for all $\varepsilon_2 (0, \varepsilon)$,

$$\|Eu\|_{H^2(\mathbb{R})} \leq C\|u\|_{H^2(\Omega_\varepsilon)}, \quad u \in H^2(\Omega_\varepsilon).$$  \hspace{1cm} (8.69)

**Lemma 8.14** (Embedding lemma). For fixed $\varepsilon > 0$, there exists $C > 0$ depending only on $\varepsilon$ such that, for all $\varepsilon_2 (0, \varepsilon)$,

$$\|u\|_{L^\infty(\Omega_\varepsilon)} \leq C\|u\|_{H^2(\Omega_\varepsilon)}, \quad u \in H^2(\Omega_\varepsilon).$$  \hspace{1cm} (8.70)

**Lemma 8.15** (A priori elliptic estimate). For fixed $\varepsilon > 0$ and $f \in L^2(\Omega_\varepsilon)$, let

$$-u'' + u = f \text{ in } \Omega_\varepsilon.$$  \hspace{1cm} (8.71)

Then, the following estimate holds:

$$\|u\|_{H^2(\Omega_\varepsilon)} \leq C\|f\|_{L^2(\Omega_\varepsilon)},$$  \hspace{1cm} (8.72)

the constant $C > 0$ is independent of $u$, $f$ and $\varepsilon_2 (0, \varepsilon)$.

**Proof of Proposition 8.1.** We first prove (8.67). Let the contrary be true. Then there exist $\{\varepsilon_n\}_n^{\infty}$ and $\phi_n \in H^2_{\epsilon,\nu}(\Omega_\varepsilon)$ such that

$$\begin{align*}
\varepsilon_n \to 0, \quad &\|\tilde{L}_{\varepsilon_n}\phi_n\|_{L^2(\Omega_n)} \to 0, \text{ as } n \to \infty, \\
\|\phi_n\|_{H^2(\Omega_n)} = 1, \quad &n = 1, 2, \cdots.
\end{align*}$$  \hspace{1cm} (8.73)

Then, each $\phi_n$ can be extended to an element of $H^2(\mathbb{R})$ by the extension lemma. For simplicity, let us denote the extended function $E\phi_n$ by $\phi_n$. Note that $\|\phi_n\|_{H^2(\mathbb{R})} \leq M$ holds for some constant $M > 0$ independent of $n$. Hence, we can pick up a subsequence (we denote the subsequence by $\{\phi_n\}$ simply), such that,

$$\begin{align*}
\phi_n \to \phi &\text{ in } H^2(\mathbb{R}), \\
\phi_n \to \phi &\text{ in } L^2_{\text{loc}}(\mathbb{R}) \text{ and } L^\infty_{\text{loc}}(\mathbb{R}),
\end{align*}$$  \hspace{1cm} (8.74, 8.75)

as $n \to \infty$, for some $\phi \in H^2(\mathbb{R})$, where $\to$ means the weak-limit. Let us denote $\delta_n$ and $\Omega_n$ corresponding to $\varepsilon_n$ by $\delta_n$ and $\Omega_n$, respectively. Recall that $\delta_n \to \delta_0$ as $n \to \infty$. We claim that

**Claim:** For any $\varphi \in C_0^\infty(\mathbb{R})$, it holds that

$$\langle \tilde{L}_{\varepsilon_n}\phi_n, \varphi \rangle_{L^2(\Omega_n)} \to \langle L_{\delta_n}\phi, \varphi \rangle_{L^2(\mathbb{R})}, \quad (n \to \infty).$$  \hspace{1cm} (8.76)
Indeed, let $K := \text{supp}(\varphi)$ for $\varphi \in C_0^\infty (\mathbb{R})$. We may assume $\Omega_n \supset K$ considering $n$ is large enough. Then,

\[
(\tilde{I}_{\varepsilon_n} \phi_n, \varphi)_{L^2(\Omega_n)} \nonumber
\]

\[
= \int_K \phi''_n \varphi - \int_K \phi_n \varphi + \int_K \frac{2\overline{m}_{\varepsilon_n} \phi_n \varphi}{T[w_{\varepsilon_n}](1 + \delta_n \alpha_D^{-2} \overline{m}_{\varepsilon_n})^2} - \int_K \frac{\overline{m}_{\varepsilon_n}^2 (T^*[w_{\varepsilon_n}]\phi_n) \varphi}{T[w_{\varepsilon_n}](1 + \delta_n \alpha_D^{-2} \overline{m}_{\varepsilon_n})^2}.
\]

Let us consider each term. We first notice that

\[
\int_K \phi''_n \varphi - \int_K \phi_n \varphi \rightarrow \int_K (\phi'' - \phi) \varphi, \quad (n \to \infty).
\]

Recall $\overline{m}_{\varepsilon_n}(y) = \alpha_D w_{\varepsilon_n}(y) \chi \left( \frac{\varepsilon_n}{y} \right)$. For each $y \in K$, we have

\[
\frac{2\overline{m}_{\varepsilon_n}(y) \phi_n(y) \varphi(y)}{T[w_{\varepsilon_n}](1 + \delta_n \alpha_D^{-2} \overline{m}_{\varepsilon_n}(y))^2} = \frac{2\alpha_D w_{\varepsilon_n}(y) \chi \left( \frac{\varepsilon_n}{y} \right) \phi_n(y) \varphi(y)}{T[w_{\varepsilon_n}][\varepsilon_n y](1 + \delta_n w_{\varepsilon_n}(y) \chi \left( \frac{\varepsilon_n}{y} \right))^2} \nonumber
\]

\[
\rightarrow \frac{2w_{\varepsilon_0}(y) \chi \left( \frac{\varepsilon_0}{y} \right) \phi_n(y) \varphi(y)}{(1 + \delta_0 w_{\varepsilon_0}(y))^2} = f'_0 \phi_n(y) \varphi(y),
\]

as $n \to \infty$. By applying Lebesgue’s convergence theorem, we can see that

\[
\int_K \frac{2\overline{m}_{\varepsilon_n} \phi_n \varphi}{T[w_{\varepsilon_n}](1 + \delta_n \alpha_D^{-2} \overline{m}_{\varepsilon_n})^2} \rightarrow \int_K f'_0 \phi_n \varphi, \quad (n \to \infty).
\]

Next, for each $y \in K$, let

\[
(T^*[w_{\varepsilon_n}]\overline{\phi}_{\varepsilon_n})(y) = 2c_{\varepsilon_n} \int_{-1}^1 G_D(\varepsilon_n y, z) w_{\varepsilon_n}(z) \phi_n \left( \frac{z}{\varepsilon_n} \right) dz
\]

\[
= \frac{2\alpha_D}{T[w_{\varepsilon_n}]} \int_{-1/\varepsilon_n}^{1/\varepsilon_n} G_D(\varepsilon_n y, \varepsilon_n z) w_{\varepsilon_n}(z) \chi \left( \frac{\varepsilon_n}{r_0} \right) \phi_n(z) dz
\]

\[
= \frac{2\alpha_D}{T[w_{\varepsilon_n}]} \left\{ \int_{-1/\varepsilon_n}^{1/\varepsilon_n} G_D(\varepsilon_n y, 0) w_{\varepsilon_n}(z) \chi \left( \frac{\varepsilon_n}{r_0} \right) \phi_n(z) dz
\]

\[
+ \int_{-1/\varepsilon_n}^{1/\varepsilon_n} [G_D(\varepsilon_n y, \varepsilon_n z) - G_D(\varepsilon_n y, 0)] w_{\varepsilon_n}(z) \chi \left( \frac{\varepsilon_n}{r_0} \right) \phi_n(z) dz \right\}.
\]

We notice that

\[
\frac{2\alpha_D}{T[w_{\varepsilon_n}]} \int_{-1/\varepsilon_n}^{1/\varepsilon_n} G_D(\varepsilon_n y, 0) w_{\varepsilon_n}(z) \chi \left( \frac{\varepsilon_n}{r_0} \right) \phi_n(z) dz \rightarrow \frac{2}{T[w_{\varepsilon_n}]} \int_{\mathbb{R}} w_{\varepsilon_0}(z) \phi(z) dz
\]

as $n \to \infty$ for each $y \in K$. By the same estimate as was used in the proof of Lemma 8.9, the following estimate holds:

\[
\frac{2\alpha_D}{T[w_{\varepsilon_n}]} \int_{-1/\varepsilon_n}^{1/\varepsilon_n} [G_D(\varepsilon_n y, \varepsilon_n z) - G_D(\varepsilon_n y, 0)] w_{\varepsilon_n}(z) \chi \left( \frac{\varepsilon_n}{r_0} \right) \phi_n(z) dz
\]

\[
\leq C \varepsilon_n \int_{\Omega_n} \left( |y| + |z| \right) w_{\varepsilon_n}(z) |\phi_n(z)| dz
\]

\[
\leq C \varepsilon_n \|y\|_{L^2(\Omega_n)} + \|z w_{\varepsilon_n} \|_{L^2(\Omega_n)} \|\phi_n\|_{L^2(\Omega_n)}
\]

\[
\leq C' \varepsilon_n (1 + |y|),
\]

139
for some constants $C, C' > 0$ independent of $n$. Hence, for each $y \in K$, it holds that
\[
(T^t[w_{\epsilon_n}]\phi_n)(y) \to 2\frac{\int_{\mathbb{R}} w_{\delta_0}\phi}{\int_{\mathbb{R}} w_{\delta_0}^2}, \quad (n \to \infty).
\] (8.77)

Noting (8.77), we can see by Lebesgue’s convergence theorem that
\[
\lim_{n \to \infty} \int_{\mathbb{K}} w_{\epsilon_n} (T^t[w_{\epsilon_n}]\phi_n) \phi = 2\frac{\int_{\mathbb{R}} w_{\delta_0}\phi}{\int_{\mathbb{R}} w_{\delta_0}^2} \int_{\mathbb{K}} w_{\delta_0} \phi, \quad (n \to \infty).
\]

By these observations, the claim is verified.

On the other hand, we notice that
\[
(J_n \phi_n, \phi)_{L^2(\Omega_{\epsilon_n})} \leq \| J_n \phi_n \|_{L^2(\Omega_{\epsilon_n})} \| \phi \|_{L^2(\Omega_{\epsilon_n})} \to 0
\] (8.78)
as $n \to \infty$ for any $\phi \in C_0^\infty(\mathbb{R})$. Combining (8.76) and (8.78), we have
\[
(L_{\delta_n} \phi, \phi)_{L^2(\mathbb{R})} = 0 \quad \text{for any } \phi \in C_0^\infty(\mathbb{R}).
\] (8.79)

Therefore, $L_{\delta_n} \phi = 0, \phi \in H^2(\mathbb{R})$. Thus we conclude $\phi = 0$ by Lemma 8.5.

Next, we claim that
\[
\| \phi_n \|_{H^2(\Omega_{\epsilon_n})} \to 0 \quad \text{as } n \to \infty.
\] (8.80)

Indeed, by Lemma 8.15, we have
\[
\| \phi_n \|_{H^2(\Omega_{\epsilon_n})} \leq C \left\{ \| J_n \phi_n \|_{L^2(\Omega_{\epsilon_n})} + \frac{2w_{\epsilon_n}\phi_n}{T[w_{\epsilon_n}](1 + \delta_n \alpha_D^2 w_{\epsilon_n}^2)} \right\} + \frac{\| w_{\epsilon_n}^2 (T^t[w_{\epsilon_n}]\phi_n) \|_{L^2(\Omega_{\epsilon_n})}}{T[w_{\epsilon_n}](1 + \delta_n \alpha_D^2 w_{\epsilon_n}^2)} \right\} \quad (8.81)
\]

By (8.73), $I \to 0$ as $n \to \infty$. Moreover, by the exponentially decay estimate of Lemma 8.1(iv) and the fact $\phi_n \to \phi = 0$ in $L^\infty(\mathbb{R})$ and $L^2_{\text{loc}}(\mathbb{R})$, we can see that $II, III \to 0$ as $n \to \infty$.

However, (8.80) contradicts $\| \phi_n \|_{H^2(\Omega_{\epsilon_n})} = 1$. Thus (8.67) is verified.

Next, we show (8.68). We note that (8.67) implies the range of $\tilde{L}_{\epsilon}$ is closed. Hence, by a general theory of the functional analysis, $\text{Ran}(\tilde{L}_{\epsilon}) = L^2_{\text{loc}}(\Omega_{\epsilon})$ if and only if $\tilde{L}_{\epsilon}^*$ is one to one. However, by the same argument as was used in the proof of (8.67), we can show that $\tilde{L}_{\epsilon}^*$ is one to one for sufficiently small $\epsilon$ under the assumption where $\delta_{\epsilon} \to \delta_0 \in [0, \delta_2)$ as $\epsilon \to 0$. Therefore, we omit the details.

At last, we construct a solution to (8.1) and complete the proof of Theorem 8.1. Let us find $\phi \in H_{r,\nu}^2(\Omega_{\epsilon})$ such that $S[w_{\epsilon} + \varepsilon \phi] = 0$ for sufficiently small
there exists $\varepsilon > \lambda$ such that, for $\varepsilon \in (0, \varepsilon_1)$, $M_\varepsilon$ is a contraction mapping on $B$.

Proof. For $\phi \in B$, note that $M_\varepsilon(\phi) \in H^2_{2, \nu}(\Omega_\varepsilon)$. Moreover, by Lemma 8.10, (8.63) and (8.70), we can estimate as follows:

\[
\|M_\varepsilon(\phi)\|_{H^2(\Omega_\varepsilon)} \leq \frac{1}{\varepsilon \lambda} \left\{ \|S[\varpi_\varepsilon + \varepsilon \phi] - S[\varpi_\varepsilon] - S'[\varpi_\varepsilon](\varepsilon \phi)\|_{L^2(\Omega_\varepsilon)} + \|S[\varpi_\varepsilon + \varepsilon \phi] - S[\varpi_\varepsilon] - S'[\varpi_\varepsilon](\varepsilon \phi)\|_{L^2(\Omega_\varepsilon)} \right\} \\
\leq \frac{1}{\varepsilon \lambda} \left\{ C_1 \varepsilon + C_2 \varepsilon^2 (\|\phi\|_{L^\infty(\Omega_\varepsilon)} + \|\phi\|_{L^\infty(\Omega_\varepsilon)}) \right\} \\
\leq \frac{1}{\lambda} \left\{ C_1 + C_2 \varepsilon \right\} \\
\leq \frac{1}{\lambda} \left\{ C_1 + C_2 \varepsilon \right\},
\]

where $C_1, C_2 > 0$ are independent of $\varepsilon$ sufficiently small. Hence, if $\varepsilon$ is small so that $\varepsilon < \lambda^2/(8C_1 C_2)$, then $\|M_\varepsilon(\phi)\|_{H^2(\Omega_\varepsilon)} < \frac{2C_1}{\lambda}$ for $\phi \in B$. Therefore, $M_\varepsilon$ is a mapping form $B$ into itself for sufficiently small $\varepsilon$.

For $\phi_1, \phi_2 \in B$, by (8.63), (8.64) and (8.70), we can estimate as follows:

\[
\|M_\varepsilon(\phi_1) - M_\varepsilon(\phi_2)\|_{H^2(\Omega_\varepsilon)} \\
\leq \frac{1}{\varepsilon \lambda} \|S[\varpi_\varepsilon + \varepsilon \phi_1] - S[\varpi_\varepsilon + \varepsilon \phi_2] - S'[\varpi_\varepsilon](\varepsilon \phi_1) + S'[\varpi_\varepsilon](\varepsilon \phi_2)\|_{L^2(\Omega_\varepsilon)} \\
\leq \frac{1}{\varepsilon \lambda} \left\{ \|S[\varpi_\varepsilon + \varepsilon \phi_1] - S[\varepsilon \phi_1 - \varepsilon \phi_2]\| + \|S[\varepsilon \phi_1 - \varepsilon \phi_2]\| \right\} \\
\leq \frac{C}{\varepsilon \lambda} \left\{ \|\phi_1 - \phi_2\|_{H^2(\Omega_\varepsilon)} + \|\phi_1 - \phi_2\|_{H^2(\Omega_\varepsilon)} \right\} \\
\leq \left\{ C_1 \varepsilon \|\phi_1 - \phi_2\|_{H^2(\Omega_\varepsilon)} + C_2 \varepsilon \|\phi_1 - \phi_2\|_{H^2(\Omega_\varepsilon)} \right\},
\]

where $C_1, C_2 > 0$ are independent of $\varepsilon$ sufficiently small. Therefore, $M_\varepsilon$ is a contraction mapping on $B$ provided $\varepsilon$ is small enough. \qed
Proof of Theorem 8.1. By Proposition 8.2, $M_\varepsilon$ has a unique fixed point in $B$ if $\varepsilon$ is sufficiently small. Let $\phi_\varepsilon \in B$ be the fixed point. Then $\phi_\varepsilon$ satisfies $S[w_\varepsilon + \varepsilon \phi_\varepsilon] = 0$. As we observed in Section 3.2, by putting $A_\varepsilon(x) := c_\varepsilon(w_\varepsilon(x) + \varepsilon \phi_\varepsilon(x))$ and $H_\varepsilon(x) := c_\varepsilon T[w_\varepsilon + \varepsilon \phi_\varepsilon(x)]$, we obtain a solution to (8.1). We can see that this $(A_\varepsilon, H_\varepsilon)$ satisfies (8.10)-(8.12). Thus we complete the proof.

8.6 Construction of a solution for $\sigma_0 > 0$

In this section, we construct a solution to (8.1) in the case $\sigma_0 > 0$ and prove Theorem 8.2. Let us treat $\sigma_0$ as a parameter. To lead precise estimates, we fix $\sigma > 0$ arbitrarily, and we will consider $\sigma_0 \in (0, \sigma)$. Let $\phi_\varepsilon \in B$ be a unique fixed point of $M_\varepsilon$ given in the proof of Theorem 8.1. Put $U_\varepsilon(y) := w_\varepsilon(y) + \varepsilon \phi_\varepsilon, y \in \Omega_\varepsilon$, and we define an operator $\tilde{L}_\varepsilon$ on $L^2(\Omega_\varepsilon)$ with $\text{Dom}(\tilde{L}_\varepsilon) = H^{2,\nu}(\Omega_\varepsilon)$ by

$$\tilde{L}_\varepsilon \phi := S[w_\varepsilon + \sigma \varepsilon] \phi, \phi \in \text{Dom}(\tilde{L}_\varepsilon).$$

We note that $\|\sigma\|_{H^2(\Omega_\varepsilon)} = \|\sigma\|_1 \|H^2(\Omega_\varepsilon) = \frac{\sigma_0}{\sqrt{2\varepsilon}} = \frac{\varepsilon \sigma_0 \int_\mathbb{R} w_\varepsilon^2(y)dy}{\sqrt{2\varepsilon}} < C \sqrt{\varepsilon} \sigma_0$

holds for some constant $C > 0$ independent of $\varepsilon$ sufficiently small. Thus, we may assume $U_\varepsilon + \sigma_\varepsilon \in B_\varepsilon(\Omega_\varepsilon)$ for sufficiently small $\varepsilon$ and $\sigma_0 \in (0, \sigma)$. Then we have the following proposition.

**Proposition 8.3.** There exists $\varepsilon_0 > 0$ depending on $\sigma_0$ such that, for $\varepsilon \in (0, \varepsilon_0)$ and $\sigma_0 \in (0, \sigma)$, $\tilde{L}_\varepsilon$ has a bounded inverse $\tilde{L}_\varepsilon^{-1} : L^2(\Omega_\varepsilon) \to H^{2,\nu}(\Omega_\varepsilon)$, and the following estimate holds:

$$\|\tilde{L}_\varepsilon^{-1} \phi\|_{H^2(\Omega_\varepsilon)} \leq \frac{2}{\lambda} \|\phi\|_{L^2(\Omega_\varepsilon)}, \phi \in L^2(\Omega_\varepsilon),$$

where $\lambda$ is a constant given in Proposition 8.1.

**Proof.** Let $\phi \in L^2(\Omega_\varepsilon)$ be given. For $\psi \in H^{2,\nu}(\Omega_\varepsilon)$, the following equations are equivalent:

$$\tilde{L}_\varepsilon \psi = \phi,$n
$$\tilde{L}_\varepsilon \psi - (\tilde{L}_\varepsilon \psi - \tilde{L}_\varepsilon \psi) = \phi,$n
$$\psi - \tilde{L}_\varepsilon^{-1}(\tilde{L}_\varepsilon \psi) = \tilde{L}_\varepsilon^{-1} \phi.$$ (8.86)

Noting $\tilde{L}_\varepsilon \psi - \tilde{L}_\varepsilon \psi = S'[\xi_\varepsilon] \psi - S'[U_\varepsilon + \sigma_\varepsilon] \psi$, we can regard $K_\varepsilon$ is a mapping.
from $L^2_\sigma(\Omega_\varepsilon)$ into itself. We can estimate so that

$$\|K_\varepsilon \psi\|_{L^2(\Omega_\varepsilon)} \leq \|K_\varepsilon \psi\|_{H^2(\Omega_\varepsilon)} \leq \frac{1}{\lambda} \|\tilde{L}_\varepsilon \psi - \tilde{L}_\varepsilon \psi\|_{L^2(\Omega_\varepsilon)}$$

$$= \frac{1}{\lambda} \|S'[\overline{\sigma}_\varepsilon + \varepsilon \phi_\varepsilon + \sigma_\varepsilon] \psi - S'[\overline{\sigma}_\varepsilon] \psi\|_{L^2(\Omega_\varepsilon)}$$

$$\leq \frac{C}{\lambda} (\|\varepsilon \phi_\varepsilon + \sigma_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\varepsilon \phi_\varepsilon + \sigma_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}) \|\psi\|_{L^2(\Omega_\varepsilon)}$$

$$\leq C' (\varepsilon + \sqrt{\sigma}) \|\psi\|_{L^2(\Omega_\varepsilon)},$$

by Lemma 8.12, where $C' > 0$ is independent of $\sigma_0 \in (0, \overline{\sigma})$ and $\varepsilon$. Therefore, $\|K_\varepsilon\|_{L^2(\Omega_\varepsilon) \rightarrow L^2(\Omega_\varepsilon)} \leq 1/2$ holds provided $\varepsilon$ is small enough. Hence, by the Neumann series theory, $(I - K_\varepsilon)^{-1}: L^2(\Omega_\varepsilon) \rightarrow L^2(\Omega_\varepsilon)$ exists. Thus, we have $\psi = (I - K_\varepsilon)^{-1} \phi \equiv \tilde{L}_\varepsilon \phi$. Moreover, from (8.86), we have $\psi \in H^2_{\varepsilon, \nu}(\Omega_\varepsilon)$ and the estimate:

$$\|\psi\|_{H^2(\Omega_\varepsilon)} \leq \|K_\varepsilon \psi\|_{H^2(\Omega_\varepsilon)} + \|\tilde{L}_\varepsilon^{-1} \psi\|_{H^2(\Omega_\varepsilon)}$$

$$\leq \frac{1}{2} \|\psi\|_{H^2(\Omega_\varepsilon)} + \frac{1}{\lambda} \|\phi\|_{L^2(\Omega_\varepsilon)}.$$

Hence, $\|\psi\|_{H^2(\Omega_\varepsilon)} \leq \frac{2}{\lambda} \|\phi\|_{L^2(\Omega_\varepsilon)}$ follows. $\square$

We put

$$\tilde{B} := \{ \phi \in H^2_{\varepsilon, \nu}(\Omega_\varepsilon): \|\phi\|_{H^2(\Omega_\varepsilon)} < \overline{\sigma} \},$$

and fix $\gamma \in (0, 1/2)$ arbitrarily. Let us find $\phi \in \tilde{B}$ such that $S[U_\varepsilon + \sigma_\varepsilon + \varepsilon^\gamma \phi] + \sigma_\varepsilon = 0$. We note that this is equivalent to the following:

$$-\varepsilon^\gamma \tilde{L}_\varepsilon \phi = S[U_\varepsilon + \sigma_\varepsilon + \varepsilon^\gamma \phi] - S[U_\varepsilon + \sigma_\varepsilon] - S[U_\varepsilon + \sigma_\varepsilon] + S[U_\varepsilon + \sigma_\varepsilon] + \sigma_\varepsilon,$n

$$\phi = -\frac{1}{\varepsilon^\gamma} L_\varepsilon^{-1} \left( S[U_\varepsilon + \sigma_\varepsilon + \varepsilon^\gamma \phi] - S[U_\varepsilon + \sigma_\varepsilon] - S'[U_\varepsilon + \sigma_\varepsilon] \varepsilon^\gamma \phi \right)$$

$$+ S[U_\varepsilon + \sigma_\varepsilon] + \sigma_\varepsilon =: M_{\varepsilon, \sigma}(\phi).$$

**Proposition 8.4.** For $\overline{\sigma}$ and $\gamma$, there exists $\tilde{\varepsilon}_1 > 0$ such that, for $\varepsilon \in (0, \tilde{\varepsilon}_1)$ and $\sigma_0 \in (0, \overline{\sigma})$, $M_{\varepsilon, \sigma}$ is a contraction mapping from $\tilde{B}$ into itself.

**Proof.** By Proposition 8.3, we have

$$\|M_{\varepsilon, \sigma}(\phi)\|_{H^2(\Omega_\varepsilon)} \leq \frac{2}{\varepsilon^\gamma \lambda} \{ \|S[U_\varepsilon + \sigma_\varepsilon + \varepsilon^\gamma \phi] - S[U_\varepsilon + \sigma_\varepsilon] - S'[U_\varepsilon + \sigma_\varepsilon] \varepsilon^\gamma \phi\|_{L^2(\Omega_\varepsilon)}$$

$$+ \|S[U_\varepsilon + \sigma_\varepsilon]\|_{L^2(\Omega_\varepsilon)} + \|\sigma_\varepsilon\|_{L^2(\Omega_\varepsilon)} \}$$

$$\equiv \frac{2}{\varepsilon^\gamma \lambda} (I + II + III).$$

Moreover, by Lemmas 8.12, 8.14, we can see that the following estimates hold: for $\phi \in \tilde{B}$,

$$I \leq 2C \varepsilon^{2\gamma} \|\phi\|_{H^2(\Omega_\varepsilon)}^2 \leq 2C \varepsilon^{2\gamma} \sigma^2,$$
Thus we complete the proof.

In this section, we give the proofs of Lemmas 8.11, 8.12, 8.13, 8.14 and 8.15. We first prove Lemmas 8.13, 8.14, 8.15.

**Proof of Lemma 8.13.** In general, \( H^2(\mathbb{R}) \)-extensions denoted by \( E_1u \) and \( E_2v \) of \( u \in H^2(0,\infty) \) and \( v \in H^2(-\infty,0) \) are given by

\[
E_1u(x) = \begin{cases} u(x), & x > 0, \\ 3u(-x) - 2u(-2x), & x < 0, \end{cases} \quad E_2v(x) = \begin{cases} 3v(-x) - 2v(-2x), & x > 0, \\ u(x), & x < 0, \end{cases}
\]

respectively, and

\[
\|E_1u\|_{H^2(\mathbb{R})} \leq C\|u\|_{H^2(0,\infty)}, \quad \|E_2v\|_{H^2(\mathbb{R})} \leq C\|v\|_{H^2(-\infty,0)},
\]

hold for some \( C > 0 \) independent of \( u \) and \( v \). By translation, we see that there exists \( H^2(\mathbb{R}) \)-extensions denoted by \( \tilde{E}_1u \) and \( \tilde{E}_2v \) of \( u \in H^2(-\frac{1}{2},\infty) \) and \( v \in H^2(-\infty,-\frac{1}{2}) \). Because translation does not change the \( H^2 \)-norm,

\[
\|\tilde{E}_1u\|_{H^2(\mathbb{R})} \leq C\|u\|_{H^2(-\frac{1}{2},\infty)}, \quad \|\tilde{E}_2v\|_{H^2(\mathbb{R})} \leq C\|v\|_{H^2(-\infty,-\frac{1}{2})},
\]

**8.7 Appendix**

In this section, we give the proofs of Lemmas 8.11, 8.12, 8.13, 8.14 and 8.15. We first prove Lemmas 8.13, 8.14, 8.15.

**Proof of Lemma 8.13.**
hold for the same constant $C$ as that in (8.87). Now, let $\varphi \in C^\infty(\mathbb{R})$ be a function such that, $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $x \leq -\frac{1}{2}$, $\varphi(x) = 0$ for $x > \frac{1}{2}$. Moreover, we take $\varphi$ so that

$$1 - \varphi(x) = \varphi(-x), \ x \in \mathbb{R},$$

(8.89)

holds. We define $\varphi_\varepsilon(x) := \varphi(\varepsilon x)$. Then, for fixed $\varepsilon > 0$, we note that the estimates

$$\sup_{x \in \mathbb{R}} |\varphi_\varepsilon(x)|, \sup_{x \in \mathbb{R}} |\varphi_\varepsilon''(x)| \leq M, \ \varepsilon \in (0, \tau),$$

(8.90)

hold for some constant $M > 0$ depending only on $\varphi$ and independent of $\varepsilon \in (0, \tau)$. For $u \in H^2_0(\Omega_\varepsilon)$, if we regard $(\varphi_\varepsilon u)(x) = 0$ for $x \in [-\frac{1}{2}, \infty)$, then $\varphi_\varepsilon u \in H^2(-\frac{1}{2}, \infty)$. Note that $\|\varphi_\varepsilon u\|_{H^2(-\frac{1}{2}, \infty)} \leq C\|u\|_{H^2(\Omega_\varepsilon)}$ holds for all $\varepsilon \in (0, \tau)$ by (8.90). We extend $\varphi_\varepsilon u$ by $\tilde{E}_1$, $\tilde{E}_1(\varphi_\varepsilon u) \in H^2(\mathbb{R})$. Similarly, we can regard $(1 - \varphi_\varepsilon)u \in H^2(-\infty, \frac{1}{2})$, and $\tilde{E}_2((1 - \varphi_\varepsilon)u) \in H^2(\mathbb{R})$. Define $Eu := \tilde{E}_1(\varphi_\varepsilon u) + \tilde{E}_2((1 - \varphi_\varepsilon)u)$. Then, this $E$ is a desired extension operator from $H^2_0(\Omega_\varepsilon)$ into $H^2(\mathbb{R})$. Indeed, we can easily see that $Eu$ gives the $H^2(\mathbb{R})$-extension of $u$. Moreover, $Eu$ is radially symmetric by our construction. Note the estimate

$$\|Eu\|_{H^2(\mathbb{R})} \leq \|\tilde{E}_1(\varphi_\varepsilon u\|_{H^2(\mathbb{R})} + \|\tilde{E}_2((1 - \varphi_\varepsilon)u)\|_{H^2(\mathbb{R})} \leq C\|\varphi_\varepsilon u\|_{H^2(-\frac{1}{2}, \infty)} + \|(1 - \varphi_\varepsilon)u\|_{H^2(-\infty, \frac{1}{2})} \leq 2CC'\|u\|_{H^2(\Omega_\varepsilon)}.$$

Thus we complete the proof.

Proof of Lemma 8.14. Let $E$ be the extension operator given by Lemma 8.13. By Morrey’s inequality, we have

$$\|u\|_{L^\infty(\Omega_\varepsilon)} = \|Eu\|_{L^\infty(\Omega_\varepsilon)} \leq \|Eu\|_{L^\infty(\mathbb{R})} \leq C\|u\|_{H^2(\mathbb{R})} \leq CC''\|u\|_{H^2(\Omega_\varepsilon)},$$

the constants $C, C'' > 0$ are independent of $\varepsilon \in (0, \tau)$.

Proof of Lemma 8.15. We first note that

$$\|u\|_{H^2(\Omega_\varepsilon)} \leq \|f\|_{L^2(\Omega_\varepsilon)},$$

(8.91)

holds. This can be easily confirmed by multiplying (8.71) by $u$ and integrating over $\Omega_\varepsilon$. For fixed $\varepsilon \in (0, \frac{1}{2})$, let $\varphi_1, \varphi_2 \in C^\infty_0(\mathbb{R})$ be functions such that

$$\varphi_1(x) = \begin{cases} 1, & |x + 1| < c, \\ 0, & |x + 1| > 2c, \end{cases} \quad \varphi_2(x) = \begin{cases} 1, & |x - 1| < c, \\ 0, & |x - 1| > 2c. \end{cases}$$

Define $\varphi_0(x) := 1 - (\varphi_1(x) + \varphi_2(x))$ for $x \in (-1, 1)$. Let

$$\varphi_0^j(x) := \varphi_j(\varepsilon x), \ j = 0, 1, 2.$$

Then, note that $\frac{d^n\varphi_0^j}{dx^n}(x)$, $j = 0, 1, 2$, $n = 1, 2$, are bounded uniformly with respect to $\varepsilon \in (0, \tau)$. Now, $\varphi_0^j u$ solves the following equation:

$$-(\varphi_0^j u)'' + (\varphi_0^j u) = -(\varphi_0^j)'u - 2(\varphi_0^j)'u' + \varphi_0^j f =: g_0^j \text{ in } \Omega_\varepsilon.$$  (8.92)
Note that $\|g_0\|_{L^2(\Omega_\varepsilon)} \leq C\|f\|_{L^2(\Omega_\varepsilon)}$ holds for some constant $C > 0$ independent of $\varepsilon \in (0, \varepsilon)$ by (8.91). We extend $\varphi_0^\varepsilon u$ and $g_0^\varepsilon$ as 0 in $\Omega_\varepsilon$. Then $\varphi_0^\varepsilon u \in H^2(\mathbb{R})$ and $g_0^\varepsilon \in L^2(\mathbb{R})$ satisfy the same equation as that in (8.92) over $\mathbb{R}$. Thus, by using a priori estimate of solutions for elliptic equations in whole space, we have

$$\|\varphi_0^\varepsilon u\|_{H^2(\Omega_\varepsilon)} = \|g_0^\varepsilon\|_{H^2(\mathbb{R})} \leq C\|f\|_{L^2(\Omega_\varepsilon)} \leq C_0\|f\|_{L^2(\Omega_\varepsilon)},$$  \hspace{1cm} (8.93)

for some constant $C_0 > 0$. Next, we consider $\varphi_1^\varepsilon u$. We extend $\varphi_1^\varepsilon u = 0$ for $x \geq \frac{1}{2}$. Then $\varphi_1^\varepsilon u \in H^2((\frac{1}{2}, \infty))$. Moreover, we extend it to $H^2(\mathbb{R})$-function by reflection (this extension is possible since $u$ satisfies $u'(-\frac{1}{2}) = 0$). Then we notice that $\varphi_1^\varepsilon$ satisfies

$$-(\varphi_1^\varepsilon)^{\prime\prime} - (\varphi_1^\varepsilon)^\prime u = -(\varphi_1^\varepsilon)^{\prime\prime} u - 2(\varphi_1^\varepsilon)^\prime u + \varphi_1^\varepsilon f \text{ in } \mathbb{R}. \hspace{1cm} (8.94)$$

Hence, by the same argument as was used for $\varphi_0^\varepsilon u$, we have $\|\varphi_1^\varepsilon u\|_{H^2(\Omega_\varepsilon)} = C''\|f\|_{L^2(\Omega_\varepsilon)}$ for some constant $C'' > 0$ independent of $\varepsilon \in (0, \varepsilon)$. We can estimate for $\varphi_2^\varepsilon u$ in the same way. Thus we have

$$\|u\|_{H^2(\Omega_\varepsilon)} \leq \|\varphi_0^\varepsilon u\|_{H^2(\Omega_\varepsilon)} + \|\varphi_1^\varepsilon u\|_{H^2(\Omega_\varepsilon)} + \|\varphi_2^\varepsilon u\|_{H^2(\Omega_\varepsilon)} \leq C''\|f\|_{L^2(\Omega_\varepsilon)} \hspace{1cm} (8.95)$$

for some constant $C'' > 0$ independent of $u, f$ and $\varepsilon \in (0, \varepsilon)$.

**Proof of Lemma 8.11.** It is easily to see that the Fréchet derivative of $T$ at $u \in L^2(-1, 1)$ is given by $T'[u]$ of (8.57). Hence, we only show the inequalities (8.58) and (8.59). Noting that $c \leq G_D(x, z) \leq C, x, z \in (-1, 1)$, holds for some $C, c > 0$, we can estimate as follows:

$$|T[u + h](y) - T[u](y) - (T[u + h] - T[u])(y)| = \left| \int_{-1}^1 G_D(\varepsilon y, z)h(z)dz \right|$$

$$= \frac{1}{f_{\varepsilon, z}} \int_{-1}^{1/\varepsilon} G_D(\varepsilon y, \varepsilon z)h(z)dz$$

$$\leq C''\|h\|_{L^2(\Omega_\varepsilon)}, \hspace{1cm} y \in \Omega_\varepsilon,$$

and

$$|(T[u + h] - T[u])(y)| = 2\varepsilon c \int_{-1}^{1} G_D(\varepsilon y, z)u(z)h(z)dz$$

$$\leq C''\|u\|_{L^2(\Omega_\varepsilon)}\|h\|_{L^2(\Omega_\varepsilon)}$$, \hspace{1cm} y \in \Omega_\varepsilon,

for any $u, h \in L^2(\Omega_\varepsilon)$, where $C'' > 0$ is independent of $\varepsilon$ sufficiently small. Thus we complete the proof.

**Proof of Lemma 8.12.** Let us show the inequalities (8.63) and (8.64). We first note that $\|u\|_{L^\infty(\Omega_\varepsilon)} \leq C_\tau$ holds for any $u \in B_\tau(\Omega_\varepsilon)$, where $C_\tau > 0$ is some constant independent of $u$ and $\varepsilon$ sufficiently small. For simplicity of notation, we put

$$g_\varepsilon(t) := \frac{t^2}{1 + \delta_\varepsilon \alpha_D^{-2}I^2}, \hspace{1cm} (8.96)$$

146
Let \( u \in B_r(\mathcal{V}_x), \ h \in H^2(\Omega_x), \ \|h\|_{H^2(\Omega_x)} < 1, \) and
\[
S[u + h] - S[u] = g_e(u + h) - g_e(u) \quad \frac{T[u + h]}{T[u]} \quad + \quad \frac{g'_e(u)h}{T[u]^2}
\]
\[
= \frac{1}{T[u + h]} \left( g_e(u + h) - g_e(u) - g'_e(u)h \right) + g_e(u) \left\{ \frac{1}{T[u + h]} - \frac{1}{T[u]} \right\} + g'_e(u) \left\{ \frac{1}{T[u + h]} - \frac{1}{T[u]} \right\}
\]
\[
\equiv I + II + III.
\]

Note that
\[
|g_e(u + h) - g_e(u) - g'_e(u)h| = \left| \int_0^1 \{ g'_e(u + th) - g'_e(u) \} dt \cdot h \right| \leq M|h|^2
\]
holds for some constant \( M > 0 \) independent of \( \varepsilon \) sufficiently small. By this,
\[
\|I\|_{L^2(\Omega_x)} \leq C \left( \int_{\Omega_x} h^4 \right)^{\frac{1}{2}} \leq C\|h\|_{L^\infty(\Omega_x)}\|h\|_{L^2(\Omega_x)}
\]
holds for some constant \( C > 0 \) independent of \( \varepsilon \) sufficiently small. Next, let
\[
II = \frac{g_e(u)}{T[u + h]T[u]} \left\{ \frac{T[u]}{T[u + h]} - T[u + h] \right\} + \frac{g'_e(u)}{T[u + h]T[u]} \left\{ \frac{T'[u]}{T[u + h]} - \frac{T'[u]}{T[u]} \right\}.
\]

Note that
\[
\left\| \frac{g_e(u)}{T[u + h]T[u]} \right\|_{L^2(\Omega_x)}, \quad \left\| \frac{T[u]}{T[u + h]} \right\|_{L^\infty(\Omega_x)}
\]
are bounded independently of \( u \) and \( \varepsilon \) sufficiently small. Hence, by applying (8.58) and (8.59), we can estimate so that
\[
\|II\|_{L^2(\Omega_x)} \leq C'|\|h\|_{L^2(\Omega_x)}^2
\]
for some constant \( C' > 0 \) independent of \( \varepsilon \) sufficiently small. By the same estimate, we have \( \|III\|_{L^2(\Omega_x)} \leq C''\|h\|_{L^2(\Omega_x)}^2 \) for some constant \( C'' > 0 \) independent of \( \varepsilon \) sufficiently small. By these estimates, we obtain (8.63).
Let $u \in B_{r}(\varpi_{\varepsilon})$, $h \in H^{2}(\Omega_{\varepsilon})$, $\|h\|_{H^{2}(\Omega_{\varepsilon})} << 1$, $\phi \in H^{2}(\Omega_{\varepsilon})$, and

$$
\|S'[u+h]\phi - S'[u]\phi\|_{L^{2}(\Omega_{\varepsilon})}
= \left\| \left( \frac{g'_{\varepsilon}(u+h)}{T[u+h]} - \frac{g'_{\varepsilon}(u)}{T[u]} \right)\phi + \left( \frac{T'[u+h]\phi}{T[u+h]} - \frac{T'[u]\phi}{T[u]} \right)g_{\varepsilon}(u+h) - \frac{T'[u]\phi}{T[u]}g_{\varepsilon}(u) \right\|_{L^{2}(\Omega_{\varepsilon})}
\leq \left\| \left( \frac{g'_{\varepsilon}(u+h)}{T[u+h]} - \frac{g'_{\varepsilon}(u)}{T[u]} \right)\phi \right\|_{L^{2}(\Omega_{\varepsilon})} + \left\| \left( \frac{T'[u+h]\phi}{T[u+h]} - \frac{T'[u]\phi}{T[u]} \right)g_{\varepsilon}(u+h) - \frac{T'[u]\phi}{T[u]}g_{\varepsilon}(u) \right\|_{L^{2}(\Omega_{\varepsilon})}
+ \left\| g_{\varepsilon}(u) \left( \frac{T'[u+h]\phi}{T[u+h]} - \frac{T'[u]\phi}{T[u]} \right) \right\|_{L^{2}(\Omega_{\varepsilon})}
= IV + V + VI.
$$

Let us estimate each term. By applying Lemma 8.11 and the mean value theorem,

$$
IV \leq \|\phi\|_{L^{2}(\Omega_{\varepsilon})} \left\{ \left\| \frac{g'_{\varepsilon}(u+h)}{T[u+h]} - \frac{1}{T[u]} \right\|_{L^{\infty}(\Omega_{\varepsilon})} + \left\| \frac{1}{T[u]} \left( g_{\varepsilon}(u+h) - g_{\varepsilon}(u) \right) \right\|_{L^{\infty}(\Omega_{\varepsilon})} \right\}
\leq C\|\phi\|_{L^{2}(\Omega_{\varepsilon})} \left\{ \|T[u+h] - T[u]\|_{L^{\infty}(\Omega_{\varepsilon})} + \|g_{\varepsilon}(u+h) - g_{\varepsilon}(u)\|_{L^{\infty}(\Omega_{\varepsilon})} \right\}
\leq C'\|\phi\|_{L^{2}(\Omega_{\varepsilon})} \left\{ \|h\|_{L^{2}(\Omega_{\varepsilon})} + \|h\|_{L^{\infty}(\Omega_{\varepsilon})} \right\},
$$

and

$$
V \leq C\|T'[u+h]\phi\|_{L^{\infty}(\Omega_{\varepsilon})} \|g_{\varepsilon}(u+h) - g_{\varepsilon}(u)\|_{L^{2}(\Omega_{\varepsilon})} \leq C\|\phi\|_{L^{2}(\Omega_{\varepsilon})} \|h\|_{L^{2}(\Omega_{\varepsilon})},
$$

$$
VI \leq C\|T'[u+h]\phi\|_{L^{\infty}(\Omega_{\varepsilon})} \|T[u]\|_{L^{\infty}(\Omega_{\varepsilon})}^2 - \|T'[u+h]\phi\|_{L^{\infty}(\Omega_{\varepsilon})} \|T[u]\|_{L^{\infty}(\Omega_{\varepsilon})}^2 - \|T'[u+h]\phi\|_{L^{\infty}(\Omega_{\varepsilon})} \|T[u+h]\phi\|_{L^{\infty}(\Omega_{\varepsilon})} + \|T[u+h]\phi\|_{L^{\infty}(\Omega_{\varepsilon})} \|T'[u+h]\phi\|_{L^{\infty}(\Omega_{\varepsilon})}
\leq C''\|\phi\|_{L^{2}(\Omega_{\varepsilon})} \|h\|_{L^{2}(\Omega_{\varepsilon})},
$$

holds for some constants $C, C' > 0$ independent of $\varepsilon$ sufficiently small. Here, we used the fact that we may assume there exists a constant $M' > 0$ independent of $\varepsilon$ such that

$$
\|T[u]\|_{L^{\infty}(\Omega_{\varepsilon})}, \|T[u+h]\|_{L^{\infty}(\Omega_{\varepsilon})} \leq M'
$$

holds as long as $u \in B_{r}(\varpi_{\varepsilon})$ and $\|h\|_{H^{2}(\Omega_{\varepsilon})} << 1$. Indeed, for example,

$$
|T[u](x)| = c_{u} \int_{-1}^{1} G_{D}(x,z)u^{2}(z)dz = \frac{1}{J_{\Omega_{\varepsilon}}} \left| \int_{\Omega_{\varepsilon}} G_{D}(x,z)u^{2}(z)dz \right| \leq CC_{T}^{2}.
$$

By these estimates, we complete the proof. □
Chapter 9

Stability analysis for general shadow systems

9.1 Introduction and main results

We consider the following general shadow system:

\[
\begin{aligned}
\frac{\partial A}{\partial t} &= \varepsilon^2 \Delta A + f(A, \xi) \text{ in } \Omega \times (0, \infty), \\
\tau \frac{\partial \xi}{\partial t} &= \frac{1}{|\Omega|} \int_{\Omega} g(A, \xi) dx \text{ in } (0, \infty), \\
\frac{\partial A}{\partial \nu} &= 0 \text{ in } (0, \infty),
\end{aligned}
\]

(9.1)

W.-M. Ni, P. Poláčik and E. Yanagida [73] proved that, for general one-dimensional shadow systems with \( \Omega = (0, 1) \), only monotone stationary solutions could be stable. Then we have a question that whether like this property holds also in higher dimension. One of the answer is given by Y. Miyamoto [63]. He showed that, for two-dimensional general shadow systems, if \( \Omega \) is a disc in \( \mathbb{R}^2 \), then a stationary solution \((A, \xi)\) which has more than two peaks at the boundary is unstable under some conditions on \( f \) and \( g \). Moreover, it was proven by F. Li, K. Nakashima and W.-M. Ni [52] that, if \( \Omega \) is convex domain in \( \mathbb{R}^N \) with smooth boundary, then a non-constant stationary solution \((A, \xi)\) is unstable for all large \( \tau \) without restrictions on \( f \) on \( g \). In addition, they showed that, if \( N \geq 2 \) and \( \Omega \) is a ball or an annulus, then a non-constant radially symmetric stationary solution \((A, \xi)\) is unstable. Inspired with their works, we will consider the instability of some solution to the general shadow system on cylindrical domain in \( \mathbb{R}^N \).

We consider the stability of weak stationary solutions \((A, \xi)\) of class \( L^\infty(\Omega) \cap H^1(\Omega) \times \mathbb{R} \) to the general shadow system (9.1) on the domain in \( \mathbb{R}^N \), \( N \geq 2 \), such that

\[(A10) \ \Omega \text{ is a cylindric domain in } \mathbb{R}^N \text{ such that } \Omega = G \times (-T, T) \text{ where } G \text{ is a smooth bounded domain in } \mathbb{R}^{N-1} \text{ and } T \in (0, \infty).\]
We assume the following conditions on $f$ and $g$.

\textbf{(A11)} $f(s, t)$ and $g(s, t)$ are of class $C^2$ on some open subset in $\mathbb{R}^2$.

\textbf{(A12)} $f_\xi < 0$ and $g_\xi < 0$ hold on $D$, and $g_A(A(x), \xi)/f_\xi(A(x), \xi)$ is independent of $x$, namely, $h(\xi) := g_A(A(x), \xi)/f_\xi(A(x), \xi)$.

Here, we say that $(A, \xi)$ is a weak stationary solution to (9.1) when it holds that

$$
\begin{cases}
\varepsilon^2 \int_\Omega \nabla A \cdot \nabla \phi dx = \int_\Omega f(A, \xi) \phi dx, \\
\int_\Omega g(A, \xi) \eta dx = 0
\end{cases}
$$

for any $\phi \in H^1(\Omega)$. Note that, under these conditions, if $(A, \xi)$ is a weak stationary solution to (9.1), then $A \in H^3(\Omega)$ and $\frac{\partial A}{\partial \nu} = 0$ on $\partial \Omega$ in the trace sense. Indeed, it is easy to see that $f(A, \xi) \in H^1(\Omega)$ under the conditions above. Hence, we can see that $A \in H^3(\Omega)$ by using the Regularity Lemma in [25]. $\frac{\partial A}{\partial \nu} = 0$ on $\partial \Omega$ is verified by the usual argument.

Let us define the word “$P_N$-symmetric” for a simplicity of statements.

\textbf{Definition 9.1.} If a function $u$ on a cylindric domain $\Omega = G \times (-T, T)$ is invariant for the reflection transform with respect to the hyperplane $P_N := \{x_N = 0\}$, then we say that $u$ is $P_N$-symmetric.

In particular, $P_N$-symmetry of a function $u$ means $u(x', x_N) = u(x', -x_N)$, $(x', x_N) \in G \times (-T, T)$.

We are ready to state an instability result.

\textbf{Theorem 9.1.} Assume (A10)-(A12). Let $(A, \xi)$ be a weak stationary solution to (9.1). If $A(x)$ is $P_N$-symmetric and $\frac{\partial A}{\partial x_N} \neq 0$, then $(A, \xi)$ is unstable for all $\tau > 0$.

\textbf{Remark 9.1.} We remark that the assumption (A12) means the conditions $p = r - 1$ and $\kappa = 0$ for the shadow system of the Gierer-Meinhardt system.

\textbf{Remark 9.2.} If $G \subset \mathbb{R}^{N-1}$ is axially symmetric with respect to the $x_{N-1}$-axis, we can construct a stationary $P_N$-symmetric solution to the shadow system of the Gierer-Meinhardt system, which has peaks on some points on the base $\partial G \times \{x_N = -1\} \cup \{x_N = T\}$. Indeed, we can use the reflection argument also on a cylindric domain as we did in Chapter 3. Hence, by a similar argument to the one in Chapters 3 and 6, we can construct multi-peak solutions on a cylindric domains.

\subsection{9.2 Proof of Theorem 9.1}

Let us give the proof of Theorem 9.1. Let $(A(x), \xi)$ be a weak stationary solution to (9.1). Let us consider the following linearized eigenvalue problem:

$$
\begin{cases}
L \phi + f_\xi(A, \xi) \eta = \lambda \phi \text{ in } \Omega, \\
\frac{1}{\Omega} \int_\Omega g_A(A, \xi) \phi + g_\xi(A, \xi) \eta dx = \tau \lambda \eta, \\
\frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial \Omega,
\end{cases}
$$
where
\[ L := \varepsilon^2 \Delta + f_A(A, \xi). \] (9.4)

It is known that, if the problem (9.3) has an eigenvalue \( \lambda \) with a positive real part, then \((A, \xi)\) is unstable.

We first consider the eigenvalue problem:
\[ \begin{cases}
L \psi = \mu \psi \text{ in } \Omega, \\
\frac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{cases} \] (9.5)

Let \( \mu_1 > \mu_2 \geq \cdots \) denote the eigenvalues of (9.5), and \( \psi_1, \psi_2, \cdots \) denote the corresponding normalized eigenfunctions, i.e.,
\[ \langle \psi_i, \psi_j \rangle := \int_{\Omega} \psi_i \psi_j \, dx = \begin{cases} 
0 \text{ if } i \neq j, \\
1 \text{ if } i = j.
\end{cases} \] (9.6)

These eigenvalues \( \{\mu_j\} \) are completely determined in [52]. Moreover, it is known that, if the second eigenvalue \( \mu_2 \) of (9.5) is positive, then there exists a positive eigenvalue \( \lambda \) for any \( \tau > 0 \), under the assumptions (A11) and (A12). This fact was established in [63]. Thus, to show Theorem 9.1 it suffices to show the following theorem.

**Theorem 9.2.** Assume (A10) and (A11). Let \((A, \xi)\) be a weak stationary solution to (9.1). If \( A(x) \) is \( P_N \)-symmetric and \( \frac{\partial A}{\partial x_N} \neq 0 \). Then the second eigenvalue \( \mu_2 \) of (9.5) is positive.

**Remark 9.3.** Theorem 9.2 do not need the assumption (A12).

We define
\[ H[\psi] := \int_{\Omega} \left( -\varepsilon^2 |\nabla \psi|^2 + f_A(A, \xi) \psi^2 \right) \, dx, \quad \psi \in H^1(\Omega). \] (9.7)

Then it is known that
\[ \mu_1 = \sup \left\{ \frac{H[\psi]}{\int_{\Omega} \psi^2 \, dx} : \psi \in H^1(\Omega) \setminus \{0\} \right\}, \] (9.8)

and for \( n \geq 1 \),
\[ \mu_{n+1} = \sup \left\{ \frac{H[\psi]}{\int_{\Omega} \psi^2 \, dx} : \langle \psi_i, \psi \rangle = 0, 1 \leq i \leq n, \psi \in H^1(\Omega) \setminus \{0\} \right\}. \] (9.9)

We need the following lemma to show Theorem 9.2.

**Lemma 9.1.** Assume (A10) and (A11). Let \((A, \xi)\) be a weak stationary solution to (9.1). Let \( A(x) \) be \( P_N \)-symmetric. For the eigenvalue problem (9.5), if \( \mu \) is a simple eigenvalue, then the corresponding eigenfunction \( \psi \) is \( P_N \)-symmetric. In particular, \( \psi_1 \) is \( P_N \)-symmetric.
Proof. Put \( \tilde{\psi}(x', x_N) := \psi(x', -x_N), \) \( x' = (x_1, \cdots, x_{N-1}) \). Then we can see that
\[
\begin{cases}
\varepsilon^2 \Delta \tilde{\psi} + f_A(A, \xi) \tilde{\psi} = \mu \tilde{\psi} & \text{in } \Omega, \\
\frac{\partial \tilde{\psi}}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\] (9.10)
holds by the symmetry of \( A \). Hence, \( \tilde{\psi} \) is also the corresponding eigenfunction for \( \mu \). Because \( \mu \) is simple, it holds that \( \tilde{\psi} = c \psi \) for some constant \( c \neq 0 \). However, noting that
\[
\psi(x', 0) = \tilde{\psi}(x', 0) = c \psi(x', 0),
\]
we see that \( c = 1 \). Therefore, \( \psi(x', x_N) = \psi(x', -x_N) \) holds.

Now, let us prove Theorem 9.2.

Proof of Theorem 9.2. Let \( A_{x_N} \) denote \( \frac{\partial A}{\partial x_N} \). By the characterization of \( \mu_2 \), it suffices to show the following:

(i) \( H(A_{x_N}) = 0 \).

(ii) \( \langle \psi_1, A_{x_N} \rangle = 0 \).

(iii) \( A_{x_N} \) is not an eigenfunction of (9.5).

We show (i). As we observed in Chapter 2, \( A \in H^3(\Omega) \). Hence, we can see that
\[
\varepsilon^2 \Delta A_{x_N} + f_A(A, \xi) A_{x_N} = 0 \quad \text{in } \Omega,
\] (9.11)
form \( \varepsilon^2 \Delta A + f(A, \xi) = 0 \). We integrate (9.11) after multiplying by \( A_{x_N} \), and use Green’s formula, then we have
\[
-\varepsilon^2 \int_{\Omega} |\nabla A_{x_N}|^2 dx + \varepsilon^2 \int_{\partial \Omega} \frac{\partial A_{x_N}}{\partial \nu} A_{x_N} d\sigma + \int_{\Omega} f_A(A, \xi) A_{x_N}^2 dx = 0.
\] (9.12)
Hence,
\[
H[A_{x_N}] = -\varepsilon^2 \int_{\partial \Omega} \frac{\partial A_{x_N}}{\partial \nu} A_{x_N} d\sigma.
\] (9.13)
Here, note that,
\[
\frac{\partial A_{x_N}}{\partial \nu} = 0 \quad \text{on } \partial G \times (-T, T),
\] (9.14)
\[
A_{x_N} = 0 \quad \text{on } G \times \{x_N = -T\} \cup \{x_N = T\}.
\] (9.15)
Thus, \( H[A_{x_N}] = 0 \) holds.

We show (ii). Note that
\[
\psi(x', x_N) A_{x_N}(x', x_N) = -\psi(x', -x_N) A_{x_N}(x', -x_N)
\]
by Lemma 9.1 and the symmetry of \( A \). Then the assertion (ii) is easily verified.

We show (iii). Suppose that the contrary is true, namely, \( A_{x_N} \) is an eigenfunction of (9.5). Then, by (9.11), the corresponding eigenvalue is 0, and it holds that
\[
\begin{cases}
\varepsilon^2 \Delta A_{x_N} + f_A(A, \xi) A_{x_N} = 0 & \text{in } \Omega, \\
\frac{\partial A_{x_N}}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\] (9.16)
However, recall the property (9.15). Then $A_{x_N} \equiv 0$ must hold by the unique continuation property. This contradicts our assumption. Thus $A_{x_N}$ is not an eigenfunction.

Acknowledgment

I would like to take this opportunity to thank my advisor, Professor Kazuhiro Kurata, for his guidance and warm encouragement. I am also indebted to Professor Daishin Ueyama of Meiji University for his useful comments and guidance about numerical simulation.
Bibliography


